Chapter 1

Introduction

1.1 History and Development of Options

Options are a class of *derivative securities* whose market value is derived from an underlying financial instrument or commodity. Other well known derivatives include, futures, swaps, and forward rate agreements (FRAs). There are two basic types of options, the *call* and *put* option. A call option gives the buyer (holder) the right but not the obligation to *buy* the underlying security at some specified time in the future known as the *expiry date* for an agreed price known as the *strike* price or *exercise* price. The expiry date is also known as the the *option maturity*. On most organized exchanges the expiry dates are set normally on the last Thursday of every month except if the day falls on a holiday. The call option seller (writer) is paid a *premium* in exchange for this option. A put option gives the holder the right but not the obligation to *sell* the underlying security for the agreed strike price at some *expiry date* in the future. Similarly the put option seller will receive a premium in exchange for the put option. The earliest options developed were called European options. The most popular kinds of options traded on most exchanges are American options. The advantage of American options over their European counterparts is the ability of the option to be exercised at any time on or before the expiry date.

Options markets and their trading can be traced back to the seventeenth century, where traders in Amsterdam used options in the trading of Dutch East India Company shares. In the early 1900s, the first options market was created by the Put and Call Brokers and Dealers Association. Associated members would link up traders who wanted to buy options on common stock with traders who wished to sell. If a seller could not be found for the buyer, or vice versa, the associate member would take the opposite position until the member could offload it. Since this market was primarily over-the-counter, it was susceptible to many risks for the option holder. Firstly, the option holder did not have the opportunity to sell the option to someone else prior to maturity of the contract. Secondly, the option sellers' commitments were guaranteed only by the broker-dealer firm. If the writer or associate member firm was unable to make its financial obligations, then this was a risk the option holder had to bear. Third, there were large transaction costs due to the high credit risks of options sellers and associate members.

In 1973, the Chicago Board of Trade (CBOT), the largest and oldest exchange developed a market devoted exclusively for options trading. This exchange known as the the Chicago Board Option Exchange (CBOE), is still the largest organised options exchange in the world. In conjunction with the options clearing house, the CBOE traded *standardised* options contracts, which offered greater liquidity and lower credit risk that resulted in lower transaction costs for participants. The American Stock Exchange (AMEX), the Philadelphia Stock Exchange (PHLX), the Pacific Stock Exchange (PSE) and the New York Stock Exchange (NYSE) followed the CBOE's lead soon after. In 1976 the Australian Stock Exchange began trading in stock and index options and continues to grow annually in terms of market turnover.

1.2 Modern Theory of Options Pricing

In the same year as the opening of the CBOE, Fisher Black and Myron Scholes published their analytical options pricing formula [5] which became the most widely accepted model to price options. Practitioners, market participants and regulators have accepted the model for valuing options for each of their individual purposes, from managing risks to speculating on the future movements of the underlying security. Simplicity was the major advantage of the Black-Scholes formula. The model depends on four basic parameters which are readily accessible or measurable by the market participants. The formula is dependent upon the exercise price and maturity date that defines the option. In addition, the current asset price, and the risk-free interest rate are the remaining two parameters. The asset price volatility is the only non-observable parameter in the Black-Scholes formula. The volatility can be estimated from a historical or implied basis. The historical volatility is calculated by measuring the annualised variance of the underlying asset's log return movements. The length of the historical time series used is a contentious issue since there are advantages and disadvantages for choosing longer (shorter) time series. Implied volatility is the more common estimate, Figlewski [23] purports that implied volatilities are superior predictors of actual future volatilities. The implied volatility method involves obtaining the variance from comparing a series of option prices of the same type and maturity but not necessarily the same exercise price, then imputing the variance iteratively using the original Black-Scholes pricing formula. Merton [46] extended the Black-Scholes model to assets that pay dividends. Cox and Ross [21] introduced another valuation method in terms of risk-neutral valuation.

1.3 American Option Pricing Models

The pricing formula developed by Black & Scholes was primarily based on *European* options, which are options that can only be exercised at maturity date. *American* options on the other hand allow the holder to exercise at any time prior to its maturity date. Thus conceptually, an American option commands a greater price than an otherwise equivalent European option. American option pricing can be categorized into two distinct approaches: the partial differential equation approach and the expectations approach.

In the celebrated paper of Black & Scholes, and the addition of Merton's modification, the governing equation for the price of a European option that has value $V(x,t)$ is given by

$$\mathcal{L}V = 0, \quad V(x,T) = f(x) \tag{1.1}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (r-q)x\frac{\partial}{\partial x} - r \tag{1.2}$$

and $f(x)$ is the European option's payoff condition at its maturity date $t = T$. Solving the second order homogeneous partial differential equation leads to a closed-form analytical solution. The partial differential equation for the European option can be modified for the American option to take into consideration of the early-exercise feature. An additional boundary condition is imposed on the American option problem, with the introduction of the *high contact* condition. At any point t between the current time and the option's maturity date, there may exist a *critical* asset price $c(t)$ such that early exercise is optimal. If the critical asset price were a known function, then the partial differential equation can be solved subject to a time-dependent boundary condition. However *a priori* $c(t)$ is unknown and must be determined as part of the solution. In this thesis the $c(t)$ term is represented as the critical asset price after a change in variable has been made, usually in the case where the governing Black-Scholes partial differential equation is transformed into the standard diffusion (heat) equation. The original variable for the critical asset price function is denoted as $S^*(t)$.

The framework for the expectations approach to options pricing was firstly introduced by Cox and Ross [21] which uses the concept of risk-neutrality. The European option price $V(x,t)$ at time t is represented as the expected payoff at the maturity date T of the option discounted by the risk-free rate of interest r (assumed constant throughout this thesis).

$$V(x,t) = e^{-r(T-t)}\mathbb{E}_{\mathcal{Q}}[f(X_T) \mid X_t = x] \tag{1.3}$$

Harrison and Kreps [26] and later Harrison and Pliska [27] consider the option valuation theory in terms of *martingale* measures. In their rigorous work, options are priced as the discounted expected payoff, the expectation is taken under a *risk-neutral measure*. It is this measure that the discounted asset price is a *martingale*. By the Feynman-Kac formula[1] it is shown to be equivalent to the partial differential equation method described briefly above. The value of an American option $V(x,t)$ at time t on the other hand, can be written as

$$V(x,t) = \max_{\tau} \mathbb{E}_{\mathcal{Q}}[e^{-r(T-\tau)}f(X_\tau) \mid X_t = x] \tag{1.4}$$

[1]See Appendices for a description of the Feynman-Kac formula

where the maximum is taken over all possible stopping times $t \leq \tau \leq T$ and $\mathcal{Q}$ is the equivalent martingale measure such that under this measure:

$$X_s = xe^{(r-\frac{1}{2}\sigma^2)(T-s)+\sigma\sqrt{T-s}Z} \tag{1.5}$$

where Z is a standard normal distribution $Z \sim \mathcal{N}(0,1)$. Bensoussan [4] as well as Karatzas [36] examined the expectation approach to provide completeness by proving that no-arbitrage will guarantee that the method will provide the fair price.

In terms of formulating the American option problem, the earliest approach was by McKean [44] who solves the option price as a *free boundary value* or *Stefan* type problem. McKean uses the theory of Kolodner [40] to arrive at an integral representation of the free boundary which in turn is used to solve the differential equation giving the price of the option. McKean's preliminary exploration of the free boundary value problem has prompted many authors to continue the development of pricing options in this framework. Notable authors include van Moerbeke [56] who proved the existence and uniqueness of McKean's solutions as an optimal stopping problem. This then led Kim [37] to derive valuation formulae for the American call and put option and examine the behaviour of the optimal exercise boundary. Carr, Jarrow and Myneni [18], Jamshidian [33] and Jacka [31] independently develop the McKean solution for the American option price $V(x,t)$, to express the solution into the value of an equivalent European option $V^E(x,t)$ and an additional *early exercise premium* $V^X(x,t)$ or

$$V(x,t) = V^E(x,t) + V^X(x,t). \tag{1.6}$$

where the early exercise premium $V^X(x,t)$ was expressed as an integral dependent on the critical asset price $c(t)$.

Other pricing methods include the linear complementarity formulation introduced by Jalliet *et al.* [32] and later Wilmott *et al.* [58] who treat the linear complementarity formulation as an "obstacle problem" used in mechanics. Barles *et al.* [2] examined the pricing of options and the behaviour of the critical asset price for times close to expiry which was then rigorously examined by Kuske and Keller [41] who gave an expression for the asymptotic solution to the integral equation representing the critical asset price which was further investigated by Chen et. al [19]. The general expression was found to be of the form

$$\alpha(\tau) = \gamma\sigma\sqrt{\tau \mid \log\tau \mid}. \tag{1.7}$$

Further asymptotic analysis was conducted by Knessl [38], [39] and later work by Evans *et al.* [22] corrected some of the earlier work by the authors mentioned and provided explicit expressions for the critical exercise boundary near the expiry date.

1.4 American Option Valuation Methods

The earliest types of numerical solutions to the free boundary value problem was conducted by Brennan and Schwartz [10] using finite difference methods to solve the partial differential equation corresponding to the price of the American put. The convergence of this method was proven by Jaillet *et al* [32]. This earlier numerical approach was improved with the use of Crank-Nicholson and Projected Successive Over Relaxation schemes that provided stable solutions. Further work was carried out by Parkinson [50] who obtains an explicit formula for the American put option. Cox, Ross and Rubinstein [20] and Rendleman and Bartter [51] produced the binomial lattice method which is used by many practitioners and academics alike as a benchmark pricing model. It will also be used for comparison with existing models as well as the new numerical pricing method to be introduced in this thesis. Quasi-analytic approximations for pricing American options have been introduced by Geske and Johnson [25], McMillan [45], Omberg [49] and Barone-Adesi and Whaley [3]. The Geske-Johnson approach provides an exact analytical solution where the compound option formula is expressed as integrals with infinite folds and is exact if the period between each option approaches zero in the limit case. The quadratic method of MacMillan and Barone-Adesi and Whaley are based on exact solutions to approximations of the Black-Scholes partial differential equation.

Broadie and Detemple [11] made comparisons of approximation techniques based on speed and accuracy while they introduced their own technique labeled the lower and upper bound approximation (LUBA). Carr and Faguet [17] introduced the analytic method of lines with the use of the 3-point Richardson extrapolation method, where the time derivative is approximated and the resulting ordinary differential equation is solved analytically at each time step. While Meyer and Van der Hoek [48] produced a similar scheme using a Ricatti transformation of the partial differential equation. Later Carr [15] uses a randomization approach, where the time to maturity is randomized to create an approximation of the American option price using the mean maturity time. Johnson [35] developed a semi-analytical form for the pricing of American options on non-dividend paying assets that was extended by Blomeyer [6] for dividends. The method worked only for options with shorter maturity dates.

With the need to calculate hedge parameters considered to be as equally important as the option price, the integral representation for the early-exercise value $V^X(x,t)$ discussed in the previous section is worthwhile to explore, due to the attractiveness of obtaining analytical expressions and potentially providing fast and accurate pricing and hedging values. Huang *et al.*[28] use the expression of Carr, Jarrow and Myneni [18] for the integral representing $V^X(x,t)$. The early exercise boundary $c(t)$ is recursively computed, where the integrand is approximated at some chosen points and then estimating the entire boundary using Richardson extrapolation to speed up the process without reducing the accuracy. In recent literature, the use of simulation by way of Monte Carlo methods has become increasingly popular, especially for the pricing of multi-asset, or high-dimensional pricing problems. The earliest work was done by Boyle [8] and Boyle *et al.* [9]. The main drawback of this method is the slowness of convergence, as it requires the simulation of a large number of stock price paths. This was partially overcome initially by Tilley's [55] "bundling algorithm" and then by Broadie and Glasserman [12] who generate estimates of the asset price based on the random samples of future sample paths and refined approximations to optimal exercise decisions. Longstaff and Schwartz [43] uses a least squares approach to estimate the conditional expected payoff for the option holder from the point of continuation of the option.

1.5 Cubic Spline Method

The main contribution of this thesis is to apply the method of Cubic Splines to the analysis of the Kolodner-McKean and Jamshidian representations for pricing the American call and put option. Both methods require the integral representation of the free boundary to be solved initially before the valuation of the option price. For clarification, the use of ‘free boundary’ can be interchanged with *critical exercise boundary* when used to value American options. The solution consists of the European option value $V^E(x,t)$ plus the early exercise value $V^X(x,t)$. It is shown that the critical exercise boundary is a monotonic increasing function for the call and a monotonic decreasing function for the put.

When evaluating the early exercise boundary, difficulties occur in producing an accurate smooth function. The binomial lattice methods of Cox, Ross and Rubinstein when applied to American options produce an early exercise boundary that looks “step-like” in nature attributed mainly to the discreteness of this particular method. When the binomial method’s number of steps is increased such that the time step is very small, the discreteness is reduced but is still prominent especially near the option’s expiry. A similar behaviour is prevalent in the finite difference approaches as this method can also be regarded as a form of lattice approach. When producing the critical exercise boundary using the semi-analytic approaches of MacMillan and Barone-Adesi and Whaley, the monotonic properties of the boundary breaks down if applied for longer maturity dated options.

In most of the approaches mentioned there are drawbacks in the computation of the critical exercise boundary or inaccuracies which restrict the practitioner from pricing a wide range of options with different maturity dates, and parameter ranges. Therefore there is a need to construct a method that produces the critical exercise boundary in a more accurate manner that correctly adheres to the monotonic property as well as producing a smooth function with respect to the time left to maturity. The use of cubic splines to evaluate the critical exercise boundary is explored as it provides not only an accurate approximation but also provide the "smoothness" of the boundary that is not available under the lattice approaches. The initial step towards the use of the cubic spline approach is to attempt to solve a free-boundary problem that has a known analytic solution. The Stefan problem in physics is such an example where the solution can be solved analytically but its free boundary can also be expressed as an integral that can be solved by iterative methods. The main advantage of the cubic spline method is that it solves the whole critical exercise boundary using just a few knot points, which adds to the efficiency. When the knot points are known, two adjacent knot points form the ends of the boundary segment that can be represented as a cubic polynomial with respect to time.

Using the cubic spline approach under the Kolodner-McKean framework involves the derivative of the critical exercise boundary $\dot{c}(t)$ in the integral solution. This is another advantage of the cubic spline approach with its ability to easily handle the first derivative. The Jamshidian solution does not require a derivative but as part of the the numerical process there is a need for both approaches to handle the behaviour of the critical exercise boundary at $t = 0$. A simple change of the time variable to $\sqrt{t}$ will introduce a singularity that will neatly handle the "correct" behaviour of the boundary. That is the behaviour of the boundary is monotonic decreasing when the risk free rate is less than the dividend $r < q$ for American puts and monotonic increasing when $r > q$ for American calls. This has been reinforced in the literature by Evans *et al.* [22] and Chen *et al.* [19]. To handle the singularity in the integral, another variable transformation is made for easier numerical implementation. It is also shown that the cubic spline method can easily handle the valuation of hedging parameters.

1.6 Numerical Comparisons

With the cubic spline method introduced, it is important to determine the relative efficiency with respect to existing numerical methods. To measure the efficiency, the cubic spline method along with other numerical methods to price American options were constructed in MATLAB. Various numerical methods discussed earlier were subject to calculating a series of option prices given a standard set of financial parameters such as the interest rate and volatility and the CPU times of each method were recorded and compared. Similarly the accuracy of each method was tested and compared. The benchmark solution to test the accuracy of each method was based on the binomial method calculated with $N = 50,000$ time steps. The results of these tests indicate that the cubic spline methods were superior in efficiency and accuracy relative to other existing methods described in the thesis.

1.7 Early Exercise Boundary

The main advantages of the cubic spline method are not only for its accuracy and efficiency of the valuation of American call and puts but the efficiency in which it determines the early exercise boundary $c(t)$. The cubic spline method can accurately evaluate the boundary for shorter to longer dated options. Unlike some of the semi-analytical approaches[2], where the accuracy is restricted for shorter dated options, and the lattice approaches where the free boundary is oscillatory and "step-like" in appearance, a simple variation of the spline knot points will provide smooth monotonic increasing(decreasing)

[2]The approximation of the boundary by Zhu [60] does not suffer from the "step-like" appearance.

function for $c(t)$. Since the boundary is represented as a spline of the third order and the derivative is of the second order, the boundary's singularity is well handled near $t = 0$. The accuracies of the cubic spline method will be explored in Chapter 8. It is intended to show that the cubic spline method will markedly improve the accuracy and efficiency of American option valuation. Assumptions such as continuous dividends, constant interest rates and constant volatility can be lifted to ascertain whether the proposed method handles variations of the assumptions.

1.8 Recent Advances in Literature

At the time of production of this thesis, Zhu [59] has produced an explicit solution for the American pricing problem. The method, based on homotopy theory, is to take the original free boundary problem and solve it under a different topology. In practice, an extra variable denoted by p where $0 \leq p \leq 1$, is introduced such that $p = 0$ corresponds to the European price, and $p = 1$ to the American price. Zhu obtains the American price as a Taylor series about $p = 0$. This is indeed an interesting innovation since it is known that there is no Taylor series in terms of the original parameters.

1.9 Scope of Work

Chapter 2 will introduce the concept of arbitrage and the relevant arbitrage boundaries associated with option pricing. The preliminary notation used throughout this thesis will also be introduced.

Chapter 3 begins with the assumptions of the Black-Scholes pricing model for European options. The Black-Scholes partial differential equation is described and then the method of solving subject to the boundary and initial conditions with the eventual presentation of the closed form analytical solution for the European call and put option. An extension to the model is applied for underlying assets that pay continuous dividends and also a symmetry relationship between the European call and put option is derived. The equivalent representation is also derived using the expectation approach under a risk-neutral martingale measure. This chapter is included in the thesis largely for completeness and to emphasize some of the differences between European and American options.

Chapter 4 describes the additional conditions applied to American options, in particular the high-contact conditions that are important for valuation. The partial differential equation for the American option is stated and the associated symmetry between the American call and put price is shown with the key symmetry relationship between the call and put critical exercise price derived. This chapter also explains the behaviour of the critical exercise price boundary close to expiry of the American option.

Chapter 5 explores the option valuation methods that are currently used or have been proposed by authors researching in this particular field. The theory of layer potentials is discussed first, then working through the fourier transforms and volume potentials of Kolodner, McKean and Jamshidian. The method of valuation by risk neutral methods follows where the binomial method or Cox, Ross and Rubinstein is used as the main pricing example. Linear complimentary formulation, quadratic approximation, analytical method of lines, method of interpolation between bounds and randomization techniques will also be described in this chapter. The chapter ends with simulation methods discussed by Tilley.

Chapter 6 introduces the numerical procedures involved with the valuation methods discussed in the previous chapter. The computational structure is set out for each method to be used as comparison for efficiency and accuracy with the cubic spline method.

Chapter 7 introduces the method of Cubic Splines, providing the background on splines and discussing the numerical procedure for pricing options in the Kolodner-McKean and Jamshidian models. The hedging parameters are also discussed.

Chapter 8 compares the efficiency and accuracy of the cubic spline method against existing pricing methods. Tabulated results are shown for particular examples, for both the American call and American put option price. The chapter also compares the methods of determining the early exercise boundaries, where it is shown that the cubic spine method is superior to the existing literature for both short and long dated options.

The thesis concludes with Chapter 9 containing a brief summary of the thesis outcomes.

The thesis contains several original contributions to the field of American option pricing and computation. These include:

- New insights into the Layer Potential method (see Chapter 5) for pricing American options and a proof (see Appendix A) that the layer potential method is equivalent to the PDE and Fourier Transform formulations.

- A new robust and high accuracy numerical method (see Chapter 7) for computing the early exercise boundary over all reasonable parameter ranges.

It is important to emphasize that this thesis is basically a comparative study of computational methods for pricing American options. The study includes comparisons of many different methods including: high speed semi-analytic approximations, low-accuracy discrete recursive methods and high accuracy continuously monitored algorithms. The cubic spline method of Chapter 7 is specifically designed to fit in the latter category, where numerical instability is endemic to the problem and therefore a serious issue. The semi-analytic approximations and discrete recursive methods while computationally efficient, are however limited in both accuracy and applicable parameter ranges.

Chapter 2

Arbitrage Boundaries

The boundaries that European and American option prices must satisfy in order to be free of arbitrage possibilities will be presented. This will include all the possible cases that apply for calls and puts with varying maturity, strike prices, risk free rates, dividends and importantly the underlying asset price. By focusing on these arbitrage boundaries, a range can be established within which an option should be priced. These conditions are important in the determination of the exact option prices that will be discussed in later chapters.

2.1 The Concept of Arbitrage

Arbitrage is simply defined as risk-free profit obtained by simultaneous purchase and sale of a tradeable asset that exists in two or more markets. More formally, an arbitrage opportunity involves a trading strategy that is *self financing*, requiring no initial investment of capital, having zero probability of negative return and yet having the possibility of a positive payoff. In the context of most American option pricing methods discussed, the theory and methodology is based on an arbitrage free market.

2.2 Notation and Preliminaries

A basic summary of the notation relevant to this chapter is:

$$
\begin{aligned}
S &= \text{Stock price} \\
K &= \text{Strike price} \\
r &= \text{annualised risk-free interest rate} \\
q &= \text{annualised continuous dividend yield} \\
Q_\tau &= \text{Discrete cash dividend at time } \tau \text{ in } T \\
T &= \text{Expiry date of the option expressed in years} \\
\tau &= \text{Time remaining to expiry} \\
t &= T - \tau = \text{Time now} \\
C(S,t) &= \text{American call price} \\
c(S,t) &= \text{European call price} \\
P(S,t) &= \text{American put price} \\
p(S,t) &= \text{European put price}
\end{aligned}
$$

A **call option** is a contract that gives the holder (*buyer*) the right to purchase the underlying security between now and the **expiry date** at an agreed price known as the option's **strike** price.

A **put option** is a contract that gives the holder the right to sell the underlying security between now and the **expiry date** at the option's **strike** price.

In exchange for holding this option, the buyer will pay the option seller (*writer*) an amount of money known as the **option premium**. The holder of a **European** option can only exercise their right at expiry but may sell

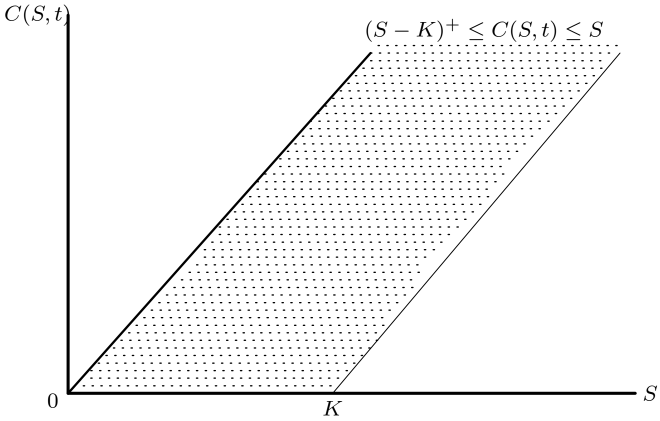


Figure 2.1: Minimum and maximum bounds for an American call

the option for a premium at any time, however for the **American** option holder, the option can be exercised and sold at any time on, or before expiry.

2.3 Call Option Boundaries

Minimum and Maximum Value of a Call

The earliest investigation into option pricing boundaries was performed in detail by Jarrow and Rudd [34]. The following details refer from their published book.

American calls should at least sell for the non-negative difference between the underlying asset price S and the strike price K. Secondly, any call option should be priced at no higher than the stock price S. That is:

$$(S-K)^+ \leq C(S,t) \leq S \ . \tag{2.1}$$

At any stage where the call price $C(S,t)$ falls outside the shaded region of figure 2.1, then an arbitrage opportunity will arise. In the case of a European call $c(S,t)$ the minimum and maximum values are given by:

$$(S - Ke^{-rt})^+ \leq c(S,t) \leq C(S,t) \leq S \ , \quad \forall \, t \leq T. \tag{2.2}$$

If the stock price exhibits *continuous* dividends at any time prior to expiration of the option then the lower bound of the European option is given by:

$$(Se^{-qt} - Ke^{-rt})^+ \leq c(S,t) \ . \tag{2.3}$$

In the case of American call options, the lower bound would simply be identical to equation (2.3). Consider the case where the value of the continuous dividend q is zero. The early exercise of American options on non-dividend paying assets will be non-optimal. By exercising the option early, the holder forgoes the delay of payment of the exercise price K at the time of expiry and equally important, the holder loses the insurance value and therefore undermines the use of the option.

Influence of Maturity

At the maturity date T of an American and European option, their values are given by:

$$C(S,T) = c(S,T) = (S - K)^+ \ . \tag{2.4}$$

The price of an American call option with maturity T_2 must be greater than, if not equal to, an American call option[1] with maturity T_1 where $T_2 > T_1$:

$$C(S,T_2) \geq C(S,T_1) \ . \tag{2.5}$$

If this inequality is violated, i.e. when $C(S,T_2) < C(S,T_1)$ then an arbitrageur could firstly buy $C(S,T_2)$ and then sell $C(S,T_1)$, to receive a possible cashflow now, and no liabilities in the future. If $C(S,T_1)$ is exercised,

[1] It is assumed that both call options refer to the same underlying asset

then the arbitrageur can cover this by exercising $C(S, T_2)$ at time T_1. For European calls the same inequality should apply:

$$c(S, T_2) \geq c(S, T_1) \ . \tag{2.6}$$

If $c(S, T_2) < c(S, T_1)$ then the arbitrageur can buy $c(S, T_2)$ and then sell $c(S, T_1)$, to obtain a positive cashflow. If $c(S, T_1)$ is exercised at its maturity to receive proceeds of $S_{T_1} - K$, the arbitrageur can sell the $c(S, t; T_2)$ option receiving proceeds worth at least $S_{T_1} - Ke^{-rT_2} > S_{T_1} - K$ or $K(1 - e^{-rT_2}) > 0$ thereby realising a positive profit.

If the stock price pays continuous or discrete dividends, applying the same arbitrage argument may not lead to a positive cash flow. For example if the stock pays a continuous annualised dividend of q, and the shorter term call $c(S, T_1)$ is exercised at maturity T_1, the minimum value of $c(S, T_2)$ is $S_{T_1} e^{-qT_2} - Ke^{-rT_2}$. If q is greater than the risk-free rate r:

$$S_{T_1} e^{-qT_2} - Ke^{-rT_2} < S_{T_1} - K \quad \text{if } q > r \ ,$$

which means that the arbitrageur cannot cover the written $c(S, T_1)$ call. If the stock pays a large discrete dividend Q_τ where $\tau < T_2$, then the minimum value of $c(S, T_2)$ may be insufficient to cover for the written $c(S, T_1)$ call:

$$S_{T_1} - Q_\tau e^{-r\tau} - Ke^{-rT_2} < S_{T_1} - K \quad \text{if } Q_\tau \text{ is large} \ .$$

Influence of the Strike Price

Suppose there are two call options on the same stock S, and with the same maturity date T, but with two different strike prices K_1 and K_2 where $K_2 > K_1$. Denote $C(K)$ and $c(K)$ to be the American and European calls respectively with an arbitrary strike price equal to K, then the following inequalities:

$$C(K_1) \geq C(K_2) \tag{2.7}$$

$$c(K_1) \geq c(K_2) \tag{2.8}$$

must hold in order to be arbitrage-free. If $C(K_1) < C(K_2)$ then an arbitrageur would buy the $C(K_1)$ option and sell the $C(K_2)$ option and is guaranteed a risk-free profit. If the written (sold) $C(K_2)$ option were to be exercised, the arbitrageur would exercise the $C(K_1)$ option and would still receive non-negative proceeds equal to the difference between the strikes K_2 and K_1, i.e

$$S - K_1 - (S - K_2) = K_2 - K_1 > 0.$$

Finally let $K_1 < K_2$ be two different strike prices for call options on the same spot price S. It can be shown that for both European and American calls:

$$\begin{array}{rcl} C(K_1) & \geq & C(K_2) \\ C(K_1) - C(K_2) & \leq & K_2 - K_1 \\ C(K) & \leq & (1-\lambda)C(K_1) + \lambda C(K_2); \quad 0 \leq \lambda \leq 1 \\ K & = & (1-\lambda)K_1 + \lambda K_2 \\ C(T_1) & \leq & C(T_2) \end{array} \tag{2.9}$$

The third inequality above implies that the call option (European or American) price is a convex decreasing function with respect to the strike price K.

2.4 Put Option Boundaries

Minimum and Maximum Values of a Put

An American put option $P(S, t)$ should sell for at least the non-negative difference between the strike K and the spot price S. Secondly, an American

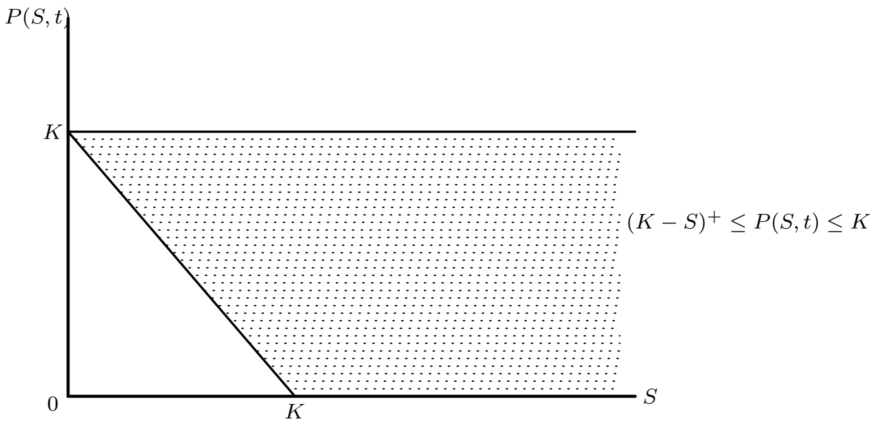


Figure 2.2: Minimum and maximum bounds for an American put

put should have a maximum value equal to its strike price K. This can be summarised as follows:

$$(K - S)^+ \le P(S,t) \le K \,. \tag{2.10}$$

For a European put $p(S,t)$, the minimum and maximum values are given by:

$$(Ke^{-rt} - S)^+ \le p(S,t) \le P(S,t) \le Ke^{-rt} \,. \tag{2.11}$$

The boundaries for a European put on an asset with continuous dividends is given by:

$$(Ke^{-rt} - Se^{-qt})^+ \le p(S,t) \,. \tag{2.12}$$

The boundary of the American put option is similar to equation (2.12). In the case of American put options on non-dividend paying assets, the gain in receiving an amount equal to the intrinsic value of the put may be more substantial than holding the option to maturity. This is particularly prevalent in the cases where the asset price S has sufficiently fallen below the put exercise price.

Influence of Maturity

American and European puts have values at maturity T given by

$$P(S,T) = p(S,T) = (K - S)^{+} \,. \tag{2.13}$$

By no-arbitrage, the price of an American put option with maturity T_2 must be greater than if not equal to an American put option with maturity T_1, where $T_2 > T_1$:

$$P(S,T_2) \geq P(S,T_1) \,. \tag{2.14}$$

However for European puts, the above inequality does not apply. Consider two European puts $p(S,T_2)$ and $p(S,T_1)$ with $T_2 > T_1$. It is also assumed that $p(S,T_2) < p(S,T_1)$ with the arbitrageur buying $p(S,T_2)$ and selling $p(S,T_1)$. If at time T_1 the $p(S,T_1)$ option is exercised, the arbitrager will be unable to cover this short position using the long $p(S,T_2)$ option, since the exercise proceeds of $p(S,T_1)$ will exceed the minimum value of the $p(S,T_2)$ option:

$$Ke^{-rT_2} - S_{T_1} < K - S_{T_1} \,.$$

If the stock price exhibits a very large leakage, then the arbitrage strategy may be successful:

$$Ke^{-rT_2} - S_{T_1}L < K - S_{T_1} \,,$$

where L could take the values of e^{-qT_2} or $\left(1 - \frac{Q_\tau}{S_{T_1}} e^{-r\tau}\right)$ for continuous or discrete dividends respectively.

Influence of the Strike Price

Analogous to section 2.3, the following inequality must hold for two American puts on the same stock and equal maturity, but with two different strike prices

K_1 and K_2 where $K_2 > K_1$:

$$P(K_2) \geq P(K_1)\,. \tag{2.15}$$

Otherwise an arbitrage possibility will arise. For instance should $P(K_2) < P(K_1)$, then an arbitrager will buy $P(K_2)$ and sell $P(K_1)$ to realise a risk-free profit. If the written put is exercised the arbitrageur is obliged to pay $K_1 - S$, which will be covered by exercising the long $P(K_2)$ put to receive proceeds of $K_2 - S$:

$$K_2 - S - (K_1 - S) = K_2 - K_1 > 0\,.$$

Inequality (2.15) should similarly apply for European put options:

$$p(K_2) \geq p(K_1)\,. \tag{2.16}$$

Similar to call options, the convexity property will apply for put options. Letting $K_1 < K_2$, be two different exercise prices for the put option and observe that:

$$\begin{array}{rcl} P(K_1) & \leq & P(K_2) \\ P(K_2) - P(K_1) & \leq & K_2 - K_1 \\ P(K) & \leq & (1-\lambda)P(K_1) + \lambda P(K_2); \quad 0 \leq \lambda \leq 1 \\ K & = & (1-\lambda)K_1 + \lambda K_2 \\ P(T_1) & \leq & P(T_2) \end{array} \tag{2.17}$$

These inequalities imply that the price for European and American put options are convex increasing functions of K.

2.5 Put-Call Parity Relations

European put option prices are related to call option prices by way of a parity equation. For non-dividend paying assets the parity relation for European options is given by:

$$p = c - S_t + Ke^{-r\tau}. \tag{2.18}$$

A simple arbitrage argument proves this parity relationship. Firstly consider Portfolio $\mathcal{A}$ to contain one European call, a cash amount $Ke^{-r\tau}$ invested at the risk-free rate of return r for the period to maturity T and finally short-selling one unit of the underlying asset S. Portfolio $\mathcal{B}$ contains only the European put option. The two portfolios can be represented as:

$$\begin{array}{rcl} \text{Portfolio } \mathcal{A} & : & c - S_t + Ke^{-r\tau} \\ \text{Portfolio } \mathcal{B} & : & p \end{array}$$

At the expiry date T of the options, both portfolios will have identical value namely:

$$\text{Portfolio } \mathcal{A}, \mathcal{B} : \begin{array}{lll} 0 & \text{if} & S_T > K \\ K - S_T & \text{if} & S_T < K \end{array}$$

Since these options are European the parity condition will hold through to expiry.

Assuming that the risk-free rate of interest r is constant, the parity relationship between an American call and put is given by an inequality rather than an equality

$$C - S_t + Ke^{-r\tau} \leq P \leq C - S_t + K. \tag{2.19}$$

For assets paying a continuous dividend yield q the relationship extends to

$$C - S_t e^{-q\tau} + Ke^{-r\tau} \leq P \leq C - S_t e^{-q\tau} + K. \tag{2.20}$$

Chapter 3

Derivation of the European Option Equation and Formulae

The framework of the Black-Scholes Model for pricing European options will be described, this includes the derivation of the partial differential equation, the analysis and formulation of the closed-formed solution commonly known as the Black-Scholes Option Pricing Formula. In addition, the extension of pricing formula will be discussed to take into consideration of assets paying a constant dividend yield.

3.1 Assumptions of the Black-Scholes Model

The key assumptions of the Black-Scholes model are:

1. **The stock price follows a log-normal random walk**

   The stock price S can be modelled by the following stochastic differential equation:

$$dS = \mu S d\tau + \sigma S dW_\tau \,, \tag{3.1}$$

   where dS is the stochastic change in the stock price S over a very small time interval τ to $\tau + d\tau$. The parameters μ and σ are the constant drift rate and volatility terms respectively, while $\{W_\tau, \tau \geq 0\}$ is a standard

Weiner process. For fixed τ, W_τ is a normally distributed random variable with zero mean and variance τ for all $\tau > 0$. To show that stock prices are log-normally distributed, a simple change of variable to $\log S$ and applying Itô's Lemma[1] yields

$$d(\log S) = \left(\mu - \tfrac{1}{2}\sigma^2\right) d\tau + \sigma dW_\tau \, . \tag{3.2}$$

After integration, equation (3.2) becomes:

$$\log \frac{S}{S_0} \sim \mathcal{N}\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right] \, , \tag{3.3}$$

where S_0 is the stock price at time $\tau = 0$.

2. **The risk-free interest rate r is a known constant**

There will be no loss of generality if the risk free rate is time varying i.e. $r(t)$, however for simplicity this assumption has been made.

3. **The asset volatility σ is a known constant**

This is the initial assumption made by Black and Scholes where the asset volatility is assumed to be unaffected by the price fluctuations of the underlying asset. Later work notably by Hull and White [29] focus on stochastic volatilities and discusses the market observed "smile" characteristics. With the constant assumption lifted, a more accurate pricing model is possible. In the stochastic volatility model, the σ is

[1]For any random variable G, say, described by the stochastic differential equation of the form

$$dG = A(G,t)dW + B(G,t)dt.$$

and any given $f(G,t)$, Itô's Lemma states that

$$df = A\frac{\partial f}{\partial G}dW + \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial G} + \tfrac{1}{2}A^2\frac{\partial^2 f}{\partial G^2}\right)dt.$$

replaced with a function $\sqrt{\nu_\tau}$ that is a measure of the variance of the asset. This variance function is itself modelled as brownian motion of the form

$$d\nu_\tau = \alpha(S,\tau)d\tau + \beta(S,\tau)dB_\tau$$

for some arbitrary functions of α and β.

4. **There are no transaction costs or taxes**
Transaction costs incorporated in the valuation model will impact the final option price. As transaction costs vary, the model can be adjusted with discretion.

5. **There are no dividends payable on the stock**
This assumption was lifted in later work by Merton[46] who modified the Black Scholes formula to account for underlying assets that pay continuous paying dividends. Later Roll[52], Geske[24] and Whaley[57] evaluated options on assets that pay a discrete dividend. This assumption is also lifted in the later chapters.

6. **Short selling is permitted without penalty**
Short selling of the underlying and the options securities is permitted on the major exchanges on the condition that a margin account is maintained.

7. **There are no riskless arbitrage possibilities**
The fundamental assumption that underlies the Black-Scholes framework.

8. **The option is European**

That is, there is no early exercise permitted.

3.2 Derivation of the Black-Scholes Partial Differential Equation

Suppose $V(S,t)$ is the value of an option which depends on the stock price S and time remaining to maturity τ. Following the derivation of Wilmott *et al.* [58] and using the assumptions of the Black-Scholes model that were previously outlined above and applying Itô's Lemma, the option value $V(S,t)$ is described by the stochastic differential equation

$$dV(S,\tau) = \sigma S \frac{\partial V}{\partial S} dW_\tau + \left(\mu S \frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial \tau} \right) d\tau \ . \tag{3.4}$$

Consider a self-financing[2] portfolio which consists of one option and a fixed number $-\Delta$ of the underlying stock S. The value of this portfolio is given by

$$\Pi = V - \Delta S \ . \tag{3.5}$$

The instantaneous change in the value of this portfolio can be described by a stochastic differential equation by combining equations (3.1) and (3.4):

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dW_\tau + \left(\mu S \frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial \tau} - \mu \Delta S \right) d\tau \ . \end{aligned}$$

Choosing

$$\Delta = \frac{\partial V}{\partial S} \tag{3.6}$$

[2]A portfolio is self-financing if the changes to its value only depends on the change in the asset prices. Harrison and Pliska [27]

will result in a portfolio value whose increment is wholly deterministic

$$d\Pi = \left(\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial \tau} \right) d\tau \, . \tag{3.7}$$

To maintain assumption 7 of no arbitrage possibilities, the right hand side of equation (3.7) must equal to the growth of an amount equivalent to Π if it were invested in a riskless asset such as a Government bond with a yield of r. That is

$$r\Pi d\tau = \left(\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial \tau} \right) d\tau \, . \tag{3.8}$$

Finally substituting equations (3.5) and (3.6) into the above equation yields the Black-Scholes partial differential equation for the value of an option

$$\frac{\partial V}{\partial \tau} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \, . \tag{3.9}$$

This is a second order linear *backward parabolic* partial differential equation. Throughout this thesis, the *forward parabolic* equations obtained by a simple change of time variable $t = T - \tau$, will be used:

$$\mathcal{B}\{V(S,t)\} \equiv \frac{\partial V}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{3.10}$$

where $\mathcal{B}\{\cdot\}$ is the Black-Scholes linear differential operator.

3.3 Boundary and Initial Conditions for European Options

Consider the boundary and initial conditions for a European call. As mentioned in the previous section, the forward parabolic Black-Scholes equation will be solved, and therefore the expiry condition for the European call becomes an *initial* condition of the form

$$c(S,0) = (S-K)^+ \, . \tag{3.11}$$

The boundary conditions for the European call are applied at the two extremes of the asset price.

When $S = 0$

$$c(0, t) = 0 \, . \tag{3.12}$$

Recall the stochastic differential equation (3.1). When S is zero then dS is also zero, and therefore the value of S will never change. If $S = 0$ at expiry, the payoff of the European call option is zero. Therefore the call option will be worthless when $S = 0$ for all $t \leq T$.

As $S \to \infty$

$$c(S, t) \sim S - Ke^{-rt} \tag{3.13}$$

When S increases to infinity, the value of the European call option will be deep in the money as at the option expiry, it will surely be exercised. The value of the European call is simply the present value of a forward contract on the same underlying asset.
For European puts, the initial condition is given by

$$p(S, 0) = (K - S)^+ \, . \tag{3.14}$$

When $S = 0$ and $S \to \infty$, the put option has value equal to

$$p(0, t) = Ke^{-rt} \, , \tag{3.15}$$

and

$$p(S, t) \to 0 \quad \text{as } S \to \infty \tag{3.16}$$

respectively.

3.4 The Black-Scholes Option Pricing Formula

The Black-Scholes partial differential equation can be solved analytically in various ways. One basic method is to transform (3.10) into the standard diffusion equation which has a known solution. The partial differential equation problem corresponding to the European call $c(S,t)$ and put $p(S,t)$ are given below:

$$\mathbb{C}_E \left\{ \begin{array}{rcl} \mathcal{B}\{c(S,t)\} & = & 0 \quad \text{in } 0 < S < \infty;\ t > 0 \\ c(S,0) & = & (S-K)^+ ; \\ c(0,t) & = & 0; \end{array} \right. \tag{3.17}$$

$$\mathbb{P}_E \left\{ \begin{array}{rcl} \mathcal{B}\{p(S,t)\} & = & 0 \quad \text{in } 0 < S < \infty;\ t > 0 \\ p(S,0) & = & (K-S)^+ ; \\ p(\infty,t) & = & 0 \end{array} \right. \tag{3.18}$$

Firstly solve the European call problem $\mathbb{C}_E$. Let

$$\mathcal{H}\{u(x,t)\} = u_t - u_{xx} \tag{3.19}$$

denote the linear heat equation operator. Define the transformations:

$$c(S,t) = e^{-rt}u(x,t) \quad \text{and } x = \left[\log \tfrac{S}{K} + (r-\hat{\sigma}^2)t\right]/\hat{\sigma}\ , \tag{3.20}$$

where $\hat{\sigma} = \frac{\sigma}{\sqrt{2}}$. Then the problem $\mathbb{C}_E$ will be transformed to a standard heat equation problem of the form

$$\mathbb{C}_E \left\{ \begin{array}{rcl} \mathcal{H}\{u(x,t)\} & = & 0 \quad \text{in } -\infty < x < \infty;\ t > 0 \\ u(x,0) & = & K(e^{\hat{\sigma}x}-1)^+ ; \\ u(-\infty,t) & = & 0 \end{array} \right. \tag{3.21}$$

It is well known that this problem has the unique solution

$$u(x,t) = \int_{-\infty}^{\infty} u(\xi,0)G(x,\xi,t)d\xi\ . \tag{3.22}$$

where

$$G(x,\xi,t) = \frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi t}} \tag{3.23}$$

is the Green's Function of the heat equation. Following Cannon [14], the solution exists provided that the $u(\xi, 0)$ term has growth conditions imposed on it. Back-transformation of equation (3.22) to the original financial variables and the use of the normal distribution function

$$\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy \ ,$$

leads to a *closed form* solution for the Black-Scholes European call option formula

$$c(S,t) = S\mathcal{N}(d_1) - Ke^{-rt}\mathcal{N}(d_2) \ , \tag{3.24}$$

where

$$d_{1,2} = \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \ . \tag{3.25}$$

By an analogous method to the one described above, the European put option problem

$$\mathbb{P}_E \begin{cases} \mathcal{H}\{u(x,t)\} &= 0 \quad \text{in } -\infty < x < \infty;\ t > 0 \\ u(x,0) &= K(1 - e^{\hat{\sigma}x})^+ \ ; \\ u(\infty,t) &= 0 \end{cases} \tag{3.26}$$

can be solved to yield the Black-Scholes European put option formula

$$p(S,t) = -S\mathcal{N}(-d_1) + Ke^{-rt}\mathcal{N}(-d_2) \ . \tag{3.27}$$

3.5 Extensions of the Formula

If assumption 5 of the Black-Scholes model is relaxed, a closed form analytical formula of the European option on an asset that pays a continuous dividend can be obtained. Let q denote the constant dividend yield, which represents the proportion of the asset price S that will be paid to the stock holder over time interval $d\tau$. This introduces a change in the stock price dynamics described by equation (3.1) into

$$dS = (\mu - q)Sd\tau + \sigma S dW_\tau \ . \tag{3.28}$$

Applying a similar riskless hedging argument to the one described in section 3.2, the differential of the portfolio $\Pi = V - \Delta S$ is given by

$$d\Pi = dV - \Delta S - q\Delta S d\tau \, . \tag{3.29}$$

The last term accounts for the fact that the short position in the asset must make full restitution for any dividends paid. To arrive at the above differential, the portfolio $\Pi(\tau; t)$; t changes value after a small time $d\tau$. This is expressed as

$$\begin{aligned} \Pi(\tau; t) &= V(\tau) - \Delta S(\tau) \\ \Pi(\tau + d\tau; t) &= V(\tau + d\tau) - \Delta S(\tau + d\tau) - q\Delta S d\tau \end{aligned} \tag{3.30}$$

Taking the difference yields equation (3.29). The choice of $\Delta = \dfrac{\partial V}{\partial S}$ removes the uncertainty, and therefore must earn the riskless rate of return $r\Pi d\tau$. Equation (3.7) is combined with $qS\dfrac{\partial V}{\partial S}d\tau$ on the right hand side with the risk-free return of the portfolio value on the left hand side yields:

$$\begin{aligned} r\left(V - S\frac{\partial V}{\partial S}\right) d\tau &= \left(\frac{\partial V}{\partial \tau} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S}\right) d\tau \\ \text{or} \qquad \frac{\partial V}{\partial \tau} &= rV - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} \, , \end{aligned} \tag{3.31}$$

which by substituting $\tau = -t$ leads to the forward parabolic partial differential equation:

$$\mathcal{B}\{V(S,t)\} \equiv \frac{\partial V}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV = 0 \, . \tag{3.32}$$

The analytical formula is solved using the same method in section 3.4 with the call and put formulas given by

$$c(S,t) = Se^{-qt}\mathcal{N}(\hat{d}_1) - Ke^{-rt}\mathcal{N}(\hat{d}_2) \tag{3.33}$$

$$p(S,t) = Ke^{-rt}\mathcal{N}(-\hat{d}_2) - Se^{-qt}\mathcal{N}(-\hat{d}_1) \tag{3.34}$$

$$\text{where} \quad \hat{d}_{1,2} = \frac{\log(S/K) + (r - q \pm \tfrac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \, .$$

The European put-call parity condition becomes

$$p(S,t) = c(S,t) - Se^{-qt} + Ke^{-rt} \,. \tag{3.35}$$

3.6 Put-Call Symmetry

Using equations (3.33) and (3.34) we consider a European call and put option with identical financial parameters S, K, r, q, σ and T.

Let $c(S,K,r,q,t)$ and $p(S,K,r,q,t)$ denote the price of the European call and put respectively. Then

$$p(S,K,r,q,t) = c(K,S,q,r,t). \tag{3.36}$$

By substituting (K,S,q,r) for the variables (S,K,r,q) in equation (3.33) and using the properties

$$\begin{aligned} d_1(K,S,q,r,t) &= \frac{\log(K/S) + (q - r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \\ &= -\frac{\log(S/K) + (r - q + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} + \sigma\sqrt{t} \\ &= -(d_1(S,K,r,q,t) - \sigma\sqrt{t}) \\ &= -d_2(S,K,r,q,t) \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} d_2(K,S,q,r,t) &= \frac{\log(K/S) + (q - r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \\ &= -\frac{\log(S/K) + (r - q - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} + \sigma\sqrt{t} \\ &= -(d_2(S,K,r,q,t) - \sigma\sqrt{t}) \\ &= -d_1(S,K,r,q,t) \end{aligned} \tag{3.38}$$

will give the desired equation. An extension of this property will be applied to American options in the next chapter.

3.7 Alternative Derivation Under a Risk Neutral Measure

An alternative method used for the derivation of option prices employs the *risk-neutral measure* or *equivalent martingale measure.* Harrison and Pliska[27] show that options are priced as the discounted expected payoff. The expectation is taken under the *risk-neutral measure.* This risk-neutral measure is the measure such that the discounted asset price is a *martingale.* Let S_t be the stock price at any time t and its risk-neutral value can be represented as

$$S_0 = e^{-rt}\mathbb{E}[S_t] \tag{3.39}$$

where r is the risk-free rate of interest. By the Feynman-Kac formula, the expectations approach is shown to be equivalent to the arbitrage free method described in section 3.2. The risk-neutral measure that is equivalent to assumption 1 described in section 3.1 is

$$dS = \mu S dt + \sigma S dW_t^{\mathcal{P}} \tag{3.40}$$

In addition, if the initial stochastic differential equation is applied under a **real-world measure** $\mathcal{P}$, then under the different **equivalent martingale measure** $\mathcal{Q}$ there will exist a different Weiner process namely:

$$dS = rSdt + \sigma S dW_t^{\mathcal{Q}} \tag{3.41}$$

The two stochastic differential equations are equivalent if and only if

$$dW_t^{\mathcal{Q}} = dW_t^{\mathcal{P}} + \frac{(\mu - r)}{\sigma} dt \tag{3.42}$$

which is possible by applying Girsanov's Theorem. Applying Itô's Lemma shows that equation (3.41) is equivalent to

$$\begin{aligned} S &= S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z} \\ Z &\sim N(0,1) \quad \text{under } \mathcal{Q} \end{aligned} \tag{3.43}$$

It can be easily shown that the discounted stock price is a martingale under this measure and therefore $S_0 = e^{-rt}\mathbb{E}[S_t]$.

For a call option, the cash flow will be the payoff at expiry $t = 0$, $C(S, 0) = [S - K]^+$. Therefore the value of the call option under the risk-neutral measure is represented as

$$C_0 = e^{-rt}\mathbb{E}\left[(S-K)^+\right] \tag{3.44}$$

It then follows from (3.43) that

$$\begin{aligned} C_0 &= \frac{e^{-rt}}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \left(S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}z} - K\right)^+ e^{-z^2/2}dz \\ &= \frac{e^{-rt}}{\sqrt{2\pi}} \int_{z=-\frac{\log(S_0/K)+(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}}^{\infty} \left(S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}z} - K\right) e^{-z^2/2}dz \end{aligned} \tag{3.45}$$

The evaluation of this integral leads to the familiar Black-Scholes formula

$$c(S,t) = S_0\mathcal{N}(d_1) - Ke^{-rt}\mathcal{N}(d_2)\ , \tag{3.46}$$

where

$$d_{1,2} = \frac{\log(S_0/K) + (r \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\ . \tag{3.47}$$

Chapter 4

American Options

4.1 American Option Characteristics

Recall that the distinctive feature of an American option is that exercise (receiving a known payoff) is permitted at any time during the life of the option. This gives the holder of an American option more rights than a holder of a European option and therefore the American option price should at least equal if not command a higher price than a European option.

American Options on a Non-Dividend Paying Asset

For a non-dividend paying asset the American call option should not be exercised early. If a holder of an American option on a non-dividend paying asset were to exercise, they would lose the time value of the option. However, the early exercise of an American put on a non-dividend paying asset may be attractive if the proceeds K received upon exercise exceeds the time value of the put.

American Options on Assets Paying a Continuous Dividend Yield

Consider the European option value $c(S,t)$ on an asset S paying a continuous dividend payment q. When the asset value is trading at a very high price, the European call option will have a high certainty of being exercised at expiry, therefore the option's value is described as follows. As $S \to \infty$, the values of the normal distribution terms $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$ will approach unity. The Black-Scholes European pricing formula (3.33) will reduce to

$$c(S,t) \sim Se^{-qt} - Ke^{-rt} \quad \text{when} \quad S \to \infty. \tag{4.1}$$

It can be deduced that the price of a European call may be below the intrinsic value when S is sufficiently high due to the e^{-qt} term. Being a European call option, the holder cannot take advantage of early exercise unlike the holder of an American call who can exercise the option at any time prior to expiry. This early exercise can be described graphically in figure 4.1. The value of the American call option $C(S,t)$ may at some critical asset price represented by $S^*(t)$, touch tangentially the line representing the intrinsic value of the call option $S - K$. The term critical asset price can be interchangeably used with other terms such as critical exercise price, or critical price throughout this thesis without loss of generality. When $S > S^*(t)$, the American call will be exercised and the value of the call option is simply $S - K$. Therefore a collection of points of $S^*(t)$ $\forall t \in (0,T]$ in the (S,t) plane will constitute the critical exercise boundary.

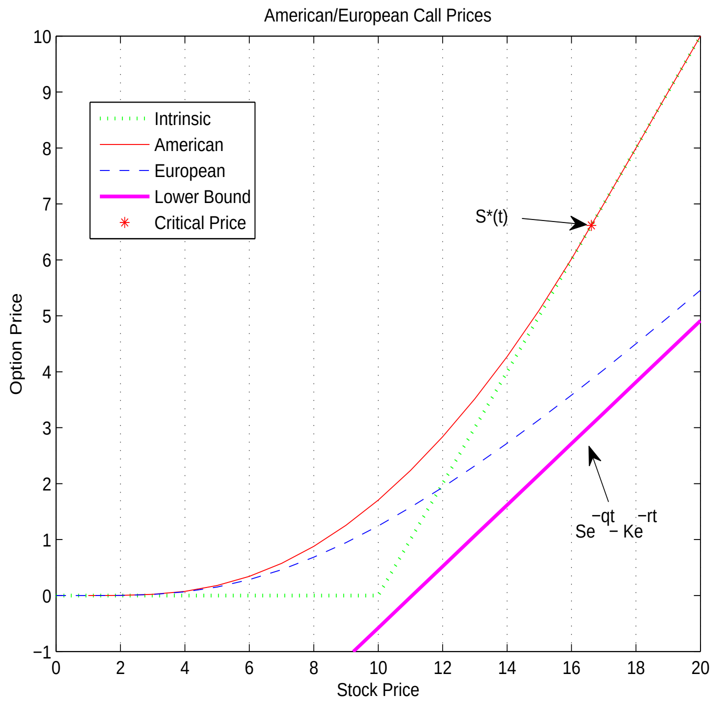


Figure 4.1: The American call option's contact condition.

4.2 High Contact Conditions for American Calls and Puts

When $S = S^*(t)$, the value of an American call option is $S^*(t) - K$. This can be defined as a known boundary condition:

$$C(S^*(t), t) = S^*(t) - K. \tag{4.2}$$

Similarly for the American put option

$$P(S^*(t), t) = K - S^*(t). \tag{4.3}$$

Using the assumptions in the beginning of chapter 3, the continuity of the asset path would imply that the path represented by the critical exercise

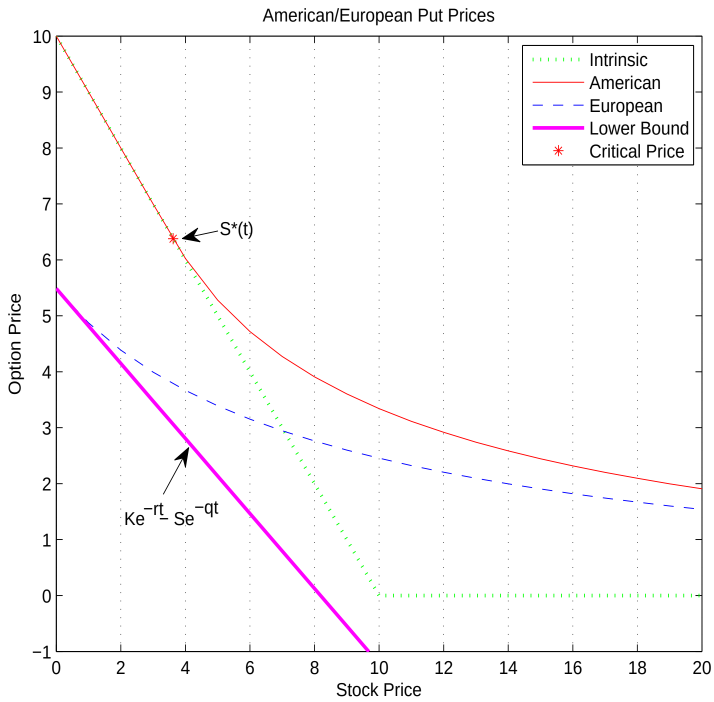


Figure 4.2: The American put option's contact condition.

boundary should also be continuous for $t > 0$.

If $S^*(t)$ were a known function, the pricing model for the American call would simply be a boundary value problem with a time dependent boundary. However, $S^*(t)$ is not known in advance and therefore must be determined as part of the solution. The continuity of the asset path implies that the solution $C(S,t)$ is also continuous. Therefore the partial derivative with respect to the asset price S (the delta) is also continuous and is of the form:

$$\frac{\partial C}{\partial S}(S^*(t),t) = 1. \tag{4.4}$$

Equation (4.4) is commonly called the *high contact condition*. A similar argument is applied to American puts. This condition was initially explored by van Moerbeke [56], who proved the continuity of the optimal exercise boundary. By the knowledge that the slope of the intrinsic value for an American put is -1, the continuity of the delta of the put at $S = S^*(t)$ is given by

$$\frac{\partial P}{\partial S}(S^*(t), t) = -1. \tag{4.5}$$

These two conditions can be interpreted as the tangential points where the value of the American option touches the intrinsic value. These intrinsic values are described equations (4.2) and (4.3). There is no more incentive to hold as the option value equals the intrinsic value without any additional early exercise premia. Thus the decision for earlier exercise can be made. A no-arbitrage condition can also be applied to arrive at the same result. This was explored by Merton [47]. Since it is optimal to exercise an American put option early if $S < S^*(t)$, if the critical asset price is less than the exercise price $S^*(t) < K$, the slope of the payoff function $(K - S)^+$ at the contact point is -1. Consider three possibilities for the slope of the option $\frac{\partial P}{\partial S}$ at the contact point $S = S^*(t)$ to be

$$\frac{\partial P}{\partial S}\begin{cases} < & -1 \\ > & -1 \\ = & -1 \end{cases}$$

Suppose the first case is true. Then as the asset price S increases from the critical asset price $S^*(t)$, the option value $P(S, t)$ drops below the payoff $(K - S)^+$ in order for the slope to be more negative. This case contradicts the arbitrage bound that $P(S, t) \geq (K - S)^+$.

Now suppose the second case is true. The option values with slope greater than -1 would be inconsistent with the Black-Scholes arbitrage free model and the constraint $P(S,t) \geq (K-S)^+$. For the option holder, they must decide how far the asset price S should fall before exercising the option. If the slope is greater than -1 at $S = S^*(t)$ the value of the option can be increased by choosing a smaller value for $S^*(t)$. This leads to an increase in the payoff value and therefore decreases the slope $\frac{\partial P}{\partial S}$. The option will be misvalued. Therefore the only correct case is when $\frac{\partial P}{\partial S} = -1$.

4.3 The American Option Partial Differential Equation

Recall the derivation argument described in chapter 3 for the Black-Scholes partial differential equation for European options. A similar argument is derived for American options beginning with equation (3.31)

$$\mathcal{B}\{V(S,t)\} \equiv \frac{\partial V}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV = 0, \tag{4.6}$$

however American calls and puts satisfy the above p.d.e. but in different domains, with different initial (expiry) conditions and different free boundary conditions. Firstly the American call satisfies the free boundary problem:

$$\mathbb{C}_A \left\{ \begin{array}{rcl} \mathcal{B}\{C(S,t)\} &=& 0 \quad \text{in } S < S^*(t);\ t > 0 \\ C(S,0) &=& (S-K)^+; \quad C(0,t) = 0 \\ C(S^*(t),t) &=& S^*(t) - K; \quad \frac{\partial C}{\partial S}(S^*(t),t) = 1 \\ C(S,t) &=& S - K \quad \text{in } S > S^*(t);\ t > 0 \end{array} \right. \tag{4.7}$$

The American put satisfies the following free boundary problem:

$$\mathbb{P}_A \left\{ \begin{array}{rcl} \mathcal{B}\{P(S,t)\} &=& 0 \quad \text{in } S > S^*(t);\ t > 0 \\ P(S,0) &=& (K-S)^+; \quad P(\infty,t) = 0 \\ P(S^*(t),t) &=& K - S^*(t); \quad \frac{\partial P}{\partial S}(S^*(t),t) = -1 \\ P(S,t) &=& K - S \quad \text{in } S < S^*(t);\ t > 0 \end{array} \right. \tag{4.8}$$

The critical stock price $S^*(t)$ is of course different for the call and put, but it can be shown that there exists a parity relation between them. This parity relation is described later in this chapter.

4.4 Optimal Exercise Boundary and Symmetry

In the previous section, we have shown that $C(S, t)$ is a continuous function of t. It has also been shown that a longer dated American call's option price curve will be above that of a shorter dated counterpart for all values of the asset price S. The longer dated American option call curve will cut the intrinsic value tangentially at a higher critical asset price than another American call with shorter expiry date. This can be expressed mathematically as follows:

Let t_1 and t_2 be two expiry dates of two American call options with all other parameters equal and $t_1 < t_2$. Then

$$S^*(t_1) < S^*(t_2). \tag{4.9}$$

Hence the optimal exercise curve is a monotonic increasing function. The monotonicity property for $S^*(t)$ can also be explained by considering that the time value lost on the strike price K is more significant for a longer dated American call. The call must be deep in-the-money for it to be exercised early and therefore the call will be exercised at a higher critical value of $S^*(t)$ than a shorter dated American call.

Let t_1 and t_2 be two expiry dates of two American put options with all other parameters equal and $t_1 < t_2$. Then

$$S^*(t_1) > S^*(t_2). \tag{4.10}$$

The optimal exercise curve of an American put is a monotonic decreasing function.

The upper bounds of an American call and the lower bounds of an American put can be determined by solving problems $\mathbb{C}$ and $\mathbb{P}$ by taking the limit as $t \to \infty$. These problems are known as perpetual option problems and can be easily solved by setting the time derivative component $\frac{\partial V}{\partial t} = 0$ and solving the resulting ordinary differential equation. By letting $B = S^*(\infty)$ and allowing $\beta = r - q - \frac{1}{2}\sigma^2$, the perpetual option solution for the American call and put are given by the explicit expressions:

$$\mathbb{C}_\infty \begin{cases} C(S) &= \begin{cases} \frac{B}{\alpha}\left(\frac{S}{B}\right)^\alpha & \text{in} \quad S < B \\ S - K & \text{in} \quad S > B \end{cases} \\ B &= \frac{\alpha K}{\alpha - 1} \\ \alpha &= [-\beta + \sqrt{\beta^2 + 2r\sigma^2}]/\sigma^2 \end{cases} \tag{4.11}$$

for the perpetual call option; and correspondingly for the put option:

$$\mathbb{P}_\infty \begin{cases} P(S) &= \begin{cases} \frac{B}{\alpha}\left(\frac{B}{S}\right)^\alpha & \text{in} \quad S > B \\ K - S & \text{in} \quad S < B \end{cases} \\ B &= \frac{\alpha K}{\alpha + 1} \\ \alpha &= [\beta + \sqrt{\beta^2 + 2r\sigma^2}]/\sigma^2 \end{cases} \tag{4.12}$$

Detailed analysis at arriving at the solution of the perpetual option can be found in the appendices.

The determination of the critical exercise boundary for an American call option can lead to determining the corresponding American put using the put-call symmetry relation. Recall the symmetry relation between European puts and calls in section 3.6. It can be shown that a similar symmetry property exists for American options

$$P(K, S, q, r, t) = C(S, K, r, q, t). \tag{4.13}$$

However, the American put-call symmetry relation does not have a clear analytic relationship that the European put and call have due to the early exercise nature of the American option. The proof of this symmetry relationship was developed by Carr and Chesney [16] as follows.

The American put option price $P(S, K, r, q, t)$ satisfies the Black-Scholes equation as described by equation (4.8). Suppose that the put price denoted by $P(K, S, q, r, t)$ also satisfies the Black-Scholes equation with the auxiliary conditions established. By the linear homogeneity property in S and K for $P(K, S, q, r, t)$ it is shown that $P(K, S, q, r, t)$ is equivalent to $SKP(\frac{1}{S}, \frac{1}{K}, q, r, t)$. By allowing $P(S', t) = P(\frac{1}{S}, \frac{1}{K}, q, r, t)$ for simplicity, the corresponding derivatives can be provided as follows:

$$S\frac{\partial}{\partial S}SKP(S',t) = SKP(S',t) - K\frac{\partial P}{\partial S'}(S',t) \tag{4.14}$$

$$S^2\frac{\partial^2}{\partial S^2}SKP(S',t) = S'K\frac{\partial^2 P}{\partial S'^2}(S',t). \tag{4.15}$$

and then observe that

$$\begin{aligned}
&\tfrac{\partial}{\partial t}SKP(S',t) - \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2}{\partial S^2}SKP(S',t) \\
&-(r-q)S\tfrac{\partial}{\partial S}SKP(S',t) + rSKP(S',t) \\
= \;& SK\left[\tfrac{\partial P}{\partial \tau}(S',t) - \tfrac{1}{2}\sigma^2 S'^2 \tfrac{\partial^2 P}{\partial S'^2}(S',t) - (q-r)S'\tfrac{\partial P}{\partial S'}(S',t) + qP(S',t)\right]
\end{aligned} \tag{4.16}$$

The bracketed term on the right-hand side is equal to zero since the price function $P(S', t)$ satisfies the Black-Scholes differential equation with the variables q and r interchanged. Therefore the price function $SKP(S', t)$ is also shown to satisfy the Black-Scholes equation with its auxiliary conditions given by

$$\begin{aligned} SKP(S',0) &= SK(\tfrac{1}{K}-\tfrac{1}{S})^+ = (S-K)^+ \\ SKP(S',t) &\geq (S-K)^+ \text{ for } t>0 \end{aligned} \tag{4.17}$$

which are identically the auxiliary conditions for the corresponding American call option $C(S,K,r,q,t)$. This concludes the proof. □

Moreover, a symmetry relationship can be established for the critical exercise prices of the American call and put option. Again by homogeneity of the call option price function, it can be expressed as a function of the American put price. Following Kwok, [42]

$$C(S,K,r,q,t) = P(K,S,q,r,t) = \frac{S}{K}P(\frac{K^2}{S},K,q,r,t). \tag{4.18}$$

The critical exercise $S_c^*(t;r,q)$ must satisfy

$$\begin{aligned} C(S,K,r,q,t) &= \frac{S}{K}P(\frac{K^2}{S},K,q,r,t) = S-K; \quad S \geq S_c^*(t,r,q) \\ C(S,K,r,q,t) &= \frac{S}{K}P(\frac{K^2}{S},K,q,r,t) > S-K; \quad S < S_c^*(t,r,q) \end{aligned}$$

which can be reexpressed as

$$\begin{aligned} P(\frac{K^2}{S},K,q,r,t) &= K-\frac{K^2}{S}; \quad S \geq S_c^*(t,r,q) \\ P(\frac{K^2}{S},K,q,r,t) &> K-\frac{K^2}{S}; \quad S < S_c^*(t,r,q). \end{aligned}$$

Now allowing $\tilde{S} = \frac{K^2}{S}$, the conditions become

$$\begin{aligned} P(\tilde{S},K,q,r,t) &= K-\tilde{S}; \quad \tilde{S} \leq \frac{K^2}{S_c^*(t,r,q)} \\ P(\tilde{S},K,q,r,t) &> K-\tilde{S}; \quad \tilde{S} > \frac{K^2}{S_c^*(t,r,q)}. \end{aligned}$$

This result implies that $\frac{K^2}{S_c^*(t,r,q)}$ is the critical exercise boundary for the American put option with an interest rate q and dividend r, namely

$$S_p^*(t;q,r) = \frac{K^2}{S_c^*(t;r,q)}. \tag{4.19}$$

There is also an importance to observe the behaviour of the American option free boundary close to expiry. Firstly by examining the behaviour of the American call, observe that when $t = 0$ and $S > K$, its terminal payoff function $C(S, 0) = S - K$, when substituted into the Black-Scholes equation yields

$$\begin{aligned} \frac{\partial C}{\partial t}(S, 0) &= (r - q)S - r(S - K) \\ &= rK - qS \quad S > K. \end{aligned} \tag{4.20}$$

For the American call to remain alive until expiry, $\frac{\partial C}{\partial t}(S, 0) \geq 0$. The value of S at which $\frac{\partial C}{\partial t}$ changes sign is $S = \frac{r}{q}K$, where this value lies in the interval $S > K$ only when $q < r$. When $q \geq r$, $\frac{r}{q}K$ will be less than K implying that the asymptotic optimal exercise price S_c^* cannot be greater the strike price K and therefore early exercise is not possible. Kim [37] deduced that the optimal exercise for American calls can be expressed as

$$\lim_{t \to 0^+} S_c^*(t) = \begin{cases} \frac{r}{q}K & q < r \\ K & q \geq r \end{cases} = \max\left(\frac{r}{q}K, K\right). \tag{4.21}$$

It can be shown that when $q = 0$, as $t \to 0^+$, the optimal exercise price $S_c^*(t) \to \infty$ for all values of t. This result is consistent with non-optimality of early exercise for American call options on non-dividend paying assets prior to expiry. Using relation (4.19), the optimal exercise boundary for the American put is shown to be a monotonically decreasing function of t

$$\begin{aligned} \lim_{t \to 0^+} S_p^*(t; r, q) &= \frac{K^2}{\lim_{t \to 0^+} S_c^*(t; r, q)} \\ &= \frac{K^2}{\max(K, \frac{q}{r}K)} \\ &= \min(K, \frac{r}{q}K). \end{aligned} \tag{4.22}$$

Similar analysis of S_p^* as $t \to \infty$ when $q = 0$ shows that the value of the put will approach K. This implies that there is always a possibility of early exercise of an American put when the underlying asset pays no dividend provided that $r \neq 0$.

4.5 Asymptotic Behaviour of the Critical Exercise Price Near Expiry

The behaviour of the critical exercise boundary $S_c^*(t)$ or $S_p^*(t)$ in the vicinity of $t = 0^+$ is an important aspect of the pricing of the American option and subsequently for the remaining sections of this thesis. For clarity, the critical exercise price can be simply represented as $S^*(t)$. McKean [44] incorrectly assumed that $S^*(0) = K$ for both the call and the put, and was later corrected by Kim [37] who showed that from the integral representations of the free boundary developed by Jamshidian [33], the free boundary value $S^*(0)$ is given by (4.21) and (4.22). The other important behaviour concerns how $S^*(t)$ approaches $S^*(0)$ in the limit. Let $S^*(t) = S^*(0)[1 \pm \alpha(t)]$ for some t near zero. The minus sign applies to the American put option while the plus sign corresponds to the American call. Barles *et al* [2] and Kuske and Keller [41] have shown that the $\alpha(t)$ is of the form

$$\alpha(t) = \gamma\sigma\sqrt{t|\log t|} \tag{4.23}$$

where γ = constant for the case where the dividend yield $q = 0$. However there are differing representations for the behaviour of the critical exercise price for the case where $q > r$ for puts and $q < r$ for calls. In order to explore the behaviour further, a local analysis of the free boundary must be made. The methodology involves an understanding of the pricing framework developed by Koldoner [40] and McKean [44] that will be described in the next chapter. Further work by Evans *et al* [22], Wilmott *et al* [58] and Knessl [38] [39] rigorously examine the behaviour of the critical exercise price $S_c^*(t)$ and

$S_p^*(t)$ by finding the asymptotic solution that is valid near expiry for cases where $r \lesseqgtr q$ and $r = q$. The analysis of these authors is beyond the scope of this thesis, however the importance of their results can be summarised below.

For the critical exercise price of American put option $S_p^*(t)$ near $t = 0$:

$$S_p^*(t) \sim K(1 - \sigma\sqrt{t\log[\sigma^2/(8\pi t(r-q)^2)]}); \quad 0 \leq q < r \tag{4.24}$$

$$S_p^*(t) \sim K(1 - \sigma\sqrt{2t\log[1/(4\sqrt{\pi}qt)]}); \quad q = r \tag{4.25}$$

$$S_p^*(t) \sim \frac{r}{q}K(1 - \sigma\alpha_0\sqrt{2t}); \quad q > r \tag{4.26}$$

where the parameter α_0 is a numerical constant that is determined by the transcendental equation

$$-\alpha_0^3 e^{\alpha_0^2}\int_{\alpha_0}^{\infty} e^{-u^2}du = \frac{1}{4}(1 - 2\alpha_0^2) \tag{4.27}$$

and has the solution

$$\alpha_0 \approx 0.4517... \tag{4.28}$$

Similarly, the behaviour of the American call option near $t = 0$ is given by the following:

$$S_c^*(t) \sim K(1 + \sigma\sqrt{t\log[\sigma^2/(8\pi(t)(r-q)^2)]}); \quad q > r \tag{4.29}$$

$$S_c^*(t) \sim K(1 + \sigma\sqrt{2t\log[1/(4\sqrt{\pi}qt)]}); \quad q = r \tag{4.30}$$

$$S_c^*(t) \sim \frac{r}{q}K(1 + \sigma\alpha_0\sqrt{2t}); \quad 0 \leq q < r \tag{4.31}$$

which has the same numerical constant α_0 that appears in the American put option.

These results above show at present that there is no uniform asymptotic representation for the early exercise boundary close to the option expiry date.

Chapter 5

Valuation Methods

In this section, various methods of solution to the American option pricing problem will be described that will form the basis of a new numerical method for pricing. This new numerical method will be discussed in chapter 7.

5.1 The American Option Partial Differential Equation Solution

Transformation to the Heat Equation

Recalling the basic equations for the American call (4.7) and American put (4.8), these equations can be converted into equivalent heat equation problems. For ease of construction, let $V(S,t)$ to denote either the American call option price $C(S,t)$ or American put option price $P(S,t)$. Let

$$\mathcal{H}\{u(x,t)\} = u_t - u_{xx} \tag{5.1}$$

denote the canonical heat equation operator. Define the transformations

$$V(S,t) = e^{-rt}u(x,t) \quad ; \; x = [\log \frac{S}{K} + (r - q - \hat{\sigma}^2)t]/\hat{\sigma} \tag{5.2}$$

where $\hat{\sigma} = \frac{\sigma}{\sqrt{2}}$. The stock price S can be expressed in terms of a new variable x, denoted by the function $S(x,t)$ where

$$S(x,t) = Ke^{\hat{\sigma}x-(r-q-\hat{\sigma}^2)t}. \tag{5.3}$$

The free boundary $S = S^*(t)$ transforms to $x = c(t)$ where $S^*(t) = S(c(t),t)$. Define domains $\mathbb{D}_c^{\pm}$ and $\mathbb{D}_c$ (see Fig. 5.1) by:

$$\mathbb{D}_c^{\pm} = \{(x,t)|x \gtrless c(t); t > 0\}; \quad \mathbb{D}_c = \mathbb{D}_c^{+} \cup \mathbb{D}_c^{-} \tag{5.4}$$

$\mathbb{D}_c$ represents all real x and $t > 0$ with the boundary $x = c(t)$ removed.

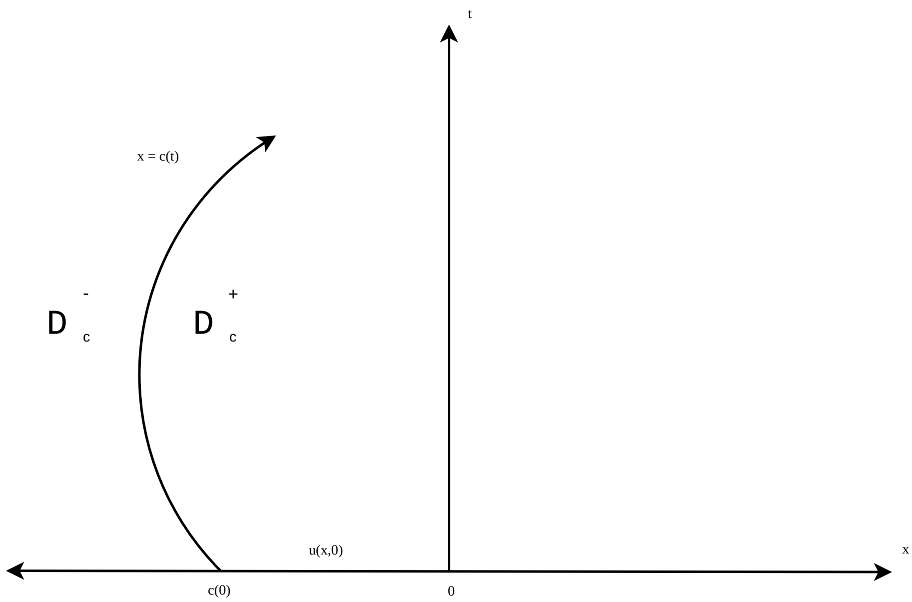


Figure 5.1: The domain defined by $\mathbb{D}_c^{\pm}$.

The American call and put problems $\mathbb{C}$ and $\mathbb{P}$ are transformed into the equivalent free boundary heat equation or Stefan-type problems:

$$\mathbb{C}_A \begin{cases} \mathcal{H}\{u(x,t)\} &= 0 \quad \text{in } \mathbb{D}_c^- \\ u(x,0) = u_0(x) &= K(e^{\hat{\sigma}x} - 1)^+ \text{ for } x < c_0 \\ u(c(t),t) = -f(t) &= e^{rt}[S(c(t),t) - K] \\ u_x(c(t),t) = -g(t) &= \hat{\sigma}e^{rt}S(c(t),t) \\ u(-\infty,t) &= 0 \quad \text{for all } t > 0 \\ u(x,t) = v(x,t) &= e^{rt}[S(x,t) - K] \quad \text{in } \mathbb{D}_c^+ \end{cases} \tag{5.5}$$

where $c(t) \geq 0$; $\forall t > 0$ for the call and

$$\mathbb{P}_A \begin{cases} \mathcal{H}\{u(x,t)\} &= 0 \quad \text{in } \mathbb{D}_c^+ \\ u(x,0) = u_0(x) &= K(1 - e^{\hat{\sigma}x})^+ \text{ for } x > c_0 \\ u(c(t),t) = f(t) &= e^{rt}[K - S(c(t),t)] \\ u_x(c(t),t) = g(t) &= -\hat{\sigma}e^{rt}S(c(t),t) \\ u(\infty,t) &= 0 \quad \text{for all } t > 0 \\ u(x,t) = v(x,t) &= e^{rt}[K - S(x,t)] \quad \text{in } \mathbb{D}_c^- \end{cases} \tag{5.6}$$

where $c(t) \leq 0$; $\forall t > 0$ is the condition of the boundary for the put. The auxiliary equations relate to the contact conditions expressed in (4.7) and (4.8) and as such the call option conditions defined by $f(t)$ and $g(t)$ will differ to that for the put. The functions $v(x,t)$ relate to the early exercise payoff and this will exist in the domain $\mathbb{D}_c^+$ for the call option and in $\mathbb{D}_c^-$ for the put.

Layer Potential Theory

The fundamental solution of the heat equation operator $\mathcal{H}$ in the domain $x \in \mathbb{R}$ is

$$G(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} H(t) \tag{5.7}$$

more commonly known as the Green's Function, where H denotes the unit step function. The two main properties of the Green's Function are:

$$\mathcal{H}\{G(x,t)\} = \delta(x)\delta(t) \quad \text{and} \quad G(x,0) = \delta(x) \tag{5.8}$$

where $\delta(\cdot)$ is the Dirac delta-function. Now define a modified Green's Function in order to remove the singularity at $t = 0$. That is

$$G_0(x,t) = \begin{cases} \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases} \tag{5.9}$$

Two special solutions of the heat equation for a regular trajectory $x = c(t)$ and an arbitrary function $\phi(t)$ are the single and double layer potentials,

$$S_c\{\phi\}(x,t) = \int_0^t G(x - c(\tau), t - \tau)\phi(\tau)d\tau \tag{5.10}$$

and

$$D_c\{\phi\}(x,t) = -\int_0^t G_x(x - c(\tau), t - \tau)\phi(\tau)d\tau. \tag{5.11}$$

Kolodner [40] used layer potential theory to solve a free boundary problem arising in physics. The theory was applied to the *change of phase* process such as melting, evaporation, recrystallization and dissolution. The similarity of this physical process with the problem of American options led McKean [44] to apply the theory to the American option free boundary problem. Kolodner defines the domain $\mathbb{D}_c$ as described in (5.4) as the union of domain $\mathbb{D}_c^+$ and the complementary domain $\mathbb{D}_c^-$. The single layer potential solution $u = S_c\{\phi\}(x,t)$ is the unique solution of the saltus problem:

$$\mathbb{S}_1 \begin{cases} \mathcal{H}\{u(x,t)\} &= 0 \quad \text{in } \mathbb{D} \\ u(x,0) &= 0; \quad u(\pm\infty, t) = 0 \\ [u]_c &= 0; \quad [u_x]_c = -\phi(t) \end{cases} \tag{5.12}$$

where $[u]_c$ denotes the jump in $u(x,t)$ across the trajectory $x = c(t)$ represented by

$$[u]_c(t) = u(c^+(t), t) - u(c^-(t), t) \tag{5.13}$$

Similarly, the double layer potential $u = D_c\{\phi\}(x,t)$ is the unique solution of the saltus problem:

$$\mathbb{S}_2 \begin{cases} \mathcal{H}\{u(x,t)\} &= 0 \quad \text{in } \mathbb{D} \\ u(x,0) &= 0; \quad u(\pm\infty, t) = 0 \\ [u]_c &= \phi(t); \quad [u_x]_c = -\dot{c}(t)\phi(t) \end{cases} \tag{5.14}$$

where $\dot{c}(t)$ is the derivative $\frac{dc}{dt}$. The single layer potential is therefore continuous across $x = c(t)$ and

$$\frac{\partial S_c \{\phi\}}{\partial x} = -D_c \{\phi\} (x, t) \tag{5.15}$$

Unlike the single layer potential, the double layer potential has a finite jump of magnitude $\phi(t)$ across $x = c(t)$. Cannon[pg. 218][14]proves that

$$D_c \{\phi\} (c^{\pm}, t) = D_0 \{\phi\} (c, t) \pm \tfrac{1}{2}\phi(t) \tag{5.16}$$

where D_0 denotes the *regularisation* of D_c defined by (5.9). The regularisation term in the single layer potential is denoted by $S_0 \{\phi\} (x, t)$. The derivative of D_c is given by the expression

$$\frac{\partial D_c \{\phi\}}{\partial x} = \phi(t)\delta(x - c(t)) - \phi_0 G(x - c_0, t) - S_c\{\dot{\phi}\} - D_c\{\dot{c}\phi\} \tag{5.17}$$

By combining equations (5.12) and (5.14), the general saltus problem:

$$\mathbb{S} \begin{cases} \mathcal{H}\{u(x,t)\} &= 0 \quad \text{in } \mathbb{D}_c \\ u(x,0) &= v_0(x); \quad u(\pm\infty, t) = 0 \\ [u]_c &= f(t); \quad [u_x]_c = g(t) \end{cases} \tag{5.18}$$

has the solution given by

$$u(x, t) = v_0(x, t) + D_c\{f\} - S_c\{\dot{c}f + g\} \tag{5.19}$$

where

$$v_0(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t) v_0(\xi) d\xi \tag{5.20}$$

arises from the non-zero initial value $u(x, 0) = v_0(x)$.

Application of Layer Potentials to American Options

With the method for solving the general saltus problem developed by Kolodner, the next step is to apply the method for solving the American option valuation problem. The similarity of the problem arising in physics and the American option problem is that both are free boundary problems and requires the solution to the heat equation subject to an unknown free boundary condition $x = c(t)$. Kolodner describes a phase change from one physical state to the next such as melting, evaporation, recrystallization and dissolution by defining the two domains $\mathbb{D}_c^+$ and $\mathbb{D}_c^-$ as the two different physical states separated by moving boundary layer $x = c(t)$. Similarly the price of an American option is determined by it's position in two regions, namely the *early exercise* region, or the *holding* region which is separated by the contact condition i.e. when $S = S^*(t)$.

American Puts

Recall that the transformed put problem $\mathbb{P}_A$ is a free boundary problem in $\mathbb{D}_c^+$ and is analogous with the general saltus problem $\mathbb{S}$ in $\mathbb{D}_c$. The condition

$$u(x,t) \equiv 0 \quad \text{in } \mathbb{D}_c^-, \tag{5.21}$$

ensures that problems $\mathbb{P}_A$ and $\mathbb{S}$ have the same solution in $\mathbb{D}_c^+$. The justification for the condition (5.21) can be described by visualizing the solution of the American put option $P(S,t)$ as the stock price reaches the contact point. Let $S^{*-}(t)$ and $S^{*+}(t)$ be the infinitesimal points on either side of the boundary $S = S^*(t)$, where the $\pm$ indicates the holding region $\mathbb{D}_c^+$ and early exercise region $\mathbb{D}_c^-$ respectively. As $S \to S^{*-}(t)$ the put option value remains a solution to $\mathcal{B}\{P(S,t)\} = 0$, where $\mathcal{B}\{\cdot\}$ is the Black-Scholes differential operator. Upon reaching $S^{*+}(t)$, the option has jumped across to the early exercise region and therefore a different pricing condition applies.

After transformation to the standard heat equation problem and its variables, the solution to $\mathbb{S}$ for all $(x,t) \in \mathbb{D}_c$ was found to be (5.19). The solution to $\mathbb{P}_A$ for all $(x,t) \in \mathbb{D}_c^+$ can easily be obtained. For condition (5.21) to be satisfied at $t=0$, the initial value $u_0(\xi) = v_0(\xi)H(\xi - c_0)$ must be set for the put option. An integral equation can be obtained for the unknown free boundary function $c(t)$ by setting $x = c^+(t)$ (or $x = c^-(t)$) using the double layer potential (5.16) and the solution to the general saltus problem $\mathbb{S}$. Using the theorem

$$D_c\{f(t)\}(c^+(t),t) = D_0\{f(t)\}(c^+(t),t) + \tfrac{1}{2}f(t) \tag{5.22}$$

and apply it to the first contact condition of the put problem in $\mathbb{P}_A$ and allowing $c = c^+(t)$, $f(t) = f$ and $g(t) = g$ for brevity

$$\begin{aligned} u(c,t) = f &= v_0(c) + \tfrac{1}{2}f + D_0\{f\}(c,t) - S_0\{\dot{c}f + g\}(c,t) \\ \tfrac{1}{2}f &= \bar{u}_0(c,t) + D_0\{f\}(c,t) - S_0\{\dot{c}f + g\}(c,t) \end{aligned} \tag{5.23}$$

where

$$u_0(x) = \bar{u}_0(x,t) = \int_{c_0}^{\infty} G(x-\xi,t)v_0(\xi)d\xi. \tag{5.24}$$

Equation 5.23 is an integral equation for the boundary function $c(t)$. Since $u_0(x) \equiv 0$ for $x > 0$ and $c_0 \leq 0$, it can be seen that $\bar{u}_0(x,t) = 0$ if $c_0 = 0$ but is non-zero if $c_0 < 0$. The derivative of the solution $u_x(x,t)$ at $x = c^\pm(t)$ can be evaluated using (5.15), (5.16), (5.17) and (5.24):

$$\begin{aligned} u_x(c,t) = g &= -f_0G(c-c_0,t) + \bar{u}_{0_x}(c,t) - [D_0\{\dot{c}f\} + \tfrac{1}{2}\dot{c}f](c,t) \\ &\quad - S_0\{\dot{f}\}(c,t) + [D_0\{\dot{c}f + g\} + \tfrac{1}{2}(\dot{c}f + g)](c,t) \end{aligned}$$

to give a second integral equation for $c(t)$ which has the form

$$\tfrac{1}{2}g + f_0G(c-c_0,t) = \bar{u}_{0x}(c,t) + D_0\{g\}(c,t) - S_0\{\dot{f}\}(c,t). \tag{5.25}$$

American Calls

The transformed call problem $\mathbb{C}$ is a free boundary problem in $\mathbb{D}_c^-$ and is associated with the general saltus problem $\mathbb{S}$ in $\mathbb{D}_c$, and

$$u(x,t) \equiv 0 \quad \text{in } \mathbb{D}_c^+. \tag{5.26}$$

A similar analysis that was applied to the American put yields the following equations for the American call beginning with the first integral representation for the free boundary $c(t)$:

$$-\tfrac{1}{2}f = \bar{u}_0(c,t) + D_0\{f\}(c,t) - S_0\{\dot{c}f + g\}(c,t), \tag{5.27}$$

and followed by the second integral representation

$$-\tfrac{1}{2}g + f_0 G(c - c_0, t) = \bar{u}_{0x}(c,t) + D_0\{g\}(c,t) - S_0\{\dot{f}\}(c,t) \tag{5.28}$$

where

$$\bar{u}_0(x,t) = \int_{-\infty}^{c_0} G(x-\xi,t)u_0(\xi)d\xi. \tag{5.29}$$

Again, since $u_0(x) \equiv \bar{u}_0(x,t)$ for $x < 0$ and $c_0 \geq 0$ we see that $u_0(x,t) = 0$ if $c_0 = 0$ and is non-zero otherwise. In both the formulations shown above, the first step is to evaluate the free boundary function $c(t)$. To evaluate the American option price, the solution $u(x,t)$ is determined then it is back-transformed to the original variables $V(S,t)$. The free boundary function $c(t)$ is also transformed back into $S^*(t)$.

Method of Fourier Transforms (McKean's Solution)

The free boundary function $c(t)$ can be obtained using the method of Fourier Transforms to solve the problems (5.5) and (5.6). This method was firstly introduced by McKean [44]. Consider the problem $\mathbb{P}$ of (5.6) and define

$$\bar{u}(x,t) = u(x,t)H(x - c(t)) \tag{5.30}$$

The Fourier Transform of $\bar{u}(x,t)$ is the complex valued function

$$\mathcal{F}\{\bar{u}\} = \hat{u}(\xi,t) = \int_{-\infty}^{\infty} \bar{u}(x,t)e^{i\xi x}dx = \int_{c(t)}^{\infty} u(x,t)e^{i\xi x}dx. \tag{5.31}$$

Let $(\hat{u})_t$ denote the t-derivative of $\hat{u}$ and $\hat{u}_t$ denote the Fourier Transform of u_t. By differentiating (5.31) with respect to t and using the property $u(c,t) = f(t)$ yields

$$\begin{aligned}(\hat{u})_t &= \int_{c(t)}^{\infty} u_t(x,t)e^{i\xi x}dx - \dot{c}(t)f(t)e^{i\xi c(t)} \\ (\hat{u})_t &= \hat{u}_t - \dot{c}(t)f(t)e^{i\xi c(t)}. \end{aligned} \tag{5.32}$$

Then integrating by parts twice the Fourier Transform of $\bar{u}_{xx}$ and using the second free boundary property $u_x(c,t) = g(t)$ leads to

$$\hat{u}_{xx} = \int_{c(t)}^{\infty} u_{xx}(x,t)e^{i\xi x}dx \tag{5.33}$$

$$= ge^{i\xi c(t)} - i\xi[u(x,t)e^{i\xi x}]_{c(t)}^{\infty} + (i\xi)^2 \int_{c(t)}^{\infty} u(x,t)e^{i\xi x}dx$$

$$\hat{u}_{xx} = -\xi^2\hat{u} + (i\xi f(t) - g(t))e^{i\xi c(t)}. \tag{5.34}$$

The problem $\mathbb{P}$ of equation (5.6) therefore has the equivalent Fourier representation:

$$\hat{\mathbb{P}}\left\{\begin{array}{rcl} (\hat{u})_t + \xi^2\hat{u} &=& [i\xi f - (\dot{c}f + g)]e^{i\xi c} \\ \hat{u}(\xi,0) &=& \hat{u}_0(\xi) = \int_{c_0}^{\infty} u_0(x)e^{i\xi x}dx \end{array}\right. \tag{5.35}$$

This problem is a first order ordinary differential equation with given initial value and integrates to the solution

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-\xi^2 t} + \int_0^t e^{-\xi^2(t-\tau)+i\xi c(\tau)}[i\xi f(\tau) - h(\tau)]d\tau \tag{5.36}$$

with $h(t) = \dot{c}(t)f(t) + g(t)$. Applying the Inverse Fourier Transform to $\hat{u}(x,t)$

$$\bar{u}(x,t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{-i\xi x}d\xi \tag{5.37}$$

$$= \tfrac{1}{4\pi}\int_0^t ds \int_{-\infty}^{\infty} e^{-\xi^2(t-\tau)-i\xi(x-c(\tau))}[i\xi f(\tau) - h(\tau)]d\xi, \tag{5.38}$$

and noting that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(t-\tau)\xi^2 - i\xi x} d\xi = G(x, t-\tau) = \frac{e^{-x^2/2(t-\tau)}}{\sqrt{4\pi(t-\tau)}} \quad (5.39)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} i\xi e^{-(t-\tau)\xi^2 - i\xi x} d\xi = -G_x(x, t-\tau) = \frac{x}{4(t-\tau)} G(x, t-\tau) \quad (5.40)$$

the solution is now of the form

$$\bar{u}(x,t) = \bar{u}_0(x,t) + \int_0^t [-h(\tau)G(x - c(\tau), t-\tau) - f(\tau)G_x(x - c(\tau), t-\tau)]d\tau \quad (5.41)$$

with $\bar{u}_0(x,t)$ given by (5.29). The solution is equivalent to

$$\bar{u}(x,t) = \bar{u}_0(x,t) - S_c\{\dot{c}f + g\}(x,t) + D_c\{f\}(x,t) \quad (5.42)$$

in complete agreement with the Kolodner's solution solved by layer potentials. It can be similarly shown that the solution to the American call option can be solved by Fourier Transforms by taking

$$\bar{u}(x,t) = u(x,t)H(c(t) - x) \quad (5.43)$$

with the solution given by

$$\bar{u}(x,t) = \bar{u}_0(x,t) + \int_0^t [h(\tau)G(c(\tau) - x, t-\tau) - f(\tau)G_x(c(\tau) - x, t-\tau)]d\tau. \quad (5.44)$$

which is also in agreement with Kolodner's solution.

Volume Potential Method

The problems $\mathbb{P}$ and $\mathbb{C}$ described by equations (5.6) and (5.5) were treated in a mathematically equivalent form by Jamshidian [33]. The author treats the American option problem as an inhomogeneous heat equation:

$$\mathbb{J}\begin{cases} \mathcal{H}\{u(x,t)\} = w(x,t) \text{ in } \mathbb{D}_c \\ u(x,0) = u_0(x); \quad u(\pm\infty, t) = 0 \\ u, u_x \quad \text{continuous at } x = c(t) \end{cases} \quad (5.45)$$

The function $u_0(x) = K(1-e^{\hat{\sigma}x})^+$ for the put options and $u_0(x) = K(e^{\hat{\sigma}x}-1)^+$ for the call. The inhomogeneous term $w(x,t)$ is given respectively by:

$$\begin{aligned} w(x,t) &= \mathcal{H}\{e^{rt}(K - S(x,t))\}H(c(t) - x) \\ &= e^{rt}[rK - qS(x,t)]H(c(t) - x) \end{aligned} \tag{5.46}$$

for the put option and

$$\begin{aligned} w(x,t) &= \mathcal{H}\{e^{rt}(S(x,t) - K)\}H(x - c(t)) \\ &= e^{rt}[qS(x,t) - rK]H(x - c(t)) \end{aligned} \tag{5.47}$$

for the call option.

For the put option, it is clearly observable that $\mathcal{H}\{u\} = 0$ in $\mathbb{D}_c^+$ and $u = e^{rt}(K - S(x,t))$ in $\mathbb{D}_c^-$ with continuity of u and u_x across the free boundary $x = c(t)$. For the call option, $\mathcal{H}\{u\} = 0$ in $\mathbb{D}_c^-$ and $u = e^{rt}(S(x,t) - K)$ in $\mathbb{D}_c^+$ with continuity of u and u_x across the free boundary $x = c(t)$. The problem $\mathbb{J}$ can be solved in the standard form of an inhomogeneous heat equation problem with a given initial value. The solution can be found using Green's functions of equation (5.7):

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G(x-\xi,t)u_0(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} G(x-\xi,t-\tau)w(\xi,\tau)d\tau d\xi \\ &= u^E(x,t) + u^X(x,t) \end{aligned} \tag{5.48}$$

where the first integral $u^E(x,t)$ corresponds to the equivalent European option, while the second double integral $u^X(x,t)$, otherwise known as the *volume potential*, relates to the value of the early-exercise premium. This decomposition of the solution into $u^E(x,t)$ and $u^X(x,t)$ had been derived independently by Carr, Jarrow and Myneni [18]. Their solution followed the method by Mckean [44] and Samuelson's [53] equilibrium framework.

Transformation to Financial Variables

The solution of the American put can be described explicitly

$$u^E(x,t) = K\int_{-\infty}^{0} G(x-\xi,t)(1-e^{\hat{\sigma}\xi})d\xi \quad (5.49)$$

$$u^X(x,t) = K\int_{0}^{t} d\tau \int_{-\infty}^{c(\tau)} G(x-\xi,t-\tau)\{re^{rt} - qe^{\hat{\sigma}\xi+(q+\hat{\sigma}^2)\tau}\}d\xi \quad (5.50)$$

Using the following result

$$\begin{aligned}\int_{-\infty}^{b} e^{a\xi}G(x-\xi,s)d\xi &= \tfrac{1}{\sqrt{4\pi s}}\int_{-\infty}^{b} e^{-[(x-\xi)^2/2s-a\xi]}d\xi \\ &= \tfrac{1}{\sqrt{4\pi s}}\int_{-\infty}^{b} e^{-[(\xi-(x+2as))^2-2(ax+a^2s)(2as)]/2s}d\xi \\ &= \tfrac{e^{ax+a^2s}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{b-x}{\sqrt{2s}}-a\sqrt{2s}} e^{-\frac{1}{2}v^2}dv \\ &= e^{ax+a^2s}\mathcal{N}[\tfrac{b-x}{\sqrt{2s}} - a\sqrt{2s}], \end{aligned} \quad (5.51)$$

the solution for $u^E(x,t)$ and $u^X(x,t)$ can be simplified in terms of normal distribution functions by using the property that $\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}dy$. The solution to the American put option can be back-transformed to original variables to obtain the following representation:

$$\begin{cases} P(S,t) = P^E(S,t) + P^X(S,t) \\ P^E(S,t) = Ke^{-rt}\mathcal{N}(-d_2) - Se^{-qt}\mathcal{N}(-d_1) \\ P^X(S,t) = \int_0^t\{rKe^{-r\tau}\mathcal{N}(-z_2) - qSe^{-q\tau}\mathcal{N}(-z_1)\}d\tau \end{cases} \quad (5.52)$$

where

$$d_{1,2}(t) = \{\log[\tfrac{S}{K}] + (r-q\pm\tfrac{1}{2}\sigma^2)t\}/\sigma\sqrt{t} \quad (5.53)$$

and

$$z_{1,2}(\tau) = \{\log[\tfrac{S}{S^*(t-\tau)}] + (r-q\pm\tfrac{1}{2}\sigma^2)\tau\}/\sigma\sqrt{\tau} \quad (5.54)$$

where $S = S^*(t)$ is the critical stock price below which early exercise of the option occurs. Now let the critical stock price of an American put option be denoted by $S = S_p^*(t)$ while $S = S_c^*(t)$ denotes the critical stock price for the American call. The term $P^E(S,t)$ denotes the Black-Scholes European put option price. The integral equation for the critical exercise boundary $S_p^*(t)$ is found by using the continuity of $P(S,t)$ and $\frac{\partial P}{\partial S}(S,t)$ at $S = S_p^*(t)$. From equation (4.3) where $P(S_p^*(t),t) = K - S_p^*(t)$, it follows that the first integral equation for $S_p^*(t)$ is given by

$$K - S_p^*(t) = P^E(S_p^*(t),t) + P^X(S_p^*(t),t). \tag{5.55}$$

It can be also shown that a second integral equation can be found using the continuity condition of the American put at $\frac{\partial p}{\partial S}(S_p^*(t),t) = -1$.

The integral representation for the American call option is given by

$$u^E(x,t) = K\int_{-\infty}^{0} G(x-\xi,t)(e^{\hat{\sigma}\xi}-1)d\xi \tag{5.56}$$

$$u^X(x,t) = K\int_0^t d\tau \int_{c(\tau)}^{\infty} G(x-\xi,t-\tau)\{qe^{\hat{\sigma}\xi+(q+\hat{\sigma}^2)\tau} - re^{r\tau}\}d\xi. \tag{5.57}$$

Using the result and by symmetry of the normal distribution function,

$$\int_b^{\infty} e^{a\xi}G(x-\xi,s)d\xi = e^{ax+a^2s}\mathcal{N}[\tfrac{x-b}{\sqrt{2s}} + a\sqrt{2s}] \tag{5.58}$$

and transforming back into the original variables yields

$$C(S,t) = C^E(S,t) + C^X(S,t) \tag{5.59}$$

where $C^E(S,t)$ is the Black-Scholes European call option price

$$C^E(S,t) = Se^{-qt}\mathcal{N}(d_1) - Ke^{-rt}\mathcal{N}(d_2) \tag{5.60}$$

and

$$C^X(S,t) = \int_0^t \{qSe^{-q\tau}\mathcal{N}(z_1) - rKe^{-r\tau}\mathcal{N}(z_2)\}d\tau \tag{5.61}$$

is the early exercise premium. The integral equation for the critical exercise boundary $S = S_C^*(t)$ is given implicitly by:

$$S_C^*(t) - K = C^E(S_C^*(t), t) + C^X(S_C^*(t), t). \tag{5.62}$$

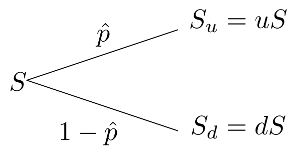
More of these analytic methods will be discussed in 7, however some other methods of pricing American options will now be reviewed for the remainder of the chapter.

5.2 Binomial Methods

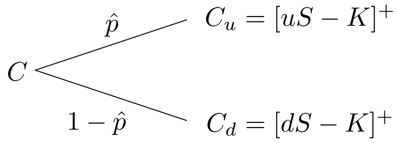
The binomial method or lattice approach to pricing American options was firstly introduced by Cox, Ross and Rubinstein [20] as well as independently by Rendleman and Bartter [51].

European Option Pricing Formula

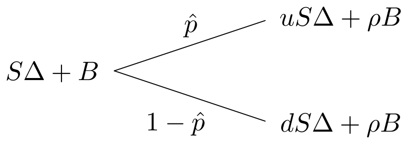
Unlike the Black-Scholes model, the binomial model is evaluated in discrete time, however it can be proven that the model converges to the Black-Scholes formula when the discrete time interval $\Delta t \to 0$.



The first assumption to consider is the stock S follows a multiplicative binomial process over discrete time periods. Thus at time t the stock at the end of one discrete time period $t + \Delta t$ will either become uS with probability $\hat{p}$ or dS with probability $1 - \hat{p}$ under the *real-world measure*. The second assumption is that the interest rate r is constant, and that the inequality condition $u > 1 + r\Delta t > d$ holds, where u and d are the up and down factors respectively. To value a call option in the one period model, let C be the current value of the call, C_u be the call value after one time period if the stock price moves up to uS and C_d be its value after one time period if the stock price moves down to dS. Define $\rho = 1 + r\Delta t$ to be the risk free future value factor after time period Δt. The payoffs of the call option after one period are shown below:



Now suppose a portfolio consisting of Δ shares of stock and the dollar amount B in riskless bonds. The cost of this portfolio is $S\Delta + B$. At the end of one period the value of the portfolio becomes



Selecting the values of the end period portfolios to be equal to the value of the call for each possible outcome will require

$$\begin{aligned} uS\Delta + \rho B &= C_u \\ dS\Delta + \rho B &= C_d \end{aligned} \tag{5.63}$$

Solving these equations for Δ and ρB yields

$$\begin{aligned} \Delta &= \frac{C_u - C_d}{(u-d)S} \\ \rho B &= \frac{uC_d - dC_u}{(u-d)} \end{aligned} \tag{5.64}$$

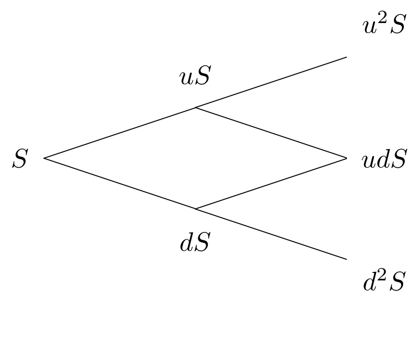
Under an arbitrage free conditions, the current value of the call C must be no less than the current value of the equivalent portfolio, $S\Delta + B$. Otherwise one can take advantage of an arbitrage opportunity by buying the call option and selling the portfolio. Similarly the value of the call C must be no more than the value of the portfolio otherwise another arbitrage opportunity will arise. Then it must be true that

$$\begin{aligned} C &= S\Delta + B \\ &= \frac{C_u - C_d}{(u-d)} + \frac{uC_d - dC_u}{(u-d)\rho} \\ &= \left[\left(\frac{\rho - d}{u-d} \right) C_u + \left(\frac{u-\rho}{u-d} \right) C_d \right] / \rho. \end{aligned} \tag{5.65}$$

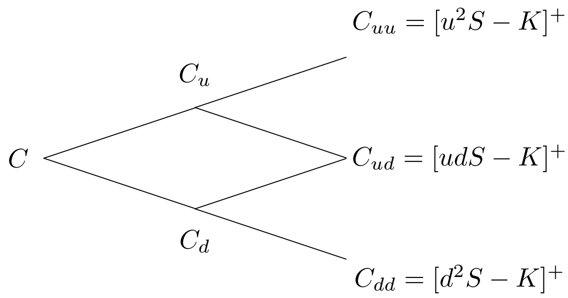
By defining the coefficients (known as the *risk-neutral probabilities*) of C_u and C_d in equation (5.65) to be $p \equiv (\rho - d)/(u-d)$ and $1 - p = (u - \rho)/(u - d)$ then the value of the 1-period call option can be written as

$$C = [pC_u + (1-p)C_d]/\rho \tag{5.66}$$

It can be seen that the formula is completely independent of the probability $\hat{p}$. This shows that investor estimates of directional (i.e. up/down) probability are not required in pricing and the only important determinants are the underlying price S and the risk-free rate r. Now consider a call with two periods remaining to expiry. Following the standard binomial process, the stock can now take three possible values after two periods.



Then the 2-period stock tree for the call is given by



After one time period the calculation of values C_u and C_d can be done using similar analysis as the one-period model. These two values are represented by

$$C_u = [pC_{uu} + (1-p)C_{ud}]/\rho \tag{5.67}$$

$$C_d = [pC_{du} + (1-p)C_{dd}]/\rho \tag{5.68}$$

A portfolio of $S\Delta$ stock and B bonds that will have a one time period value equal to C_u if the stock price moves to uS or C_d if the stock price moves to dS. Substituting the representation of C_u and C_d into (5.66) to obtain the value of a two period call option

$$C = [p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}]/\rho^2. \tag{5.69}$$

It follows that the valuation of the 2-period model is similar to the 1-period case with the use of determinants S, K, u, d and ρ. A recursive procedure can be developed to determine the value of a call option with n remaining time periods. Starting from the expiration date and working recursively backwards the general valuation formula for any n is given by:

$$C = \rho^{-n}\left\{\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}[u^jd^{n-j}S - K]^+\right\} \tag{5.70}$$

where

$$\left(\begin{array}{c} n \\ j \end{array} \right) = \frac{n!}{j!(n-j)!} \tag{5.71}$$

are the usual binomial coefficients.

Extension to American Options

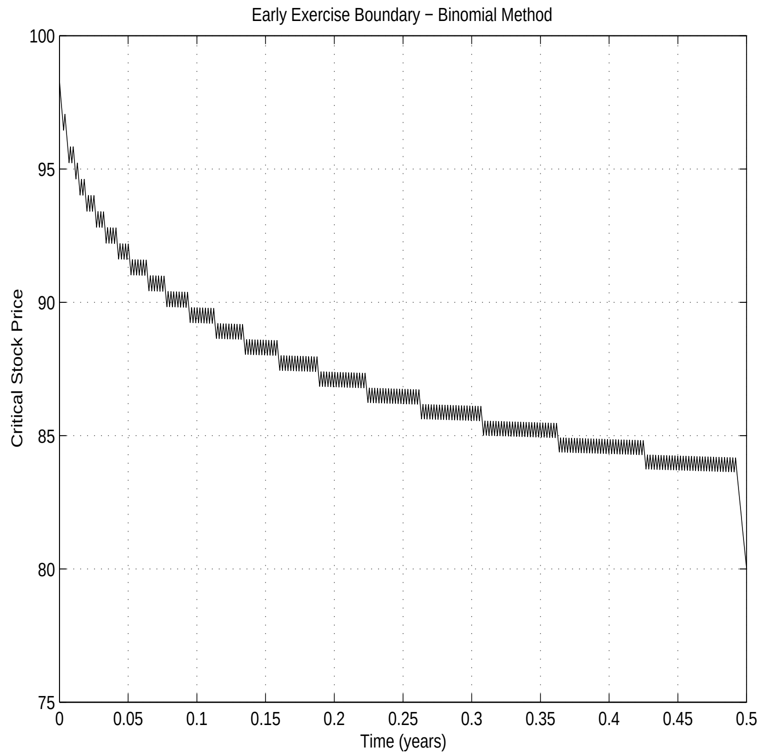


Figure 5.2: The discrete nature of the exercise boundary evaluated under the binomial method.

The early exercise feature of American option can be added to the valuation of the binomial method. A recursive procedure can be constructed for both the American call and put. At each time node beginning with the expiry date, the following formulas can be computed:

$$C^{i+1} \quad = \quad \max[S^{i+1} - K, \tfrac{1}{\rho}(pC_u^i + (1-p)C_d^i)] \tag{5.72}$$

$$P^{i+1} \quad = \quad \max[K - S^{i+1}, \tfrac{1}{\rho}(pP_u^i + (1-p)P_d^i)]. \qquad (5.73)$$

This will ensure that the call and put prices will always remain greater than their intrinsic values. It is optimal to hold the option at expiry at any time node if the discounted expected value of the option's payoff is greater than the intrinsic value. If the intrinsic value of the call at any node point is equal to $(S^i - K)$, then the call should be exercised. Similarly the put option will be exercised if the value equals to $(K - S^i)$. The calculation at each node point will allow the formation of a critical exercise boundary albeit being discrete and when plotted against time left to expiry will appear "step-like" in nature. This non-smoothness problem can be corrected as discussed by Zhu and Francis [61]. Figure 5.2 is an example of the discrete behaviour of the critical exercise boundary of an American put option found by binomial methods.

5.3 Linear Complementarity Formulation

Recall from equation (4.2), the stock price S reaches the optimal exercise boundary when $C(S^*(\tau), \tau) = S^*(\tau) - K$, the partial differential equation of the American option as described by Jamshidian [33] can be represented as

$$\begin{aligned} \mathcal{B}\{C(S,t)\} &= 0 && \text{for} \quad S \leq S^*(t);\ t > 0 \quad \text{(holding region)} \\ C(S,t) &= S - K && \text{for} \quad S \geq S^*(t);\ t > 0 \quad \text{(exercise region)} \end{aligned} \qquad (5.74)$$

where $\mathcal{B}\{\cdot\}$ is the Black-Scholes partial differential operator. Define a riskless hedging portfolio consisting of a bought call option and a short position in the underlying asset of quantity ΔS. As an equation, this is represented as

$$\Pi = C - \Delta S = C - \frac{\partial C}{\partial S} S. \qquad (5.75)$$

The American option problem can be solved by initially showing that it is equivalent to

$$d\Pi \leq r\Pi dt. \tag{5.76}$$

This means that the American call option is optimally exercised when the rate of return from the hedging portfolio is less than the riskless interest rate. In addition, when the asset price S is less than the critical price $S^*(\tau)$ the value of the American call satisfies the Black-Scholes equation. Therefore with this knowledge, the condition (5.74) can be used to re-express the Black-Scholes partial differential equation as an inequality

$$\mathcal{B}\{C(S,t)\} \geq 0, \quad S > 0. \tag{5.77}$$

However, the American call option will have a value above the intrinsic value $S - K$ when $S < S^*(\tau)$ and will be equal to the intrinsic value when $S \geq S^*(\tau)$, that is

$$C(S,\tau) \geq S - K, \quad S > 0. \tag{5.78}$$

Combining these two inequalities into a single equation yields a linear complementarity problem for the American call option

$$\begin{cases} \mathcal{B}\{C(S,t)\} \cdot [C - (S - K)] = 0, \ S > 0 \\ \mathcal{B}\{C(S,t)\} \geq 0, \quad C(S,\tau) \geq S - K \end{cases} \tag{5.79}$$

And similarly for the put option

$$\begin{cases} \mathcal{B}\{P(S,t)\} \cdot [P - (K - S)] = 0, \ S > 0 \\ \mathcal{B}\{P(S,t)\} \geq 0, \quad P(S,\tau) \geq K - S \end{cases} \tag{5.80}$$

There exists special computational methods to solve linear complementarity problems. It will not be pursued further in this thesis but has been discussed for the sake of a more complete exposition.

Computational Method

The numerical method to solve linear complementarity problems involve the use of finite differences. The basic methodology comprehensively described by Wilmott *et al.* [58] is to replace the partial derivatives by finite-difference expansions. Furthermore a matrix inversion technique can be applied to solve the complementarity problem when solved by LU Decomposition techniques. The difficulties faced with the method of finite differences is due to the stability and efficiency of the standard implicit and explicit finite difference schemes. The stability problem occurs due to the use of finite precision computer arithmetic to solve the difference equations. Rounding errors are introduced into the numerical solution. For improved efficiency, LU Decomposition or Successive-Over-Relaxation (SOR) techniques are applied to improve convergence. A more comprehensive explanation of finite difference methods will be discussed in the next chapter.

5.4 Quadratic Approximation

McMillan [45] and Barone-Adesi & Whaley [3] developed this method to price American options.

Methodology

McMillan deduced the approximate value of an option which has a constant dividend q and interest rate r. The governing equation for this model is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV. \tag{5.81}$$

With the volatility of the asset price given by σ. Define $e(S,\tau)$ to be the early exercise premium of a call option expressed as

$$e(S,\tau) = C(S,\tau) - c(S,\tau). \tag{5.82}$$

The governing p.d.e (5.81) will hold for both $C(S,\tau)$ and $c(S,\tau)$. Since the differential equation is linear, the exercise premium $e(S,\tau)$ will also satisfy the p.d.e. Let $k_1 = \frac{2r}{\sigma^2}$ and $k_2 = \frac{2(r-q)}{\sigma^2}$ and define

$$e(S,\tau) = Q(\tau)f(S,Q). \tag{5.83}$$

Equation (5.81) can be transformed into

$$S^2\frac{\partial^2 f}{\partial S^2} + k_2 S\frac{\partial f}{\partial S} - k_1 f\left[1 + \frac{1}{rQ}\frac{dQ}{d\tau}\left(1 + \frac{Q}{f}\frac{\partial f}{\partial Q}\right)\right] = 0. \tag{5.84}$$

Choosing $Q(\tau) = 1 - e^{r\tau}$ and substituting into equation (5.84)

$$S^2\frac{\partial^2 f}{\partial S^2} + k_2 S\frac{\partial f}{\partial S} - \frac{k_1}{Q}\left[f + (1-Q)Q\frac{\partial f}{\partial Q}\right] = 0 \tag{5.85}$$

where the factor $(1-Q)Q$ becomes zero at $\tau = 0$ and $\tau \to \infty$. The factor will have a maximum value of $1/4$ at $Q = 1/2$. If the last term containing $(1-Q)Q$ in the previous equation is dropped, the equation can be transformed to an ordinary differential equation. Therefore assuming that $Q > 0$, the approximate equation for f now becomes

$$S^2\frac{\partial^2 f}{\partial S^2} + k_2 S\frac{\partial f}{\partial S} - \frac{k_1}{Q}f = 0 \quad Q > 0 \tag{5.86}$$

The general solution for $f(S)$ is given by

$$f(S) = w_1 S^{y_1} + w_2 S^{y_2}, \tag{5.87}$$

where w_1 and w_2 are arbitrary constants and y_1 and y_2 are roots of the auxiliary equation

$$y^2 + (k_2 - 1)y - \frac{k_1}{Q} = 0 \tag{5.88}$$

Solving the above quadratic equation

$$y_1 = -\frac{1}{2}\left[(k_2 - 1) + \sqrt{(k_2-1)^2 + 4\frac{k_1}{Q}}\right] < 0 \tag{5.89}$$

$$y_2 = \frac{1}{2}\left[-(k_2 - 1) + \sqrt{(k_2-1)^2 + 4\frac{k_1}{Q}}\right] > 0 \tag{5.90}$$

The negative root y_1 may be discarded, as by definition one must have $f(S) \rightarrow 0$ as $S \rightarrow 0$. Therefore the approximate value of the option is given by

$$C(S,\tau) \approx \widetilde{C}(S,\tau) = c(S,\tau) + w_2 Q S^{y_2}. \tag{5.91}$$

Finally the arbitrary constant w_2 is determined by using the boundary condition at $S = S^*$ which is $\widetilde{C}(S^*,\tau) = S^* - K$. However since S^* is yet to be determined, another important condition of the problem must be used. This is the high-contact condition $\frac{\partial \widetilde{C}}{\partial S}(S^*,\tau) = 1$. Therefore the following equations can be solved

$$S^* - K = c(S^*,\tau) + w_2 Q S^{y_2} \tag{5.92}$$

$$1 = e^{-q\tau}\mathcal{N}(d_1(S^*)) + w_2 y_2 Q S^{y_2 - 1} \tag{5.93}$$

where

$$d_1(S^*) = \frac{\log S^*/K + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}. \tag{5.94}$$

The constant w_2 is removed by elimination in equations (5.92) and (5.93). A non-linear algebraic equation for $S^*(\tau)$ is expressed as

$$S^* - K = c(S^*,\tau) + \left[1 - e^{-q\tau}\mathcal{N}(d_1(S^*))\right]\frac{S^*}{y_2}. \tag{5.95}$$

It follows that for $q > r$, the approximate value of an American commodity call option is given by

$$\widetilde{C}(S,\tau) = c(S,\tau) + \frac{S^*}{y_2}\left[1 - e^{-q\tau}\mathcal{N}(d_1(S^*))\right]\left(\frac{S}{S^*}\right)^{y_2}, \quad S < S^* \tag{5.96}$$

The value of S^* is found by solving (5.95). The second term of (5.96) is known as the early exercise premium which is positive for $q > r$. When $q \leq r$, the American call will never be exercised and therefore its value is simply the value of the equivalent European option. A similar derivation yields the value of the American put option given by

$$\widetilde{P}(S,\tau) = p(S,\tau) - \frac{S^*}{y_1}\left[1 - e^{-q\tau}\mathcal{N}(-d_1(S^*))\right]\left(\frac{S}{S^*}\right)^{y_1}, \; S > S^* \tag{5.97}$$

Similarly the critical asset price $S^*(\tau)$ is found by solving the following non-linear equation

$$K - S^* = p(S^*,\tau) + \left[1 - e^{-q\tau}\mathcal{N}(-d_1(S^*))\right]\frac{S^*}{y_1} \tag{5.98}$$

Computational Method

As described above, the value of the critical asset price $S^*(\tau)$ must be solved first in order to price the American option. Barone-Adesi and Whaley described an algorithm to find the critical exercise price S^* using a non-linear scheme. More details will follow in the next chapter.

5.5 Analytical Method of Lines

The analytic method of lines of Carr and Faguet [17] begins with a semi-discretization of the $\frac{\partial P}{\partial \tau}$ of the Black-Scholes partial differential equation by approximation using a finite difference operator. Meyer and Van der Hoek [48] improve the efficiency of Carr and Faguet's method. Meyer and Van der Hoek's method begins with the Black-Scholes partial differential equation for the American put with the option price represented as $P(S,\tau)$ and the asset price S normalized by the exercise price K, namely $V = P/K$ and $x = S/K$ with

$$\frac{\partial V}{\partial \tau} = \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r-q)x\frac{\partial V}{\partial x} - rV \tag{5.99}$$

subject to the following initial and boundary conditions:

$$\left\{\begin{array}{rcll} s(0) & = & \min\left[\frac{r}{q},1\right] & s(\tau):\ 0\le\tau\le T \\ V(x,0) & = & (1-x)^+ & x\ge 0 \\ V(x,\tau) & = & 1-x & \text{if}\quad 0\le x\le s(\tau) \\ \frac{\partial V}{\partial x}(s(\tau),\tau) & = & -1 & 0\le\tau\le T \\ V(x,\tau) & > & 1-x & \text{if}\quad x>s(\tau) \end{array}\right. \tag{5.100}$$

Implementation of MOL

Using the initial condition $V(x,0) = (1-x)^+$ known for all $x \geq 0$, the time interval $0 \leq \tau \leq T_{max}$ can be discretized into n time-steps of size $\Delta\tau = T_{max}/n$. Write each time-step as $\{\tau_0, \tau_1, \cdots, \tau_n\}$ with $\tau_0 = 0, \tau_n = T$. Then at τ_n allow the value of the option $V(x,0) = (1-x)^+$. In order to determine $V(x,\tau_{n-1})$, let $u(x) \simeq V(x,\tau_n)$ and define the backward difference approximation

$$\frac{V(x,\tau_j) - V(x,\tau_{j-1})}{\Delta\tau} \simeq \frac{\partial V}{\partial\tau}(x,\tau_j); \quad 1 \leq j \leq n. \tag{5.101}$$

Equation (5.99) can be approximated by

$$\mathcal{L}u(x) = g(x) \tag{5.102}$$

where

$$\mathcal{L}u(x) \equiv \tfrac{1}{2}\sigma^2 x^2 u''(x) + (r-q)xu'(x) - (r + \tfrac{1}{\Delta\tau})u(x) \tag{5.103}$$

with

$$g(x) = \frac{-V(x,\tau_{n-1})}{\Delta\tau} \quad \text{for} \quad x > s(\tau) \tag{5.104}$$

Now let $\bar{s} \equiv s_n$ be the critical price at time τ_n. Therefore the solution for u must be determined subject to

$$\left\{\begin{array}{rcl} u(\bar{s}) & = & 1-\bar{s} \\ u'(\bar{s}) & = & -1 \\ u(x) & \to & 0 \quad \text{as} \quad x\to\infty \\ u(x) & = & 1-x \quad \text{for} \quad 0\le x\le\bar{s} \end{array}\right. \tag{5.105}$$

Two solutions need to be determined in this problem. Firstly the critical exercise price $\bar{s}$ must be found such that

$$\begin{aligned} \bar{s}_n &\leq \bar{s}_{n-1} && (5.106) \\ u(x) &\geq 1-x \quad \text{for} \quad x \geq \bar{s} > s_\infty && (5.107) \end{aligned}$$

and secondly, the problem (5.102) can be solved by reducing it into a first-order system of equations and applying a Ricatti transform. Let

$$\begin{aligned} u'(x) &= v(x) \\ v'(x) &= a(x)u(x) - b(x)v(x) + c(x) \\ v(\bar{s}) &= -1 \end{aligned} \qquad (5.108)$$

where the coefficients $a(x), b(x)$ and $c(x)$ are defined by

$$a(x) = \frac{r + \frac{1}{\Delta\tau}}{\frac{1}{2}\sigma^2 x^2}; \quad b(x) = \frac{(r-q)x}{\frac{1}{2}\sigma^2 x^2}; \quad c(x) = \frac{g(x)}{\frac{1}{2}\sigma^2 x^2}. \qquad (5.109)$$

Now apply the Ricatti transform, by letting

$$u(x) = R(x)v(x) + w(x) \qquad (5.110)$$

where $R(x)$, $w(x)$ are solutions to the first-order differential equation that are determined by substituting (5.110) into $v(x)$. The solution $v(x)$ is therefore given by:

$$\begin{aligned} v(x) &= R'(x)v(x) + a(x)[R^2(x)v(x) + R(x)w(x)] \\ &\quad -b(x)R(x)v(x) + R(x)c(x) + w'(x), \end{aligned} \qquad (5.111)$$

and rearranging this yields

$$\begin{aligned} v(x) &+ b(x)R(x)v(x) - a(x)R^2(x)v(x) \\ &= R'(x)v(x) + a(x)R(x)w(x) + R(x)c(x) + w'(x). \end{aligned} \qquad (5.112)$$

The earlier condition of $u(x) \to 0$ as $x \to \infty$ can be approximated by setting $u(\bar{X}) = 0$ for some $\bar{X} > K$. This condition is used when separating (5.112) into two first order differential equations namely:

$$R'(x) = 1 + b(x)R(x) - a(x)R^2(x); \quad R(\bar{X}) = 0 \tag{5.113}$$

$$w'(x) = -a(x)R(x)w(x) - R(x)c(x); \quad w(\bar{X}) = 0 \tag{5.114}$$

The Ricatti equation (5.113) has a uniformly bounded non-positive solution on the interval $[s_\infty, \bar{X}]$ and equation (5.114) has an exponentially decaying fundamental solution. Each of these equations can be solved numerically. The solution is given by

$$R(x) = \frac{x}{2\beta}\left[\alpha - 1 + \sqrt{\gamma}\tan^{-1}\left\{\tfrac{1}{2}\left(2\alpha\tan\left[\frac{1-\alpha}{\sqrt{\gamma}}\right] + \sqrt{\gamma}\log\bar{X} - \sqrt{\gamma}\log x\right)\right\}\right] \tag{5.115}$$

$$w(x) = \frac{1}{I(x)}\left[-\int_x^{\bar{X}} G(s)ds\right] \tag{5.116}$$

$$I(s) = \rho^{\alpha\int_x^{\bar{X}}\frac{R(s)}{s^2}ds} \tag{5.117}$$

$$G(s) = I(s)c(s)R(s) \tag{5.118}$$

where

$$\gamma = -1 + 2\alpha - \alpha^2 - 4\beta \tag{5.119}$$

$$\alpha = \frac{r + \frac{1}{\Delta\tau}}{\frac{1}{2}\sigma^2} \tag{5.120}$$

$$\beta = \frac{(r-q)}{\frac{1}{2}\sigma^2} \tag{5.121}$$

are constants.

Once $R(x)$ and $w(x)$ are determined, they are used to find the solution for $v(x)$ defined by equation (5.111)

$$v'(x) = \{a(x)R(x) - b(x)\} v(x) + a(x)w(x) + c(x) \quad (5.122)$$

$$v(\bar{s}) = -1. \quad (5.123)$$

This equation is solved as a first order differential equation and it has a solution

$$v(x) = \frac{1}{J(x)} \left[\int_{\bar{s}}^{x} \{a(\xi)w(\xi) + c(\xi)\} J(\xi) d\xi + v(\bar{s}) \right] \quad (5.124)$$

where

$$J(x) = e^{-\int_{\bar{s}}^{x} (a(\eta)R(\eta) - b(\eta)) d\eta}. \quad (5.125)$$

To find the critical asset price $\bar{s}$ for the problem of the American put option, set

$$\phi(x) \equiv -R(x) + w(x) - (1 - x) \quad (5.126)$$

$$\phi(\bar{X}) \equiv \bar{X} - K > 0; \quad \phi(0) < 0. \quad (5.127)$$

There will exist a value of x such that $\phi(x) = 0$. In addition, at each time level τ_n there will exist a value $\bar{s}$ such that $\bar{s}_n \leq \bar{s}_{n-1}$. The solution $v(x)$ is substituted back into the solution for $u(x)$

$$u(x) = \begin{cases} R(x)v(x) + w(x) & \text{for} \quad \bar{s} \leq x \leq \bar{X} \\ 1 - x \quad \text{for} \quad 0 \leq x \leq \bar{s} \end{cases} \quad (5.128)$$

5.6 Method of Interpolation between Bounds

Johnson [35] developed a method of pricing an American option on a non-dividend paying asset with the exercise price K increasing at a constant riskless interest rate of r. The lower bound for the American put value is given by

$$P(S, \tau; Ke^{r\tau}) \geq \max(K - S, 0) \tag{5.129}$$

This indicates that it would not be optimal to exercise this American put prematurely. Observing that the financial benefit due to the time value will be lost if the American put is prematurely exercised given that the exercise price also grows at the riskless interest rate. Therefore in this case, the American put option is equivalent to its European counterpart. Since the American put price is an increasing function of the strike price, the following must hold

$$p(S, \tau; K) \leq P(S, \tau; K) \leq P(S, \tau; Ke^{r\tau}) = p(S, \tau; Ke^{r\tau}). \tag{5.130}$$

This establishes the lower and upper bound of the American put value. It follows that

$$P(S, \tau; K) = \alpha p(S, \tau; Ke^{r\tau}) + (1 - \alpha) p(S, \tau; K), \quad 0 \leq \alpha \leq 1 \tag{5.131}$$

where α is defined as the interpolating parameter for the upper and lower bounds for the option value. The parameter α is not constant but dependant upon the parameters S, K, r, τ and σ. The first and second terms of the equation above contain European option values that can be easily computed. However, the difficulty is determining the estimate of the interpolating parameter α. Following Johnson, he proposed that when the American option is at-the-money, then $\frac{S}{K} \sim 1$, and the parameter α lies between 0.2 and 0.25. When $r\tau$ is small, α is also small. Therefore a natural choice for the representation of α is given by

$$\alpha = \frac{r\tau}{a_0 r\tau + a_1}, \tag{5.132}$$

with $a_0 \approx 4$ and $a_1 \ll 1$. For some known at-the-money American put option values, the parameter α can be estimated by applying linear regression. Johnson obtained the parameters for $a_0 = 3.9649$ and $a_1 = 0.032325$ with the regression coefficient $R^2 = 0.9998$. However for in-the-money and out-of-the-money puts, (5.132) will give values that are too large or too small. The equation must be modified to allow α to become larger when the asset price S becomes smaller whilst giving the same values as before when $S = K$. To counteract this, Johnson modified the α parameter to

$$\begin{aligned} \alpha &= \left(\tfrac{r\tau}{a_0 r\tau + a_1}\right)^l \\ l &= \tfrac{\log S/S^*}{\log K/S^*} \end{aligned} \tag{5.133}$$

where S^* is the critical asset price. Determining the critical asset price requires solving

$$K - S^* = P(S^*, \tau; K). \tag{5.134}$$

Since the American put pricing equation is yet to be determined, the critical asset price S^* must be estimated. Firstly observing that when $\sigma^2\tau = 0$, $S^* = K$. And when $\sigma^2\tau \to \infty$ (i.e American perpetual put), then $S^* = \frac{\gamma}{1+\gamma}K$ where $\gamma = \frac{2r}{\sigma^2}$. Therefore a choice for an estimate of S^* can be given by

$$S^* = \left(\frac{\gamma}{1+\gamma}\right)^m K \tag{5.135}$$

where the exponent m depends on $\sigma^2\tau$ and takes the form

$$m = \frac{\sigma^2\tau}{b_0\sigma^2\tau + b_1} \tag{5.136}$$

These estimates satisfy the conditions $\sigma^2\tau = 0$ and $\sigma^2\tau = \infty$. To determine the components b_0 and b_1, (5.135) is re-expressed in terms of the value m

$$m = \frac{\sigma^2\tau}{b_0\sigma^2\tau + b_1} = \frac{\log S^*/K}{\log \frac{\gamma}{1+\gamma}} \tag{5.137}$$

and the expression for α defined in (5.133) can be re-expressed as

$$l = \frac{\log (S/K)(K/S^*)}{\log K/S^*} = \frac{\log \alpha}{\log \frac{r\tau}{a_0 r\tau + a_1}} \tag{5.138}$$

Johnson obtained the values from applying regression of in-the-money and out-of-the money put prices obtained using other numerical methods and thereby arriving at $b_0 = 1.04083$ and $b_1 = 0.00963$ with $R^2 = 0.9975$. Furthermore, Johnson observed that (5.131) will work well for small values of $r\tau$, but the error increases when $r\tau$ is large. The observed value was determined to be $\sigma^2\tau = 0.25$, and therefore (5.131) will provide good valuation only if $r\tau \leq 0.125$.

Extension to Assets Paying Dividends

Blomeyer [6] extended Johnson's method to assets paying one dividend prior to the expiration date. An American put on an asset that goes ex-dividend at expiration will not be exercised prior to expiration if the cash dividend Q, is sufficiently large enough to satisfy

$$Q \geq K[e^{r\tau} - 1] \tag{5.139}$$

where τ is the time left to expiry, r is the risk-free rate of interest and K is the exercise price. The American put option is equivalent to a European option

$$P(Q, \tau_Q) = p(Q, \tau_Q) \tag{5.140}$$

where τ_Q is the time to ex-dividend. The American put option value of an asset that has just gone ex-dividend with an amount Q will be at least as valuable as an American put option on an asset that pays a dividend at the exact time as the option expiry T with payment of an amount Q. This can be stated as

$$P(Q, T) \geq P(Q, \tau_Q) = p(Q, \tau_Q). \tag{5.141}$$

If the asset pays a dividend at a date τ prior to option expiration, the American put will be at least valuable as an American put that goes ex-dividend at expiration. In addition, an American put option on an asset that has just gone ex-dividend will be at least as valuable as an American put option that goes ex-dividend prior to expiration, namely

$$P(Q,T) \geq P(Q,\tau) \geq p(Q,\tau_Q). \tag{5.142}$$

The European put value in the above inequality can be calculated directly from the Black-Scholes formula, by escrowing the discounted dividend. Similarly the American put value $P(Q,T)$ is calculated by subtracting the dividend amount from the asset price. The remaining put value $P(Q,\tau)$ is found by linear interpolation

$$P(Q,\tau) = p(Q,\tau_Q) + \left[\frac{\tau_Q - \tau}{\tau_Q}\right]\{P(Q,T) - p(Q,\tau_Q)\} \tag{5.143}$$

The American put option value is an increasing function with respect to the dividend size. If the dividend Q is larger than the actual dividend Q_a then

$$P(Q,\tau) > P(Q_a,\tau) > P(0,\tau) \tag{5.144}$$

where $P(0,\tau)$ is the American put option value on an asset that pays no dividends. By interpolation $P(Q,\tau)$ can be found using

$$P(Q,\tau) = P(0,\tau) + [Q_a/Q][P(Q,\tau) - P(0,\tau)] \tag{5.145}$$

If the actual cash dividend is such that $Q_a \geq Q$, then (5.143) can be used instead of the above equation.

5.7 Randomization Techniques

Carr [15] uses a technique called *randomization* to price American options on an asset that pays dividends. The randomization technique is applied to the expiry date of the American option. This means the holder of this option can exercise at any time up to and including its random maturity. The expiry date is determined by the waiting time to a prespecified number of jumps of a standard Poisson process, which is assumed to be independent of the asset price process.

The simplest expression occurs when the randomized American option matures at the first jump of a Poisson process. The maturity date is exponentially distributed. Whilst this leads to simple approximations for American options, the error will be too high to be used in practice. Instead it is assumed that the time to maturity is subdivided into n independent exponential subperiods. This implies that the randomized American put option will mature at the nth jump of the Poisson process. The maturity time is found to be Erlang distributed with the mean equal to the fixed maturity date of the true American option.

Visually, the exercise boundary for this case takes the form of a staircase, with each of the levels determined by optimizing each of the subperiod. This results in an expression for the randomized option value as a triple sum, that involves no other special functions other than the natural logarithmic function.

As the number of subperiods chosen becomes large, the variance of the random maturity will approach zero, and therefore the Erlang distribution governing the maturity of the American option will approach a Dirac delta function centered on the option's fixed maturity. However increasing the number of periods decreases computational efficiency. The introduction of Richardson extrapolation will allow the option value to converge to the true value in a more efficient manner. Let τ denote the random maturity time. The random maturity time τ is assumed to be exponentially distributed with scale parameter λ

$$\mathbb{P}\{\tau \in dt\} = \lambda e^{-\lambda t} dt \tag{5.146}$$

Since the mean of τ is reciprocal of λ, by setting $\lambda = \frac{1}{T}$ the maturity of the randomized American option becomes T which is identical to the maturity of the true American option. Now denote $P^{(1)}(S)$ to be the value of the randomized American put option that expires at the first jump of the Poisson process with intensity $\lambda = \frac{1}{T}$. The Poisson process governing the maturity is independent of the stock price process. In addition, the Poisson process is uncorrelated with any market factor. The randomized value can be calculated by risk-neutral methods. This is represented as

$$P^{(1)}(S) = \sup_{B} E_{0,S}\{e^{-r(\tau_B \wedge \tau)}[K - S_{\tau_B \wedge \tau}]^+\}, \quad S > S^* \tag{5.147}$$

where S^* is the unknown optimal exercise boundary. The supremum is only taken over the time-stationary boundaries B rather than functions of time $B(t)$. This implies that the passage of time has no effect on the randomized option value or the optimal exercise boundary. When the Poisson process governing maturity reaches its first jump, the randomized option value will jump down to the intrinsic value for the American put $(K - S)^+$. The time decay of the put option is released at the jump time, that causes the exercise boundary to jump up from S^* to K. This crudely approximates the behaviour of the true exercise boundary. The above problem can be evaluated in closed form and the result can be maximized by using the analytical formulas of barrier options. The above equation can be re-written as an iterated expression in order to obtain the American put option value

$$P^{(1)}(S) = \sup_{B} E_{0,S}\{E_{0,S}\{e^{-r(\tau_B \wedge t)}[K - S_{\tau_B \wedge t}]^+ \mid \tau = t\}\}, \quad S > S^* \tag{5.148}$$

The first expectation is taken over the random maturity, while the second expectation is taken over the future stock price at a given realization of the random maturity. The following relationship between random and fixed maturity put option values using equations (5.146) and (5.148)

$$P^{(1)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t} D(0, S; t; B) dt \tag{5.149}$$

where $D(0, S; t; B)$ is the initial value of a down-and-out put with fixed maturity t, a barrier B, and a rebate equal to $K - B$

$$D(0, S; t; B) = E_{0,S}\{e^{-r(\tau_B \wedge t)}[K - S_{\tau_B \wedge t}]^+\}, \quad S > B \tag{5.150}$$

This relationship representing the American put value is known as a type of Laplace-Carson transform of a fixed maturity barrier put, which is maximized over barriers. Down-and-out put option values satisfy the Black-Scholes partial differential equation, therefore applying the Laplace-Carson transform to the original Black-Scholes p.d.e. to arrive at a simpler ordinary differential equation will yield:

$$\frac{\sigma^2}{2} S^2 P_{SS}^{(1)}(S) + rSP_S^{(1)}(S) - rP^{(1)}(S) = \lambda[P^{(1)}(S) - (K - S)^+], \quad S > S^* \tag{5.151}$$

subject to

$$\begin{cases} \lim_{S\to\infty} P^{(1)}(S) &= 0 \\ \lim_{S\to S^*} P^{(1)}(S) &= K - S^* \\ \lim_{S\to S^*} P_S^{(1)}(S) &= -1 \end{cases} \tag{5.152}$$

By standard solving techniques of ODEs, the randomized American put option value can be written as

$$P^{(1)}(S) = \begin{cases} p^{(1)}(S) + b^{(1)}(S) & \text{if } S > S_0^* \equiv K \\ KR - S + c^{(1)}(S) + b^{(1)}(S) & \text{if } S \in (S_1^*, S_0^*) \\ K - S & \text{if } S \leq S_1^* \end{cases} \tag{5.153}$$

where $p^{(1)}(S)$ is the randomized value of a European put paying $(K - S)^+$ at the first jump period,

$$p^{(1)}(S) = \left(\frac{S}{K}\right)^{\gamma-\epsilon} (qKR - \hat{q}K), \quad S > K \tag{5.154}$$

with $\gamma \equiv \frac{1}{2} - \frac{r}{\sigma^2}, R \equiv \frac{1}{1+rT}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2 T}}$, and

$$p \equiv \frac{\epsilon-\gamma}{2\epsilon}, q \equiv 1-p, \hat{p} \equiv \frac{\epsilon-\gamma+1}{2\epsilon}, \text{ and } \hat{q} \equiv 1-\hat{p} \tag{5.155}$$

The $b^{(1)}(S)$ term is the present value of interest received below the critical stock price S_1^* until the first jump time,

$$b^{(1)}(S) = \left(\frac{S}{S_1^*}\right)^{\gamma-\epsilon} qKRrT \tag{5.156}$$

The randomized value of a European call paying $(S-K)^+$ at the first jump period is given by

$$c^{(1)}(S) = \left(\frac{S}{S_1^*}\right)^{\gamma-\epsilon} (\hat{p}K - pKR), \quad S < K \tag{5.157}$$

The equation representing the European put value (5.154) is simpler than the standard Black Scholes formula as it does not involve any normal distribution functions. However, the randomized option formula only holds for out-of-the-money values ($S > K$) as it does not correctly value the put when $S < K$. The put call parity condition must be used in order to evaluate in-the-money put value for European options with random maturity. For stocks that are trading below the critical asset price S_1^*, the put value is simply the payoff $K - S$. It follows by imposing a value matching on (5.153) at the critical asset price S_1^* will yield the balance equation

$$c^{(1)}(S_1^*) = pKRrT \tag{5.158}$$

where the left hand side represents the randomized value of the European call when the asset price is at the critical point. The right hand side represents the randomized value of a claim paying interest on the exercise price at all asset prices above the current asset price level. The critical asset price is found such that the call value just matches the present value of the interest payment received above the boundary. The stationarity in the values involved implies that the exercise boundary is flat until the jump time. Equation (5.157) can be solved explicitly to determine the first approximation to the exercise boundary S_1^*

$$S_1^* = K \left(\frac{pRrT}{\hat{p} - Rp} \right)^{\frac{1}{\gamma+\epsilon}} \tag{5.159}$$

This explicit expression for the critical asset price will lose explicitness when constant proportional dividends are applied.

5.8 Monte Carlo Methods

Following the work of Boyle [8] on the pricing of European options, Bossaerts [7] investigated the use of simulation on the optimal early exercise of American options. This was proceeded by Longstaff and Schwartz [43] who used a least squares approach with Monte Carlo methods. Tilley [55] also applied Monte Carlo methods to American option pricing and his treatment of the valuation problem will be discussed in this section. Although simulation methods are classified as numerical methods, the framework of the model is discussed in this section and the numerical algorithm will be described in chapter 6.

Methodology

Consider options that are exercisable only at specified calendar time points $t_1, t_2, ..., t_N$. The origin of time is denoted by t_0 when $t = 0$. These time points are indexed according to their subscripts $0, 1, 2, ..., N$. The options are considered to be first exercisable at the time node 0 or at node 1. The path of asset prices can be regarded as the sequence $S(0), S(1), S(2), ..., S(N)$ in which the arguments of S refers to the time nodes at which the asset prices occur. All price paths emanate from the initial stock price $S(0)$. The simulation process involves randomly generating a finite sample of R stock price paths and the estimation of the option price value form that sample. The k-th path is represented as $S(0), S(k,1), S(k,2), ..., S(k,N)$ where the first index represents the path and the second represents the time node.

Let $d(k,t)$ be the present value at time node t on path k of a \$1 payment

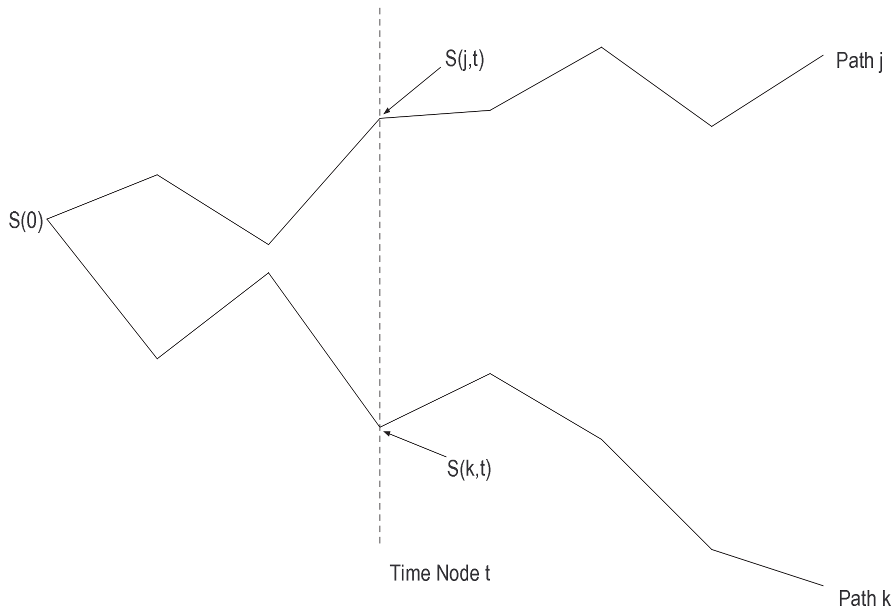


Figure 5.3: The stock price paths of the continuous state model.

that will occur at the next time node $t+1$ on path k. Then let $D(k,t)$ be the present value at time $t=0$ of a \$1 payment that occurs at time t on path k. This can be calculated as the product of the discount factors $d(k,s)$ from $s=0$ to $s=t-1$. Now assume that the option to be priced has strike prices that depend on the date of exercise, and not on the stock price at the time of exercise. Let $K(1), K(2), ...K(N)$ denote the sequence of exercise prices at time nodes $1, 2, ...N$ respectively. The intrinsic value $I(k,t)$ of the option on path k at time node t is defined as:

$$I(k,t) = \begin{cases} \max\left[0, S(k,t) - K(t)\right] \text{ for calls or} \\ \max\left[0, K(t) - S(k,t)\right] \text{ for puts} \end{cases} \quad (5.160)$$

Let $z(k,t)$ be the *exercise-or-hold* indicator variable, which is 0 when the option is not exercised at time t on path k, and takes the value 1 if exercised. Clearly there should be at most one time t^* along the path k where the option is exercised. Otherwise, there exists no exercise points along the path k. The price of an asset can be determined at time node 0 if its cash flows are known at all time nodes along all the possible asset price paths. By Monte Carlo methods, the first step is to compute the present value at time 0, the asset's cash flows along each path k using the path specific discount factors $D(k,t)$. The second step is to average the present values across all the price paths that were computed in the first step. For a given stock price path, the cash flow for an option will be 0 at every time node other than the node at which the option is exercised. At exercise, the options' cash flow will be equal to its payoff or intrinsic value. If the assumption holds that all randomly sampled stock price paths are equally likely with probability weight R^{-1} then the option premium can be expressed as

$$V_{Premium} = R^{-1} \sum_{\text{all paths } k} \sum_{\text{all nodes } t} z(k,t)D(k,t)I(k,t) \quad (5.161)$$

Computational Method

Tilley uses a backward induction algorithm to estimate the exercise-or-hold indicator $z(k,t)$. The algorithm starts at the latest time the option can be exercised which is the expiry date of the option represented as the time node N. If the option is still alive (unexercised), on path k at time N then the option will be exercised if and only if $I(k,N) > 0$. The general rule for this backward induction procedure is for an arbitrary time node t, one must determine whether it is optimal to hold the option for possible exercise beyond t or to exercise the option immediately at time t. The decision process is determined by comparing the option's holding value to the intrinsic value. The intrinsic value will be known for every time node t and path k as the price of the asset is known. The option's holding value at time t is the present value of the expected option value at time $t + 1$. A more comprehensive description of the method will be discussed in the next chapter.

5.9 Critical Appraisal

Each of the various valuations discussed in this chapter contain a characteristic feature that allows each of these methods to be categorized. For instance the binomial method is clearly a unique but robust method for pricing options. The pricing methodology is clear and simple to follow and implement numerically which is a reason for its popularity for practitioners and academics alike. The early exercise feature as well as the potential discrete dividend payments can be handled with relative ease. However, one disadvantage is computational inefficiency when there is an objective to obtain high levels of accuracy. There is a trade off between accuracy and efficiency in the binomial model and had been prevalent in cases prior to the arrival of faster computer processors. Similarly for linear complementarity formulation, the progressive step is to price using finite-difference schemes, a form of lattice method. The pricing of American options is treated as an "obstacle" problem, the set of equations are discretized into finite difference approximations and solved with LU Decomposition and Successive -Over-Relaxation (SOR). Moreover, the Crank-Nicholson scheme will address the stability concerns due to finite precision computer arithmetic and improving convergence.

The determination of the free boundary is relatively straight forward, but as with the binomial method, the numerical pricing is computationally intensive and the methodology is more complex. Some semi-analytical methods such as Barone-Adesi and Whaley [3] and Johnson [35] claim to efficiently price options with some accuracy but this will vary according to the expiry time used. In general these semi-analytic methods are only able to price short-term options with accuracy but become inaccurate with longer dated options. Randomization techniques are different to the two state pricing of the binomial method the disadvantage would be the numerical implementation is not simple. Lastly, simulation methods are conceptually straight forward for numerical implementation, but a high degree if simulation is required for obtaining accuracy.

It is evident that there is a need to find a new pricing method that will be both accurate as well as efficient. Several authors noted in this chapter and the next attempt to address this. The next chapter will discuss in some detail some of the algorithms used and techniques implemented to price American options. The other matter of interest is the determination of the critical exercise boundary $c(t)$, it will be evident later that approaches such as the binomial, and method of lines exhibit the monotonicity of the function, but in the case of the binomial method, the appearance of “steps” is clearly due to the discreteness of the method. With the method of lines, there is oscillatory behaviour when the boundary is observed near time to expiry $t \approx 0$.

Chapter 6

Numerical Methods

The numerical algorithms for the valuation methods described in the previous chapter are discussed. These numerical techniques will be compared for accuracy and efficiency against a new numerical method to be described in chapter 7.

6.1 Lattice Methods

Binomial Method

From the theory described in section 5.2, a binomial tree with n time steps will have computing time has convergence of order $O(\frac{1}{\sqrt{n}})$. Therefore the computing time increases as the choice of n increases. However it is not necessary to have the entire binomial tree but only the information related to the current time step is required. The method involves computing the asset price values Su^jd^{n-j} recursively . Firstly, the factors namely u and d respectively, are calculated using the given financial parameters S, K, T, r, σ and q and the number of time steps n.

Selection of Up and Down Factors u and d

Under the discrete binomial random walk, the expected value and variance of the asset price S at time time step $n+1$ or S^{n+1} given the asset price at time step n or S^n are chosen so that they would be equal to the expected value and variance of an asset following a continuous random walk namely

$$\mathbb{E}_{cont}[S^{n+1}|S^n] = \int_0^\infty \hat{S} f(S^n, n\Delta t; \hat{S}, (n+1)\Delta t) d\hat{S} = e^{r\Delta t} S^n \tag{6.1}$$

where the probability density function is given by:

$$f(S,t;\hat{S},\hat{t}) = \frac{1}{\sigma\hat{S}\sqrt{2\pi(\hat{t}-t)}}\; e^{-[\log(\hat{S}/S)-(r-\frac{1}{2}\sigma^2)(\hat{t}-t)]^2/2\sigma^2(\hat{t}-t)} \tag{6.2}$$

and the expected value of the binomial random walk

$$\mathbb{E}_{binom}[S^{n+1}|S^n] = (\hat{p}u + (1-\hat{p})d)S^n. \tag{6.3}$$

Equating the two expected values yields

$$\hat{p}u + (1-\hat{p})d = e^{r\Delta t}. \tag{6.4}$$

The variance of S^{n+1} given S^n is represented as

$$\mathrm{Var}[S^{n+1}|S^n] = \mathbb{E}[(S^{n+1})^2|S^n] - \mathbb{E}[S^{n+1}|S^n]^2. \tag{6.5}$$

Therefore using (6.1) it is obvious that

$$\mathbb{E}_{cont}[(S^{n+1})^2|S^n] = \int_0^\infty \hat{S}^2 f(S^n, n\Delta t; \hat{S}, (n+1)\Delta t) d\hat{S} = e^{(2r+\sigma^2)\Delta t}(S^n)^2 \tag{6.6}$$

which has the same probability density function described by (6.2). It follows that the variance of the continuous process is

$$\mathrm{Var}_{cont}[S^{n+1}|S^n] = e^{2r\Delta t}(e^{\sigma^2\Delta t} - 1)(S^n)^2. \tag{6.7}$$

Now under the discrete binomial process it is observed that

$$E_{binom}[(S^{n+1})^2|S^n] = (\hat{p}u^2 + (1-\hat{p})d^2)(S^n)^2, \tag{6.8}$$

and therefore the variance is given by

$$\mathrm{Var}_{binom}[S^{n+1}|S^n] = (\hat{p}u^2 + (1-\hat{p})d^2 - e^{2r\Delta t})(S^n)^2 \tag{6.9}$$

where the equality (6.4) is used to allow $\mathbb{E}_{binom}[S^{n+1}|S^n] = S^n e^{r\Delta t}$. The equating of the two variances will yield

$$\hat{p}u^2 + (1-\hat{p})d^2 = e^{(2r+\sigma^2)\Delta t}. \tag{6.10}$$

There are two popular choices for determining the value for u, d. One choice is by using the conditions that $u, d > 0$ and $0 \leq \hat{p} \leq 1$, and letting

$$u = \frac{1}{d}. \tag{6.11}$$

It then follows that the use of the expressions (6.4), (6.10) and (6.11) will lead to an expression for the probability p

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{(2r+\sigma^2)\Delta t} - d^2}{u^2 - d^2} \tag{6.12}$$

and an expression for $u + d$

$$u + d = \frac{e^{(2r+\sigma^2)}\Delta t - d^2}{e^{r\Delta t} - d} \tag{6.13}$$

leading to the quadratic equation $d^2 - 2\beta d + 1 = 0$ where

$$\beta = \tfrac{1}{2}(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}). \tag{6.14}$$

Hence the value of d can be found along with u and p

$$d = \beta - \sqrt{\beta^2 - 1},\ u = \beta + \sqrt{\beta^2 - 1},\ p = \frac{e^{r\Delta t} - d}{u - d}. \tag{6.15}$$

A second choice is by solving for u and d given that $p = \frac{1}{2}$. Following Buchen [13], the expectation of the asset price S^{n+1} given S^n is simply the asset S^n earning the riskless rate $1 + r\Delta$ after the change in time Δt, or expressed in terms of the factor u as

$$\begin{aligned} \mathbb{E}_{binom}[S^{n+1}|S^n] &= (\tfrac{1}{2}u + \tfrac{1}{2}d)S^n = (1 + r\Delta t)S^n \\ u &= 2(1 + r\Delta t) - d. \end{aligned} \tag{6.16}$$

The other equation used to solve for the up and down factors is the for the variance

$$\begin{aligned} \mathrm{Var}_{binom}[S^{n+1}|S^n] &= (\tfrac{1}{2}u^2 + \tfrac{1}{2}d^2)(S^n)^2 - [2(1 + r\Delta t)(S^n)]^2 \\ &= 2\sigma^2(S^n)^2\Delta t. \end{aligned} \tag{6.17}$$

Substituting the value of u will produce a quadratic equation for d with the solution

$$d = 1 + r\Delta t - \sigma\sqrt{\Delta t} \tag{6.18}$$

and therefore the value for u is given by

$$u = 1 + r\Delta t + \sigma\sqrt{\Delta t}. \tag{6.19}$$

The payoff function for the option depends on the underlying asset price at expiry, therefore if M represents the last time step occurring at expiry, the payoff of a call or put option can be found

$$C_j^M = \left[S_j^M - K\right]^+,\ j = 0, 1, ..., M \tag{6.20}$$

$$P_j^M = \left[K - S_j^M\right]^+,\ j = 0, 1, ..., M \tag{6.21}$$

where C_j^M, P_j^M and S_j^M are the j-th possible values of of the call and put option and asset price at time step M respectively. When the above payoffs are known, the expected value of the option at the time step prior to expiry $(M-1)\Delta t$ using the possible set of asset prices S_n^{M-1} for $n = 0, 1, ..., M-1$ moving to S_{n+1}^M with known probability p or moving to S_n^M with probability $(1-p)$ Thus by the risk neutral argument, the value of the option can be found at each asset price at time step $M-1$. Once $M-1$ is known then the time step $M-2$ is evaluated and so forth until the time step reaches back to 0 where the current value of the option is found. For standard European options, the calculation of the option value at time step $m\Delta t$ is the discounted expected value of the option values calculated at time step $(m+1)\Delta t$ with the discount factor $\rho = 1 + r$ namely

$$C_n^m = [pC_{n+1}^{m+1} + (1-p)C_n^{m+1}]/\rho \tag{6.22}$$

$$P_n^m = [pP_{n+1}^{m+1} + (1-p)P_n^{m+1}]/\rho. \tag{6.23}$$

The values of C_n^m and P_n^m are calculated recursively for each $n = 0, 1, ..., m$ for $m < M$ to arrive at the current values C_0^0 and P_0^0.

For American options, the addition of the early exercise can be easily be incorporated into the binomial pricing model. Since the option can be exercised prior to expiry, (6.22) and (6.23) can be modified to

$$C_n^m = \max(S_n^m - K, [pC_{n+1}^{m+1} + (1-p)C_n^{m+1}]/\rho) \tag{6.24}$$

$$P_n^m = \max(K - S_n^m, [pP_{n+1}^{m+1} + (1-p)P_n^{m+1}]/\rho). \tag{6.25}$$

To accommodate for constant dividend yield q paid on the underlying asset, the terms β, u, d and p will be modified to become

$$\beta = \tfrac{1}{2}(e^{-(r-q)\Delta t} + e^{(r-q+\sigma^2)\Delta t}) \tag{6.26}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}. \tag{6.27}$$

Computing the Critical Exercise Boundary

The computation of the critical exercise boundary $S^*(t)$ from the binomial method is easily attainable. The basic method is to attempt to find through the iterative process, starting at the expiry date of the option, the existence of the critical exercise price at each time node $j = 0, 1, ..., M$. The larger the number of time steps chosen, then the finer the lattice created by the random process. For example to find the critical exercise price for the American call at each time node, which is known to be a monotonic increasing function with respect to time, one must find the stopping and continuation regions. In the case of call options, this corresponds to the minimum value of the possible prices of S_j at time node j that will result in the value of C_j being equal to the early exercise payoff $S_j - K$. For put options it would correspond to the maximum value of the possible prices of S_j such that the value of the put option P_j is equal to the early exercise payoff of $K - S_j$. Figure 6.1 shows a comparison of critical exercise boundaries calculated for varying step sizes. The smaller the step sizes the more accurate the calculation of the boundary.

Finite Difference Methods

Recall the linear complementarity problem for the American call option in chapter 5, equations (5.79) and (5.80) generalize to

$$\begin{cases} \mathcal{B}\{V(S,t)\} \cdot [V - g(S,K)] = 0, \; S > 0 \\ \mathcal{B}\{V(S,t)\} \geq 0, \quad V(S,\tau) \geq g(S,K) \end{cases} \tag{6.28}$$

where $\mathcal{B}\{\cdot\}$ is the Black-Scholes partial differential equation operator. The following problem can be solved by finite difference methods as described by Wilmott et al.[58]. The first step of this method is to express the Black-

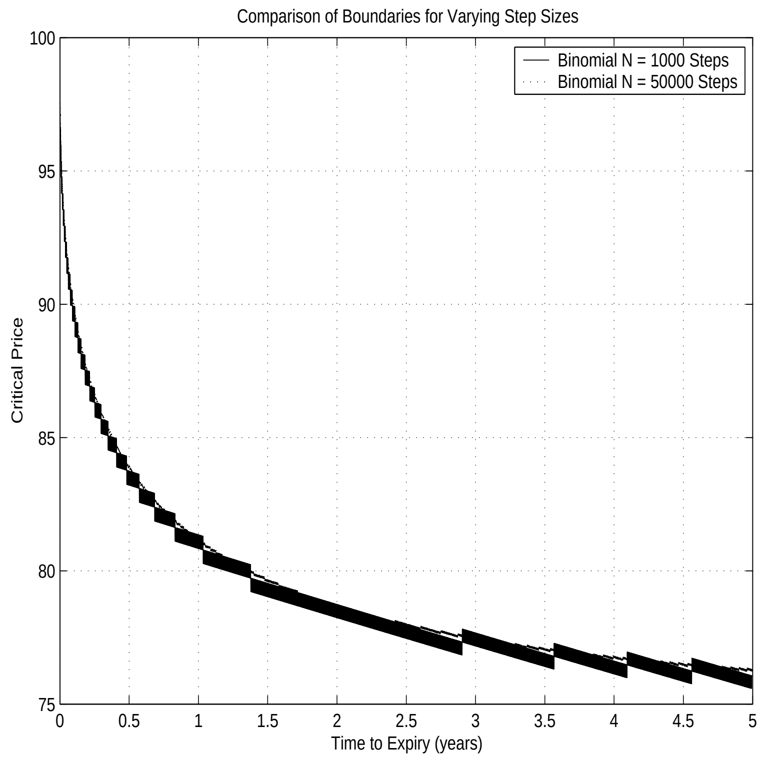


Figure 6.1: Comparison between two critical exercise boundaries for varying step sizes.

Scholes equation (3.10) adjusted for assets paying a dividend q

$$\mathcal{B}\{V(S,t)\} \equiv \frac{\partial V}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV = 0, \tag{6.29}$$

in terms of an average of the implicit and explicit discretisation. Let $V(n\Delta S, m\Delta t) = V_n^m$. Defining $N^{\pm}$ to be the minimum and maximum values of the discretised spatial component S. The discretisation of the Black-Scholes partial differential equation will be given by:

$$\begin{aligned} \frac{V_n^{m+1} - V_n^m}{\Delta t} &= \tfrac{1}{4}\sigma^2 n^2 \left[V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1} + V_{n+1}^m - 2V_n^m + V_{n-1}^m\right] \\ &\quad + \tfrac{1}{4}(r-q)n\left[V_{n+1}^{m+1} - V_{n-1}^{m+1} + V_{n+1}^m - V_{n-1}^m\right] - rV_n^m \end{aligned} \tag{6.30}$$

Collecting like terms the above equation can be rearranged as:

$$
\begin{aligned}
LHS &= V_{n-1}^{m+1}\left[-\tfrac{1}{4}(\sigma^2 n^2 - (r-q)n)\Delta t\right] + V_n^{m+1}\left[1+\tfrac{1}{2}\sigma^2 n^2 \Delta t\right] \\
&\quad + V_{n+1}^{m+1}\left[-\tfrac{1}{4}(\sigma^2 n^2 - (r-q)n)\Delta t\right] \\
RHS &= V_{n-1}^{m}\left[\tfrac{1}{4}(\sigma^2 n^2 - (r-q)n)\Delta t\right] \\
&\quad + V_n^m\left[1-(r+\sigma^2 n^2)\Delta t\right] + V_{n+1}^m\left[\tfrac{1}{4}(\sigma^2 n^2 + (r-q)n)\Delta t\right]
\end{aligned}
\tag{6.31}
$$

The coefficients are then extracted such that

$$
\begin{aligned}
A_n &= \left[\tfrac{1}{4}(\sigma^2 n^2 - (r-q)n)\Delta t\right] \\
B_n &= \left[1-(r+\sigma^2 n^2)\Delta t\right] \\
C_n &= \left[\tfrac{1}{4}(\sigma^2 n^2 + (r-q)n)\Delta t\right] \\
D_n &= \left[1+\tfrac{1}{2}\sigma^2 n^2 \Delta t\right]
\end{aligned}
\tag{6.32}
$$

which simplifies (6.30) to

$$
D_n V_n^{m+1} - A_n V_{n-1}^{m+1} - C_n V_{n+1}^{m+1} = Z_n^m \tag{6.33}
$$

where

$$
Z_n^m = A_n V_{n-1}^m + B_n V_n^m + C_n V_{n+1}^m. \tag{6.34}
$$

By applying the **Projected SOR Method** to the discretised expression, this particular iterative scheme is used in the following set of equations:

$$
V_n^{m+1} = \frac{1}{D_n}\left[Z_n^m + A_n V_{n-1}^{m+1} + C_n V_{n+1}^{m+1}\right] \tag{6.35}
$$

$$
Y_n^{m+1,k+1} = \frac{1}{D_n}\left[B_n^m + A_n V_{n-1}^{m+1,k+1} + C_n V_{n+1}^{m+1,k}\right] \tag{6.36}
$$

$$
V_n^{m+1,k+1} = \max\left(g_n^m, V_n^{m+1,k} + \omega(Y_n^{m+1,k+1} - V_n^{m+1,k})\right) \tag{6.37}
$$

where g_n^m is the payoff of the option if exercised early

$$
g_n^m = \begin{cases} [S_n^m - K]^+ & \text{Calls} \\ [K - S_n^m]^+ & \text{Puts} \end{cases} \tag{6.38}
$$

This scheme attempts to find the value of V_n^{m+1} by iterating $V_n^{m+1,k}$ so that it converges to V_n^{m+1}. The task is to solve $\mathbf{M}\mathbf{V}^{m+1} = \mathbf{b}^m$ of the constrained matrix problem

$$\begin{aligned} &\mathbf{M}\mathbf{V}^{m+1} \geq \mathbf{b}^m, \mathbf{V}^{m+1} \geq \mathbf{g}^{m+1}, \\ &\left(\mathbf{V}^{m+1} - \mathbf{g}^{m+1}\right) \cdot \left(\mathbf{M}\mathbf{V}^{m+1} - \mathbf{b}^m\right) = 0 \end{aligned} \tag{6.39}$$

where $\mathbf{M}$ is a tridiagonal matrix containing the coefficients A_n, B_n and C_n:

$$\mathbf{M} = \begin{pmatrix} B_0 & C_0 & 0 & \cdots & 0 \\ A_1 & B_1 & C_1 & & \vdots \\ 0 & A_2 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & C_{N-1} \\ 0 & \cdots & 0 & A_N & B_N \end{pmatrix} \tag{6.40}$$

and

$$\mathbf{b}^m = \begin{pmatrix} b_{N^-+1} \\ \vdots \\ b_0^m \\ \vdots \\ b_{N^+-1}^m \end{pmatrix} = \begin{pmatrix} Z_{N^-+1} \\ \vdots \\ Z_0^m \\ \vdots \\ Z_{N^+-1}^m \end{pmatrix} + \begin{pmatrix} g_{N^-}^{m+1} \\ 0 \\ \vdots \\ 0 \\ g_{N^+}^{m+1} \end{pmatrix} \tag{6.41}$$

The constraint $\mathbf{V}^{m+1} \geq \mathbf{g}^{m+1}$ is also included in (6.37), and therefore the algorithm is to iterate with respect to k until the error $\|\mathbf{V}^{m+1,k+1} - \mathbf{V}^{m+1,k}\|$ is negligible.

Locating the Free Boundary

The critical asset price can be determined as part of the solution using the projected SOR method. Within the iteration scheme, an early exercise opportunity will occur at the value of S_n^m where

$$V_n^m = g_n^m \tag{6.42}$$

6.2 Simulation Methods

Recalling the asset price structure from 5.8, the Monte Carlo method discussed by Tilley [55] begins with partitioning the simulated asset price paths of size R at each time nodes into Q bundles each containing P paths. This is illustrated below. The assumption is that the paths within a bundle Q are

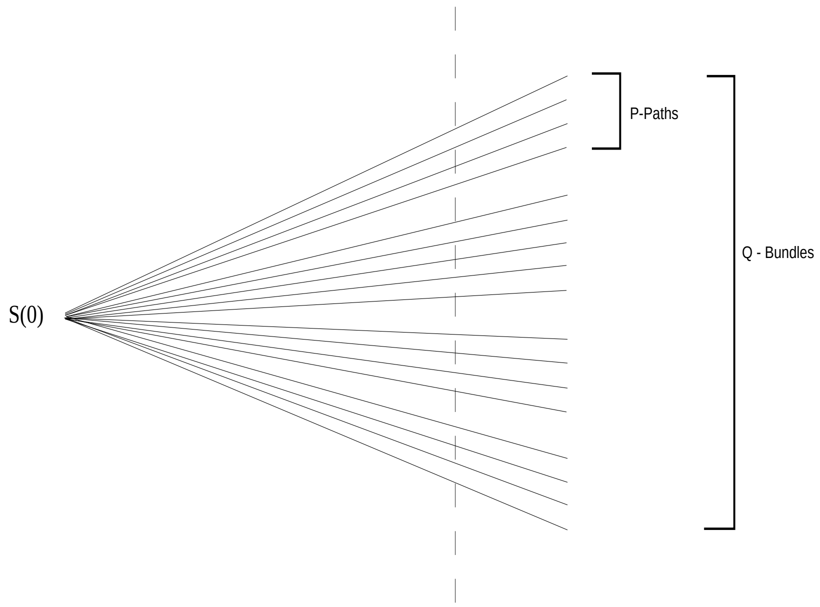


Figure 6.2: The partitioned stock price paths.

sufficiently similar that they will be considered to have the same expected one-period ahead option value. This means that Q must be sufficiently large. In addition, the number of price paths P in each bundle Q must also be sufficiently large as the required estimation of the one-period ahead option value is found by taking the average of the one period ahead option values of each price path P per bundle. When the expected one-period ahead option value is evaluated for each bundle, there exists one bundle for which the decision for some paths is to hold the option to the next time node, while the rest of the paths in the bundle is for early exercise. It is from this step that the early exercise boundary can be determined by finding the *sharp boundary* at each time node t. The algorithm involves the following steps which are performed at each time node t by backward induction.

Reorder Asset Price Paths

After the sample asset price paths are generated, each stock price path is reordered from lowest price to highest for evaluating call options, while the opposite is applied for puts. The paths are then reindexed from 1 to R according to the reordering.

Compute the Intrinsic Value of the Option

At each path k of the newly ordered asset price paths, the intrinsic value $I(k,t)$ is calculated. This is given by

$$I(k,t) = \begin{cases} [S(k,t) - K]^+ & \text{Calls} \\ \\ [K - S(k,t)]^+ & \text{Puts} \end{cases} \tag{6.43}$$

for each path $k = 1, 2, ..., R$.

Partition the Ordered Paths

The set of R ordered paths are partitioned into Q distinct bundles each of size P paths. Therefore the first P paths are assigned to the first bundle, the second P paths are to the next bundle and so forth, until the final P paths are assigned to the Q-th bundle. Thus it can be seen that P and Q are integer factors of R, $R = PQ$.

Calculate the Holding Value

At each path k, the option's holding value $H(k,t)$ is calculated as the expectation taken over all the paths in the bundle Q containing the path k:

$$H(k,t) = \frac{d(k,t)}{P} \sum_{\forall j \in Q} V(j,t+1) \tag{6.44}$$

where $V(k,t)$ is the current value of the option. The definition of $V(k,t)$ is given in the proceeding steps. At the time node $t = N$, $V(k,N) = I(k,N)\ \forall k$.

Tentatively Determine Exercise Possibility

For each path k, the holding value $H(k,t)$ is compared to the intrinsic value $I(k,t)$ and an indicator variable $x(k,t)$ is created such that

$$x(k,t) \begin{cases} 1 & \text{if} \quad I(k,t) > H(k,t) \quad \text{Exercise} \\ 0 & \text{if} \quad H(k,t) \geq I(k,t) \quad \text{Hold} \end{cases} \tag{6.45}$$

Determine Transition Zone

After producing a sequence of 0 and 1's for each path $x(k,t); k = 1, ...R$. Determine the first sequence of consecutive 1's such that the length of which exceeds the length of every subsequent string of 0's. This locates the sharp boundary between the decision to hold and the decision to exercise. Let $k_*(t)$ be the path index from the ordered sample representing the leading 1 of the string of consecutive 1's.

Boundary
↓
00...01100011111001..11

Define the Exercise-Hold Indicator

The variable $y(k,t)$ is defined as the new exercise-or-hold indicator where

$$y(k,t)\begin{cases} 1 & \text{for} \quad k \geq k_*(t) \\ \\ 0 & \text{for} \quad k < k_*(t) \end{cases} \tag{6.46}$$

this means that values of $y(k,t) = 1$ will mean that the option will be exercised early, while the option will be held when $y(k,t) = 0$. This leads to the final step.

The Current Value of the Option

The current value of the option $V(k,t)$ is defined as

$$V(k,t)\begin{cases} I(k,t) & \text{if} \quad y(k,t) = 1 \\ \\ H(k,t) & \text{if} \quad y(k,t) = 0 \end{cases} \tag{6.47}$$

The algorithm is iterated from the last time node N to the first node, and thus for American options where early exercise is permitted, the following indicator variable for $t \leq N$ is estimated as

$$z(k,t)\begin{cases} 1 & \text{if } y(k,t) = 1 \text{ and } y(k,s) = 0 \ \forall s < t \\ \\ 0 & \text{otherwise} \end{cases} \tag{6.48}$$

The term $z(k,t)$ is used in (5.161) to calculate the value of the option.

6.3 Approximation Methods

Recall that in order to price the option by the method described by McMillan [45] and Barone-Adesi & Whaley [3], the critical asset price S^* needs to be found subject to equations (5.95) and (5.98)

$$\text{Calls:} \qquad S^* - K = c(S^*, \tau) + [1 - e^{-q\tau}\mathcal{N}(d_1(S^*))]\,\tfrac{S^*}{y_2}$$

$$\text{Puts:} \qquad K - S^* = p(S^*, \tau) + [1 - e^{-q\tau}\mathcal{N}(-d_1(S^*))]\,\tfrac{S^*}{y_1}$$

For the case of the American call option, evaluate both sides of (5.95) at a seed value $\hat{S}_i$ and

$$\text{LHS}(\hat{S}_i) = \hat{S}_i - K, \quad \text{and} \tag{6.49}$$

$$\text{RHS}(\hat{S}_i) = c(\hat{S}_i, \tau) + \left[1 - e^{-q\tau}\mathcal{N}(d_1(\hat{S}_i))\right]\frac{\hat{S}_i}{y_2} \tag{6.50}$$

where $d_1(\hat{S}_i) = \left[\log \hat{S}_i/K + (r - q + \frac{1}{2}\sigma^2)\tau\right]/\sigma\sqrt{\tau}$. Starting at $i = 1$, the values of the LHS($\hat{S}_i$) and RHS($\hat{S}_i$) will unlikely be equal, therefore a second guess must be made at $i+1$.To develop the next guess at $\hat{S}_{i+1}$, the slope m_i of the RHS($\hat{S}_i$) needs to be determined

$$m_i = e^{-q\tau}\mathcal{N}(d_1(\hat{S}_i))(1 - 1/y_2) + \left[1 - e^{-q\tau}n(d_1(\hat{S}_i))/\sigma\sqrt{\tau}\right]/y_2 \tag{6.51}$$

where $n(\cdot)$ is the univariate normal density function. The next step is to use the Newton-Rapson method to find the line tangent to the curve RHS($\hat{S}_i$), i.e. where the curve intersects the pay-off of the American call

$$\text{RHS}(\hat{S}_i) + m_i(S - \hat{S}_i) = S - K. \tag{6.52}$$

Rearranging the above equation to find $\hat{S}_{i+1}$:

$$\hat{S}_{i+1} = \left[K + \text{RHS}(\hat{S}_i) - m_i\hat{S}_i\right]/(1 - m_i). \tag{6.53}$$

This equation will provide the subsequent estimates for $\hat{S}$ computed by iteration. The final value will be found when the absolute error falls to an acceptable level. Barone-Adesi & Whaley uses

$$|\text{LHS}(\hat{S}_i) - \text{RHS}(\hat{S}_i)|/K < 0.00001. \tag{6.54}$$

Choice of the Initial Value

The efficiency of this method can be improved by taking the initial starting value $\hat{S}$ closer to the true solution. This is done by finding the approximate value of the critical asset price using (5.95). At time to expiry, if the value of the American call option is equal to 0, the critical asset price above which the call will be exercised will be the exercise price of the option K. In the case of a perpetual option where the expiry time T is infinite, the critical asset price can be found by substituting $T = \infty$ in (5.95) which yields

$$S^*(\infty) = \frac{K}{1 - 1/q_2(\infty)}, \tag{6.55}$$

where $q_2(\infty) = \frac{1}{2}\left[-(k_2 - 1) + \sqrt{(k_2-1)^2 + 4k_1}\right]$, $k_1 = \frac{2(r-q)}{\sigma^2}$ and $k_2 = \frac{2r}{\sigma^2}$.

Consider the time $\Delta\tau$ to be an arbitrary small time period to expiry of the option. If the call option is exercised at time $\Delta\tau$, the proceed is $S(\Delta\tau) - K$ which will earn interest up to expiry of the option at $\tau = 0$. Alternatively the holder of the call option will choose to keep the option alive, therefore the option is worth $\mathbb{E}[S(0) - K|S(0) > K]$. Therefore the critical asset price which the holder will choose to exercise an American call option will be given by

$$[S^*(\Delta\tau) - K](1 + r\Delta\tau) = \mathbb{E}[S^*(0) - K|S^*(0) > K] \tag{6.56}$$

The right hand side of the above equation can be evaluated using Cox-Ross and Rubinstein binomial method. Recall in the binomial method, the random rate of return of an asset S is equal to the $(r-q)\Delta\tau$ plus or minus the stochastic component $\sigma\sqrt{\Delta\tau}$ each with equal probability. Therefore the expected value of holding the call option can be given by

$$\mathbb{E}[S^*(0) - K|S^*(0) > K] = \tfrac{1}{2}[S^*(\Delta\tau)(1 + (r-q)\Delta\tau + \sigma\sqrt{\Delta\tau}) - K] \tag{6.57}$$

Substituting (6.57) into (6.56) leads to an equation where the critical asset price $S^*(\Delta t)$ in the equation

$$[S^*(\Delta\tau) - K](1 + r\Delta\tau) = \tfrac{1}{2}[S^*(\Delta\tau)(1 + (r-q)\Delta\tau + \sigma\sqrt{\Delta\tau}) - K] \tag{6.58}$$

can be isolated

$$\begin{aligned} S^*(\Delta\tau) &= [K(1+2r\Delta\tau)]/[1+(r-q)\Delta\tau-\sigma\sqrt{\Delta\tau}] \\ &\approx K(1+(r-q)\Delta\tau+\sigma\sqrt{\Delta\tau})+\mathcal{O}(\Delta\tau)^2 \end{aligned} \tag{6.59}$$

since $\Delta\tau$ is assumed to be small enough that early exercise opportunities will not exist. Therefore (6.59) will hold for values of $\tau \to 0$. To find an approximation for S^* for any time to expiration, the value of $S^*(0)$ is expanded around $S^*(\Delta\tau)$

$$S^*(0) = S^*(\Delta\tau) + \frac{\partial S^*}{\partial \tau}|_{\tau=\Delta\tau}\Delta\tau \tag{6.60}$$

Substituting the approximation (6.59) into the equation above and with $S^*(0) = K$ it follows that the critical asset price will satisfy the differential equation

$$\frac{\partial S^*}{\partial \tau} = S^*(0)((r-q)+\frac{\sigma}{\sqrt{\Delta\tau}}) \tag{6.61}$$

and after integration this yields a solution of exponential form with exponent $((r-q)\tau+2\sigma\sqrt{\tau})$. Using the characteristics of the critical asset price function $S^*(\tau)$ at $\tau = 0$ and $\tau = \infty$ and along with the slope of the function, the function can be appropriately expressed as

$$S^*(\tau) = K + [S^*(\infty)-K](1-e^{l_2}) \tag{6.62}$$

where $l_2 = -((r-q)\tau+2\sigma\sqrt{\tau})\frac{K}{S^*(\infty)-K}$.

6.4 Semi-Analytical Methods

Accelerated Recursive Method

Huang, Subrahmanyam and Yu [28] follow the McKean's methodology of solving an American option as a free boundary problem described in (5.49), (5.50), (5.56) and (5.57) . For the American option value to be found, the critical asset price $S^*(\tau)$ must be predetermined and then used as part of the solution. To determine the critical exercise boundary, recall the expressions of the value of the American put and call option evaluated at the critical asset price $S_P^*(t)$ and $S_C^*(t)$ respectively:

$$\begin{aligned} K - S_P^*(t) &= P^E(S_P^*(t), t) + P^X(S_P^*(t), t) \\ S_C^*(t) - K &= C^E(S_C^*(t), t) + C^X(S_C^*(t), t) \end{aligned} \tag{6.63}$$

The authors suggested to approximate the early exercise boundary by computing it using recursive methods. In the simple case the free boundary expression for the call and put can be found by dividing the interval $[0, T]$ into n subintervals and evaluating n implicit integral equations $S^*(t_i)$ for $t_0, t_1, ..., t_{n-1}$, while the value of $S^*(t_N)$ is given by

$$S^*(t_N) = \begin{cases} \min[K, \frac{rK}{q}] & \text{Puts} \\ \max[K, \frac{rK}{q}] & \text{Calls} \end{cases} \tag{6.64}$$

It is fairly evident that this type of recursive method would be computationally inefficient especially for longer dated options as there needs to be a large number of points on the boundary to obtain an accurate approximation. To overcome this inefficiency, the authors apply a Richardson extrapolation method to estimate the early exercise boundary, and once found, it would be used to calculate the option values and if necessary the option hedge ratios. Following the Geske-Johnson [25] method of Richardson extrapolation, their formula for extrapolating the American put option value P_0 is given by

$$\hat{P}_0 = \tfrac{1}{2}(P_1 - 8P_2 + 9P_3) \tag{6.65}$$

where $P_i, i = 1, 2, 3$ denotes the price of an i-times exercisable option and $\hat{P}_0$ is the estimate of the American put value. An option that is i-times exercisable is known as a Bermudan option where exercise can only occur at predetermined periods of the option's life. Recall the expression (5.52) for the value of an American put option in terms of an integral equation is given by

$$\begin{cases} P(S,t) = P^E(S,t) + P^X(S,t) \\ P^E(S,t) = Ke^{-rt}\mathcal{N}(-d_2) - Se^{-qt}\mathcal{N}(-d_1) \\ P^X(S,t) = \int_0^t \{rKe^{-r\tau}\mathcal{N}(-z_2) - qSe^{-q\tau}\mathcal{N}(-z_1)\}d\tau \end{cases} \quad (6.66)$$

It follows that the integral term $P^X(S,t)$ above is replaced by a summation over the exercisable points

$$\begin{aligned} P_1 &= p_0 \\ P_2 &= p_0 + \tfrac{rKt}{2}e^{-\frac{rt}{2}}\mathcal{N}(-d_2(S,S^*(\tfrac{t}{2}),\tfrac{1}{2}t)) - \tfrac{qSt}{2}e^{-\frac{qt}{2}}\mathcal{N}(-d_2(S,S^*(\tfrac{t}{2}),\tfrac{1}{2}t)) \\ P_3 &= p_0 + \tfrac{rKt}{3}\left[e^{-\frac{rt}{3}}\mathcal{N}(-d_2(S,S^*(\tfrac{t}{3}),\tfrac{t}{3})) + e^{-\frac{2rt}{3}}\mathcal{N}(-d_2(S,S^*(\tfrac{2t}{3}),\tfrac{2t}{3}))\right] \\ &\quad -\tfrac{qSt}{3}\left[e^{-\frac{qt}{3}}\mathcal{N}(-d_2(S,S^*(\tfrac{t}{3}),\tfrac{t}{3})) + e^{-\frac{2qt}{3}}\mathcal{N}(-d_2(S,S^*(\tfrac{2t}{3}),\tfrac{2t}{3}))\right] \end{aligned} \quad (6.67)$$

It is evident that only three boundary points $S^*(\frac{t}{2}), S^*(\frac{t}{3})$ and $S^*(\frac{2t}{3})$ are needed to evaluate P_1, P_2 and P_3 and therefore $\hat{P}_0$

6.5 Gaussian Quadrature

Sullivan[54] uses Gaussian quadrature to evaluate the risk-neutral expectations, and maintains a log-normal probability density rather than replacing with a binomial. Assume the asset price S follows geometric Brownian motion with volatility σ, and a risk-free rate r. Assume also that the option may be exercised at n dates which are separated by time intervals of length $\Delta t = M/n$. For a put option, the value is the greater of either the intrinsic value on early exercise or the value of holding the option. Holding the option will be the risk-neutral expectation of the option's value at the next exercise date discounted by the risk free rate r. The expectation is taken over the changes in the log of the asset price z that are normally distributed with mean $u_z = (r - \frac{1}{2}\sigma^2)\Delta t$ and variance $\sigma_z^2 = \sigma^2 \Delta t$. Therefore for an option with remaining maturity τ

$$P(S,\tau) = \max(K - S, e^{-r\Delta t}\mathbb{E}\left[Se^z, \tau - \Delta t\right]) \quad \tau = \Delta t, 2\Delta t, ..., M \tag{6.68}$$

At option maturity the put option's value will be $P(S,0) = [K-S]^+$. The critical asset price $S^*(\tau)$ is defined as the largest price where the put option value equals the intrinsic value, i.e that satisfies

$$K - S^*(\tau) = P(S^*(\tau), \tau) \tag{6.69}$$

Expressing (6.68) in terms of an integral representation, with $f(z)$ being the probability density function of the log normal stock price. Separate the integral at the critical asset price $S^*(\tau - \Delta t)$ when $S > S^*(\tau)$

$$\begin{aligned} P(S,\tau) &= e^{-r\Delta t} \int_{-\infty}^{\log(S^*(\tau-\Delta t)/S)} (K - Se^z) f(z) dz \\ &\quad + e^{-r\Delta t} \int_{\log(S^*(\tau-\Delta t)/S)}^{\infty} P(Se^z, \tau - \Delta t) f(z) dz \\ &\equiv U(S,\tau) + W(S,\tau) \end{aligned} \tag{6.70}$$

The term $U(S,\tau)$ is simply the Black-Scholes value for the European put option, which has a known analytic formula but the difference is that the possible time to exercise is Δ and the exercise boundary will be less than the exercise price except at maturity.

$$U(S,\tau) = Ke^{-r\Delta t}\mathcal{N}(d_2(S, S^*(\tau - \Delta t))) - S\mathcal{N}(d_1(S, S^*(\tau - \Delta t))) \tag{6.71}$$

The second term $W(S,\tau)$ can be determined recursively by numerical integration. To estimate the integral of a given function $g(x)$ over an interval

$[\beta, \alpha]$ the Gaussian quadrature method approximates the integral with a linear combination of function values within the interval. The general rule specifies a set of abscissas x_i and associated weights w_i for estimating the value of the integral by a weighted sum

$$\int_{\beta}^{\alpha} g(x)dx \approx \sum_i g(x_i)w_i \tag{6.72}$$

The classical forms of the quadrature method for approximate integration are the trapezoidal rule and the Simpson's rule which use a fixed number of equally spaced abscissae and the weights are chosen to achieve maximum order. The interval of integration is divided into k subintervals of equal size. The rule is applied to each subinterval and the sum of these will be the estimate of the integral. As k increases the quadrature method will converge to the value of the actual integral as the integrand will become a continuous function. The problem may be the speed of convergence if a large value of k is used, there will be a large number of function valuations. The Gaussian quadrature rule chooses k abscissae and k weights that produces a $2k-1$ order rule. The Gauss-Legendre quadrature rule over the interval $[-1, 1]$ is the solution to

$$x_1^j w_1 + x_2^j w_2 + ... + x_k^j w_k = \int_{-1}^{1} x^j dx \quad j = 0, 1, ..., 2k-1 \tag{6.73}$$

The abscissae and weights that solve the above equation for various k can be found in Abramowitz and Stegun [1]. For an arbitrary finite interval, these various results are rescaled. For a finite interval $[\beta, \alpha]$ the abscissae are $z_i = (x_i(\alpha - \beta) + \alpha + \beta)/2$ while the weights are given by $y_i = w_i(\alpha - \beta)/2$. This type of Gaussian rule converges to the actual integral as $q \to \infty$. Applying Gaussian quadrature to evaluate $W(S, m)$, observing that the upper limit is $+\infty$ while the lower limit is finite. Truncating these limits to an interval that depends on the current stock price, i.e. $\beta = \max[\log(S(m - \Delta)/S), u_z - 6\sigma_z]$ and $\alpha = u_z + 6\sigma_z$. To value an option that has n exercise dates, the method works backwards from the maturity date. At each step the unexercised option value is found, then the location of the exercise boundary above which exercise is not optimal. At $m = \Delta$ an option that is not exercised will have the same value as a European option which is easily found using the Black-Scholes formula. To find the exercise boundary, $S^*(\Delta)$ must be determined such that it satisfies

$$K - S^*(\Delta) = U(S^*(\Delta), \Delta) \tag{6.74}$$

At $m = 2\Delta$, $U(S, 2\Delta)$ can be found using equation (6.71). For $W(S, 2\Delta)$, the Gauss-Legendre rule is applied with the abscissae z_i and weights y_i are chosen as described above. The approximation of the integral is given by

$$W(S, 2\Delta) \approx \sum_{i=1}^{q} U(Se^{z_i}, \Delta) f(z_i) y_i \tag{6.75}$$

It follows that the next value of the exercise boundary $S^*(2\Delta)$ can be determined by finding the solution to

$$K - S^*(2\Delta) = U(S^*(2\Delta), 2\Delta) \tag{6.76}$$

Function Approximation

When directly applied, the quadrature method described will become computationally intensive when the time to maturity increases. One alternative to improve the efficiency without the detriment of accuracy is to use function approximations. Following Sullivan[54], he assumes that $h(y)$ is a continuous function. The Weierstrass theorem states that if $h(y)$ is continuous in a finite interval, then there exists a polynomial of sufficiently high degree that reduces the maximum approximation error in the interval to a small value. The Chebyshev approximation can be used to fit the function where the theory states that the polynomial approximations come from set of all possible linear combinations of the p Chebyshev polynomials

$$T_j(x) = \cos(j \cos^{-1}(x)), j = 0, 1, ..., p-1 \text{ where } -1 \leq x \leq 1 \tag{6.77}$$

where the best approximation is the one where it equals $h(y)$ at the set of p zeros of $T_p(x), x_j = \cos(\pi(j-1/2)/p), j = 1, 2, ..., p$. To find coefficients c_j for the approximating function, choose the finite interval $[b, a]$ in which to approximate $h(y)$. Evaluate the function at p points, $y_j = (x_j(a-b)+a+b)/2$ which is scaled from the original interval $[-1, 1]$ to match the values of x_j and solve the system of equations

$$c_0 T_0(x_j) + c_1 T_1(x_j) + ... + c_{p-1} T_{p-1}(x_j) = h(y_j) \text{ for } j = 1, 2, ..., p \tag{6.78}$$

It follows that the function $h(y)$ is approximated at an arbitrary y by an initial rescaling to $x = (2y - (a+b))/(a-b)$ then the p-term polynomial is evaluated by

$$h(y) \approx \sum_{j=0}^{p-1} c_j T_j(x) \tag{6.79}$$

In order to reduce the size of the errors across the estimation interval, the option value function is approximated in terms of logs, where the log of option prices are fitted to log stock prices. The lower boundary interval will naturally be the exercise boundary. The upper boundary is chosen as the asset price at which the option value will be very small. Let $A(m)$ be this small asset price such that it satisfies $P(A(m), m) = P^*$, where P^* is the arbitrarily small value of the put option. The estimation interval becomes $[\log(S^*(m)), \log(A(m))]$. In this interval, locate the p points that will correspond to the zeros of $T_p(x)$, this allows the valuation of the option price by quadrature. The corresponding option prices are converted to logs, and the coefficients to the approximating polynomial are determined.

Accuracy and Efficiency

The important parameters that control this method are the number of quadrature points, the number of terms used in the Chebyshev approximation, and the number of exercise dates. Sullivan[54] tested the method for various values of p, q and n. A limitation of his method is to do with approximating the Chebyshev polynomials, a substantial increase in extrapolation errors occur when $p > 8$. He found that by introducing more exercise dates will require a higher degree of quadrature. A comparison of these results will be outlined further in Chapter 8.

Chapter 7

A New Algorithm

In the previous chapter, a selection of numerical methods was discussed for the pricing of American options, and in most instances the determination of the early exercise price was also described. The main issues to address when developing pricing methods would be the ability for the method to provide an efficient yet very accurate pricing of American options. Some problems that arise in the binomial method for example is that accuracy improves only with an increase in the number of time steps computed and therefore may not be a practical method in a fast dynamic option market. Analytic approximations are generally only accurate for shorter maturity dates. Quadrature rules require a high degree of precision for the prices to be accurate.

This chapter introduces a new algorithm to price American options. The algorithm uses the expression of the early exercise boundary and the American option integral equation in the Kolodner-McKean as well as the Jamshidian framework. Unlike existing algorithms such as the Recursive Method of Huang et. al [28] that require dividing the time interval $[0, T]$ into n subintervals and thus evaluating n implicit integral equations, the new algorithm requires only the determination of a small set of points within the time interval $[0, T]$. With each of these points called *knot points*, the solution to the integral corresponding to the free boundary is evaluated by numerical integration. By a least squares iterative scheme, the exercise boundary is solved to a high degree of accuracy and eventually used as part of the pricing solution. The monotonic function for the critical exercise boundary $S^*(t)$ will be represented as a cubic spline with respect to t. It is from cubic splines that the vast improvement in efficiency arises from the computation of the boundary and also the valuation of associated option prices can be made highly accurate.

7.1 Method of Cubic Splines in the Kolodner-McKean Framework

Recall in Chapter 5 that the solution to the general saltus problem $\mathbb{S}$ is given by

$$\tfrac{1}{2}f(t) = \bar{u}_0(c,t) + D_0\{f\}(c,t) - S_0\{\dot{c}f + g\}(c,t) \tag{7.1}$$

where for the case of the American put option

$$\begin{cases} f(t) &= e^{rt}[K - b(t)] \\ g(t) &= -\hat{\sigma}e^{rt}b(t) \\ c(t) &= [\log \frac{b(t)}{K} + \beta t]/\hat{\sigma} \\ \dot{c}(t) &= [\frac{\dot{b}(t)}{b(t)} + \beta]/\hat{\sigma} \\ \beta &= r - q - \hat{\sigma}^2; \quad (\hat{\sigma} = \sigma/\sqrt{2}) \end{cases} \tag{7.2}$$

Using the definitions for the single and double layer potentials

$$S_c\{\phi\}(x,t) = \int_0^t G(x - c(\tau), t - \tau)\phi(\tau)d\tau \tag{7.3}$$

and

$$D_c\{\phi\}(x,t) = -\int_0^t G_x(x - c(\tau), t - \tau)\phi(\tau)d\tau \tag{7.4}$$

where the Green's function $G(x,t)$ is defined by (5.9) and it's derivative by $G_x(x,t) = -(x/2t)G(x,t)$. The equivalent integral form is therefore

$$\tfrac{1}{2}f(t) - \bar{u}_0(c,t) = \int_0^t \frac{e^{-(t-\tau)\phi^2(t,\tau)}}{\sqrt{4\pi(t-\tau)}}\{\phi(t,\tau)f(\tau) - h(\tau)\}d\tau \qquad (7.5)$$

where

$$\phi(t,\tau) = \left[\frac{\log\{b(t)/b(\tau)\}}{t-\tau} + \beta\right]/2\hat{\sigma}^2 \qquad (7.6)$$

The term $\bar{u}_0(c,t)$ is given by equation (5.24) with the initial term $u_0(\xi) = K(1-e^{\hat{\sigma}\xi})^+$ for the American put option. The term $\bar{u}_0(c,t)$ can be evaluated analytically in terms of normal distribution functions. This leads to the result:

$$\begin{aligned}
\bar{u}_0(c,t) &= K\int_{-\infty}^{c_0} G(c(t)-\xi,t)(1-e^{\hat{\sigma}\xi})^+ d\xi \\
&= \frac{K}{\sqrt{4\pi t}}\int_{c_0}^{0} e^{-(c(t)-\xi)^2/4t}(1-e^{\hat{\sigma}\xi})d\xi; \quad u_0(\xi) = 0 \text{ for } \xi > 0 \\
&= I_1 - I_2
\end{aligned}$$

$$\begin{aligned}
I_1 &= \frac{K}{\sqrt{4\pi t}}\int_{c_0}^{0} e^{-(c(t)-\xi)^2/4t}d\xi \\
&= \frac{K}{\sqrt{2\pi}}\int_{\frac{c(t)-c(0)}{\sqrt{2t}}}^{\frac{c(t)}{\sqrt{2t}}} e^{-\frac{1}{2}v^2}dv; \quad \text{letting } v = (c(t)-\xi)/\sqrt{2t}, \\
&= \frac{K}{\sqrt{2\pi}}\left[\int_{-\infty}^{-\frac{c(t)}{\sqrt{2t}}} e^{-\frac{1}{2}v^2}dv - \int_{-\infty}^{-\frac{c(t)-c(0)}{\sqrt{2t}}} e^{-\frac{1}{2}v^2}dv\right] \\
&= K\left[\mathcal{N}(-y_2) - \mathcal{N}(-z_2)\right] \qquad (7.7)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{K}{\sqrt{4\pi t}}\int_{c_0}^{0} e^{-(c(t)-\xi)^2/4t+\hat{\sigma}\xi}d\xi \\
&= \frac{K}{\sqrt{4\pi t}}\int_{c_0}^{0} e^{-(\xi^2-2\xi(c(t)+\sqrt{2}\sigma t)+c^2(t))/4t}d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{K}{\sqrt{4\pi t}} e^{\frac{\sigma}{\sqrt{2}}(c(t)-\frac{\sigma}{\sqrt{2}}t)} \int_{c_0}^{0} e^{(\xi-(c(t)+\sqrt{2}\sigma t))^2/4t} \\
&= \frac{K}{\sqrt{2\pi}} e^{\frac{\sigma}{\sqrt{2}}(c(t)-\frac{\sigma}{\sqrt{2}}t)} \left[\int_{-\infty}^{-\frac{c(t)+\sqrt{2}\sigma t}{\sqrt{2t}}} e^{-\frac{1}{2}v^2} dv - \int_{-\infty}^{-\frac{c(0)-c(t)+\sqrt{2}\sigma t}{\sqrt{2t}}} e^{-\frac{1}{2}v^2} dv \right] \\
&= b(t)e^{(r-q)t} \left[\mathcal{N}(-y_1) - \mathcal{N}(-z_1) \right] \qquad (7.8)
\end{aligned}
$$

with

$$
\begin{aligned}
y_2(b,t) &= [\log\{b(t)/K\} + \beta t]/\sigma\sqrt{t} \\
y_1(b,t) &= y_2(b,t) + \sigma\sqrt{t}
\end{aligned} \qquad (7.9)
$$

and

$$
\begin{aligned}
z_2(b,t) &= [\log\{b(t)/b(0)\} + \beta t]/\sigma\sqrt{t} \\
z_1(b,t) &= z_2(b,t) + \sigma\sqrt{t}
\end{aligned} \qquad (7.10)
$$

A similar analysis for the American call option leads to an expression for $\bar{u}_0(c,t)$:

$$
\bar{u}_0(c,t) = b(t)e^{(r-q)t} \left[\mathcal{N}(y_1) - \mathcal{N}(z_1) \right] - K \left[\mathcal{N}(y_2) - \mathcal{N}(z_2) \right]. \qquad (7.11)
$$

It is also worth noting that $I_1 = I_2 = 0$ when $b(0) = K$.

Implementation of Cubic Spline

Recall that the critical exercise boundary $S^*(t) = b(t)$ was shown to be a monotonic increasing function in Chapter 4. The integral equation approach, requires the critical asset price boundary $S^*(t)$ to be determined as part of the solution to value the American option. Some of the methods described have been to solve the boundary using recursive methods by taking discrete points and solving the critical asset value at each discrete time step. As the number of discrete time step increases the accuracy of the solution improves, but computational efficiency will decrease. This leads to a new method of determining the critical asset price boundary using cubic splines. The basic method by cubic splines is to determine the values of the critical asset price at time knots selected between the option's starting date to expiry. There are two problems arising from using this approach. Firstly the integrand in equation (7.5) is singular in the vicinity of $\tau = t$ and secondly, $b(t)$ has an infinite slope at $t = 0$. Consider equation (7.6) as $\tau \to t$. Using L'Hôpital's rule to find the limit:

$$\lim_{\tau \to t} \dot{\phi}(t, \tau) = \tfrac{1}{2} \left[\frac{\dot{b}(t)}{2tb(t)} + \beta \right] / \hat{\sigma} \tag{7.12}$$

Clearly as $t \to 0$, $\dot{\phi}(t, \tau) \to \infty$. Therefore allowing a change of variables through

$$T = \sqrt{t}; \quad S = \sqrt{\tau}; \quad d\tau = 2SdS \tag{7.13}$$

to introduce a singularity at $t = 0$. Once this change is made, the exercise boundary $b(t)$ will now become

$$b(t) = b(T^2) = \tilde{b}(T) \tag{7.14}$$

and evaluated as a cubic spline in the variable T. A further change in variable follows the first step where allowing the the transformation

$$S = T \sin\theta; \qquad \frac{dS}{\sqrt{T^2 - S^2}} = d\theta \tag{7.15}$$

will remove the singularity and infinite slope to allow for easier integration and reduce equation (7.5) to

$$\tfrac{1}{2}\tilde{f}(T) - \tilde{u}_0(T) = \tfrac{1}{\sqrt{\pi}} \int_0^{\pi/2} e^{-(T^2 - S^2)\tilde{\phi}^2(T,S)} \{S\tilde{f}(S)\tilde{\phi}(S,T) - \tilde{h}(S)\} d\theta \tag{7.16}$$

with

$$\tilde{h}(S) = \dot{\tilde{b}}(S)\tilde{f}(S)/2\hat{\sigma}\tilde{b}(S) + S\tilde{g}(S) \tag{7.17}$$

and

$$\tilde{\phi}(T, S) = \left[\frac{\log\{\tilde{b}(T)/\tilde{b}(S)\}}{T^2 - S^2} + \beta \right] / 2\hat{\sigma}. \tag{7.18}$$

With the singularities addressed, the next step is to represent the exercise boundary $\tilde{b}(T)$ in terms of a cubic spline. It is important to understand the uses of cubic splines with some background information.

A Note on Cubic Splines

Assume for $N+1$ data points $\{(x_k, y_k)\}$ such that for a set of knot points,

$$a = x_0 < ... < x_N = b. \tag{7.19}$$

The function $S(x)$ is called a cubic spline interpolation if there exists N cubic polynomials $S_k(x)$ with coefficients $s_{k,i}$ $0 \leq i \leq 3$ such that the following hold

$$\begin{aligned}
&1. \quad S(x) &&= S_k(x) = \sum_{i=0}^{3} s_{k,i}(x - x_k)^i \quad \forall x \in [x_k, x_{k+1}]\ 0 \leq k \leq N-1 \\
&2. \quad S(x_k) &&= y_k \quad 0 \leq k \leq N \\
&3. \quad S_k(x_{k+1}) &&= S_{k+1}(x_{k+1}) \quad 0 \leq k \leq N-2 \\
&4. \quad S'_k(x_{k+1}) &&= S'_{k+1}(x_{k+1}) \quad 0 \leq k \leq N-2 \\
&5. \quad S''_k(x_{k+1}) &&= S''_{k+1}(x_{k+1}) \quad 0 \leq k \leq N-2
\end{aligned} \tag{7.20}$$

The cubic spline on the fixed set of knots will form a vector space for cubic spline addition and multiplication. The cubic spline interpolates the data $\{(x_k, y_k)\}$ and also matches the first and second derivative at the selected knot points. The result at each node will lead to a linear system of equations consisting of $4N$ coefficients and $4N$ linear conditions. A tridiagonal system can be formed with the coefficients $s_{k,3}$ the variables to be determined. Once solved, the remaining coefficients are easily found. There can be different choices of endpoints used, with the common choice being $S''(a) = S''(b) = 0$, or otherwise called the *natural spline*. A *clamped spline* gives fixed values $S'(a) = u, S'(b) = v$ for given values of u, v. In the case of using the splines numerically in MATLAB, the choices of end points follows the *not-a-knot* spline where the requirement is that the third derivative of the spline is continuous at the values x_1 and x_{N-1}.

Summary of Numerical Procedure

There are advantages of using MATLAB [30] for this method as it provides a good cubic spline function, numerical integration functions as well as various non-linear least squares solvers, all of which are required in this procedure. After the variable changes from (t, τ) to $(\sqrt{t}, \sqrt{\tau})$ and S to θ using $S = T \sin\theta$, the next step is to solve the function $\tilde{b}(T)$ at the selected knot points by applying a least-squares algorithm iteratively to equation (7.16). In order to apply the least squares algorithm to find $\tilde{b}(T)$, a starting value must be used in the parameter input to reduce the number of iterations to convergence. A convenient starting function would be to use a simple exponential in $\sqrt{t}$ that will have the correct value at expiry $t = 0$ and is asymptotic to the known perpetual solution as $t \to \infty$. The starting function used in this procedure is given by

$$\gamma_0(t) = \begin{cases} \left(\max\left[K, \frac{rK}{q}\right] - \frac{\mu}{1-\mu}\right) e^{-\varepsilon t} + \frac{\mu}{1-\mu} & \text{for calls} \\ \left(\min\left[K, \frac{rK}{q}\right] - \frac{\mu}{1+\mu}\right) e^{-\varepsilon t} + \frac{\mu}{1+\mu} & \text{for puts} \end{cases} \tag{7.21}$$

The seed values of $\gamma_0(t)$ are converted to a cubic spline using Matlab's **spline** M-function. This will produce a set of coefficients that when applied for values of t between two consecutive knot points, will create a spline curve that is the equivalent to the initial seed function. The derivative of $\tilde{b}(T)$, or $\dot{\tilde{b}}(T)$ is simply a piece-wise quadratic function of $\sqrt{t}$ which is obtained also from

Matlab using the **spline** and **mkpp** M-functions. The left hand side of equation (7.16) involves the function $\tilde{b}(T)$ and $\tilde{u}_0(T)$ which consists of the early exercise payoff and European option value evaluated at the critical exercise price. The computation of this part involves only analytical formulas related to the Black-Scholes formula. The right hand side requires integration.

Cubic Spline Representation

The boundary $b(t)$ can be represented by a cubic spline in the variable S,

$$b_S = b_0 + b_1 S + b_2 S^2 + b_3 S^3 \tag{7.22}$$

and its derivative $\dot{b}(t)$ is represented as

$$\dot{b}_S = b_1 + 2b_2 S + 3b_3 S^2 \tag{7.23}$$

which is well behaved near $S = 0$. For values near $S = T(\theta = \frac{\pi}{2})$, the value of $\tilde{\phi}(T, S)$ will also be well behaved as shown in (7.12).

For dealing with cubic splines in MATLAB, let $U = \sqrt{u}$ denote the left-hand knot when T is the right-hand knot and $U \leq S \leq T$. Then

$$\begin{aligned} b_S &= b_0 + b_1(S-U) + b_2(S-U)^2 + b_3(S-U)^3 \\ b_T &= b_0 + b_1(T-U) + b_2(T-U)^2 + b_3(T-U)^3 \end{aligned} \tag{7.24}$$

Hence

$$\tilde{\phi}(T, S) = \frac{b_T - b_S}{T^2 - S^2} \tag{7.25}$$

$$\begin{aligned} &= \frac{b_1 - 2Ub_2 + 3U^2 b_3}{T+S} + (b_2 - 3Ub_3) + \frac{b_3(T^2 + S^2 + TS)}{T+S} \\ &= \frac{b_1 - 2Ub_2 + 3U^2 b_3}{T(1+\sin\theta)} + (b_2 - 3Ub_3) + \frac{b_3 T(1 + \sin\theta + \sin^2\theta)}{T(1+\sin\theta)} \end{aligned} \tag{7.26}$$

which is also well behaved at $\theta = \pi/2$. The value for $\tilde{\phi}(T, S)$ will be evaluated using (7.25) when S is not near T and (7.26) when S is near T.

7.2 Application to a Toy Problem

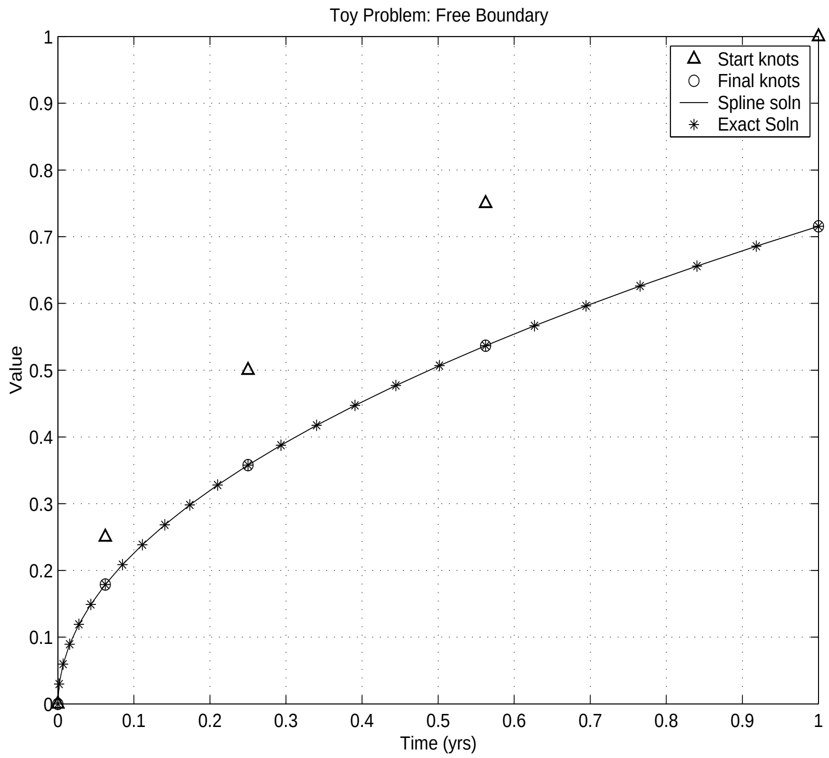


Figure 7.1: Comparison of exact (stars) and numerical (solid) solutions. Spline knots are denoted in circles; start solution by triangles

To test the accuracy of the algorithm, it is applied to a free-boundary problem that has a known analytical solution. Consider the "toy" problem $\mathbb{T}$ defined by

$$\mathbb{T}\left\{\begin{array}{rcl} \mathcal{H}\{u(x,t)\} & = & 0 \quad \text{in } \mathbb{D}_c^- \\ u(x,0) & = & 0 \quad \text{in } x < c_0 = 0 \\ u(c(t),t) & = & 1 \\ u_x(c(t),t) & = & \dot{c}(t) \\ u(-\infty,t) & = & 0 \quad \text{for all } t > 0 \end{array}\right. \tag{7.27}$$

where $\mathcal{H}\{\cdot\}$ is the heat equation operator operating in the domain $\mathbb{D}_c^- = \{(x,t)|x < c(t); t > 0\}$. The problem can be solved analytically by using a similarity solution $u(x,t) = U(\xi);\ \xi = \frac{x}{\sqrt{t}}$ and the free boundary $c(t) = a\sqrt{t}$. Then determining

$$\begin{aligned}
u_t &= -\tfrac{1}{2}\xi\frac{1}{t}U'(\xi) && (7.28)\\
u_{xx} &= \frac{1}{t}U''(\xi) && (7.29)\\
u_t = u_{xx} \Rightarrow U''(\xi) &= -\tfrac{1}{2}\xi U'(\xi) && (7.30)
\end{aligned}$$

The last expression can be solved by order reduction methods by letting $y = U'(\xi)$ and

$$y' + \tfrac{1}{2}\xi y = 0. \quad (7.31)$$

This first order differential equation has solution given by

$$y(\xi) = Ae^{-\frac{1}{4}\xi^2}. \quad (7.32)$$

Then back transforming into $U(\xi)$ yields the integral

$$U(\xi) = A\int_{-\infty}^{\xi} e^{-\frac{1}{4}v^2}dv \quad (7.33)$$

with the boundary conditions $U(a) = 1;\ U'(a) = \frac{1}{2}at^{-\frac{1}{2}}$:

$$\begin{aligned}
U(a) = 1 &= A\int_{-\infty}^{a} e^{-\frac{1}{4}v^2}dv \\
1 &= A\sqrt{2}\int_{-\infty}^{a/\sqrt{2}} e^{-\frac{1}{2}z^2}dz \quad \text{i.e. } v = \frac{z}{\sqrt{2}} \\
1 &= A\sqrt{2}\sqrt{2\pi}\mathcal{N}(\frac{a}{\sqrt{2}}) && (7.34)\\
U'(a) = \tfrac{1}{2}at^{-\frac{1}{2}} &= Ae^{-\frac{1}{4}a^2}\frac{\partial\xi}{\partial x} \\
A &= \tfrac{1}{2}ae^{\frac{1}{4}a^2} && (7.35)
\end{aligned}$$

The solution can be given explicitly by the following:

$$\begin{cases} u(x,t) &= a\sqrt{\pi}e^{\frac{1}{4}a^2}\mathcal{N}(\frac{x}{\sqrt{2t}}) \\ c(t) &= a\sqrt{t} \\ 1 &= a\sqrt{\pi}e^{\frac{1}{4}a^2}\mathcal{N}(\frac{a}{\sqrt{2}}) \\ a &= 0.715669 \end{cases} \tag{7.36}$$

The corresponding Kolodner-McKean integral equation for the free-boundary function $c(t)$ is:

$$\tfrac{1}{2} = \int_0^t \frac{e^{-(t-\tau)\phi^2(t,\tau)}}{\sqrt{4\pi(t-\tau)}}\{2\dot{c}(\tau) - \phi(t,\tau)\}d\tau \tag{7.37}$$

with

$$\phi(t,\tau) = \frac{c(t) - c(\tau)}{2(t-\tau)} \tag{7.38}$$

Figure 7.2 displays the results of this comparison. In this particular example, spline knots were chosen uniformly in $\sqrt{t}$ between $t = 0$ and $t = 1$. The starting solution used was $c_0(t) = \sqrt{t}$. The resulting solution of the toy problem by cubic splines matches exactly with the true solution determined by analysis. This result can be used as motivation to determine solutions of more complicated forms of the free boundary problem, namely $\mathbb{C}_A$ and $\mathbb{P}_A$ described in chapter 5.

7.3 The Cubic Spline Method in the Jamshidian Framework

The integral equations arising from the Jamshidian [33] volume potential framework were derived in chapter 5 and are given by

$$\begin{cases} K - S^*(t) &= P^E(S^*(t),t) + P^X(S^*(t),t) \\ P^E(S^*(t),t) &= Ke^{-rt}\mathcal{N}(-d_2) - S^*(t)e^{-qt}\mathcal{N}(-d_1) \\ P^X(S^*(t),t) &= \int_0^t\{rKe^{-r\tau}\mathcal{N}(-z_2) - qS^*(t)e^{-q\tau}\mathcal{N}(-z_1)\}d\tau \end{cases} \tag{7.39}$$

and

$$\begin{cases} S^*(t) - K &= C^E(S^*(t), t) + C^X(S^*(t), t) \\ C^E(S^*(t), t) &= S^*(t)e^{-qt}\mathcal{N}(d_1) - Ke^{-rt}\mathcal{N}(d_2) \\ C^X(S^*(t), t) &= \int_0^t \{qS^*(t)e^{-q\tau}\mathcal{N}(z_1) - rKe^{-r\tau}\mathcal{N}(z_2)\}d\tau \end{cases} \tag{7.40}$$

where

$$d_{1,2}(t) = \{\log[\tfrac{S^*(t)}{K}] + (r - q \pm \tfrac{1}{2}\sigma^2)t\}/\sigma\sqrt{t} \tag{7.41}$$

and

$$z_{1,2}(t) = \{\log[\tfrac{S^*(t)}{S^*(t-\tau)}] + (r - q \pm \tfrac{1}{2}\sigma^2)t\}/\sigma\sqrt{t} \tag{7.42}$$

It is clear from these representations that the derivative of the critical asset boundary $\dot{S}^*(t)$ is not required and therefore would clearly simplify the numerical algorithm.

7.4 Comparison between the Kolodner-McKean and Jamshidian Cubic Spline Valuation

The numerical method using the Kolodner-McKean and the Jamshidian framework is introduced, beginning with a comparison of prices and computational time between the two methods. Table 7.1 displays the values of an American put option that has an exercise price of $K = \$100$, an expiry of $T = 1$ year, a risk-free rate of $r = 8\%$, a volatility of $\sigma = 40\%$ and a continuous dividend yield of $q = 0\%$.

Both the Kolodner-McKean and Jamshidian methods exhibit identical early exercise boundaries and it can be shown analytically (see below) that the two integral equations are indeed equivalent.

Equivalence of Solutions

Recall the Jamshidian solution for the American put option is given by equation (5.48). Let $\hat{G}(x, \xi; t, \tau) = G(x - \xi, t - \tau)H(c(\tau) - \xi)$ and $v(x, t) =$

S	Koldner-McKean	Jamshidian
80	22.8752	22.8751
90	17.0368	17.0368
100	12.5992	12.5992
110	9.2675	9.2676
120	6.7915	6.7915
130	4.9654	4.9654
140	3.6262	3.6262
150	2.6477	2.6478
160	1.9345	1.9345

Table 7.1: Comparison of American put option values using the Kolodner-McKean and Jamshidian methods for varying asset price S.

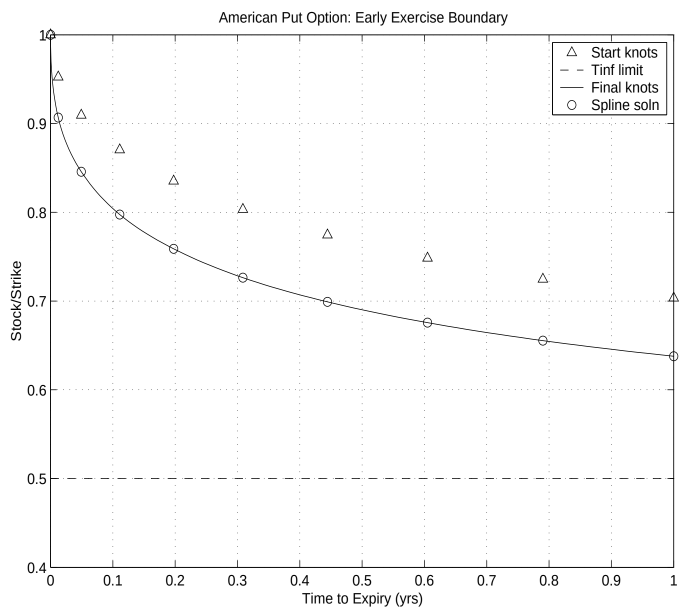


Figure 7.2: Early exercise boundary for an American put option for $r = 8\%$, $q = 0\%$, $\sigma = 40\%$, $K = \$100$ and $T = 1$ year

$e^{rt}(K - S(x,t))$, so that $w(x,t) = \mathcal{H}\{v\}H(c(t) - x)$. Then the double inte-

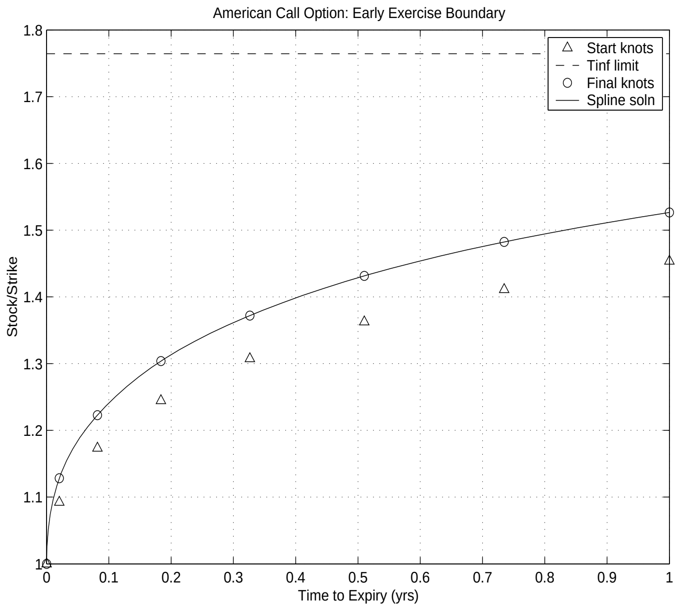


Figure 7.3: Early exercise boundary for an American call option for $r = 8\%$, $q = 15\%$, $\sigma = 40\%$, $K = \$100$ and $T = 1$ year

gral in equation (5.48) can be expressed in the form:

$$I = \int_0^\infty \int_{-\infty}^\infty \hat{G}(v_\tau - v_{\xi\xi}) d\tau d\xi \tag{7.43}$$

Integrating by parts once with respect to τ and twice with respect to ξ will result in an equivalent representation:

$$I = \int_{-\infty}^{c_0} G(x-\xi, t) v_0(\xi) d\xi - \int_0^\infty d\tau \int_{-\infty}^\infty [\hat{G}_\tau + \hat{G}_{\xi\xi}] v(\xi, \tau) d\xi \tag{7.44}$$

Differentiating $\hat{G}(x, \xi; t, \tau)$ with respect to τ and ξ will obtain

$$\hat{G}_\tau + \hat{G}_{\xi\xi} = -[G_t - G_{xx}] H(c(\tau) - \xi) + (G\dot{c}(\tau) + G_x)\delta(c(\tau) - \xi) + G\delta'(c(\tau) - \xi) \tag{7.45}$$

where $G = G(x - \xi, t - \tau)$. Using the result $G_t - G_{xx} = \delta(x - \xi)\delta(t - \tau)$,(a property obtained from the Green's function from equation (5.8)), standard integral properties of Dirac delta functions and the definitions $v(c,\tau) = f(\tau), v_x(c,\tau) = g(\tau)$, will yield the result

$$u(x,t) = \phi(x,t) + \bar{u}_0(x,t) + D_c\{f\}(x,t) - S_c\{\dot{c}f + g\}(x,t) \tag{7.46}$$

with $\phi(x,t) = v(x,t)H(c(t) - x)$. Observe the limits of the first integral in equation (5.48) is over $(-\infty, \infty)$ that combines with the first integral of (7.44) that has limits over $(-\infty, c_0)$ to produce an integral with limits over (c_0, ∞) for $\bar{u}_0(x,t)$ which is defined by (5.24). Equation (7.46) is the transformed Jamshidian solution and is seen to be identical to the Kolodner/McKean solution (5.19) except for the additional $\phi(x,t)$ term. However this term is identically zero in the active domain $\mathbb{D}_c^+$ where the equivalence of the solution is proven. A similar analysis can confirm the equivalence of the Kolodner/McKean and Jamshidian solutions for American call options. More details of this analysis is available in the appendices.

7.5 Comparative Statics

As with most pricing valuations, the determination of the hedging parameters are equally as important as the American option price. The Kolodner-McKean and Jamshidian methods readily produce these hedge parameters. Consider again the expressions from chapter 5 for the American put option:

$$\begin{cases} P(S,t) &= P^E(S,t) + P^X(S,t) \\ P^E(S,t) &= Ke^{-rt}\mathcal{N}(-d_2) - Se^{-qt}\mathcal{N}(-d_1) \\ P^X(S,t) &= \int_0^t \{rKe^{-r\tau}\mathcal{N}(-z_2) - qSe^{-q\tau}\mathcal{N}(-z_1)\}d\tau \end{cases} \tag{7.47}$$

where

$$d_{1,2}(t) = \{\log[\tfrac{S}{K}] + (r - q \pm \tfrac{1}{2}\sigma^2)t\}/\sigma\sqrt{t} \tag{7.48}$$

and

$$z_{1,2}(\tau) = \{\log[\tfrac{S}{S^*(t-\tau)}] + (r - q \pm \tfrac{1}{2}\sigma^2)\tau\}/\sigma\sqrt{\tau} \tag{7.49}$$

The American put option **delta** $\Delta_P(S,t)$ is defined as the rate of change of the option price relative to the underlying price

$$\begin{aligned}
\Delta_P(S,t) = \frac{\partial P}{\partial S} &= \frac{\partial}{\partial S}[P^E(S,t) + P^X(S,t)] \\
&= \frac{\partial P^E}{\partial S} - \frac{rK}{\sigma S}\int_0^t \frac{e^{-r(t-\tau)}}{\sqrt{t-\tau}} n[z_2] d\tau \\
&\quad + \frac{q}{\sigma}\int_0^t \frac{e^{-q(t-\tau)}}{\sqrt{t-\tau}} n[z_1] d\tau - q\int_0^t e^{-q(t-\tau)}\mathcal{N}[-z_1] d\tau.
\end{aligned} \tag{7.50}$$

where

$$\frac{\partial P^E}{\partial S} = \frac{-Ke^{r(t-\tau)}}{\sigma\sqrt{t-\tau}} n[d_2] + \frac{Se^{-q(t-\tau)}}{\sigma\sqrt{t-\tau}} n[d_1] - e^{-q(t-\tau)}\mathcal{N}[-d_1] \tag{7.51}$$

and $n[x] = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Now observing the first term from (7.51) that:

$$\frac{-Ke^{-r(t-\tau)}}{\sigma\sqrt{t-\tau}} n[d_2] \equiv \frac{-Ke^{-r(t-\tau)}}{\sigma\sqrt{2\pi(t-\tau)}} e^{-\frac{1}{2}\{\log[\frac{S}{K}]+(r-q-\frac{1}{2}\sigma^2)(t-\tau)\}^2/\sigma^2(t-\tau)} \tag{7.52}$$

$$\begin{aligned}
&= \frac{-Ke^{-r(t-\tau)}}{\sigma\sqrt{2\pi(t-\tau)}} e^{-\frac{1}{2}\{\log[\frac{S}{K}]+(r-q-\frac{1}{2}\sigma^2)(t-\tau)\}^2/\sigma^2(t-\tau) - \frac{1}{2}\sigma^2(t-\tau) - (r-q+\frac{1}{2}\sigma^2)(t-\tau) + \log[\frac{S}{K}]} \\
&= \frac{-Se^{-q(t-\tau)}}{\sigma\sqrt{2\pi(t-\tau)}} e^{-\frac{1}{2}\{\log[\frac{S}{K}]+(r-q+\frac{1}{2}\sigma^2)(t-\tau)\}^2/\sigma^2(t-\tau)} \\
&= \frac{-Se^{-q(t-\tau)}}{\sigma\sqrt{(t-\tau)}} n[d_1].
\end{aligned}$$

The first two terms cancel and simplifies (7.51) into (a well known result):

$$\frac{\partial P^E}{\partial S} = -e^{-q(t-\tau)}\mathcal{N}[-d_1] \tag{7.53}$$

A similar analysis for the first term of the integral in equation (7.50)

$$\frac{-rKe^{-r(t-\tau)}}{\sigma S\sqrt{2\pi(t-\tau)}} n[z_2] \equiv \frac{-rKe^{-q(t-\tau)}}{\sigma S^*(t)\sqrt{2\pi(t-\tau)}} n[z_1] \tag{7.54}$$

and therefore the American put option delta $\Delta_P(S,t) = \frac{\partial P}{\partial S}$ is re-expressed as:

$$\Delta_P(S,t) = -e^{-qt}\mathcal{N}[-d_1] - \int_0^t e^{-q(t-\tau)} \left\{ \frac{rK - qS^*(\tau)}{\sigma S^*(t)\sqrt{t-\tau}} n[z_1] + q\mathcal{N}[-z_1] \right\} d\tau. \tag{7.55}$$

By differentiating $\Delta_P(S,t)$ further with respect to the asset price S yields the option **gamma** $\Gamma_P(S,t)$, which is the rate of change of delta with respect to the asset price:

$$\begin{aligned} \Gamma_P(S,t) &= \frac{\partial}{\partial S}\Delta_P(S,t) \\ \frac{\partial^2 P}{\partial S^2} &= \frac{e^{-qt}}{\sigma S\sqrt{t}} n[d_1] - \frac{rK}{\sigma^2 S}\int_0^t \frac{e^{-q(t-\tau)}}{S^*(\tau)(t-\tau)} n[z_1] z_1 d\tau \\ &\quad + \frac{q}{\sigma^2 S}\int_0^t \frac{e^{-q(t-\tau)}}{(t-\tau)} n[z_1] z_1 d\tau - \frac{1}{\sigma S}\int_0^t \frac{e^{-q(t-\tau)}}{\sqrt{t-\tau}} n[z_1] z_1 d\tau \\ &= \frac{e^{-qt}}{\sigma S\sqrt{t}} n[d_1] - \frac{1}{\sigma^2 S}\int_0^t \frac{e^{-q(t-\tau)}}{(t-\tau)} n[z_1] \left\{ \frac{rKz_1}{S^*(\tau)} - qz_2 \right\} d\tau. \end{aligned} \tag{7.56}$$

Differentiation of equation (7.47) with respect to the volatility parameter σ yields the **vega** which is interpreted as the option price's change relative to the volatility:

$$\begin{aligned} \frac{\partial P}{\partial \sigma} &= Se^{-(r-q)t} n[d_1]\sqrt{t} \\ &\quad + S\int_0^t e^{-(r-q)(t-\tau)} n[z_1] \left\{ \frac{rKz_1 - (r-q)S^*(\tau)z_2}{\sigma S^*(\tau)} \right\} d\tau. \end{aligned} \tag{7.57}$$

The change in option price with respect to the strike price K and the time to maturity t are explicit functions of the delta and gamma functions namely:

$$\kappa = \frac{\partial P}{\partial K} = \frac{1}{K}\left\{P(S,t) - S\Delta_P(S,t)\right\} \tag{7.58}$$

and

$$\Theta = \frac{\partial P}{\partial t} = \tfrac{1}{2}\sigma^2 S^2 \Gamma_P(S,t) + (r-q)S\Delta_P(S,t) - rP(S,t). \tag{7.59}$$

Chapter 8

A Comparison of Valuation Methods

This chapter demonstrates the efficiency and accuracy of the cubic spline method. Initially a comparison of efficiency is made with existing numerical methods, with an emphasis on the computational speed of each. Following this comparison are tests of accuracy relative to a benchmark model. The aim is to determine whether the cubic spline method is a computationally efficient and accurate method for valuing American options, hence could be used in practice. Another comparison is made on the valuation of the early exercise boundary in particular the behaviour of the boundary close to expiry. Some concluding remarks will be made on the cubic spline method.

8.1 Comparison of Efficiency

In order to test the efficiency of the cubic spline method, the numerical methods discussed in chapter 6 have been coded specially in MATLAB following actual descriptions described by the various literature. The code used for the comparisons can be accessed by CD in the appendices.

Measure of Efficiency

The experiment conducted to compare the efficiency of the numerical methods will involve measuring the computing time (CPU time) taken to price a series of call or put options over varying stock price S, given the known financial parameters r, q, K, σ, T, or interest rate, dividend yield, strike price, volatility and option maturity respectively. The computer used to conduct the efficiency test operates using an Intel 1.73 GHz processor.

The MATLAB code for each numerical method have been prepared such that it would produce the required call and put option prices for a range of stock prices $S = 80, 90, 100, 110, 120$. The first set of results are displayed in table 8.1 where $r < q$. The interest rate $r = 8\%$, dividend yield $q = 12\%$, strike price $K = 100$, volatility $\sigma = 20\%$ and maturity $T = 0.25$ years remain constant. The second set of results are shown in table 8.2 where the financial parameters remain the same as for table 8.1 except that $r > q$, namely $r = 12\%$ and $q = 8\%$. The CPU time shown in tables 8.1 and 8.2 are measured from the beginning of the code until the end using the MATLAB timing functions **tic** and **toc**.

Overview of Numerical Methods in MATLAB

Each of the numerical methods to price American options in table 8.1 have been coded in MATLAB according to the exact description of the relevant literature which was described in chapter 5 and 6.

The binomial method uses $N = 10,000$ time steps with the real-world probability $\hat{p} = 0.5$. The binomial method will also be used later as the benchmark solution ($N = 50,0000$) to compare the accuracy of the existing pricing methods including the new methods by cubic spline.

The finite-difference method uses the Crank-Nicholson Projected Successive Over-Relaxation (SOR) method which overcomes the stability problems of standard finite-difference schemes such as the implicit or explicit finite-difference methods. The stock price and time mesh (S, τ) has intervals of size $ds = 0.01$ or 1 cent and $dt = 0.2/365$ of 1 year respectively.

The critical exercise price $S^*(t)$ used in the valuation by quadratic approximation of McMillan[45] is found using a non-linear least squares minimisation, **lsqnonlin** and once found is used to value the American call or put option subject to the approximation formula given by equations (5.95) and (5.98).

Similarly the recursive integration methods of Huang *et al.* [28], require the critical exercise price to be determined prior to the valuation of the American option by the 3-point Richardson extrapolation methods. Evaluating $S^*(t)$ specifically at $\tau = \frac{T}{2}, \frac{T}{3}, \frac{2T}{3}$ can be done by interpolating from the predetermined critical exercise boundary that has $n = 1000$ time steps between $\tau = 0$ to $\tau = T$. The method of lines code in MATLAB was converted from the original FORTRAN code supplied by Meyer, however sections of the fortran code involving loops were vectorized in MATLAB.

The valuation by Monte Carlo methods follows the one described by Tilley [55], where the optimal choice of Q bundles each containing P sample paths where the total number of paths $R = QP$ is selected such that $P = 70$, $Q = 72$ and $R = PQ = 5040$.

Finally, the specifications for the Cubic Spline Method (CSM) of Jamshidian, and Kolodner-McKean for the purposes of the comparison will be as follows. The tolerance factor used in the non-linear least squares calculation has been set at 10^{-10}. An experiment has been conducted to find the optimal tolerance level at which the option prices evaluated using CSM has the lowest root-mean squared error (RMSE) relative to the benchmark option prices. These results are shown in the later section of this chapter. The other important consideration is the number of spline-knots to be used in the evaluation. The selection of only a few knot points is one of the main advantages of the CSM for computational efficiency. For this section the number of spline-knots used in the Jamshidian and Kolodner-McKean methods is 4 knots, but an optimal choice of spline-knots can be found, one can investigate if the spline-knots $\kappa(t)$ may be a function of time to maturity. The observation can be made that a smaller κ can be used for shorter maturities T. As the initial calculation of the critical exercise boundary $S^*(t)$ is independent of the American option valuation, once $S^*(t)$ is known up to a time to maturity T, then the pricing of American call and puts $C(S,\tau)$ and $P(S,\tau)$ can be evaluated for maturities $\tau \leq T$. This is the other advantage of this type of method, as the possibility of future work may involve the valuation of $S^*(t)$ for varying values of r, q, σ and K, to create a mesh for which the pricing of $C(S,r,q,\sigma,K,\tau)$ and $P(S,r,q,\sigma,K,\tau)$ can be readily obtained.

Observations from Efficiency Test

In the case where $r < q$, the CSMs are quickest followed by the binomial method with $N = 10000$ time steps. The finite difference, quadratic approximation and method of lines follow next. The recursive integration and Monte

	Stock Price	Binomial N=10000	Finite Difference	Quadratic Approx.	Method of Lines	Recursive Integration	Monte Carlo	Jamshidian Method	Kolodner McKean
Puts	80	20.4140	20.4140	20.4211	20.4128	20.4138	21.0278	20.4133	20.4140
	90	11.2496	11.2471	11.2515	11.2485	11.2498	11.9695	11.2498	11.2498
	100	4.3966	4.3903	4.3969	4.3951	4.3964	5.1735	4.3964	4.3964
	110	1.1176	1.1205	1.1179	1.1171	1.1178	1.2515	1.1178	1.1178
	120	0.1844	0.1844	0.1845	0.1843	0.1844	0.2043	0.1844	0.1844
CPU	Time(s)	8.054	11.204	26.844	61.859	175.375	283.016	3.516	4.672
Calls	80	0.0294	0.0295	0.0308	0.0294	0.0291	0.0346	0.0294	0.0294
	90	0.5801	0.5778	0.5807	0.5802	0.5719	0.5903	0.5803	0.5803
	100	3.5251	3.5186	3.4778	3.5248	3.4797	3.1887	3.5250	3.5250
	110	10.3565	10.3534	10.1024	10.3572	10.4558	9.8618	10.3578	10.3581
	120	20.0000	20.0000	20.0000	20.0000	19.9820	19.2825	20.0000	20.0000
CPU	Time(s)	8.070	15.764	12.937	61.859	175.375	65.312	3.250	3.250

Table 8.1: Comparison of American put and call option values using the Binomial, Finite difference, Quadratic Approximation, Method of Lines, Recursive Integration, Monte Carlo and the two new methods: Jamshidian and Kolodner-McKean methods. $S = 80, 90, 100, 120$, $r = 8\%$, $q = 12\%$, $\sigma = 20\%$, $K = 100$ and $T = 0.25$ years

	Stock Price	Binomial N=10000	Finite Difference	Quadratic Approx.	Method of Lines	Recursive Integration	Monte Carlo	Jamshidian Method	Kolodner McKean
Puts	80	20.0000	20.0000	20.0000	20.0000	19.9605	19.5596	20.0000	20.0000
	90	10.1978	10.19491	9.8116	10.1980	10.3032	10.1995	10.1995	10.1978
	100	3.5251	3.5185	3.4516	3.5248	3.4797	3.5346	3.5251	3.5249
	110	0.7832	0.7805	0.7763	0.7830	0.7718	0.8830	0.7834	0.7833
	120	0.1124	0.1123	0.1131	0.1124	0.1112	0.1125	0.1125	0.1125
CPU	Time(s)	5.999	11.360	12.938	58.266	266.156	430.531	2.266	2.890
Calls	80	0.0516	0.0520	0.0517	0.0517	0.0517	0.0249	0.0517	0.0517
	90	0.8406	0.8383	0.8410	0.8405	0.8408	0.5856	0.8408	0.8408
	100	4.3966	4.39036	4.3972	4.3951	4.3964	3.3604	4.3964	4.3964
	110	11.5460	11.5432	11.5489	11.5450	11.5462	10.0559	11.5461	11.5462
	120	20.6905	20.6901	20.6993	20.6982	20.6905	19.2862	20.6905	20.6903
CPU	Time(s)	5.985	11.813	14.391	61.188	173.031	54.578	5.627	6.563

Table 8.2: Comparison of American put and call option values using the Binomial, Finite difference, Quadratic Approximation, Method of Lines, Recursive Integration, Monte Carlo and the two new methods: Jamshidian and Kolodner-McKean methods. $S = 80, 90, 100, 120$, $r = 12\%$, $q = 8\%$, $\sigma = 20\%$, $K = 100$ and $T = 0.25$ years

Carlo methods fared badly, with the recursive integration requiring 175.375 seconds to calculate the required prices. The Monte Carlo methods have a large variation of efficiency between the valuation of puts and calls, with the calls taking 65.312 seconds and the puts taking 283.016 seconds to compute.

For $r > q$ the computing times mirror the previous case, however the call prices seem to to take slightly longer to compute than the put options. Overall the speed of the CSM is superior to the existing numerical methods.

Testing the accuracy of the Cubic Spline Method

To test the accuracy of the CSM, a benchmark valuation was created using the binomial method calculated at $N = 50,000$ time-steps for a range of call and put options. Table 8.3 displays the root mean-squared errors of each of the methods relative to the benchmark binomial method for the same set of parameters used in tables 8.1 and 8.2. Taking the square of the differences between the value of the benchmark method (Binomial $N = 50,000$) versus the quantified method, then taking the square root of the average mean square error yields the following table for the cases where $r < q$ and $r > q$ respectively.

Although most of the numerical methods exhibit very good accuracy relative to the benchmark, the Jamshidian and the Kolodner McKean methods have very small errors comparable to the binomial method with $N = 10,000$ time-steps. Combined with the computational efficiency of the cubic spline methods where they were shown to be at least 40-50% efficient, these new methods offers the best tradeoff in terms of accuracy and efficiency.

	Binomial N=10000	Finite Difference	Quadratic Approx.	Method of Lines	Recursive Integration	Monte Carlo	Jamshidian Method	Kolodner McKean
RMSE $r < q$	0.000158	0.407167	0.105587	0.000642	0.044633	0.631072	0.000572	0.000616
RMSE $r > q$	0.000163	0.003655	0.16055	0.003186	0.04973	0.960569	0.000702	0.000135

Table 8.3: Root Mean Squared Errors of the the Binomial, Finite difference, Quadratic Approximation, Method of Lines, Recursive Integration, Monte Carlo and the two new methods: Jamshidian and Kolodner-McKean methods relative to the benchmark Binomial method using $N = 50,000$ time steps.

8.2 Calculating the Early Exercise Boundary

Determining the early exercise boundary is still a matter of interest because most valuation methods require it a priori to evaluate the price of the American option. Secondly, it gives a visual aspect of a variable that represents the decision of holding the option to its maturity date or exercising the option early to take advantage of the option's intrinsic value. The solid bold line in

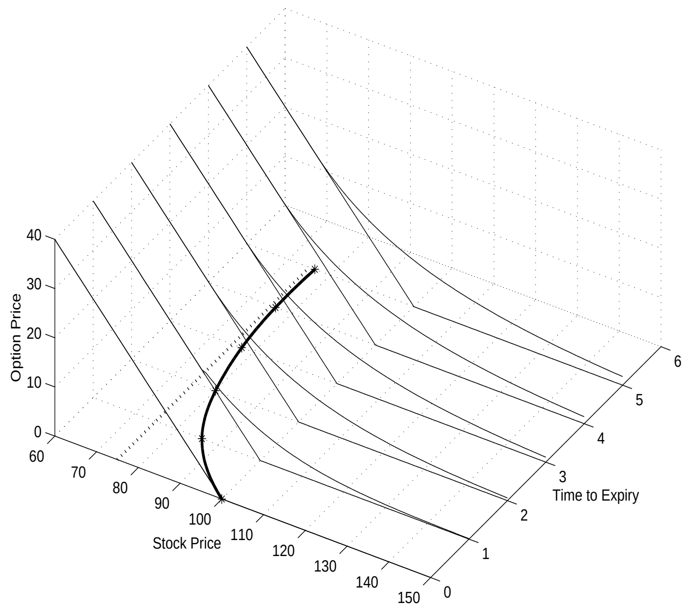


Figure 8.1: Formation of the early exercise boundary for an American put option

figure 8.1 is the result of plotting the early exercise value on the stock price versus time to expiry spatial grid. The early exercise price at each time to expiry segment, is the point where the American option value curve crosses the intrinsic value line. It is clearly visible that the curve is a monotonic decreasing function and as time increases, the early exercise boundary reaches the known perpetual value represented by the faint dotted line. In the example $S^*(\infty) = 75$, when the early exercise boundary for the American put option is evaluated at time $T = 0, 1, 2, 3, 4, 5$ years with $r = 12\%$, $q = 8\%$, $\sigma = 20\%$, $K = 100$.

Comparison of Early Exercise Boundaries

Almost all of the valuation methods discussed in this thesis can produce the early exercise boundary for American call and put options. The binomial and finite-difference methods do not require the computation of the early exercise boundary to price the American options, however the boundary can be obtained using dynamic programming, checking the existence of early exercise at each price step. In the quadratic approximation method, a function approximation for the early exercise price $S^*(\tau)$ was found but is only accurate for options with shorter time to expiry. By the method of lines, there is a requirement to refine the spatial grid to improve the accuracy of the early exercise boundary, but this will mean a loss of computational efficiency. Similarly the recursive integration technique will require a reasonably sized time steps to successfully price the American option. To test the accuracy of each method, a comparison of early exercise boundaries was superimposed and plotted in figures 8.2 and 8.3. This includes the cubic spline methods following the valuation method of Jamshidian and Kolodner-McKean.

In the case of $r > q$, all the numerical methods used to determine the early exercise boundary agree with the exception of the quadratic approximation where it appears in figure 8.2 to overvalue the critical price $S^*(t)$ as $T \to 0.25$. For $r < q$, the method of lines computation has difficulty calculating the boundary near $T = 0$. The finite difference scheme can evaluate the free boundary and produces the smooth monotonic solution albeit with

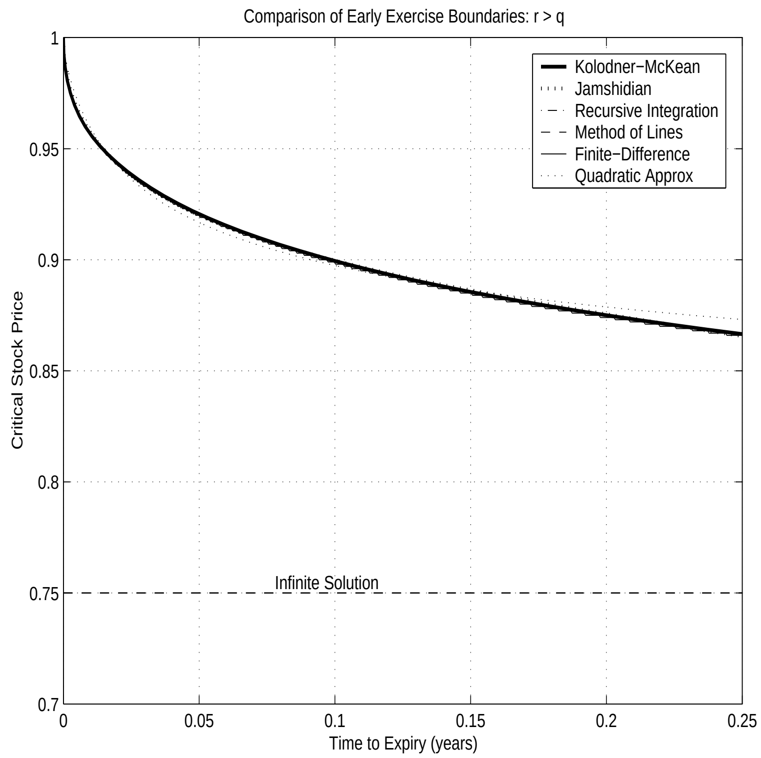


Figure 8.2: Comparison of the early exercise boundaries for an American put option for the case when $r > q$.

less efficiency than that of the CSM of Jamshidian and Kolodner-McKean.

Early Exercise Boundary for Longer Maturity Dates

So far the early exercise boundaries that have been examined are for maturities of $T = 0.25$ years or less. The computation of the boundary was amenable by all the methods tested with the CSM providing the most efficient evaluation. This is due to the use of the spline-knots $\kappa(t)$ to evaluate the boundary. The function $\kappa(t)$ is not to be confused with the term used in the previous chapter discussing the comparative statics of the option price. For maturity dates less than $T = 0.25$, $\kappa(T) = 4$ is sufficient. However as the

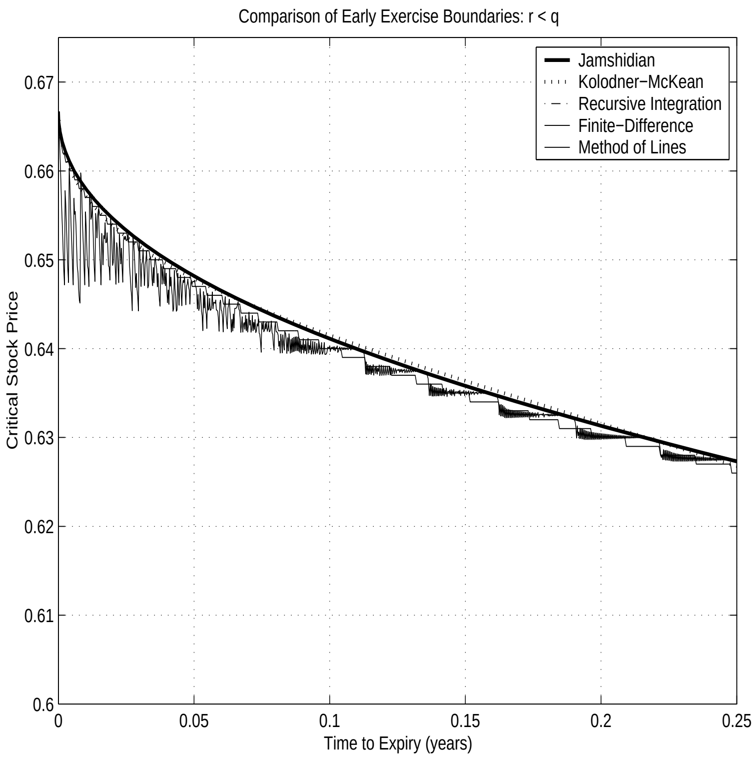


Figure 8.3: Comparison of the early exercise boundaries for an American put option for the case when $r < q$.

time to maturity lengthens, this may not be the case as shown in figure 8.4. By close observation, increasing the number of spline knots $\kappa(t)$ will allow the free boundary to converge to the "true" solution which for the purposes of conducting this particular test, the Kolodner-McKean cubic spline method was used with $\kappa = 12$ knots. To find the relationship of spline knots with respect to time, the root mean squared error (RMSE) of each the 4,5,6,7,8 knot points used to solve the boundary are found for a range of varying maturity dates ranging from $T = 0.25$ to $T = 5$ years. By observation the choice of $\kappa(t)$ is as follows:

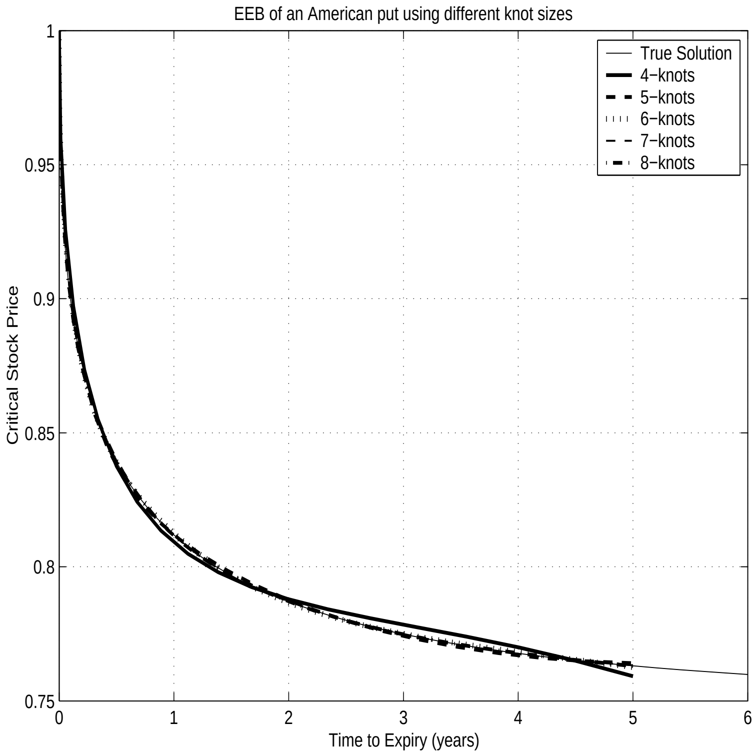


Figure 8.4: The Early-exercise boundaries for an American put for various choices of spline knots.

Time to Maturity T (years)	Knots $\kappa(t)$
$T < 0.5$	4
$0.5 < T < 1$	5
$1 < T < 1.5$	6
$1.5 < T < 2.5$	7
$T > 2.5$	8

Table 8.4: Choice of knot sizes for various maturity dates.

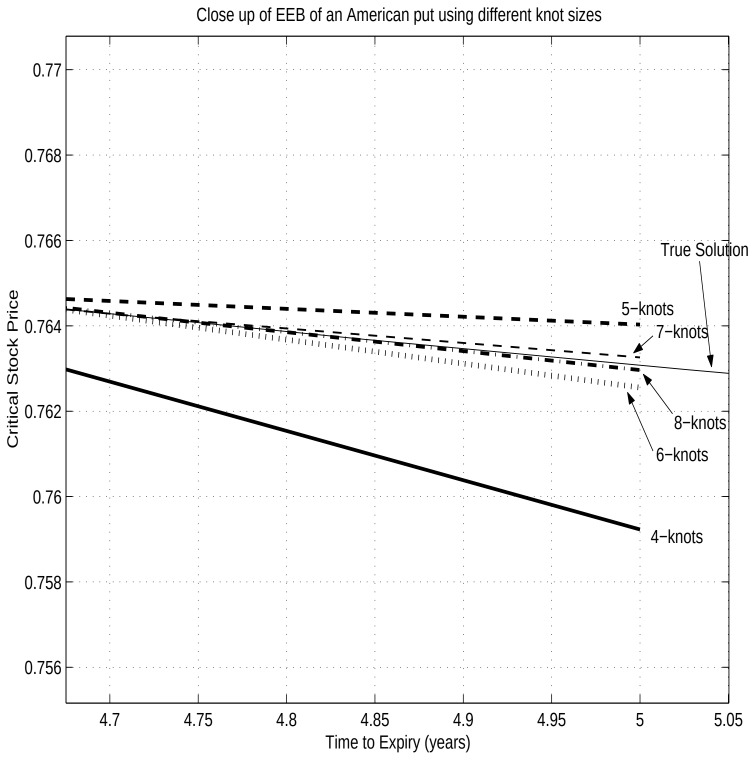


Figure 8.5: A close up of the spline knots compared with the true solution.

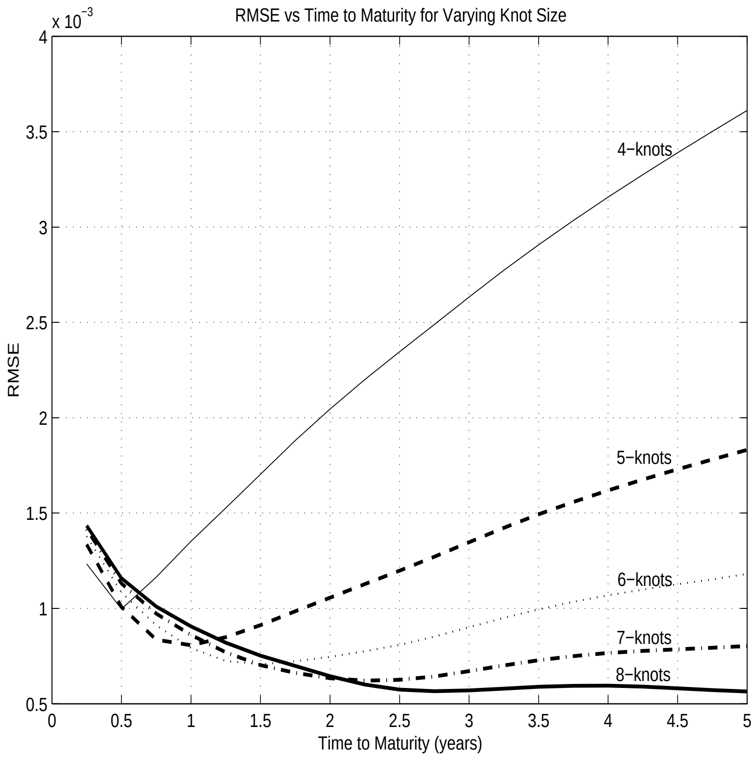


Figure 8.6: The root mean squared errors for the early exercise boundary of an American put for various choices of spline knots.

Chapter 9

Conclusion

The pricing of options plays an important role in many aspects of finance. Therefore practitioners, market participants and academics have developed various option pricing methods with a high regard to accuracy and efficiency. The analytical valuation of European options introduced by Black and Scholes paved the way for academics, to pursue the pricing of American options. Earlier progress included binomial and finite difference methods that provided accurate prices but lacked the computational efficiency and stability. The arrival of approximation methods improved the efficiency but still lacked a high level of accuracy in particular when pricing options with longer dated maturities. Similarly Monte Carlo methods will require large number of simulations to reach acceptable accuracy.

Motivated by the need to find a pricing method that efficiently prices American options with a high degree of accuracy, the introduction of the Cubic Spline method in this thesis achieves this desired objective. The Cubic Spline method is an efficient and accurate way of pricing American options in practice. It has been demonstrated to be a significantly faster approach than all of the popular numerical methods such as the binomial, finite-difference, method of lines and Monte Carlo methods. Unlike other methods such as the Quadratic approximation, the cubic spline method accurately solves option prices for shorter to longer term maturities based on the solution to the Jamshidian or Kolodner-McKean integral equation. In addition, hedge parameters are easily attainable.

The cubic spline method is accurate for shorter and longer maturity options. However with longer term options, a higher number of spline knots should be selected in order to provide a more accurate solution. Despite this, the cubic spline method is still an efficient method for all parameter ranges and correctly handles the behaviour of the critical exercise boundary near the time to expiry.

When compared against a benchmark solution, the cubic spline method was shown to have very small differences, measured by root mean squared error. Other methods have a higher deviation from the benchmark solution reinforcing the cubic spline methods practicality.

Appendix A

Proofs

A.1 Feynman-Kac Formula

The Feynman-Kac formula connects partial differential equations (PDE) to stochastic processes. Assume that a standard PDE is given by

$$\frac{\partial v}{\partial t} + \mu(x,t)\frac{\partial v}{\partial x} + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2 v}{\partial x^2} = 0 \tag{A.1}$$

which is subject to the terminal condition $v(x,T) = \psi(x)$. The coefficients of μ, σ and ψ are known functions with $v(x,t)$ the solution to be solved. This particular PDE is known as a forward one-dimensional Fokker-Planck or Kolmogorov equation.

The Feynman-Kac formula expresses the solution for $v(x,t)$ as an expectation given by:

$$v(x,t) = \mathbb{E}[\psi(X_T) \mid X_t = x] \tag{A.2}$$

where X is known as an Ito process represented by the stochastic differential equation

$$dX = \mu(X,t)dt + \sigma(X,t)dW \tag{A.3}$$

where $W(t)$ is a standard Wiener process and the initial condition is $X(t) =$

x. By applying Ito's Lemma to $v(x,t)$,

$$dv = \left(\mu(x,t)\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2 v}{\partial x^2} \right) dt + \sigma(x,t)\frac{\partial v}{\partial x} dW. \tag{A.4}$$

the term in the brackets is zero, therefore an integration of both sides of the equation yield

$$\int_t^T dv = v(x,T) - v(x,t) = \int_t^T \sigma(x,t)\frac{\partial v}{\partial x} dW. \tag{A.5}$$

This leads to taking the expected value of both sides of the above equation

$$f(x,t) = \mathbb{E}[f(x,T)] - \mathbb{E}\left[\int_t^T \sigma(x,t)\frac{\partial v}{\partial x} dW \right]. \tag{A.6}$$

The stochastic integral containing the Wiener process will be zero which leads to

$$v(x,t) = \mathbb{E}[v(x,T)] = \mathbb{E}[\psi(x)] = \mathbb{E}[\psi(X_T) \mid X_t = x]. \tag{A.7}$$

A.2 Equivalence of the PDE and Layer Potential Methods

Proof:

Let $\hat{G} = G\left[x-\xi, t-s\right] \ \mathcal{H}(\xi - b(t))$ and consider the second term in Equation (5.48):

$$\begin{aligned} I &= \int_0^t \int_{-\infty}^{\infty} \hat{G}\left\{v_s - v_{\xi\xi}\right\} d\xi ds \\ &= I_1 - I_2\ . \\ I_1 &= \int_0^t ds \int_{-\infty}^{\infty} \hat{G} v_s d\xi \end{aligned}$$

$$= \int_{-\infty}^{\infty} d\xi \left\{ \left[\hat{G}v\right]_0^t - \int_0^t \hat{G}_s v ds \right\}$$

$$= \int_{-\infty}^{\infty} \{G[x-\xi,0]v(\xi,t)\mathcal{H}(\xi-b(t)) - G[x-\xi,t]v(\xi,0)\mathcal{H}(\xi-b_0)\} d\xi$$
$$- \int_0^t ds \int_{-\infty}^{\infty} \hat{G}_s v(\xi,s) d\xi ds \ .$$

But $G[x-\xi,0] = \delta(x-\xi)$, therefore

$$I_1 = v(x,t)\mathcal{H}(x-b(t))$$
$$- \int_{b_0}^{\infty} G[x-\xi,t]u_0(\xi)d\xi - \int_0^t ds \int_{-\infty}^{\infty} \hat{G}_s v(\xi,s) d\xi ds \ .$$

Now $$I_2 = \int_0^t ds \int_{-\infty}^{\infty} \hat{G} v_{\xi\xi} d\xi$$

$$= \int_0^t ds \left\{ \left[\hat{G}v_\xi\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \hat{G}_\xi v_\xi d\xi \right\}$$

$$= \int_0^t ds \left\{ - \left[\hat{G}_\xi v\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \hat{G}_{\xi\xi} v d\xi \right\}$$

$$= \int_0^t ds \int_{-\infty}^{\infty} \hat{G}_{\xi\xi} v(\xi,s) d\xi \ .$$

Hence

$$I = I_1 - I_2$$
$$= v(x,t)\mathcal{H}(x-b(t)) - \int_{b_0}^{\infty} G[x-\xi,t]u_0(\xi)d\xi$$
$$- \int_0^t ds \int_{-\infty}^{\infty} \left[\hat{G}_s + \hat{G}_{\xi\xi}\right] v(\xi,s) d\xi \ . \qquad (A.8)$$

Now we evaluate the third term in Equation (A.8), and recall that $\hat{G} = G[x-\xi,t-s]\mathcal{H}(\xi-b(t))$, then

$$\hat{G}_s = G_s\mathcal{H}(\xi-b(t)) - \dot{b}(s)G[x-\xi,t-s]\delta(\xi-b(t))$$

$$= \quad -G_t\mathcal{H}(\xi - b(t)) - \dot{b}(s)G^b\delta(\xi - b(t))$$

$$\text{where} \quad G^b = G[x - b(s), t - s]$$

$$\begin{aligned} \hat{G}_\xi &= -G_x\mathcal{H}(\xi - b(t)) + G^b\delta(\xi - b(t)) \\ \hat{G}_{\xi\xi} &= G_{xx}\mathcal{H}(\xi - b(t)) - G^b_x\delta(\xi - b(t)) + G^b\delta'(\xi - b(t)) \; . \end{aligned}$$

The Green's function $G[x-\xi, t-s]$ is the fundamental solution which satisfies the PDE, hence $G_t - G_{xx} = 0$ and

$$\hat{G}_s + \hat{G}_{\xi\xi} = -\left(\dot{b}G^b + G_x\right)\delta(\xi - b(t)) + G^b\delta'(\xi - b(t)) \; .$$

Therefore

$$\begin{aligned} J &= \int_0^t ds \int_{-\infty}^{\infty} \left[\hat{G}_s + \hat{G}_{\xi\xi}\right] v(\xi, s)d\xi \\ &= -\left(\dot{b}G^b + G^b_x\right) v(b(s), s) - G^b v_x\,(b(s), s) \\ &= -\left(\dot{b}G^b + G^b_x\right) f(s) - G^b g(s) \; . \end{aligned}$$

Expressing G^b_x as $-G^b\phi$ where $\phi = \dfrac{x - b(s)}{2(t - s)}$, we arrive at

$$J = -\left[\left(\dot{b}f + g\right) - \phi f\right] G^b \; . \tag{A.9}$$

Hence

$$\begin{aligned} I &= v(x,t)\mathcal{H}(x - b(t)) - \int_{b_0}^{\infty} G[x - \xi, t]u_0(\xi)d\xi \\ &\quad + \int_0^t G^b \left[\left(\dot{b}(s)f(s) + g(s)\right) - \phi f(s)\right] ds \; . \end{aligned} \tag{A.10}$$

When recombining the terms, Equation (5.48) reduces to:

$$u(x,t) \;=\; v(x,t)\mathcal{H}(x - b(t)) + \int_{-\infty}^{b_0} G[x - \xi, t]u_0(\xi)d\xi$$

$$+\int_0^t G^b\left[\left(\dot{b}(s)f(s)+g(s)\right)-\phi f(s)\right]ds\ . \qquad \text{(A.11)}$$

By substitution of $x = b(t)$ into Equation (A.11), and using

$$\begin{aligned} u(b(t),t) &= f(t) \quad \text{and} \\ v(x,t)\mathcal{H}(x-b(t)) &\rightarrow \tfrac{1}{2}f(t) \quad \text{as} \quad x \rightarrow b(t)\ , \end{aligned}$$

then by letting $h(t) = \dot{b}(t)f(t) + g(t)$:

$$\begin{aligned} \tfrac{1}{2}f(t) &= \int_{-\infty}^{b_0} G[b(t)-\xi,t]u_0(\xi)d\xi \\ &\quad + \int_0^t G\left[b(t)-b(s),t-s\right]\left\{h(s)-\phi f(s)\right\}ds\ . \end{aligned} \qquad \text{(A.12)}$$

This integral equation for the free - boundary is identical to the one found using the layer potential and Fourier transform methods. □

A.3 Derivation of the American Perpetual Option Solution

In Chapter 4 the free boundary problem of the American put option was given by

$$\mathbb{P}_A \begin{cases} \mathcal{B}\{P(S,t)\} = 0 \quad \text{in } S > S^*(t);\ t > 0 \\ P(S,0) = (K-S)^+; \quad P(\infty,t) = 0 \\ P(S^*(t),t) = K - S^*(t); \quad \frac{\partial P}{\partial S}(S^*(t),t) = -1 \\ P(S,t) = K - S \quad \text{in } S < S^*(t);\ t > 0 \end{cases} \qquad \text{(A.13)}$$

where the first equation $\mathcal{B}\{P(S,t)\} = 0$ is equivalent to

$$\frac{\partial P}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - (r-q)S\frac{\partial P}{\partial S} + rP = 0, \qquad \text{(A.14)}$$

subject to the conditions above. As $P(S,t)$ and $\frac{\partial P}{\partial S}$ is continuous at $S = S^*(t)$, when the limit $t \to 0$ is taken on the partial derivative $\frac{\partial P}{\partial t}$ the Black-Scholes partial differential equation reduces to

$$\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - (r-q)S\frac{\partial P}{\partial S} + rP = 0, \tag{A.15}$$

which is a non-homogeneous second order differential equation. A solution to this will have the form $P = S^{-\alpha}$ that would exist in the region $S > S^*(t)$. Substituting this form into the above equation yields

$$\tfrac{1}{2}\sigma^2\alpha(\alpha-1) - (r-q)\alpha - r = 0 \tag{A.16}$$

or

$$\tfrac{1}{2}\sigma^2\alpha^2 - (r-q-\tfrac{1}{2}\sigma^2)\alpha - r = 0 \tag{A.17}$$

which is a quadratic equation solved for α and has the solution

$$\alpha = [\beta \pm \sqrt{\beta^2 + 2r\sigma^2}]/\sigma^2; \quad \beta = r - q - \tfrac{1}{2}\sigma^2.$$

Substituting the solution P into the boundary and contact conditions with $B = S^*(\infty)$:

$$\begin{array}{lllll} P(B) & = & B^{-\alpha} & = & K - B \\ P'(B) & = & \alpha B^{-\alpha-1} & = & 1 \\ & & 1 & = & \alpha(K-B)B^{-}1 \\ & & B & = & \alpha(K-B) \\ & & B & = & \frac{\alpha K}{\alpha+1}. \end{array} \tag{A.18}$$

This leads to the solution for the perpetual American put option

$$\mathbb{P}_\infty \left\{ \begin{array}{lll} P(S) & = & \left\{ \begin{array}{lll} \frac{B}{\alpha}\left(\frac{B}{S}\right)^\alpha & \text{in} & S > B \\ K - S & \text{in} & S < B \end{array} \right. \\ B & = & \frac{\alpha K}{\alpha+1} \\ \alpha & = & [\beta + \sqrt{\beta^2 + 2r\sigma^2}]/\sigma^2 \end{array} \right. \tag{A.19}$$

An "optimality" solution exists for the perpetual option price as mentioned (by private communication) by Jeff Dewynne. He explains that the price of the American put option $V(x,t)$ will have two forms namely

$$V(x,t) = \begin{cases} Ax^{-\alpha} & \text{for} \quad x > c \\ k - x & \text{for} \quad x < c \end{cases} \tag{A.20}$$

For continuity:

$$\begin{aligned} V(c,t) : Ac^{-\alpha} &= k - c \\ A &= kc^{\alpha} - c^{1+\alpha} \end{aligned}$$

The maximum of $V(x,t)$ occurs when $\frac{\partial V}{\partial c} = 0$,

$$\frac{\partial A}{\partial c} = 0 \Rightarrow \begin{aligned} \alpha k c^{\alpha-1} - (1+\alpha)c^{\alpha} &= 0 \\ c^{-1} &= \tfrac{1+\alpha}{\alpha k} \\ c &= \tfrac{\alpha k}{1+\alpha} \end{aligned} \tag{A.21}$$

Appendix B

Notation

Term	Meaning
S	Stock Price.
K	Exercise or strike price of an option.
r	Risk - free rate of return.
T	Expiry date of the option expressed in years.
t	$= T - \tau$. Time to maturity.
τ	Some general time between expiry ($\tau = t$) and present ($\tau = 0$).
σ	Stock volatility.
q	Dividend yield rate.
$S^*(\tau)$	Critical stock price.
$C(S, \tau)$	American call option price.
$c(S, \tau)$	European call option price.
$P(S, \tau)$	American put option price.
$p(S, \tau)$	European put option price.
$V(S, \tau)$	Option contract value.
$\hat{\sigma}$	$\frac{\sigma}{\sqrt{2}}$.
$x(S, \tau)$	$\frac{1}{\hat{\sigma}}\left[\log \frac{S}{K} + (r - q - \hat{\sigma}^2)\tau\right]$.
$u(x, \tau)$	Solution to the diffusion equation.
$u(x, 0), u_0(x)$	Solution of the diffusion equation at $\tau = 0$.
$c(\tau)$	$\dfrac{S^*(\tau)}{K}$.
log	Natural logarithm.
$b(\tau)$	$\frac{1}{\hat{\sigma}}\left[\log c(t) + (r - q - \hat{\sigma}^2)\tau\right]$ is the location of the free - boundary.

Term	Meaning
β	$r - q - \frac{1}{2}\sigma^2$.
$b(0), b_0$	Value of $b(t)$ at time $\tau = 0$.
$\dot{b}(\tau)$	Time derivative of $b(\tau)$.
$\mathbb{D}_c^+$	$\{(x,t) \| x > c(t); t > 0\}$.
$\mathbb{D}_c^-$	$\{(x,t) \| x < c(t); t > 0\}$.
$\mathbb{D}$	$\mathbb{D}_c^+ \cup \mathbb{D}_c^- = \{(x,t) \| x \gtrless c(t); t > 0\}$.
$G\left[x,t\right]$	$\frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$, Green's function for the Heat Equation.
$G_x\left[x,t\right]$	$\frac{x}{2t}\frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$, derivative of Green's function with respect to x.
$\mathcal{S}^b[h]$	$\int_0^t G\left[x,t\right] h(\tau) d\tau$, single - layer potential.
$\mathcal{D}^b[h]$	$\int_0^t G_x\left[x,t\right] h(\tau) d\tau$, double - layer potential.
$[\cdot]$	The "jump" across the boundary $x = b(t)$.
$\mathcal{L}$	Linear operator.
$H(x,t)$	$\int_{-\infty}^{\infty} G\left[x-\xi,t\right] u_0(\xi) d\xi$.
$H_c(x,t)$	$\int_{-\infty}^{b_0} G\left[x-\xi,t\right] u_0(\xi) d\xi$, used for pricing American calls.
$H_p(x,t)$	$\int_{b_0}^{\infty} G\left[x-\xi,t\right] u_0(\xi) d\xi$, used for pricing American puts.
$\delta(x)$	Dirac delta - function.
$\mathcal{H}(\cdot)$	Heaviside step function. $\mathcal{H}(x) = \int_{-\infty}^{x} \delta(z) dz = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$.
$\mathcal{F}\{h(x)\}$	$\hat{h}(\lambda) = \int_{-\infty}^{\infty} h(x) e^{i\lambda x} dx$, Fourier transform of a function $h(x)$.
$\mathcal{N}(\cdot)$	Cumulative standard normal distribution function. $\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx$.
$[\cdot]^+$	Positive value function. $[x]^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$.

Appendix C

Computer Code

The MATLAB code used in the numerical work for this thesis is available on CD. It is broken up into ten folders labeled as:

- Final Binomial
- Final Finite Diff
- Final Jamshidian
- Final Kolodner McKean
- Final Method Interp
- Final MOL
- Final Monte Carlo
- Final Quad Approx
- Final Recursive Int Method
- Toy Problem

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