

A COMPARATIVE STUDY
OF AMERICAN OPTION VALUATION
AND COMPUTATION

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Abstract

For many practitioners and market participants, the valuation of financial derivatives is considered of very high importance as its uses range from a risk management tool, to a speculative investment strategy or capital enhancement. A developing market requires efficient but accurate methods for valuing financial derivatives such as American options.

A closed form analytical solution for American options has been very difficult to obtain due to the different boundary conditions imposed on the valuation problem. Following the method of solving the American option as a free boundary problem in the spirit of the “no-arbitrage” pricing framework of Black-Scholes, the option price and hedging parameters can be represented as an integral equation consisting of the European option value and an early exercise value dependent upon the optimal free boundary.

Such methods exist in the literature and along with risk-neutral pricing methods have been implemented in practice. Yet existing methods are accurate but inefficient, or accuracy has been compensated for computational speed. A new numerical approach to the valuation of American options by cubic splines is proposed which is proven to be accurate and efficient when compared to existing option pricing methods. Further comparison is made to the behaviour of the American option’s early exercise boundary with other pricing models.

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Contents

Abstract	i
Acknowledgments	ii
List of Figures	vii
List of Tables	ix
1 Introduction	1
1.1 History and Development of Options	1
1.2 Modern Theory of Options Pricing	3
1.3 American Option Pricing Models	4
1.4 American Option Valuation Methods	7
1.5 Cubic Spline Method	9
1.6 Numerical Comparisons	12
1.7 Early Exercise Boundary	12
1.8 Recent Advances in Literature	13
1.9 Scope of Work	13
2 Arbitrage Boundaries	17
2.1 The Concept of Arbitrage	17
2.2 Notation and Preliminaries	18

2.3	Call Option Boundaries	19
2.4	Put Option Boundaries	22
2.5	Put-Call Parity Relations	25
3	Derivation of the European Option Equation and Formulae	27
3.1	Assumptions of the Black-Scholes Model	27
3.2	Derivation of the Black-Scholes Partial Differential Equation .	30
3.3	Boundary and Initial Conditions for European Options	31
3.4	The Black-Scholes Option Pricing Formula	33
3.5	Extensions of the Formula	34
3.6	Put-Call Symmetry	36
3.7	Alternative Derivation Under a Risk Neutral Measure	37
4	American Options	39
4.1	American Option Characteristics	39
4.2	High Contact Conditions for American Calls and Puts	41
4.3	The American Option Partial Differential Equation	44
4.4	Optimal Exercise Boundary and Symmetry	45
4.5	Asymptotic Behaviour of the Critical Exercise Price Near Expiry	50
5	Valuation Methods	52
5.1	The American Option Partial Differential Equation Solution .	52
5.2	Binomial Methods	65
5.3	Linear Complementarity Formulation	70
5.4	Quadratic Approximation	72
5.5	Analytical Method of Lines	75
5.6	Method of Interpolation between Bounds	79
5.7	Randomization Techniques	83
5.8	Monte Carlo Methods	88

5.9	Critical Appraisal	91
6	Numerical Methods	93
6.1	Lattice Methods	93
6.2	Simulation Methods	102
6.3	Approximation Methods	105
6.4	Semi-Analytical Methods	108
6.5	Gaussian Quadrature	110
7	A New Algorithm	116
7.1	Method of Cubic Splines in the Kolodner-McKean Framework	117
7.2	Application to a Toy Problem	124
7.3	The Cubic Spline Method in the Jamshidian Framework . . .	126
7.4	Comparison between the Kolodner-McKean and Jamshidian Cubic Spline Valuation	127
7.5	Comparative Statics	130
8	A Comparison of Valuation Methods	133
8.1	Comparison of Efficiency	133
8.2	Calculating the Early Exercise Boundary	141
9	Conclusion	148
A	Proofs	151
A.1	Feynman-Kac Formula	151
A.2	Equivalence of the PDE and Layer Potential Methods	152
A.3	Derivation of the American Perpetual Option Solution	155
B	Notation	158

List of Figures

2.1	Minimum and maximum bounds for an American call	19
2.2	Minimum and maximum bounds for an American put	23
4.1	The American call option's contact condition.	41
4.2	The American put option's contact condition.	42
5.1	The domain defined by \mathbb{D}_c^\pm	53
5.2	The discrete nature of the exercise boundary evaluated under the binomial method.	69
5.3	The stock price paths of the continuous state model.	89
6.1	Comparison between two critical exercise boundaries for vary- ing step sizes.	99
6.2	The partitioned stock price paths.	102
7.1	Comparison of exact (stars) and numerical (solid) solutions. Spline knots are denoted in circles; start solution by triangles .	124
7.2	Early exercise boundary for an American put option for $r =$ 8% , $q = 0\%$, $\sigma = 40\%$, $K = \$100$ and $T = 1$ year	128
7.3	Early exercise boundary for an American call option for $r =$ 8% , $q = 15\%$, $\sigma = 40\%$, $K = \$100$ and $T = 1$ year	129

8.1 Formation of the early exercise boundary for an American put option 141

8.2 Comparison of the early exercise boundaries for an American put option for the case when $r > q$ 143

8.3 Comparison of the early exercise boundaries for an American put option for the case when $r < q$ 144

8.4 The Early-exercise boundaries for an American put for various choices of spline knots. 145

8.5 A close up of the spline knots compared with the true solution. 146

8.6 The root mean squared errors for the early exercise boundary of an American put for various choices of spline knots. 147

List of Tables

7.1	Comparison of American put option values using the Kolodner-McKean and Jamshidian methods for varying asset price S .	128
8.1	Comparison of American put and call option values using the Binomial, Finite difference, Quadratic Approximation, Method of Lines, Recursive Integration, Monte Carlo and the two new methods: Jamshidian and Kolodner-McKean methods. $S = 80, 90, 100, 120$, $r = 8\%$, $q = 12\%$, $\sigma = 20\%$, $K = 100$ and $T = 0.25$ years	137
8.2	Comparison of American put and call option values using the Binomial, Finite difference, Quadratic Approximation, Method of Lines, Recursive Integration, Monte Carlo and the two new methods: Jamshidian and Kolodner-McKean methods. $S = 80, 90, 100, 120$, $r = 12\%$, $q = 8\%$, $\sigma = 20\%$, $K = 100$ and $T = 0.25$ years	138
8.3	Root Mean Squared Errors of the the Binomial, Finite difference, Quadratic Approximation, Method of Lines, Recursive Integration, Monte Carlo and the two new methods: Jamshidian and Kolodner-McKean methods relative to the benchmark Binomial method using $N = 50,000$ time steps.	140
8.4	Choice of knot sizes for various maturity dates.	145