Modelling complex cracks with finite elements: a kinematically enriched constitutive model

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Abstract A continuum constitutive framework with embedded cohesive interface model is presented to describe the failure of quasi-brittle materials. Both cohesive behaviour for cracking inside the fracture process zone and elastic bulk behaviour are treated at integration points making implementation straightforward. In this sense, the proposed approach is simpler than existing approaches that focus on element enrichments, such as the extended finite element method, while share similarities with smeared crack models, and offers the capability to correctly model quasi-brittle failure in post-peak regime at constitutive level. In this work, the formulation is established, numerical algorithms described and static and dynamic fracture simulations with complex crack patterns are conducted to demonstrate the capability and advantage of the proposed approach.

Keywords cohesive crack · finite elements · kinematically enriched constitutive model · quasi-brittle · cohesive interface elements

1 Introduction

Localised failures are usually encountered in quasi-brittle materials such as concrete, rock and ceramics. The deformation usually localises on narrow bands where most inelastic deformation takes place and the surrounding bulk material typically unloads elastically. Localised failures have traditionally been modelled using continuous and discontinuous approaches. Notable models belonging to the former approach include nonlocal constitutive models Bažant (1991), gradient enhanced damage models Peerlings et al (1996), viscous or rate-dependent models Needleman (1988), smeared crack models Rashid (1968); Bazant and Oh (1983); Rots (1991), phase-field models Miehe et al (2010); Hofacker and Miehe (2012); Borden et al (2014); Amiri et al (2014) and other enrichments at constitutive level Pietruszczak and Mroz (2001); Pietruszczak and Xu (1995); Nguyen et al (2014b). The key characteristic of continuous approaches is that the failure progression is modelled by the degradation of the material and therefore the incorporation of a length scale is made at the constitutive level or, alternatively,
at the integration point and/or element level in a numerical method framework, such as the finite element method. Its name reflects one of the drawbacks of the continuous approach - true separation cannot be captured since the continuum model, even though cracked, always behaves like a continuum because cracks are not explicitly represented and their effects are only parametrically taken into account by changes in the effective properties of the continuum.

On the other hand, discontinuous approaches employ explicit crack representations and hence allow true material separation to be accurately reproduced as a geometrical discontinuity (often referred to as strong discontinuities). Some discontinuous approaches for localised failures include zero-thickness cohesive interface elements Ngo and Scordelis (1967); Schellekens and Borst (1993); Xu and Needleman (1994); Nguyen et al (2014c) in combination with a discontinuous Galerkin formulation that helps to avoid an artificially high dummy stiffness Mergheim et al (2004a); Radovitzky et al (2011); Nguyen (2014a), or elements with embedded discontinuities Ortiz et al (1987); Simo et al (1993); Belytschko et al (1988); Armero and Linder (2009); Dvorkin et al (1990); Dias-da-Costa et al (2009), or elements with partition of unity based discontinuous enrichment via the extended finite element method (XFEM) Moës et al (1999); Strouboulis et al (2001); Wells and Sluys (2001); Simone et al (2006). For completeness particle/meshless and peridynamics methods e.g., Rabcezuk and Belytschko (2004); Rabcezuk et al (2007b,a); Zhuang et al (2012); Yang et al (2015); Silling (2000) are also capable of handling complex cracking problems with intersecting cracks and branching cracks. Usually cohesive zone models Dugdale (1960); Barenblatt (1962) are employed in the discontinuous approach to model fractures that result from various mechanisms such as void nucleation, crack shielding due to micro-cracks, crack deflection, aggregate bridging etc. A comprehensive coverage of all fracture models is far beyond the scope of this study that focuses on a new continuum approach with embedded cohesive behaviour for the modelling of complex fracture problems. For more details we refer to review articles Jirasek (2000); Dias-da-Costa et al (2010); de Borst et al (2004); Mosler and Meschke (2004); Rabcezuk (2013), references therein and recent works on multiscale fracture modelling Budarapu et al (2014); Nguyen et al (2011); Budarapu et al (2015). Both continuous and discontinuous approaches have their own advantages, disadvantages and applications. It is widely accepted that discrete crack models are best suitable for problems with dominant cracks while continuous methods handle well distributed cracks. From a computational perspective, continuous approaches are easy to implement, efficient and robust, while discontinuous methods are (i) hard to implement especially with complex crack patterns even in two dimensions (2D) and particularly in commercial FE packages, see e.g., Giner et al (2009); Bordas et al (2007); Wyatt et al (2008), (ii) computationally expensive due to extra degrees of freedom and costs to handle the representation of discrete cracks and (iii) less robust (e.g., crack close to a node in XFEM Moës et al (1999)). The major drawback of continuous techniques is that a true discontinuity cannot be properly represented. This led to the development of continuous-discontinuous approaches, where a continuum description of cracking is used until the final stage of failure which is modelled by a discrete approach, see e.g., Simone et al (2003). However to the best of our knowledge, there is no consensus in defining a transition procedure between continuous and discontinuous models, especially when anisotropy induced by cracking is important. This continuous-discontinuous approach is therefore not well within the scope and hence not discussed in this study.

Nguyen et al (2014b) developed a new constitutive model for strain localisation. The basic idea is that the Representative Volume Element (RVE), over which the stress/strain field is defined, is considered as a composite material consisting of an inelastic localisation band of finite thickness embedded in a bulk. By using a mixture theory and homogenisation concepts, a stress-strain relationship of the RVE was obtained by including a length scale and the kinematics of strain localisation. This constitutive relationship was then used at the integration points in a standard FE formulation. As the kinematics of localisation bands are included in the stress-strain equation, this model can be referred to as a kinematic enrichment at the constitutive level. A concrete specification of the general model in which the localisation band is described by an isotropic damage model for quasi-brittle materials was presented, and numerical results were satisfactory: the crack patterns were similar to experiments and the load-displacement curves were independent of the numerical spatial discretisation, which is an essential feature of localised failures. Compared to nonlocal Bažant (1991) and gradient-enhanced damage models Peerlings et al (1996), the model is more efficient and avoids the complications of nonlocal damage models (e.g., averaging across a discontinuity) and gradient enhanced damage models (e.g., boundary conditions of the nonlocal field). Note, however, that the constitutive model remains local. Recently, the approach was extended to hydro-mechanical multiphysics problems Nguyen et al (2014a) and mod-

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1 One exception is the cohesive interface elements which are available in many FE packages such as Abaqus and LS-Dyna.

Despite many successes in modelling quasi-brittle failure, existing discrete methods (cohesive interface elements, XFEM, elements with embedded discontinuities) are too complicated to handle problems with multiple branching and intersecting cracks even in 2D, while other higher order continuous approaches (nonlocal and gradient models) are too costly. All these issues render them impractical for complex cracking problems with several branching and intersecting cracks. In addition, all these approaches are also not easy to implement in existing finite codes, due to special requirements for the enrichment at the element levels. This motivates developments of a computationally efficient approach that can handle complex failure problems with multiple branching and intersecting cracks, while being simple and straightforward for implementation. The aim of this paper is twofold. First, the constitutive model from Nguyen et al (2014b) will be adapted to the case of cohesive cracks as the width of the localisation band approaches zero. As will be shown, the model is different from the original smeared crack model of Rashid (1968) but shares some similarities with the model of Rots (1991), even though the starting point and operation are different. Although not being the focus of the paper, we do provide a qualitative comparison with experiment for a mixed mode fracture test. Second, we present an assessment of the computational efficiency and robustness of the model in comparison with a discontinuous approach for the modelling of quasi-brittle failure with complex crack patterns. For the discontinuous approach, cohesive interface elements will be used in a discontinuous Galerkin framework Mergheim et al (2004b); Radovitzky et al (2011); Nguyen (2014b). These two different approaches will be applied to two different problems: (1) two and three dimensional mesoscale failure simulations of fibre-reinforced composite materials, where both matrix and fibres are explicitly represented; and (2) mesoscale dynamic tensile failure of concretes. The fact that the simulations using these approaches are carried out on the same computer using the same FE solver will facilitate the comparison of their performance.

The paper is organised as follows. Section 2 presents the kinematically enriched constitutive model for strain localisation. Both general case of a finite thickness localisation band (Section 2.1) and the case of a zero thickness cohesive crack (Section 2.2) are discussed. This is followed by Section 3, where cohesive interface elements with intrinsic and extrinsic cohesive laws (also referred to as initially elastic and initially rigid cohesive laws) are given. Representative simulations are given in Section 4 and discussions are presented in Section 5. The paper ends with some conclusions and remarks in Section 6.

2 Kinematically enriched constitutive model for strain softening

In this section, the kinematically enriched constitutive model developed in Nguyen et al (2012, 2014b) for modelling localisation bands is briefly presented. We focus on small displacements and small strains, which are suitable for the failure of quasi-brittle materials explored here, although the ideas can be carried forward to large displacements with moderate strains.

We consider a material volume $\Omega$ with a localisation band of thickness $h$ ($\Omega_i$) surrounded by a bulk material $\Omega_o$ (Fig. 1). Subscripts "i" and "o" are used for quantities inside and outside the localisation band, respectively. The stresses and strains are denoted by ($\sigma_i$, $\epsilon_i$) and ($\sigma_o$, $\epsilon_o$) for materials inside the localisation band and outside homogeneous bulk, respectively. It is assumed that dissipative processes take place only inside the localisation band, while the outside bulk is undergoing elastic unloading. This is a typical situation in quasi-brittle failure, and we do not include in this work the initiation of secondary cracks at the same point due to change in the loading paths. The question is how to devise a model, in terms of the averaged (or macro) stress $\sigma$ and averaged strain $\epsilon$ defined over the domain $\Omega$, by coupling the different responses of the materials inside and outside the localised region i.e., in terms of ($\sigma_i$, $\epsilon_i$), ($\sigma_o$, $\epsilon_o$), $\eta$, $h$ and $H$.

Key equations are presented here, while further details can be found in Nguyen et al (2012, 2014b). We view the volume element crossed by a localisation band as a composite material consisting of two phases. Following mixture theory, the volume-averaged total strain rate can be expressed as

$$\dot{\epsilon} = \eta \dot{\epsilon}_i + (1 - \eta) \dot{\epsilon}_o$$

(1)

where $\eta$ is the volume fraction of the localisation band: $\eta = h/H$ and the dot indicates rates.

For a very thin localisation band in quasi-brittle failure ($h \ll H$), the rate of the inelastic strain inside the band can be approximated as

$$\dot{\epsilon}_i \approx \frac{1}{h} \left( n \otimes [\dot{u}] \right)^{sym} = \frac{1}{2h} (n \otimes [\dot{u}] + [\dot{u}] \otimes n)$$

(2)

Einstein summation convention does not apply here.

2 Reasons for this choice will be given later in the paper.
where \( \mathbf{n} \) denotes the unit normal vector to the localisation band and \( \otimes \) is the dyadic product. This enrichment allows an additional constitutive behaviour for the material inside the localisation band to be introduced. We can write the behaviour inside and outside the band in a generic rate form as:

\[
\dot{\sigma}_o = \mathbf{a}_o : \dot{\varepsilon}_o, \quad \dot{\sigma}_i = \mathbf{a}_i : \dot{\varepsilon}_i
\]  

(3)

where \( \mathbf{a}_o \) and \( \mathbf{a}_i \) are the fourth-order material tangents of the outside and inside materials, respectively. For quasi-brittle failure, the outside material can be adequately described by a linear elastic model (\( \mathbf{a}_o \) is thus the elasticity tensor) and the localisation band can be modelled by a (local) damage model or a cohesive model. To model failure of geomaterials (e.g., soils and sands) a softening plasticity model can be adopted for both the outside and inside materials (Nguyen et al. 2015a).

The connection between the behaviour of the constituents (localisation band and the remaining part) and the macroscopic (or homogenised) behavior is defined by the Hill-Mandel equation (Hill 1965; Mandel 1971) that reads

\[
\sigma : \dot{\varepsilon} = \eta \sigma_o : \dot{\varepsilon}_o + (1 - \eta) \sigma_i : \dot{\varepsilon}_i
\]  

(4)

Using Eqs.(1)-(4), it can be shown that (i) the macro homogenised stress coincides with the stress describing the behaviour of the material outside the localisation zone, and (ii) the traction must be continuous across the boundary of the localisation zone:

\[
\sigma = \sigma_o, \quad \sigma_i \cdot \mathbf{n} = \sigma_o \cdot \mathbf{n}
\]  

(5)

The above traction continuity condition together with Eqs.(1-3) are the keys to determine the velocity jump vector from a given macro strain rate (Nguyen et al. 2014b)

\[
[u] = \left[ \frac{\eta}{h} \mathbf{A}_o + \frac{1 - \eta}{h} \mathbf{A}_i \right]^{-1} (\mathbf{a}_o : \dot{\varepsilon}) \mathbf{n} = C^{-1} \cdot (\mathbf{a}_o : \dot{\varepsilon}) \cdot \mathbf{n}
\]  

(6)

where \( \mathbf{C} \) is the tensor in the square brackets and the acoustic tensors \( \mathbf{A}_i/o \) are defined as \( \mathbf{A}_i/o = \mathbf{n} \cdot \mathbf{a}_i/o \cdot \mathbf{n} \).

Note that the term \( \mathbf{a}_o : \dot{\varepsilon} \) plays a role of a stress tensor and hence \( (\mathbf{a}_o : \dot{\varepsilon}) \cdot \mathbf{n} \) is a traction-like term. This interpretation will be shown to be useful in casting the above into matrix notation for computer implementation.

From Eqs.(1),(3) and (5), the stress strain relationship, in rate form, can be obtained as

\[
\dot{\sigma} = \frac{1}{1 - \eta} \mathbf{a}_o \left[ \dot{\varepsilon} - \frac{\eta}{h} (\mathbf{n} \otimes (C^{-1} \cdot (\mathbf{a}_o : \dot{\varepsilon}) \cdot \mathbf{n}))^{\text{sym}} \right]
\]  

(7)

Note that it is the second term that accounts for the cracks by reducing the stress field. In other words, cracking is modelled as a material degradation process in the same manner as damage models or smeared crack models. Since cracking is dealt with by the constitutive model, or at the integration points in a FEM context, element technologies are standard. This is the key advantage of the continuous approach relative to the discontinuous formulations, which depend on the type of elements. The connection with the spatial discretisation is through the element/grid size \( H \), and the model behaviour in this case is intrinsically size dependent.

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Fig. 1: Numerical discretisation and a localisation zone (darkened) after Nguyen et al (2012) (a) and corresponding material responses inside and outside the localisation zone (b).
that band is simply a surface/line (3D/2D), which indicates cohesive law being used.

\[ \eta = \frac{h}{A_e} = \frac{h}{H} \quad H = \frac{A_e}{l} \quad l = \frac{1}{2} |\overline{AB}| \]

Fig. 2: “Exact” way to determine \( H \) for three-node triangle elements. The element area is denoted by \( A_e \) and points A, B are the intersections of the crack with the element.

2.2 Embedding cohesive cracks

2.2.1 Stress-strain relationship

In the case of cohesive cracking, instead of working with \( (\sigma_t, \varepsilon_t) \), one uses \( (t, [u]) \) (defined in the global coordinate system) or \( (t_{cr}, [u]_{cr}) \) (defined in the local coordinate system of the crack) of which relation is given by an initially rigid cohesive law

\[ \dot{t}_{cr} = K_{cr}[\dot{u}]_{cr} \quad (8) \]

where subscript \( cr \) is used to indicate that the cohesive law is defined in the local coordinate system that is attached to the crack. Note that an explicit expression for \( K_{cr} \) is not provided as it depends on the specific cohesive law being used.

In the case of cohesive cracking, the localisation band is simply a surface/line (3D/2D), which indicates that \( h \rightarrow 0 \) and thus \( \eta \rightarrow 0 \). Hence, Eq. (1) is simplified to

\[ \dot{\varepsilon} = \eta \dot{\varepsilon}_t + (1 - \eta) \dot{\varepsilon}_o = \frac{1}{H} (n \otimes [\dot{u}])^{\text{sym}} + \dot{\varepsilon}_o \quad (9) \]

where use was made of Eq. (2) in the second equality.

The traction continuity \( \sigma_o \cdot n = t \) reads

\[ \left[ a_o : \left( \dot{\varepsilon} - \frac{1}{H} (n \otimes [\dot{u}])^{\text{sym}} \right) \right] \cdot n = \frac{R^T \cdot K_{cr} \cdot R \cdot [\dot{u}]}{K} \]

where \( R \) is the transformation matrix (between the global coordinate system and the local coordinate system). Equation (10) is used to compute the velocity jump as follows

\[ [\dot{u}] = \frac{1}{H} A_o + K \quad (10) \]

Note that matrix \( C \) is rarely singular and thus \( C \) is invertible for practical problems. We refer to Appendix 2 in Nguyen et al (2015b) for a discussion on the singularity of \( C \) in one dimension.

The final \( \sigma - \dot{\varepsilon} \) relationship is obtained from Eq. (7)

\[ \dot{\sigma} = a_o : \left[ \dot{\varepsilon} - \frac{1}{H} \left( n \otimes \left( C^{-1} \cdot (a_o \cdot \dot{\varepsilon}) \cdot n \right) \right)^{\text{sym}} \right] \quad (12) \]

which can be rewritten in the following matrix notation

\[ \dot{\sigma} = a_o - \frac{1}{H} a_o n C^{-1} n^T \quad \dot{a}_o \quad (13) \]

where the stress/strain rates are stored as column vectors using the Voigt notation; \( n \) is given by

\[ n_{2D} = \begin{bmatrix} n_x & 0 \\ 0 & n_y \end{bmatrix}, \quad n_{3D} = \begin{bmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & n_z & 0 \end{bmatrix} \quad (14) \]

Although the equations derived here share some similarities with more recent smeared crack models (e.g., Rots (1991)), they have been derived using a completely different approach and, unlike the smeared crack models, can directly incorporate any cohesive law.

2.2.2 Crack initiation and crack direction

Crack initiation in quasi-brittle materials is herein identified when the maximum principal stress exceeds the tensile strength and the crack is perpendicular to the principal direction. This so-called Rankine criterion was chosen purely due to its simplicity, according to which the following eigenvalue problem has to be solved

\[ (\sigma - \sigma_l) X = 0 \quad (15) \]

where \( \sigma \) is the \( 3 \times 3 \) stress matrix, \( \sigma_l (l = 1, 2, 3) \) denotes the principal stresses and \( X \) are the principal
directions. The normal vector \( \mathbf{n} \) equals \( \mathbf{X}_f \) for the
maximum principal stress \( \sigma_f \). The rotation matrix \( \mathbf{R} \) then reads (for two dimensional problems)

\[
\mathbf{R} = \begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix}
\] (16)

Crack initiation is checked only at the end of a load increment/time step as in Wells and Sluys (2001). At
\begin{align*}
\text{crack initiation one also needs to compute the cohesive tangent } \mathbf{K}_{\text{cr}} \text{ corresponding with } [\mathbf{u}] = 0 \text{ as it is needed at the next load/time step.}
\end{align*}

2.3 Stress update

Stress update refers to the procedure in which the stresses are updated given the total strain increment \( \Delta \mathbf{e} \) and the previous state of the material point under consideration. In this section we present two stress update algorithms—an explicit and an implicit one. The explicit algorithm is easy to implement and best suited for model verification (and also serves to verify the implementation of the more elaborate implicit algorithm). However its accuracy is only ensured with sufficiently small strain increments. For simplicity we assume that the bulk is elastic with elasticity matrix \( \mathbf{D}_e \) i.e., \( \mathbf{a}_o = \mathbf{D}_e \).

2.3.1 Explicit stress update

The displacement jump increment is determined using Equation (11)

\[
\Delta [\mathbf{u}] = \mathbf{C}^{-1} \mathbf{n}^T (\mathbf{a}_o \Delta \mathbf{e})
\] (17)

where \( \mathbf{C} \) is computed based on quantities from the previous time step. This jump increment is then fed to the cohesive law to update response of the cohesive crack (for example \( \mathbf{K}_{\text{cr}} \) and internal variables of the cohesive law).

Finally the (averaged) stress increment is computed according to

\[
\Delta \mathbf{\sigma} = \mathbf{a}_o \Delta \mathbf{\epsilon}_o = \mathbf{a}_o \left( \Delta \mathbf{\epsilon} - \frac{1}{H} \mathbf{n} \Delta [\mathbf{u}] \right)
\] (18)

and the stresses are updated \( \mathbf{\sigma} \leftarrow \mathbf{\sigma} + \Delta \mathbf{\sigma} \). The stress update procedure is summarised in Algorithm 1. Step 11 is only needed for implicit analyses. As can be seen, in addition to other standard internal variables of the cohesive law, one has to store \( \mathbf{K} \) and a binary number either 0 (for an un-cracked point) or 1 (for a cracked one). In order to improve the performance of this explicit stress update \( \Delta \mathbf{\sigma} \) can be divided into a number of sub-increments Pérez-Foguet et al (2001) and for each sub-increment the procedure given in Algorithm 1 is applied.

**Algorithm 1 Explicit stress update.**

1: if not yet localised (or cracked) then
2: Stress update follows the standard procedure
3: Check localisation, if yes determine \( \mathbf{n} \) and compute \( \mathbf{a}_o \), \( \mathbf{a}_o = \mathbf{n}^T \mathbf{a}_o \mathbf{n} \)
4: else
5: Compute \( \mathbf{C} = \frac{1}{H} \mathbf{a}_o + \mathbf{K}, \mathbf{K} = \mathbf{R}^T \mathbf{K}_{\text{cr}} \mathbf{R} \)
6: Compute \( \Delta [\mathbf{u}] = \mathbf{C}^{-1} \mathbf{n}^T \mathbf{a}_o \Delta \mathbf{e} \)
7: Compute \( \Delta [\mathbf{u}]_{\text{cr}} = \mathbf{R} \Delta [\mathbf{u}] \)
8: Use \( \Delta [\mathbf{u}]_{\text{cr}} \) in the cohesive law to update the cohesive crack
9: Compute \( \Delta \mathbf{\sigma} = \mathbf{a}_o \left( \Delta \mathbf{\epsilon} - \frac{1}{H} \mathbf{n} \Delta [\mathbf{u}] \right) \)
10: Update stresses \( \mathbf{\sigma} \leftarrow \mathbf{\sigma} + \Delta \mathbf{\sigma} \)
11: Compute tangent \( \mathbf{D} = \mathbf{a}_o - \frac{1}{H} \mathbf{a}_o \mathbf{n} \mathbf{C}^{-1} \mathbf{n}^T \mathbf{a}_o \)
12: end if

As can be seen in Algorithm 1, the stress return algorithm can be considered as a two-level process in which the interface with finite elements is performed in the outer level, via macro stress-strain relationship, while the inner level involves the stress update of the cohesive model that takes the increment of the displacement jump obtained from the outer level as input. The snapback instability condition that usually appears as negative plastic multiplier in traditional smeared crack models casted in the framework of plasticity (e.g., Schreyer et al (2006)) never happens in this approach, since there is no scaling in the inner level of the cohesive stress return algorithm.

2.3.2 Implicit stress update

It is evident that the explicit stress update algorithm yields inaccurate results with large steps as the traction continuity is not exactly enforced. In the proposed implicit stress update an explicit stress update is firstly carried out and the traction continuity condition is checked. If this condition is violated, refinement iterations are performed. To this end, let us define the following residual vector

\[
\mathbf{r} = \mathbf{\sigma} \cdot \mathbf{n} - \mathbf{t}
\] (19)
A first order Taylor expansion of the residual vector at the state of the last iteration is given by

\[ r_k = r_{k-1} + \delta \sigma \cdot n - \delta t \]

\[ = r_{k-1} - \frac{1}{H} \left[ a_o : (n \otimes \delta [u]_{k})^{\text{sym}} \right] \cdot n - K \cdot \delta [u]_{k} \]

(20)

where \( k \) denotes the iteration number. In the second equality, the strain increment \( \Delta \epsilon \) has been applied in the explicit step and is therefore zero during these refinement iterations.

By zeroing \( r_k \) one can compute \( \delta [u]_{k} \) as follows

\[ \delta [u]_{k} = \left[ \frac{1}{H} A_o + K \right]^{-1} \cdot r_k \]

(21)

This jump increment is then fed to the cohesive law to update the response of the cohesive crack.

Finally the (averaged) stress increment is computed according to

\[ \delta \sigma_k = -\frac{1}{H} \left[ a_o : (n \otimes \delta [u]_{k})^{\text{sym}} \right] \]

(22)

And the stresses are updated \( \sigma \leftarrow \sigma + \delta \sigma_k \). The process is performed until convergence is attained. The implicit stress update is summarised in Algorithm 2.

**Algorithm 2 Implicit stress update.**

1. if not yet localised (or cracked) then
2. Stress update follows the standard procedure
3. Check localisation, if yes determine \( n \) and compute \( a_o, A_o = n^T a_o n \)
4. else
5. Compute \( C = \frac{1}{H} A_o + K \)
6. Compute \( \Delta [u] = C^{-1} n^T a_o \Delta \epsilon \)
7. Compute \( \Delta [u]_{cr} = r \cdot \Delta [u] \)
8. Use \( \Delta [u]_{cr} \) in the cohesive law to update the cohesive crack \( (t_{cr}, K_{cr}) \)
9. Compute \( \Delta \sigma = a_o \left[ \Delta \epsilon - \frac{1}{H} n^T \Delta [u] \right] \)
10. Update stresses \( \sigma = \sigma + \Delta \sigma \)
11. Compute residual \( r = \sigma \cdot n - t, t = R^T t_{cr} \)
12. while \( ||r|| > \epsilon \) do
13. \( C = (1/H) A_o + K, K = R^T K_{cr} R \)
14. Compute \( \delta [u] = C^{-1} r, \) then \( \delta [u]_{cr} \)
15. Use \( \delta [u]_{cr} \) in the cohesive law to update the cohesive crack \( (t_{cr}, K_{cr}) \)
16. Compute \( \delta \sigma = -(1/H) a_o n \delta [u] \)
17. Update stresses \( \sigma = \sigma + \delta \sigma \)
18. Compute residual \( r = \sigma \cdot n - t \)
19. end while
20. end if

The tolerance is denoted by \( \epsilon \). Alternatively one can use \( ||r|| > \epsilon ||t|| \).

### 2.3.3 Tangent stiffness of the constitutive model

When a static formulation is adopted one needs the tangent stiffness. Here, we use a continuum tangent stiffness obtained from the general model (cf. Equation (7)) by letting \( \eta = 0 \) and \( \eta/h = 1/H \):

\[ \dot{\sigma} = \left[ a_o - \frac{1}{H} a_o n C^{-1} n^T a_o \right] \dot{\epsilon} \]

(23)

Note that this tangent is only symmetric if both \( a_o \) and \( K_{cr} \) are symmetric. The former is always symmetric as we assumed linear elasticity for the bulk. The latter depends on the cohesive law.

### 3 Discontinuous approach using cohesive interface elements

In this work we use cohesive interface elements to model complex crack patterns. We chose cohesive interface elements rather than XFEM or embedded discontinuity methods because they are straightforward in implementation, robust and phenomena like crack branching and merging are the natural outcome of the original boundary value problem. Note that other techniques such as XFEM require ad hoc fracture criteria for detecting crack branching and sophisticated algorithms to deal with crack merging and intersection. Furthermore, most FE packages\(^4\) available to engineers do not yet offer them. These zero-thickness interface elements (IEs) are inserted at every element boundaries prior to the simulation. Although crack are restrained to the element boundaries, using unstructured meshes and statistically distributed material properties (e.g., a Weibull distribution for the tensile strength) has significantly reduced the mesh influences on the crack trajectories Zhou and Molinari (2004); Tijssens et al (2001). Fig. 3 gives a 2D illustration where the interface thickness was exaggerated for visibility by shrinking the bulk elements. A priori insertion of IEs simplifies the implementation at the expenses of increased degrees of freedom. Adaptive insertion of IEs can dramatically reduce the unknowns (see e.g., Park et al (2012)) but with an intricate implementation. For modelling interfacial cracking such as debonding of material interfaces and composite delamination intrinsic cohesive laws (or traction-separation laws) are used (Fig. 4-left). On the other hand, extrinsic cohesive laws (Fig. 4-right) are used to model matrix cracking. We use the discontinuous Galerkin (dG) method to prevent the IEs from separating prior

\(^4\) Exceptions include Abaqus of Dassault Systemes and System of ESI.
Kinematically enriched constitutive model for crack modelling to crack initiation in a variationally consistent manner contrary to the commonly used penalty method using a high stiffness $K$.

![Fig. 3: Cohesive interface elements are inserted at every element edges. Solid elements are three-node triangular elements and interface elements are four-node linear ones.](image)

3.1 Intrinsic cohesive interface elements

The weak formulation reads: finding the displacement field $u$ such that

$$
\int_{\Omega} \delta u \cdot b d\Omega + \int_{\Gamma_t} \delta u \cdot \bar{t} d\Gamma_t = \int_{\Omega} \delta \varepsilon : \sigma(u) d\Omega + \int_{\Gamma_d} \delta [u] \cdot t([u]) d\Gamma_d
$$

be satisfied for any admissible displacement field $\delta u$ subject to the Dirichlet boundary conditions on $\Gamma_u$. The discrete equations of the weak form given in Eq. (24) are

$$
\mathbf{f}^{\text{ext}} - \mathbf{f}^{\text{int}} - \mathbf{f}^{\text{coh}} = 0
$$

where $\mathbf{f}^{\text{ext}}$ is the external force vector, the internal force vector is denoted as $\mathbf{f}^{\text{int}}$, and the cohesive force vector $\mathbf{f}^{\text{coh}}$. The elemental external and internal force vectors are computed from contributions of continuum elements and are given by

$$
\mathbf{f}^{\text{int}}_e = \int_{\Omega_e} \mathbf{B}^T \sigma d\Omega_e
$$

$$
\mathbf{f}^{\text{ext}}_e = \int_{\Omega_e} \rho N^T \bar{b} d\Omega_e + \int_{\Gamma^c_t} N^T \bar{t} d\Gamma^c_t
$$

where $\Omega_e$ is the element domain, $\Gamma^c_t$ is the element boundary that overlaps with the Neumann boundary, $\bar{b}$ and $\bar{t}$ are the body forces and traction vector, respectively. The shape function matrix and the strain-displacement matrix are denoted by $\mathbf{N}$ and $\mathbf{B}$; $\sigma$ is the Cauchy stress vector.

The cohesive force vector is computed by assembling the contribution of all interface elements. It is given by for a general interface element $ie$

$$
\mathbf{f}^{\text{coh}}_{ie,+} = +\int_{\Gamma_{ie}} N^T_{\text{int}} \bar{t} d\Gamma
$$

$$
\mathbf{f}^{\text{coh}}_{ie,-} = -\int_{\Gamma_{ie}} N^T_{\text{int}} \bar{t} d\Gamma
$$

in which $\mathbf{N}_{\text{int}}$ represents the shape function matrix of interface elements and $\Gamma_{ie}$ is the interface element domain which is chosen to be the mid-surface of the interface element that makes the formulation also valid for large displacements. The subscripts +/- denote the upper and lower faces of the interface element. Details can be found in Nguyen (2014b).
3.2 Extrinsic cohesive interface elements

The weak form is given by Radovitzky et al (2011); Nguyen (2014a): Find $u$ such that

$$
(1 - \beta) \left[ - \int_{\Gamma_a} [\delta u]^T n(\sigma) d\Gamma + \theta_{\text{DG}} \int_{\Gamma_a} (\sigma(\delta u))^T n^T [u] d\Gamma \right. \\
+ \int_{\Gamma_a} \alpha [\delta u]^T [u] d\Gamma \Big] + \int_{\Omega} (\epsilon(\delta u))^T \sigma d\Omega \\
+ \beta \int_{\Gamma_a} [\delta u]^T t([u]) d\Gamma = \int_{\Gamma_a} (\delta u)^T t d\Gamma + \int_{\Omega} (\delta u)^T b d\Omega
$$

(29)

be satisfied for any admissible displacement field $\delta u$ subject to the Dirichlet boundary conditions on $\Gamma_u$. In the above, $\{ \cdot \}$ is the average operator and $\beta$ is a binary number that takes a value of zero at an integration point when this point is not yet cracked and a value of unity otherwise. The terms in the square brackets are called the dG terms, while $\theta_{\text{DG}}$ refers to different dG formulations: Symmetric Interior Penalty Method if $\theta_{\text{DG}} = -1$ (unless otherwise stated, this is the method adopted in the numerical examples), and Non-Symmetric Interior Penalty Method if $\theta_{\text{DG}} = 1$, and Incomplete Interior Penalty Method if $\theta_{\text{DG}} = 0$. Basically, the dG terms are replaced by the cohesive term when a fracture criterion is met. Discretisation of this weak form by finite elements, that is quite similar to the intrinsic cohesive interface elements, was given in Nguyen (2014a) and is not reported here. Issues of intrinsic cohesive models, such as ill-conditioning of the global stiffness matrix for implicit methods, or very small critical time step for explicit methods, are avoided due to the dG terms Radovitzky et al (2011); Nguyen (2014a).

4 Examples

In this section, the proposed method is tested using two sets of numerical examples. In the first set, cf. Section 4.1, one static and one dynamic fracture test are designed using two dominant non-intersecting cracks. This set serves to demonstrate the mesh insensitivity of the proposed enriched constitutive model. Qualitative comparison with experiments is also provided. The second set of examples, Section 4.2, deals with cracking of fiber-reinforced composite and concrete materials. Microstructures are explicitly modelled and complex fracture patterns have to be dealt with. Solutions are compared with the ones obtained with the discontinuous Galerkin cohesive interface elements presented in Section 3 to demonstrate that the proposed constitutive model can be an alternative to the costly cohesive interface element approach. Finite element meshes are generated using Gmsh Geuzaine and Remacle (2009) and post-processing is performed in Paraview Henderson (2007). A Newton-Raphson method is used to solve the nonlinear equations in static simulations, uniform material properties are assumed unless otherwise stated, the square-root definition of $H$ and the implicit stress return are used. As for the cohesive law, the model of Turon et al (2006), which is a mixed-mode initially elastic bilinear cohesive law, was used.

4.1 Static and dynamic problems with dominant cracks

4.1.1 Mixed-mode static fracture test

This test concerns a mixed mode cracking of a double edge notched (DEN) specimen (Fig. 5). Experiments on DEN specimens were carried out by Nooru-Mohamed et al (1993) $^5$. Only biaxial loading was considered in this exercise (path 2a in Nooru-Mohamed et al (1993)), in which the axial tensile $P_a$ and lateral compressive shear load $P_s$ were applied to the specimen so as to keep the ratio $\delta_a/\delta_s$ unchanged throughout the test ($\delta_a/\delta_s$ equals to 1.0 in load path 2a). Following Jefferson (2003), the upper left edge and top edge of the specimen are prescribed with $\delta_a$ and $\delta_s$, respectively, while the lower right edge and bottom edge are kept fixed in both directions. A plane stress state (thickness is 50 mm) was assumed. Three unstructured meshes (generated by Gmsh Geuzaine and Remacle (2009)) consisting of Q4 elements were considered, see Fig. 6.

The obtained crack pattern (with the mesh consisting of roughly 7000 elements) is depicted in Fig. 7. Good qualitative agreement with experiment can be observed. Before turning our attention to the global behavior—the load-displacement response, we emphasise that our focus is on the numerical performance of the proposed model rather than replicating the experiments. Actually many previous works fail in this attempt e.g., Gasser and Holzapfel (2005). The load-displacement curves obtained with three different finite element meshes are given in Fig. 8 plus the ones from XFEM (435 bilinear elements) and superimposed with the experimental ones. Note that the XFEM result was obtained using a different code, a different cohesive law and a different mesh (both element type and number).

$^5$ The load-displacement curves were extracted from Nooru-Mohamed et al (1993) using the software plotdigitizer which is freely available at http://plotdigitizer.sourceforge.net.
et al (2000b); Ruiz et al (2001); Zi et al (2005); Sam
studied by many researchers, for example Belytschko
tal value of
e from the midspan for
ure mode–crack grows from the notch for
↵
erent crack patterns for di
erent values of
↵
John and Shah (1990). The problem configuration is
loading of a three point bending beam carried out by
We consider the mixed mode fracture under impact
4.1.2 Mixed-mode dynamic fracture test

Next, a preliminary study of three dimensional crack
modelling within the proposed approach is given. To
this end we use a mesh of linear tetrahedra elements
(34397 nodes and 182788 elements, Fig. 9a). Cracking is
only allowed in the refined region (marked as yellow do-
main). Similarly to what was adopted in 2D, parameter
H was defined for 3D as $H = \sqrt{V}$, where $V$ denotes
the element volume. Three dimensional study of this test
was considered by other researchers, for example Gasser
and Holzapfel (2005) where XFEM was used. The same
cohesive law used for the 2D analyses was adopted for
the 3D analysis. The calculated crack pattern is de-
picted in Fig. 9b and the load-displacement curve is
given in Fig. 9c. The 3D elastic response matched the
2D response well, but the post-peak did not. This could
be due to 3D vs. 2D issues: a single crack in 2D is across
the whole thickness, while in 3D there may be several
cohesive cracks at different stages across the thickness
(e.g. they do not crack at same time through the thick-
ness).

4.1.2 Mixed-mode dynamic fracture test

We consider the mixed mode fracture under impact
loading of a three point bending beam carried out by
John and Shah (1990). The problem configuration is
given in Fig. 10, where an offset notch from the midspan
(with varying locations depending on the dimensionless
parameter $\gamma$) is made to study mixed mode fracture.
The authors in John and Shah (1990) observed different
crack patterns for different $\gamma$ and interestingly there
exists a transition value $\gamma_t$ that defines a change in fail-
ure mode–crack grows from the notch for $\gamma < \gamma_t$ and
from the midspan for $\gamma > \gamma_t$ (Fig. 11). The experimen-
tal value of $\gamma_t$ is 0.77. This test has been numerically
studied by many researchers, for example Belytschko
et al (2000b); Ruiz et al (2001); Zi et al (2005); Sam
et al (2001); Silani et al (2014) presented 3D simula-
tions of this test. Different authors reported slightly
different values of $\gamma$: $\gamma_t = 0.6$ in Ruiz et al (2001),
$\gamma_t = 0.635$ in Zi et al (2005) and $\gamma_t = 0.734$ in Be-
lytschko et al (2000b). Herein we study this example
using the proposed embedded cohesive crack model for a
plane stress condition. The bulk is assumed to be lin-
ear elastic with Young’s modulus and Poisson’s of $E =
31.37$ GPa and $\nu = 0.2$, respectively. Parameters for the
(rate-independent) cohesive model are $f_t = 10.45$ MPa
and $G_f = 19.58 \times 10^{-3}$ N/mm following Zi et al (2005).

The imposed velocity is given by

$$v(t) = \begin{cases} 
\frac{v_1}{t_0} t & \text{if } t \leq t_0 \\
v_1 & \text{otherwise}
\end{cases} \quad (30)$$

where $t_0 = 196 \mu s$ and $v_1 = 60 \text{ mm/s}$. In the FE code
we impose the displacement which is given by (after integrating the imposed velocity)

$$u(t) = \begin{cases} 
\frac{v_1}{2t_0} t^2 & \text{if } t \leq t_0 \\
v_1(t - \frac{t_0}{2}) & \text{otherwise}
\end{cases} \quad (31)$$

and the central difference time integration scheme Be-
lytschko et al (2000a) was used with a constant time
step $\Delta t = 10^{-8} \mu s$.

Firstly we study the influence of mesh refinement
by considering the case $\gamma = 0.5$. Three unstructured
meshes consisting of 6043 triangular elements (3116
nodes), 11045 elements (5650 nodes) and 20605 (10457
nodes) were used. The load histories are plotted in Fig. 12.
It is obvious that two peak loads are present which
is in good agreement with the experiment John and
Shah (1990) and the numerical study in Ruiz et al
(2001). Crack initiation took place at about 600 $\mu s$.
Only slight differences were found between the results
obtained with different meshes and therefore, in subse-
quent analyses the coarse mesh was adopted. The crack
pattern is depicted in Fig. 13 which indicates that fail-
ure is at the offset notch. Note that the small notch at
the midspan was introduced to initiate crack growth
from the midspan as done in Belytschko et al (2000b);

Next, various simulations with $\gamma = \{0.5, 0.64, 0.65\}$
were carried out and the crack patterns are given in
Fig. 14. One can conclude from these simulations that
the transition location $\gamma_t$ is $\gamma_t = 0.64$, which is similar
4.2 Tests with complex fracture patterns

In this section, we study cracking of fibre-reinforced composite materials and concretes at mesoscale where the microstructures (e.g., fibres, matrix and aggregates) play a significant role in the fracture mechanism. As these problems usually involve complex crack paths, it is difficult to have a robust simulation with discrete crack models (e.g., XFEM) and implementation is particularly intricate in 3D. Herein we demonstrate that our simple constitutive model can provide an alternative.

4.2.1 Microcrackings of fibre-reinforced composites

Fibre-reinforced polymer (FRP) composites have been used in many modern technologies mainly due to their attractive properties characterised by the high strength weight ratio. Failure of such materials is not yet well understood. The problem herein presented concerns the transverse fracture of unidirectional fibre-reinforced composite plies. For simplicity thermal residual stresses and plastic deformation are not considered. The matrix and fibres are modelled as linear elastic materials with properties given in Table 1.

A plane stress condition is assumed. It should be noted that to model the matrix/fibre debonding one needs to use a discontinuous approach (i.e. interface elements). If a continuous approach is to be used to model the debonding, then a thin layer around the fibres is needed. Finding the thickness of this thin layer is not trivial and...
its meshing can result in many elements\(^6\). Therefore, we opted for using interface elements for modelling matrix/fibre interface debonding. In what follows, the term ‘continuous approach’ means that a continuous model (i.e., our proposed model) is used for the matrix cracks and the term ‘discontinuous approach’ indicates that a discrete crack model is used for the matrix cracks. In both approaches, discrete cracks are used to model the interface debonding process.

The model is a fibre-reinforced composite sample with one single fibre as shown in Fig. 15 because the crack trajectory is known \textit{a priori} from experiments. The square domain has a side of 1 mm and the fibre diameter is 0.5 mm (i.e., fibre volume fraction 19.6\%). A symmetric, displacement-controlled loading is imposed on both the left and right edges, while the top and bottom edges are stress free. The unstructured three node triangle (T3) mesh is given in Fig. 16 from which two other additional meshes are created using the preprocessor developed in Nguyen (2014b): the first one (used with the continuous model) is obtained by inserting interface elements at the matrix/fibre interface and the second one (used with the discontinuous model) is obtained by inserting interface elements at every element boundaries except for the fibre, since it is assumed the fibre will not crack. The number of nodes of the mesh that employs IEs everywhere (except the fibre) is about six times the nodes of the mesh used in the continuous approach.

The crack patterns obtained by two methods are given in Fig. 17. Both methods produced a correct crack pattern experimentally observed: the cracking process starts with the debonding of the fibre/matrix interface and then these interfacial cracks kink into the matrix Paris et al (2006). The load-displacement curves measured at the right edge of the sample show similarity as well, cf. Fig 18. The discontinuous simulation was prematurely stopped due to divergence of the Newton-Raphson solver. The continuous simulation took about 170 seconds while the discontinuous one took 7784 seconds which is 75 times slower. Note that the displacement increments for the discontinuous analysis are half of the ones adopted in the continuous model to ensure convergence of the former. This fact contributes largely to the number 75 aforementioned. There is no significant difference between the square-root-of-area criterion and the exact one in calculation of \(H\) as in our previous works Nguyen et al (2014b).

\subsection*{Failure of concretes at mesoscale}

This section considers failure of concretes at the so-called mesoscale where concrete is usually modelled as a three-phase material with matrix, hard aggregates of circular shape and an interfacial transition zone (ITZ) surrounding each aggregate. The ITZ is modelled by initially elastic cohesive IEs and thus is assumed to have a zero thickness. Material properties are given in Table 2 according to Tijssens et al (2001). The sample is generated by randomly placing circles of different radii into a box. Portion of the geometry which is within a chosen window is next imported into Gmsh to create FE meshes. The considered sample is shown in Fig. 19 together with a part of the mesh\(^7\). Since no cracking is allowed in the aggregates, large elements were used. Similar to previous examples, uniaxial tension test under a dynamic condition is carried out under a plane stress condition (unit thickness).

\(^6\) In Nguyen et al (2015b) we presented simulations with finite thin interfaces to indicate that our model can be used without interface elements.

\(^7\) Periodicity was not considered simply because our microstructure generator is unable to generate periodic structures.
Fig. 9: Double-edge notched test: three dimensional case; (a) utilised mesh and (b) obtained crack pattern.

Fig. 10: John-Shah test: problem description. The width of the notch is chosen to be 4. Dimensions are in millimeters.

A fine mesh consisting of 41032 T3 elements and 20757 nodes was made and then two other meshes are created: mesh 1 (adopted in the continuous model) contains 41032 T3 elements, 2354 IEs and 23093 nodes and mesh 2 (used in the discontinuous model) contains 41032 T3 elements, 30470 IEs and 74909 nodes (3.25 times mesh1 nodes).

The crack patterns are shown in Fig. 20 and the load-displacement curves in Fig. 21. The crack patterns are different locally but quite similar globally, which resulted in load-displacement curves close to each other. The runtime of the continuous simulation was 1408 seconds while the discontinuous simulation took 4457 seconds (on the same computer and the same code) which is about 3.2 times slower.

5 Discussions

Based on the numerical simulations in Section 4, the following observations can be summarised.

– Continuous approach is compatible with any existing numerical codes, as it deals with everything at the constitutive (integration points) level and interacts with the numerical scheme for solving boundary value problems via a simple size $H$. Its implementation in any numerical code is therefore straightforward.

– Discrete crack modelling lacks robustness due to divergence of the iterative Newton-Raphson solver probably due to the penalty method to avoid crack penetration. Regarding this issue, modelling complex crack problems are ideal tests for recent advancements in non-iterative solvers such as Graça-e-Costa et al (2013, 2012); Rots and Invernizzi (2004).
Fig. 11: John-Shah test: different failure modes (failure at the notch when the notch is closed to the midspan i.e., $\gamma < \gamma_t$ and failure at the midspan if the notch is far from the midspan i.e., $\gamma > \gamma_t$).

<table>
<thead>
<tr>
<th>$E_m$</th>
<th>$\nu_m$</th>
<th>$E_f$</th>
<th>$\nu_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 GPa</td>
<td>0.4</td>
<td>40 GPa</td>
<td>0.33</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G_{IC}$</th>
<th>$G_{IIc}$</th>
<th>$F_{2t}$</th>
<th>$F_{12}$</th>
<th>$\mu$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05 N/mm</td>
<td>0.05 N/mm</td>
<td>10.0 MPa</td>
<td>10.0 MPa</td>
<td>2.0</td>
<td>$10^6$ N/mm$^3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G_{IC}$</th>
<th>$G_{IIc}$</th>
<th>$F_{2t}$</th>
<th>$F_{12}$</th>
<th>$\mu$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25 N/mm</td>
<td>0.25 N/mm</td>
<td>30.0 MPa</td>
<td>30.0 MPa</td>
<td>2.0</td>
<td>$10^6$ N/mm$^3$</td>
</tr>
</tbody>
</table>

Table 1: Fibre-reinforced composite sample: material properties. Subscripts $m$ and $f$ indicate matrix and fibres, respectively.

<table>
<thead>
<tr>
<th>$E_m$</th>
<th>$\nu_m$</th>
<th>$\rho_m$</th>
<th>$E_a$</th>
<th>$\nu_a$</th>
<th>$\rho_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 GPa</td>
<td>0.2</td>
<td>2200 kg/m$^3$</td>
<td>55 GPa</td>
<td>0.2</td>
<td>2200 kg/m$^3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G_{IC}$</th>
<th>$G_{IIc}$</th>
<th>$F_{2t}$</th>
<th>$F_{12}$</th>
<th>$\mu$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 N/m</td>
<td>3 N/m</td>
<td>2.1 MPa</td>
<td>2.1 MPa</td>
<td>2.0</td>
<td>$10^6$ N/mm$^3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G_{IC}$</th>
<th>$G_{IIc}$</th>
<th>$F_{2t}$</th>
<th>$F_{12}$</th>
<th>$\mu$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19 N/m</td>
<td>19 N/m</td>
<td>5.0 MPa</td>
<td>5.0 MPa</td>
<td>2.0</td>
<td>$10^6$ N/mm$^3$</td>
</tr>
</tbody>
</table>

Table 2: Mesoscale concrete sample: material properties. Subscripts $m$ and $a$ indicate the cement paste and aggregates, respectively.

On the other hand, continuous approach to cracking is more robust;

- Discrete crack modelling is computationally expensive whereas continuum modeling of crack is much cheaper and thus more suitable for large-scale simulations;
- For all cases considered in this manuscript the results obtained with the continuous approach do not deviate significantly from the discontinuous ones. Therefore they provide a more practical tool at least for preliminary studies;

6 Conclusions

A new approach to cohesive crack modelling in quasi-brittle solids was presented in this manuscript. The biggest advantage was its simplicity and compatibility with existing numerical codes, thanks to the kinematic enrichment at constitutive (or integration point) level. Although similar to smeared crack models in the overall behavior, the proposed model directly uses a traction-separation law instead of a crack stress-crack strain relationship. Cohesive cracking is taken into account by a new composite constitutive model that consists of a standard cohesive law and a standard continuum con-
The constitutive model. Incorporation of our method into existing finite element codes is straightforward by writing a new subroutine for the stress update in which existing stress return subroutines for cohesive models and continua are performed and combined to drive the composite behaviour of the continuum model. In particular, this is a two-level stress update process in which the outer level interfaces with finite elements and the inner level involves the stress update for the cohesive crack model. We demonstrated the capability of the model in capturing quasi-brittle cracking in both static and dynamic regimes. The results are objective with respect to the numerical spatial discretisation size and correct crack patterns were captured for selected examples.

The performance of our model was also compared with an existing discontinuous approach in the simulation of the failure of a fibre-reinforced composite material and concretes at mesoscale. We have shown that our kinematically enriched constitutive model provides a tool that balances accuracy and efficiency for modelling complex cracking problems. For all examples considered, the kinematically enriched constitutive model yields results comparable to the ones obtained by a discrete method employing cohesive models but with less computational demands (particularly for quasi-static problems).

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References


Fig. 14: John-Shah test: different failure modes for $\gamma = 0.64$ and $\gamma = 0.65$.

Fig. 15: Fibre-reinforced composite sample. Subscripts $m$ and $f$ indicate matrix and fibres, respectively.

Fig. 16: Fibre-reinforced composite sample-FE mesh: the original mesh consists of 8878 three-node triangle elements and 4566 nodes. For the continuous simulation, the mesh consists of 8878 bulk elements, 100 interface elements and 4666 nodes. For the discontinuous simulation, the mesh contains 8878 bulk elements, 12550 interface elements and 25700 nodes.

Fig. 17: Fibre-reinforced composite sample in uniaxial tension: crack patterns obtained by two methods. On the left is the crack pattern with the discontinuous approach with a contour plot of $\sigma_{xx}$ and on the right is the damage pattern with the continuous approach. A plot of damage was given to clearly visualise the crack.

Fig. 18: Fibre-reinforced composite sample in uniaxial tension: load-displacement curves obtained with two methods. Legend ‘square root’ refers to the square-root-of-area formula to determine $H$ and legend ‘exact’ indicates that $H$ was calculated using the crack length.


Fig. 19: Mesoscale concrete sample: geometry and FE discretisation (zoom in).


Fig. 20: Mesoscale concrete sample: crack patterns with a magnification factor of 50. On the left is the pattern obtained with the discontinuous approach. On the right is the pattern obtained with the continuous approach.
Fig. 21: Mesoscale concrete sample: load-displacement curves.


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