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ESTIMATION OF AN ORIGIN-DESTINATION TRIP MATRIX FROM LINK TRAFFIC COUNTS FOR LARGE NETWORKS

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Abstract

Existing estimation approaches for estimating origin-destination trip matrices (O-D) from traffic counts are limited by large networks. These approaches concentrate on achieving a target O-D matrix obtained elsewhere (eg from previous data or sample data survey). With the inconsistency in flow sampling and bias target matrix estimation, they may lead to incorrect results. This paper realises that specifying a target matrix in the objective function of some existing models may not resolve the problem of uniqueness for congested networks. As an alternative approach to overcome the bias estimation probably caused by specifying a target trip matrix, the paper formulates a non-linear programming model which incorporates production and attraction information and inconsistency in the traffic counts, and a network model for extending sampled traffic counts to network population flows. Heuristics for solving the formulated models are presented.

1. INTRODUCTION

Over the last two decades, research on the estimation of origin-destination trip matrices based on link traffic counts has been increasing. This can in part be attributed to the increasing feasibility and convenience of obtaining link traffic counts. With the installation of counting devices in many cities, traffic counts can be obtained less expensively and quicker than via the traditional O-D data survey technique. Although an O-D sample survey will be executed from time to time to obtain behavourial data on the units being surveyed (eg. passengers and commodities), the sample sizes are often inadequate for revealing O-D trip profiles. Traffic counts are a less expensive and more easily updated data souce. Trip matrix estimation from link traffic counts is evolving as a promising approach to trip matrix estimation.

Approaches to the estimation of O-D trip matrices from link traffic counts on a network have been classified by a number of authors (eg. Sherali et el., 1994) into three broad categories: *statistical estimation methods*, *models based on maximum entropy/minimum-information theory*, and *network equilibrium based techniques*. These models however can be generalised as two mathematical forms. The first is formulated as follows (Yang et al., 1992)

$$\min_{t} D(t,\bar{t}) \quad \textit{subject to} \quad M(t) = \bar{v} \tag{1}$$

where t is the O-D trip matrix to be estimated, represented in a column vector, $\bar{t}$ is a column vector representing a target O-D trip matrix and $\bar{v}$ is a column vector representing observed traffic volumes on a subset of links of a network.

$D(t,\bar{t})$ is a function of the generalised distance between the estimated O-D trips t and the target O-D trip matrix $\bar{t}$. Typical forms of the function are entropy (Van Zuylen and Willumsen, 1980) and L_p norm (Cascetta, 1984).

M(t) is referred to as the assignment map (or operator) from the O-D trip matrix t to the observed link flows $\bar{v}$ (Cascetta and Nguyen, 1988). In general, the inverse of *M(t)* may not be unique. Therefore, a target O-D trip matrix $\bar{t}$ is proposed to minimise the generalised distance between the estimated and target O-D trip matrices.

The estimation of the O-D trip matrix in this type of model is constrained by the observed link counts, referred to herein as a counts-constrained (CC) model.

Earlier research has concentrated on uncongested networks, with a linear map of primary proposition (Turnquist and Gur, 1979; Van Zuylen and Willumsen, 1980; Bell, 1984; Cascetta, 1984; Brenninger-Gothe et al., 1989): *M(t) = Pt*, where *P* is a proportion matrix whose element is a proportion of trips of an O-D pair using an observed link.

Due to the fact that the linear mapping is not appropriate for congested networks where the trip matrix is generally assigned to the network with user equilibrium, *M(t)* is specified as a user equilibrium assignment operator consistent with Wardrop's first principle (Wardrop, 1952). The user equilibrium assignment based approaches include Nguyen's (1977) model in which the trip matrix is estimated to reproduce the observed O-D travel times, Fisk's (1988) entropy maximisation model with user-equilibrium constraints, and distribution/assignment calibration model of Fisk and Boyce (1983). Further discussions and applications of these models can be found in Nguyen (1984), LeBlanc and Farhangian (1982), Sheffi (1985), Sheffi and Barnhart (1982).

No matter what the form of the assignment operator *M(t)*, the CC model has an inherent limitation for inconsistent traffic counts. In reality, inconsistency in traffic counts is not unusual because traffic counts cannot be guaranteed error free. Even though the traffic counts are collected error free, the actual flow may not exactly conform to a user equilibrium solution. A number of methods for eliminating and preprocessing inconsistencies have been proposed (Van Zuylen and Branston, 1982; Bell, 1983; Van Zuylen and Willumsen, 1984; and Carey and Revelli, 1986; Jornsten and Wallace, 1993). However, to date there has been no method which can efficiently and systematically resolve the inconsistency problem, especially for a congested network. It is arguable that even with consistent traffic counts, there might not exist a trip matrix that exactly reproduces the link flows because the methods for obtaining the assignment operator *M(t)* are mostly heuristic based.

Many researchers have realised the inconsistency problem in the CC model and have proposed alternative models. These models are classified herein into the second mathematical form:

$$\min_{t,v} D(t,\bar{t}) + E(v,\bar{v}) \quad \textit{subject to} \quad M(t) = v \tag{2}$$

where, $E(v,\bar{v})$ is a function of the generalised distance between estimated flow v and observed traffic counts $\bar{v}$ on the set of observed links. Typical forms include generalised least squares (Bell, 1984), total expected penalty (Jornsten and Wallace, 1993) and L_p norm.

This model is constrained by estimated *link flows* instead of observed *link counts*, and is referred to as a flows-constrained (FC) model. The FC model will produce a trip matrix identical to the one obtained by the CC model if and only if the estimated link flows v in the FC model is such that $E(v,\bar{v}) = 0$.

With the FC model, the generalised least squares approach (Bell, 1984; Cascetta, 1984; Bell, 1991) and the stochastic programming approach (Jornsten and Wallace, 1993) are not applicable for congested networks because they use the proportional assignment of flows for the estimated trip matrix. Although the bilevel programming approach (Yang, 1992) decomposes *M(t)* from the FC model into a lower level

decision making which evolves from solving a traffic assignment problem, it is computationally limiting for large networks.

Both CC and FC models specify a target O-D trip matrix for the estimated trip matrix. This specification limits the ability to apply statistical procedures to the application of trip estimation models. Specifying a target trip matrix will require extensive study of travel data which is not always readily available. It will be shown below that biased target information will lead to biased estimate. In certain circumstances this implies that specifying a target O-D trip matrix is a much more important task than the task of estimating an O-D trip matrix from traffic counts. Estimating an O-D trip matrix is precisely what we hope to produce (Barbour and Fricker, 1994)

Given the difficulty in specifying a target trip matrix, Barbour and Fricker (1994) proposed the SHAPE-2 algorithm, a heuristic method based on shortest augmenting paths. Sherali et al. (1994) proposed a preliminary linear programming model which can be modified to optionally incorporate the inconsistent traffic counts and uniqueness by adding the target matrix constraint. The model requires traffic counts on *all* links of a network and has computational limitations for large networks. No systematic method has been proposed, to our knowledge, for extending the sample traffic counts to entire network flows.

This paper proposes an alternative approach to the trip estimation problem using traffic counts by eliminating the specification of a target matrix. The objective of the paper is to identify the problem of inconsistencies in traffic counts, the possible biased estimation in the presence of a pre-specified target trip matrix, and the computational limitations associated with large networks. The paper is organised as follows. Section 2 introduces all notation and discusses some key factors in O-D trip estimation using traffic counts. Section 3 establishes the O-D trip estimation model and proposes an heuristic method for the model. Section 4 develops a systematic approach to extend sample traffic counts into entire network flows and discusses optimal sampling strategies. The remaining sections present a case study and the major conclusions.

2. NOTATION

Define a directed and connected network by (N, Ω), where N is the set of nodes and Ω is the set of links. The set of links with observed traffic counts are denoted as A and the set of links without observed traffic counts are denoted as B. A link from node i to node j is given as (i, j). A path from node i to node j is defined as P_{ij}.

Demand zones are classified according to geographic and socio-economic criteria, as an overlay for the network. Each zone contains a set of nodes of the network. These zones are mutually exclusive and in aggregate they form all the nodes of the network. Using set notation, if the urban area is divided into m zones $Z_1, Z_2, \ldots, Z_m$ and each zone Z_i contains a subset of nodes of the network $(i = 1, 2, \ldots, m)$, Z_i and Z_j must be

mutually exclusive: $Z_i \cap Z_j = \varnothing \, (i \neq j, 1, 2, \ldots, m)$ and the total union of these zones contains all the nodes of the network: $\bigcup_{i=1}^{m} Z_i = N$.

A shortest free flow travel time (SFT) path between two nodes is defined as a shortest travel time path with free flows on each link of the path. Speed limits on each link differentate the shortest distance path based on the link lengths of the path. Drivers prefer using the shortest time routes rather than the shortest distance routes. For a congested network, the total travel time passing the SFT path may not represent the actual shortest travel time path between the two nodes.

When the SFT path is saturated, i.e. at least one link's traffic volume has reached its capacity, the paths with the total free flow travel time less than or equal to the saturated travel time are defined as feasible paths. The SFT path is one of the feasible paths. Travellers will still continue to travel on feasible paths even when the SFT path is saturated. Unless the SFT path is over-saturated, the routes travelled by trip makers according to the Wardrop's first equilibrium principle (Wardrop, 1952) must be feasible routes. If we assume that only feasible paths are chosen by trip makers, we may drastically reduce the route choice set for the trip makers for a large network. Let us denote the set of feasible paths between nodes i and j as K_{ij} and the set of infeasible paths as $\overline{K}_{ij}$. $K_{ij} \cup \overline{K}_{ij}$ forms all paths connecting nodes i and j, denoted as *K*.

The O-D trip matrix estimation task using traffic counts is to estimate the number of trips from zone Z_i to Z_j $(i, j = 1, 2, \ldots, m)$ to reproduce the traffic counts on links in *A*. Without additional information, we may not be able to derive a unique solution, even though the traffic counts are obtained in a user equilibrium pattern for the entire network. Consider the following simple network with only three nodes and three links. Each node also represents a zone.

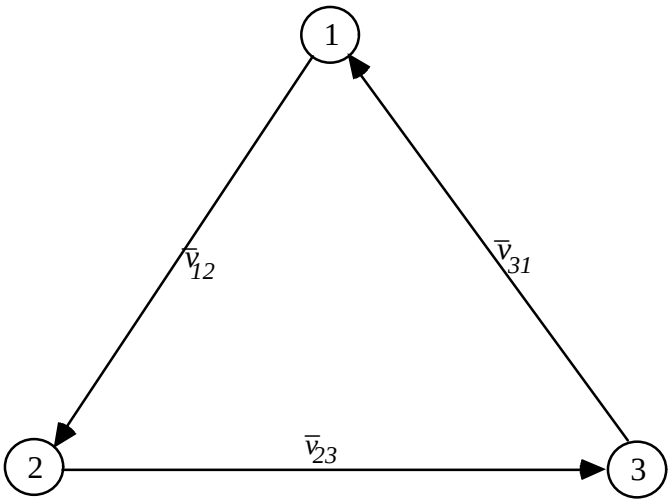

Figure 1. Example network with three nodes and three links

With full traffic counts on the network, we can verify that the O-D trip matrix { t_{ij} } satisfying the following equations will be able to reproduce the observed traffic counts.

$$\begin{cases} t_{12} + t_{13} + t_{32} = \bar{v}_{12} \\ t_{23} + t_{21} + t_{13} = \bar{v}_{23} \\ t_{31} + t_{32} + t_{21} = \bar{v}_{31} \end{cases} \tag{3}$$

The O-D trip matrices satisfying Equations (3) are not unique. To overcome the problem of uniqueness, most existing approaches specify a target O-D matrix in the objective function. However, specifying a target trip matrix does not solve the problem for the following reasons.

(a) The above example shows that with a target trip matrix, intra-zonal trips will most likely match exactly the specified target intra-zonal trips.

(b) Some inter-zonal trips will match exactly the specified target inter-zonal trips if traffic counts are observed only on a subset of network links. In the above example, if there was not a traffic count on link (3,1), the number of trips from node 3 to node 1 would exactly match the specified target.

(c) As discussed in the Appendix, with a general user-equilibrium assignment for congested networks, both counts and flows constrained models may not produce a unique O-D trip matrix even if a target matrix is specified.

The above point suggests that specifying a target trip matrix may lead to biased estimation of a correct trip matrix. Thus in practical applications more effort may be devoted to examining a target matrix rather than the trip estimation itself in order to avoid bias estimation. If justified, the significance of estimating a trip matrix from traffic counts might be neglected. In view of this, we formulate the estimation problem in a similar model to the FC model but with (i) omission of a target matrix and (ii) the addition of production and attraction constraints:

$$\begin{aligned} &\min_{t,v} E(v, \bar{v}) \\ &\textit{subject to} \\ &M(t) = v \\ &\sum_{j \in D} t_{ij} = o_i \qquad i \in O \\ &\sum_{i \in O} t_{ij} = d_j \qquad j \in D \end{aligned} \tag{4}$$

where O is the set of origin zones and D the set of destination zones; o_i is the total number of trips from the origin zone i, or trip production, and d_j is the total number of trips to the destination zone j, or trip attraction. The production and attraction constraints are also called row and column constraints (Carey, 1981).

Whether (4) is a better model system than the FC model needs confirmation. From an application perspective, it is easy to implement because trip production and attraction are easier to obtain. The estimation of trip production and attraction requires a major investigation on population and employment densities. A realistic application requires that an O-D trip matrix satisfy the trip production and attraction constraints.

Substituting the flow constraint $v = M(t)$ into the objective function, we obtain the following transportation problem with a general objective function where traffic assignment is involved.

$$\begin{aligned} &\min_{t,v} E(M(t),\bar{v}) \\ &\textit{subject to} \\ &\sum_{j\in D} t_{ij} = o_i \qquad i \in O \\ &\sum_{i\in O} t_{ij} = d_j \qquad j \in D \end{aligned} \tag{5}$$

With transportation constraints, the model will always have feasible solutions and thus eliminate the inconsistencies. For an uncongested network, $M(t)$ can take the linear form $M(t) = Pt$. If we assume the objective function is quadratic, then it can be written as $E(M(t),\bar{v}) = (Pt-\bar{v})' Q(Pt-\bar{v})$, where Q is a positive finite weighting matrix. Q can also be interpreted as a dispersion matrix (Maher, 1983) or the variance-covariance matrix (Brenninger and Jornsten, 1989) of the random error term $Pt-\bar{v}$ where the objective is interpreted as generalised least squares regression model. For the quadratic objective function and transportation constraints, a number of algorithms can be applied to derive an optimal global solution (Carey 1981).

Although many researchers have proposed methods for solving the transportation problem (see Carey (1981) for a review), without exception they all assume the convexity of the objective function. When the general user equilibrium assignment is involved, the objective function of model (5) may not be convex. Furthermore, it is unknown whether $M(t)$ is a continuously differentiable function because there is no explicit form for the user equilibrium assignment operator $M(t)$. With this type of general objective function, the problem becomes very complex. Therefore, heuristics are utilised.

In searching for a heuristic method for the trip estimation model (5), the strategy is to combine a method which can solve the transportation problem and be a user equilibrium algorithm. The travelling costs from zones to zones for the transportation problem are the travelling times that are the weighted average of travel time using the current estimated traffic flow v and the travelling time used for the transportation in the last iteration. That is, if we denote $\tilde{u}_{ij}$ as the travel time from zone i to zone j using the estimated traffic flows and u_{ij} as the travelling time used for the transportation in the last iteration, then the travelling time used for the current transportatoin problem is updated as $u_{ij} \leftarrow u_{ij} + l\,(\tilde{u}_{ij} - u_{ij})$, where $0 \le l \le 1$ is a constant such that the updated travelling time u_{ij} is positive for all zone pairs (i, j). The algorithm is described below.

STEP 0 For the observed traffic flows $\bar{v}$, determine the O-D travel time $\bar{u}_{ij}$ using a shortest path algorithm for each zone pair (i, j). Use the determined O-D travel time as the initial travel time for the transportation problem, ie. $u_{ij} = \bar{u}_{ij}$.

STEP 1 Estimate the O-D trips t_{ij} using a method for the transportation problem using the travel time u_{ij}.

STEP 2 For the estimated O-D trips { t_{ij} }, calculate traffic flow v_{xh} for each link $(x, h) \in A$ and zonal travel time $\tilde{u}_{ij}$ for each zone pair (i, j) using a user equilibrium algorithm.

STEP 3 If one of the following stopping criteria is satisfied, then stop. Otherwise, update the generalised impedance $u_{ij} = u_{ij} + l\,(\tilde{u}_{ij} - u_{ij})$ and go to STEP 1.

Stopping Criteria

(1) Number of iterations exceeds a specified number.

(2) The difference of values of the objective function $E(v,\bar{v})$ between the current run and previous run is within a specified number.

(3) The maximum difference of O-D trips between the current run and previous run is within a specified number.

Remark 1. The proposed heuristic can efficiently solve for large networks because no additional variables have to be introduced. In the execution of the above method, the travel time u_{ij} used for the transportation problem is constantly updated during each iteration. It is updated from the previous iteration in the way that if the estimated flow v_{xh} on the path from zone i to zone j is less than the observed flow $\bar{v}_{xh}$, more trips will be assigned between zone i and zone j with the reduced impedance u_{ij}, it is increased if estimated flow v_{xh} is greater than the observed flow $\bar{v}_{xh}$.

Remark 2. In step 1 of the above method, it involves selecting a solution procedure for solving the transportation problem while in step 2 it involves using a method for traffic assignment. The traffic assignment method selected for the case study presented in Section 4 is Frank and Wolfe's linear approximation method (Frank and Wolfe, 1956), implemented in the EMME/2 package (INRO, 1994). The method selected for solving the transportation problem is the double factor method (Kruithof, 1937) for the transportation model with an entropy objective function. The advantage of the double factor method is that it is efficient and the convergence is guaranteed (Gorman, 1963; Evan and Kirby, 1974; Robillard and Stewart, 1974). In applying the double factor method, we substitute the target trips with travel times as the balancing parameters:

Modified double factor method

***Initialisation*:** $b_j^0 = 1$ for all j, $t_{ij}^0 = 0$ for all (i, j), and set n=0.

1° Perform row balancing

$$a_i^n = \frac{1}{\sum_j b_j^{n-1} d_j / u_{ij}} \qquad \text{for all i}$$

and column balancing

$$b_j^n = \frac{1}{\sum_i a_i^n o_i / u_{ij}} \qquad \text{for all j}$$

2° Estimate the intermediate O-D trips

$$t_{ij}^n = \frac{a_i^n b_j^n o_i d_j}{u_{ij}} \qquad \text{for all (i, j)}$$

If the $\max_{i,j} \left| t_{ij}^n - t_{ij}^{n-1} \right| < e \quad (e > 0)$, then stop. Otherwise increase n by 1 and repeat steps *1°* and *2°*.

3. EXTENSION OF TRAFFIC COUNTS TO THE ENTIRE NETWORK FLOW

In reality, traffic counts are obtained for a subset of links of a network, given resource constraints. Direct application of the estimation models using the subset of traffic counts may lead to unexpected results due to the fact that the collected traffic counts cannot be guaranteed error free or in an equilibrium pattern. Preprocessing of the observed traffic counts reveals a necessary step. One of the methods to calibrate the traffic counts is to extend the sample traffic counts into the entire network in a user-equilibrium pattern. During the process of extension, validation and calibration can be undertaken consistent with user-equilibration. With the extended link flows, real traffic counts on unobserved links may also be collected to conduct validation and calibration activities.

The model for extending the observed traffic counts in *A* to the whole network traffic flows (ie. estimate the flows on links in *B*) is formulated as follows:

$$\begin{aligned}
&\textit{Minimize} \quad \sum_{i,j\in N} \sum_{P_{ij},P'_{ij}\in K_{ij}} D_P(P_{ij}, P'_{ij}) \\
&\textit{subject to} \\
&\sum_{\substack{j\in N \\ (i,j)\in\Omega}} v_{ij} = \sum_{\substack{j\in N \\ (j,i)\in\Omega}} v_{ji} \qquad \textit{for all } i\in N \\
&0 \le v_{ij} \le C_{ij} \qquad \textit{for all } (i,j)\in\Omega
\end{aligned} \tag{6}$$

where,

P_{ij} and P'_{ij} are feasible paths from nodes i to j (i, j N), i.e. $P_{ij}, P'_{ij} \in K_{ij}$.

$D_P(P_{ij}, P'_{ij})$ is the function of general distance between the travel times $T(P_{ij})$ and $T(P'_{ij})$ parsing paths P_{ij} and P'_{ij} respectively. For example, $D_P(P_{ij}, P'_{ij})$ can be expressed as $D_P(P_{ij}, P'_{ij}) = ((T(P_{ij}) - T(P'_{ij}))^2$.

v_{ij} is the traffic flow on link (i, j) . When (i, j) B, v_{ij} is known as a decision variable in the model. Otherwise it a known as a constant in the model which represents an observed traffic count.

If a path P_{ij} from node i to node j in the network is represented by a set of links,

$$P_{ij} = \{(i, n_1), (n_2, n_3), \ldots, (n_r, j)\} \tag{7}$$

with $n_0 = i$ and $n_{r+1} = j$, the travel time of traversing path P_{ij} can be represented by

$$T(P_{ij}) = \sum_{k=0}^{r} T(v_{n_k n_{k+1}}) \tag{8}$$

where v_{ij} is the traffic flow on link (i, j).

$T(v)$ is the volume delay function of the link traffic volume v. There are a number of specifications in the literature for the volume delay function, a well known form is that introduced by the Bureau of Public Roads (BPR, 1964):

$$T(v) = T_0(1 + a(v/C)^b) \tag{9}$$

where T_0 is the free-flow travel time, C is the capacity of the link, a and b are parameters.

Davidson (1966, 1978) proposed a volume-delay function which is widely used in Australia:

$$T(v) = \begin{cases} T_0(1 + J\frac{v/C}{1-v/C}) & v/C < r \\ T_0(1 + J\frac{r}{1-r} + \frac{J}{(1-r)^2}(v/C - r) & v/C \geq r \end{cases} \tag{10}$$

where J is a delay parameter reflecting the road type (eg. level of internal friction within the traffic stream) and r is a predetermined constant, usually in the range of [0.85, 0.95] (Taylor 1992).

Akcelik (1991) proposed a statistically based time-independent volume delay function based on Davidson's function:

$$T(v) = T_0\left\{1 + 0.25r\left[(v-1) + \sqrt{(v-1)^2 + \frac{8J}{r}v}\right]\right\} \tag{11}$$

where, r is the ratio of flow period to free flow travel time and J is the delay parameter as defined in the Davidson function.

The flow conservation constraints of the model ensure that the flows are conserved at intermediate nodes of the network. If a node is connected to a centroid of a zone, then the flows from and to the zone should also be included in the conservation constraints. However, the centroids are not present in the conservation constraints. The results of the models will also contain the flows from and to zones and therefore production and attraction of each zone can be obtained. These estimated productions and attractions can be compared with the real productions and attractions to gain the confidence about the quality of the production and attraction data.

The model may not produce feasible solutions for inconsistent traffic counts. Especially when $A =$, the traffic counts collected for the entire network cannot be guaranteed to satisfy the conservation equations. Therefore, the model needs to be modified to accommodate the inconsistencies in traffic counts:

$$\begin{aligned} &\textit{Minimize} \quad \sum_{i,j\in N} \sum_{P_{ij},P'_{ij}\in K_{ij}} D_P(P_{ij}, P'_{ij}) + g\sum_{i\in N} d_i^+ + d_i^- \\ &\textit{subject to} \\ &\sum_{\substack{j\in N \\ (i,j)\in\Omega}} v_{ij} - \sum_{\substack{j\in N \\ (j,i)\in\Omega}} v_{ji} + d_i^+ - d_i^- = 0 \qquad \textit{for all } i \in N \\ &0 \leq v_{ij} \leq C_{ij} \qquad \textit{for all } (i,j) \in \Omega \\ &d_i^+, d_i^- \geq 0 \qquad \textit{for all } i \in N \end{aligned} \tag{12}$$

where d_i^+ and d_i^- are non-negative artificial decision variables to permit positive or negative deviations in the conservation constraints, borrowing the idea from goal programming. The parameter g in the objective function represents a scaling constant which shows the importance of the total error term $\sum_{i\in N} d_i^+ + d_i^-$ in the objective

function. The value of g should be in the range of $0 \le \sqrt{g} \le \sum_{(i,j)\in\Omega} T_{ij}(C_{ij})$, where $T_{ij}(C_{ij})$ is the volume delay function for link (i, j) with traffic flow $v_{ij} = C_{ij}$, the capacity of link (i, j) (i.e. the link is saturated).

The model will produce a feasible solution $v_{ij} = 0$ for all links (i, j) ∈ Ω when *A* = Ø. However, this may not be unique because flows may exist satisfying the conservation constraints which may also achieve the minimisation of the objective function. Figure 2 gives an illustration.

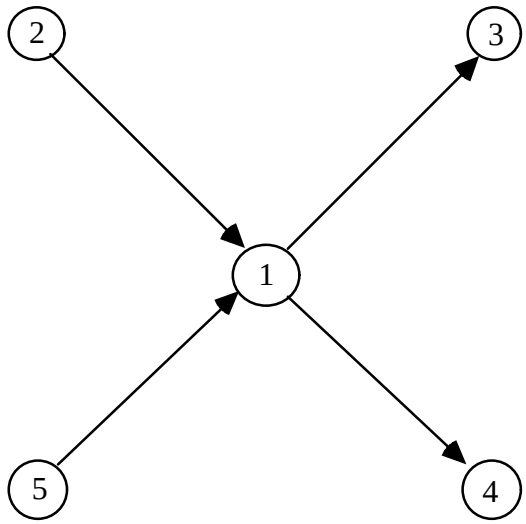

Figure 2. Simple network showing the multiple solutions to the flow extension

In the above network (Figure 2), we assume nodes 2, 3, 4, 5 are centroids. Any flow solutions that satisfy the conservation constraint at node 1 will achieve the minimisation of the objective function because there is no second feasible path from nodes to nodes. A unique solution can be obtained only if traffic counts are observed for three of the four links.

This begs the question: what is the minimum number of links in the observed link set *A* for a general network so that the model will produce a unique solution, and what type of the structure should these links be?

The answer depends on the structure of the network. In the above example, we can conclude that there should be n-2 links for the star type network of n nodes. With the line network as depicted in Figure 3, only one link is sufficient.

Figure 3. A line network

For other types of networks, further research is needed. In this paper, we recommend using a minimum spanning tree algorithm to choose n-1 links for sample traffic counts. The search of the minimum spanning tree is guided by the weights associated with each link. The weights may be associated with the cost of obtaining traffic counts or the importance of links in the entire network.

Given a minimum spanning tree with traffic counts, any unobserved link (i, j) B of the network will form a loop with links in the spanning tree. Thus an alternative path from node i to node j exists and the minimisation of the generalised distance between the travel times of paths can be carried out given that the link (i, j) and the alternative path are feasible paths. In many cases, the conservation constraints will ensure the model produces a unique solution even if the alternative path is not feasible. A loop network with a large number of nodes is a good example (Figure 4).

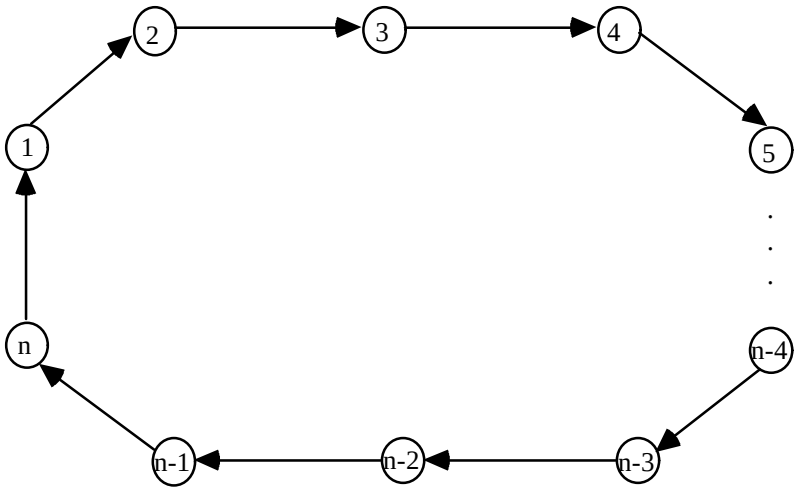

Figure 4. A loop network

Solving the flow extension model (9) involves enumerating feasible paths and deriving travel times for each feasible path. With a large network, it is almost impossible to use non-linear optimisation techniques. A sequential solution procedure is proposed which involves three steps:

We assume that the links with traffic counts are in the form of a minimum spanning tree. Then for each unobserved link, there must be an alternative path with observed counts connecting the end points of the link.

Denote ’ as the set of links with estimated link flows and *B’* as the set of unestimated links. Initially set ’ = *A* and *B’* = *B*.

STEP I *Node Conservation*. Find a node i ∈ *N* with only one link (i, j) (or (j, i) of *B’* from (or to) the node and solve the conservation equation for the link at node i with $d_{ij}^{+} = d_{ij}^{-} = 0$. Remove the link from *B’*: *B’* = *B’* - { (i, j) } and Ω’ = Ω’ ≈ { (i, j) } (or *B’* = *B’* - { (j, i) } and Ω’ = Ω’ ≈ { (j, i) }). Repeat the process until no such node is found.

STEP II *Objective minimisation*.

(1) Find a link (i, j) ∈ *B’* with the minimum difference between the link length and the length of the shortest path *P Î Kij* from i to j in the network (*N*, Ω’).

(2) Solve the following optimisation problem with one variable v_{ij} for link (i, j):

$$\begin{aligned} &\textit{Minimize} \quad (T_{ij}(v_{ij}) - T(P))^2 \\ &\textit{subject to} \\ &\qquad 0 \le v_{ij} \le C_{ij} \end{aligned}$$

where, $T_{ij}(v_{ij})$ is the volume delay function for link (i, j) with capacity C_{ij}.

(3) Remove (i, j) from *B'*: *B'* = *B'* - { (i, j) } and Ω' = Ω' ≈ { (i, j) }, and go back to (1) until *B'* = Ø.

Because of the minimum spanning tree of the observed links, each unobserved link will eventually be assigned an estimated flow by the non-linear programming model in (2). However, the obtained flows may not satisfy the conservation constraint on each node. The next step is to balance the conservation for flows at each node.

STEP III *Node conservation balance*

(1) Find a node with the maximum difference in out-flows and in-flows, e.g. to find a node i such that

$$\left| \sum_{\substack{j \in N \\ (i,j) \in \Omega}} v_{ij} - \sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji} \right| = \max_{i' \in N} \left| \sum_{\substack{j \in N \\ (i',j) \in \Omega}} v_{i'j} - \sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji'} \right| .$$

(2) Two cases:

CASE 1

If $\sum_{\substack{j \in N \\ (i,j) \in \Omega}} v_{ij} > \sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji}$, then we try to decrease v_{ij} and increase v_{ji} so as to minimise $\sum_{\substack{j \in N \\ (i,j) \in \Omega}} v_{ij} - \sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji}$.

(i) Sort V_i = { (i, j) } according to the descending order of { v_{ij} } and sort U_i = { (j, i) } according to the descending order of { $C_{ji} - v_{ji}$ }. The resulting link lists are still denoted as V_i and U_i respectively.

(ii) If $\sum_{\substack{j \in N \\ (i,j) \in \Omega}} (C_{ij} - v_{ij}) < \sum_{\substack{j \in N \\ (j,i) \in \Omega}} (C_{ji} - v_{ji})$, then for the link (j, i) on the top of U_i, calculate the flow increment for link (j, i):

$$\Delta_{ji} = \min\left\{ C_{ji} - v_{ji},\ \sum_{\substack{j' \in N \\ (i,j') \in \Omega}} v_{ij'} - \sum_{\substack{j' \in N \\ (j',i) \in \Omega}} v_{j'i},\ \min\left\{ 0,\ \sum_{\substack{j' \in N \\ (j',j) \in \Omega}} v_{j'j} - \sum_{\substack{j' \in N \\ (j,j') \in \Omega}} v_{jj'} \right\} \right\}$$

and set $v_{ji} = v_{ji} + \Delta_{ji}$ and U_i = U_i - { (j, i) }.

Otherwise, for the link (i, j) on the top of V_i calculate the flow decrement for link (i, j).

$$d_{ij} = \min\left\{ v_{ij},\ \sum_{\substack{j' \in N \\ (i,j') \in \Omega}} v_{ij'} - \sum_{\substack{j' \in N \\ (j',i) \in \Omega}} v_{j'i},\ \min\left\{ 0,\ \sum_{\substack{j' \in N \\ (j',j) \in \Omega}} v_{j'j} - \sum_{\substack{j' \in N \\ (j,j') \in \Omega}} v_{jj'} \right\} \right\}$$

and set $v_{ij} = v_{ij} - d_{ij}$ and V_i = V_i - { (i, j) }.

(iii) Repeat (ii) until the flows are conserved at node i: $\sum_{\substack{j \in N \\ (i,j) \in \Omega}} v_{ij} = \sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji}$), or U_i and V_i are empty.

CASE 2

if $\sum_{\substack{j \in N \\ (i,j) \in \Omega}} v_{ij} < \sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji}$, increase v_{ij} and decrease v_{ji} so as to minimise $\sum_{\substack{j \in N \\ (j,i) \in \Omega}} v_{ji} - \sum_{\substack{j \in N \\ (i,j) \in \Omega}} v_{ij}$.

(i) Sort V_i = { (i, j) } according to the descending order of { $C_{ij} - v_{ij}$ } and sort U_i = { (j, i) } according to the descending order of { v_{ji} }. The resulting link lists are still denoted as V_i and U_i respectively.

(ii) If $\sum_{\substack{j \in N \\ (i,j) \in \Omega}} (C_{ij} - v_{ij}) > \sum_{\substack{j \in N \\ (j,i) \in \Omega}} (C_{ji} - v_{ji})$, then for the link (i, j) on the top of V_i, calculate the flow increment for link (i, j):

$$\Delta_{ij} = \min\left\{ C_{ij} - v_{ij},\ \sum_{\substack{j' \in N \\ (j',i) \in \Omega}} v_{j'i} - \sum_{\substack{j' \in N \\ (i,j') \in \Omega}} v_{ij'},\ \min\left\{ 0,\ \sum_{\substack{j' \in N \\ (j,j') \in \Omega}} v_{jj'} - \sum_{\substack{j' \in N \\ (j',j) \in \Omega}} v_{j'j} \right\} \right\}$$

and set $v_{ij} = v_{ij} + \Delta_{ij}$ and V_i = V_i - { (i, j) }.

Otherwise, for the link (j, i) on the top of U_i calculate the flow decrement for link (j, i).

$$d_{ji} = \min\left\{ v_{ji},\ \sum_{\substack{j' \in N \\ (j',i)\in\Omega}} v_{j'i} - \sum_{\substack{j' \in N \\ (i,j')\in\Omega}} v_{ij'},\ \min\left\{ 0,\ \sum_{\substack{j' \in N \\ (j,j')\in\Omega}} v_{jj'} - \sum_{\substack{j' \in N \\ (j',j)\in\Omega}} v_{j'j} \right\} \right\}$$

and set $v_{ji} = v_{ji} - d_{ji}$ and $U_i = U_i$ - { (j, i) }.

(iii) Repeat (ii) until the flows are conserved at node i: $\sum_{\substack{j \in N \\ (i,j)\in\Omega}} v_{ij} = \sum_{\substack{j \in N \\ (j,i)\in\Omega}} v_{ji}$), or U_i and V_i are empty.

(3) Repeat (1) and (2) until the stopping criteria are met at all nodes:

$$\left| \sum_{\substack{j \in N \\ (i,j)\in\Omega}} v_{ij} - \sum_{\substack{j \in N \\ (j,i)\in\Omega}} v_{ji} \right| \le e \quad (e > 0).$$

5. A CASE STUDY

A case study was conducted to examine the proposed methods. The data was drawn from the Future Directions Study (FDS) Network for Sydney, created by the Roads and Traffic Authority (RTA) of New South Wales, Australia. The proposed methods were applied to the FDS network, with the original 78 zones aggregated into 14 contiguous zones. The links of the FDS network were aggregated into a single link between each zone pair (Hensher et al., 1995). The resulting network is called a zonal network, depicted as follows (Figure 5).

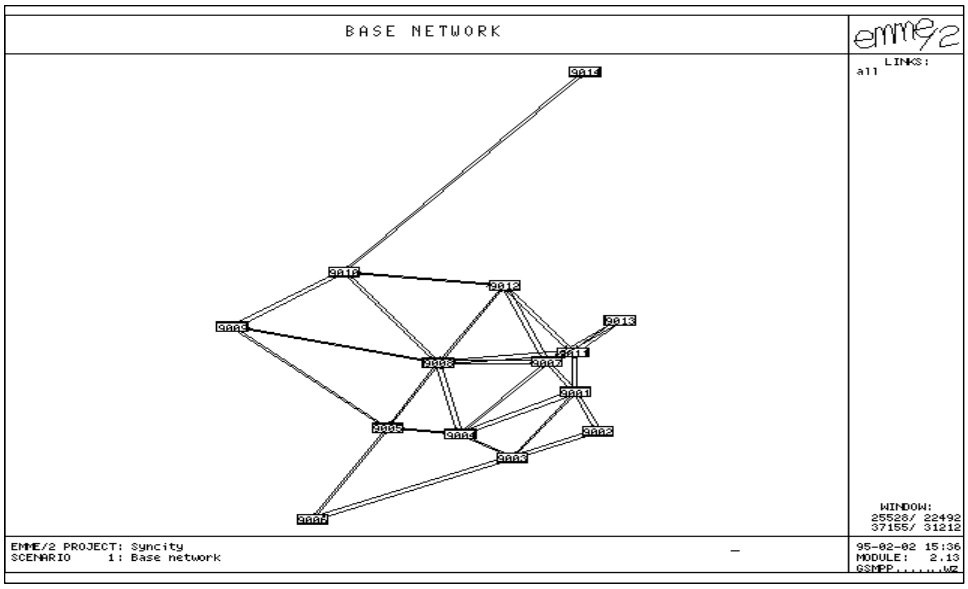

Figure 5. Sydney zonal network

The observed traffic counts were obtained from the equilibrium results produced by the EMME/2 package (INRO, 1994) with the following O-D trip matrix (Table 1). The O-D trip matrix is obtained by aggregating the detailed zonal trips and modified to produce the traffic flows which are in the time equilibrium pattern by Wardrop's first Principle. Therefore, the obtained traffic counts are in the equilibrium pattern.

Table 1. 'True' O-D trip matrix for Sydney zonal network

	01	**02**	**03**	**04**	**05**	**06**	**07**	**08**	**09**	**10**	**11**	**12**	**13**	**14**
01	24784	6060	5968	3693	3479	1555	4375	4277	1499	2682	5596	2923	1694	1257
02	12403	7336	4188	2190	2325	1152	2510	2710	1047	1816	3581	1942	1137	908
03	14388	5060	13581	5172	4957	2810	3409	5177	1982	3446	5338	3207	1778	1711
04	8011	2310	4778	4997	3577	1455	2696	4070	1230	2295	3660	2123	1136	987
05	8233	2599	4742	3723	8556	3407	2980	6071	2527	3736	4912	3466	1601	1723
06	3270	1158	2342	1299	2910	4128	1121	2101	1009	1463	1962	1364	669	763
07	10141	2812	3224	2787	3001	1299	5927	4972	1373	2625	6330	2995	1696	1078
08	6892	2080	3357	2868	4213	1687	3362	8606	1776	3716	4638	3403	1350	1293
09	3337	1088	1724	1172	2505	1131	1279	2422	3865	2797	2222	1577	754	1104
10	6294	2004	3195	2313	3680	1666	2571	5304	2806	6206	4203	3054	1361	1978
11	10316	3122	3900	2928	3884	1773	4976	5452	1880	3378	16426	5211	3532	1567
12	6915	2164	3020	2187	3479	1574	2980	5022	1667	3115	6045	6772	1761	1393
13	4279	1354	1780	1233	1752	841	1783	2172	893	1516	4879	2046	3632	792
14	3056	1033	1622	1003	1711	884	1070	1829	1164	1922	2025	1398	740	6430

To apply the developed method to estimate the O-D trip matrix based on the equilibrium traffic flows produced by the EMME/2 package, the L_2 norm function is used as the generalised distance between the observed and estimated traffic flows in the objective function. We set $l = 0.5$ and use the BPR function (BPR, 1964) with $a = 2.5$ and $b = 2.5$ for the application of the proposed method for estimating the O-D trip matrix based on the equilibrium traffic counts. The EMME/2 package was used to produce the equilibrium traffic flows for the estimated O-D trip matrix during the iteration of the algorithm. The stopping criteria set for the termination of the algorithm are trip and flow difference between two successive iterations, set as 1 and 10 respectively. The actual running of the algorithm involved only six iterations and was terminated by the flow difference of 4.24. The algorithm is not sensitive to the value of l .The resulting estimated O-D trip matrix is presented in Table 2.

Table 2. Estimated O-D trip matrix for Sydney zonal network

	1	**2**	**3**	**4**	**5**	**67**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1	25774	6239	6008	3624	3412	1521	4280	4079	1434	2593	5626	2869	1650	733
2	12862	7592	4170	2169	2293	1079	2484	2625	1010	1773	3605	1916	1115	551
3	14884	5147	13922	5270	4984	2594	3408	5149	1946	3437	5333	3146	1748	1048
4	7554	2217	5005	5332	3710	1483	2760	4106	1235	2342	3726	2137	1141	577
5	7930	2516	4782	3748	9017	3439	2951	6390	2680	3897	4749	3559	1580	1041
6	3144	1103	2128	1292	2931	4634	1111	2163	1060	1517	1912	1390	668	505
7	10075	2822	3224	2760	2972	1289	6305	4891	1360	2636	6521	3098	1696	611
8	6661	2028	3398	2887	4364	1713	3242	9290	1808	3904	4388	3542	1321	693
9	3309	1078	1732	1180	2582	1158	1268	2520	4114	2945	2152	1578	744	618
10	6233	1991	3238	2342	3790	1712	2545	5572	2988	6730	4070	3069	1350	1006
11	10598	3205	3948	2936	3676	1706	5187	5060	1800	3243	17155	5304	3600	928
12	6865	2147	2975	2160	3490	1571	3013	5174	1677	3135	6354	6899	1830	804
13	4377	1384	1796	1235	1693	827	1819	2088	872	1488	4906	2078	3877	510
14	2052	712	1098	633	1115	633	663	1080	733	1073	1320	895	522	13360

Two statistical measures of 'closeness' for both the estimated trips and flows were employed: the root mean squared error (RMSE) and mean absolute error (MAE) (Sherali et al, 1994):

$$RMSE_t = \sqrt{\sum_{(i,j)\in OD} (t_{ij} - \bar{t}_{ij})^2 / |OD|} \qquad MAE_t = \sum_{(i,j)\in OD} |t_{ij} - \bar{t}_{ij}| / |OD|$$

for the estimated O-D trips, and

$$RMSE_v = \sqrt{\sum_{(i,j)\in A} (v_{ij} - \bar{v}_{ij})^2 / |A|} \qquad MAE_v = \sum_{(i,j)\in A} |v_{ij} - \bar{v}_{ij}| / |A|$$

for the estimated network flows.

$\bar{t}_{ij}$ and $\bar{v}_{ij}$ are the true trips (real target) and observed flows respectively. OD and A are the sets of O-D pairs and observed network links respectively. Relative errors can be obtained by dividing the sbsolute errors by the average of the true values (Yang 1992), i.e.

$$RMSE(\%)_t = \frac{RMSE_t}{\sum_{(i,j)\in OD} t_{ij} / |OD|} \qquad MAE(\%)_t = \frac{MAE_t}{\sum_{(i,j)\in OD} t_{ij} / |OD|}$$

for the estimated trips, and

$$RMSE(\%)_v = \frac{RMSE_v}{\sum_{(i,j)\in A} \bar{v}_{ij} / |A|} \qquad MAE(\%)_v = \frac{MAE_v}{\sum_{(i,j)\in A} \bar{v}_{ij} / |A|}$$

Similar definitions of RMSE and MAE can also applied to the travel times between zones. Table 3 summarises the performance of the proposed method on these measures for the estimated network flows.

Table 3. Statistical measures for the proposed method applied to the Sydney network

	IterDif	**ObsrDif**	**RMSE**	**MAE**	**RMSE(%)**	**MAE(%)**
Travel time	0.39	1646.77	117.63	42.64	0.71	0.26
Trip	2.40	7882.96	563.07	198.07	0.17	0.06
Flow	4.24	14542.64	1979.00	1040.65	0.10	0.05

In Table 3, *IterDif* stands for the Euclidean distance of attibutes (travel time, trip, and flows) between the current and previous iterations and *ObsrDif* is the Euclidean distance between the estimated and observed attributes. From Table 3, we can see that the both absolute and relative errors are small for the three attributes. This indicates that the method developed in this paper produces satisfactory result for the Sydney network.

6. CONCLUSION

Taking into account the inconsistency in traffic counts and the disadvantage of specifying a target matrix in estimating an origin-destination (O-D) trip matrix from traffic counts, this paper formulates a non-linear programming model for the trip matrix estimation problem from traffic counts without the specification of a target trip matrix. Rather, the model incorporates the trip production and attraction information which is relatively easier to collect than the target trip matrix. The formulated model has the advantages that it incorporates inconsistent traffic counts, it does not require a full set of network counts, and it will always produce feasible solutions. A heuristic method combining the traditional double factor method and user-equilibrium assignment is proposed for the model.

A network model is formulated to extend sampled traffic counts to network population flows in the way that they are specified in a user-equilibrium pattern consistent with Wardrop's first principle. For the extended network flows, validation and calibration can be carried out with the real network flows. A heuristic algorithm is proposed for the network model which considers the balance of travel time between a link and the alternative shortest path associated with the end points of the link, and the node flow conservation balance. A sampling strategy utilising the minimum spanning tree algorithm is outlined for an optimal traffic counting.

The proposed approach is applied to the Sydney network. Satisfactory results are obtained based on indices of root mean square error (RMSE) and the mean absolute error (MAE) for both network flows and O-D trip matrices.

Further research to compare our approach with the target matrix approach for real networks is recommended. In addition, there is a need to (i) investigate the convergence of the proposed heuristic methods, (ii) evaluate alternative heuristics to obtain more accurate estimates, and (iii) to examine the statistical variability of the sample counts.

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APPENDIX: SOLUTION UNIQUENESS OF EXISTING MODELS FOR ESTIMATING A TRIP MATRIX FROM LINK TRAFFIC COUNTS

Both CC and FC models will produce unique solutions if the objective functions are convex and the constraints are concave. For congested networks, the non-linear user-equilibrium assignment operator *M(t)* may not turn out to be concave since there is no explicit form for *M(t)*. Therefore the uniqueness of the solutions to both either CC or FC model is questionable. Further research may be needed to show that there are non-linear user-equilibrium assignment operators for congested networks that are concave.

For uncongested networks, when linear assignment is applied: M(t) = Pt and quadratic objective functions are used if it is known that both CC and FC models will produce unique solutions since the objective functions become convex and the constraints concave. This result can also be proven in matrix form as follows.

Counts constrained (CC) models:

$$\begin{aligned} &\textit{Minimize} \quad (t-\bar{t})^T Q(t-\bar{t}) \\ &\textit{subject to} \quad Pt = \bar{v} \end{aligned}$$

where, *Q* is a finite positive definite matrix.

The problem may be solved by forming the Lagrangian equation

$$L(t, l) = (t-\bar{t})^T Q(t-\bar{t}) + l^T (Pt - \bar{v})$$

where l is the vector of Lagrangian multipliers. Differentiate with respect to *t* and l, the necessary and sufficient conditions for a solution is given by:

$$\frac{\partial L}{\partial t} = Q(t - \bar{t}) + P^T l = 0 \tag{13}$$

$$\frac{\partial L}{\partial l} = Pt - \bar{v} = 0 \tag{14}$$

Drive *t* from (13): $t = \bar{t} + Q^{-1}P^T l$, and substitute into (14), we have

$$PQ^{-1}P^T l = \bar{v} - P\bar{t}$$

If we assume that the rows of the proportion matrix *P* are linearly independent, then the matrix $PQ^{-1}P^T$ is non-singular. Therefore l , thus *t*, can be uniquely solved.

Flows constrained (FC) models:

$$\begin{aligned} &Minimize \quad (t-\bar{t})^T Q(t-\bar{t}) + (v-\bar{v})^T R(v-\bar{v}) \\ &subject\ to \quad Pt = v \end{aligned}$$

where, Q and R are finite positive definite matrices.

Substitute the constraints into the objective function and differentiate with respect to *t*, the necessary and sufficient condition for a solution is given by

$$Q(t - \bar{t}) + P^T R(Pt - \bar{v}) = 0$$

or

$$(Q + P^T RP)t = Q\bar{t} + P^T R\bar{v}$$

Since $P^T RP$ is non-negative definite, $Q + P^T RP$ is positive definite. Therefore, *t* is uniquely determined.

From the above proofs, it can be seen that the uniqueness of a solution to the CC model requires that the rows of the proportion matrix are linearly independent while the FC model does not require this constraint. Therefore it is clear that the FC model can incorporate the inconsistencies in both flows and constraints. This appendix also derives solutions to both CC and FC models with a quadratic objective function and linear constraints for uncongested networks.