



WORKING PAPER

ITLS-WP-18-14

**Steady-state link travel time methods:
formulation, derivation, and classification**

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August 2018

ISSN 1832-570X

**INSTITUTE of TRANSPORT and
LOGISTICS STUDIES**

The Australian Key Centre in
Transport and Logistics Management

The University of Sydney

Established under the Australian Research Council's Key Centre Program.

NUMBER: Working Paper ITLS-WP-18-14

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ABSTRACT: Travel times are one of the most important outputs of transport planning models and this is unlikely to change in the future. It is therefore paramount that the methods that underpin the construction of travel times are well understood. However, while there exist many different travel time formulations to date, their relation to each other is not well researched, especially in the context of the three main types of macroscopic modelling paradigms: dynamic, semi-dynamic, and static traffic assignment. In this work, we provide consistent and general link travel time formulations across these three modelling paradigms, assuming steady state flow rates and by directly deriving them from a recent state-of-the-art continuous time macroscopic dynamic network loading model. We do so from two different perspectives; an experienced perspective, which actively tracks the tail of a physical queue, and a functional perspective, which does not. Based on the existing literature and our generalised link travel time formulations, a classification framework is proposed allowing one to compare existing (and future) methods in the literature in an objective fashion. We provide a number of explicit derivations of existing model formulations that can be considered special cases of our unified approach. In addition a number of representative existing methods in the literature has been classified based on the above mentioned framework for the reader's convenience.

KEY WORDS: *Link Travel time, Decomposition, Link Transmission Model, Classification framework, Steady state, Network loading*

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DATE: August 2018

1 Introduction

Since travel times are among the most important outputs of transport planning models it is somewhat surprising that, to date there has been no attempt to construct a unified formulation of travel time, not even under the assumption of steady-state conditions in the context of a single link. In this work we attempt to fill this gap, by deriving semi-dynamic and static link travel time formulations under steady-state conditions from a recent macroscopic dynamic continuous time model. Some of the benefits of having a unified formulation are to be found in the ability to, for example, compare different existing formulations in an objective fashion, pinpointing their (dis)benefits, and highlight why they might or might not be suitable for particular application contexts. Also, it allows one to verify if different modelling approaches, such as for example a dynamic model used for short term planning (traffic management, road works etc.) and a static longer term planning model (mode choice, destination choice, location choice) are consistent or not. This matters because in practice, models with a shorter planning horizon are typically constructed from the - coarser - longer term models, and only by maintaining consistency across these different models, results remain meaningful.

Currently, both in practice and the literature, a plethora of different approaches exist with respect to constructing travel times for a road section, i.e. *link*. Traditionally, link travel times were often constructed according to a particular functional form with attractive mathematical properties. These properties proved beneficial when finding solutions to the transport planning model as a whole (on a network level), but at the same time these approaches compromise the capability of the model to accurately reflect traffic conditions, especially in congested situations. In recent decades, more realistic travel time- and traffic flow propagation - methods have been developed that more closely match reality, but they do come at the price of a higher computational cost and less attractive mathematical properties. Still, due to the increase in computational power and the reliance on simulation based solution schemes rather than analytical models, these approaches, over time, have quickly become more popular, both in dynamic, as well as static planning models.

Travel time formulations are often directly associated with a particular type of model. Models with an explicit time dimension for example, i.e. dynamic models, construct travel times often as a post-processing step, based on the resulting densities/speeds and/or cumulative inflow and outflows on a link. Static models on the other hand, do not do this. In traditional static models, travel times are constructed based on so called link performance functions, where the flow rate uniquely determines the travel time and queues are modelled only implicitly. In the past decades, attempts have been made to model queues explicitly in static models by supplementing, or replacing, the link performance function with something that more accurately reproduces the delay due to the formation of queues. Semi-dynamic models reside between the static and dynamic modelling paradigm, they are effectively static models that transfer residual queues and/or demand between periods. Semi-dynamic models come with their own challenges in formulation (link) travel time formulations, due to the interactions between periods.

Currently, due to the existence of all these different modelling approaches, it is not straightforward to compare existing, or novel, travel time formulations in a meaningful way, nor is it readily apparent how to quantify the capability of each of these formulations. In this work, we attempt not only to overcome this lack of insight, but provide generalised link travel time formulations for the continuous time, semi-dynamic, and static modelling paradigms

under steady-state conditions, based on which we subsequently classify and/or derive existing formulations in the literature as special cases.

To be able to do so we make a number of simplifying assumptions: (i) we only consider a single road section, i.e. a *link*, (ii) each link is assumed to be in a steady-state, where steady-state is defined as the link having stable inflow/outflow rates for the period considered, (iii) only a single user class (mode) is considered, (iv) traffic flow propagation is assumed to be consistent with any general concave two-regime Fundamental Diagram (FD), i.e. a macroscopic perspective of traffic flow is adopted, adhering to the First-In-First-Out (FIFO) principle, (v) only first order effects are considered such that any transition between two traffic flow states results in an immediate change in speed, density, and flow rate.

Throughout this paper, we explore two conceptually different perspectives when constructing link travel times. First, one can take the perspective of a (virtual) vehicle driver. The driver experiences a portion of the link in free flow conditions, while the remainder is traversed in congested conditions. Both components yield a constant, but different, travel time per distance unit travelled. Therefore, in these models, we must explicitly track the physical location of the tail of the queue to be able to make this distinction. This perspective is termed *experienced travel time decomposition*. It has the benefit that one can attribute different utilities to the free flowing and queuing component. This can be important for route choice, as it is well-known that drivers experience delay in a different way than they do free flow conditions (Hensher, 2001). The other, second, perspective adopts a more mathematical, or functional, approach, where one constructs the travel time by taking the minimum link travel time and supplements it with additional delay that is governed by the two branches of the FD. Here, there is no explicit tracking of a queue needed. This approach is termed *functional travel time decomposition*. This, currently, is the most common way to formulate link travel time. It has the benefit of being less cumbersome than an experienced approach, but lacks the capability to attribute different weights to the difference in experience between congestion and free flow.

The static and semi-dynamic generalised link travel time formulations are directly derived from a state-of-the-art continuous time traffic flow propagation model by Bliemer and Raadsen (in press) and Raadsen and Bliemer (in press) assuming steady-state conditions. This model solves the well-known LWR model (Lighthill and Witham, 1955; and Richards, 1956) and is referred to as the event-based Generalised Link Transmission Model (eGLTM).

1.1 Contributions and outline

There are several contributions made in this paper: (i) we formulate two different perspectives on link travel times, i.e. a functional and experienced perspective. We prove that, when derived from eGLTM under steady-state conditions, they yield identical results, (ii) we derive semi-dynamic link travel time formulations that are consistent with first order traffic flow theory and do the same for the static modelling paradigm, (iii) we demonstrate that depending on the assumptions made, existing travel time formulations in the literature appear as special cases by deriving them explicitly, (iv) we provide a classification framework that allows one to categorise future and existing link travel time formulations in an objective manner and conducted this classification for a comprehensive number of existing studies for the readers convenience.

The remainder of this paper is organised as follows: Section 2 discusses the existing literature and introduces the link travel time classification framework and its categories. In Section 3 the

general concave two-regime FD and the continuous time link model components, required for our link travel time derivation, are discussed. In Section 4, the experienced link travel time decomposition is formulated for the different time dimensions considered, while Section 5 does the same for the functional approach. Section 6 proves that the two perspectives yield identical results under a consistent initial state. Then, in Section 7 we show how relaxing some of the conditions for a consistent link travel time function can lead to a link performance function oriented approach. Then, in Section 8, we explicitly derive two special cases in the current literature, demonstrating how the general formulation unifies existing methods. Finally, we draw some conclusions and discuss research gaps in Section 9.

2 Link travel time: existing formulations and classification framework

Let us discuss existing model formulations and their link travel time formulations in a static, semi-dynamic, and dynamic context.

2.1 Static link travel time in the literature

Seminal traditional static approaches such as described in Beckman *et al.*, (1956) adopt link performance functions such as the BPR function (Bureau of Public Roads, 1964) and only model queues implicitly. As a result they are only capable of modelling the uncongested branch of the FD, where density increases with increasing flow. They allow flow rates above capacity and, which is not realistic and are therefore mainly suitable for uncongested conditions, yet they do represent steady-state conditions and are heavily used in practice, so we do consider them here as a special case, see also Section 7.2. Other traditional static approaches were often formulated as optimisation problems, where delay was merely used as a mathematical construct to find a solution to the problem posed (Bell *et al.* 1995, Yang and Yagar, 1994). In those approaches, delay is not related to the actual flow rates in the period considered and does not have a physically meaningful interpretation. To address this issue, models emerged that constructed delay based on explicit queues following from the assignment. Some models retained a link performance function, but supplemented it with a penalty for violating the physical road capacity, introducing a queuing delay (Nakayama and Connors, 2012; Lam and Zhang, 2000; Hungerink, 1989). This queuing delay approach stems from the non-random delay, or persistent delay, formulation used in intersection modelling (Akçelik and Roupail, 1993; van Vliet, 1982)¹. It is constructed by dividing the number of vehicles in the queue, by the saturation flow rate, i.e. outflow rate. Combining a link performance function with queuing delay can lead to potential double counting of link travel time. Other approaches opted to replace the link performance function with a flow invariant, free flow travel time (Smith 1987; Payne and Thompson, 1975), or a (capacity constrained) flow dependent free flow travel time (Bliemer *et al.* 2014). None of the aforementioned models accounts for blocking back, i.e. spillback. They also lack restrictions on the inflow, even when the link has a predefined maximum capacity, in turn impacting on the resulting link travel times, which can become unrealistically large. Attempts to capture blocking back, limiting the maximum travel time that one can experience on a single link, can be found in (Brederode *et al.*, 2018; Bliemer and Raadsen, 2017; Smith, 2013; Smith *et al.* 2013, Bundschuh, 2006, Bakker *et al.*, 1994). Capacity based inflow restrictions, which avoid constructing queues inside bottlenecks, were first introduced in Bliemer *et al.* (2014), but remain a rarity in static assignment models to date.

¹ In this work persistent delay is equivalent to the hypercritical delay of the functional perspective, see Section 5.

2.2 *Semi-dynamic link travel time in the literature*

To date, semi-dynamic models either adopt traditional link travel time functions of their static counterparts (Nakayama *et al.*, 2012), or they do consider explicit residual queues that are transferred between the periods, attempting to account for the resulting queuing delay (Nakayama and Connors, 2012; Davidson *et al.*, 2011; Akamatsu *et al.*, 1998 (cited in Kanamori *et al.*, 2007)). Compared to the static and dynamic approaches the semi-dynamic literature is somewhat lagging. Either the adopted model form is based on traditional static methods, or the models are solely described as algorithms, without any clear description of the underlying model. We hope our formulations can contribute in providing new opportunities in this area by providing a more rigorous starting point.

2.3 *Dynamic link travel times in the literature:*

The earliest macroscopic dynamic models, like semi-dynamic models, extended traditional static models, adopting link travel time functions, but now adopting a continuous time formulation (Astarita; 1996; Wie *et al.*, 1994; Friesz *et al.*, 1993, 1989). These models are no longer considered suitable to construct accurate link travel times, because they lack the capability to properly reflect traffic conditions in congested situations, can be inconsistent with LWR, might violate FIFO, and are computationally costly². For this reason, in practice, these models are mostly abandoned in favour of more capable (simulation based) models adopting the *cell transmission* solution scheme, or the *link transmission* solution scheme, both of which solve the LWR model.

The cell transmission model was among the first efficient macroscopic dynamic solution methods that could solve LWR in a reasonable amount of time. (Daganzo, 1995, 1994). To extract link travel times in this model, one typically constructs average densities, or speeds over a predefined period of time (a few minutes), by aggregating the simulation time steps (of a few seconds) and extract travel times based on this information. In (time discretised) link transmission models (van der Gun *et al.*, 2017; Himpe *et al.*, 2016; Gentile, 2010; Yperman, 2007) link travel times are also constructed as a post-processing step, only now by comparing the time a (virtual) vehicle enters the link, with the moment it leaves, see for example Szeto and Lo (2005), for a general discussion on this topic. This approach has the benefit of incurring less smoothing errors than a cell transmission approach (Raadsen *et al.*, 2016). Another, more recent work investigating link travel times in a dynamic context is found in Long *et al.* (2011), who specifically explore the effect of discretising continuous time dynamic link travel time formulations. In eGLTM however, all smoothing errors – on the link level – are absent due to the fact that this link transmission model type is formulated in continuous time, rather than discretised. To the best of the authors' knowledge, none of the current literature provides a comprehensive overview of link travel time formulations across the different modelling paradigms, nor derives them mathematically from a common base model that is consistent with LWR and has been demonstrated to not exhibit any modelling error (on the link level).

2.4 *Categorisation and Classification*

Based on the current literature, we present the reader with our link travel time classification framework, see also Table 1, to objectively categorise existing (and future) link travel time formulations as special cases of our forthcoming generalised link travel time formulation.

² These models construct outflows for example as a function of the link performance function, leading to the strange phenomenon that they cannot construct a steady-state situation when the link is oversaturated. Also, a link performance function only reflects the uncongested branch of a FD, making it unsuitable to model congestion properly.

Formulations are classified based on the fact if their formulation is dynamic (**D**), semi-dynamic (**Sd**) or static (**St**) in nature. While our proposed link travel time formulation is consistent with LWR, adopting a two-regime FD, and is therefore both capacity and storage constraint (**CSC**), i.e. it considers capacity constraints as well as spillback, existing formulations might not be. Hence, we also provide a capacity constrained category (**Cc**), i.e. spillback is not considered, and a capacity restrained category (**Cr**), i.e. no capacity nor storage constraints. The latter alludes to link performance functions which deter the usage of congested links, but do not withhold any flow. We denote capacity restrictions that are only imposed on one of the two link boundaries as follows, where **-/Cc** reflects no inflow restrictions, but the outflow is restricted by capacity, i.e. queues emerge inside the bottleneck rather than in front of it. For an in-depth discussion on the properties of traffic assignment models in general, rather than pertaining to link travel times specifically, we refer the reader to Bliemer *et al.* (2017).

Table 1: Categories and category options in constructing link travel time formulations.

General categories			Steady-state compliant categories				
Category - options	Time dimension	inflow/outflow restrictions	Steady-state compliance	Link travel time perspective	Link travel time decomposition	Queue	
						Scenario	Initial
Category - options	D (Dynamic)	CSC (Capacity + Storage constrained)	Ss (steady-state)	F (Functional)	C (Consistent)	G (Growing)	Y (Yes)
	Sd (Semi-dynamic)	Cc (Capacity constrained)	NSs (not steady-state)	E (Experienced)	O (Overlap)	Sh (Shrinking)	N (No)
	St (Static)	Cr (Capacity restrained)			G (Gaps)	St (Stable)	
		- (unconstrained)				I (Implicit)	

Interestingly, while most model formulations are capable of constructing steady-state conditions regarding inflow and outflow rates (**Ss**), we found that some of the earlier formulations cannot (Astarita, 1996; Wie *et al.*, 1994; Friesz *et al.* 1993). This is due to the outflow being a function of a link performance function rather than the other way around. Hence these models are classified as non-steady-state compliant (**NSs**). Models capable of constructing steady-state conditions are further differentiated by how link travel time is decomposed, following a functional (**F**) or experienced (**E**) approach. While the two perspectives decompose the link travel time differently, in both cases, the sum of the decomposed components should reflect the total link travel time. If this is the case the formulation is considered consistent (**C**), as per Figure 1(a).

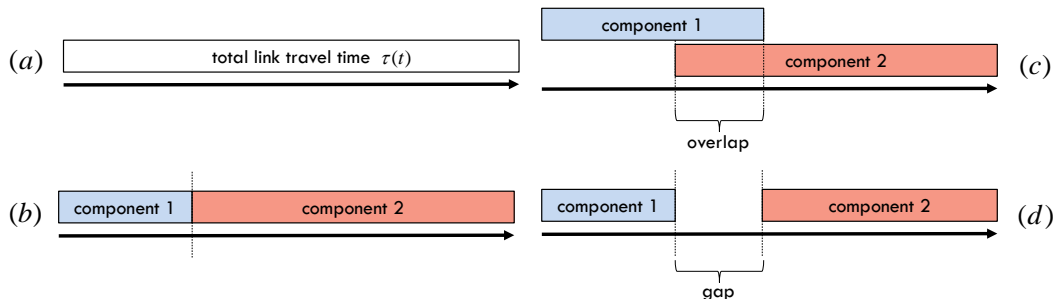


Figure 1: Decomposing the link travel time, (a) total travel time, (b) consistently, (c) with overlapping components, (d) with gaps.

However, there exist formulations where the components are not disjunct, i.e. they overlap (**O**), leading to overestimation of travel time as per Figure 1(b), or alternatively there can be gaps (**G**) between the components, resulting in an underestimation, as per Figure 1(c). Note that if the link travel time is based on a link performance function, the travel time as a whole is inconsistent with traffic flow theory and depending on the parameter settings either over or underestimates the actual travel time, hence they are referred to as **G/O**. Existing formulations also make different assumptions on how queues are constructed (or not), queues either grow (**G**), shrink (**Sh**), or remain stable (**St**). Stable queues can alternatively be thought of as a *super steady-state*, where not only flow rates are stable, but also the queue itself. Lastly, we also determine if the formulation accounts for the possibility of an initial queue, or not (**Y/N**).

Utilising this classification with respect to our base model, i.e. eGLTM, from a functional perspective (Section 5.1), yields the following classification **D:CSC:Ss:F:C:G+Sh+St:Y**. Alternatively, the formulation in Smith *et al.* (2013) is classified as **St:-/CSC:Ss:E:C:St:Y**, while Bliemer *et al.* (2014) is defined through **St:Cc:Ss:F:C/O:Gr:N**. A comprehensive, but inevitably incomplete, overview of existing link travel time formulations and their respective classifications can be found in the table provided in Appendix A.

3 Fundamental diagram and dynamic network loading link model

In order to derive the generalised link travel time formulations, we adopt a generalised concave FD. This FD has two regimes, or branches; an uncongested *hypocritical* branch where flow increases with increasing density, and a congested *hypercritical* branch where flow decreases with increasing density (Cascetta, 2009). The choice of fundamental diagram and its underlying properties play an important role in the modelling of traffic flow propagation and therefore the travel time formulation. The FD uniquely describes the relationship between a flow rate q [veh/h], density k [veh/km] and speed \mathcal{G} [km/h] on a cross section of the road. In addition, each link has a length ℓ [km], maximum throughput, termed capacity, q^{\max} [veh/h], maximum vehicle speed \mathcal{G}^{\max} [km/h], and maximum density k_{jam} [veh/km]. This notation largely follows the two-regime concave fundamental diagram formulation of Bliemer and Raadsen (in press). Most traffic flow models, including eGLTM, assume links to be homogeneous such that the FD is representative for the entire link instead of a single point.

The (uncongested) hypocritical branch, is captured by flow-density function $\Phi_I^{-1}(q) < k^{\text{crit}}$ [veh/km], while the (congested) hypercritical branch, is captured by $\Phi_H^{-1}(q) \geq k^{\text{crit}}$ [veh/km], see also Figure 2(a)³. Flow dependent hypocritical and hypercritical speeds are formulated similarly via $\mathcal{G}_I(q), \mathcal{G}_H(q)$ [km/h], respectively such that:

$$\mathcal{G}_I(q) = \frac{q}{\Phi_I^{-1}(q)} \quad \text{and} \quad \mathcal{G}_H(q) = \frac{q}{\Phi_H^{-1}(q)}, \quad q \in [0, q^{\max}]. \quad (1)$$

The shockwave speed that separates two flow states can traverse a link in either direction and is obtained via $\eta(q^{\text{hypo}}, q^{\text{hyper}})$ [km/h]. In steady-state conditions, the shockwave speed, together with the time and location of its inception, determines the location of the tail of a queue.

³ We explicitly refer to the density functions as inverses to highlight that traditionally one converts density into flow rather than the other way around. Having two separate functions allows for this – in our context - more convenient formulation.

Graphically, the shockwave speed equates to the slope connecting the two flow states on the FD, see Figure 2(b), and is given by:

$$\eta(q^{\text{hypo}}, q^{\text{hyper}}) = \frac{q^{\text{hyper}} - q^{\text{hypo}}}{\Phi_{II}^{-1}(q^{\text{hyper}}) - \Phi_{I}^{-1}(q^{\text{hypo}})}, \quad q^{\text{hyper}}, q^{\text{hypo}} \in [0, q^{\text{max}}]. \quad (2)$$

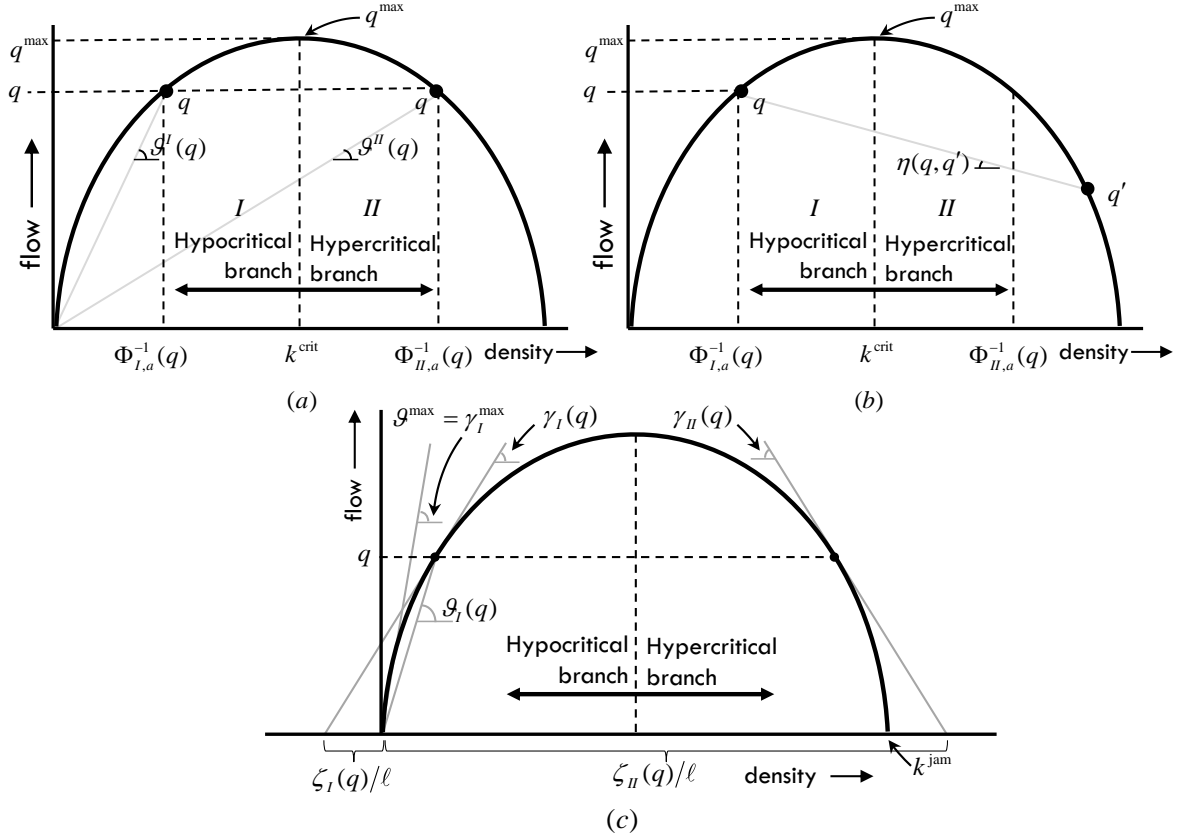


Figure 2: (a) two-regime concave FD with hypocritical and hypercritical branches, (b) same FD picturing the shockwave speed given two flow states, (c) wave speeds and effect of difference between vehicle speed and wave speed.

3.1 Steady-state continuous time traffic flow propagation

We now briefly revisit the underpinnings of eGLTM, adopting the above FD, to introduce the notation required to derive the link travel time formulations. Link transmission models like eGLTM are formulated in terms of a cumulative inflow $U(t)$ [veh] and cumulative outflow $V(t)$ [veh] on their respective upstream and downstream link boundary (Bliemer and Raadsen, in press; Raadsen and Bliemer, in press). Each cumulative curve is the result of integrating their respective inflow rates $u(t)$ [veh/h], and outflow rates $v(t)$ [veh/h] across the considered time period, as per Equation (3):

$$U(t) = \int_0^t u(\omega) d\omega, \quad \text{and} \quad V(t) = \int_0^t v(\omega) d\omega. \quad (3)$$

In this work we assume steady-state conditions in the period $t^{\text{start}} \leq t \leq t^{\text{end}}$ [h], i.e. $u(t) = u(t')$, and $v(t) = v(t')$, $\forall (t, t') \in [t^{\text{start}}, t^{\text{end}}]$. Therefore, Equation (3) simplifies⁴ to:

⁴ We could also refrain from making the flow rate time dependent due to its steady-state, but since we also derive semi-dynamic and static versions of this formulation later, we refrain from doing so, to more clearly distinguish between the flow rate variables across each model type.

$$U(t) = U(t^{\text{start}}) + u(t)(t - t^{\text{start}}), \quad \text{and} \quad V(t) = V(t^{\text{start}}) + v(t)(t - t^{\text{start}}), \quad t^{\text{start}} \leq t \leq t^{\text{end}}. \quad (4)$$

The interaction between the two cumulative curves is governed by the aforementioned two-regime FD. Following Bliemer and Raadsen (in press), two additional “projected” cumulative curves are also defined. First, $\bar{U}(t)$ [veh], constructs the maximum potential outflow curve assuming hypocritical conditions, while $\bar{V}(t)$ [h] represents the maximum potential inflow curve under hypercritical conditions, i.e. spillback. They are defined as follows:

$$\bar{U}(t) = \min_{q \in [0, q^{\text{max}}]} \left\{ U \left(t - \frac{\ell}{\gamma_I(q)} \right) + \xi_I(q) \right\}, \quad \text{with} \quad \xi_I(q) = \ell q \left(\frac{1}{\gamma_I(q)} - \frac{1}{\mathcal{G}_I(q)} \right), \quad (5)$$

$$\bar{V} \left(t - \frac{\ell}{\gamma_{II}(q)} \right) = \min_{q \in [0, q^{\text{max}}]} \left\{ V(t) + \xi_{II}(q) \right\}, \quad \text{with} \quad \xi_{II}(q) = \ell q \left(\frac{1}{\mathcal{G}_{II}(q)} - \frac{1}{\gamma_{II}(q)} \right), \quad (6)$$

where $\gamma_I(q), \gamma_{II}(q)$, are the characteristic wave speeds given by the tangent of the FD in each branch. Compensation factors $\xi_I(q), \xi_{II}(q)$, represent the number of vehicles that one would encounter if one would traverse the link with the characteristic wave speed consistent with q , see also Figure 2(c). In the special case the FD would be triangular (Newell, 1993), $\xi_I(\cdot) = 0$, and $\xi_{II}(\cdot) = k_{\text{jam}}$. Only in that case it holds that $\mathcal{G}_I(\cdot) = \gamma_I(\cdot)$, otherwise $\mathcal{G}_I(\cdot) \leq \gamma_I(\cdot)$. This generally leads to an underestimation of the link travel time (assuming $\mathcal{G}_I(\cdot) = \mathcal{G}^{\text{max}}$).

The projected flow rates that go with each curve are defined as sub-derivatives $\bar{u}(t) \in \partial \bar{U}(t)$, $\bar{v}(t) \in \partial \bar{V}(t)$, i.e. these are the flow rates q that are compatible with the specified minimisation. However, under steady-state conditions $u(t) = \bar{u}(t)$, and $v(t) = \bar{v}(t)$, $\forall t^{\text{start}} \leq t \leq t^{\text{end}}$, simplifying Equations (5) and (6) to:

$$\bar{U}(t) = U \left(t - \frac{\ell}{\gamma_I(u(t))} \right) + \xi_I(u(t)) = U(t - \tau_I(u(t))), \quad (7)$$

$$\bar{V}(t) = V \left(t - \frac{\ell}{\gamma_{II}(v(t))} \right) + \xi_{II}(v(t)) = V(t - \tau_{II}(v(t))), \quad (8)$$

with:

$$\tau_I(q) = \frac{\ell}{\mathcal{G}_I(q)}, \quad \text{and} \quad \tau_{II}(q) = \frac{\ell}{\mathcal{G}_{II}(q)}, \quad q \in [0, q^{\text{max}}], \quad (9)$$

where $\tau_I(q), \tau_{II}(q)$ [h] represent the hypocritical and hypercritical link travel times under steady-state conditions. Let us demonstrate this with an example as per Figure 3(a). The total link travel time $\tau(t)$ [h] is the time difference between entering a link and leaving it, i.e. $U(t) = V(t + \tau(t))$. Under hypocritical conditions Equation (7) must hold such that $U(t) = \bar{U}(t + \tau(t)) = V(t + \tau(t)) \Rightarrow \tau(t) = \tau_I(u(t))$. Similarly, we find that in a spillback state, see Figure 3(b), that $V(t) = \bar{V}(t - \tau(t)) = U(t - \tau(t)) \Rightarrow \tau(t) = \tau_{II}(v(t))$. Special cases of the hypocritical and hypercritical link travel times are the absolute minimum link travel time τ^{min} and the absolute maximum link travel time τ^{max} :

$$\tau^{\min} = \tau_I(0) = \frac{\ell}{g^{\max}}, \quad \text{and} \quad \tau^{\max} = \tau_{II}(0) = \infty, \quad q \in [0, q^{\max}]. \quad (10)$$

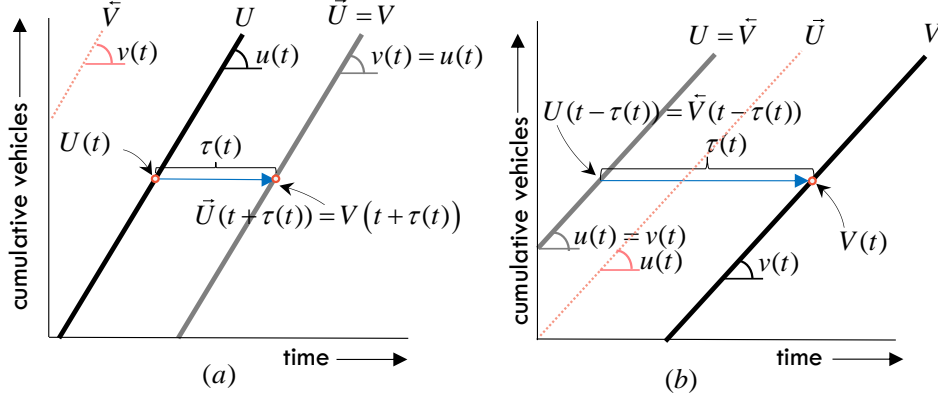


Figure 3: Cumulative curves under (a) hypocritical free flow conditions, i.e. inflow curve is restricting the outflow rate, (b) hypercritical spillback conditions, i.e. outflow curve is restricting the inflow rate.

Let us now derive the generalised formulations for the functional and experienced perspectives under steady-state conditions. We start with the latter in continuous time and subsequently derive its semi-dynamic and static counterparts.

4 Experienced link travel time perspective

The generalised link travel time formulation that adopts the experienced perspective decomposes the total link travel time in two distinct components; the experienced free flowing component, denoted $\tau^{\text{free}}(\cdot)$ [h], representing the portion of the link that the traveller experiences in free flow, and the experienced queuing component $\tau^{\text{queue}}(\cdot)$ [h], for the remainder of the link that is traversed in congestion. Unless the location of tail of the queue is stable, this requires an active tracking of the queue's location over time. We do so via $\lambda(t) \in [0, 1]$, where $\lambda(t) = 1$ reflects a link that is in a hypocritical steady-state, while $\lambda(t) = 0$ signifies spillback conditions. Further, we assume that $\lambda^{\text{start}} = \lambda(t^{\text{start}})$ is known and given, such that the number of vehicles present in an initial queue $N^{\text{queue, start}} = N^{\text{queue}}(\lambda^{\text{start}})$ [veh], with:

$$N^{\text{queue}}(\lambda) = (\ell(1-\lambda)) \cdot \Phi_{II}^{-1}(v(t)), \quad \text{with} \quad \lambda \in \{\lambda(t) \mid \forall t^{\text{start}} \leq t \leq t^{\text{end}}\}, \quad (11)$$

where we assume that the density $\Phi_{II}^{-1}(v(t))$ in the queue is consistent with the current outflow rate. This is an unavoidable assumption since we do not know how this queue was constructed since its build up occurred outside the period of consideration.

4.1 Experienced link travel time: continuous time perspective

The total link travel time in the dynamic continuous time formulation is constructed from the free flowing and congested portion of the link such that:

$$\tau(t) = \tau^{\text{free}}(u(t), \lambda(t)) + \tau^{\text{queue}}(v(t), \lambda(t)), \quad t^{\text{start}} \leq t \leq t^{\text{end}}, \quad (12)$$

where both components depend on their respective inflow and outflow rates as well as the location of the tail of the queue. In steady-state conditions, a queue that grows or shrinks, does

so in a linear fashion because the shockwave separating the free flow portion of the link from the congested portion of the link, i.e. the tail of the queue, travels at a constant speed, see Equation (2). Further, when the link is in a hypocritical state there is no queue, i.e. $\tau^{\text{queue}}(\cdot) = 0, \lambda(t) = 1$, hence $\tau(\cdot) = \tau^{\text{free}}(\cdot) = \tau_I(\cdot)$. Knowing that the location of the queue only changes linearly, if at all, we find that⁵:

$$\tau^{\text{free}}(q, \lambda) = \lambda \cdot \tau_I(q), \quad q \in [0, q^{\text{max}}], \lambda \in \{\lambda(t) \mid t^{\text{start}} \leq t \leq t^{\text{end}}\}. \quad (13)$$

We construct the experienced queuing travel time $\tau^{\text{queue}}(\cdot)$ in an identical fashion. In steady-state hypercritical conditions $\tau^{\text{free}}(\cdot) = 0, \lambda(t) = 0$, hence $\tau(\cdot) = \tau^{\text{queue}}(\cdot) = \tau_{II}(\cdot)$. Only now the travel time is inversely related to λ , resulting in:

$$\tau^{\text{queue}}(q, \lambda) = (1 - \lambda) \cdot \tau_{II}(q), \quad q \in [0, q^{\text{max}}], \lambda \in \{\lambda(t) \mid t^{\text{start}} \leq t \leq t^{\text{end}}\}. \quad (14)$$

Let us now determine $\lambda(t)$ for the three different steady-state scenarios (growing, shrinking, or stable queue), either in absence, or the presence of an initial queue, as depicted in Figure 4.

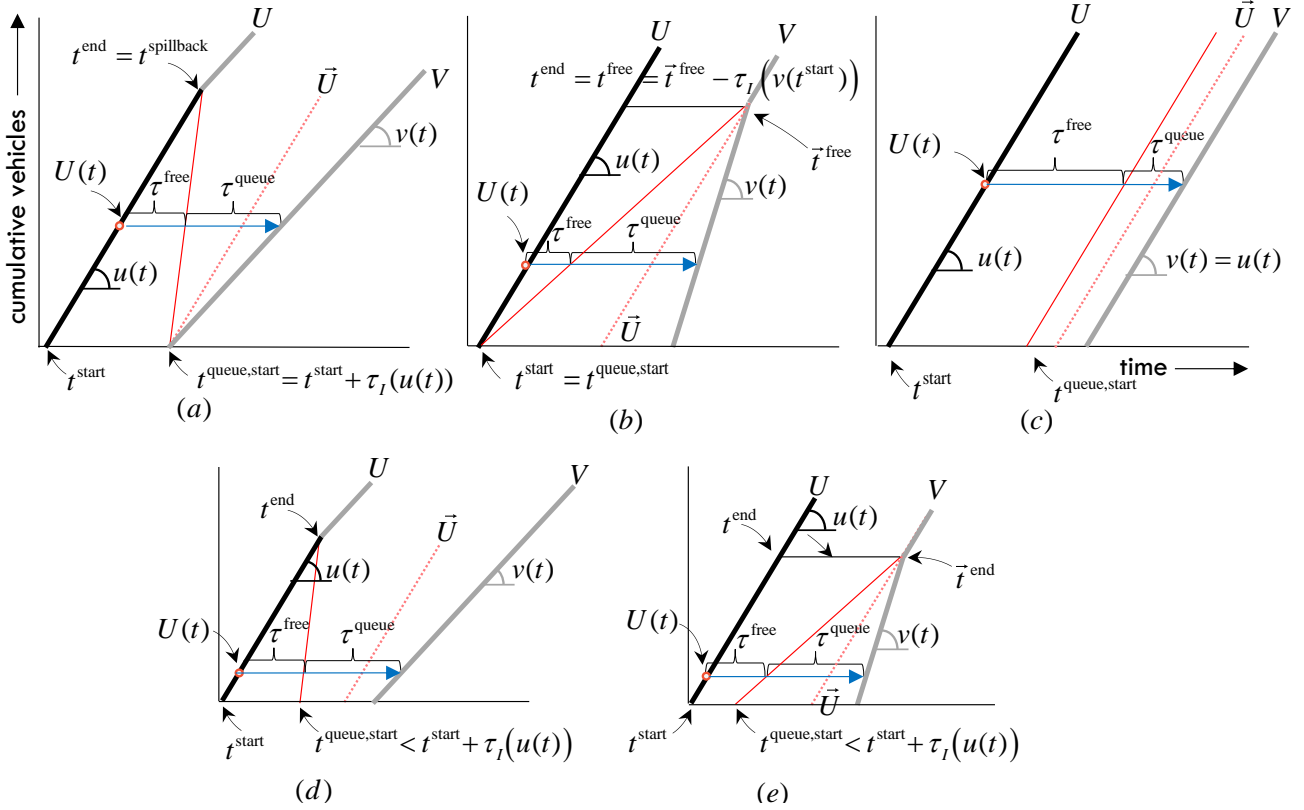


Figure 4: Experienced link travel time – tail of the queue as red solid line – (a) growing queue starting in free flow state, (b) shrinking queue starting in spillback state, (c) stable queue scenario, (d) growing queue starting in partially congested state, (e) shrinking queue starting in partially congested state.

First, depending on the scenario, the queue moves in different directions (or not at all), resulting in separate formulations of $\lambda(t)$ depending on the scenario. Secondly, the steady-state period is necessarily finite under a growing or shrinking queue scenario, because if the period does not end once the link reaches spillback, or free flow conditions, one of the flow rates necessarily

⁵ To construct a linear function, one only requires a point and its rate of change, both of which we know.

violates our steady-state assumption. Therefore, under a growing queue scenario $u(t) > v(t)$, $t^{\text{start}} \leq t \leq t^{\text{spillback}}$, with $t^{\text{end}} \leq t^{\text{spillback}}$. Similarly, under a shrinking queue, $u(t) < v(t)$, $t^{\text{start}} \leq t \leq t^{\text{free}}$, with $t^{\text{end}} \leq t^{\text{free}}$, where $t^{\text{spillback}}$, t^{free} , denote the moment the link first reaches spillback, or free flow respectively. Observe from Figure 4(a) and (b) that $t^{\text{spillback}}$ and t^{free} are defined in relation to the upstream link boundary, that way, we are able to geometrically interpolate $\lambda(t)$ via:

$$\lambda(t) = \begin{cases} \lambda^{\text{start}} \left(1 - \left(\frac{t - t^{\text{start}}}{t^{\text{spillback}} - t^{\text{start}}} \right) \right), & \text{if } u(t) > v(t), \\ \lambda^{\text{start}} \left(1 + \left(\frac{t - t^{\text{start}}}{t^{\text{free}} - t^{\text{start}}} \right) \right), & \text{else if } u(t) < v(t), \\ \lambda^{\text{start}}, & \text{otherwise,} \end{cases} \quad (15)$$

with $t^{\text{start}} \leq t \leq t^{\text{end}}$. The first two cases relate to a growing or shrinking queue, respectively, where depending on the scenario, the free flow portion of the link shrinks or grows linearly. Under a stable queue, or super steady-state, $(\lambda(t) = \lambda^{\text{start}}) \geq 0$, hence $u(t) = v(t)$ and this steady-state is the only state that has no limit on the period end time, see Figure 4(c). Only in this special case, the experienced travel time components are constant and directly determined by λ^{start} .

To determine $t^{\text{spillback}}$, we first construct $t^{\text{queue,start}}$, denoting the time the first vehicle reaches the tail of the queue. From that moment in time, the growing queue needs to traverse the free flow portion of the link in the opposite direction of the traffic flow to reach the upstream link boundary, resulting in:

$$t^{\text{spillback}} = t^{\text{queue,start}} + \frac{\lambda^{\text{start}} \ell}{-\eta(u(t^{\text{start}}), v(t^{\text{start}}))}, \quad \text{with } t^{\text{queue,start}} = t^{\text{start}} + \tau^{\text{free}}(u(t^{\text{start}}), \lambda(t^{\text{start}})). \quad (16)$$

To determine t^{free} we follow the same approach, only now the queue is shrinking. Since t^{free} is an upstream point of reference and the queue dissipates downstream, we first construct the ‘‘projected’’ downstream time \bar{t}^{free} the link first enters a hypocritical state and then find the time the corresponding exiting vehicle, entered, i.e. $V(\bar{t}^{\text{free}}) = \bar{U}(\bar{t}^{\text{free}}) = U(\bar{t}^{\text{free}} - \tau_H(u(t)))$ following Equation (7). This then, results in:

$$t^{\text{free}} = \bar{t}^{\text{free}} - \tau_I(u(t^{\text{start}})), \quad \text{with } \bar{t}^{\text{free}} = t^{\text{queue,start}} + \frac{(1 - \lambda^{\text{start}}) \ell}{\eta(u(t^{\text{start}}), v(t^{\text{start}}))}. \quad (17)$$

4.2 Experienced link travel time: semi-dynamic perspective

Let us now derive the semi-dynamic link travel time consistent with the continuous time formulation of the previous section. In a semi-dynamic context, we consider multiple periods $i \in I$, where a period does not consider the time dimension explicitly, but implicitly. We therefore only obtain an *average* link travel time per period, replacing $\tau(t)$ with $\bar{\tau}_i$ and replacing $\tau^{\text{free}}(t)$, $\tau^{\text{queue}}(t)$, with $\bar{\tau}_i^{\text{free}}$ and $\bar{\tau}_i^{\text{queue}}$ per period i , respectively:

$$\bar{\tau}_i = \bar{\tau}_i^{\text{free}} + \bar{\tau}_i^{\text{queue}}, \quad \forall i \in I. \quad (18)$$

Since we assumed steady-state conditions in continuous time, it holds that $u(t) = \bar{u}_i, v(t) = \bar{v}_i, t_i^{\text{start}} \leq t \leq t_i^{\text{end}}$, where \bar{u}_i and \bar{v}_i are the average flow rates within steady-state period i , demarcated by $[t_i^{\text{start}}, t_i^{\text{end}}]$. Further, the initial queue location in each period becomes dependent on the final queue of the preceding period like the following:

$$\lambda_i^{\text{start}} = \begin{cases} \lambda^{\text{start}}, & i = 1, \\ \lambda_{i-1}^{\text{end}}, & \text{otherwise,} \end{cases} \quad (19)$$

with $i \in I$, with $\lambda_i^{\text{end}} = \lambda_i(t_i^{\text{end}})$. Since the travel time changes linearly over time (or not at all) under steady state conditions, we can construct $\bar{\tau}_i^{\text{free}}$ and $\bar{\tau}_i^{\text{queue}}$ by taking the average of the two extreme points in continuous time within the considered period. For $\bar{\tau}_i^{\text{free}}$ this results in:

$$\bar{\tau}_i^{\text{free}} = \frac{1}{2} (\tau^{\text{free}}(\bar{u}_i, \lambda_i^{\text{start}}) + \tau^{\text{free}}(\bar{u}_i, \lambda_i^{\text{end}})) = \frac{(\lambda_i^{\text{start}} + \lambda_i^{\text{end}}) \cdot \tau_I(\bar{u}_i)}{2}, \quad i \in I. \quad (20)$$

Then, for $\bar{\tau}_i^{\text{queue}}$, utilising Equation (14), yields:

$$\bar{\tau}_i^{\text{queue}} = \frac{1}{2} (\tau^{\text{queue}}(\bar{v}_i, \lambda_i^{\text{start}}) + \tau^{\text{queue}}(\bar{v}_i, \lambda_i^{\text{end}})) = \tau_{II}(\bar{v}_i) - \frac{(\lambda_i^{\text{start}} + \lambda_i^{\text{end}}) \cdot \tau_{II}(\bar{v}_i)}{2}, \quad i \in I. \quad (21)$$

This leaves us with determining λ_i^{end} . This requires replacing t with the fixed t_i^{end} in Equation (15), as well as replacing the other constants with their period based counterparts, yielding:

$$\lambda_i^{\text{end}} = \begin{cases} \lambda_i^{\text{start}} \left(1 - \left(\frac{t_i^{\text{end}} - t_i^{\text{start}}}{t_i^{\text{spillback}} - t_i^{\text{start}}} \right) \right), & \text{if } \bar{u}_i > \bar{v}_i, \\ \lambda_i^{\text{start}} \left(1 + \left(\frac{t_i^{\text{end}} - t_i^{\text{start}}}{t_i^{\text{free}} - t_i^{\text{start}}} \right) \right), & \text{else if } \bar{u}_i < \bar{v}_i, \\ \lambda_i^{\text{start}}, & \text{otherwise,} \end{cases} \quad (22)$$

where in the first case $t_i^{\text{end}} \leq t_i^{\text{spillback}}$ while in the second case $t_i^{\text{end}} \leq t_i^{\text{free}}$. We then construct $t_i^{\text{spillback}}$ and t_i^{free} following Equations (16) and (17), replacing the continuous time variables with their semi-dynamic counterparts, yielding:

$$t_i^{\text{spillback}} = t_i^{\text{queue, start}} + \frac{\lambda_i^{\text{start}} \ell}{-\eta(\bar{u}_i, \bar{v}_i)}, \quad \text{with } t_{i,a}^{\text{queue, start}} = t_i^{\text{start}} + \tau^{\text{free}}(\bar{u}_i, \lambda_i^{\text{start}}), \quad i \in I, \quad (23)$$

and:

$$t_i^{\text{free}} = \bar{t}_i^{\text{free}} - \tau^I(\bar{u}_i), \quad \text{with } \bar{t}_i^{\text{free}} = t_i^{\text{queue, start}} + \frac{(1 - \lambda_i^{\text{start}}) \ell}{\eta(\bar{u}_i, \bar{v}_i)}, \quad i \in I. \quad (24)$$

In case the adopted scenario results in a stable queue, i.e. $\lambda_{i-1}^{\text{end}} = \lambda_i^{\text{start}} = \lambda_i^{\text{end}}$. Then, $\bar{\tau}_i^{\text{free}}, \bar{\tau}_i^{\text{queue}}$, can be simplified further to:

$$\bar{\tau}_i^{\text{free}} = \lambda_i^{\text{start}} \cdot \tau_I(\bar{u}_i), \quad \text{and } \bar{\tau}_i^{\text{queue}} = (1 - \lambda_i^{\text{start}}) \tau_{II}(\bar{v}_i), \quad i \in I. \quad (25)$$

We point out that this formulation does introduce a (slight) inconsistency between periods, because the last vehicle of period $i-1$, which is the same vehicle as the first vehicle in period i , receives a different travel time depending on which period we allocate it to. This is due to the fact that the density in the queue is assumed to be consistent with the flow rate of the period under consideration. For example, when transferring a queues that emerged in the current period to a new period; the current queue is consistent with the current period's flow rate (and density), but when it is transferred, it is assumed that it is consistent with the new period's flow rate (and density), altering the link travel time. While we acknowledge this discrepancy, we also point out that semi-dynamic models assume rather large periods, often up to an hour or more, and therefore we expect the effect of this discrepancy on the overall average travel time in the entire period to be rather small. Also, to the best of the authors' knowledge, no semi-dynamic models to date either have mentioned this inconsistency, nor do they address it. Therefore, for the sake of simplicity and to prevent an unnecessarily complex notation, we acknowledge this fact, but refrain from addressing it in this paper, leaving it for further research.

4.3 Experienced link travel time: static perspective

In a static context, periods are aggregated into a single (large) time period, for which we assume $t^{\text{start}} = 0$. The semi-dynamic period based flow rates $\bar{u}_i, \bar{v}_i, i \in I$ collapse into a single average inflow and outflow rate, denoted \bar{u}, \bar{v} , respectively. The link travel time of Equation (18) then reduces to:

$$\bar{\tau} = \bar{\tau}^{\text{free}} + \bar{\tau}^{\text{queue}}, \quad (26)$$

where $\bar{\tau}^{\text{free}}, \bar{\tau}^{\text{queue}}$, are the static equivalents of $\bar{\tau}_i^{\text{free}}, \bar{\tau}_i^{\text{queue}}$, respectively. We derive $\bar{\tau}^{\text{free}}$ and $\bar{\tau}^{\text{queue}}$ by abstracting out period i in Equations (20) and (21), yielding:

$$\bar{\tau}^{\text{free}} = \frac{(\lambda^{\text{start}} + \lambda^{\text{end}}) \cdot \tau_I(\bar{u})}{2}, \quad (27)$$

and:

$$\bar{\tau}^{\text{queue}} = \tau_{II}(\bar{v}) - \frac{(\lambda^{\text{start}} + \lambda^{\text{end}}) \cdot \tau_{II}(\bar{v})}{2} = \tau_{II}(\bar{v}) \left(1 - \frac{(\lambda^{\text{start}} + \lambda^{\text{end}})}{2} \right), \quad (28)$$

where λ^{start} is readily available as before, leaving λ^{end} to be constructed from λ_i^{end} in Equation (22). Given $t^{\text{start}} = 0$, we find:

$$\lambda^{\text{end}} = \begin{cases} \lambda^{\text{start}} \left(1 - \frac{t^{\text{end}}}{t^{\text{spillback}}} \right), & \text{if } \bar{u} > \bar{v}, \\ \lambda^{\text{start}} \left(1 + \frac{t^{\text{end}}}{t^{\text{free}}} \right), & \text{else if } \bar{u} < \bar{v}, \\ \lambda^{\text{start}}, & \text{otherwise,} \end{cases} \quad (29)$$

where in the first case $t^{\text{end}} \leq t^{\text{spillback}}$ while in the second case $t^{\text{end}} \leq t^{\text{free}}$. Variables $t^{\text{spillback}}$ and t^{free} are found by abstracting out the period and plugging in the steady-state flow rates in Equations (23) and (24), yielding:

$$t^{\text{spillback}} = \tau^{\text{free}}(\bar{u}, \lambda^{\text{start}}) + \frac{\lambda^{\text{start}} \ell}{-\eta(\bar{u}, \bar{v})}, \quad (30)$$

and:

$$t^{\text{free}} = \bar{t}^{\text{free}} - \tau_I(\bar{u}), \quad \text{with } \bar{t}^{\text{free}} = \tau^{\text{free}}(\bar{u}, \lambda^{\text{start}}) + \frac{(1 - \lambda^{\text{start}}) \ell}{\eta(\bar{u}, \bar{v})}. \quad (31)$$

Similar, to the semi-dynamic case, the scenario with a stable queue allows for a further simplification, because $\lambda^{\text{end}} = \lambda^{\text{start}}$ and consequently $\bar{\tau}^{\text{free}}, \bar{\tau}^{\text{queue}}$, in Equations (27) and (28) reduce to:

$$\bar{\tau}^{\text{free}} = \lambda^{\text{start}} \cdot \tau_I(\bar{u}), \quad \text{and} \quad \bar{\tau}^{\text{queue}} = (1 - \lambda^{\text{start}}) \cdot \tau_{II}(\bar{v}). \quad (32)$$

In, for example, Smith *et al.* (2013), this particular super-steady-state scenario is more informally referred to as “peak of the peak” period and in the context of that particular work λ^{start} is alternatively referred to as “shrinkage factor”, because it represents a reduction of the original free flowing portion of the link.

5 Functional link travel time perspective

The functional link travel time perspective refrains from establishing a 1:1 relationship between how a traveller experiences travel time and the decomposition of the travel time formulation. Instead it adopts a more abstract approach akin to queuing theory. The premise here is that one assumes one can traverse the link in uncongested conditions under all circumstances, and in case some of the inflow that cannot leave the link, due to outflow restrictions, this excess demand is placed in a (vertical) queue causing additional hypercritical delay. Therefore, this approach is sometimes referred to as a point queue based method, although in this general formulation the storage constraints of the link are also taken into account, making it effectively irrelevant how one pictures the physical presence of this delay.

In this approach, two types of delay are formulated in addition to the absolute and unchanging minimum link travel time τ^{min} . These two additional delay components are directly governed by the underlying FD, where we distinguish between *hypocritical delay*, denoted $d^{\text{hypo}}(\cdot)$ [h], and *hypercritical delay*, denoted $d^{\text{hyper}}(\cdot)$ [h]. Concretely, this perspective, and the hypercritical delay in particular, relies on the cumulative flow curves on the link boundaries, rather than tracking the location of the queue explicitly, resulting in a rather different travel time formulation.

In the experienced perspective we captured the initial state via the number of vehicles in the physical link queue through $N^{\text{queue,start}}$, obtained via $\lambda(t^{\text{start}})$, recall Equation (11). However, in this perspective, we no longer actively track the tail of the queue. Also, the number of excess vehicles on a link compared to a free flowing (hypocritical) state, denoted $N^{\text{hyper,start}}$, is generally not the same as $N^{\text{queue,start}}$. We therefore adopt the more suitable $N^{\text{hyper,start}}$ [veh] in the context of this perspective. The interested reader is referred to Appendix B, demonstrating that, in general, it holds that $N^{\text{hyper,start}} \leq N^{\text{queue,start}}$.

One direct benefit of adopting a functional approach, over an experienced one, is found in the fact that regardless if a queue is growing, shrinking, or stable, the formulation of the travel time components and its parameters do not change. We also find that one can not only construct a functional travel time decomposition by adding delays to the minimum travel time, but one can alternatively invert this formulation to construct link travel times by subtracting excess delay, denoted $d^{\text{excess}}(\cdot)$, from the (steady-state) hypercritical link travel time $\tau_{II}(q)$, (see Section 5.1.1). To the best of the author's knowledge such a formulation has to date never been used in conjunction with a semi-dynamic or static formulation, but might in some situations be more intuitive than the conventional approach of adding queuing delay, for example when a link is moving to a spillback state.

5.1 Functional link travel time: continuous time perspective

As mentioned, the continuous time link travel time function under a functional perspective, consists of three components, where all but the hypercritical delay component are constant under steady-state conditions:

$$\tau(t) = \tau^{\min} + d^{\text{hypo}}(u(t)) + d^{\text{hyper}}(v(t), N^{\text{hyper}}(t)), \quad t^{\text{start}} \leq t \leq t^{\text{end}}. \quad (33)$$

Since the hypercritical delay $d^{\text{hyper}}(\cdot)$ only captures the additional delay compared to a hypocritical state, the link travel time under free flow conditions equates to $\tau(t) = \tau^{\min} + d^{\text{hypo}}(u(t))$. Recall from Section 3.1 that in this situation $U(t) = \vec{U}(t + \tau(t)) = V(t)$, as per Figure 3(a), where $\tau(t) = \tau_I(u(t))$. Therefore, the hypocritical delay must necessarily be given by:

$$d^{\text{hypo}}(q) = \tau^I(q) - \tau^{\min}, \quad q \in [0, q^{\max}]. \quad (34)$$

Let us now consider the example in Figure 5(a) to graphically illustrate how we construct hypercritical delay $d^{\text{hyper}}(\cdot)$. First, we construct $N^{\text{hyper}}(t)$ [veh] describing the excess number of vehicles on the link compared to free flow conditions at time t . We do so by taking the difference between the maximum potential cumulative outflow and the actual cumulative outflow for the vehicle departing at time t , utilising Equation (4) this results in:

$$\begin{aligned} N^{\text{hyper}}(t) &= \vec{U}(t + \tau_I(u(t))) - V(t + \tau_I(u(t))), \quad \text{with } \tau_I(u(t)) = \tau^{\min} + \tau^{\text{hypo}}(u(t)), \\ &= \vec{U}(t + \tau_I(u(t))) - V(t') - v(t)(t - t'), \quad \text{with } t' = t^{\text{start}} + \tau^I(u(t)), \\ &= \vec{U}(t') - V(t') + (u(t'') - v(t''))(t'' - t'), \quad \text{with } t'' = t + \tau^I(u(t)), \\ &= N^{\text{hyper,start}} + (u(t) - v(t))(t - t^{\text{start}}), \end{aligned} \quad (35)$$

with $t^{\text{start}} \leq t \leq t^{\text{end}}$ and $N^{\text{hyper,start}} = \vec{U}(t') - V(t')$, resulting in a convenient flow rate based formulation. Figure 5(a) demonstrates that hypercritical delay is found by the time it takes for $N^{\text{hyper}}(t)$ to dissipate under flow rate $v(t)$, or more generally:

$$d^{\text{hyper}}(q, N) = \frac{N}{q}, \quad q \in [0, q^{\max}], N \in \{N^{\text{hyper}}(t) \mid t^{\text{start}} \leq t \leq t^{\text{end}}\}. \quad (36)$$

This formulation is agnostic to whether the queue is growing, shrinking, or stable, see Figure 5(b) and (c). Only the eligible period depends on the adopted steady-state flow rate conditions such that:

$$t^{\text{end}} = \begin{cases} t^{\text{spillback}}, & \text{if } u(t) < v(t), \\ t^{\text{free}}, & \text{else if } u(t) > v(t), \\ \infty, & \text{otherwise,} \end{cases} \quad (37)$$

where in the first case $t^{\text{start}} \leq t \leq t^{\text{spillback}}$, while for the second case $t^{\text{start}} \leq t \leq t^{\text{free}}$.

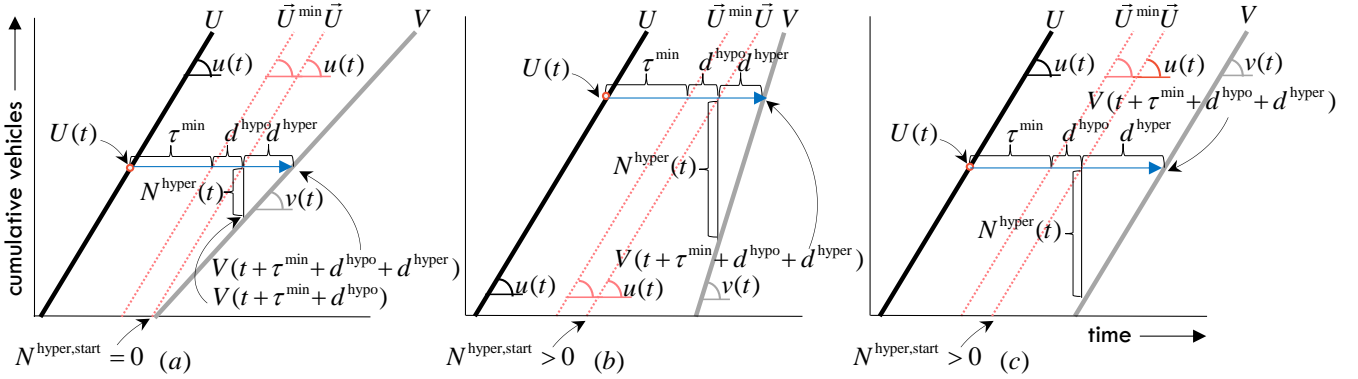


Figure 5: Functional link travel time perspective under (a) growing queue starting in free flow state, (b) shrinking queue starting in spillback state, (c) stable queue scenario⁶.

Since the functional decomposition approach utilises cumulative curves to derive the link travel time formulation, we formulate $t^{\text{spillback}}$ and t^{free} in a similar fashion adopting the formulation in Raadsen and Bliemer (in press)⁷:

$$t^{\text{spillback}} = \operatorname{argmin} \{t \in (t^{\text{start}}, \infty] \mid U(t) - \bar{V}(t) = 0\}, \quad (38)$$

and:

$$t^{\text{free}} = \operatorname{argmin} \{t \in (t^{\text{start}}, \infty] \mid V(t) - \bar{U}(t) = 0\}, \quad (39)$$

which for $t^{\text{spillback}}$ should be interpreted as identifying the moment the inflow rate must be reduced to the outflow rate due to spillback, i.e. the first moment the projected inflow curve dictates the actual inflow and they therefore coincide. The same holds for t^{free} representing the first time the projected downstream curve coincides with the actual downstream curve.

5.1.1 Alternative functional decomposition based on excess travel time

We now demonstrate it is also possible to construct link travel times by subtracting excess travel time, instead of supplementing the minimum travel time with delay. In this situation, the link is assumed to be in a hypercritical state, such that $\tau(t) = \tau_H(v(t))$. However, this might not actually be the case and this requires us to subtract any *excess delay*, denoted $d^{\text{excess}}(\cdot)$ [h], compensating for our overestimation. This yields the following travel time function:

⁶ The cumulative curve of \bar{U}^{min} relates to the projected curve under the fixed travel time of $\frac{\ell}{g_{\text{max}}}$ and is the lower bound curve identical to $\bar{U}(0)$.

⁷ Although one could also retain the earlier shockwave based formulation, if so desired.

$$\tau(t) = \tau_{II}(v(t)) - d^{\text{excess}}(v(t), \bar{N}^{\text{excess}}(t)), \quad t^{\text{start}} \leq t \leq t^{\text{end}}, \quad (40)$$

We can construct $d^{\text{excess}}(\cdot)$ in a similar fashion as the hypercritical delay, only now utilising The number of projected excess vehicles $\bar{N}^{\text{excess}}(t)$, instead of $N^{\text{hyper}}(t)$. Consider the Example in Figure 6(a), as can be seen, the number of excess vehicles is given by the difference between the projected potential cumulative inflow $\bar{V}(\cdot)$ and the actual cumulative inflow $U(\cdot)$, such that:

$$\begin{aligned} \bar{N}^{\text{excess}}(t) &= \bar{V}(t - d^{\text{excess}}(v(t), \bar{N}^{\text{excess}}(t))) - U(t - d^{\text{excess}}(v(t), \bar{N}^{\text{excess}}(t))) \\ &= \bar{V}(t^{\text{start}} - d^{\text{excess}}(u(t), N^{\text{excess,start}})) - U(t^{\text{start}} - d^{\text{excess}}(u(t), N^{\text{excess,start}})) \\ &\quad + (u(t) - v(t))(t - t^{\text{start}}), \\ &= \bar{N}^{\text{excess,start}} + (u(t) - v(t))(t - t^{\text{start}}), \end{aligned} \quad (41)$$

Where $\bar{N}^{\text{excess,start}}$ is assumed given, so all but the flow rates conveniently drop out. The excess delay, analogous to Equation (36), is found via:

$$d^{\text{excess}}(q, N) = \frac{N}{q}, \quad q \in [0, q^{\text{max}}], N \in \{\bar{N}^{\text{excess}}(t) | t^{\text{start}} \leq t \leq t^{\text{end}}\}, \quad (42)$$

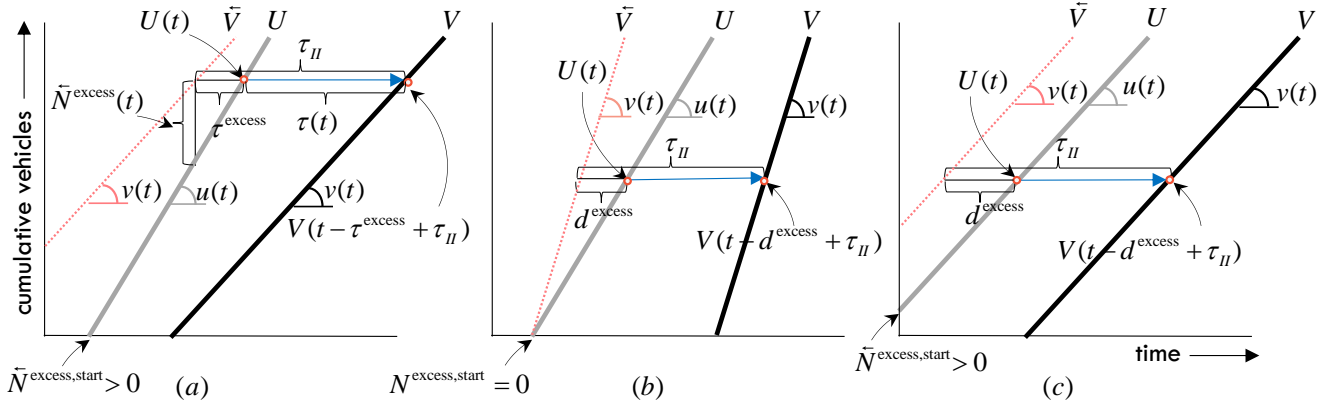


Figure 6: Functional link travel time (excess) perspective under (a) growing queue starting in free flow state, (b) shrinking queue starting in spillback state, (c) stable queue scenario.

Alternatively, under a stable queue, i.e. $u(t) = v(t), t^{\text{start}} \leq t \leq t^{\text{end}}$, it holds that $N^{\text{excess,start}} = \bar{N}^{\text{excess}}(t), t^{\text{start}} \leq t \leq t^{\text{end}}$, see Figure 6(c). While this alternative link travel time formulation is equally valid as supplementing the minimum travel time with additional delay, we only derive semi-dynamic and static formulations for latter approach. The reason being that there are no existing models (yet) that follow this novel excess delay based approach.

5.2 Functional link travel time: semi-dynamic perspective

Analogous to Section 4.2, we consider multiple time periods $i \in I$, and their respective steady-state flow rates, start, and end times. The link travel time formulation in Equation (33) is then replaced with its average period based counterpart:

$$\bar{\tau}_i = \tau^{\text{min}} + \bar{d}_i^{\text{hypo}} + \bar{d}_i^{\text{hyper}}, \quad \text{with } \bar{d}_i^{\text{hypo}} = d^{\text{hypo}}(u_i), \quad i \in I, \quad (43)$$

where the absolute minimum travel time τ^{\min} , remains constant and the average hypocritical delay, due to steady state conditions, is given by $\bar{d}_i^{\text{hypo}} = d^{\text{hypo}}(u_i), i \in I$. The hypercritical delay however, depends on $N^{\text{hyper}}(t)$, for which a semi-dynamic based alternative is required. Analogous to Equation (19) we discretise $N^{\text{hyper}}(t)$, on a per period basis, yielding N_i^{hyper} through:

$$N_i^{\text{hyper}} = \begin{cases} N^{\text{hyper,start}}, & i = 1, \\ N_{i-1}^{\text{hyper}} + N_{i-1}^{\text{hyper},\Delta}, & \text{otherwise,} \end{cases} \quad (44)$$

where N_i^{hyper} is the generalised per period version of $N^{\text{hyper,start}}$, while $N_i^{\text{hyper},\Delta}$, reflects the change in the queue during period i , which based on Equation (35) yields:

$$N_i^{\text{hyper},\Delta} = (\bar{u}_i - \bar{v}_i)(t_i^{\text{end}} - t_i^{\text{start}}), \quad i \in I. \quad (45)$$

Then, analogous to constructing Equations (21) and (22), we utilise Equation (36) to construct the following average hypercritical delay per period:

$$\begin{aligned} \bar{d}_i^{\text{hyper}} &= \frac{1}{2} \left(d^{\text{hyper}}(\bar{v}_i, N_i^{\text{hyper}}(t_i^{\text{start}})) + d^{\text{hyper}}(\bar{v}_i, N_i^{\text{hyper}}(t_i^{\text{end}})) \right) \\ &= \frac{1}{2} \left(\frac{N_i^{\text{hyper}}}{\bar{v}_i} + \frac{N_i^{\text{hyper}} + N_i^{\text{hyper},\Delta}}{\bar{v}_i} \right) = \frac{N_i^{\text{hyper}} + \frac{1}{2} N_i^{\text{hyper},\Delta}}{\bar{v}_i}, \quad i \in I. \end{aligned} \quad (46)$$

The period end times t_i^{end} , follow from Equation (37), yielding:

$$t_i^{\text{end}} = \begin{cases} t_i^{\text{spillback}}, & \text{if } \bar{u}_i < \bar{v}_i, \\ t_i^{\text{free}}, & \text{else if } \bar{u}_i > \bar{v}_i, \\ \infty, & \text{otherwise,} \end{cases} \quad (47)$$

with $i \in I$. Rewriting Equations (38) and (39) yield:

$$t_i^{\text{spillback}} = \operatorname{argmin}_{t \in (t_i^{\text{start}}, \infty]} \{ t \mid N_i^{\text{excess}} + (t - t_i^{\text{start}})(\bar{u}_i - \bar{v}_i) = 0 \}, \quad \text{with } N_i^{\text{excess}} = U_i^{\text{start}} - \bar{V}_i^{\text{start}}, \quad (48)$$

and:

$$t_i^{\text{free}} = \operatorname{argmin}_{t \in (t_i^{\text{start}}, \infty]} \{ t \mid N_i^{\text{hyper}} - (t - t_i^{\text{start}})(\bar{v}_i - \bar{u}_i) = 0 \}, \quad \text{with } N_i^{\text{hyper}} = V_i^{\text{start}} - \bar{U}_i^{\text{start}}, \quad (49)$$

with $U_i^{\text{start}} = \bar{U}(t^{\text{start}} + \tau_l(\bar{u}_i)), V_i^{\text{start}} = V(t^{\text{start}} + \tau_l(\bar{u}_i))$ which makes N_i^{hyper} consistent with Equation (35). We leave it to the reader to observe that $N_i^{\text{excess}} = U_i^{\text{start}} - \bar{V}_i^{\text{start}}$ can be derived in an identical fashion as N_i^{hyper} . Similar to the semi-dynamic experienced travel time formulation (see Section 4.2), an inconsistency arises here in the form of the underlying assumption that N_i^{hyper} is constructed in conjunction with v_i , while in reality, it is consistent with preceding periods' flow rates. As stated before, we do not pursue this any further in this work, for the sake of keeping a clear focus.

5.3 Static functional link travel time decomposition

Following the same process as discussed in Section 4.3, the link travel time function of Equation (43) reduces to its static counterpart through:

$$\bar{\tau} = \tau^{\min} + \bar{d}^{\text{hypo}} + \bar{d}^{\text{hyper}}, \quad \text{with } \bar{d}^{\text{hypo}} = \bar{d}^{\text{hypo}}(\bar{u}), \quad (50)$$

where the static hypercritical delay \bar{d}^{hyper} is derived from Equation (46), yielding:

$$\bar{d}^{\text{hyper}} = \frac{1}{2} \left(\bar{d}^{\text{hyper}}(\bar{v}, N^{\text{hyper,start}}) + \bar{d}^{\text{hyper}}(\bar{v}, N^{\text{hyper,start}} + N^{\text{hyper},\Delta}) \right) = \frac{N^{\text{hyper}} + \frac{1}{2} N^{\text{hyper},\Delta}}{\bar{v}}, \quad (51)$$

with $N^{\text{hyper},\Delta}$ obtained from Equation (45) and reduced to:

$$N^{\text{hyper},\Delta} = t^{\text{end}}(\bar{u} - \bar{v}). \quad (52)$$

This completes the two generalised steady-state link travel formulations.

6 Experienced travel time versus functional travel time

Theorem: decomposing travel time based on a functional or experienced perspective yields identical travel times such that:

$$\left(\tau^{\text{free}}(\cdot) + \tau^{\text{queue}}(\cdot) \right) - \left(\tau^{\min} + d^{\text{hypo}}(\cdot) + d^{\text{hyper}}(\cdot) \right) = 0, \quad (53)$$

with $(u(t), v(t)) \in [0, q^{\max}]$, $t \in [t^{\text{start}}, t^{\text{end}}]$, $\forall N^{\text{start}}$, where N^{start} refers to some valid initial state, reflected through $N^{\text{hyper,start}}$ or $N^{\text{queue,start}}$, depending on the perspective, which are assumed to be consistent with each other, even though they carry different information.

The above can be interpreted as follows: constructing link travel times by actively tracking the free flowing and congested portion of a link yields the exact same result as when one would supplement the absolute minimum travel time with a hypocritical delay and hypercritical delay, conditional on both formulations having the same initial state and the formulations being derived from eGLTM.

Proof: the proof consists of two parts: (i) demonstrating the derivatives of the functional and experienced components in Equation (53) - towards time - are identical, i.e. both perspectives model any changes to the link travel time identically, (ii) for a known feasible link state, Equation (53) holds, i.e. both experienced and functional travel time yield the same link travel time for a particular, consistent, input. When (i) and (ii) hold, Equation (53) holds.

We verify condition (ii) of the proof by considering any free flowing hypocritical state, irrespective of the actual flow rates. From an experienced perspective this means that no initial queue exists, hence $N^{\text{queue,start}} = 0 \Rightarrow \lambda^{\text{start}} = \lambda^{\text{end}} = 1$. Therefore, $\tau(t) = \tau^{\text{free}}(u(t), 1) = \tau_i(u(t))$. From a ‘‘functional’’ perspective, this same situation implies $N^{\text{hyper,start}} = 0$, yielding $\tau(t) = \tau^{\min} + d^{\text{hypo}}(u(t)) = \tau_i(u(t))$. Therefore, the hypocritical link travel times of both perspectives, under consistent inputs, yield identical link travel times.

Condition (i) of the proof requires the construction of each perspective's derivative towards time; for the functional approach we find:

$$\frac{\partial \tau(t)}{\partial t} = \frac{\partial (\tau^{\min} + d^{\text{hypo}}(t) + d^{\text{hyper}}(t))}{\partial t} = \frac{u(t) - v(t)}{v(t)}, \quad (54)$$

with $(u(t), v(t)) \in [0, q^{\max}]$, $t \in [t^{\text{start}}, t^{\text{end}}]$, $\forall N^{\text{hyper, start}}$. For the experienced travel time some algebraic manipulation is required to demonstrate it equates to Equation (54):

$$\begin{aligned} \frac{d\tau(t)}{dt} &= \frac{d(\tau^{\text{free}}(\cdot) + \tau^{\text{queue}}(\cdot))}{dt} = \frac{d(\lambda(t)\tau_I(u(t)) + (1-\lambda(t))\tau_{II}(v(t)))}{dt} \\ &= -\frac{\lambda^{\text{start}}\tau_I(u(t))}{t^{\text{spillback}} - t^{\text{start}}} + \frac{\lambda^{\text{start}}\tau_{II}(v(t))}{t^{\text{spillback}} - t^{\text{start}}} \\ &= (\tau_{II}(v(t)) - \tau_I(u(t))) \left(\frac{\lambda^{\text{start}}}{t^{\text{spillback}} - t^{\text{start}}} \right) \\ &= (\tau_{II}(v(t)) - \tau_I(u(t))) \left(\frac{\lambda^{\text{start}}}{\lambda^{\text{start}}\tau_I(u(t)) + \frac{\lambda^{\text{start}}\ell}{-\eta(u(t), v(t))}} \right) \\ &= \frac{\tau_{II}(v(t)) - \tau_I(u(t))}{\tau_I(u(t)) + \frac{\ell}{-\eta(u(t), v(t))}} \\ &= \frac{\tau_{II}(v(t)) - \tau_I(u(t))}{\frac{\ell\Phi_I^{-1}(u(t))}{u(t)} - \frac{\ell(\Phi_{II}^{-1}(v(t)) - \Phi_I^{-1}(u(t)))}{v(t) - u(t)}} \\ &= \frac{\left(\frac{\ell\Phi_{II}^{-1}(v(t))}{v(t)} \right) - \left(\frac{\ell\Phi_I^{-1}(u(t))}{u(t)} \right)}{\left(\frac{\ell\Phi_I^{-1}(u(t))(v(t) - u(t))}{u(t)(v(t) - u(t))} \right) - \left(\frac{\ell(\Phi_{II}^{-1}(v(t)) - \Phi_I^{-1}(u(t)))u(t)}{u(t)(v(t) - u(t))} \right)} \\ &= \frac{\left(\frac{-\ell(\Phi_I^{-1}(u(t))v(t) - \Phi_{II}^{-1}(v(t))u(t))}{v(t)u(t)} \right)}{\left(\frac{\ell(\Phi_I^{-1}(u(t))v(t) - \Phi_{II}^{-1}(v(t))u(t))}{u(t)(v(t) - u(t))} \right)} = -\frac{u(t)(v(t) - u(t))}{v(t)u(t)} = \frac{u(t) - v(t)}{v(t)}. \end{aligned} \quad (55)$$

This concludes the proof. ■

7 Impact of violating fundamental diagram consistency

So far, our generalised link travel time formulations are consistent with a general concave two-regime FD and therefore link travel times comply with physical link capacity restrictions as well as storage constraints and classify as **SCS**, recall Section 2.4. However, most models violate either the capacity constrained condition, the storage capacity condition, or both. We now illustrate how to derive a capacity constrained, but non-storage constrained formulation (**Cs**), and non-capacity constrained, non-storage constrained, i.e. a capacity restrained formulation (**Cr**) by relaxing some of the existing conditions.

7.1 Violating storage constraints

The violation of storage constraints only arises naturally under a growing queue. To move from a CSC to a Cs classification, and allow the queue to grow beyond what is physically possible, one simply removes the restriction of $t_i^{\text{end}} \leq t_i^{\text{spillback}}, t^{\text{end}} \leq t^{\text{spillback}}$, for a semi-dynamic, or static model, respectively. Hence, t_i^{end} , and/or t^{end} become free variables. In the functional perspective this results in a vertical queuing - or point queue - approach without any additional effort. In an experienced link travel time setting however, where we track the queue explicitly, removing aforementioned condition is not feasible because it can result in a negative free flow and queuing travel times whenever $\lambda^{\text{end}} \geq 1$, which is possible since now $\lambda^{\text{end}} \in [0, \infty)$. In Smith *et al.* (2013), they therefore instead choose an artificially increased link length to make sure spillback doesn't occur in this situation. We do point out that this is far from ideal, since this artificial and arbitrary length does impact on the travel time, so one ends up with a formulation with an additional variable that needs calibration.

We do not discuss a shrinking queue scenario in this context, because in that situation the conservation of flow is violated, i.e. it leads to the disappearance of vehicles and eventually results in a negative number of vehicles on a link. Given that any proper network loading model prevents this behaviour there is little point in constructing such a formulation.

7.2 Violating capacity constraints and storage constraints

When one not only allows for storage constraint violations, but also allows for capacity constraints to be violated, i.e. flow can exceed capacity, the underlying FD is no longer considered consistent with a two-regime approach. In other words, the hypercritical branch where flow decreases with increasing density is absent. In that case, the hypocritical branch continues to grow with increasing density, as is the case with well-known link performance functions, such as the BPR function. This has two important impacts: (i) flow is not actively held back and inflow rates become identical to the outflow rates, i.e. $u(t) = v(t), \forall t \geq t^{\text{start}}$, (ii) queues are implicit rather than explicit and do not take up physical space, i.e. $N^{\text{queue,start}} = N^{\text{hyper,start}} = 0$.

From an experienced link travel time perspective, this results in $\lambda(t) = 1, \forall t \geq t^{\text{start}}$, because there is no explicit queue. Hence, Equation (12) reduces to $\tau(t) = \tau^{\text{free}}(u(t), 1) = \tau_l(u(t)), \forall t \geq t^{\text{start}}$, and the fundamental diagram is replaced by the link performance function of choice, which in case of the BPR function results in:

$$\tau_l(q) = \tau^{\text{min}} \left(1 + \alpha \left(\frac{q}{q_{\text{max}}} \right)^\beta \right), \quad q \in [0, \infty), \quad (56)$$

with α and β being parameters that require calibration. In case of a functional link travel time formulation, the hypercritical travel time component is absent ($d^{\text{hyper}}(\cdot) = 0$), resulting in $\tau(t) = \tau^{\text{min}} + d^{\text{hypo}}(u(t)) = \tau_l(u(t)), \forall t \geq t^{\text{start}}$. Hence, both perspectives collapse to the same link travel time formulation, however, since most link performance functions supplement the minimum link travel time with additional delay, we do classify them as functional (F) in our classification of existing models in Appendix A, although they are in no way consistent with traffic flow theory.

8 Special cases resulting in existing models

Let us now derive two recent formulations explicitly from our generalised link travel time formulations. The first adopts an experienced (**E**) perspective and is discussed in the work of Smith *et al.* (2013), the second adopts a functional form (**F**) and is proposed in the work of Bliemer *et al.* (2014). Both are static models (**St**). Also, the former is storage constrained (**CSC**), while the latter is not (**Cs**). We refer the reader to Appendix A for other existing model classifications.

Smith *et al.* (2013) term their model quasi-dynamic, where there is some initial queue, $N^{\text{queue,start}} \geq 0$, and $\bar{u} = \bar{v}$, i.e. the queue is stable. Their model is both capacity and storage constrained, however, the storage constraint itself is not necessarily consistent with the FD and is a free variable. Their free flow component $\bar{\tau}^{\text{free}}(\cdot)$ is identical to Equation (27), while their queuing component $\bar{\tau}^{\text{queue}}$ is formulated as follows:

$$\bar{\tau}^{\text{queue}} = \frac{N^{\text{queue,start}}}{\bar{v}}. \quad (57)$$

They can do this, because this is a special case where the number of vehicles in the queue does not change, i.e. $\lambda^{\text{start}} = \lambda^{\text{end}}$. In that case, we can indeed derive Equation (57) from the more general Equation (28), utilising the fact that $\lambda^{\text{start}} = 1 - \frac{N^{\text{queue,start}}}{\ell \Phi_{II}^{-1}(\bar{v})}$ from Equation (11), yielding:

$$\begin{aligned} \bar{\tau}^{\text{queue}} &= \tau_{II}(\bar{v}) \left(1 - \frac{(\lambda^{\text{start}} + \lambda^{\text{end}})}{2} \right) = \tau_{II}(\bar{v}) (1 - \lambda^{\text{start}}) = \tau_{II}(\bar{v}) \left(1 - \left(1 - \frac{N^{\text{queue,start}}}{\ell \Phi_{II}^{-1}(\bar{v})} \right) \right) \\ &= \frac{\tau_{II}(\bar{v}) N^{\text{queue,start}}}{\ell \Phi_{II}^{-1}(\bar{v})} = \frac{\frac{\ell}{g_{II}(\bar{v})} N^{\text{queue,start}}}{\frac{\ell \bar{v}}{g_{II}(\bar{v})}} = \frac{N^{\text{queue,start}}}{\bar{v}}. \end{aligned} \quad (58)$$

The other special case is the (static) model by Bliemer *et al.* (2014). Unlike Smith *et al.* (2013), this model does not assume a stable queue, but only stable flow rates. This model is special in the sense that it is one of the few models that places the queues in front of the bottleneck, so the outflow is constrained by the capacity of the next link and not its own link, therefore the inflow can exceed the outflow without violating the current link's capacity constraints. This model does not impose storage constraints, i.e. it constructs a vertical queue and is therefore of the **Cs** family. It adopts a functional perspective and its minimum and hypocritical link delay comply Equation (50). However, hypercritical delay is formulated differently via:

$$\bar{d}^{\text{hyper}} = \frac{t^{\text{end}}}{2} (\alpha - 1), \quad \text{with } \alpha = \begin{cases} \frac{\bar{u}_a}{\bar{v}}, & \bar{u} > \bar{v}, \\ 1, & \text{otherwise.} \end{cases}, \quad (59)$$

with $N^{\text{queue,start}} = 0$, so a queue is non-existent or grows, i.e. $N^{\text{hyper},\Delta} \geq 0$, restricting Equation (52) to $N^{\text{hyper},\Delta} = \max(0, t^{\text{end}}(\bar{u} - \bar{v}))$. Under these conditions, we find that this indeed is a special case of Equation (51):

$$\begin{aligned}
 \bar{d}^{\text{hyper}} &= \frac{N^{\text{hyper,start}} + \frac{1}{2} N^{\text{hyper},\Delta}}{\bar{v}} = \frac{N^{\text{hyper,start}} + \frac{1}{2} \max\{0, (\bar{u} - \bar{v}) t^{\text{end}}\}}{\bar{v}} = \\
 &= \frac{0 + \frac{1}{2} \max\{0, (\bar{u} - \bar{v}) t^{\text{end}}\}}{\bar{v}} = \max\left\{0, \frac{t^{\text{end}}}{2} \left(\frac{\bar{u} - \bar{v}}{\bar{v}}\right)\right\} = \\
 &= \frac{t^{\text{end}}}{2} (\alpha - 1), \quad \text{with } \alpha = \begin{cases} \frac{\bar{u}}{\bar{v}}, & \bar{u} \geq \bar{v}, \\ 1, & \text{otherwise.} \end{cases}
 \end{aligned} \tag{60}$$

9 Observations and research gaps

In this work we derived general steady-state link travel time formulations from a state-of-the-art macroscopic dynamic continuous time first order link model consistent with LWR. These formulations adopted either a functional or experienced perspective. Both perspectives have (dis)benefits depending on the application context, where the latter is attractive for attaching different utilities to the driver experience depending on if he/she is in a queue or not, while the former is less cumbersome to execute and more widely accepted. We proved that, when properly defined, both perspectives yield identical results. Consistent formulations are derived for both for a semi-dynamic and static (quasi-dynamic) modelling perspective as well and we demonstrated that existing formulations can be derived explicitly as special cases. Further, a comprehensive classification of existing models is provided based on a newly presented classification framework allowing one to objectively compare the consistency and capabilities of existing models with respect to one of their most values outcomes; their predicted link travel times. Based on our findings we also make a number of observations:

- Early continuous time dynamic model formulations adopting link performance functions are incapable of constructing steady-state link conditions,
- Almost all existing model formulations seem to prefer a functional formulation of link travel time over an experienced perspective, even though they are equally viable.
- To date, there exists hardly any semi-dynamic nor static models that are fully consistent with state-of-the-art dynamic continuous time formulations regarding their link travel times.
- Link performance functions are still used frequently, and inconsistently, even when one models queues explicitly. We demonstrated that this leads to overestimation of link travel times unless link inflows are restricted to the physical link capacity.

These findings open up opportunities for future research:

- The lack of experienced link travel time formulations in assignment models presents an opportunity to construct new models that attribute congestion and free flow travel times differently. For example by exploring the impact of such models compared to the current practice of considering travel time “as a whole”.
- There exist, to the best of the authors’ knowledge, no semi-dynamic model formulation that are capable of dealing with the inconsistency in travel times when transferring queues between (static) periods.
- There exist, to the best of the authors’ knowledge, hardly any applications that adopt semi-dynamic and static models with link travel time formulations consistent with the ones presented here. The few that do exist, are currently highly theoretical rather than practically applied.

Appendix A

	General			Link travel time		Queue	
	Time dimension	Inflow/outflow restrictions	Steady-state compliance	Perspective	Decomposition	Scenario	Initial
Dynamic:							
Friesz (1989)	D	Cr	NSs	-	-	-	-
Friesz (1993)	D	Cr	NSs	-	-	-	-
Wie <i>et al.</i> (1994)	D	Cr	NSs	-	-	-	-
Daganzo (1995,1994)	D	CSC	Ss	^{4,15}	G ³	Gr+Sh+St	Y ⁵
Astarita (1996)	D	-/Cc ¹	NSs	-	-	-	-
Huang and Lam (2002)	D	-/Cc ²	Ss	F	G ³	Gr+Sh+St	Y ⁵
Himpe <i>et al.</i> (2016), Yperman (2007)	D	CSC	Ss	^{4,15}	G ³	Gr+Sh+St	Y ⁵
van der Gun <i>et al.</i> (2017), Gentile (2010)	D	CSC	Ss	^{4,15}	C	Gr+Sh+St	Y ⁵
Bliemer and Raadsen (in press), Raadsen and Bliemer (in press)	D	CSC	Ss	⁴	C	Gr+Sh+St	Y ⁵
Semi-dynamic:							
Hall <i>et al.</i> (1980), van Vliet (1982)	Sd	-/Cc ²	Ss	F	G ³	Gr+Sh+St	Y
Akamatsu <i>et al.</i> (1998), Kanamori <i>et al.</i> (2007)	Sd	-/Cc ²	Ss	F	O ⁶	Gr+Sh+St	Y
Davidson <i>et al.</i> (2011)	Sd	-/CSC	Ss	n/a ¹⁰	n/a ¹⁰	Gr+Sh+St	Y
Nakayama <i>et al.</i> (2012)	Sd	Cr	Ss	F	O/G ⁹	I	N
Nakayama and Connors (2012)	Sd	-/Cc ²	Ss	F	O ⁶	Gr+Sh+St	Y
Static:							
Bureau of Public Roads (1964)	St	Cr	Ss	F	O/G ⁹	I	N
Smit (1987), Payne and Thompson (1975)	St	-/Cc ²	Ss	F	G ³	I/St ¹¹	Y
Hungerink (1989)	St	-/Cc ²	Ss	F	O ⁶	Gr	N
Akçelik and Roupail (1993)	St	n/a ¹²	Ss	F	n/a + C ¹²	Gr+Sh	Y
Bakker <i>et al.</i> (1994) (in Dutch)	St	-/CSC ^{2,7}	Ss	F	n/a + O ^{6,8}	Gr	N
Bell (1995, 1997)	St	-/Cc ²	Ss	F	G ³	I/St ¹¹	Y
Bifulco and Crisalli (1998)	St	-/CC ²	Ss	n/a ¹⁰	n/a ¹⁰	Gr	N
Lam and Zhang (2000)	St	-/CC ²	Ss	F	O ⁶	Gr	N
Bundschuh (2006)	St	-/CSC ^{2,7}	Ss	F	n/a ¹⁰	Gr	N
Bliemer <i>et al.</i> (2012), Brederode <i>et al.</i> (2018)	St+D ¹⁴	CSC	Ss	⁴	C ¹⁴	Gr	N
Smith (2013)	St	-/CSC ^{2,7}	Ss	F	G ³	I/St ¹¹	Y
Smith <i>et al.</i> (2013)	St	-/CSC ^{2,7}	Ss	E	C	I/St ¹¹	Y
Bliemer <i>et al.</i> (2014)	St	Cc	Ss	F	C	Gr	N
Bliemer and Raadsen (2017)	St	CSC	Ss	F	C	Gr	N

¹outflow rates based on link performance function that in the limit results in capacity outflows, so while capacity restrained, the net effect is an outflow that does not exceed capacity.

²Inflow not restricted by link capacity.

³The free flow travel time is (partly or completely) constant, irrespective of the hypocritical flow rate, this is only correct under the assumption of a linear hypocritical branch of the FD. However, we assume a generalised concave

FD, hence the hypocritical branch can be non-linear, generally leading to an underestimation of the travel time under this simplified formulation.

⁴Link travel time is only computed as a total travel time based on cumulative curves (link transmission models), or average densities/speeds (cell transmission models), no explicit decomposition formulated.

⁵Network starts empty in formulation, but formulation can be easily altered to support an initial state that has a queue.

⁶hypocritical delay is based on link performance function capturing part of delay beyond hypocritical delay and the fact that this is supplemented with a separate hypercritical queuing delay, generally leads to an overestimation of link travel time.

⁷Storage capacity that is a free variable or heuristic inconsistent with an FD.

⁸Extension of Hungerink (1989), but now with inconsistent storage capacity⁷, so unknown what the effect on the link travel time consistency is on top of the originally overestimating formulation.

⁹Link travel time based solely on link performance function, leading to inconsistent link travel times which either over or underestimate depending on the parameter calibration.

¹⁰Information not provided in paper.

¹¹Queue is assumed to be present and stable during the modelled period and not the result of the modelling period itself, because during the modelled period inflow is assumed to be equal to the outflow.

¹²Intersection delay, perspective is that on the stop line of a junction, instead of a whole link, so only formulation of (functional) hypercritical delay is formulated here and that bit is consistent.

¹³Storage capacity is not formally defined. However, if it were to be consistent with the adopted FD, then the link travel times could be consistent with our general formulation. This depends on if the capacity restrained link performance function (also not defined) represents a valid uncongested branch of a general concave two-regime FD.

¹⁴Hybrid model, where a capacity constrained static assignment is complemented with a dynamic model based on LTM to propagate the vertical queues backward. In this approach, travel times remain consistent on the link level, but not on the path level.

¹⁵Solution scheme is time discretised, resulting in smoothing errors in the resulting travel time, compared to a continuous time formulation.

Appendix B

The difference between $N^{\text{queue,start}}$ and $N^{\text{hyper,start}}$ is intuitively best understood by considering a link in spillback, then $N^{\text{queue,start}} > 0, N^{\text{hyper,start}} > 0$. To make this more concrete, we establish a density based, formulation of both variables to make them comparable. The density on a link in spillback is given by $\Phi_H^{-1}(\bar{v})$, while the free flow density is defined through $\Phi_I^{-1}(\bar{u})$. Recall that $N^{\text{hyper,start}}$ represents the excess vehicles on the link compared to free flow, i.e. $\ell(\Phi_H^{-1}(\bar{v}) - \Phi_I^{-1}(\bar{u}))$. At the same time, in spillback, all vehicles on the link traverse the queue physically, i.e. $\lambda^{\text{start}} = \lambda^{\text{end}} = 0$. Hence, following Equation (11) $N^{\text{queue,start}} = \ell\Phi_H^{-1}(\bar{v})$. So, in this particular scenario it indeed holds that $N^{\text{hyper,start}} \leq N^{\text{queue,start}}$. Based on Theorem 1 in Section 6, this now necessarily also holds in general.

References

- Akçelik, R., Roupail, N.M., 1993. Estimation of delays at traffic signals for variable demand conditions. *Transp. Res. Part B Methodol.* 27, 109–131. doi:10.1016/0191-2615(93)90003-S.
- Akamatsu, T., Makino, Y., and Takahashi, E., 1998. (in Japanese) Semi-dynamic Traffic Assignment Models with Queue Evolution and Elastic OD Demands, *Infrastructure Planning Review*, Vol.15, 535-545.
- Astarita, V., 1996. A continuous time link model for dynamic network loading based on travel time function. *Transp. traffic theory. Proc. 13th Int. Symp. Transp. traffic theory* 79–101.

- Bakker, D., Mijjer, P.H., Hofman, F., 1994. (in Dutch) QBLOK: een toedelingsmethodiek voor het modelleren van de afhankelijkheid tussen knelpunten en de voorspelling van blokkades. Proceedings of Colloquium Vervoersplanologisch Speurwerk, Delft, 313-332.
- Beckmann, M., McGuire, C.B., Winsten, B.W., 1956. Studies in the economics of transportation. Yale University Press, New Haven CT, USA.
- Bell, M.G.H., 1995. Stochastic user equilibrium assignment in networks with queues. *Transp. Res. Part B Methodol.* doi:10.1016/0191-2615(94)00030-4.
- Bliemer, M.C.J., Raadsen, M.P.H., (in press). Continuous-time general link transmission model with simplified fanning (Part I Theory and link mode formulation).
- Bliemer, M.C.J., Raadsen, M.P.H., 2017. Static traffic assignment with residual queues and spillback, in: Proceedings of the 17th Swiss Transport Research Conference. Monte Verita, pp. 1--32.
- Bliemer, M.C.J., Raadsen, M.P.H., Brederode, L.J.N., Bell, M.G.H., Wismans, L.J.J., Smith, M.J., 2017. Genetics of traffic assignment models for strategic transport planning. *Transp. Rev.* 37, 56–78. doi:10.1080/01441647.2016.1207211.
- Bliemer, M.C.J., Raadsen, M.P.H., Smits, E.-S., Zhou, B., Bell, M.G.H., 2014. Quasi-dynamic traffic assignment with residual point queues incorporating a first order node model. *Transp. Res. Part B Methodol.* 68, 363–384. doi:10.1016/j.trb.2014.07.001.
- Bliemer, M.C.J., Brederode, L., Wismans, L., Smits, E., 2012. Quasi-dynamic traffic assignment: static traffic assignment with queuing and spillback. *Transp. Res. Rec.*
- Brederode, L., Pel, A., Wismans, L., de Romph, E., Hoogendoorn, S., 2018. Static Traffic Assignment with Queuing: model properties and applications. *Transp. A Transp. Sci.* 0, 1–36. doi:10.1080/23249935.2018.1453561.
- Bundschuh, M., Vortisch, P., van Vuuren, T., Mott McDonald, 2006. Modelling queues in static traffic assignment. *Eur. Transp. Conf. Proc.*
- Bureau of Public Roads, 1964 Traffic Assignment Manual. U.S. Dept. of Commerce, Urban Planning Division, Washington D.C.
- Cascetta, E., 2009. Transportation Systems analysis: models and applications. Springer, New York (NY), USA.
- Davidson, P., Thomas, A., Teye-Ali, C., 2011. Clocktime assignment: a new mesoscopic junction delay highway assignment approach to continuously assign traffic over the whole day, in: European Transport Conference Proceedings.
- Daganzo, C.F., 1995. The cell transmission model, part II: Network traffic. *Transp. Res. Part B Methodol.* 29, 79–93. doi:10.1016/0191-2615(94)00022-R.
- Daganzo, C.F., 1994. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. *Transp. Res. Part B* 28, 269–287. doi:10.1016/0191-2615(94)90002-7.
- Friesz, T.L., Bernstein, D., Smith, T.E., Tobin, R.L., Wie, B.W., 1993. A Variational Inequality Formulation of the Dynamic Network User Equilibrium Problem. *Oper. Res.* 41, 179–191. doi:10.1287/opre.41.1.179.
- Friesz, T.L., Luque, J., Tobin, R.L., Wie, B., 1989. Dynamic Network Traffic Assignment Considered As a Continuous Time Optimal Control Problem. *Oper. Res.*
- Gentile, G., 2010. The General Link Transmission Model for Dynamic Network Loading and a comparison with the DUE algorithm, in: New Developments in Transport Planning: Advances in Dynamic Traffic Assignment (Chapter 8). pp. 1615–1620.
- van der Gun, J.P.T., Pel, A.J., van Arem, B., 2017. Extending the Link Transmission Model with non-triangular fundamental diagrams and capacity drops. *Transp. Res. Part B Methodol.* 98, 154–178. doi:10.1016/j.trb.2016.12.011.
- Hensher, D.A., 2001. Measurement of the valuation of travel time savings. *J. Transp. Econ. Policy* 35, 71–98.

- Himpe, W., Corthout, R., Tampère, M.J.C., 2016. An efficient iterative link transmission model. *Transp. Res. Part B Methodol.* 92, 170–190. doi:10.1016/j.trb.2015.12.013.
- Hungerink G.J., 1989. G. Hungerink. Q-Net: Assignment on Over-Congested Networks by Link Inflow Constraint, in: *Proc., U.S.-Ita/y Joint Seminar on Urban Traffic Networks: Dynamic Control and Flow Equilibrium*. Capri.
- Kanamori, R., Miwa, T., Morikawa, T., 2007. Application of Time-Dependent Stochastic Equilibrium Assignment Model Considering Activity Choices to Nagoya Metropolitan Area. *J. East. Asia Soc. Transp. Stud.* 7.
- Lam, W., Zhang, Y., 2000. Capacity-constrained traffic assignment in networks with residual queues. *J. Transp. Eng.* 121–128.
- Lighthill, M.J., Whitham, G.B., 1955. On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads. *Proc. R. Soc. A Math. Phys. Eng. Sci.* 229, 317–345. doi:10.1098/rspa.1955.0089.
- Long, J., Gao, Z., Szeto, W.Y., 2011. Discretised link travel time models based on cumulative flows: Formulations and properties. *Transp. Res. Part B Methodol.* 45, 232–254. doi:10.1016/j.trb.2010.05.002.
- Nakayama, S., Connors, R., 2014. A quasi-dynamic assignment model that guarantees unique network equilibrium. *Transp. A Transp. Sci.* 10, 669–692. doi:10.1080/18128602.2012.751685.
- Nakayama, S., Takayama, J., Nakai, J., Nagao, K., 2012. Semi-dynamic traffic assignment model with mode and route choices under stochastic travel times. *J. Adv. Transp.* 46, 269–281. doi:10.1002/atr.208.
- Newell, G.F., 1993. A simplified theory of kinematic waves in highway traffic, part II: Queueing at freeway bottlenecks. *Transp. Res. Part B Methodol.* 27, 289–303. doi:10.1016/0191-2615(93)90039-D.
- Payne, H.J., Thompson, W.A., 1975. Traffic assignment on transportation networks with capacity constraints and queueing. In: *Paper Presented at the 47th National ORSA Meeting/TIMS 1975 North-American Meeting, Chicago, IL*.
- Raadsen, M.P.H., Bliemer, M.C.J., in press. Continuous-time general link transmission model with simplified fanning, part II: Event based algorithm for networks. *Transp. Res. Part B Methodol.* 0, 1–31. doi:https://doi.org/10.1016/j.trb.2018.01.003.
- Raadsen, M.P.H., Bliemer, M.C.J., Bell, M.G.H., 2016. An efficient and exact event-based algorithm for solving simplified first order dynamic network loading problems in continuous time. *Transp. Res. Part B Methodol.* 92, 191–210. doi:10.1016/j.trb.2015.08.004.
- Richards, P.I., 1956. Shock Waves on the Highway. *Oper. Res.* 4, 42–51. doi:10.1287/opre.4.1.42.
- Smith, M.J., 2013. A link-based elastic demand equilibrium model with capacity constraints and queueing delays. *Transp. Res. Part C Emerg. Technol.* 29, 131–147. doi:10.1016/j.trc.2012.04.011.
- Smith, M.J., Huang, W., Viti, F., 2013. Equilibrium in Capacitated Network Models with Queueing Delays, Queue-storage, Blocking Back and Control. *Procedia - Soc. Behav. Sci.* 80, 860–879. doi:10.1016/j.sbspro.2013.05.047.
- Smith, M.J., 1987. Traffic control and traffic assignment in a signal-controlled network with queueing, in: Gartner, N.H., Wilson M. (Eds.), *Proceedings of the Tenth International Symposium on Transportation and Traffic Theory*. Elsevier, pp. 61–68.
- Szeto, W.Y., Lo, H.K., 2005. Properties of dynamic traffic assignment with physical queues. *J. East. Asia Soc. Transp. Stud.* 6, 2108–2123.
- van Vliet, D., 1982. SATURN: a modern assignment model. *Traffic Eng. Control* 23, 575–581.

- Wie, B., Tobin, R., Friesz, T.L., 1994. The augmented Lagrangian method for solving dynamic network traffic assignment models in discrete time. *Transp. Sci.* 28, 204–220.
- Yang, H., Yagar, S., 1994. Traffic assignment and traffic control in general freeway-arterial corridor systems. *Transp. Res. Part B Methodol.* 28, 463–486. doi:10.1016/0191-2615(94)90015-9.
- Yperman, I., 2007. The Link Transmission Model for Dynamic Network Loading. Katholieke Universiteit Leuven.