To Alice,
the love of my life.
Preface

Abstract

Fock and Goncharov’s coordinates parametrise decorated characters relative to the fundamental group of a punctured surface with negative Euler characteristic. We extend these coordinates to $\text{PGL}(3, \mathbb{C})$–characters of hyperbolic once-punctured torus bundles using monodromy ideal triangulations. In particular, we give a description of the decorated character variety as set of fixed points of edge flip functions in $(\mathbb{C} \setminus \{0, -1\})^8$. From there, we find a special character that can be realised as the holonomy of a branched CR structure. We conclude by explicitly constructing such a geometric structure on every hyperbolic once-punctured torus bundle.

Acknowledgement

Firstly, I would like to thank my supervisor Stephan Tillmann. Among the list of things I am grateful for, I will only mention one: being a role model. As a mathematician and as a human being, an exceptionally great man to listen to, and to laugh and spend time with. The one I hope to be one day.

Next, I would like to thank my wife Alice Giovannini. For everything. My love and my thesis are dedicated to you.

Thanks to my family, for your limitless support. Although the distance, you are a firm point of reference I can always go back to when I feel lost. You have never let me down, and never will.

Thanks to my friends, from within and outside the mathematics department, for filling these years with joy and smiles. Honourable mentions go to Andrea Compatangelo, Brett Latham, Zac Chodos and Dominic Tate.

Finally, I would like to thank the Commonwealth of Australia for an International Postgraduate Research Scholarship and the School of Mathematics and Statistics at USyd for financially supporting my studies.
Statement of Originality and Authorship

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes. I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

This thesis is the last of three major projects in my PhD studies. The first one being a joint work with Luo and Tillmann on hyperbolic cone-manifold structures, published in the Proceedings of the American Mathematical Society [5]. The second project, on convex projective structures and Fock-Goncharov’s coordinates, resulted in a manuscript with Tate and Tillmann, currently under review for publication [6]. References to [6] are made in this thesis. I certify that my contribution to each of these works was substantial.

Alex Casella
# Contents

## Preface
- Abstract .......................................................... i
- Acknowledgement .................................................. i
- Statement of Originality and Authorship ...................... ii

## Introduction

### 1 Fock-Goncharov Coordinates On The Once-Punctured Torus
- 1.1 Ideal triangulations and the Farey tessellation ............. 6
- 1.2 Flags and ratios .............................................. 8
  - 1.2.1 Triple ratio ........................................... 9
  - 1.2.2 Cross ratio ........................................... 9
  - 1.2.3 Quadruple ratio ....................................... 11
  - 1.2.4 Configuration of flags ................................ 11
- 1.3 Representation variety, character variety and decorations .... 14
  - 1.3.1 Representations and characters ....................... 14
  - 1.3.2 Decorated varieties .................................. 15
- 1.4 Fock-Goncharov Coordinates ................................. 16
  - 1.4.1 The canonical isomorphism ............................ 16
  - 1.4.2 A useful convention .................................. 18
  - 1.4.3 Example: the standard positive ideal triangulation .... 19
  - 1.4.4 Change of coordinates ................................ 21

### 2 Fock-Goncharov Coordinates On Once-Punctured Torus Bundles
- 2.1 Once-punctured torus bundles $M_f$ .......................... 25
- 2.2 The monodromy ideal triangulation .......................... 26
  - 2.2.1 The flip sequence of $M_f$ ........................... 26
  - 2.2.2 The triangulation ..................................... 27
  - 2.2.3 Combinatorics around the edges ....................... 29
  - 2.2.4 Example: the figure eight knot complement ............ 30
- 2.3 FG coordinates on $M_f$ ..................................... 31
  - 2.3.1 A parametrisation of $\mathcal{X}^+(M_f)$ ............... 31
  - 2.3.2 Two special decorated characters ..................... 34
# CONTENTS

3 Branched Cauchy-Riemann Structures .................................................. 37
   3.1 CR geometry ................................................................. 38
      3.1.1 CR simplices ....................................................... 39
      3.1.2 Two fundamental pieces: the standard symmetric tetrahedron and
            the slab ............................................................ 41
   3.2 Branched CR structures on once-punctured torus bundles ............... 46
      3.2.1 Finite geometric realisations .................................. 47
      3.2.2 The figure eight knot complement ............................. 48
      3.2.3 General case ...................................................... 52
      3.2.4 Branch locus ...................................................... 59

A Appendix ....................................................................................... 63
   A.1 A different set of coordinates .............................................. 63
      A.1.1 The deformation variety ........................................ 63
      A.1.2 Relationship between the two coordinate systems ........ 65

References ....................................................................................... 69
Introduction

A geometry or geometric structure \((G, X)\) is a homogeneous space \(X\) together with a transitive action on \(X\) by a Lie group \(G\), which acts as the symmetry group of the geometry. This concept was originally introduced by Klein in his celebrated Erlangen program [25], and rapidly developed by Ehresmann [8] and many others afterwards. When \(X\) and \(G\) are chosen appropriately, one recovers many classical geometries like hyperbolic \((\text{SO}(1, n), \mathbb{H}^n)\), Euclidean \((\mathbb{R}^n \rtimes \text{O}(n), \mathbb{E}^n)\) or spherical \((\text{O}(n + 1), \mathbb{S}^n)\) geometry. A \((G, X)\)–manifold \(M\) is a manifold endowed with a \((G, X)\)–structure, namely an atlas of charts in the model space \(X\), whose transition functions are restrictions of elements of \(G\). Associated to every \((G, X)\)–structure is a developing map and holonomy representation

\[
\text{dev} : \widetilde{M} \rightarrow X \quad \text{and} \quad \text{hol} : \pi_1(M) \rightarrow G,
\]

such that

\[
\text{hol}(\gamma) \cdot \text{dev}(x) = \text{dev}(\gamma \cdot x), \quad \gamma \in \pi_1(M), \quad x \in \widetilde{M}.
\]

The developing pair \((\text{dev}, \text{hol})\) is uniquely determined up to conjugation by an element of \(G\), and so it is an invariant of the \((G, X)\)–structure on \(M\). Furthermore, the Ehresmann-Thurston principle implies that sufficiently nearby \((G, X)\)–structures are completely determined by their holonomy representations [35]. On the other hand, it is a rather complicated task to determine whether a representation \(\rho : \pi_1(M) \rightarrow G\) arises as the holonomy of some \((G, X)\)–structure.

As more and more connections between topology and geometry were discovered, \((G, X)\)–structures have become a central topic in the study of manifolds. Among many contributors, William Thurston is one of the most celebrated pioneers. Not only for his major breakthroughs in hyperbolic 3–manifolds, like the Perelman-Thurston Geometrisation Theorem ([26], [36]), but also for the exceptional techniques that he has developed. For example, in [35] he develops a way to construct hyperbolic structures on cusped 3–manifolds using ideal triangulations, namely decompositions into tetrahedra whose vertices are removed. The strategy consists in realising these simple pieces as hyperbolic objects, that glue up coherently in the manifold \(M\). On the one hand, the shape of a hyperbolic ideal tetrahedron is completely described by a complex number, namely the cross-ratio of its four vertices on the sphere at infinity. On the other, consistency of the gluings can be encoded in a system of two sets of equations in those complex variables: edge equations and consistency equations. A solution to this system corresponds to a unique complete hyperbolic metric.
of $M$. When consistency equations are dropped and one considers a system of just edge equations, completeness is lost and the solution set parametrises more general hyperbolic structures together with representations of $\pi_1(M)$ into $\text{PSL}(2, \mathbb{C}) \cong \text{SO}(1, 3)$, the group of orientation preserving isometries of $\mathbb{H}^3$.

Since Thurston, many authors have studied and further developed his technique ([5], [7], [30], [34], [39], et al.). In a recent paper [14], Fock and Goncharov put together ideal triangulations and Thurston’s ideas to fill in the gap between representations and geometry of surfaces. Given a surface $S$ and a group $G$, they use flags to parametrise decorated representations of $\pi_1(S)$ in $G$. Decorated representations are representations enriched with some geometry, which are only one step away from being $(G, X)$–structures. When $G = \text{PGL}(3, \mathbb{R})$, they show that most of these representations can be promoted to holonomies of convex projective structures on $S$, geometric structures modelled on projective ($\text{PGL}(3, \mathbb{R}), \mathbb{RP}^2$) geometry [15]. Generalisations of these flag parametrisations to 3–manifolds are the Ptolemy coordinates of Garoufalidis, Thurston and Zickert [17] for $G = \text{SL}(m, \mathbb{C})$, and the shape coordinates of Garoufalidis, Goerner and Zickert [16] for $G = \text{PGL}(m, \mathbb{C})$. Some of these coordinate systems were independently developed by Bergeron, Falbel and Guilloux [3] for $G = \text{PGL}(3, \mathbb{C})$, following the work of Falbel on Cauchy-Riemann ($\mathbb{S}^3, \text{PU}(2, 1)$) structures (CR in short) [9].

In this thesis, we propose a different way to extend Fock and Goncharov’s coordinate system to dimension three, for a special class of manifolds. The spaces we are interested in are punctured surface bundles, orientable manifolds which are the interior of compact 3–manifolds with boundary a union of tori. They are fiber bundles over the circle, with fiber space a punctured surface.

This thesis will concentrate on the case where the surface is a once-punctured torus. The figure eight knot complement is one such example. Most of these manifolds are hyperbolic [31], and exhibit important combinatorial properties. In particular, Floyd and Hatcher showed that each hyperbolic once-punctured torus bundle admits a canonical realisation as an ideal triangulation, called the monodromy ideal triangulation [13]. The importance of this decomposition relies on its rich combinatorial structure, but also on its geometric properties. For instance, it was employed by Guéritaud [21], together with Casson’s volume maximisation principle for angle structures, to prove hyperbolicity.

We show that a subset of the set of conjugacy classes of decorated representations is a subvariety of $(\mathbb{C} \setminus \{0, -1\})^8$, using the fact that the monodromy ideal triangulation is constructed by layering tetrahedra over a once-punctured torus. In particular, we give a concrete description of this algebraic variety in terms of fixed points of some explicit functions, called edge flips. This provides a coordinate system in eight complex variables of the character variety of a hyperbolic once-punctured torus bundle. Although related to the work in [3], our approach gives a different point of view on the matter. As an immediate consequence, we find that every hyperbolic once-punctured torus bundle has a special representation $\rho_P$, whose decorated character $[\rho_P]$ is shown to have special properties.
Introduction

The representation $\rho_P$ is irreducible, and its image lies inside a subgroup of $\text{PGL}(3, \mathbb{C})$, sometimes called the *Eisenstein-Picard modular group*. It is the subgroup of $\text{PU}(2, 1)$ with entries in the ring of integers in the imaginary quadratic number field $\mathbb{Q}[\sqrt{-3}]$. In particular, this implies that the image of $\rho_P$ is discrete and one might wonder if it could be realised as the holonomy of a geometric structure.

In [9], Falbel finds the same representation $\rho_P$ in the case of the figure eight knot complement $K_8$, the simplest hyperbolic once-punctured torus bundle. He shows that $\rho_P$ is the only representation of $\pi_1(K_8)$ in $\text{PU}(2, 1)$ whose restriction to the peripheral subgroup is faithful and purely parabolic. Moreover, he constructs a branched CR structure on $K_8$ whose holonomy is $\rho_P$. CR structures are modelled on the three-sphere $S^3 \subset \mathbb{C}^2$ together with the action of $\text{PU}(2, 1)$, its group of biholomorphic transformations. They are branched when the charts are locally branched coverings.

Inspired by the work of Falbel and Thurston, we modify the monodromy ideal triangulation of each once-punctured torus bundle to a new ideal cell decomposition. This decomposition is made up of tetrahedra and 3–cells that we call slabs, CW complexes obtained by deformation retracting the base of a square pyramid onto one of its sides. In the case of the figure eight knot complement, Falbel uses one of these slabs implicitly, as part of a *generalised tetrahedron*. The CR structure thus constructed consists of charts that are not embeddings of the tetrahedra, and it does not generalise further. On the other hand, we geometrically realise each ideal cell by embedding it in CR space and use the malleability of slabs to build CR structures on almost all once-punctured torus bundles. For this to work, six geometrically different types of slabs will be defined. A collection of the main results is summarised in the following theorem.

**Theorem.** Let $M_f$ be a hyperbolic once-punctured torus bundle. Then $M_f$ admits an ideal cell decomposition $D_f$ that is geometrically realisable in CR space. It corresponds to a branched CR structure, whose branch locus is the set of edges of $D_f$.

Moreover, the restriction of its associated decorated holonomy to the fundamental group $(\alpha, \beta)$ of the base once-punctured torus does not depend on the monodromy automorphism $f$. It is the decorated character $[\rho_P, \Phi_P]$, where

$$
\rho_P(\alpha) = \begin{bmatrix}
\bar{\omega} & 0 & 0 \\
-1 & \omega & -\omega \\
\bar{\omega} & 0 & -1
\end{bmatrix}, \\
\rho_P(\beta) = \begin{bmatrix}
1 & -\omega & -1 \\
0 & \bar{\omega} & 0 \\
0 & \bar{\omega} & -\omega
\end{bmatrix}, \\
\text{and} \quad \omega = -\frac{1}{2}(1 + \sqrt{-3}).
$$

In particular, its Fock-Goncharov coordinates are

$$
\Psi_f([\text{hol}_f, \text{dev}_f^{(0)}]) = (\omega, \omega, \omega, \omega, \omega, \omega, \omega).
$$

In the end, we also analyse the branch locus and give simple descriptions of the ramification orders in terms of the combinatorics of the ideal cell decomposition.

The work done in this project has the potential to further extend to more general punctured surface bundles, as they also admit layered triangulations. Even though the number of
coordinates (and the complexity of the problem) increases as the Euler characteristic of the punctured surface decreases, we do not see any theoretical barriers in the parametrisations of decorated characters. On the contrary, it is not clear whether one could construct CR structures in a similar way, as the new cell decompositions here described rely on the fact that the base surface is a once-punctured torus. We intend to address this problem in future work using the veering triangulations of Agol [1].

The content of this thesis is organised as follows. The first chapter mainly focusses on the once-punctured torus, with background material on ideal triangulations, flags, decorated varieties and Fock-Goncharov coordinates. Most of the material is based on [5]. In the second chapter we move on to dimension three. A large part is devoted to the construction of the monodromy ideal triangulation and the study of its relevant combinatorial properties. The remainder concerns our new coordinate system for the decorated character variety. In particular, we show how the special representation $\rho_P$ naturally arises, and some of its distinguishing characteristics. Finally, the last chapter is completely devoted to branched Cauchy-Riemann structures. We begin by familiarising the reader with CR geometry and some of its fundamental objects, then dive straight into the proof of the main result. An explicit example is also provided. We conclude the thesis with an appendix, where we compare our coordinate system with the one of Bergeron, Falbel and Guilloux [3].
Chapter 1

Fock-Goncharov Coordinates On The Once-Punctured Torus

Fock and Goncharov’s coordinates were introduced in [14] to parametrise decorated representations of the fundamental group of a punctured surface $S$ of negative Euler characteristic into a split semisimple algebraic group $G$ over $\mathbb{Q}$, with trivial centre. For $G = \text{PGL}(3, \mathbb{R})$, the same coordinates encode framed marked properly convex projective structures on $S$, with minimal or maximal ends ([15], [6]). They are constructed by choosing a decomposition of $S$ into triangles, whose vertices are decorated by geometric objects called flags. This chapter is devoted to the necessary background material on Fock and Goncharov’s work. The surface of interest is the once-punctured torus $T_0$, and the group is $\text{PGL}(3, \mathbb{C})$. We briefly recall the notions of ideal triangulations §1.1, flags §1.2 and decorated varieties §1.3, then proceed to construct Fock-Goncharov coordinates for conjugacy classes of decorated representations of $\pi_1(T_0)$ in $\text{PGL}(3, \mathbb{C})$ §1.4.

Let $\mathbb{Z}^2$ be the integer lattice in the real plane $\mathbb{R}^2$. The group $\mathbb{Z}^2$ naturally acts on $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \mathbb{Z}^2$ by translations. The quotient spaces are

$$T := \mathbb{R}^2 / \mathbb{Z}^2 \quad \text{and} \quad T_0 := \left( \mathbb{R}^2 \setminus \mathbb{Z}^2 \right) / \mathbb{Z}^2,$$

respectively a torus and once-puncture torus. The former $T$ is a compact orientable surface of genus one, unique up to diffeomorphism, while the latter $T_0$ is the complement of a point in $T$. We call it the puncture of $T_0$. Both $T$ and $T_0$ are said to be positively oriented when their orientation is induced by the positive standard basis $(i, j)$ of $\mathbb{R}^2$. Otherwise they are negatively oriented. To simplify some notation, we will sometimes identify $T_0$ with the square spanned by $i$ and $j$, with side pairings and vertices removed.

The fundamental group of the once-puncture torus $\pi_1(T_0)$ is a free group in two generators. It has a standard presentation $\pi_1(T_0) = \langle \alpha, \beta \rangle$, where $\alpha$ and $\beta$ correspond to the homotopy types of curves with direction vectors $i$ and $j$ respectively.
1.1 Ideal triangulations and the Farey tessellation

An essential arc in $T_0$ is the intersection with $T_0$ of a simple arc whose interior is embedded in $T$, it intersects the puncture only at the endpoints and is not homotopic (relative to the puncture) to a point in $T$. An ideal triangulation $T$ of $T_0$ is a maximal collection of pairwise disjoint and non-homotopic (relative the puncture) essential arcs. It is well known that every ideal triangulation of $T_0$ always comprises three essential arcs, called ideal edges, and divides the surface into two ideal triangles. All of these ideal triangulations are combinatorially equivalent, but they can be distinguished by that they are not isotopic via an isotopy of $T$ fixing the puncture.

Without loss of generality, one can assume that ideal triangulations of $T_0$ are straight, in the sense that each ideal edge is the intersection with $T_0$ of the quotient of a straight line through the origin in $\mathbb{R}^2$. In a straight triangulation $T$, the slope of an edge is the slope of the corresponding straight line. Since edges start and terminate at the puncture, their slopes must be rational, hence there is a bijection between ideal edges and $\mathbb{Q} \cup \{\infty\}$. In what follows, we adopt the convention that for any $p/q \in \mathbb{Q}$, $p, q$ are coprime and $q > 0$, and $-\infty = \frac{-1}{0} < \frac{p}{q} < \frac{1}{0} = \infty$.

Floyd and Hatcher [13] found a very elegant way of encoding the set of isotopy classes of ideal triangulations as the vertices of a tree. This tree is dual to a tessellation of the hyperbolic plane by ideal triangles. The ideal vertices of this tessellation are the set of slopes of ideal edges $\mathbb{Q} \cup \{\infty\}$ in the circle at infinity.

Two ideal vertices $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ are joined by a geodesic if and only if $p_1q_2 - p_2q_1 = \pm 1$. In this case, they are said to form a Farey pair, and they correspond to arcs in $T_0$ that can be isotoped off each other. Three ideal vertices $\frac{p_1}{q_1} < \frac{p_2}{q_2} < \frac{p_3}{q_3}$ of the tessellation form a triangle if and only if $\frac{p_2}{q_2} = \frac{p_1}{q_1} \oplus \frac{p_3}{q_3}$, where the operation $\oplus$ is the Farey sum $\frac{p_1}{q_1} \oplus \frac{p_3}{q_3} := \frac{p_1q_1 + p_3q_3}{q_1q_3}$.

For every Farey pair there are precisely two other ideal vertices satisfying the Farey sum. It follows that the geodesics divide the hyperbolic plane into ideal triangles, forming the required tessellation, which is known as the Farey tessellation $F$. The dual graph of $F$ is a trivalent tree $F^*$, also called the Farey tree. A beautiful treatment of this topic can be found in [4].

The ideal vertices of a triangle in $F$ correspond to the slopes of three disjoint non-homotopic properly embedded arcs in $T_0$, and hence to an ideal triangulation. Thus, there is one vertex of the dual tree $F^*$ for each isotopy class of ideal triangulation of the once-punctured torus, and every such ideal triangulation is uniquely determined by a triplet of slopes satisfying the Farey sum. By adopting the convention that $0$ and $\infty$ are neither negative nor positive, we will say that an ideal triangulation is positive (resp. negative) if at least one if its slopes is positive (resp. negative). The standard positive (resp. negative) ideal triangulation of $T_0$ is the triangulation $T_+ \ (resp. \ T_-)$ with slopes $\{0, 1, \infty\}$ (resp. $\{0, -1, \infty\}$).

Two vertices of the dual tree $F^*$ are joined by an edge if and only if their corresponding ideal triangulations differ by a single slope. Passing from one triangulation to the other is usually called edge flipping, as it involves removing one edge, resulting in a square with side
identifications, and then inserting the other diagonal of the square. As $F^*$ is a tree, every two ideal triangulations of $T_0$ differ by a unique minimal sequence of edge flips.

Edge flips are of three types, depending on the slope we are flipping over. A right flip $\mathcal{R}$ (resp. left flip $\mathcal{L}$) is an edge flip of the largest (resp. smallest) slope. The remaining flip will be referred to as a middle flip $\mathcal{M}$. For example, starting from the standard positive triangulation $\{0, 1, \infty\}$ of $T_0$, a right flip produces the triangulation $\{0, \frac{1}{2}, 1\}$, a left flip gives $\{1, 2, \infty\}$, and a middle flip gives $\{0, -1, \infty\}$.

One can visualise the dynamics of edge flips on the dual tree $F^*$ as follows. Let $T_m$ be a positive ideal triangulation (different from the standard one) and let $T_+, T_1, \ldots, T_{m-1}$ be the sequence of triangulations along the unique shortest path between the standard positive triangulation and $T_m$. By definition, a middle flip kills the middle slope, hence it corresponds to a back-track towards $T_+$, and transforms $T_m$ into $T_{m-1}$, contradicting the minimality of the path. If you exclude back-tracking, one can move along $F^*$ in only two other ways, corresponding to a right or left flip. By orienting the hyperbolic plane with its standard positive orientation, a right (resp. left) flip corresponds exactly to turning right (resp. left) at $T_m$ (cf. Figure 1.1). A perfectly analogous arguments works if we replace $T_m$ with a negative ideal triangulation.

![Figure 1.1: The Farey tree is dual to the Farey tessellation of the hyperbolic plane. Every vertex corresponds to an ideal triangulation of the once-punctured torus, and every edge corresponds to an edge flip.](image)

The following lemma is stated for future reference. It is a direct consequence of the above discussion.

**Lemma 1.** Let $T$ be a positive (resp. negative) ideal triangulation different from the positive (resp. negative) standard one $T_0$. The unique sequence of edge flips from $T_0$ to $T_m$ does not contain any middle flips. Conversely, the sequence of flips from $T_m$ to $T_0$ only contains middle flips.
1.2 Flags and ratios

We consider the complex projective plane \( \mathbb{CP}^2 \) and the group of projective linear transformations \( \text{PGL}(3, \mathbb{C}) \). Points and lines of \( \mathbb{CP}^2 \) are denoted by column and row vectors, respectively. In particular, a line \([a : b : c] \) corresponds to the set of points \([x : y : z]^t \) of \( \mathbb{CP}^2 \), satisfying \( ax + by + cz = 0 \). Thus a point \( P \) belongs to a line \( \eta \) if and only if \( \eta(P) = 0 \).

The matrix group \( \text{PGL}(3, \mathbb{C}) \) naturally acts on the set of points and the set of lines \( (\mathbb{CP}^2)^* \) of \( \mathbb{CP}^2 \). In particular, for every line \( \eta \in (\mathbb{CP}^2)^* \) and matrix \( A \in \text{PGL}(3, \mathbb{C}) \),

\[
A \cdot \eta := \eta A^{-1} \in (\mathbb{CP}^2)^*.
\]

We say that \( m \) points (resp. \( m \) lines) of \( \mathbb{CP}^2 \) are in general position if no three are collinear (resp. no three are incident). An ordered quadruple of points in general position is often referred as a projective basis, because:

- \( \text{PGL}(3, \mathbb{C}) \) is simply transitive on ordered 4–tuples of points in general position;
- \( \text{PGL}(3, \mathbb{C}) \) is simply transitive on ordered 4–tuples of lines in general position.

We fix the following dual map between points and lines of \( \mathbb{CP}^2 \):

\[
\perp: \mathbb{CP}^2 \rightarrow (\mathbb{CP}^2)^*,
\]

\[
P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto P^\perp = [x : y : z].
\]

A flag \( \mathcal{F}_j := (V_j, \eta_j) \) of \( \mathbb{CP}^2 \) is a pair consisting of a point \( V \in \mathbb{CP}^2 \) and a line \( \eta \subset \mathbb{CP}^2 \) passing through \( V \). The space of flags is denoted \( \text{FL} \). An \( m \)–tuple of flags \( \{\mathcal{F}_1, \ldots, \mathcal{F}_m\} \) is in general position if

- points and lines are in general position, respectively;
- \( \eta_i(V_j) = 0 \iff i = j \).

Henceforth, we will denote by \( \langle \mathcal{F}_1, \ldots, \mathcal{F}_m \rangle \) a cyclically ordered \( m \)–tuple of flags, as opposed to an ordered \( m \)–tuple \( (\mathcal{F}_1, \ldots, \mathcal{F}_m) \). The group of projective transformations acts on the space of flags via

\[
A \cdot \mathcal{F}_j := (A \cdot V_j, A \cdot \eta_j), \quad A \in \text{PGL}(3, \mathbb{C}).
\]

The action naturally extends to \( m \)–tuples, ordered \( m \)–tuples and cyclically ordered \( m \)–tuples of flags.
1.2 Flags and ratios

1.2.1 Triple ratio

Suppose $\mathfrak{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ is a cyclically ordered triple of flags in general position. Following [15], we define the triple ratio of $\mathfrak{F}$ as

$$\mathcal{J}(\mathfrak{F}) := \frac{\eta_0(V_1) \cdot \eta_1(V_2) \cdot \eta_2(V_0)}{\eta_0(V_2) \cdot \eta_1(V_0) \cdot \eta_2(V_1)},$$

where $V_j$ and $\eta_j$ are representatives of the projective equivalence classes. The above definition is well-defined, as it is independent of the choice of representatives, and manifestly invariant under a cyclic permutation of the flags. The following property is easy to verify.

**Lemma 2.** $\mathcal{J}$ is invariant under projective transformations, i.e. for all $A \in \operatorname{PGL}(3, \mathbb{C})$

$$\mathcal{J}(A \cdot \mathfrak{F}) = \mathcal{J}(\mathfrak{F}).$$

1.2.2 Cross ratio

Let $l \subset \mathbb{C}P^2$ be a line. Given $P_0, P_1, P_2, P_3 \in l$ with $P_0, P_1, P_2$ pairwise distinct, let $A : l \to \mathbb{C}P^1$ be the unique projective map such that $A(P_0) = \infty$, $A(P_1) = -1$ and $A(P_2) = 0$. Then the cross ratio of the ordered quadruple $(P_0, P_1, P_2, P_3)$ is

$$\operatorname{CR}(P_0, P_1, P_2, P_3) = A(P_3).$$

Just as for the triple ratio, it is easy to check that the cross ratio is invariant under projective transformation. Moreover, if $x_0, x_1, x_2, x_3$ are local coordinates for $P_0, P_1, P_2, P_3 \in l$, then

$$\operatorname{CR}(P_0, P_1, P_2, P_3) = \frac{(x_0 - x_1)(x_2 - x_3)}{(x_0 - x_3)(x_1 - x_2)}.$$ 

It follows that, if $\sigma$ is a permutation on four symbols and $\lambda = \operatorname{CR}(P_0, P_1, P_2, P_3)$, then

$$\operatorname{CR}(P_{\sigma(0)}, P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}) \in \left\{ \lambda, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda - 1}{\lambda - 1} \right\}.$$ 

In particular $\operatorname{CR}(P_3, P_2, P_1, P_0) = \operatorname{CR}(P_0, P_1, P_2, P_3)$.

Similarly, we can define the cross ratio of four incident lines via duality. Let $P \in \mathbb{C}P^2$ be a point and $l_0, l_1, l_2, l_3$ lines through $P$ with $l_0, l_1, l_2$ pairwise distinct. Then the cross ratio of the ordered quadruple $(l_0, l_1, l_2, l_3)$ is

$$\operatorname{CR}(l_0, l_1, l_2, l_3) := \operatorname{CR}(l_0^1, l_1^1, l_2^1, l_3^1).$$

A straightforward argument shows:

**Lemma 3.** Let $m$ be a line intersecting the lines $l_0, l_1, l_2, l_3$ transversely in $Q_0, Q_1, Q_2, Q_3$ respectively, then

$$\operatorname{CR}(l_0, l_1, l_2, l_3) = \operatorname{CR}(Q_0, Q_1, Q_2, Q_3).$$
The convention for the cross ratio used here was originally suggested by Fock and Goncharov [14, pg. 253], principally motivated by the geometry of the real case.

Henceforth, if \( P, Q \in \mathbb{CP}^2 \) are two points and \( m, l \subset \mathbb{CP}^2 \) are two lines, we will denote by \( PQ \) the line passing through \( P \) and \( Q \), and by \( lm \) the point of intersection between \( l \) and \( m \).

**Lemma 4.** Let \( \mathcal{F} = \langle \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \rangle \) be a cyclically ordered triple of flags in general position where \( \mathcal{F}_j = (V_j, \eta_j) \). Then

\[
\text{CR}(\eta_0, V_0V_1, V_0(\eta_1\eta_2), V_0V_2) = \beta(\mathcal{F}).
\]

**Proof.** Both cross ratio and triple ratio are projectively invariant, and the points \( V_0, V_1, V_2, \eta_1\eta_2 \) are in general position so we may assume, without loss of generality, that

\[
V_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \eta_1\eta_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

It follows that

\[
\eta_1 = \begin{bmatrix} 1 : 0 : -1 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 0 : 1 : -1 \end{bmatrix}.
\]

The projective line \( \eta_0 \) passes through \( V_0 \) but does not pass through \( V_2, V_1, \eta_1\eta_2 \), so \( \eta_0 = [t : 1 : 0] \) for some \( t \in \mathbb{C} \setminus \{0, -1\} \) (cf. Figure 1.2). By Lemma 3,

\[
\text{CR}(\eta_0, V_0V_1, V_0(\eta_1\eta_2), V_0V_2) = \text{CR}(\eta_0\eta_2, (V_0V_1)\eta_2, (V_0(\eta_1\eta_2))\eta_2, (V_0V_2)\eta_2).
\]

A direct calculation shows that

\[
\beta(\mathcal{F}) = t = \text{CR}(\eta_0\eta_2, (V_0V_1)\eta_2, (V_0(\eta_1\eta_2))\eta_2, (V_0V_2)\eta_2).
\]

This completes the proof. \( \blacksquare \)
1.2 Flags and ratios

Figure 1.3: The quadruple ratio $\hat{A}(\vec{\eta})$ can be read off both in terms of $\text{CR}(\eta_0, V_0V_3, V_0V_2, V_0V_1)$ and $\mathcal{B} (\langle V_0, \eta_0, V_2V_3 \rangle, \langle V_1, V_1V_2 \rangle)$.

1.2.3 Quadruple ratio

Let $\vec{\eta} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be an ordered quadruple of flags, $\mathcal{F}_j = (V_j, \eta_j)$. We say that $\vec{\eta}$ is in special position if

- the triples of flags $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ and $\{\mathcal{F}_0, \mathcal{F}_2, \mathcal{F}_3\}$ are in general position,
- the quadruple of points $\{V_0, V_1, V_2, V_3\}$ is in general position.

We underline that an ordered quadruple of flags in general position is always in special position, but the converse is not true. For example, special position allows $\eta_1(V_3) = 0$. It follows from the definition that $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is in special position if and only if $(\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_0, \mathcal{F}_1)$ is in special position, while $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_0)$ might not be.

Special position is a necessary and sufficient condition to define the quadruple ratio. If $\vec{\eta}$ is in special position, the quadruple ratio of $\vec{\eta}$ is

$$\hat{A}(\vec{\eta}) := \text{CR}(\eta_0, V_0V_3, V_0V_2, V_0V_1).$$

The ratio $\hat{A}(\vec{\eta})$ is sometimes referred to as the edge ratio with respect to $V_0V_2$. We give an intuitive description of this definition in §1.2.4. It follows from Theorem 4 that

$$\hat{A}(\vec{\eta}) = \mathcal{B} (\langle V_0, \eta_0, V_2V_3 \rangle, \langle V_1, V_1V_2 \rangle).$$

1.2.4 Configuration of flags

Let $\text{Conf}_3$ be the space of cyclically ordered triples of flags in general position, and let $\text{Conf}_4^*$ be the space of ordered quadruples of flags in special position, both modulo the action of $\text{PGL}(3, \mathbb{C})$.

We give parametrisations of $\text{Conf}_3$ and $\text{Conf}_4^*$ using triple ratios and quadruple ratios respectively. This will used in the proof of Theorem 7. Define the map $\tilde{\mathcal{B}} : \text{Conf}_3 \to \mathbb{CP}^1$ as follows. For $[\vec{\eta}] \in \text{Conf}_3$, choose a representative $\vec{\eta} = \langle \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \rangle$. Then $\tilde{\mathcal{B}}([\vec{\eta}]) := \mathcal{B}(\eta_1(V_3))$ is well-defined by Lemma 2. Let $\mathbb{C}^\dagger := \mathbb{C} \setminus \{0, -1\}$. 
Theorem 5 ([14]). The map $\mathcal{F}$ establishes a bijection between $\text{Conf}_3$ and $\mathbb{C}^\dagger$.

Proof. Let $[\mathcal{F}] \in \text{Conf}_3$ and fix a representative $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{F}_j = (V_j, \eta_j)$. As in Theorem 4, after a projective transformation, we can assume

$$V_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \eta_1 \eta_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\eta_1 = [1 : 0 : -1], \quad \eta_2 = [0 : 1 : -1], \quad \eta_0 = [t : 1 : 0], \quad t \in \mathbb{C}^\dagger.$$

Recall that $\mathcal{F}(\mathcal{F}) \in \mathbb{C}^\dagger$ because $\eta_0$ is a line through $V_0$ which is disjoint from $V_2, V_1$ or $\eta_1 \eta_2$. Furthermore, $[\mathcal{F}]$ is uniquely determined by $\mathcal{F}(\mathcal{F})$, so bijectivity is immediate.

Now we are going to show something similar for $\text{Conf}_4^*$ using both triple ratios and quadruple ratios. We define $g : \text{Conf}_4^* \to (\mathbb{C}P^1)^4$ as follows. For $[\mathcal{F}] \in \text{Conf}_4^*$, choose a representative $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$, with $\mathcal{F}_j = (V_j, \eta_j)$. Then $g([\mathcal{F}]) := (t_{012}, t_{023}, e_{02}, e_{20})$, where

$$t_{012} := \mathcal{F}(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2),$$
$$t_{023} := \mathcal{F}(\mathcal{F}_0, \mathcal{F}_2, \mathcal{F}_3),$$
$$e_{02} := \mathcal{A}(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \mathcal{F}((V_0, \eta_0), (V_3, V_2V_3), (V_1, V_1V_2)),$$
$$e_{20} := \mathcal{A}(\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_0, \mathcal{F}_1) = \mathcal{F}((V_2, \eta_2), (V_1, V_0V_1), (V_3, V_3V_0)).$$

The function $g$ is well-defined due to the projective invariance of triple ratio and $\mathcal{F}$ is in special position. One may visualise $[\mathcal{F}]$ as in Figure 1.4, with an additional edge crossing from $V_0$ to $V_2$. This motivates why $\mathcal{A}(\mathcal{F})$ is also called the edge ratio of the oriented edge $V_0V_2$. 

Figure 1.4: The edge ratio $e_{02}$ is relative to the edge oriented from $V_0$ to $V_2$, adjacent to the two triangles with triple ratios $t_{012}$ and $t_{023}$. 

![Figure 1.4](image-url)
1.2 Flags and ratios

Let \(\text{Conf}_4\) changing the starting flag of an ordered quadruple of points. As in Theorem 5, we assume without loss of generality that \(V_0, V_2, V_3\) and \(\eta_2, \eta_0\) are fixed at an arbitrary generic quadruple of points.

Having fixed these points, the lines \(\eta_2\) and \(\eta_0\) are determined. By Theorem 5, the line \(\eta_3\) is uniquely determined by \(t_{023} \in \mathbb{C}^1\), and any assignment to \(t_{023}\) gives rise to a unique, well-defined element of \(\text{Conf}_3\). This already ensures the injectivity of \(g\).

Recall from §1.2.3 that the quadruple ratios \(e_{02}\) and \(e_{20}\) can be expressed as triple ratios:

\[
e_{02} = \mathcal{J} (((V_0, \eta_0), (V_3, V_2 V_3), (V_1, V_1 V_2)) \quad \text{and} \quad e_{20} = \mathcal{J} (((V_2, \eta_2), (V_1, V_1 V_0), (V_3, V_3 V_0)) \right).
\]

As in Theorem 5, the lines \(V_0 V_1\) and \(V_2 V_1\) are uniquely determined by \(e_{02}, e_{20} \in \mathbb{C}^1\), so \(V_1\) is uniquely determined (cf. Figure 1.5). Given that \(V_1, \mathcal{F}_0\) and \(\mathcal{F}_2\) are now fixed, the line \(\eta_1\) is uniquely determined by \(t_{012} \in \mathbb{C}^1\), once again appealing to Theorem 5. This concludes the proof that \(g\) surjects onto \((\mathbb{C}^4)^4\).

Let \(\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\) be an ordered quadruple of flags in special position, \(\mathcal{F}_j = (V_j, \eta_j)\). We define the function \(\lambda(x, y, z) := xyz + xy + x + 1\). It is a straightforward computation to show that \(\mathcal{F}\) is in general position if and only if:

1. \(\eta_1(V_3) \neq 0\), in coordinates \(\lambda(e_{20}, t_{012}, e_{02}) \neq 0\);
2. \(\eta_3(V_1) \neq 0\), in coordinates \(\lambda(e_{02}, t_{023}, e_{20}) \neq 0\);
3. \(\eta_0(\eta_1 \eta_3) \neq 0\), in coordinates \(\lambda(t_{023}, e_{20}, t_{012}) \neq 0\);
4. \(\eta_2(\eta_1 \eta_3) \neq 0\), in coordinates \(\lambda(t_{012}, e_{02}, t_{023}) \neq 0\).

Let \(\text{Conf}_4 \subset \text{Conf}_4\) be the subset of quadruples in general position. Then \(\text{Conf}_4\) is Zariski open in \(\text{Conf}_4 \cong (\mathbb{C}^4)^4\). The cyclic group of order four \(C_4 := \langle \alpha \mid \alpha^4 = 1 \rangle\), acts on \(\text{Conf}_4\) by changing the starting flag of an ordered quadruple \((\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\) (cf. Figure 1.6). More precisely,

\[
\alpha \cdot (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) := (\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_0, \mathcal{F}_1).
\]
Figure 1.6: The change of coordinates for the action of $C_4$. The quadrilateral formed by $V_j$, $j = 0, \ldots, 3$ is depicted so that the ordering of the flags is anticlockwise.

The corresponding action of $C_4$ on $(\mathbb{C}^\dagger)^4$ can be thought of as the change of coordinates:

$$
\alpha \cdot (t_{012}, t_{023}, e_{02}, e_{20}) \ := \ (t_{123}, t_{013}, e_{13}, e_{31}),
$$

where

$$
t_{123} = \frac{t_{012} \lambda(e_{02}, t_{023}, e_{20})}{\lambda(e_{20}, t_{012}, e_{02})}, \quad t_{013} = \frac{t_{023} \lambda(e_{20}, t_{012}, e_{02})}{\lambda(e_{02}, t_{023}, e_{20})},
$$

$$
e_{13} = \frac{e_{20} + 1}{(e_{02} + 1)t_{012}e_{20}}, \quad e_{31} = \frac{e_{02} + 1}{e_{02}t_{023}(e_{20} + 1)}.
$$

This change of coordinates will be analysed again in §1.4.4.

1.3 Representation variety, character variety and decorations

For the rest of this section, $M$ will denote the interior of a compact manifold with a single boundary component, and $\Gamma = \pi(M)$ the fundamental group of $M$. Among these manifolds, we will mainly focus on the cases where $M$ is either the once-punctured torus or a once-punctured torus bundle.

1.3.1 Representations and characters

Let $\mathcal{R} = \mathcal{R}(\Gamma, \text{PGL}(3, \mathbb{C}))$ be the set of all representations from $\Gamma$ to PGL(3, $\mathbb{C}$). We recall that $\mathcal{R}$ can be viewed as an affine algebraic variety, as $\Gamma$ is finitely presented and PGL(3, $\mathbb{C}$) is affine algebraic (see for example [19]). This space is called the representation variety.

The action of PGL(3, $\mathbb{C}$) on itself by conjugation is an algebraic action. Therefore it defines an algebraic action on $\mathcal{R}$ which induces an algebraic action on its regular functions. The character variety is the algebraic quotient:

$$
\mathcal{X} = \mathcal{X}(\Gamma, \text{PGL}(3, \mathbb{C})) := \mathcal{R} / \text{PGL}(3, \mathbb{C}).
$$
1.3 Representation variety, character variety and decorations

It comes equipped with a quotient map \( \pi : R \to \mathcal{X} \), which is a regular map.

The name of this variety comes from its links with the set of characters. The character of a representation \( \rho \in R \) is the trace function

\[ \chi_{\rho} : \Gamma \to \mathbb{C}, \]

defined by \( \chi_{\rho}(\gamma) := \text{tr}^1(\overline{\rho(\gamma)}) \), where \( \rho(\gamma) \in \text{SL}(3, \mathbb{C}) \) is any lift of \( \rho(\gamma) \). The character \( \chi_{\rho}(\gamma) \) does not depend on the chosen lift, hence it is well defined. It is a result of Lawton [28] that the map

\[ \mathcal{X} \to \{ \text{characters of } \Gamma \} \]

induced by \([\rho] \mapsto \chi_{\rho}\) is a bijection.

1.3.2 Decorated varieties

The group \( \Gamma \) acts on the universal cover \( \tilde{M} \) by deck transformations and on the set of boundary components \( C \) of \( \tilde{M} \) by permutations. Given a representation \( \rho \in R \), we also have an action of \( \Gamma \) on the space of flags \( \mathcal{F}L = \{(V, \eta) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \mid \eta(V) = 0\} \), through \( \rho \).

A decoration of \( \tilde{M} \) is a map

\[ \Phi : C \to \mathcal{F}L, \]

that assigns a flag to each boundary component of \( \tilde{M} \). A decorated representation is a pair \((\rho, \Phi)\) of a representation and a decoration, such that the decoration is \( \rho \)-equivariant. In symbols,

\[ \Phi(\gamma \cdot C_j) = \rho(\gamma) \cdot \Phi(C_j), \quad \forall \gamma \in \Gamma \quad \text{and} \quad C_j \in C. \]

The space of decorations is \( \text{Dec} \), and the space of decorated representations is \( R^\times = R^\times(\Gamma, \text{PGL}(3, \mathbb{C})). \) If \( C_0 \in C \) is a boundary component of \( \tilde{M} \), and \( \Gamma_0 < \Gamma \) is the stabiliser of \( C_0 \), then a decoration \( \Phi \) of a representation \( \rho \) is uniquely determined by a choice of a flag \( \Phi(C_0) \) that is invariant under \( \rho(\Gamma_0) \). We remind the reader that \( M \) has a single boundary component. This shows that \( R^\times \) is an algebraic variety, hence we call it the decorated representation variety.

The group \( \text{PGL}(3, \mathbb{C}) \) acts naturally on decorated representations, by conjugation on the representation and left multiplication on the flags. This action is also algebraic, hence the decorated character variety is the algebraic quotient

\[ \mathcal{X}^\times = \mathcal{X}(\Gamma, \text{PGL}(3, \mathbb{C})) := R^\times // \text{PGL}(3, \mathbb{C}). \]

In the cases we are interested in, \( \Gamma_0 \) is always infinite abelian, thus \( \rho(\Gamma_0) \) has generically exactly three distinct global fixed points in \( \mathbb{CP}^2 \). This allows a maximum of six choices of invariant flags for \( \Phi(C_0) \), corresponding to all possible orderings of the set of eigenvectors. It follows that the algebraic map \( r_1 : R^\times \to R \) defined by removing the decoration is a rational branched covering of degree 6. The branching locus is the subvariety of decorated representations where \( \rho(\Gamma_0) \) has fewer than three distinct fixed points. We add that \( r_1 \) is also
1. Fock-Goncharov Coordinates On The Once-Punctured Torus

\[
\begin{array}{ccc}
\mathcal{R} & \xleftarrow{r_1} & \mathcal{R}^x \\
\downarrow & & \downarrow \\
\mathcal{X} & \xleftarrow{r_2} & \mathcal{X}^x \\
\mathcal{X}^x & \xrightarrow{s_1} & \text{Dec} \\
\downarrow & & \downarrow \\
\mathcal{X}^x & \xrightarrow{s_2} & \text{Dec}/\text{PGL}(3, \mathbb{C})
\end{array}
\]

Figure 1.7: Diagram of maps between representation varieties, character varieties and decorations.

surjective, as \( \rho(\Gamma_0) \) always admits at least a global fixed point and an invariant subspace. Analogous considerations are true for the forgetful map \( r_2 : \mathcal{X} \to \mathcal{X}^x \).

Let \( s_1 : \mathcal{R}^x \to \text{Dec} \) be map defined by forgetting the representation. This map is not surjective, but generally injective. Indeed, suppose two representations \( \rho_1, \rho_2 \) share the same decoration \( \Phi \). An element of \( \text{PGL}(3, \mathbb{C}) \) is uniquely determined by where it maps a projective basis, hence \( \rho_1 = \rho_2 \) when the image of \( \Phi \) contains at least three flags in general position. But the set of decorated representations whose decoration does not contain more than two flags in general position is an algebraic subvariety of \( \mathcal{R}^x \), hence its complement is Zariski open. Once again, similar statements are true for the forgetful map \( s_2 : \mathcal{X}^x \to \text{Dec}/\text{PGL}(3, \mathbb{C}) \).

The natural quotient maps and the forgetful maps form the commutative diagram of Figure 1.7.

1.4 Fock-Goncharov Coordinates

Let \( \mathbb{C}^\dagger := \mathbb{C} \setminus \{0, -1\} \), and let \( \mathcal{X}^x(\mathbb{T}_0) \) be the decorated character variety of the once-puncture torus \( \mathbb{T}_0 \). Given an ideal triangulation \( \mathcal{T} \) of \( \mathbb{T}_0 \) and an orientation \( \nu \), Fock and Goncharov [14] use triple ratios and quadruple ratios to construct a parametrisation of \( \mathcal{X}^x(\mathbb{T}_0) \).

1.4.1 The canonical isomorphism

For a fixed ideal triangulation \( \mathcal{T} \) of \( \mathbb{T}_0 \), let \( \Delta \) be the set of all ideal triangles and \( E \) be the set of all oriented ideal edges. We denote by \( \tilde{\mathcal{T}} \) the lift of \( \mathcal{T} \) to the universal cover \( \tilde{\mathbb{T}}_0 \) of \( \mathbb{T}_0 \).

Given a point in \( \mathcal{X}^x \), the decoration associates to each ideal vertex of \( \tilde{\mathcal{T}} \) both a point in \( \mathbb{P}^2 \) and a line though that point, a flag. Whence to each ideal triangle there is an associated triple of flags and (after choosing a cyclic order depending on the orientation \( \nu \)) a triple ratio. Similarly, to each oriented edge, there is an associated quadruple ratio of flags. This gives rise to a map \( \mathcal{X}^x(\mathbb{T}_0) \to (\mathbb{C}^\dagger)^{\Delta \cup E} \). The proof of the below theorem will make the association precise and turn it into a well-defined birational isomorphism. The Fock–Goncharov moduli space is then the set of all functions \( (\mathbb{C}^\dagger)^{\Delta \cup E} \Rightarrow \{\Delta \cup E \to \mathbb{C}^\dagger\} \).

We note that \( |\Delta \cup E| = 8 \).
**Theorem 7** (Fock-Goncharov 2006, [14]). Let $\mathcal{X}(\mathcal{T}_0)$ be the decorated character variety of the once-puncture torus $\mathcal{T}_0$. For each ideal triangulation $\mathcal{T}$ and orientation $\nu$ of $\mathcal{T}_0$, there is a canonical birational isomorphism

$$
\Psi_{\mathcal{T},\nu} : \mathcal{X}(\mathcal{T}_0) \to (\mathbb{C}^*)^{\Delta \cup E}.
$$

By fixing an orientation of $\mathcal{T}_0$, we may suppress $\nu$ from the notation and simply write $\Psi_{\mathcal{T}}$. We will call $\Psi_T([\rho, \Phi])$ the Fock–Goncharov coordinate of $[\rho, \Phi] \in \mathcal{X}(\mathcal{T}_0)$. For completeness and future reference, we include a proof of Theorem 7 which is an adaptation of the one in [6].

**Proof.** We first define $\Psi_T$. Let $[\rho, \Phi] \in \mathcal{X}(\mathcal{T}_0)$ be a decorated character. Let $t_j \in \Delta$ be an ideal triangle. Any lift $\tilde{t}_j$ of $t_j$ to $\tilde{T}$ is assigned an element $[\tilde{\gamma}_j]$ of Conf$^*_t$, cyclically ordered according to the orientation induced on $t_j$ by $\mathcal{T}_0$. Hence define

$$
\Psi_T([\rho, \Phi])(t_j) := \tilde{\gamma}([\tilde{\gamma}_j]) \in \mathbb{C}^*.
$$

There is a choice of $\tilde{t}_j$ in this definition. Any two such choices differ by a deck transformation, but since $\Phi$ is $\rho$–equivariant, any two resulting triples of flags represent the same element in Conf$^*_t$.

Let $e_j \in E$ be an oriented edge. Choose a lift $\tilde{e}_j$ of $e_j$. $\tilde{e}_j$ is shared by two triangles in $\tilde{T}$, each of which is decorated. Thereby we can consider $\tilde{e}_j$ to be uniquely associated to an element $[\tilde{\gamma}_j] := [(\mathcal{F}_{j,0}, \mathcal{F}_{j,1}, \mathcal{F}_{j,2}, \mathcal{F}_{j,3})]$ of Conf$^*_3$, ordered according to the orientation induced by $\mathcal{T}_0$, where $\mathcal{F}_{j,0}$ and $\mathcal{F}_{j,2}$ represent the vertices at the tail and head of $\tilde{e}_j$ respectively. We define

$$
\Psi_T([\rho, \Phi])(e_j) := \tilde{\gamma}([\tilde{\gamma}_j]) \in \mathbb{C}^*.
$$

As above, the choice of $\tilde{e}_j$ only changes $\tilde{\gamma}_j$ by a projective transformation and does not change its class in Conf$^*_3$.

The map $\Psi_T$ just constructed is defined almost everywhere on $\mathcal{X}(\mathcal{T}_0)$, but where the triples (reps. quadruples) of flags were not in general (resp. special) position. The set of such decorated characters is a Zariski closed subset of $\mathcal{X}(\mathcal{T}_0)$, hence we have a well-defined rational morphism

$$
\Psi_T : \mathcal{X}(\mathcal{T}_0) \to (\mathbb{C}^*)^{\Delta \cup E}.
$$

It remains to show that this map admits a rational inverse.

Let $x \in (\mathbb{C}^*)^{\Delta \cup E}$. Then $x$ assigns to each triangle and each oriented edge in $\mathcal{T}$ a number in $\mathbb{C}^*$. Lift these assignments to $\tilde{T}$. We will explicitly construct a decorated representation $(\rho, \Phi)$, whose corresponding decorated character’s image under $\Psi_T$ is $x$.

Let $t \in \Delta$. Fix a lift $\tilde{t}$ of $t$ to $\mathcal{T}_0$, with orientation inherited from $\mathcal{T}_0$. By Theorem 5, the number assigned to $\tilde{t}$ uniquely determines an element of Conf$^*_3$. Hence fix a cyclically ordered representative $\tilde{\gamma}\tilde{T}$. These three flags are representatives of the image under $\Phi$ of the vertices of $\tilde{T}$. The rest of the construction of $\Phi$ proceeds as in the proof of Theorem 6 where, given a triple of flags, a fourth flag of an adjacent triangle is uniquely determined by two edge ratios and a triple ratio. Iterating this procedure, we obtain a decoration $\Phi$ of $\mathcal{T}_0$. 

1.4 Fock-Goncharov Coordinates 17
We now define the representation $\rho : \pi_1(\mathcal{T}_0) \to \text{PGL}(3, \mathbb{C})$. For each $\gamma \in \pi_1(S)$, $\tilde{\gamma}' := \gamma(\tilde{t})$ is another lift of $t$, which is assigned another triple of flags $\tilde{\gamma}'$ with the same triple ratio as $\tilde{\gamma}$. Once again, Theorem 5 applies to provide a unique projective transformation $A_\gamma \in \text{PGL}(3, \mathbb{C})$ such that

$$A_\gamma \cdot \tilde{\gamma} = \tilde{\gamma}'$$

Hence we define the representation

$$\rho(\gamma) := A_\gamma, \quad \forall \gamma \in \pi_1(S).$$

Because $\Phi$ is defined exclusively in terms of triangle parameters and edge parameters, which are invariant under projective transformations, it follows that $(\rho, \Phi)$ is a decorated representation. In constructing $(\rho, \Phi)$, we chose a lift $\tilde{t}_j$ and a cyclically ordered triple of flags $\tilde{\gamma}$. Different choices determine decorated representations that are conjugate to $(\rho, \Phi)$, hence we have a well defined map $x \mapsto [\rho, \Phi]$. But clearly $\Psi_T((\rho, \Phi)) = x$, therefore $\Psi_T$ is a birational isomorphism.

From the proof of Theorem 7, one deduces that $\Psi_T$ is surjective, as numbers in $\mathbb{C}^+$ always defines triples of flags in general position and quadruple of flags in special position (Theorem 5 and 6), but $\Psi_T^{-1}$ is not. Indeed $\Psi_T$ is only defined on a Zariski open subset of $X$, whose complement $\mathcal{D}$ can be explicitly computed.

Let $C_0$ be a vertex of $\overline{T}$ and $\Gamma_0 \leq \pi_1(\mathcal{T}_0) = \langle \alpha, \beta \rangle$ the stabiliser of $C_0$. Given a decorated representation $(\rho, \Phi)$, let $A := \rho(\alpha)$, $B := \rho(\beta)$ and $(V_0, l_0) := \Phi(C_0)$. Then $\mathcal{D}$ is the union of the following Zariski closed subsets of $X$:

\[
\begin{align*}
\mathcal{D}_0 &= \{[\rho, \Phi] \mid A \cdot V_0 \in l_0\}, & \mathcal{D}_1 &= \{[\rho, \Phi] \mid B \cdot V_0 \in l_0\}, \\
\mathcal{D}_2 &= \{[\rho, \Phi] \mid A^{-1} \cdot V_0 \in l_0\}, & \mathcal{D}_3 &= \{[\rho, \Phi] \mid B^{-1} \cdot V_0 \in l_0\}, \\
\mathcal{D}_4 &= \{[\rho, \Phi] \mid B^{-1} A \cdot V_0 \in l_0\}, & \mathcal{D}_5 &= \{[\rho, \Phi] \mid A^{-1} B \cdot V_0 \in l_0\}, \\
\mathcal{D}_6 &= \{[\rho, \Phi] \mid [V_0, A \cdot V_0, B \cdot V_0] \text{ collinear}\}, & \mathcal{D}_7 &= \{[\rho, \Phi] \mid [A B \cdot V_0, A \cdot V_0, B \cdot V_0] \text{ collinear}\}, \\
\mathcal{D}_8 &= \{[\rho, \Phi] \mid [B^2 \cdot V_0, A \cdot V_0, B \cdot V_0] \text{ collinear}\}, & \mathcal{D}_9 &= \{[\rho, \Phi] \mid [B^2 \cdot V_0, A \cdot V_0, AB \cdot V_0] \text{ collinear}\}, \\
\mathcal{D}_{10} &= \{[\rho, \Phi] \mid [V_0, A \cdot V_0, AB \cdot V_0] \text{ collinear}\}, & \mathcal{D}_{11} &= \{[\rho, \Phi] \mid [V_0, B \cdot V_0, AB \cdot V_0] \text{ collinear}\}, \\
\mathcal{D}_{12} &= \{[\rho, \Phi] \mid [A^2 \cdot V_0, A \cdot V_0, B \cdot V_0] \text{ collinear}\}, & \mathcal{D}_{13} &= \{[\rho, \Phi] \mid [A^2 \cdot V_0, B \cdot V_0, AB \cdot V_0] \text{ collinear}\}.
\end{align*}
\]

Each $\mathcal{D}_j$ is well defined as it does not depend on a chosen representative for $[\rho, \Phi]$. They are Zariski closed subsets of $X$, corresponding to all possible configurations that do not to produce FG coordinates.

### 1.4.2 A useful convention

In this paragraph we describe a useful convention that we will adopt to simplify most of the notation in what follows.

Let $\mathcal{T}_s$ be the standard positive ideal triangulation of $\mathcal{T}_0$, with ideal triangles $\triangle_0$ and oriented edges $E_0$. Let $t_1, t_2$ be the ideal triangles in $\mathcal{T}_s$ with vertices $(0,0), (1,1), (0,1)$
1.4 Fock-Goncharov Coordinates

Figure 1.8: The once-punctured torus $T_0$ endowed with its standard positive ideal triangulation $T_+$. Decorated characters are parametrised by eight complex numbers, associated to oriented edges and triangles.

and $(0,0), (1,0), (1,1)$ respectively. Let $e_\infty, e_1, e_0$ be the ideal edges of $T_+$ with respective slopes $\infty, 1, 0$, oriented away from the origin (cf. Figure 1.8). We denote by $T$ the edge $e$ with opposite orientation. Then there is an isomorphism of vector spaces

$$\eta_0 : (C^\dagger)^{\triangle \cup E_0} \rightarrow (C^\dagger)^8$$

$$\varphi \mapsto (a, b, c, d, e, f, g, h),$$

defined by

$$a := \varphi(e_\infty), \quad b := \varphi(e_\infty), \quad c := \varphi(t_1), \quad d := \varphi(e_1),$$

$$e := \varphi(t_2), \quad f := \varphi(t_2), \quad g := \varphi(e_0), \quad h := \varphi(e_0).$$

If $T$ is another ideal triangulation, with ideal triangles $\triangle$ and oriented edges $E$, there are six orientation-preserving combinatorial isomorphisms $T \rightarrow T_+$. We consider the unique one that matches edges oriented away from the origin with largest, smallest and middle slopes, respectively. The induced bijection $\triangle \cup E \rightarrow \triangle_0 \cup E_0$ translates into an isomorphism of vector spaces $(C^\dagger)^{\triangle \cup E} \rightarrow (C^\dagger)^{\triangle_0 \cup E_0}$, which may be composed with $\eta$ to obtain the identification

$$\eta : (C^\dagger)^{\triangle \cup E} \rightarrow (C^\dagger)^8.$$

Through $\eta$, one can rewrite the Fock-Goncharov isomorphism of Theorem 7 in a canonical way as a map

$$\Psi_T : \mathcal{X}_c(T_0) \rightarrow (C^\dagger)^8.$$

1.4.3 Example: the standard positive ideal triangulation

In this paragraph, we analyse the image of a generic point $x \in (C^\dagger)^8$ under $\Psi^{-1}_{T_+}$, for the standard positive ideal triangulation $T_+$. In particular, we give an explicit description of the decorated character $[\rho, \Phi] := \Psi^{-1}_{T_+}(x)$ in terms of the complex coordinates. See [23] for a similar calculation.

Let $T_0$ be endowed with its positive standard ideal triangulation $T_+$. Recall that $\pi_1(T_0) = \langle \alpha, \beta \rangle$. For ideal triangles, edges and coordinates, we follow the notation and conventions

\begin{align*}
(0,1) & \quad (1,1) \\
(0,0) & \quad (1,0)
\end{align*}
of §1.4.2. Hence decorated characters of \( \tilde{T}_0 \) are parametrised by eight complex coordinates \((a, b, c, d, e, f, g, h) \in (\mathbb{C}^\ast)^8\), as in Figure 1.8. Let \( \tilde{t}_2 \) be a lift of the ideal triangle \( t_2 \in \Delta_0 \) to the universal cover \( \tilde{T}_0 \). Let \( \tilde{t}_1 \) be a lift of the ideal triangle \( t_1 \in \Delta_0 \), such that \( \tilde{t}_2 \) is adjacent to \( \tilde{t}_1 \), \( \alpha \cdot \tilde{t}_1 \) and \( \beta^{-1} \cdot \tilde{t}_1 \). We refer to Figure 1.9 for a local picture of \( \tilde{T}_0 \). Each \( C_j \) in Figure 1.9 is an ideal vertex of the lift \( \tilde{T}_0 \), namely a lift of the boundary component \( C \) of \( T_0 \).

As in Theorem 7, we can fix a projective basis of \( \mathbb{R}P^2 \) and assume that \( \Phi(C_j) = \mathcal{F}_j := (V_j, \eta_j) \), where

\[
V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \eta_1 \eta_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

and hence

\[
\eta_1 = \begin{bmatrix} 0 : 1 : -1 \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} 1 : 0 : -1 \end{bmatrix}.
\]

By means of Theorem 5, one finds that \( \eta_2 = \begin{bmatrix} f : 1 : 0 \end{bmatrix} \). \( V_3 \) is uniquely determined by the edge ratios \( d \) and \( e \), while \( \eta_3 \) also depends on \( c \). Explicitly

\[
V_3 = \begin{bmatrix} e(d + 1) \\ e + 1 \\ e(d + 1) + 1 \end{bmatrix} \quad \text{and} \quad \eta_3 = \begin{bmatrix} \lambda(d, c, e) : (c + 1)d + 1 : -\lambda(d, c, e) \end{bmatrix},
\]

where \( \lambda \) is the function \( \lambda(x, y, z) := xyz + xy + x + 1 \) that was mentioned in §1.2.4. In a similar fashion, one uses the ratios \( a, b \) and \( g, h \) to compute

\[
V_5 = \begin{bmatrix} a + 1 \\ -af \\ a(bf + 1) + 1 \end{bmatrix} \quad \text{and} \quad V_6 = \begin{bmatrix} -g \\ fg(h + 1) \\ fg(h + 1) + 1 \end{bmatrix}.
\]

Using the fact that \( \Phi \) is \( \rho \)-equivariant, it follows that \( \rho(\gamma) \) is the unique projective transformation mapping the three flags at the vertices of \( \tilde{t}_1 \) to the three flags of \( \gamma \cdot \tilde{t}_1 \). Letting

\( \mathcal{F}_j := (V_j, \eta_j) \),
1.4 Fock-Goncharov Coordinates

\( \gamma \in \{ \alpha, \beta \} \), we compute

\[
\rho(\alpha) = \begin{bmatrix}
ad & a(d + 1) + 1 & -a(d + 1) - 1 \\
-adf & -af(d + 1) & af(d + 1) \\
(ad(bf(ce + c + 1) + 1) & abf(cd + d + 1) + ad + a + 1 & abf(\lambda(c, d, e) + ad + a + 1)
\end{bmatrix},
\]

and

\[
\rho(\beta) = \begin{bmatrix}
-e f(cdh + dh + h + d + 1) & e(cdfgh - d - 1) & -cdefgh \\
-(e + 1) & cdfg & 0 \\
-f(e cdh + dh + h + d + 1) + 1) & e(cdfgh - d - 1) - 1 & -cdefgh
\end{bmatrix}.
\]

It is possible to use this description of \([\rho, \Phi]\) to show that \(\Psi_{T_0}^{-1}\) is injective.

Consider the map composition \(\psi : (C^1)^8 \to X(T_0)\) obtained by composing \(\Psi_{T_0}^{-1}\) and the forgetful map. This map is a 6-to-1 branched covering onto its image. As the forgetful map is surjective, we deduce that \(\psi\) also maps into a Zariski open subset of \(X(T_0)\). It is not surjective, as for example there is no inverse image of the trivial character. It is worth noticing that the image of \(\psi\) contains some abelian non-trivial characters, as for example the one obtained from the parameters

\[
a = -\frac{22}{13}, \quad b = -\frac{45}{88}, \quad c = \frac{13}{20}, \quad d = -\frac{16}{55}, \quad e = \frac{11}{9}, \quad f = \frac{20}{13}, \quad g = -\frac{143}{48}, \quad h = \frac{12}{11}.
\]

On the other hand, not every irreducible character is parametrised by \(\psi\). One example is the character \([\rho]\) with representative:

\[
\rho(\alpha) = \begin{bmatrix}
-4 & -3 & 1 \\
-1 & -2 & 1 \\
-\frac{1}{2} & -\frac{3}{2} & 1
\end{bmatrix}, \quad \rho(\beta) = \begin{bmatrix}
\frac{1}{2} & \frac{3}{2} & -1 \\
\frac{1}{2} & \frac{5}{2} & -2 \\
\frac{1}{2} & 3 & -\frac{3}{2}
\end{bmatrix}.
\]

The commutator of this representation is parabolic, therefore there is a unique choice of flag \((V_j, \eta_j)\) for each \(C_j\). It is a simple check that \(\rho(\alpha)^{-1}V_j, \rho(\beta)V_j\) and \(V_j\) are collinear, and therefore \(\rho\) cannot be parametrised by triple ratios and quadruple ratios.

1.4.4 Change of coordinates

Having fixed a triangulation \(T\) and orientation \(\nu\), Theorem 7 provides a canonical parametrisation of \(X^x(T_0)\) with complex numbers. A different choice of \(\nu\) or \(T\) may be interpreted as a change of coordinates.

Transition maps for a different orientation

The transition map associated to a switch in the orientation of \(T_0\) is simple to describe. Denote by \(-\nu\) the opposite orientation of \(\nu\). Then for all \(q \in \triangle \cup E\),

\[
\Psi_{T,-\nu}([\rho, \Phi])(q) = \frac{1}{\Psi_{T,\nu}([\rho, \Phi])(q)}.
\]
1. Fock-Goncharov Coordinates On The Once-Punctured Torus

Figure 1.10: The change of coordinates for a right, left and middle flip on the positive standard ideal triangulation $T_0$. 

Indeed triple ratios are computed with respect to flags with the opposite cyclical order, and edge ratios are computed after permuting the second and final arguments.

Transition maps for a different triangulation

The transition map induced by a change of triangulation is slightly more complicated. Henceforth we fix an orientation $\nu$ on $T_0$ and simplify the notation to $\Psi_T = \Psi_{T,\nu}$. Recall from §1.1 that any two ideal triangulations $T$ and $T'$ of $T_0$ differ by a finite sequence of edge flips. Edge flips are of three types, right $R$, left $L$ or middle $M$, depending on the slope of the edge we are flipping. Let $\Delta \cup E$ and $\Delta' \cup E'$ denote triangles and oriented edges of the triangulations $T$ and $T'$ respectively. Let $\eta: (C^\dagger)^{\Delta \cup E} \rightarrow (C^\dagger)^8$ and $\eta': (C^\dagger)^{\Delta' \cup E'} \rightarrow (C^\dagger)^8$ be the isomorphisms described in §1.4.2. Finally, let $\Theta_X: (C^\dagger)^8 \rightarrow (C^\dagger)^8$ be the coordinate change induced by a $X$ flip, for $X \in \{R, L, M\}$. Figure 1.10 shows how coordinates are rearranged for the positive standard ideal triangulation of $T_0$. As it was already underlined at the end of §1.2.4, the map $\Theta_X$ is not defined on all $(C^\dagger)^8$ but on a Zariski open subset. This is due to the difference between flags in special position and general position. We remark that the map $\Theta_X$ was studied in [15], where it was shown to preserve the subset $(\mathbb{R}_{>0})^8 \subset (C^\dagger)^8$ of positive real numbers.

If $w := \chi_0 \cdots \chi_m$ is the sequence of edge flips in order to get from $T$ to $T'$, then the coordinate change between $T$ and $T'$ is the composition map $\Theta_w := \Theta_{\chi_m} \circ \cdots \circ \Theta_{\chi_0}$. The corresponding commuting diagram is depicted in Figure 1.11.
We explicitly give $\Theta_R, \Theta_L$ below. As we explained in §1.1, a middle flip is usually the inverse of a right flip or a left flip, hence $\Theta_M$ is one of $\Theta_R^{-1}, \Theta_L^{-1}$. The only cases where that is not true, is when we perform a middle flip to go from the positive standard triangulation $T_+$ to the negative standard triangulation $T_-$, and vice-versa. Those middle flips are inverses of each others. We display $\Theta_M$ below, for the change of coordinates going from $T_+$ to $T_-$. We remind the reader that $\lambda$ is the function $\lambda(x, y, z) = xyz + xy + x + 1$ that was introduced in §1.2.4.

\[ \Theta_R(a, b, c, d, e, f, g, h) = \left( \frac{a^2(b+1)df}{(a+1)\lambda(a, f, b)} \cdot \frac{(a+1)b^2ce}{(b+1)\lambda(b, c, a)} \cdot \frac{c\lambda(a, f, b)}{\lambda(b, c, a)} \cdot \frac{a+1}{a(b+1)f}, \right. \]

\[ \left. \frac{b+1}{(a+1)bc} \cdot \frac{f\lambda(b, c, a)}{\lambda(a, f, b)} \cdot \frac{(b+1)\lambda(a, f, b)g}{a+1} \cdot \frac{(a+1)\lambda(b, c, a)h}{b+1} \right), \tag{1.1} \]

\[ \Theta_L(a, b, c, d, e, f, g, h) = \left( \frac{ac(g+1)h^2}{(g+1)\lambda(h, c, g)} \cdot \frac{bfg^2(h+1)}{(h+1)\lambda(g, f, h)} \cdot \frac{f\lambda(h, c, g)}{\lambda(g, f, h)} \cdot \frac{h+1}{c(g+1)h}, \right. \]

\[ \left. \frac{g+1}{fg(h+1)} \cdot \frac{c\lambda(g, f, h)}{\lambda(h, c, g)} \cdot \frac{d(g+1)\lambda(h, c, g)}{h+1} \cdot \frac{e(h+1)\lambda(g, f, h)}{g+1} \right), \tag{1.2} \]

\[ \Theta_M(a, b, c, d, e, f, g, h) = \left( \frac{cd^2(e+1)g}{(d+1)\lambda(d, c, e)} \cdot \frac{e^2f(h+1)}{(e+1)\lambda(e, f, d)} \cdot \frac{f\lambda(d, c, e)}{\lambda(e, f, d)} \cdot \frac{d+1}{cd(e+1)} \right. \]

\[ \left. \cdot \frac{e+1}{ef(d+1)} \cdot \frac{c\lambda(e, f, d)}{\lambda(d, c, e)} \cdot \frac{b(e+1)\lambda(d, c, e)}{d+1} \cdot \frac{a(d+1)\lambda(e, f, d)}{e+1} \right). \tag{1.3} \]
1. Fock-Goncharov Coordinates On The Once-Punctured Torus
Chapter 2

Fock-Goncharov Coordinates On Once-Punctured Torus Bundles

Once-punctured torus bundles are the simplest class of fibred 3–manifolds. Due to many geometric and combinatorial properties, this class of manifolds is particularly interesting to study, and often a good source of examples.

In this chapter we build on Fock and Goncharov’s coordinates to parametrise some irreducible components of the decorated character variety of every hyperbolic once-punctured torus bundle $M_f$. One of the main ingredients is described in §2.2, a special ideal triangulation of $M_f$ sometimes called the monodromy ideal triangulation. We highlight some of its important combinatorial properties, then dive straight into an explicit description of the parametrisation §2.3. We conclude with a discussion on two special decorated characters that are naturally found from a simple analysis of this coordinate system.

2.1 Once-punctured torus bundles $M_f$

Let $\mathbb{T}_0$ be the once-punctured torus endowed with its differential structure and standard orientation $\nu$. The mapping class group of $\mathbb{T}_0$ is the group $\text{MCG} = \text{MCG}(\mathbb{T}_0)$ of isotopy classes of orientation preserving diffeomorphisms $f : \mathbb{T}_0 \to \mathbb{T}_0$. For $[f] \in \text{MCG}$, the once-punctured torus bundle $M_f$ is the differentiable oriented 3–manifold $M_f := \mathbb{T}_0 \times [0, 1]/\sim$, where $(x, 0) \sim (y, 1)$ if and only if $y = f(x)$ for some $x \in \mathbb{T}_0$. $M_f$ is a special fiber bundle over the circle, with fiber space $\mathbb{T}_0$, well-defined up to diffeomorphism. We observe that conjugate automorphisms of $\mathbb{T}_0$ induce diffeomorphic bundles. Indeed, if $[f] = [h^{-1}gh]$ then the map $M_f \to M_g$ defined by $(x, t) \mapsto (h(x), t)$ is a diffeomorphism, and $M_f \cong M_g$.

The natural identification of $\mathbb{T}_0$ with the square spanned by $i$ and $j$ in $\mathbb{R}^2$ induces an isomorphism $H_1(\mathbb{T}_0, \mathbb{Z}) \cong \mathbb{Z}^2$, between the first homology group and the integer lattice. Every $[f] \in \text{MCG}$ descends to an automorphism of homology $[f]_* : H_1(\mathbb{T}_0, \mathbb{Z}) \to H_1(\mathbb{T}_0, \mathbb{Z})$, 

25
hence there is a map \( \varphi : \text{MCG} \to \text{Aut}(\mathbb{Z}^2) \cong \text{GL}(2, \mathbb{Z}) \). \( \varphi \) is well known to be an isomorphism \( \text{MCG} \cong \text{SL}(2, \mathbb{Z}) \), onto the subgroup of matrices with determinant one, hence each map \([f] \in \text{MCG}\) has well-defined eigenvalues in \( \mathbb{C} \) (cf. [12]). This characterisation is fundamental to study the geometry of \( \mathcal{M}_f \), as for example it helps discerning hyperbolic bundles. Hyperbolicity of once-punctured torus bundles was first studied by Jørgensen, in an unpublished work from 1975 [22]. A complete classification can be deduced as a particular case from Thurston’s Hyperbolization Theorem [31].

**Theorem 8** (Otal, 1996 [31]). \( \mathcal{M}_f \) admits a finite volume, complete hyperbolic metric if and only if \([f]\) has two distinct real eigenvalues.

The element \([f]\) has distinct real eigenvalues if and only if \( (\text{tr}[f])^2 > 4 \). If the trace is in \( \{-1,0,1\} \), then \([f]\) has finite order and \( \mathcal{M}_f \) is Seifert fibred. While if \( \text{tr}[f] = \pm 2 \), then \( f \) preserves a non-trivial simple closed curve in the punctured torus, which defines an incompressible torus or Klein bottle in \( \mathcal{M}_f \). In both cases we get an obstruction to the existence of the hyperbolic metric. An elementary and constructive proof of the other cases can be found in [21].

### 2.2 The monodromy ideal triangulation

Let \( \mathcal{M}_f \) be a hyperbolic once-punctured torus bundle. In this section we deal with a canonical realisation of \( \mathcal{M}_f \) as an ideal triangulation, described by Floyd and Hatcher in [13], called the *monodromy ideal triangulation* of \( \mathcal{M}_f \). The rich combinatorial structure of this triangulation has led to many topological-geometric results. For example, Lackenby shows in [27] that every monodromy ideal triangulation is *geometrically canonical* in the sense of Epstein-Penner, namely it is topologically dual to the Ford-Voronoi domain of \( \mathcal{M}_f \). Closely related work can be found also in the article [2]. In [21], Guéritaud uses Casson’s approach of angle structures and volume optimisation to recover Thurston’s hyperbolicity of once-punctured torus bundles, by showing that the monodromy ideal triangulation is geometric, i.e. it can be realised by straight hyperbolic ideal tetrahedra glued via hyperbolic isometries.

For \( \mathcal{M}_f \) hyperbolic, Theorem 8 implies that the eigenvalues of \([f]\) are distinct with the same sign. To simplify the construction of the monodromy triangulation of \( \mathcal{M}_f \), we are going to make the further assumption that the eigenvalues are positive. This will not cause any loss of generality: if \([f]\) has two negative eigenvalues, then \([-f]\) has positive eigenvalues, and the monodromy triangulation of \( \mathcal{M}_f \) can be easily deduced from the monodromy triangulation of \( \mathcal{M}_{-f} \). We will make that more precise at the end of §2.2.2.

#### 2.2.1 The flip sequence of \( \mathcal{M}_f \)

Let \( f : \mathbb{T}_0 \to \mathbb{T}_0 \) be a diffeomorphism of the once-punctured torus. By acting on the set of ideal triangulations of \( \mathbb{T}_0 \), \( f \) induces an isomorphism of the Farey tree \( \mathcal{F}^* \). Every isomorphism of a simplicial tree has either a fixed point, or leaves invariant a unique copy of \( \mathbb{R} \), called *axis*. The former case happens when \( \text{tr}([f]) \in \{-1,0,1\} \) and the action is periodic.
In the latter case, let \( V_0 \) be a vertex on the axis. The unique shortest path in \( F^+ \) from \( V_0 \) to \( f(V_0) \) runs along the axis, and naturally corresponds to a sequence of edge flips. When \( \text{tr}([f])^2 = 4 \), the axis has a unique endpoint on the boundary of the hyperbolic plane, and the action is parabolic. Finally we observe that \(-f\) acts on \( F^+ \) as \( f \). Hence we will only consider automorphisms with distinct positive real eigenvalues.

After conjugating \( f \), one can assume that \( V_0 \) corresponds to the standard positive ideal triangulation \( T_0 \) and the axis does not run through any negative triangulation. It follows from Lemma 1 that \( f(T_0) \) differs from \( T_0 \) by a unique sequence \( w_f \) of right \( R \) and left \( L \) flips. Finally, when the eigenvalues of \( f \) are distinct, \( w_f \) always contains at least one right flip and one left flip. In other words, there exist \( a_j, b_j, k \in \mathbb{N} \setminus \{0\} \) and \( c \in \mathbb{N} \) such that

\[
    w_f = R^{a_0} L^{b_0} \ldots R^{a_k} L^{b_k} R^c \quad \text{or} \quad w_f = L^{a_0} R^{b_0} \ldots L^{a_k} R^{b_k} L^c.
\]

We say that \( w_f \) is the \textit{flip sequence} of \( f \) or of \( M_f \). Its \textit{length} is the total number of edge flips, namely \( c + \sum_{j=0}^{k} (a_j + b_j) \). Under the canonical isomorphism \( \text{MCG}(\mathcal{T}_0) \cong \text{SL}_2(\mathbb{Z}) \), a right flip and a left flip correspond to the matrices

\[
    [f_R] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad [f_L] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Consequently, if \( f \) is a diffeomorphism with flip sequence \( w_f = R^{a_0} L^{b_0} \ldots R^{a_k} L^{b_k} R^c \) (the other case being similar), then

\[
    [f] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_0} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b_0} \ldots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_k} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b_k} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^c.
\]

### 2.2.2 The triangulation

The following description of the monodromy ideal triangulation is adapted from [21].

The \textit{standard ideal tetrahedron} \( \sigma \) is topologically a compact tetrahedron with its vertices removed. One can picture \( \sigma \) as a square with its two diagonals, as in Figure 2.1. Oriented simplices of \( \sigma \) are determined by an ordering of the vertices, hence we will refer to them by the notation \( \sigma(i), \sigma(ij), \sigma(ijk), \sigma(ijkl) \). Sometimes we will use the same notation for the unoriented counterparts, but only when it is clear form the context that we ignore the orientation. By identifying the pair of opposite edges \( \sigma(13), \sigma(24) \) and \( \sigma(12), \sigma(34) \), the boundary of \( \sigma \) becomes the union of two \textit{pleated surfaces}, homeomorphic to the once-punctured torus \( \mathcal{T}_0 \). The \textit{top} pleated surface \( \sigma(\mathcal{T}_0)_+ \) is made up of the two ideal triangles \( \sigma(143), \sigma(124) \), while the \textit{bottom} pleated surface \( \sigma(\mathcal{T}_0)_- \) is made up of the two ideal triangles \( \sigma(123), \sigma(324) \). Thus the ideal triangulation of \( \sigma(\mathcal{T}_0)_+ \) is obtained from \( \sigma(\mathcal{T}_0)_- \) by an edge flip along \( \sigma(23) \).

Consider the once-punctured torus \( \mathcal{T}_0 \) with some ideal triangulation \( \mathcal{T} \). We say that the tetrahedron \( \sigma \) \textit{layers on} \( \mathcal{T}_0 \) if the bottom pleated surface of \( \sigma \) is glued to \( \mathcal{T}_0 \) via an
orientation-preserving combinatorial isomorphism, called the \textit{layering}. Let \( e \) be an oriented edge of \( \mathcal{T} \). We will say that \( \sigma \) layers on \( T_0 \) along \( e \), if the chosen layering identifies \( e \) with the edge \( \sigma(23) \). In general, there are six possible ways to layer \( \sigma \) on \( T_0 \), one for each oriented edge of \( \mathcal{T} \). To simplify the notation we will make a further distinction. We will say that a layering of \( \sigma \) is a \((\text{right}) R\) layering (resp. \((\text{left}) L\) layering) if \( \sigma \) layers along the edge with largest (resp. smallest) slope, oriented towards (resp. away from) the origin in \( T_0 \). The motivation behind this notation is clear: if \( \sigma \) right layers (resp. left layers) on \( T_0 \), the ideal triangulations of \( \sigma(T_0)_+ \) is obtained from \( \sigma(T_0)_- \) by a right flip (resp. left flip).

Let \( f \) be an element of \( \text{SL}_2(\mathbb{Z}) \) with two distinct positive real eigenvalues and let \( w_f \) be the flip sequence of \( f \). Suppose \( w_f \) has length \( m \). Now we describe how to construct the monodromy triangulation of the hyperbolic once-punctured torus bundle \( M_f \). Let \( T_0 \) be the once–punctured torus endowed with its negative standard ideal triangulation \{0, \(-1\), \(\infty\}\}. Let \( \sigma_0 \) be a copy of the standard ideal tetrahedron layered on \( T_0 \) along the edge of slope \(-1\), oriented as in Figure 2.2. Then the top pleated surface \( \sigma_0(T_0)_+ \) is triangulated as the positive standard ideal triangulation \( T_0 \). For each letter \( X_j \) in \( w_f \), \( j = 1, \ldots, m \), reading from left to right, we perform an \( X_j \) layering a copy of the standard ideal tetrahedron \( \sigma_j \) on \( \sigma_{j-1}(T_0)_+ \).

The space obtained by stacking these tetrahedra is naturally homeomorphic to \( T_0 \times I \). The last top pleated surface is \( \sigma_m(T_0)_+ \); its triangulation \( T_m \) is obtained from \( T_0 \) by performing the sequence of edge flips \( w_f \). It follows that \( T_m = f(T_0) \), and \( f \) induces an identification between \( \sigma_0 \) and \( \sigma_m \) which makes \( T_0 \times I \) into \( M_f \). The \textit{monodromy triangulation} of \( M_f \) is the ideal triangulation consisting of the tetrahedra \( \sigma_0, \ldots, \sigma_{m-1} \) and the face pairings inherited from the layering construction.

\textbf{Remark 9.} Recall from §2.2.1 that \( f \) and \( -f \) act in the same way on the Farey tree, hence they share the same flip sequence. It follows that the monodromy triangulation of \( M_{-f} \) differs from the one of \( M_f \) only in the way \( \sigma_0 \) and \( \sigma_m \) are identified. More precisely, one can construct \( M_{-f} \) by composing the identification \( f \) between \( \sigma_0 \) and \( \sigma_m \) with a rotation by the angle \( \pi \).

The layering construction induces a natural cyclic ordering of the tetrahedra, thus they will often be indexed modulo \( m \). Similarly, one should think of the flip sequence \( w_f \) as a
cyclic word, with a preferred starting point.

For future reference, we introduce the following notation. A tetrahedron $\sigma_j$ of the monodromy triangulation is said to be of type $R$ (resp. type $L$) if the next tetrahedron $\sigma_{j+1}$ is layered by a right (resp. left) layering. We will sometimes record the type of $\sigma_j$ by writing $\sigma_j^R$ or $\sigma_j^L$.

### 2.2.3 Combinatorics around the edges

Let $T$ be the monodromy ideal triangulation of the once-punctured torus bundle $M_f$, and let $m$ be the length of its flip sequence $w_f$. We recall from §2.2.2 that $T$ is made up of $m$ tetrahedra $\sigma_0, \ldots, \sigma_{m-1}$, glued together by the layering construction. We denote by $\pi$ the natural quotient map $\pi : \sqcup_j \sigma_j \rightarrow T \cong M_f$, defined by the face pairings.

The space $M_f$ is the interior of a compact 3–manifold with torus boundary, so its Euler characteristic is zero. It follows that $T$ has as many edges as tetrahedra, namely $m$. Nevertheless, each edge may be represented by multiple edges in each tetrahedron. The valence of an edge is the size of its inverse image under $\pi$.

We are now going to describe the local structure of the edges in $T$. We recall that each tetrahedron $\sigma_j$ is a copy of the standard ideal tetrahedron $\sigma$ via a canonical identification, hence it inherits labels at the vertices from $\sigma$.

Consider the edge $\sigma_0(14)$ of $\sigma_0$, and let $e_0$ be the edge in $T$ such that $\sigma_0(14) \in \pi^{-1}(e_0)$. Suppose that $\sigma_0 = \sigma_0^L$ is of type $L$. Let $\sigma_1^R, \ldots, \sigma_n^R$, $n_0 \geq 0$, be the (possibly empty) sequence of tetrahedra of type $R$ layered on top of $\sigma_0^L$, such that $\sigma_{n_0+1}^L$ is of type $L$. This sequence corresponds to a subsequence $LR^{n_0}L$ in the word $w_f$ (thought of as a cyclic word). By definition, $\sigma_1$ left layers on $\sigma_0$, thus $\sigma_1(12), \sigma_1(34) \in \pi^{-1}(e_0)$. For every $2 \leq j \leq n_0+1$, $\sigma_j$ right layers on $\sigma_{j-1}$, therefore $\sigma_j(12), \sigma_j(34) \in \pi^{-1}(e_0)$. Finally, $\sigma_{n_0+2}$ left layers on $\sigma_{n_0+1}$, closing up the sequence of tetrahedra around $e_0$ with the edge $\sigma_{n_0+2}(23)$. Locally around $e_0$, the tetrahedra $\sigma_0, \ldots, \sigma_{n_0+2}$ glue to form a ribbon, where $\sigma_0$ and $\sigma_{n_0+2}$ appear once, while every other tetrahedron appears twice. See Figure 2.3 for a cross section of a neighbourhood of $e_0$. The simplex $\sigma_0$ (resp. $\sigma_{n_0+2}$) is the bottom (resp. top) of the ribbon, and every other simplex $\sigma_j$ constitutes a loop on each side. We deduce that the valence of $e_0$ is $2n_0 + 4$.

An analogous picture arises when we assume that $\sigma_0$ is of type $R$, with the difference that every tetrahedron of type $R$ is now of type $L$, and vice versa (cf. Figure 2.4). Furthermore, one may replace $\sigma_0$ with any other tetrahedron in $T$ and make the same definitions. For future reference, we summarise all of the above in the following Lemma.

**Lemma 10.** Every edge $e_j$ in $T$ corresponds to a unique subsequence $LR^{n_j}L$ or $RL^{n_j}R$ in $w_f$, $n_j \geq 0$, and a unique ribbon of tetrahedra $\sigma_j, \ldots, \sigma_{j+n_j+2}$. The simplex $\sigma_j$ is the bottom of the ribbon, while $\sigma_{j+n_j+2}$ is the top of the ribbon, and every other tetrahedron in between constitutes a loop on each side. Hence the valence of $e_j$ is $2n_j + 4$.

We remark that uniqueness of the ribbon follows from the fact that the bottom of the ribbon is the only tetrahedron in $T$ whose edge (14) is a representative of $e_j$. Similarly,
the top of the ribbon is the only tetrahedron whose edge (23) belongs to $\pi^{-1}(e_j)$. A simple counting argument shows that there is a bijection between the set of tetrahedra and the set of edges, thus associating every edge to its unique ribbon.

Figure 2.3: A cross section of the ribbon around $e_0$ for $\sigma_0 = \sigma_0^L$.

Figure 2.4: A cross section of the ribbon around $e_0$ for $\sigma_0 = \sigma_0^R$.

2.2.4 Example: the figure eight knot complement

The figure eight knot complement $K_8$ is the space obtained by removing a closed tubular neighbourhood of the figure eight knot from the three-sphere. Topologically, it is homeomorphic to the once-punctured torus bundle associated to the class $[f_8] := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

The matrix $[f_8]$ has two distinct real eigenvalues, thus $K_8$ is hyperbolic (cf. Theorem 8). Its flip sequence is $w_{f_8} = \mathcal{R}\mathcal{L}$, whence the corresponding monodromy ideal triangulation $\mathcal{T}_8$ has two tetrahedra: $\sigma_0^R$ of type $\mathcal{R}$ and $\sigma_0^L$ of type $\mathcal{L}$ (see Figure 2.5). As a cyclic word, $w_{f_8}$ has a subsequence $\mathcal{L}\mathcal{R}\mathcal{L}$ and a subsequence $\mathcal{R}\mathcal{L}\mathcal{R}$, corresponding to the two edges $e_R$ and $e_L$ of $\mathcal{T}_8$ respectively (cf. Lemma 10). Both edges have valence six. The ribbon of tetrahedra around $e_R$ is $\sigma_0^R, \sigma_1^L, \sigma_0^R, \sigma_1^L$, as depicted in Figure 2.6.

Figure 2.5: The monodromy ideal triangulation of the figure eight knot complement $K_8$.

Figure 2.6: The ribbon of tetrahedra around the red edge $e_R$, viewed from the vertex $\sigma_0^R(4)$. 
2.3 FG coordinates on $M_f$

The fundamental group of the once-punctured torus is the free group in two generators $\langle \alpha, \beta \rangle \cong \pi_1(\mathbb{T}_0)$. Given a diffeomorphism $f : \mathbb{T}_0 \to \mathbb{T}_0$, we denote by $f_* : \langle \alpha, \beta \rangle \to \langle \alpha, \beta \rangle$ the induced automorphism on the fundamental group. The manifold $M_f$ is a special fiber bundle over the circle, with fiber space $\mathbb{T}_0$, hence its fundamental group is an HNN extension of $\pi_1(\mathbb{T}_0)$, relative to $f_*$. By identifying $\mathbb{T}_0$ with $\mathbb{C}$, we recall from §1.3.2 that the forgetful maps $\pi : M_f \to \mathbb{T}_0$ are generally injective, therefore $\pi_1(M_f) = \langle \alpha, \beta, \tau \mid \tau \alpha \tau^{-1} = f_*(\alpha), \tau \beta \tau^{-1} = f_*(\beta) \rangle$.

where $\tau$ is represented by the base circle of the fiber bundle. This extension comes naturally equipped with regular maps

$$
\iota_1 : \mathcal{R}(M_f, \operatorname{PGL}(3, \mathbb{C})) \to \mathcal{R}(\mathbb{T}_0, \operatorname{PGL}(3, \mathbb{C})),
\iota_2 : \mathfrak{X}(M_f, \operatorname{PGL}(3, \mathbb{C})) \to \mathfrak{X}(\mathbb{T}_0, \operatorname{PGL}(3, \mathbb{C})),
$$

defined by restricting a representation or character to the subgroup $\pi_1(\mathbb{T}_0) < \pi_1(M_f)$. Henceforth, we will drop the reference to $\operatorname{PGL}(3, \mathbb{C})$ in the notation above and simply write, for example, $\mathfrak{X}(M_f) = \mathfrak{X}(M_f, \operatorname{PGL}(3, \mathbb{C})).$

A representation $\rho \in \mathcal{R}(\mathbb{T}_0)$ extends to a representation $\rho' \in \mathcal{R}(M_f)$ if and only if there is $T \in \operatorname{PGL}(3, \mathbb{C})$ such that

$$
T \rho(\alpha) T^{-1} = \rho(f_*(\alpha)) \quad \text{and} \quad T \rho(\beta) T^{-1} = \rho(f_*(\beta)).
$$

In that case, one defines $\rho'$ as $\rho$ on $\langle \alpha, \beta \rangle$ and $\rho'(t) := T$. In general, existence or uniqueness of such $T$ are not guaranteed, therefore $\iota_j$ is neither injective nor surjective. On the other hand, the subset of $\mathcal{R}(M_f)$ where the centraliser of $\rho(f_*(\alpha))$ and $\rho(f_*(\beta))$ is trivial is Zariski closed, therefore $\iota_j$ is generally one-to-one.

Let $\varepsilon : \mathbb{T}_0 \hookrightarrow M_f$ be the embedding $x \mapsto (x, 0)$. We fix a lift $\tilde{\varepsilon} : \tilde{\mathbb{T}}_0 \hookrightarrow \tilde{M}_f$ of $\varepsilon$. Then $\tilde{\varepsilon}$ induces a bijection $\mathcal{C}(\tilde{\mathbb{T}}_0) \to \mathcal{C}(\tilde{M}_f)$, between the sets of boundary components of $\tilde{\mathbb{T}}_0$ and $\tilde{M}_f$. Whence we have a bijection between decorations of $M_f$ and decorations of $\mathbb{T}_0$, which extends $\iota_1$ and $\iota_2$ to the regular maps

$$
\iota_1^\kappa : \mathcal{R}^\kappa(M_f) \to \mathcal{R}^\kappa(\mathbb{T}_0) \quad \text{and} \quad \iota_2^\kappa : \mathfrak{X}^\kappa(M_f) \to \mathfrak{X}^\kappa(\mathbb{T}_0).
$$

We recall from §1.3.2 that the forgetful maps $\mathcal{R}^\kappa \to \operatorname{Dec}$ and $\mathfrak{X}^\kappa \to \operatorname{Dec}/\operatorname{PGL}(3, \mathbb{C})$ are generally injective, therefore $\iota_1^\kappa, \iota_2^\kappa$ are also generally injective.

2.3.1 A parametrisation of $\mathfrak{X}^\kappa(M_f)$

In §1.4.1 we recalled Fock and Goncharov’s parametrisation of the decorated character variety $\mathfrak{X}^\kappa(\mathbb{T}_0)$. As the morphism $\iota_2^\kappa$ is generally injective, one may see $\mathfrak{X}^\kappa(M_f)$ as a subvariety of $\mathfrak{X}^\kappa(\mathbb{T}_0)$ and study its coordinates. In this section we make that more precise and give an explicit description of some irreducible components of $\mathfrak{X}^\kappa(M_f)$. 
Let $T_f$ be the monodromy ideal triangulation of $M_f$, with tetrahedra $\sigma_0, \ldots, \sigma_{m-1}$. We fix a diffeomorphism $T_f \to M_f$ which identifies the top pleated surface $\sigma_0(T_0)_+$ with $T_0 \times \{0\} \subset M_f$. Then the once-punctured torus $T_0 \times \{0\}$ is equipped with its positive standard ideal triangulation $T_*$. Theorem 7 provides a birational parametrisation

$$\Psi_{T_*} : \mathcal{X}^+(T_0) \to (\mathbb{C}^*)^8,$$

that can be composed with $i^*_2$ to get the rational map

$$\Psi_f := (\Psi_{T_*} \circ i^*_2) : \mathcal{X}^+(M_f) \to (\mathbb{C}^*)^8.$$

A priori, it is possible that for an irreducible component $U$ of $\mathcal{X}^+(M_f)$ the image $i^*_2(U)$ may be contained in the Zariski closed set where $\Psi_{T_*}$ is not defined, and hence $\Psi_f(U)$ is empty.

We are now going to give a description of the union of some irreducible components of $\Psi_f(\mathcal{X}^+(M_f))$, as a subvariety of $(\mathbb{C}^*)^8$.

Let $[\rho, \Phi] \in \mathcal{X}^+(T_0)$ be a decorated character and let $w_f = \chi_1 \cdots \chi_m$ be the flip sequence of $M_f$, where $\chi_j \in \{L, R\}$. When defined, $x_0 := \Psi_{T_*}([\rho, \Phi])$ parametrises the decorated character $[\rho, \Phi]$ with respect to the pleated surface $\sigma_0(T_0)_+$, which is triangulated as $T_*$. The ideal triangulation $T_f$ of the top pleated surface $\sigma_f(T_0)_+$ differs from $T_*$ by the sequence of flips $\chi_1 \cdots \chi_j$. Thus $x_f := \Psi_{T_*}([\rho, \Phi])$ parametrises the decorated character $[\rho, \Phi]$ with respect to $\sigma_f(T_0)_+$. We recall from §1.4.3 that one may explicitly compute $x_f = \Theta_{\chi_j} \cdots \Theta_{\chi_1}(x_0)$. The ideal triangulation $T_m$ is obtained from $T_{m-1}$ by performing the last edge flip in the sequence $w_f$. Let $x_m := \Psi_{T_m}([\rho, \Phi])$ be the corresponding coordinate, namely $x_m = \Theta_{\chi_m} \cdots \Theta_{\chi_1}(x_0)$.

**Lemma 11.** Let $[\rho, \Phi]$ be a decorated character for which $x_0 = \Psi_{T_*}([\rho, \Phi])$ is defined. Then $[\rho, \Phi]$ is in the image of $i^*_2$ if and only if $x_0$ is a fixed point of the function obtained by composing the edge flips of $w_f$, i.e.

$$x_0 = \Theta_{\chi_m} \cdots \Theta_{\chi_1}(x_0) = x_m.$$

**Proof.** The diffeomorphism $f : T_0 \to T_0$ maps the ideal triangulation $T_*$ to the ideal triangulation $T_m$. It induces an automorphism $f_* : \pi_1(T_0) \to \pi_1(T_0)$ of the fundamental group and a permutation $f : C(T_0) \to C(T_0)$ of boundary components such that

$$f_* (\gamma) \cdot \tilde{f}(C_j) = \tilde{f}(\gamma \cdot C_j), \quad \forall \gamma \in \pi_1(T_0) \quad \text{and} \quad C_j \in C(T_0).$$

Consequently, we define an action of $f$ on the set of representations and decorations by

$$f \cdot \rho := \rho \circ f_* \quad \text{and} \quad f \cdot \Phi := \Phi \circ \tilde{f}.$$ 

If $(\rho, \Phi)$ is a decorated representation, then $f \cdot (\rho, \Phi) := (f \cdot \rho, f \cdot \Phi)$ is also a decorated representation, hence $f$ acts on the space of decorated representations and decorated characters. We remark that the Fock-Goncharov coordinate of $f \cdot [\rho, \Phi]$ with respect to $T_m$ is the same as the one of $[\rho, \Phi]$ with respect to $T_*$. In symbols

$$x_0 = \Psi_{T_m}([f \cdot \rho, f \cdot \Phi]).$$
Now suppose $[\rho, \Phi]$ is in the image of $\iota_2^\tau$ and let $[\rho', \Phi']$ be an inverse image. Then for all $\gamma \in \pi_1(T_0)$,
\[ f \cdot \rho(\gamma) = \rho(f_*(\gamma)) = \rho(\tau \gamma \tau^{-1}) = \rho'(\tau) \rho(\gamma) \rho'(\tau)^{-1}, \]
and for all $C_j \in C(T_0) = C(M_f)$
\[ f \cdot \Phi(C_j) = \Phi(\tilde{f}(C_j)) = \Phi(\tau \cdot C_j) = \rho'(\tau) \Phi(C_j), \]
where $\tau \in \pi(M_f)$ is the element represented by the base circle of the fiber bundle. Hence $f \cdot [\rho, \Phi] = [\rho'(\tau) \rho(\gamma) \rho'(\tau)^{-1}, \rho'(\tau) \Phi(C_j)] = [\rho, \Phi]$ and
\[ x_0 = \Psi_{T_m}([f \cdot \rho, f \cdot \Phi]) = \Psi_{T_m}([\rho, \Phi]) = x_m. \]
Conversely, suppose $x_0 = x_m$. By Theorem 7, the decorated characters $f \cdot [\rho, \Phi]$ and $[\rho, \Phi]$ are the same, hence there exists a projective transformation $T \in \text{PGL}(3, \mathbb{C})$ such that
\[ T \cdot \rho \cdot T^{-1} = f \cdot \rho \quad \text{and} \quad T \cdot \Phi = f \cdot \Phi. \]
We define $\rho' : (\alpha, \beta, \tau) \rightarrow \text{PGL}(3, \mathbb{C})$ by
\[ \rho'(\alpha) := \rho(\alpha), \quad \rho'(\beta) := \rho(\beta), \quad \rho'(\tau) = T. \]
We remark that for all $\gamma \in \pi_1(T_0)$,
\[ \rho'(\tau \gamma \tau^{-1}) = \rho'(f_*(\gamma)) = \rho(f_*(\gamma)) = f \cdot \rho(\gamma) = T \rho(\gamma) T^{-1} = \rho'(\tau) \rho'(\gamma) \rho'(\tau)^{-1}, \]
and for all $C_j \in C(T_0) = C(M_f)$,
\[ \rho'(\tau) \Phi(C_j) = T \Phi(C_j) = f \cdot \Phi(C_j) = \Phi(\tilde{f}(C_j)) = \Phi(\tau \cdot C_j), \]
therefore $(\rho', \Phi) \in \mathcal{M}_X(M_f)$ and $\iota_2^\tau(\rho', \Phi) = [\rho, \Phi]$. ■

**Corollary 12.** Let $U$ be an irreducible component of $\mathcal{X}^X(M_f)$ such that $\Psi_f(U)$ is not empty. Then $U$ is birationally isomorphic to an irreducible subvariety of $(\mathbb{C}^\tau)^8$ via $\Psi_f$.

**Proof.** Let $V$ be the subvariety of $\mathcal{X}^X(M_f)$ where $\iota_2^\tau$ has degree bigger than one, and let $V'$ be the subvariety of $\mathcal{X}^X(T_0)$ where $\Psi_{T_0}$ is not defined. The intersection $\iota_2^\tau(\mathcal{X}^X(M_f) \setminus V) \cap V'$ is a Zariski closed subset of $\iota_2^\tau(\mathcal{X}^X(M_f) \setminus V)$, possibly equal to $\iota_2^\tau(\mathcal{X}^X(M_f) \setminus V)$. If $U$ is an irreducible component of $\mathcal{X}^X(M_f)$ such that $\Psi_f(U)$ is not empty, then $\iota_2^\tau(U) \cap V'$ is a proper subvariety of $\iota_2^\tau(U)$ and the restriction of $\Psi_f$ to $U$ is generally one-to-one.

It follows from Lemma 11 that $\Psi_{T_0}(\iota_2^\tau(\mathcal{X}^X(M_f) \setminus V) \setminus V')$ is the set of fixed points of the function obtained by composing the edge flips of $w$. Since edge flips are quotients of polynomials (cf. 1.4.4), we deduce that the closure of $\Psi_{T_0}(\iota_2^\tau(\mathcal{X}^X(M_f) \setminus V) \setminus V')$ is a subvariety of $(\mathbb{C}^\tau)^8$, and $U$ is birationally isomorphic to one of its irreducible components. ■
An important remark is in order. In §1.4.4 we underlined the fact that the change of coordinates $\Theta$ due to an $\mathcal{X}$ flip is not defined on all $(\mathbb{C}^\times)^8$ but on a Zariski open subset. This is because some flags are only assumed to be in special position and not in general position. As a consequence, it may happen that $x_0$ is a fixed point of the composition function $\Theta_{w_f} := \Theta_{x_m} \circ \cdots \circ \Theta_{x_1}$, but the quantity $\Theta_{x_j} \circ \cdots \circ \Theta_{x_1}(x_0)$ is not defined at every intermediate step $0 < j < m$. It is then suggested to think of $\Theta_{w_f}$ as a single function, as opposed to a step-by-step composition.

A similar coordinate system was developed by Bergeron, Falbel and Guilloux in [3]. We refer the reader to the Appendix A for a summary of their work and a comparison with the parametrisation defined in this chapter.

### 2.3.2 Two special decorated characters

In the last section we noted that the union of some irreducible components of the decorated character variety $\mathcal{X}^\times(M_f)$ is birationally isomorphic to a subvariety of $(\mathbb{C}^\times)^8 \cong \mathcal{X}^\times(T_0)$, via the morphism $\Psi_f$. By Lemma 11, $\Psi_f(\mathcal{X}^\times(M_f))$ is the set of fixed points of the function $\Theta_{w_f} := \Theta_{x_m} \circ \cdots \circ \Theta_{x_1}$ obtained by composing the edge flips in the flip sequence $w_f$ of $M_f$. Although $\Theta_{w_f}$ becomes quite complicated for strings $w$ of length larger than two, one can easily compute the sets of fixed points for the base functions $\Theta_R$ and $\Theta_L$.

**Lemma 13.** The set of fixed points of $\Theta_R$ and $\Theta_L$ are:

$$
S_R = \left\{ \left( a, a, -\frac{1}{a+1}, -\frac{a+1}{a}, -\frac{a+1}{a+1}, g, h \right) \mid a, g, h \in \mathbb{C}^\times \right\},
$$

$$
S_L = \left\{ \left( a, b, -\frac{h+1}{h}, -\frac{1}{h+1}, -\frac{1}{h+1}, h, h \right) \mid a, b, h \in \mathbb{C}^\times \right\}.
$$

$S_R$ and $S_L$ intersect in precisely two points $\{P, \overline{P}\} = S_R \cap S_L$, complex conjugates of each other. Letting $\omega$ be the cube root of unity $\omega = -\frac{1}{2} \left( 1 + i \sqrt{3} \right)$, then

$$
P = (\omega, \omega, \omega, \omega, \omega, \omega, \omega, \omega) \quad \text{and} \quad \overline{P} = (\overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}).
$$

These are very special points because they are fixed points of the function $\Theta_{w_f}$, for every automorphism $f$. It follows from Lemma 11 that the corresponding characters $[\rho_P, \Phi_P] := \Psi_{T_0}^{-1}(P)$ and $[\rho_P, \Phi_P] := \Psi_{T_0}^{-1}(\overline{P})$ always extend to decorated characters of $M_f$. It turns out that $[\rho_P, \Phi_P] = [\overline{\rho_P}, \overline{\Phi_P}]$, that is they are complex conjugate decorated characters, hence we will only focus on $[\rho_P, \Phi_P]$ from here on.

Consider $\langle \alpha, \beta \rangle = \pi_1(T_0) < \pi_1(M_f)$. Then, with respect to the setting in §1.4.3,

$$
\rho_P(\alpha) = \begin{bmatrix}
\overline{\omega} & 0 & 0 \\
-1 & \omega & -\omega \\
\overline{\omega} & 0 & -1
\end{bmatrix} = \begin{bmatrix}
\frac{-1 + i \sqrt{3}}{2} & 0 & 0 \\
-1 & \frac{1 + i \sqrt{3}}{2} & \frac{1 + i \sqrt{3}}{2} \\
\frac{-1 + i \sqrt{3}}{2} & 0 & -1
\end{bmatrix} \in \text{PGL}(3, \mathbb{C}),
$$

(2.1)
2.3 FG coordinates on $M_f$

\[
\rho_P(\beta) = \begin{bmatrix}
 1 & -\omega & -1 \\
 0 & \omega & 0 \\
 0 & \bar{\omega} & -\omega
\end{bmatrix} = \begin{bmatrix}
 1 & \frac{1+i\sqrt{3}}{2} & -1 \\
 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\
 0 & \frac{-1+i\sqrt{3}}{2} & \frac{1+i\sqrt{3}}{2}
\end{bmatrix} \in \text{PGL}(3, \mathbb{C}). \tag{2.2}
\]

We remind the reader of some definitions. A representation into $\text{PGL}(3, \mathbb{C})$ is said to be \textit{irreducible} if it has no global fixed points in $\mathbb{C}P^2$, and \textit{strongly irreducible} if it does not preserve any finite union of proper subspaces. Furthermore, it is \textit{faithful} if it is injective. These properties are invariant under conjugation, thus one defines an \textit{irreducible} and \textit{faithful} character similarly.

\textbf{Lemma 14.} The character $[\rho_P]$ is irreducible but not strongly irreducible. Moreover, it is not faithful, but it has infinite image.

\textit{Proof.} Let $\rho_P$ be the representative described in (2.1) and (2.2). Both projective transformations $\rho_P(\alpha)$ and $\rho_P(\beta)$ have three distinct fixed points $\mathbb{C}P^2$. They are:

\[
\begin{bmatrix}
 1 & 0 \\
 -\omega & 1
\end{bmatrix}, \quad \begin{bmatrix}
 0 & 1 \\
 -\omega & 0
\end{bmatrix}, \quad \begin{bmatrix}
 1 & 0 \\
 0 & -\omega
\end{bmatrix}, \quad \begin{bmatrix}
 0 & -\omega \\
 1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
 1 & 0 \\
 0 & -\omega
\end{bmatrix}
\]

Since no three of them are contained in the same projective line, $\rho_P$ is irreducible.

On the other hand, $\rho_P$ is not faithful because both $\rho(\alpha)$ and $\rho(\beta)$ are of order 6. Nevertheless, the commutator

\[
\rho_P(\alpha^{-1} \beta^{-1} \alpha \beta) = \begin{bmatrix}
 1 & 0 & 2(\omega - 1) \\
 0 & 1 & 2(1 - \bar{\omega}) \\
 0 & 0 & 1
\end{bmatrix}
\]

has infinite order, hence the image of $\rho_P$ is infinite.

For all $c \in \mathbb{C}$, the line $\eta_c : [1, -\omega, c] \subset \mathbb{C}P^2$ is preserved by the commutator, therefore for every $\gamma \in \pi_1(T_0)$ there are $i, j \in \{0, \ldots, 5\}$ such that

\[
\rho_P(\gamma) \cdot \eta_c = \rho_P(\alpha^i \beta^j) \cdot \eta_c.
\]

We deduce that the finite union

\[
\eta_c \bigcup_{i=0}^{5} \bigcup_{j=0}^{5} \rho_P(\alpha^i \beta^j) \cdot \eta_c
\]

is preserved by the entire representation $\rho_P$, hence it is not strongly irreducible. \hfill \blacksquare

In the next result, we are going to show that the image of $\rho_P$ lies inside a special subgroup of $\text{PGL}(3, \mathbb{C})$, sometimes called the \textit{Eisenstein-Picard modular group}. The importance of this observation relies on the geometric interpretation of the Eisenstein-Picard modular group as a subgroup of isometries of the Cauchy-Riemann sphere. In fact, we will show
in the next chapter that every extension of \([\rho_P]\) is the holonomy of a (branched) geometric structure on \(M_f\).

Consider the matrix group \(U(2, 1)\) preserving the Hermitian form \(\langle z, w \rangle = w^* J z\) defined on \(\mathbb{C}^3\) by the matrix
\[
J = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]
By projectivization of \(U(2, 1) < GL(3, \mathbb{C})\) one obtains \(PU(2, 1)\) as a subgroup of \(PGL(3, \mathbb{C})\). We denote by \([A] \in PGL(3, \mathbb{C})\) the equivalence class of a matrix \(A \in GL(3, \mathbb{C})\). Observe that a matrix \(A \in GL(3, \mathbb{C})\) belongs to \(U(2, 1)\) if and only if \(A^T J A = J\). Therefore one can determine \(PU(2, 1)\) as subgroup of \(PGL(3, \mathbb{C})\) by the set of solutions to the homogeneous equation
\[
[A]^T [J][A] = [J].
\]
Let \(\mathbb{Z}[\omega]\) be the set of \textit{Eisenstein integers}, namely the ring of integers in the imaginary quadratic number field \(\mathbb{Q}[\sqrt{3}]\). The \textit{Picard modular group} for \(\mathbb{Z}[\omega]\) is \(PU(2, 1, \mathbb{Z}[\omega])\), the subgroup of \(PU(2, 1)\) with entries in \(\mathbb{Z}[\omega]\). This is also known as the \textit{Eisenstein-Picard modular group}.

**Lemma 15.** The group \(\rho_P(\pi_1(\mathbb{T}_0))\) is conjugate to a subgroup of the Eisenstein-Picard modular group \(PU(2, 1, \mathbb{Z}[\omega])\). In particular, \(\rho_P\) has discrete image.

**Proof.** Consider the following matrix of \(PGL(3, \mathbb{C})\):
\[
M = \begin{pmatrix}
0 & 0 & -1 \\
-\bar{\omega} & 1 & \bar{\omega} \\
\omega & 0 & -\omega
\end{pmatrix}.
\]
It is easy to check that
\[
(M \rho_P(\gamma) M^{-1})^T [J] \left( M \rho_P(\gamma) M^{-1} \right) = [J], \quad \gamma \in \{\alpha, \beta\}.
\]
It follows that \(M \rho_P M^{-1}\) has image in \(PU(2, 1)\). Furthermore, since \(M, \rho_P(\alpha)\) and \(\rho_P(\beta)\) have entries in the set of Eisenstein integers, \(M \rho_P M^{-1}\) maps into \(PU(2, 1, \mathbb{Z}[\omega])\). Discreteness follows from the observation that \(PU(2, 1, \mathbb{Z}[\omega])\) is a discrete subgroup of \(PU(2, 1)\) [10].
Chapter 3

Branched Cauchy-Riemann Structures

Cauchy-Riemann geometry (CR in short) is modelled on the three-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$, with the contact structure obtained by the intersection $X = T\mathbb{S}^3 \cap JT\mathbb{S}^3$, where $J$ is the multiplication by $i$ in $\mathbb{C}^2$. The operator $J$ restricted to $X$ defines the standard CR structure on $\mathbb{S}^3$. Its group of CR automorphisms is $\text{PU}(2, 1)$, thus a manifold $M$ has a (spherical) CR structure when it is endowed with a geometric $(\text{PU}(2, 1), \mathbb{S}^3)$–structure.

The fact that every 3–manifold admits a contact structure suggests that CR geometry has the potential to play an important role in three dimensional topology. Nevertheless, only few of examples of CR manifolds are known. Most of them are closed Seifert fibred manifolds [24] or obtained by Dehn surgery from the Whitehead link [32, 33]. On the other hand, some examples of 3–manifolds which have no spherical CR structures are known [18].

In [9], Falbel generalises the notion of CR structures by allowing branching. Charts are not diffeomorphisms anymore, but locally branched coverings. By relaxing this condition, one obtains a geometric structure whose developing map is locally injective everywhere except for a nowhere-dense set, the branch locus. Falbel proceeds to study representations of the fundamental group of the figure eight knot complement in $\text{PU}(2, 1)$, and shows that one of them is the holonomy of a branched CR structure.

In this chapter, we build on the work of Falbel and show that every hyperbolic once-punctured torus bundle admits a branched CR structure. The proof is constructive: we define an ideal cell decomposition and geometrically realise it in CR space. Finally, we prove that the holonomies of these structures are extensions of the character $[\rho_P]$ found in §2.3.2.

The first section 3.1 is mostly devoted to background material on CR geometry, with the exception of 3.1.2. There we define the geometric pieces that build our CR structures. In particular, we introduce six different types of 3–cells that were never used before. They are *slabs*, namely CW complexes obtained by deformation retracting the base of a square pyramid onto one of its sides. These 3–cells were specifically designed to fit nicely with
each other and the *standard symmetric tetrahedra* to form branched CR structures.

Branched CR structures are defined in §3.2. As a reference point, we apply our construction to the figure eight knot complement to recover Falbel’s branched CR structure, then move on to the general case in §3.2.3. In the same section we compute the holonomies of the structures and relate them to the character \([\rho_P]\). We conclude with a thorough analysis of the geometry around the branch locus.

### 3.1 CR geometry

The *spherical Cauchy-Riemann geometry* is modelled on the CR sphere, namely the three-sphere \(S^3\) equipped with a natural \(\text{PU}(2, 1)\) action. Most of the background material can be found in [20] and [33].

Consider the matrix group \(\text{U}(2, 1)\) preserving the following Hermitian form defined on the complex space \(\mathbb{C}^3\)

\[
\langle z, w \rangle := \bar{w}^T J z,
\]

where

\[
J := \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Let \(\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2\) be the canonical projection, and consider the following cones in \(\mathbb{C}^3\),

\[
V_0 := \{ z \in \mathbb{C}^3 \setminus \{0\} \mid \langle z, z \rangle = 0 \}, \quad V_- := \{ z \in \mathbb{C}^3 \mid \langle z, z \rangle < 0 \}.
\]

Then \(H^2_C := \pi(V_-)\) is the Siegel domain model of the *complex hyperbolic plane* and its boundary is

\[
\partial H^2_C := \pi(V_0) = \{ [x, y, z] \in \mathbb{CP}^2 \mid |x|^2 + |y|^2 + z\bar{x} = 0 \}.
\]

As a topological space, \(\partial H^2_C\) is homeomorphic to the three-sphere \(S^3\). The projective group \(\text{PU}(2, 1) := \text{U}(2, 1)/\lambda\mathbb{I}\) is the group of its biholomorphic transformations, naturally a subgroup of \(\text{PGL}(3, \mathbb{C})\). Together with the complex conjugation \(z \to \overline{z}\), it forms the group \(\hat{\text{PU}}(2, 1)\) of isometries of the complex hyperbolic space \(H^2_C\). The action of \(\text{PU}(2, 1)\) on \(S^3\) is by CR transformations.

The natural inclusion map \(S^3 \hookrightarrow \mathbb{CP}^2\) together with its first complex jet, gives an embedding \(\iota : S^3 \hookrightarrow \text{FL}\) of the CR sphere into the flag space. Every element \(P \in S^3\) is associated to a flag \(\iota(P) = (P, \eta)\), where \(\eta\) is the unique complex tangent to \(S^3\) at \(P\). In this way, one may talk about points in general position, special position, triple ratios and quadruple ratios as defined in §1.2.

The group of CR transformations acts transitively on pairs of distinct points of \(S^3\), while generic configurations of triples of points are parametrised by a real number. Given a cyclically ordered triple of points \(\langle P_1, P_2, P_3 \rangle\) in \(S^3\), its Cartan angle \(\hat{\lambda}\) is

\[
\hat{\lambda}(P_1, P_2, P_3) := \arg(-\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle) \in \mathbb{R}.
\]

**Lemma 16.** *The group PU(2, 1) is simply transitive on ordered triples of points in general position with the same Cartan angle.*
3.1 CR geometry

Up to CR transformations, a generic configuration of three points (and corresponding tangents) with Cartan angle $\Lambda(P_1, P_2, P_3) = \arctan(t)$, for $t \in \mathbb{R}$, is

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} -\frac{1+it}{2} \\ 1 \\ 1 \end{bmatrix}, \quad (3.1)$$

$$\eta_1 = [0 \ 0 \ 1], \quad \eta_2 = [1 \ 0 \ 0], \quad \eta_3 = [1 \ 1 \ -\frac{1-it}{2}].$$

We recall that $\text{PGL}(3, \mathbb{C})$ is simply transitive on ordered 4–tuples of points in general position, hence in particular it is transitive on ordered pairs of flags. It follows that given any two flags $F_1, F_2$, there is always at least one $G \in \text{PGL}(3, \mathbb{C})$ such that $G \cdot F_1$ and $G \cdot F_2$ are in $\iota(\mathbb{S}^3)$. The following result shows that this is not the case for triples of flags.

**Lemma 17.** Let $\mathcal{F} = \langle F_1, F_2, F_3 \rangle$ be a cyclically ordered triple of flags with triple ratio $z = \Lambda(\mathcal{F})$. There is $G \in \text{PGL}(3, \mathbb{C})$ such that each $G \cdot F_j$ is in $\iota(\mathbb{S}^3)$ if and only if $\frac{1-i\eta^2}{1+\eta^2}$ is purely imaginary.

**Proof.** We recall from §1.2.4 that $\text{PGL}(3, \mathbb{C})$ is simply transitive on ordered triples of flags with the same triple ratio. Up to $\text{PU}(2,1)$, a cyclically ordered triple of flags in $\iota(\mathbb{S}^3)$ is configured as in (3.1). Its triple ratio is $\frac{1-it}{1+it}$, where $\arctan(t) \in \mathbb{R}$ is its Cartan angle. It follows that we can map a triple of flags with triple ratio $z$ into $\iota(\mathbb{S}^3)$ if and only if

$$z = \frac{1-it}{1+it} \iff \frac{1-z}{1+z} = it, \quad t \in \mathbb{R}. \quad \blacksquare$$

We are now going to describe a model for $\partial \mathbb{H}^2_{\mathbb{C}}$ which will be particularly suitable for our framework. The *Heisenberg group* $\mathcal{H}$ is the space $\mathbb{C} \times \mathbb{R}$, equipped with the group law

$$(z_1, t_1) \cdot (z_2, t_2) := (z_1 + z_2, t_1 + t_2 + 2\mathfrak{F}(z_1z_2)), \quad z_1, z_2 \in \mathbb{C}, \quad t_1, t_2 \in \mathbb{R}.$$ 

In the formula above, $\mathfrak{F}(z)$ is the imaginary part of the complex number $z$. Using stereographic projection $\Lambda$, one can identify $\partial \mathbb{H}^2_{\mathbb{C}}$ with the one-point compactification $\overline{\mathcal{H}}$ of $\mathcal{H}$, thus obtaining the *Heisenberg model* of the CR sphere. In coordinates,

$$\Lambda : \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \mapsto \left( y, \frac{2x + |y|^2}{t} \right), \quad \Lambda^{-1} : (z, t) \mapsto \begin{bmatrix} \frac{it+|z|^2}{2} \\ z \\ 1 \end{bmatrix} \quad \text{and} \quad \Lambda : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \infty.$$ 

### 3.1.1 CR simplices

There are two kinds of totally geodesic submanifolds of real dimension two in $\mathbb{H}^2_{\mathbb{C}}$: complex geodesics and totally real geodesic planes. Each of these submanifolds is a model of the hyperbolic plane. Their boundaries in $\partial \mathbb{H}^2_{\mathbb{C}}$ are called $\mathbb{C}$–circles and $\mathbb{R}$–circles respectively.
In hyperbolic space, a three dimensional simplex is the convex hull of its four vertices. This notion is not well defined in CR space, hence instead we are going to make use of circles to canonically foliate simplices from a set of given vertices. In principle, one could interchangeably work with \( \mathbb{R} \)-circles or \( \mathbb{C} \)-circles, however \( \mathbb{C} \)-circles turn out to be a more optimal choice for our purpose.

**Lemma 18.** In the Heisenberg model \( \mathcal{H} \), \( \mathbb{C} \)-circles are either vertical lines or ellipses whose projections onto the \( z \)-plane are circles.

A \( \mathbb{C} \)-circle that is not vertical will be sometimes referred to as *finite*. Every finite \( \mathbb{C} \)-circle is uniquely determined by a *centre* \((z_0, t_0)\) and a *radius* \( r_0 \). Any point \((z, t)\) on it satisfies the equations

\[
\begin{cases}
|z - z_0| = r_0, \\
t = t_0 + 2\Im(z_0).
\end{cases}
\]

We remark that a complex geodesic in \( \mathbb{H}^2_\mathbb{C} \) is naturally endowed with a positive orientation given by its complex structure, hence every \( \mathbb{C} \)-circle also inherits an orientation.

**Lemma 19.** CR transformations map \( \mathbb{C} \)-circles to \( \mathbb{C} \)-circles, preserving their orientations.

Given two distinct points \( P_1 \) and \( P_2 \) in Heisenberg space \( \mathcal{H} \), there is a unique \( \mathbb{C} \)-circle between them. We define the *oriented edge* \([P_1, P_2]\) to be the segment of the \( \mathbb{C} \)-circle between \( P_1 \) and \( P_2 \), oriented towards \( P_2 \). For example, the oriented edge \([(0,0), \infty]\) is the segment \( \{(0,t) \in \mathcal{H} | t > 0\} \), oriented towards \( \infty \). Then \([P_1, P_2] \cup [P_2, P_1]\) is the whole \( \mathbb{C} \)-circle through \( P_1 \) and \( P_2 \). A disk bounded by the loop \([P_1, P_2] \cup [P_2, P_1]\) will be referred to as a *bigon*.

Suppose \( P_1, P_2, P_3 \in \mathcal{H} \) are three points in general position. For each pair, there are two possible oriented edges, for a total of eight choices of 1–skeletons defining a *triangle*. As \( \mathcal{H} \) is simply connected, we can always extend the 1–skeleton of a triangle to an embedded 2–cell, with boundary defined by that 1–skeleton. This can be done in many different ways, all equivalent up to isotopy. Inspired by the work of Falbel [9], we define the *marked triangles* \([P_1]^*, P_2, P_3\) and \([P_1]^-, P_2, P_3\) as foliations of oriented edges:

\[
[P_1]^*, P_2, P_3] := \{P \in \mathcal{H} | P \in [P_1, P] \text{ for } P_1 \in [P_2, P_3]\}, \\
[P_1]^-*, P_2, P_3] := \{P \in \mathcal{H} | P \in [P, P_1] \text{ for } P \in [P_2, P_3]\}.
\]

The vertex \( P_1 \) is the source of the foliation in the former triangle, and the sink of the foliation in the latter one (cf. Figure 3.1). By fixing \( P_1 \) to be at infinity, a marked triangle is half a cylinder with base part of a finite \( \mathbb{C} \)-circle. One of the advantages of using marked triangles is that they are uniquely determined by their vertices. The following result is a direct consequence of Lemma 19, which implies that foliations by \( \mathbb{C} \)-circles are preserved.

**Lemma 20.** Let \( P_1, P_2, P_3 \) and \( Q_1, Q_2, Q_3 \) be two triples of points of \( \mathcal{H} \) in general position. Suppose there exists \( G \in \text{PU}(2, 1) \) such that \( G(P_j) = Q_j \), for all \( j \in \{1, 2, 3\} \). Then

\[
G([P_1]^*, P_2, P_3]) = [Q_1]^*, Q_2, Q_3], \quad \star \in \{+, -\}.
\]
Given four points in the CR sphere in general position, a choice of a marked triangle for each triple will not always patch up to form the boundary of a 3–simplex. On one hand, the faces might not be compatible at the edges and have gaps between them. On the other hand, they could intersect away from the edges. One quickly finds that there is not a canonical choice of marked triangles which always works, thus three dimensional simplices need to be checked on a case by case basis. More details and examples will be studied in the next section.

3.1.2 Two fundamental pieces: the standard symmetric tetrahedron and the slab

In this section we are going to describe two fundamental objects, which will be the building blocks of the CR structures in §3.2. They are subsets of the Heisenberg space, both topologically homeomorphic to the 3–ball, but equipped with different simplicial structures.

The standard symmetric tetrahedron

Let $\omega$ be the cube root of unity $\omega = -\frac{1}{2} \left( 1 + i\sqrt{3} \right)$. We consider the following 4–tuple of points in generic position in Heisenberg space:

$$P_1 := (1, \sqrt{3}), \quad P_2 := (-\omega, \sqrt{3}), \quad P_3 := (0, 0), \quad P_4 := \infty.$$ 

They correspond to the four flags $F_j := \left( \Lambda^{-1}(P_j), \eta_j \right) \in \mathcal{F}_L$, where

$$\Lambda^{-1}(P_1) = \begin{bmatrix} -\overline{\omega} \\ 1 \end{bmatrix}, \quad \Lambda^{-1}(P_2) = \begin{bmatrix} -\overline{\omega} \\ 1 \end{bmatrix}, \quad \Lambda^{-1}(P_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Lambda^{-1}(P_4) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\eta_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} -\omega \\ 1 \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$ 

For each triple of points, we consider the following marked triangles:

1. $[P_1^-, P_2, P_3]$: the oriented segment $[P_1, P_2]$ is the shortest arc of the circle $(e^{i\theta}, \sqrt{3})$, oriented from $P_1$ to $P_2$. The triangle $[P_4^-, P_1, P_2]$ is part of a cylinder, foliated by vertical segments above $[P_1, P_2]$. 

![Diagram](image-url)
2. \([P_{4}^{-}, P_{3}, P_{1}]\): the edge \([P_{3}, P_{1}]\) is an arc of ellipse which projects onto the \(z\)-coordinate of the Heisenberg space as an arc of the unit circle with centre \(-\omega\). It is given by the parametrisation
\[
[P_{3}, P_{1}] := (-\omega + e^{is}, \sqrt{3} \cos(s) - \sin(s)), \quad s : -\frac{2\pi}{3} \mapsto -\frac{\pi}{3}.
\]
Hence \([P_{4}^{-}, P_{3}, P_{1}]\) is foliated by the vertical rays from \([P_{3}, P_{1}]\) to \(P_{4}\).

3. \([P_{4}^{-}, P_{3}, P_{2}]\): this marked triangle is obtained by a \(\frac{\pi}{3}\) clockwise rotation of the previous triangle \([P_{4}^{-}, P_{3}, P_{1}]\).

4. \([P_{4}^{-}, P_{3}, P_{1}]\) and \([P_{3}^{+}, P_{1}, P_{2}]\): the first marked triangle is foliated by oriented edges from \([P_{3}, P_{1}]\) to \(P_{2}\). For \(\varphi(t, s) := t + s + \frac{\pi}{3}\), we have
\[
[P_{4}^{-}, P_{3}, P_{1}] := \left( e^{i\varphi(t, s)} + e^{i(s-\frac{\pi}{3})} - \omega, \right.
\]
\[
\left. -\sin(\varphi(t, s)) - \sin(\varphi(t, 0)) + \sin(s) + \sqrt{3} (\cos(\varphi(t, s)) - \cos(t, 0) + \cos(s) + 1) \right),
\]
where \(s : -\frac{2\pi}{3} \mapsto -\frac{\pi}{3}\) and \(t : 0 \mapsto \frac{\pi}{3}\). The latter one instead, is foliated by oriented edges from \(P_{3}\) to \([P_{1}, P_{2}]\). It can be parametrised as
\[
[P_{3}^{+}, P_{1}, P_{2}] := \left( e^{i\varphi(t, \omega + e^{is})}, \sqrt{3} \cos(s) - \sin(s) \right), \quad s : -\frac{2\pi}{3} \mapsto -\frac{\pi}{3}, \quad t : 0 \mapsto \frac{\pi}{3}.
\]

**Lemma 21.** ([38]) The spaces
\[
[P_{4}^{-}, P_{1}, P_{2}] \cup [P_{4}^{-}, P_{3}, P_{1}] \cup [P_{4}^{-}, P_{3}, P_{2}] \cup [P_{2}^{-}, P_{3}, P_{1}], \quad (3.2)
\]
\[
[P_{4}^{-}, P_{1}, P_{2}] \cup [P_{4}^{-}, P_{3}, P_{1}] \cup [P_{4}^{-}, P_{3}, P_{2}] \cup [P_{3}^{+}, P_{1}, P_{2}], \quad (3.3)
\]
are combinatorially isomorphic to a \(3\)-simplex. In particular, they bound a \(3\)-ball on each side in \(\overline{H}\).

The standard (symmetric) tetrahedron \(\Xi_{A}\) of type \(A\) is the closure of the \(3\)-ball bounded by the \(3\)-simplex in (3.2), which is contained in the upper half of \(\overline{H}\). Similarly, the \(3\)-simplex in (3.3) is the boundary of the standard (symmetric) tetrahedron \(\Xi_{B}\) of type \(B\). Figure 3.2 shows \(\Xi_{A}\) and \(\Xi_{B}\) in the Heisenberg model.

These tetrahedra exhibit various symmetries, for example an anti-holomorphic involution swapping the vertices \(P_{1}\) with \(P_{2}\), and \(P_{3}\) with \(P_{4}\) (cf. [38]). Furthermore, the vertices of each face (taken with the correct cyclic order) have the same Cartan angle
\[
\hat{A}(P_{2}, P_{3}, P_{1}) = \hat{A}(P_{4}, P_{1}, P_{2}) = \hat{A}(P_{4}, P_{3}, P_{2}) = \hat{A}(P_{4}, P_{3}, P_{1}) = \frac{\pi}{3}.
\]
It follows that the corresponding triples of flags share the same triple ratio \(3(F_{1}, F_{2}, F_{3}) = \omega\). As a consequence of Lemma 16 and Lemma 20, we can glue faces of \(\Xi_{A}\) and \(\Xi_{B}\) pairwise by (unique) CR transformations. Consider the following matrices of \(\text{PU}(2, 1)\),
\[
G_{1} := \begin{bmatrix} -\omega & 0 & 0 \\ 1 & 1 & 0 \\ -\overline{\omega} & \omega & -\omega \end{bmatrix}, \quad G_{2} := \begin{bmatrix} 1 & 1 & \omega \\ 0 & -\overline{\omega} & \overline{\omega} \\ 0 & 0 & 1 \end{bmatrix}, \quad G_{3} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Figure 3.2: The standard symmetric tetrahedra $\mathcal{T}_A$ and $\mathcal{T}_B$ only differ along the face with vertices $\{P_1, P_2, P_3\}$. Their standard embeddings in Heisenberg space and their $\mathbb{C}$–projections are displayed here.

These are the unique CR transformations mapping:

\[
G_1 : \quad P_4 \mapsto P_2 \quad P_3 \mapsto P_3 \quad P_1 \mapsto P_1 \quad \text{hence} \quad [P_4^-, P_3, P_1] \mapsto [P_2^-, P_3, P_1],
\]

\[
G_2 : \quad P_4 \mapsto P_4 \quad P_1 \mapsto P_3 \quad P_2 \mapsto P_2 \quad \text{hence} \quad [P_4^-, P_1, P_2] \mapsto [P_4^-, P_3, P_2],
\]

\[
G_3 : \quad P_4 \mapsto P_4 \quad P_3 \mapsto P_3 \quad P_1 \mapsto P_2 \quad \text{hence} \quad [P_4^-, P_3, P_1] \mapsto [P_4^-, P_3, P_2].
\]

We remark that $G_2$ and $G_3$ are face pairings between two standard tetrahedra of any types, while $G_1$ necessarily glues onto a face of the standard tetrahedron of type $A$. Furthermore, $G_2$ and $G_3$ can be described quite nicely in Heisenberg coordinates:

\[
G_2([z, t]) = \left[ -\bar{\omega}(z - 1), \sqrt{3}\bar{\omega}(z + \bar{\omega} - 1)(z + \omega) + t \right],
\]

\[
G_3([z, t]) = [-\omega z, t].
\]

The transformation $G_2$ preserves vertical $\mathbb{C}$–circles and it restricts on the $z$–plane to a $\pi_3$ clockwise rotation around the point $-\omega$. The transformation $G_3$ is a $\pi_3$ anticlockwise rotation of $\mathcal{H}$ around the vertical $\mathbb{C}$–circle through $[0, 0]$.

Slabs

The next fundamental piece that we are going to define is of the combinatorial type of the CW complex obtained by deformation retracting the base of a square pyramid onto one of its sides. In particular, it is a 3–cell bounded by two triangular faces and two bigons. It contains a total of five 1–cells and three 0–cells.

We define the following bigons of $\mathcal{H}$:

\[
B' := \left( 1 + te^{-i\frac{\pi}{6}}, s \right), \quad t \in \mathbb{R}_{>0} \cup \{\infty\}, \quad s \in \mathbb{R} \cup \{\infty\},
\]
We remark that both \( B' \) and \( B_k \) are foliated by vertical \( \mathbb{C} \)–circles. In particular, \( B' \cap B_k = \infty \) for all \( k \). Moreover,

\[
B_{k_1} = B_{k_2} \iff k_1 = k_2 \mod 6.
\]

The CW complex obtained by attaching

\[
[P_4^+, P_1, P_2] \cup [P_4^-, P_1, P_2] \cup B' \cup B_k,
\]

is topologically a 2–sphere. For all \( k \), it bounds a 3–ball containing the point \((2, \sqrt{3}) \in \mathbb{H} \). We define the slab \( \mathcal{S}_k \) to be the closure of such 3–ball. The slabs \( \mathcal{S}_{k_1} \) and \( \mathcal{S}_{k_2} \) are geometrically equivalent if and only if \( k_1 = k_2 \mod 6 \), in the sense that there is \( G \in \text{PU}(2, 1) \) such that \( G(\mathcal{S}_{k_1}) = \mathcal{S}_{k_2} \). This is due to the fact that the 2–skeletons of \( \mathcal{S}_{k_1} \) and \( \mathcal{S}_{k_2} \) only differ along one face. Whence we defined a total of six different slabs. Two examples \( \mathcal{S}_1 \) and \( \mathcal{S}_4 \) are depicted in Figure 3.3.

As we mentioned earlier, \( \Lambda(P_4, P_1, P_2) = \Lambda(P_3, P_1, P_2) \), hence let \( G_4 \) be the (unique) element of \( \text{PU}(2, 1) \):

\[
G_4 := \begin{bmatrix}
0 & 0 & -\omega \\
0 & -\overline{\omega} & 0 \\
-\omega & 0 & 1 - \overline{\omega}
\end{bmatrix},
G_4 : [P_4, P_3, P_1] \mapsto [P_3, P_1, P_2],
G_4 : [P_4^+, P_1, P_2] \mapsto [P_3^+, P_1, P_2].
\]

For all \( k \), the CR transformation \( G_4 \) is a face pairing between the slab \( \mathcal{S}_k \) and the standard tetrahedron of type \( B \).

The use of six different slabs turns out to be necessary in the general construction of §3.2.3. The reason for the number six is due to the fact that the CR transformations \( G_1, G_2, G_3 \) and \( G_4 \) are all of order six. The connection between them and the slabs is revealed in Theorem 28.
3.1 CR geometry

We conclude this section with a definition and an observation. Let $W_1$ and $W_2$ be two CW complexes embedded in $\mathcal{H}$, and let $G \in \text{PU}(2, 1)$ be a face pairing between the faces $F_1 \subset W_1$ and $F_2 \subset W_2$. Then $G(W_1)$ and $W_2$ might intersect away from $G(F_1) = F_2$. We say that the face pairing $G$ is monotone if there are neighbourhoods $N_1, N_2$ of $F_1, F_2$ in $W_1, W_2$ respectively such that $N_2 \cap G(W_1) = G(N_1) \cap W_2 = F_2$. The following result generalises an observation by Falbel [9].

**Lemma 22.** The transformations $G_1, G_2, G_3$ are monotone face pairings of the standard symmetric tetrahedra $\mathcal{I}_A$ and $\mathcal{I}_B$, while $G_4$ is a monotone face pairing between the slab and the standard tetrahedron of type $B$.

**Proof.** The transformations $G_2$ and $G_3$ are simple to check. They preserve vertical $\mathbb{C}$–circles, therefore one only needs to check the intersection of the projections of the tetrahedra on the $z$–plane.

On the other hand, $G_1$ and $G_4$ are more tedious. We give a summary of the argument for $G_4$, and refer to [9] for $G_1$. Consider the slab $\mathcal{S}_k$ and the tetrahedron $\mathcal{I}_B$. The transformation $G_4^{-1}$ glues $\mathcal{S}_B$ to $\mathcal{S}_k$ along the face $[P_4^+, P_1, P_2] = G_4^{-1}([P_3^+, P_1, P_2])$. The remaining vertex of $\mathcal{S}_B$ is mapped to the point $G_4^{-1}(P_4) = [0, 2\sqrt{3}]$ in Heisenberg space. The projection of the 1–skeleton of $G_4^{-1}(\mathcal{I}_B)$ is displayed next to the projection of $\mathcal{S}_1$ in Figure 3.4.

![Figure 3.4: The projection of the 1–skeleton of $G_4^{-1}(\mathcal{I}_B)$ next to the projection of $\mathcal{S}_1$.](image)

Let $R$ be the region of $\mathbb{C}$–plane bounded by the straight segment from 0 to 1, and the projections of the edges $[P_1, P_2]$ and $G_4^{-1}([P_4, P_2])$. Then $G_4^{-1}(\mathcal{I}_B)$ is completely contained in the vertical cylinder of Heisenberg space with base $R$. In particular, there is a neighbourhood of the common face where $G_4^{-1}(\mathcal{I}_B)$ and $\mathcal{S}_k$ only intersect along the face, and therefore $G_4^{-1}$ is a monotone face pairing between $\mathcal{S}_k$ and $\mathcal{I}_B$. By symmetry of the definition, we conclude that $G_4$ is also monotone.

In [9], the CR transformations $G_j$ are implicitly used to glue the simplices composing the complement of the figure eight knot, to define a branched CR structure. In particular, Falbel considers a generalised standard tetrahedron, which is the union of a standard symmetric tetrahedron of type $A$ and a slab $\mathcal{S}_1$. More details on this particular example are developed in §3.2.2.
3.2 Branched CR structures on once-punctured torus bundles

Let $f$ be an automorphism of the once punctured torus with two distinct real eigenvalues, and let $M_f$ be the corresponding once-punctured torus bundle. We formalise the notion of a branched CR structure on $M_f$, a special type of geometric structure modelled on the CR space $\mathbb{H}$ and CR transformations $\text{PU}(2, 1)$. Definitions and terminology are borrowed from the work on branched analytic structures on Riemann surfaces in [29].

A locally branched covering between two manifolds is a covering map everywhere except for a nowhere-dense set, called the branch locus. For example, the map $\xi : \mathbb{H} \to \mathbb{H}$ defined by $\xi(z,t) := (z^N, t)$ is a locally branched map of ramification order $N \in \mathbb{Z} \setminus \{0\}$. In particular, $\xi$ is locally injective everywhere except at the branch locus, namely the Heisenberg $t$–axis.

A branched coordinate covering $\{U_j, \phi_j\}$ of $M_f$ consists of an open covering $\{U_j\}$ of $M_f$ together with locally branched coverings $\phi_j : U_j \to V_j$ into open subsets $V_j$ of the CR space $\mathbb{H}$. A branched CR cover is a coordinate covering $\{U_j, \phi_j\}$ such that, on each non-empty intersection $U_i \cap U_j$, there are homeomorphisms called coordinate transition functions $G_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$, that are restrictions of elements in $\text{PU}(2, 1)$. In particular they satisfy $G_{ij} \circ \phi_i = \phi_j$. A branched CR structure on $M_f$ is an equivalence class of branched CR covers, where two branched CR covers are equivalent if their union is a branched CR cover. As a brief example of a natural branched structure, we mention the hypersurface $\Sigma \subset \mathbb{C}^2$ defined by $\Sigma := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^{2N} + |z_2|^2 = 1\}$. We observe that the map $\xi' : \Sigma \to \mathbb{H}$ defined by $\xi'(z_1, z_2) = (z_1^N, z_2)$ is a branched covering, branched along the curve $z_2 = 0$.

Let $\{U_j, \phi_j\}$ be a branched CR structure on $M_f$. When the ramification order of each chart $\phi_j$ is one, they are homeomorphisms and one recovers the usual definitions of coordinate covering, CR cover and CR structure [35]. We recall that every CR structure admits a developing map and a holonomy representation,

$$\text{dev} : M_f \to \mathbb{H} \quad \text{and} \quad \text{hol} : \pi_1(M_f) \to \text{PU}(2, 1),$$

such that

$$\text{hol}(\gamma) \cdot \text{dev}(x) = \text{dev}(\gamma \cdot x), \quad \gamma \in \pi_1(M_f), \quad x \in M_f. \quad (3.4)$$

The developing map is considered up to deck transformation invariant isotopy, and the pair $(\text{dev}, \text{hol})$ is uniquely determined up to the following action of $\text{PU}(2, 1)$:

$$G \cdot (\text{dev}, \text{hol}) := (G \cdot \text{dev}, G \cdot \text{hol} \cdot G^{-1}), \quad G \in \text{PU}(2, 1).$$

Developing maps thus obtained are locally injective, as the charts $\phi_j$ are homeomorphisms. Vice versa, a locally injective developing map together with a holonomy representation...
satisfying the equivariancy condition (3.4), always defines a CR structure. We refer the reader to [37] for a full treatment in the wider context of geometric \((G, X)\) structures.

In a similar fashion, one may construct developing maps and holonomy representations for branched CR structures. The only difference being that developing maps are not locally injective but locally branched coverings. In particular, the holonomy around each connected component of the branched locus is a rotation by an integer multiple of \(2\pi\), and therefore trivial. In other words, it is a well defined representation of \(\pi_1(M_f)\).

3.2.1 Finite geometric realisations

In this chapter we construct special branched CR structures on \(M_f\), whose branch locus is a disjoint union of curves. The strategy is to use an ideal cell decomposition \(D_f\) of \(M_f\), modelled on its monodromy ideal triangulation \(T_f\). We are going to realise each ideal cell as a geometric object in Heisenberg space and each face pairing as an element of \(PU(2, 1)\), in a compatible fashion. More precisely, suppose \(D_f\) is made up of the ideal 3–cells \(\sigma_i\), with face pairings \(g_j\). We recall that a face pairing is called monotone when the paired cells only intersect along the common face in a neighbourhood of such face (cf. end of §3.1.2).

A geometric realisation \(\{\phi_i, G_j\}\) of \(D_f\) in \(\mathcal{H}\) consists of embeddings \(\phi_i : \sigma_i \to \mathcal{H}\) and CR transformations \(G_j \in PU(2, 1)\), satisfying the following condition: if \(g_j\) is the gluing map between the faces \(F_i\) and \(F_k\) of the ideal 3–cells \(\sigma_i\) and \(\sigma_k\) respectively, then \(G_j\) is a monotone CR transformation pairing \(\phi_i(F_i)\) and \(\phi_k(F_k)\) in the same combinatorial way. Then we say that \(\phi_i\) and \(G_j\) are geometric realisations of \(\sigma_i\) and \(g_j\) respectively.

A geometric realisation differs from a branched CR structure only at the edges. For each edge \(e\), consider a small oriented loop \(\gamma_e\) around \(e\) with prescribed starting point \(x \in \gamma_e\), contained in the interior of some cell. Let \(F_0^e \ldots F_N^e\) be the sequence of faces in \(D_f\) containing \(e\), ordered as they are crossed by \(\gamma_e\), starting from \(x\). As \(\gamma_e\) travels through a face \(F_j^e\), it leaves an ideal cell \(\sigma\) to enter another ideal cell \(\sigma'\) (possibly equal to \(\sigma\)). Let \(g_j^e\) be the face pairing gluing \(\sigma\) to \(\sigma'\) along \(F_j^e\), and let \(G_j^e\) be its corresponding geometric realisation. Then the geometric holonomy of \(\{\phi_i, G_j\}\) along \(\gamma_e\) is the product \(\prod_{j=0}^{N_e} G_{N_e-j}^e\). We remark that a different choice of \(\gamma_e\) only changes the geometric holonomy by conjugation or by inverse, hence whether the geometric holonomy around an edge \(e\) is trivial (namely equal to the identity) or not, does not depend on the choice of \(\gamma_e\).

In general, it is not guaranteed that the geometric holonomy is trivial because a geometric realisation does not enforce any conditions on the local structure around the edges. However, when that is the case for every edge of the cell decomposition, then a geometric realisation can be extended to a branched CR structure. More precisely, there is a branched CR structure on \(M_f\) whose set of charts include the embeddings \(\phi_i\), and the coordinate transition functions along the faces are the CR transformations \(G_j\). In particular, it is important that the maps \(G_j\) are monotone to ensure local injectivity at the faces. Furthermore, the fact that the geometric holonomy around an edge \(e\) is trivial allows the construction of a chart containing \(e\) which is a branched covering (with branched locus \(e\)) and which agrees with \(\phi_i\) around \(e\).
An example of this construction can be found in [39], in the particular case of triangulations and hyperbolic structures.

For future reference, we summarise the above discussion in the following result.

**Lemma 23.** Let \( \{ \phi_i, G_j \} \) be a geometric realisation of \( D_f \) in \( \overline{H} \). If the geometric holonomy around each edge is trivial, then \( \{ \phi_i, G_j \} \) defines a branched CR structure on \( M_f \).

In a similar fashion to ideal triangulations, the ideal cell decomposition \( D_f \) we are going to construct is the complement of the 0–skeleton of a CW complex, which is also called \( D_f \). This CW complex is topologically homeomorphic to the end-compactification of \( M_f \). It has a single vertex, which is the only non-manifold point. When talking about (ideal) cells in \( D_f \), it will be convenient to consider the 0–skeleton as a point of reference, but we will not always underline that it is not actually part of the decomposition of \( M_f \). Moreover, we are often going to drop the word “ideal” when it is clear from the context.

A finite geometric realisation of \( D_f \) in \( \overline{H} \) is a geometric realisation \( \{ \phi_i, G_j \} \) whose embeddings \( \phi_i : \sigma_i \rightarrow \overline{H} \) extend to the 0–skeleton. Let \( \overline{D}_f \) be the ideal cell decomposition of the universal cover \( \overline{M}_f \) induced by \( D_f \). If \( \{ \phi_i, G_j \} \) is a finite geometric realisation with trivial geometric holonomy around each edge, then it defines a branched CR structure, represented by some pair \( (\text{dev}, \text{hol}) \) of developing map and holonomy representation. By finiteness, the developing map \( \text{dev} : \overline{D}_f \rightarrow \overline{H} \) extends equivariantly to the 0–skeleton \( \overline{D}_f^{(0)} \).

More precisely, if \( \text{dev}^{(0)} \) is the restriction of \( \text{dev} \) to \( \overline{D}_f^{(0)} \), then

\[
\text{hol}(\gamma) \cdot \text{dev}^{(0)}(x) = \text{dev}^{(0)}(\gamma \cdot x), \quad \gamma \in \pi_1(M_f), \quad x \in \overline{D}_f^{(0)}.
\]

We recall that the set of vertices \( \overline{D}_f^{(0)} \) is naturally in bijection with the set of boundary components of \( \overline{M}_f \), and \( PU(2,1) \) naturally embeds in \( PGL(3,\mathbb{C}) \) as a subgroup. It follows that \( (\text{hol}, \text{dev}^{(0)}) \in \mathbb{R}^{\times}(M_f, PGL(3,\mathbb{C})) \) is a decorated representation, and its \( PGL(3,\mathbb{C}) \)-class \( \left[ \text{hol}, \text{dev}^{(0)} \right] \in \mathcal{X}^{\times}(M_f, PGL(3,\mathbb{C})) \) is a decorated character (cf. §1.3.1 and §2.3).

### 3.2.2 The figure eight knot complement

Let \( K_8 \) be the figure eight knot complement, as defined in §2.2.4. In this section we construct a branched CR structure on \( K_8 \), as a preliminary example for the general case in §3.2.3. The structure we are going to describe here was first discovered by Falbel [9].

Let \( \mathcal{T}_8 \) be the monodromy ideal triangulation of \( K_8 \). We refer to §2.2.4 for the notation. Let \( D_8 \) be the cell decomposition obtained from the following manipulations on \( \mathcal{T}_8 \).

1. (Figure 3.5) We subdivide the face \( \sigma^R_0 (134) \) of the tetrahedron \( \sigma^R_0 \) into two 2–cells, by introducing a 1–cell with endpoints \( \{ \sigma^R_0 (1), \sigma^R_0 (4) \} \). The two 2–cells thus obtained are combinatorially a triangle and a bigon. Similarly, we subdivide \( \sigma^R_0 (234) \) by placing a 1–cell with endpoints \( \{ \sigma^R_0 (2), \sigma^R_0 (4) \} \). Finally, we split the
3.2 Branched CR structures on once-punctured torus bundles

The tetrahedron $\sigma_0^R$ is subdivided into two 3–dimensional cells, of the combinatorial type of a tetrahedron $\hat{\sigma}_0^R$ and a slab $\hat{\sigma}_0^S$.

Figure 3.5: The tetrahedron $\sigma_0^R$ is subdivided into two 3–dimensional cells, of the combinatorial type of a tetrahedron $\hat{\sigma}_0^R$ and a slab $\hat{\sigma}_0^S$.

tetrahedron $\sigma_0^R$ into two 3–cells, by introducing a triangular 2–cell with endpoints $\{\sigma_0^R(1), \sigma_0^R(2), \sigma_0^R(4)\}$. Whence $\sigma_0^R$ is subdivided into two 3–cells: $\hat{\sigma}_0^R$ with vertices $\{\hat{\sigma}_0^R(1), \hat{\sigma}_0^R(2), \hat{\sigma}_0^R(3), \hat{\sigma}_0^R(4)\}$ is combinatorially isomorphic to a simplex, and $\hat{\sigma}_0^S$ with vertices $\{\hat{\sigma}_0^S(1), \hat{\sigma}_0^S(2), \hat{\sigma}_0^S(4)\}$ is of the combinatorial type of a slab (cf. §3.1.2).

(2) (Figure 3.6) Similar to above, we subdivide $\sigma_1^L$ into two 3–cells by introducing a 2–cell inside the tetrahedron bounded by two 1–cells with endpoints $\{\sigma_1^L(2), \sigma_1^L(4)\}$. They are embedded in the faces $\sigma_1^L(124)$ and $\sigma_1^L(234)$ respectively. Thus $\sigma_1^L$ is decomposed into two 3–cells $\hat{\sigma}_1^L \cup \hat{\sigma}_1^W$. The former, $\hat{\sigma}_1^L$ has four triangular faces and a bigon. The latter $\hat{\sigma}_1^W$ is of the combinatorial type of a wedge, the CW complex obtained by quotienting a face of a 3–simplex to a point. Its set of vertices is $\{\hat{\sigma}_0^W(2), \hat{\sigma}_0^W(4)\}$.

(3) (Figure 3.6) We deformation retract the wedge $\hat{\sigma}_1^W$ onto the bigonal face bounded by the red and the black edge. Simultaneously, we collapse the bigonal face of $\hat{\sigma}_1^L$ into the black edge, transforming $\hat{\sigma}_1^L$ back into a 3–simplex. Finally, we remove the retracted wedge from the decomposition. As a consequence, the green edge and the black edge of $\hat{\sigma}_0^S$ are now identified (cf. Figure 3.5 and Figure 3.7). We remark that this step has the exclusive role of simplifying the cell decomposition by decreasing the number of 3–cells.

Up to step (2), the subdivisions of $\sigma_0^R$ and $\sigma_1^L$ agree along the faces, hence they form a well defined cell decomposition of $T_8$. On step (3), we flatten the 3–cell $\hat{\sigma}_1^W$ and remove it. This does not change the topology of the complex because a neighbourhood of the red edge $e_R$ contains other 3–cells other than $\hat{\sigma}_1^W$. In the end we have three 3–cells $\hat{\sigma}_0^R, \hat{\sigma}_0^S, \hat{\sigma}_1^L$, two of which are of the combinatorial type of a tetrahedron and one of which is a slab (see Figure 3.7). They glue to form a CW complex $D_8$, which is a cell decomposition of $K_8$.

The slab $\hat{\sigma}_0^S$ has two bigonal faces, with endpoints $\{\hat{\sigma}_0^S(1), \hat{\sigma}_0^S(2)\}$ and $\{\hat{\sigma}_0^S(2), \hat{\sigma}_0^S(4)\}$. Since it would be ambiguous to refer to the edges of $\hat{\sigma}_0^S$ by their vertices, we fix the convention that $\hat{\sigma}_0^S(14)$ and $\hat{\sigma}_0^S(24)$ are the edges belonging to the face shared with $\hat{\sigma}_0^R$, while
3. Branched Cauchy-Riemann Structures

Figure 3.6: The tetrahedron $\sigma^L_1$ is decomposed into two 3–cells, one of which is a wedge $\hat{\sigma}^W_1$. The wedge is collapse and removed, while the other 3–cell is deformed back into a tetrahedron $\hat{\sigma}^L_1$.

Figure 3.7: The cell decomposition $D_8$ of the figure eight knot complement $K_8$.

$\hat{\sigma}^S_0(41)$ and $\hat{\sigma}^S_0(42)$ are the others. We will say more about these choices below.

We consider the following finite geometric realisation of $D_8$ in $\overline{H}$. Let $\Xi_A$, $\Xi_B$ and $\Xi_k$ be the two standard symmetric tetrahedra and the slab defined in §3.1.2. The geometric realisations of the ideal cells are the combinatorial isomorphisms defined by

\[
\begin{align*}
\phi^R_0 : \hat{\sigma}^R_0 & \rightarrow \Xi_A, & \phi^S_0 : \hat{\sigma}^S_0 & \rightarrow \Xi_1 & \phi^C : \hat{\sigma}^C_1 & \rightarrow \Xi_B \\
\phi^R_0(1) := P_1, & \phi^S_0(1) := P_1, & \phi^C(1) := P_1, \\
\phi^R_0(2) := P_2, & \phi^S_0(2) := P_2, & \phi^C(2) := P_2, \\
\phi^R_0(3) := P_3, & \phi^S_0(4) := P_4, & \phi^C(3) := P_3, \\
\phi^R_0(4) := P_4, & \phi^C(4) := P_4.
\end{align*}
\]

We remark that $\phi^S_0$ maps the edges $\hat{\sigma}^S_0(14)$ and $\hat{\sigma}^S_0(24)$ to the segments of $C$–circles going from $P_1$ and $P_2$, respectively, to $P_4$. Similarly, $\hat{\sigma}^S_0(41)$ and $\hat{\sigma}^S_0(42)$ are mapped to the segments of $C$–circles going from $P_4$ to $P_1$ and $P_2$, respectively.

The geometric realisations of the face pairings depicted in Figure 3.7 are the matrices $G_j$ defined in §3.1.2, the identity matrix $I$ and a combination thereof. More precisely,

\[A : \hat{\sigma}^C_1(124) \rightarrow \hat{\sigma}^R_0(324)\] is realised by \[G_2 : \Xi_B \rightarrow \Xi_A\],
3.2 Branched CR structures on once-punctured torus bundles

\[ B : \tilde{\sigma}_0^S(124) \to \tilde{\sigma}_1^L(123) \text{ is realised by } G_4 : \Xi_1 \to \Xi_B, \]
\[ C : \tilde{\sigma}_1^L(134) \to \tilde{\sigma}_0^R(132) \text{ is realised by } G_1 : \Xi_B \to \Xi_A, \]
\[ D : \tilde{\sigma}_0^R(134) \to \tilde{\sigma}_1^L(234) \text{ is realised by } G_3 : \Xi_A \to \Xi_B, \]
\[ E : \tilde{\sigma}_1^L(124) \to \tilde{\sigma}_0^S(124) \text{ is realised by } I : \Xi_A \to \Xi_1, \]
\[ F : \tilde{\sigma}_0^S(14) \to \tilde{\sigma}_0^S(24) \text{ is realised by } G_2 G_3 : \Xi_1 \to \Xi_1. \]

The geometric realisations of the face pairings \( A, B, C, D \) and \( E \) are well defined by construction. The product \( G_2 G_3 \), namely the geometric realisation of \( F \), maps the bigonal face \( B' \) of \( \Xi_1 \) to its other bigonal face \( B_1 \). The combinatorics of \( D_8 \) around the red \( e_R \), black \( e'_R \) and blue \( e_L \) edges are displayed in Figure 3.8. One computes that the geometric holonomies are trivial:

\[ e_R : (G_2 G_3)^{-1} G_4^{-1} G_3 G_1 G_4 = I, \quad e'_R : G_3 I^{-1} (G_2 G_3)^{-1} I G_2 = I, \]
\[ e_L : G_1^{-1} I G_4^{-1} G_2^{-1} G_3 G_2 = I. \]

Figure 3.8: The combinatorics around the red \( e_R \), black \( e'_R \) and blue \( e_L \) edges. The view is from the vertices \( \tilde{\sigma}_0^S(4), \tilde{\sigma}_0^R(4) \) and \( \tilde{\sigma}_0^L(4) \) respectively.

As per Lemma 23, this finite geometric realisation of \( D_8 \) in \( \tilde{H} \) corresponds to a branched CR structure on \( K_8 \). By developing the cells in \( \tilde{H} \), one finds that the order of the branching around the edges \( e_R \) and \( e'_R \) is one, while it is two around \( e_L \). These ramification orders were stated incorrectly in [9], and corrected later in [11, Remark 6.1].

Let \( (\text{dev}_8, \text{hol}_8) \) be a representative pair of the associated developing map and holonomy representation. It follows from finiteness of the geometric realisation that \( \text{dev}_8 \) is defined on the 0-skeleton of \( \tilde{D}_8 \), the cell decomposition of the universal cover \( \tilde{K}_8 \) induced by \( D_8 \). Hence let \( \text{dev}_8^{(0)} \) be the restriction of \( \text{dev}_8 \) to the set of vertices \( \tilde{D}_8^{(0)} \). The pair \( (\text{hol}_8, \text{dev}_8^{(0)}) \) is a decorated representation and its class \([\text{hol}_8, \text{dev}_8^{(0)}]\) under the action of \( \text{PGL}(3, \mathbb{C}) \) is a decorated character. We recall from §2.3.1 that part of the character variety \( \mathcal{X}(K_8) \) is birationally equivalent to a subvariety of \( (\mathbb{C}^*)^8 \) via the morphism \( \Psi_8 \). Then one computes

\[ \Psi_8\left([\text{hol}_8, \text{dev}_8^{(0)}]\right) = (\omega, \omega, \omega, \omega, \omega, \omega, \omega, \omega), \quad \text{for} \quad \omega = -\frac{1}{2}(1 + i\sqrt{3}). \]
We deduce that $\left[ \text{hol}_8, \text{dev}^{(0)}_8 \right]$ is the decorated character of $K_8$ obtained by extending the decorated character $[\rho_P, \Phi_P]$ of $T_0$ defined in §2.3.2.

### 3.2.3 General case

This section is devoted to prove the main result of this chapter, namely that every hyperbolic once-punctured torus bundle $M_f$ admits a branched CR structure. In particular, we construct an ideal cell decomposition $D_f$ of $M_f$, and a finite geometric realization of it in $\overline{H}$, with trivial geometric holonomy around each edge. Then we show that the associated holonomy representation arises from the decorated character $[\rho_P, \Phi_P]$ of $T_0$ defined in §2.3.2. We refer to the previous section §3.2.2 for a detailed example of this result.

#### The ideal cell decomposition

Let $f$ be an automorphism of the once punctured torus with two distinct positive real eigenvalues, and let $M_f$ be the corresponding hyperbolic once-punctured torus bundle. Suppose the flip sequence $w_f$ of $M_f$ has length $m$. Then the monodromy ideal triangulation $T_f$ of $M_f$ is made up of $m$ ideal tetrahedra $\sigma_0, \ldots, \sigma_{m-1}$. The ideal cell decomposition $D_f$ of $M_f$ is obtained from $T_f$ by performing the three manipulations described in §3.2.2 to each tetrahedron. We recall from §2.2.2 that a tetrahedron is said to be of type $R$ (resp. type $L$) if the next tetrahedron is layered by a right (resp. left) layering. Thus we modify every tetrahedron of type $R$ as in step (1), and every tetrahedron of type $L$ as in (2) and (3). We provide a synthesis of those operations to refresh the notation.

1. Every tetrahedron $\sigma^R_j$ of type $R$ is subdivided into two 3–cells, along a newly introduced triangular 2–cell with vertices $\{ \sigma^R_j(1), \sigma^R_j(2), \sigma^R_j(4) \}$. They are a tetrahedron $\hat{\sigma}^R_j$ and a slab $\hat{\sigma}^S_j$.

2. Every tetrahedron $\sigma^L_j$ of type $L$ is decomposed into two 3–cells $\hat{\sigma}^L_j \cup \hat{\sigma}^W_j$. The former $\hat{\sigma}^L_j$ has four triangular faces, and a bigon where the wedge $\hat{\sigma}^W_j$ glues to.

3. We deformation retract the wedge $\hat{\sigma}^W_j$ onto a bigonal face, then remove it. Simultaneously, we collapse the bigonal face of $\hat{\sigma}^L_j$ into one edge, transforming $\hat{\sigma}^L_j$ back into a 3–simplex.

Up to step (2), it is easy to check that the performed subdivisions agree along the faces of $T_f$, hence they form a well defined cell decomposition of $M_f$.

Now consider the wedge $\sigma^W_j$. We claim that around each of its edges there is always at least one 3–cell that is not a wedge. This is clear for two of its edges, as it glues to the tetrahedron $\sigma^F_j$. Call $e$ the remaining edge of $\sigma^W_j$. Let $\sigma^L_j$ be the simplex of $T_f$ from which $\sigma^W_j$ is obtained, and let $\sigma^L_{j+1}$ be the next tetrahedron that left layers on top of $\sigma^L_j$. If $\sigma^L_{j+1} = \sigma^L_{j+1}$ is of type $L$, then $\sigma^W_j$ glues to the wedge $\sigma^W_{j+1}$ around $e$. On the other
hand, if $σ_j^* = σ_j^R$ is of type $R$, then $σ_j^W$ glues to the slab $σ_j^S$ around $e$. Because $f$ has two distinct real eigenvalues, its flip sequence always contains at least one $R$ and one $L$ (cf. §2.2.1). It follows that around $e$ there is always at least one slab. This ends the proof of the claim.

On step (3), we flatten the wedges and remove them. It is a consequence of the claim that this does not change the topology of the complex. Thus in the end we have a CW complex $D_f$, consisting of three types of 3–cells, two of which are of the combinatorial type of a tetrahedron and one of which is a slab. The complement of the 0–skeleton is an ideal cell decomposition of $M_f$.

To avoid introducing new terminology, we are going to make the following abuse of notation. Cells of $D_f$ coming from tetrahedra of $T_f$ of type $R$ (resp. type $L$) will also be referred to as cells of type $R$ (resp. type $L$). Moreover, if a tetrahedron $σ_j$ right layers (resp. left layers) on a tetrahedron $σ_{j−1}$ in $T_f$, then also the 3–cells of $D_f$ obtained from $σ_j$ right layer (resp. left layer) on the cells obtained from $σ_{j−1}$.

Combinatorics around the edges

As mentioned in the example of figure eight knot complement, a slab $σ^S_j$ has two bigonal faces, therefore it is ambiguous to refer to its edges by the 0–skeleton. We avoid that by fixing the convention that $σ^S_j(14)$ and $σ^S_j(24)$ are the edges belonging to the face shared with the tetrahedron $σ^*, \star \in \{L, R\}$, while $σ^S_j(41)$ and $σ^S_j(42)$ are the others. The notation is motivated by the natural orientations of the edges of a geometric slab $Σ_k \subset \overline{F}$.

Recall that $π$ is the natural quotient map from the disjoint union of the simplices of $T_f$ into $T_f$, defined by the face pairings. Let $\hat{π}$ be the corresponding map for $D_f$. Then the valence of an edge in $D_f$ is the size of its inverse image under $\hat{π}$.

**Theorem 24.** Let $D_f^{(1)}$ be the set of 1–cells in $D_f$. Let $A \subset \{0, \ldots, m−1\}$ be the subset of indices such that $σ^S_j$ is a slab of $D_f$. Then the quotient map $\hat{π}$ restricts to a bijection

$$\hat{π}_j : \left\{σ^S_j(14)\right\}_{j \in [0, m−1]} \cup \left\{σ^S_j(41)\right\}_{j \in A} \to D_f^{(1)}, \quad \star \in \{L, R\}.$$

Theorem 24 allows us to canonically pick a representative for each edge in $D_f$. For example, in the case of the figure eight knot complement of §3.2.2, the chosen representatives are $σ^L_j(14)$, $σ^R_j(14)$ and $σ^S_j(41)$ (respectively the blue, black and red edge in Figure 3.7). Its proof is a consequence of the following two Lemmas, where we deduce the valence of edges in $D_f$ from their counterparts in $T_f$.

**Lemma 25.** Let $σ^L_j$ be a 3–cell of type $L$ in $D_f$, corresponding to a tetrahedron $σ^L_j$ in $T_f$. Let $2n_j + 4$ be the valence of $π\left(σ^L_j(14)\right)$. Then the equivalence class of $σ^L_j(14)$ in $D_f$ is

$$\left\{σ^L_j(14), σ^R_j(12), σ^S_j(12), σ^R_{j+k}(34)\right\}_{k=1, \ldots, n_j}, \quad σ^L_{j+n_j+1}(12), σ^L_{j+n_j+1}(34), σ^L_{j+n_j+2}(23),$$
where \( \star \in \{ \mathcal{L}, \mathcal{R} \} \). In particular \( \hat{\mathcal{R}}(\hat{\sigma}_f^L(14)) \) has valence \( 3n_j + 4 \).

**Proof.** By Lemma 10, the edge \( \sigma_j^L(14) \) corresponds to a unique subsequence \( \mathcal{L}^n / \mathcal{L} \) of \( \mathcal{w}_f \), for \( n_j \geq 0 \). In particular, \( \sigma_j^L \) is the bottom of a unique ribbon of tetrahedra

\[
\sigma_j^L \sigma_{j+1} \cdots \sigma_{j+n_j} \sigma_{j+n_j+1}^* \sigma_{j+n_j+2}^*,
\]

where \( \star \in \{ \mathcal{L}, \mathcal{R} \} \) is undetermined. Whence \( \sigma_j^L(14) \) is identified with the edges

\[
\sigma_j^L(14), \{ \sigma_{j+k}^L(12), \sigma_{j+k}^R(34) \}_{k=1, \ldots, n_j}, \sigma_{j+n_j+1}^L(12), \sigma_{j+n_j+1}^L(34), \sigma_{j+n_j+2}^*(23).
\]

The valence of its equivalence class in \( \mathcal{T}_f \) is \( 2n_j + 4 \). In \( \mathcal{D}_f \), we introduce a slab around each edge \( \sigma_{j+k}^R(12) \), while neighbourhoods of the other edges glued to \( e_j \) are unchanged (cf. Figure 3.9 for \( j = 0 \)). The statement of the Lemma follows.

**Lemma 26.** Let \( \hat{\sigma}_j^\mathcal{R} \) and \( \hat{\sigma}_j^\mathcal{S} \) be 3–cells of type \( \mathcal{R} \) in \( \mathcal{D}_f \), corresponding to a tetrahedron \( \sigma_j^\mathcal{R} \) in \( \mathcal{T}_f \). Let \( 2n_j + 4 \) be the valence of \( \pi(\sigma_j^\mathcal{R}(14)) \). Then the equivalence class of \( \hat{\sigma}_j^\mathcal{R}(14) \) in \( \mathcal{D}_f \) is

\[
\{ \hat{\sigma}_j^\mathcal{R}(14), \hat{\sigma}_j^\mathcal{R}(14), \{ \hat{\sigma}_{j+k}^\mathcal{L}(24) \}_{k=1, \ldots, n_j}, \hat{\sigma}_{j+n_j+1}^\mathcal{L}(24), \hat{\sigma}_{j+n_j+1}^\mathcal{S}(24) \}.
\]

Similarly, the equivalence class of \( \hat{\sigma}_j^\mathcal{S}(41) \) in \( \mathcal{D}_f \) is

\[
\{ \hat{\sigma}_j^\mathcal{S}(41), \{ \hat{\sigma}_{j+k}^\mathcal{L}(13) \}_{k=1, \ldots, n_j}, \hat{\sigma}_{j+n_j+1}^\mathcal{L}(42), \hat{\sigma}_{j+n_j+1}^\mathcal{S}(23) \}.
\]

In particular, both \( \hat{\sigma}_j^\mathcal{R}(14) \) and \( \hat{\sigma}_j^\mathcal{S}(14) \) have valence \( n_j + 4 \).

**Proof.** As in the proof of Lemma 25, the edge \( \sigma_j^\mathcal{R}(14) \) corresponds to a unique subsequence \( \mathcal{R}^n / \mathcal{R} \) in \( \mathcal{w}_f \), for \( n_j \geq 0 \). The ribbon of tetrahedra around its edge class in \( \mathcal{T}_f \) is

\[
\sigma_j^\mathcal{R} \sigma_{j+1} \cdots \sigma_{j+n_j} \sigma_{j+n_j+1}^\mathcal{R} \sigma_{j+n_j+2}^*.
\]

In particular \( \sigma_j^\mathcal{R}(14) \) is glued to the \( 2n_j + 4 \) edges

\[
\sigma_j^\mathcal{R}(14), \{ \sigma_{j+k}^\mathcal{L}(13), \sigma_{j+k}^\mathcal{L}(24) \}_{k=1, \ldots, n_j}, \sigma_{j+n_j+1}^\mathcal{R}(13), \sigma_{j+n_j+1}^\mathcal{R}(24), \sigma_{j+n_j+2}^*(23).
\]

In \( \mathcal{D}_f \), the cell \( \sigma_j^\mathcal{R}(14) \) splits into the bigon with boundary \( \hat{\sigma}_j^\mathcal{S}(14) \) and \( \hat{\sigma}_j^\mathcal{S}(41) \). The two loops of the ribbon of tetrahedra around \( \sigma_j^\mathcal{R}(14) \) are split and equidistributed around those two edges (cf. Figure 3.10 for \( j = 0 \)). The statement of the Lemma follows.

**Proof of Theorem 24.** First we notice that \( \hat{\pi} \) is well defined, as it is the restriction of the natural quotient map \( \hat{\pi} \). Injectivity follows from Lemma 25 and Lemma 26, because the equivalence classes of \( \hat{\sigma}_f^L(14), \hat{\sigma}_f^R(14) \) and \( \hat{\sigma}_f^S(41) \) are distinct.

By a topological argument, we deduce that the Euler characteristic of \( \mathcal{D}_f \) is zero. Therefore \( \mathcal{D}_f \) has as many 3–cells as 1–cells, and \( |\mathcal{D}_f^{(1)}| = m + s \). It follows that \( \hat{\pi} \) is an injective map between finite sets with the same sizes, thus it is a bijection.
3.2 Branched CR structures on once-punctured torus bundles

Figure 3.9: The cross section of a neighbourhood of $\hat{\pi} \left( \hat{\sigma}^L_0(14) \right)$ in $\mathcal{D}_f$, viewed from the vertex $\hat{\sigma}^L_0(4)$.

Figure 3.10: The edge splits into two edges, $\hat{\pi} \left( \hat{\sigma}^S_0(41) \right)$ on the left and $\hat{\pi} \left( \hat{\sigma}^S_0(14) \right)$ on the right. The view is from the vertex $\hat{\sigma}^S_0(4)$.

The finite geometric realisation in $\overline{\mathcal{H}}$

A finite geometric realisation of $\mathcal{D}_f$ consists of embeddings $\phi^L_j, \phi^R_j, \phi^S_j$ of the 3–cells into $\overline{\mathcal{H}}$, and geometric realisations $G_j \in \PU(2, 1)$ of the face pairings.

Let $\hat{\sigma}^*_j$ be a tetrahedron of $\mathcal{D}_f$, $\star \in \{ \mathcal{L}, \mathcal{R} \}$. The development of $\hat{\sigma}^*_j$ depends on the tetrahedron it layers on. More precisely, let $\hat{\sigma}_{j-1}^*$ be the tetrahedron in $\mathcal{D}_f$ on top of which $\hat{\sigma}^*_j$ layers. Then the geometric realisation $\phi^*_j$ of $\hat{\sigma}^*_j$ is the combinatorial isomorphism

$$
\phi^*_j : \begin{cases} 
\hat{\sigma}^*_j \to \Xi_B & \text{if } \hat{\sigma}_{j-1} = \hat{\sigma}^R_{j-1} \text{ is of type } \mathcal{R}, \\
\hat{\sigma}^*_j \to \Xi_A & \text{if } \hat{\sigma}_{j-1} = \hat{\sigma}^L_{j-1} \text{ is of type } \mathcal{L},
\end{cases}
$$

where

$$
\phi^*_j (\hat{\sigma}^*_j(1)) := P_1, \quad \phi^*_j (\hat{\sigma}^*_j(2)) := P_2, \quad \phi^*_j (\hat{\sigma}^*_j(3)) := P_3, \quad \phi^*_j (\hat{\sigma}^*_j(4)) := P_4.
$$

Now let $\hat{\sigma}^S_j$ be a slab of $\mathcal{D}_f$. Let $k_j + 4$ be the valence of the edge $\hat{\pi} \left( \hat{\sigma}^S_j(24) \right)$. Then the geometric realisation $\phi^S_j$ of $\hat{\sigma}^S_j$ is the combinatorial isomorphism

$$
\phi^S_j : \hat{\sigma}^S_j \to \Xi_{k_j}
$$

where

$$
\phi^S_j (\hat{\sigma}^S_j(1)) := P_1, \quad \phi^S_j (\hat{\sigma}^S_j(2)) := P_2, \quad \phi^S_j (\hat{\sigma}^S_j(4)) := P_4.
$$

More precisely, we require that $\phi^S_j (\hat{\sigma}^S_j(14)) = [P_1, P_4]$ and $\phi^S_j (\hat{\sigma}^S_j(24)) = [P_2, P_4]$. Thus the bigon with endpoints $\{ \hat{\sigma}^S_j(1), \hat{\sigma}^S_j(24) \}$ is developed into

$$
B' := \left( 1 + te^{-i \hat{\pi}}, s \right), \quad t \in \mathbb{R}_{>0} \cup \{ \infty \}, \quad s \in \mathbb{R} \cup \{ \infty \},
$$
3. Branched Cauchy-Riemann Structures

while the bigon with endpoints \( \{ \hat{\sigma}^S_j(2), \hat{\sigma}^S_j(4) \} \) is realised by

\[
B_{kj} := \left( -\omega + te^{-i\pi(1-2kj)}, s \right), \quad t \in \mathbb{R}_{>0} \cup \{ \infty \}, \quad s \in \mathbb{R} \cup \{ \infty \}.
\]

Both \( B' \) and \( B_k \) are foliated by vertical \( \mathbb{C} \)-circles.

Most of the geometric realisations of the face pairings are uniquely determined by Lemma 16. They are the CR transformations \( G_i \) described in §3.1.2. The remaining ones are either the identity matrix \( I \), or products of the \( G_i \)'s. We describe them in more detail below. Let \( \hat{\sigma}^*_j \) be a tetrahedron of \( D_f \), of type \( * \in \{ \mathcal{L}, \mathcal{R} \} \).

If \( \phi^*_j \left( \hat{\sigma}^*_j \right) = \Xi_A \) is the standard symmetric tetrahedron of type \( A \), then \( \hat{\sigma}^*_j \) layers on a tetrahedron \( \hat{\sigma}^\mathcal{L}_{j-1} \) of type \( \mathcal{L} \). In particular they share two pairs of faces. Let \( \Xi_X = \phi^\mathcal{L}_{j-1} ( \hat{\sigma}^\mathcal{L}_{j-1} ) \) for some \( X \in \{ A, B \} \). Then the geometric realisations of the face pairings between \( \hat{\sigma}^\mathcal{L}_{j-1} \) and \( \hat{\sigma}^*_j \) are

\[
\begin{align*}
\hat{\sigma}^\mathcal{L}_{j-1}(134) & \rightarrow \hat{\sigma}^*_j(132) \quad \text{is realised by} \quad G_1 : \Xi_X \rightarrow \Xi_A, \\
\hat{\sigma}^\mathcal{L}_{j-1}(124) & \rightarrow \hat{\sigma}^*_j(324) \quad \text{is realised by} \quad G_2 : \Xi_X \rightarrow \Xi_A.
\end{align*}
\]

Now suppose \( \phi^*_j \left( \hat{\sigma}^*_j \right) = \Xi_B \) is the standard symmetric tetrahedron of type \( B \). In this case \( \hat{\sigma}^*_j \) layers on a tetrahedron \( \hat{\sigma}^\mathcal{R}_{j-1} \) of type \( \mathcal{R} \) and on a slab \( \hat{\sigma}^\mathcal{S}_{j-1} \). Let \( \Xi_X = \phi^\mathcal{R}_{j-1} ( \hat{\sigma}^\mathcal{R}_{j-1} ) \), for some \( X \in \{ A, B \} \), and let \( \Xi_{k_{j-1}} = \phi^\mathcal{S}_{j-1} ( \hat{\sigma}^\mathcal{S}_{j-1} ) \). Then the geometric realisations of the face pairings between \( \hat{\sigma}^\mathcal{R}_{j-1}, \hat{\sigma}^\mathcal{S}_{j-1} \) and \( \hat{\sigma}^*_j \) are

\[
\begin{align*}
\hat{\sigma}^\mathcal{R}_{j-1}(134) & \rightarrow \hat{\sigma}^*_j(234) \quad \text{is realised by} \quad G_3 : \Xi_X \rightarrow \Xi_B, \\
\hat{\sigma}^\mathcal{S}_{j-1}(124) & \rightarrow \hat{\sigma}^*_j(123) \quad \text{is realised by} \quad G_4 : \Xi_{k_{j-1}} \rightarrow \Xi_B, \\
\hat{\sigma}^\mathcal{R}_{j-1}(134) & \rightarrow \hat{\sigma}^\mathcal{S}_{j-1}(124) \quad \text{is realised by} \quad I : \Xi_X \rightarrow \Xi_{k_{j-1}}.
\end{align*}
\]

These cover all cases, except for the gluing maps between the bigonal faces of the slabs. Contrary to marked triangles, bigons in Heisenberg space can be identified via many CR transformations. Earlier in this section we showed that around each edge in \( D_f \) there is at most one face pairing gluing two slabs along their bigons (cf. Lemma 25 and Lemma 26). Whence we are going to geometrically realise those face pairings so that the geometric holonomy around each edge is trivial. Under this condition, the choices turn out to be unique.

Consider the slab \( \hat{\sigma}^\mathcal{S}_j \). By Lemma 26, the equivalence class of the edge \( \hat{\sigma}^\mathcal{S}_j(14) \) is

\[
\left\{ \hat{\sigma}^\mathcal{S}_j(14), \hat{\sigma}^\mathcal{R}_j(14), \left\{ \hat{\sigma}^\mathcal{L}_{j+k}(24) \right\}_{k=1,\ldots,n_j}, \hat{\sigma}^\mathcal{R}_{j+n_j+1}(24), \hat{\sigma}^\mathcal{S}_{j+n_j+1}(24) \right\}.
\]

Let \( A_j \ldots A_{j+n_j+2} \) be the sequence of geometric realisations of the face pairings around \( \hat{\sigma} \left( \hat{\sigma}^\mathcal{S}_j(14) \right) \), starting from \( \hat{\sigma}^\mathcal{S}_j \) to \( \hat{\sigma}^\mathcal{S}_{j+n_j+1} \), travelling anticlockwise from the point of view of
Lemma 27. The matrix product $\prod_{k=0}^{n_j+2} A_{j+n_j+2-k}$ is a geometric realisation of the face pairing between $\hat{\sigma}^S_j$ and $\hat{\sigma}^S_{j+n_j+1}$. In particular, it identifies the bigon $B'$ of $\phi^S_j\left(\hat{\sigma}^S_j\right)$ with the bigon $B_{n_j}$ of $\phi^S_{j+n_j+1}\left(\hat{\sigma}^S_{j+n_j+1}\right)$.

Proof. By construction, $A_j$ and $A_{j+n_j+2}$ are the identity matrix. On the other hand, $A_{j+1} = G_3$ and $A_{j+k} = G_2$, for all $k \in \{2, \ldots, n_j + 1\}$. Therefore

$$\prod_{k=0}^{n_j+2} A_{j+n_j+2-k} = G_3^{n_j} G_3.$$ 

We recall from §3.1.2 that the CR transformations $G_3$ and $G_2$ preserve vertical $C$–circles, and restrict to rotations on the $z$–plane. In particular, $G_3$ maps $B'$ to the bigon $B_0$ and $G_2$ maps $B_k$ to $B_{k+1}$. The Lemma follows.

We remark that the face pairing $\prod_{k=0}^{n_j+2} A_{j+n_j+2-k}$ is monotone, thus this completes the construction of the finite geometric realisation of $D_f$. We conclude the section by showing that these geometric realisations are indeed branched CR structures.

Theorem 28. The geometric holonomy around each edge in $D_f$ is trivial and therefore the geometric realisation defines a branched CR structure on $M_f$.

Proof. We recall that by Theorem 24 there is a canonical representative for each edge in $D_f$.

Let $A \subset \{0, \ldots, m-1\}$ be the subset of indexes such that $\hat{\sigma}^S_j$ is a slab of $D_f$, and let $\bar{A} = \{0, \ldots, m-1\} \setminus A$ be its complement. It is a consequence of Lemma 27 that the geometric holonomy around the edges $\hat{\sigma}^S_j (14)$, for $j \in A$, is trivial.

Consider an edge $\hat{\sigma}^S_j (41)$, for $j \in A$. Let $A_j, \ldots, A_{j+n_j+1}$ be the sequence of geometric realisations of all the face pairings around $\hat{\sigma}^S_j (41)$, starting from $\hat{\sigma}^S_j$ and travelling clockwise from the point of view of the vertex $\hat{\sigma}^S_j (4)$ (cf. Figure 3.10 on the left). Then we have

$$A_j = G_4, \quad A_{j+k} = G_1 \quad \text{for} \quad k \in \{1, \ldots, n_j\},$$

$$A_{j+n_j+1} = G_3, \quad A_{j+n_j+2} = G_4^{-1}, \quad A_{j+n_j+3} = G_3 G_2^{-n_j}.$$ 

Thus the geometric holonomy around $\hat{\sigma}^S_j (41)$ is the product $G_3 G_2^{-n_j} G_4^{-1} G_3 G_1^{n_j} G_4$. Because the matrices $G_1$ and $G_2$ are of order six, one only needs to check that the product is the identity matrix for $n_j \in \{0, \ldots, 5\}$. Straight forward computation of the six products gives
the result.

An analogous argument works for the edges \( \hat{\sigma}^j \left( \hat{\sigma}^j(14) \right) \), \( j \in \bar{A} \). The geometric holonomy around them is of the form \( G_1^{-1}G_4^{-n_j}G_2^{-1}G_1G_3^nG_2 \). The matrices \( G_3 \) and \( G_4 \) are also of order six, hence one only needs to check that the cases \( n_j \in \{0, \ldots, 5\} \). The calculation is straightforward.

We apply Lemma 23 to complete the proof.

The decorated character of the structure

We conclude by showing that the decorated characters associated to the branched CR structures are extensions of the special decorated character \([\rho_P, \Phi_P]\) of the once punctured torus \( T_0 \), found in §2.3.2.

Let \( M_f \) be a hyperbolic once-punctured torus bundle. Consider the finite geometric realisation of its cell decomposition \( D_f \) and the corresponding branched CR structure. Let \((\text{dev}_f, \text{hol}_f)\) be a representative pair of the associated developing map and holonomy representation. By finiteness of the geometric realisation, the developing map extends to the 0–skeleton of \( \widetilde{D}_f \), the cell decomposition of the universal cover \( \widetilde{M}_f \) induced by \( D_f \). Denote by \( \text{dev}_f^{(0)} \) the restriction of \( \text{dev}_f \) to the set of vertices \( \widetilde{D}_f^{(0)} \). The pair \((\text{hol}_f, \text{dev}_f^{(0)})\) is a decorated representation of \( M_f \), and its \( \text{PGL}(3, \mathbb{C}) \)–class \([\text{hol}_f, \text{dev}_f^{(0)}]\) is a decorated character.

We recall that the fundamental group of \( M_f \) is an HNN extension of the free group in two generators \( \langle \alpha, \beta \rangle \). It has a standard presentation

\[
\pi_1(M_f) = \langle \alpha, \beta, \tau \mid \tau a \tau^{-1} = f_\ast(\alpha), \ \tau \beta \tau^{-1} = f_\ast(\beta) \rangle,
\]

where \( f_\ast : \langle \alpha, \beta \rangle \to \langle \alpha, \beta \rangle \) is the automorphism induced by \( f \), and \( \tau \) is represented by the base circle of the fiber bundle. As a consequence, there is a (generally) injective morphism \( \iota_\ast^\times : \mathcal{X}(M_f) \to \mathcal{X}(T_0) \) and a parametrisation \( \Psi_{T_0} : \mathcal{X}(T_0) \to (\mathbb{C}^\times)^8 \) (cf. §2.3.1).

By choosing the appropriate class representative \((\text{dev}_f, \text{hol}_f)\), one computes that

\[
\text{hol}_f(\alpha) = G_4^{-1}G_3 \quad \text{and} \quad \text{hol}_f(\beta) = G_1^{-1}G_2.
\]

This is conjugate to the representation \( \rho_P \) found in §2.3.2, by the \( \text{PGL}(3, \mathbb{C}) \) matrix

\[
M = \begin{bmatrix}
0 & 0 & -1 \\
-\omega & 1 & \omega \\
\omega & 0 & -\omega
\end{bmatrix}.
\]

More precisely \( M^{-1} \text{hol}_f M = \rho_P \). Furthermore, it is a straight forward computation to check that \( M \) maps the decoration \( \Phi_P \) to \( \text{dev}_f^{(0)} \). It follows that

\[
\iota_\ast^\times([\text{hol}_f, \text{dev}_f^{(0)}]) = [\rho_P, \Phi_P] \quad \text{and} \quad \Psi_f([\text{hol}_f, \text{dev}_f^{(0)}]) = (\omega, \omega, \omega, \omega, \omega, \omega, \omega, \omega),
\]

where \( \omega = -\frac{1}{2} (1 + i\sqrt{3}) \).
3.2.4 Branch locus

Consider the branched CR structure on the hyperbolic once-punctured torus bundle $M_f$ described in the previous section §3.2.3. Here we analyse its branch locus, namely the set of all ideal edges of the associated cell decomposition $D_f$. In particular, we show that the ramification order around each curve is related to their valence in the simplicial complex. The strategy will be to develop each curve as a vertical line in Heisenberg space, and analyse the projection onto the $\mathbb{C}$–plane of a neighbourhood. This way we can talk about angles of the projections where otherwise it would not be possible. We remind the reader that CR transformations do not preserve angles, therefore the angles we are going to talk about depend on the chosen realisations.

We recall that by Theorem 24 there is a canonical representative for each edge in $D_f$, namely $\hat{\sigma}^R_j(14), \hat{\sigma}^S(j)(41)$ and $\hat{\sigma}^C_j(14)$. Let ceiling$(x) = \lceil x \rceil$ be the ceiling function, which associates $x$ to the smallest integer greater than or equal to $x$.

**Lemma 29.** Let $n_j + 4$ be the valence of $e_j = \hat{\pi} \left( \hat{\sigma}^R_j(14) \right)$ in $D_f$. Then the ramification order around $e_j$ is $\lceil \frac{n_j+5}{6} \rceil$.

**Proof.** First, we observe that the geometric realisation $\phi^R_j$ develops the edge $\hat{\sigma}^R_j(14)$ into the vertical ray of Heisenberg space going from $P_1 = (1, \sqrt{3}i)$ to $P_4 = \infty$. Therefore we can understand the ramification order of $e_j$ by looking at the projections of the tetrahedra around $e_j$ on the $\mathbb{C}$–plane of $\mathcal{H}$.

Let $R \subset \mathbb{C}$ be the projection of the standard symmetric tetrahedron. It is a triangular region bounded by three arcs of circles (cf. Figure 3.2). We recall from §3.2.3 (cf. Figure 3.10) that the sequence of 3–cells around $e_j$ in $D_f$ is

$$\hat{\sigma}^S_j, \hat{\sigma}^R_j, \hat{\sigma}^C_{j+1}, \ldots, \hat{\sigma}^C_{j+n_j}, \hat{\sigma}^R_j, \hat{\sigma}^S_{j+n_j+1}, \hat{\sigma}^S_{j+n_j+1}.$$  

Then $\phi^R_j \left( \hat{\sigma}^R_j \right)$ projects onto $R_j := R$. The next simplex glues to $\phi^R_j \left( \hat{\sigma}^R_j \right)$ via $G_2^{-1}$, therefore its projection $R_{j+1}$ is a $\frac{\pi}{3}$ clockwise rotation of $R_j$ around the origin. After that, we have $n_j$ simplices each of which is glued to the previous one by $G_2^{-1}$. Whence each of their projections $R_{j+k}$, for $k \in \{1, \ldots, n_j + 1\}$, is a $\frac{\pi}{3}$ anticlockwise rotation of $R_{k-1}$ about the point $1$. Finally, the projections of the geometric realisations of the two slabs $\hat{\sigma}^S_j, \hat{\sigma}^S_{j+n_j+1}$ rigidly glue to $R_j$ and $R_{j+n_j+1}$ to fill in the gap. Examples for $n_j = 1$ and $n_j = 3$ are depicted in Figure 3.11.

Around $e_j$, the first region $R_j$ contributes with an angle of $\frac{2\pi}{3}$, while every other region $R_{j+k}$, for $k \in \{1, \ldots, n_j + 1\}$, contributes with an angle of $\frac{2\pi}{3}$. The first slab also adds $\frac{2\pi}{3}$. This sums up to $\left( \frac{3n_j}{6} \right) 2\pi$. The angle of the projection of the last slab around $e_j$ is a non-negative number strictly lower than $2\pi$, therefore the total angle is the next integer multiple of $2\pi$. That is $\lceil \frac{n_j+5}{6} \rceil 2\pi$. ■

**Lemma 30.** Let $n_j + 4$ be the valence of $e_j = \hat{\pi} \left( \hat{\sigma}^S_j(41) \right)$ in $D_f$. Then the ramification order around $e_j$ is $\lceil \frac{n_j+5}{6} \rceil$. 

Figure 3.11: The developments around the branch locus for \( n_j = 1 \) (on the left) and \( n_j = 3 \) (on the right). Their respective ramification orders are one and two.

**Proof.** This proof is similar to the one of Lemma 29, as the geometric realisation \( \phi_j^S \) develops the edge \( \hat{\sigma}^S_j(41) \) into the vertical ray from \( P_4 = \infty \) to \( P_1 = (1, \sqrt{3}) \). The only difference is that we are not going to consider the projections of the entire cells, since they are not as tidy as in the previous case, but only the projections of the vertices. Every 3–cell around \( e_j \) has two vertices at \( P_4 = \infty \) and \( P_1 = (1, \sqrt{3}) \), and its angle about \( e_j \) is strictly between zero and \( 2\pi \). Therefore knowing the positions of the other vertices gives us an estimate of the total angle around \( e_j \).

The sequence of 3–cells around \( e_j \) in \( D_f \) is

\[
\hat{\sigma}^S_j, \hat{\sigma}^L_{j+1}, \ldots, \hat{\sigma}^L_{j+n_j}, \hat{\sigma}^R_{j+n_j+1}, \hat{\sigma}^L_{j+n_j+2}, \hat{\sigma}^S_{j+n_j+1},
\]

for some \( \star \in \{L, R\} \). We begin by developing \( \phi_j^S(\hat{\sigma}^S_j) \), then glue every other 3–cell around \( e_j \). The vertices that are not identified with the endpoints of \( e_j \) are listed in Table 3.1. They are all positioned at the vertices of a regular hexagon of edge length \( \sqrt{3}/2 \).

We draw examples of the projections for \( n_j = 1 \) and \( n_j = 3 \) in Figure 3.12. We remark that these are projections of the vertices and edges, but not of the 2–skeletons as faces are generally not foliated by vertical rays anymore.

Up to \( \hat{\sigma}^L_{j+n_j+2} \), the total sum of the angles is strictly between \( [n_j/6] \) and \( [n_j+5/6] \). Because the angle of the projection of the last slab around \( e_j \) is a non-negative number strictly lower than \( 2\pi \), the ramification order must be \( [n_j+5/6] \).

**Lemma 31.** Let \( 3n_j + 4 \) be the valence of \( e_j = \hat{\pi}(\hat{\sigma}_j^L(14)) \) in \( D_f \). Then the ramification order around \( e_j \) is \( n_j + 1 \).

**Proof.** We follow almost verbatim the proof of Lemma 30.

First we consider the development \( \phi_j^L(\hat{\sigma}_j^L) \). This geometric realisation maps the edge \( \hat{\sigma}_j^L(14) \) into the vertical ray from \( P_1 = (1, \sqrt{3}) \) to \( P_4 = \infty \). From the point of view of the vertex \( \hat{\sigma}_j^L(4) \) (cf. Figure 3.9), starting from \( \hat{\sigma}_j^L \) and travelling anticlockwise around \( e_j \) until \( \sigma_{j+n_j+2}^L \), we encounter the 3–cells

\[
\sigma_j^L \sigma_{j+1}^R \sigma_{j+1}^S \cdots \sigma_{j+n_j}^R \sigma_{j+n_j}^S \sigma_{j+n_j+1}^L \sigma_{j+n_j+2}^L.
\]
3.2 Branched CR structures on once-punctured torus bundles

<table>
<thead>
<tr>
<th>CR face pairing</th>
<th>3–cells</th>
<th>Vertices disjoint from ( e_j )</th>
<th>( \mathbb{C} )-coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_4 )</td>
<td>( \hat{\sigma}_j^S )</td>
<td>( \hat{\sigma}_j^S(2) )</td>
<td>(-\omega)</td>
</tr>
<tr>
<td></td>
<td>( \hat{\sigma}_j^L )</td>
<td>( \hat{\sigma}_j^L(2) )</td>
<td>(-\omega)</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>( \hat{\sigma}_j^L(4) )</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>( \hat{\sigma}_j^L(1+1) )</td>
<td></td>
<td>((-1)^k \omega^{k-1} + 1)</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>( \hat{\sigma}_j^R(4) )</td>
<td></td>
<td>((-1)^n_j \omega^{n_j-1} + 1)</td>
</tr>
<tr>
<td>( G_4^{-1} )</td>
<td>( \hat{\sigma}_j^S(1) )</td>
<td></td>
<td>((-1)^n_j \omega^{n_j+1} + 1)</td>
</tr>
</tbody>
</table>

Table 3.1: The list of vertices of the 3–cells around \( e_j \) that are not identified with the endpoints of \( e_j \). We recall that \( \omega = -\frac{1}{2}(1+i\sqrt{3}) \).

The vertices of these cells that are not identified with the endpoints of \( e_j \) are listed in Table 3.2.

Similarly, if we travel clockwise around \( e_j \), we have

\[
\sigma_j^L \sigma_{j+1}^R \cdots \sigma_{j+n_j}^R \sigma_{j+n_j+1}^L \sigma_{j+n_j+2}^*.
\]

The vertices of these cells that are not identified with the endpoints of \( e_j \) are summarised in Table 3.3.

Figure 3.12: The developments around the branch locus for \( n_j = 1 \) (on the left) and \( n_j = 3 \) (on the right). Only vertices and edges are projected. The shaded areas are just guidelines to distinguish the different cells, but they are not the actual projections of the 3–cells. The respective ramification orders are one and two.

The vertices of these cells that are not identified with the endpoints of \( e_j \) are listed in Table 3.2.

Similarly, if we travel clockwise around \( e_j \), we have

\[
\sigma_j^L \sigma_{j+1}^R \cdots \sigma_{j+n_j}^R \sigma_{j+n_j+1}^L \sigma_{j+n_j+2}^*.
\]

The vertices of these cells that are not identified with the endpoints of \( e_j \) are summarised in Table 3.3.

We remark that for all \( k = 1, \ldots, n_j \), the 3-cells \( \hat{\sigma}_j^R \) and \( \hat{\sigma}_j^S \) cover a total angle of \( 2\pi \) around \( e_j \). When \( n_j = 0 \), the total angle around \( e_j \) is exactly of \( 2\pi \), hence in the general case the ramification order around \( e_j \) is \( n_j + 1 \).
### Table 3.2: The list of vertices of some of the 3–cells around \( e_j \) that are not identified with the endpoints of \( e_j \).

<table>
<thead>
<tr>
<th>CR face pairing</th>
<th>3–cells</th>
<th>Vertices disjoint from ( e_j )</th>
<th>( \Sigma )–coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>( \hat{\sigma}_j^C )</td>
<td>( \hat{\sigma}_j^C(2) )</td>
<td>(-\omega)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\sigma}_j^C(3) )</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\sigma}_{j+k}^R )</td>
<td>( k = 1, \ldots, n_j )</td>
<td>( \hat{\sigma}_{j+k}^R(3) )</td>
<td>((-1)^{k+1} \omega^{k-1} + 1)</td>
</tr>
<tr>
<td>( I )</td>
<td>( \hat{\sigma}_{j+k}^S )</td>
<td>( k = 1, \ldots, n_j )</td>
<td>((-1)^{k+1} \omega^k + 1)</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>( \hat{\sigma}_{j+n_j+1}^C )</td>
<td>( \hat{\sigma}_{j+n_j+1}^C(3) )</td>
<td>((-1)^{n_j+1} \omega^{n_j} + 1)</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( \hat{\sigma}_{j+n_j+2}^\star )</td>
<td>( \hat{\sigma}_{j+n_j+2}^\star(4) )</td>
<td>((-1)^{n_j+3} \omega^{n_j+2} + 1)</td>
</tr>
</tbody>
</table>

Table 3.3: The list of vertices of the remaining 3–cells around \( e_j \) that are not identified with the endpoints of \( e_j \).

<table>
<thead>
<tr>
<th>CR face pairing</th>
<th>3–cells</th>
<th>Vertices disjoint from ( e_j )</th>
<th>( \Sigma )–coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_2 )</td>
<td>( \hat{\sigma}_j^C )</td>
<td>( \hat{\sigma}_j^C(3) )</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\sigma}_j^C(2) )</td>
<td>(-\omega)</td>
</tr>
<tr>
<td>( \hat{\sigma}_{j+k}^R )</td>
<td>( k = 1, \ldots, n_j )</td>
<td>( \hat{\sigma}_{j+k}^R(2) )</td>
<td>((-1)^{k+1} \omega^k + 1)</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>( \hat{\sigma}_{j+n_j+1}^C )</td>
<td>( \hat{\sigma}_{j+n_j+1}^C(2) )</td>
<td>((-1)^{n_j+2} \omega^{n_j+1} + 1)</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>( \hat{\sigma}_{j+n_j+2}^\star )</td>
<td>( \hat{\sigma}_{j+n_j+2}^\star(4) )</td>
<td>((-1)^{n_j+3} \omega^{n_j+2} + 1)</td>
</tr>
</tbody>
</table>

We recall that \( \omega = -\frac{1}{2} (1 + i\sqrt{3}) \).
Appendix

A.1 A different set of coordinates

Following the work of Fock and Goncharov in [14], Bergeron, Falbel and Guilloux (BFG in short) define a set of coordinates parametrising tetrahedra of flags in [3]. In the spirit of Thurston [35], they proceed to define edge conditions and face conditions for the ideally triangulated orientable hyperbolic 3–manifold \((M, T)\). Solutions to these equations correspond to conjugacy classes of decorated representations of \(\pi_1(M)\) into \(\text{PGL}(3, \mathbb{C})\).

This work builds on a previous construction from Falbel in [9], where he parametrises representations into \(\text{PU}(2, 1)\) instead.

The same construction was independently developed by Garoufalidis, Thurston and Zickert for boundary-unipotent representations into \(\text{SL}(m, \mathbb{C})\) (Ptolemy coordinates using affine flags [17]), and by Garoufalidis, Goerner and Zickert for representations in \(\text{PGL}(m, \mathbb{C})\) (shape coordinates using projective flags [16]). On top of being more general, shape parameters and Ptolemy coordinates were shown to be exhaustive (when the triangulation is fine enough), in the sense that they completely parametrise the respective codomains.

In this section we recall BFG’s parametrisation from [3], and compare their coordinate system with the one developed in §2.

A.1.1 The deformation variety

Let \(\mathfrak{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)\) be an ordered quadruple of flags \(\mathcal{F}_j = (V_j, \eta_j)\), in general position. It is convenient to symbolically think of these flags as the vertices of the standard tetrahedron \(\sigma\), oriented as the orientation induced by the ordering of \(\mathfrak{F}\). In that case, \(\mathfrak{F}\) is referred to as a flag tetrahedron.

Bergeron, Falbel and Guilloux define a set of 16 coordinates for a flag tetrahedron, one for each face and one for each oriented edge.

- **Faces:** at each face \(\sigma(ijk)\), oriented as the boundary of the tetrahedron, we associate the triple ratio

\[
z_{ijk} := \mathfrak{A}(\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k).
\]
• **Oriented edges:** for each oriented edge $\sigma(ij)$, let $k$ and $l$ such that the permutation $(1234) \mapsto (ijkl)$ is even. The parameter associated to $\sigma(ij)$ is

$$z_{ij} := -\text{CR}(\xi_i, V_i V_k, V_i V_j, V_i V_l) = -\mathcal{A}(\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k, \mathcal{F}_l).$$

We remark that in [3], parameters of oriented edges are usually defined in terms of a different cross ratio, which is obtained from CR by changing sign and permuting the second and the third entry.

These coordinates are related by *internal relations*: for every even permutation $(ijkl)$ of $(1234)$,

$$z_{ik} = \frac{1}{1 - z_{ij}}, \quad z_{il} = 1 - \frac{1}{z_{ij}} \quad \text{and} \quad z_{ijk} = -z_{il} z_{jl} z_{kl}.$$

The configuration space of flag tetrahedra is $\text{Conf}_4$, the space of ordered quadruple of flags in general position modulo the action of $\text{PGL}(3, \mathbb{C})$. It is the subvariety of $\mathbb{C}^4$ determined by the internal relations, and it is biholomorphic to $(\mathbb{C}^\ast)^4$. In particular the four coordinates $(z_{12}, z_{21}, z_{34}, z_{43})$ are enough to determine all others [3, Proposition 2.4.1].

Let $\mathcal{T}$ be an ideal triangulation of the oriented hyperbolic 3–manifold $M$, with $m$ tetrahedra. Suppose the vertices of each tetrahedron $\sigma$ of $\mathcal{T}$ are ordered according to the orientation of $M$, and let $z_{ijk}(\sigma)$ and $z_{ij}(\sigma)$ be the coordinates realising $\sigma$ as a flag tetrahedron. The *gluing relations* are the following equations:

- for any two adjacent tetrahedra $\sigma'$ and $\sigma''$, with a common face $(ijk)$ oriented as the boundary of $\sigma'$,

$$z_{ijk}(\sigma') = z_{ikj}(\sigma'') = 1.$$

- for every sequence $\sigma_0, \ldots, \sigma_N$ of tetrahedra sharing a common edge $(ij),

$$z_{ij}(\sigma_0) \cdots z_{ij}(\sigma_N) = z_{ji}(\sigma_0) \cdots z_{ji}(\sigma_N) = 1.$$

The *deformation variety* $\text{Defor}(M, \mathcal{T})$ is the subvariety of $\mathbb{C}^{16m}$ defined by the gluing relations. Up to combinatorial choices, every point in the deformation variety corresponds to a decorated representation of $\pi_1(M)$ into $\text{PGL}(3, \mathbb{C})$ [3, Section 5]. A different initial choice gives a conjugated decorated representation, therefore there is an algebraic map

$$\text{Hol} : \text{Defor}(M, \mathcal{T}) \to X^\times(M).$$

This map is generally injective, but not surjective. For example it misses those decorated characters whose flags are not in general position. Nevertheless it is a parametrisation in complex coordinates of a Zariski open subset of $X^\times(M)$. 


A.1 A different set of coordinates

Figure A.1: The two sets of coordinates on a tetrahedron $\sigma$: BFG’s coordinates on the left, Fock-Goncharov’s coordinates on the right.

A.1.2 Relationship between the two coordinate systems

We now specialise in the case where $M = M_f$ is a hyperbolic once-punctured torus bundle, and $T = T_f$ is its monodromy ideal triangulation.

In §2.3.1 we showed that the union of some irreducible components of the character variety $X^\times(M_f)$ is birationally isomorphic to a subvariety of $(C^\times)^8$, via the morphism $\Psi_f$. In particular, the space $\text{Hol} (\text{Defor}(M, T))$ is contained in such union. If $w_f = \chi_1 \ldots \chi_m$ is the flip sequence of $M_f$, then $\Psi_f(X^\times(M_f))$ is the set of fixed points of the function obtained by composing the edge flips of $w_f$, namely $\Theta_{\chi_m} \circ \cdots \circ \Theta_{\chi_1}$ (cf. Lemma 11).

By composition of algebraic maps, we obtain a generally one-to-one morphism

$$\Psi_f \circ \text{Hol} : \text{Defor}(M_f, T) \to (C^\times)^8,$$

which is a change of coordinates between BFG’s system and the one derived in §2. The function $(\Psi_f \circ \text{Hol})$ is a birational isomorphism onto $\Psi_f(X^\times(M_f))$, but it is usually not surjective. Its image is the Zariski open subset of those points $x \in \Psi_f(X^\times(M_f))$ such that $\Theta_{\chi_t} \circ \cdots \circ \Theta_{\chi_1}(x)$ is defined for all $t \in \{1, \ldots, m\}$. In fact, as we remarked at the end of §2.3.1, for each function $\Theta_{\chi_t} \circ \cdots \circ \Theta_{\chi_1}(x)$ there is a non-empty Zariski closed subset of $(C^\times)^8$ where it is not defined.

Instead of describing $(\Psi_f \circ \text{Hol})$ explicitly, we are going to consider its birational inverse $(\Psi_f \circ \text{Hol})^{-1}$. Let $\sigma_0, \ldots, \sigma_{m-1}$ be the tetrahedra in $T_f$, ordered according to the flip sequence $w_f = \chi_1 \ldots \chi_m$. We recall that each tetrahedron $\sigma_t$ is a copy of the standard ideal tetrahedron $\sigma$ via a canonical identification, hence it inherits labels at the vertices from $\sigma$. For $x = (x_1, \ldots, x_8)$ in the image of $(\Psi_f \circ \text{Hol})$, BFG’s coordinates of $(\Psi_f \circ \text{Hol})^{-1}(x)$ are

$$z_{14}(\sigma_t) = -x^t_4, \quad z_{32}(\sigma_t) = -x^t_3 x^t_4 \left(\frac{x^t_5 + 1}{x^t_4 + 1}\right),$$

$$z_{41}(\sigma_t) = -x^t_5, \quad z_{23}(\sigma_t) = -x^t_2 x^t_6 \left(\frac{x^t_5 + 1}{x^t_6 + 1}\right).$$
where we adopted the conventions

\[ x' := \Theta_{x^t} \circ \cdots \circ \Theta_{x^1}(x), \quad \text{with} \quad x^0 = x, \]

and

\[ x' = (x'_1, \ldots, x'_8) \in (\mathbb{C}^+)^8. \]
References


[38] Pierre Will. Groupes libres, groupes triangulaires et tore épointé dans PU(2,1). Theses, Université Pierre et Marie Curie - Paris VI, November 2006.