Interactions between Ergodic Theory and Combinatorial Number Theory

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Statement of Originality and Authorship

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

This thesis contains material from my solely authored work [10] (accepted for publication), as well as a joint publication [11] and joint work [12] produced in collaboration with Dr. Alexander Fish. References to a joint work with Dr. Michael Björklund [5] (accepted for publication), which inspired some of these works, are also made in this thesis. I certify that my contribution to each of these works was substantial.

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Abstract

The seminal work of Furstenberg on his ergodic proof of Szemerédi’s Theorem gave rise to a very rich connection between Ergodic Theory and Combinatorial Number Theory (Additive Combinatorics). The former is concerned with dynamics on probability spaces, while the latter is concerned with Ramsey theoretic questions about the integers, as well as other groups. This thesis further develops this symbiosis by establishing various combinatorial results via ergodic techniques, and vice versa. Let us now briefly list some examples of such. A shorter ergodic proof of the following theorem of Magyar is given: If $B \subset \mathbb{Z}^d$, where $d \geq 5$, has upper Banach density at least $\epsilon > 0$, then the set of all squared distances in $B$, i.e., the set $\{\|b_1 - b_2\|^2 \mid b_1, b_2 \in B\}$, contains $q\mathbb{Z}_{>R}$ for some integer $q = q(\epsilon) > 0$ and $R = R(B)$. Our technique also gives rise to results on the abundance of many other higher order Euclidean configurations in such sets. Next, we turn to establishing analogues of this result of Magyar, where $\|\cdot\|^2$ is replaced with other quadratic forms and various other algebraic functions. Such results were initially obtained by Björklund and Fish, but their techniques involved some deep measure rigidity results of Benoist-Quint. We are able to recover many of their results and prove some completely new ones (not obtainable by their techniques) in a much more self-contained way by avoiding these deep results of Benoist-Quint and using only classical tools from Ergodic Theory. Finally, we extend some recent ergodic analogues of the classical Plünnecke inequalities for sumsets obtained by Björklund-Fish and establish some estimates of the Banach density of product sets in amenable non-abelain groups. We have aimed to make this thesis accesible to readers outside of Ergodic Theory who may be primarily interested in the arithmetic and combinatorial applications.

A roadmap for this thesis

This thesis is based on several original works of the author, some of which were produced in collaboration with others. After a brief introduction to Ergodic Ramsey Theory and some preliminaries, the Introduction (Chapter 1) consists of a summary of these works. The detailed proofs are then presented in latter chapters. In particular, the material summarized in Sections 1.3, 1.4, 1.5 of the Introduction is presented in Chapters 3, 4, 5 respectively. Chapter 3 is based on the work [10]
of the author, which gives a shorter ergodic proof of Magyar’s theorem [24] on distances in positive density subsets of $\mathbb{Z}^d$ and establishes the existence of some higher order Euclidean patterns therein. Chapter 4 is based on the author’s joint with Fish [12], which demonstrates and extends, via a much more elementary and self contained proof, various Twisted Recurrence results recently obtained in [7] and [5], which may be viewed as certain algebraic analogues of the aforementioned theorem of Magyar. Finally, Chapter 5 is mostly based on the author’s joint work with Fish [11], which extends certain ergodic Plünnecke inequalities established by Björklund-Fish [6]. On the other hand, we do not claim any originality in Chapter 2, which consists of precise statements and detailed proofs of various Furstenberg correspondence principles, which are scattered throughout the literature.
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CHAPTER 1

Introduction

1.1. A brief historical introduction to Ergodic Ramsey Theory

This section will consist of a very brief historical introduction to the intersection of Ergodic Theory and Combinatorial Number Theory (Additive Combinatorics), which is commonly known as Ergodic Ramsey Theory. We start with one of the most significant results in Additive Combinatorics.

**Theorem 1.1.1.** (Szemerédi’s Theorem [33]) If $B \subset \mathbb{N}$ has positive upper asymptotic density, that is,

$$d(B) := \limsup_{N \to \infty} \frac{|B \cap [1,N]|}{N} > 0,$$

then $B$ contains arbitrarily long (finite) arithmetic progressions.

This was conjectured by Erdős and Turán in 1936 [15] and was first proven by Szemerédi in 1972 [33] in a purely combinatorial but extremely complicated way. Quite remarkably, Furstenberg [17] gave a more conceptual proof of Szemerédi’s theorem via Ergodic theory, which can be described as the study of measure preserving Dynamical Systems. To illustrate how the study of patterns in sets of positive density can be pursued through the study of Dynamical Systems, we consider the following reformulation of Szemerédi’s theorem.

**Theorem 1.1.2** (Reformulation of Szemerédi’s theorem). If $B \subset \mathbb{N}$ has positive upper asymptotic density, then for each positive integer $\ell$ there exists a positive integer $n$ such that

$$B \cap (B - n) \cap \cdots \cap (B - \ell n) \neq \emptyset.$$

In other words, we expect the intersection of certain translates of a large set $B$ to be non-empty. Such phenomena are important in the study of dynamical systems. Indeed, Furstenberg proved the following **multiple recurrence theorem**, which is a dynamical analogue of Szemerédi’s theorem (this is most apparent from the reformulation given in Theorem 1.1.2) and in fact implies Szemerédi’s theorem via a classical and very fruitful **correspondence principle** introduced by Furstenberg.
Theorem 1.1.3. Let \((X, \mu, T)\) be a measure preserving system, i.e., \(X\) is a measurable space with a probability measure \(\mu\) and \(T : X \to X\) is a measure preserving map (i.e., \(\mu(T^{-1}B) = \mu(B)\) for all measurable \(B \subset X\)). Then for all \(B \subset X\), with \(\mu(B) > 0\), we have that for all positive integers \(\ell\) there exists a positive integer \(n\) such that

\[
\mu(B \cap T^{-n}B \cap \cdots \cap T^{-\ell n}B) > 0.
\]

We note that the \(\ell = 1\) case is Poincare’s Recurrence Theorem, whose proof is an easy exercise in the pigeonhole principle. The \(\ell = 2\) case is already non-trivial, and it implies Roth’s theorem [30] on 3-term arithmetic progressions (i.e., Szemerédi’s theorem for 3 term arithmetic progressions). Let us briefly give a heuristic for the correspondence between Ergodic Theory and Additive Combinatorics. Observe that the successor map \(S : \mathbb{Z} \to \mathbb{Z}\) given by \(S(x) = x + 1\) preserves the upper asymptotic density \(\overline{d}\) defined in Theorem 1.1.1 in the sense that \(\overline{d}(S^{-1}B) = \overline{d}(B)\) for all \(B \subset \mathbb{Z}\). So if we apply Furstenberg’s multiple recurrence theorem to the “measure preserving system” \((\mathbb{Z}, \overline{d}, S)\) we immediately obtain Szemerédi’s theorem (cf. the reformulation given in Theorem 1.1.2; in fact, we deduce that the intersection of translates is not only non-empty, but has positive density). Unfortunately \((\mathbb{Z}, \overline{d}, S)\) is not exactly a measure preserving system since \(\overline{d}\) is not a probability measure (it is not additive). Despite this, our useful heuristic is justified via the following correspondence principle of Furstenberg [17], which will be proven in Chapter 2, along with many generalizations and extensions.

Proposition 1.1.4 (Furstenberg’s correspondence principle). For \(B \subset \mathbb{N}\), there exists a measure preserving system \((X, \mu, T)\) and \(\tilde{B} \subset X\), with \(\mu(\tilde{B}) = \overline{d}(B)\), such that

\[
\overline{d}\left(\bigcap_{a \in A} (B - a)\right) \geq \mu\left(\bigcap_{a \in A} T^{-aB}\right) \quad \text{for all finite } A \subset \mathbb{N}.
\]

Many of these ergodic-theoretic techniques naturally gave rise to many extensions of Szemerédi’s theorem. For instance, all of the above has been carried out, by Furstenberg and Katznelson, to \(\mathbb{Z}^d\) [19]. In particular, they were the first to prove the following multidimensional Szemerédi theorem:

Theorem 1.1.5. Suppose that \(B \subset \mathbb{Z}^d\) has positive upper Banach density, that is,

\[
d^*(B) := \lim_{L \to \infty} \max_{x \in \mathbb{Z}^d} \frac{|B \cap (x + [0, L]^d)|}{L^d} > 0,
\]

then for all finite sets \(A \subset \mathbb{Z}^d\) there exists an integer \(n > 0\) and \(b \in B\) such that

\(nA \subset B - b\).
As we will shortly need it at our disposal, let us state the appropriate generalization of Furstenberg’s correspondence principle, which one may apply to reduce this multidimensional Szemerédi theorem to an ergodic-theoretic statement (the multiple recurrence theorem for commuting transformations, see [19]).

**Proposition 1.1.6** (Furstenberg’s correspondence principle). For $B \subset \mathbb{Z}^d$, there exists a measure preserving action$^1 T : \mathbb{Z}^d \curvearrowright (X, \mu)$ and $\tilde{B} \subset X$ with $\mu(\tilde{B}) = d^*(B)$ and

$$d^* \left( \bigcap_{a \in A} (B + a) \right) \geq \mu \left( \bigcap_{a \in A} T^a \tilde{B} \right) \text{ for all finite } A \subset \mathbb{Z}^d.$$

A complete proof of this, as well as many extensions, will be given in Chapter 2.

### 1.2. Preliminaries

We now turn to recalling some basic relevant notions and setting up some notation and conventions.

#### 1.2.1. Basic Ergodic Theory.

**Definition 1.2.1.** We say that an action $G \curvearrowright (X, \mu)$ of a group $G$ on a probability space $(X, \mu)$ is **measure preserving** if $\mu(gB) = \mu(B)$ for all (measurable)$^2$ $B \subset X$ and $g \in G$. We will sometimes write $T : G \curvearrowright (X, \mu)$ if we want to give an action an explicit name $T$, in which case $T^g x$ is defined to be the element in $X$ which is obtained from acting $g \in G$ on $x \in X$ (this is useful if we have many different actions or if $G$ is abelian and written in additive notation). In this setting, we have a unitary anti-action $T : G \curvearrowright L^2(X, \mu)$ given by

$$T^g f = f \circ T^g \quad \text{for } f \in L^2(X, \mu) \text{ and } g \in G.$$

We let

$$L^2(X, \mu)^T = \{ f \in L^2(X, \mu) \mid T^g f = f \text{ for all } g \in G \}$$

denote the space of $T$-invariant square integrable maps. We say that a measure preserving action $T : G \curvearrowright (X, \mu)$ is **ergodic** if $L^2(X, \mu)^T$ consists of only the constant functions. Equivalently, ergodicity may be defined as follows: A measure preserving action $T : G \curvearrowright (X, \mu)$ is **ergodic** if for all $T$-invariant $B \subset X$ (i.e., $\mu(B \triangle T^g B) = 0$ for all $g \in G$) we have that $\mu(B) \in \{0, 1\}$.

**Remark 1.2.2.** As is tradition, we will always identify two subsets $B, B' \subset (X, \mu)$ if they almost agree, i.e., $\mu(B \triangle B') = 0$. Likewise for functions in $L^p(X, \mu)$.

---

$^1$See Definition 1.2.1.

$^2$From now on, measurability will be implicit.
1.2.2. Følner sequences and density in amenable groups. In order to extend the notion of asymptotic density to groups other than \( \mathbb{Z} \), we need to introduce the notion of a Følner sequence.

**Definition 1.2.3 (Følner sequence).** Let \( G \) be a countable semigroup. Then a (left) Følner sequence in \( G \) is a sequence of finite sets \( F_1, F_2, \ldots \subset G \) such that

\[
\lim_{N \to \infty} \frac{|gF_N \cap F_N|}{|F_N|} = 1 \quad \text{for all } g \in G.
\]

We say that a countable semigroup is **amenable** if it has a Følner sequence.

In other words, a (left) Følner sequence is a sequence of finite sets that is asymptotically (left) invariant. The simplest example of a Følner sequence in \((\mathbb{Z}, +)\) is \( F_N = [1, N] \cap \mathbb{Z} \). In fact, any sequence of intervals with lengths approaching infinity is a Følner sequence. We will mostly be interested in abelian groups, but in the very rare event that our group is non-abelian, we will use the convention that a Følner sequence is a left one (rather than a right one).

**Example 1.2.4.** (Countable abelian groups are amenable) If \( G = \mathbb{Z}^d \times F \) is a finitely generated abelian group, where \( F \) is a finite abelian group, then \( F_N = (\mathbb{Z}^d \cap [1, N]^d) \times F \) is a Følner sequence. More generally, all countable abelian groups \((G, +)\) are amenable. To see this, write \( G = \bigcup_{k \geq 1} G_k \) where \( G_1 \subset G_2 \subset \cdots \) are finitely generated. Let \( F_n^k \) be a Følner sequence in \( G_k \). Enumerate \( G = \{g_1, g_2, \ldots\} \) where \( g_k \in G_k \) and choose, for each positive integer \( j \), a positive integer \( n_j \) such that

\[
\frac{|(F_n^j + g_i) \triangle F_n^j|}{|F_n^j|} \leq \frac{1}{j} \quad \text{for all } i \in \{1, \ldots, j\}.
\]

Then clearly the sequence \( F_n^j \) is a Følner sequence in \( G \).

By averaging over Følner sequences, we may obtain invariant objects in the limit. The first example of this is the construction of invariant densities on amenable groups.

**Definition 1.2.5 (Densities along Følner sequences).** Let \( G \) be a countable amenable group. Then for \( B \subset G \) and Følner sequences \((F_n)\) in \( G \), we define the lower asymptotic density along \((F_n)\) as

\[
d\_d(F_n)(B) = \liminf_{n \to \infty} \frac{|F_n \cap B|}{|F_n|}.
\]

Similarly, we may define the upper asymptotic density along \((F_n)\) as

\[
d\_u(F_n)(B) = \limsup_{n \to \infty} \frac{|F_n \cap B|}{|F_n|}.
\]
If $\mathcal{F}$ is a collection of Følner sequences in $G$ then we can define the lower and upper densities with respect to this collection as

$$d_{\mathcal{F}}(B) = \inf \left\{ d_{(F_n)}(B) \mid (F_n) \in \mathcal{F} \right\}$$

and

$$\overline{d}_{\mathcal{F}}(B) = \sup \left\{ d_{(F_n)}(B) \mid (F_n) \in \mathcal{F} \right\}.$$ 

Finally, the lower and upper Banach densities in $G$ may be defined, respectively, as

$$d_\ast = d_{\text{Følner}}(G)$$

and

$$d^\ast = \overline{d}_{\text{Følner}}(G)$$

where $\text{Følner}(G)$ denotes the collection of all Følner sequences in $G$.

We note that one does not need to look at the set of all Følner sequences in order to compute the lower and upper Banach densities. It is in fact enough to choose any Følner sequence and consider all its possible translates (in particular, this justifies the more concrete definition of the upper Banach density in $\mathbb{Z}^d$ given in Theorem 1.1.5). More precisely, we have the following basic statement.

**Lemma 1.2.6** (cf. [1]). Fix a Følner sequence $(F_N)$ in a countable amenable group $G$. Then for any $B \subset G$ there exist $t_1, t_2, \ldots \in G$ such that

$$d^\ast(B) = \overline{d}_{(F_N t_N)}(B)$$

and $s_1, s_2, \ldots \in G$ such that

$$d_\ast(B) = d_{(F_N s_N)}(B).$$

In particular, there always exist Følner sequences which attain the lower and upper Banach densities, which will be convenient for us.

**1.2.3. The Mean Ergodic Theorem.** We now give a precise statement of the von Neumann Mean Ergodic Theorem in order to stress that our choice to work with left Følner sequences is consistent with our choice to focus on unitary anti-actions.

**Theorem 1.2.7.** Let $U : G \curvearrowright \mathcal{H}$ be a unitary anti-action of a countable amenable group $G$ on a Hilbert space $\mathcal{H}$. Then for all (left) Følner $(F_N) \subset G$ and $f \in \mathcal{H}$ we have that

$$P_U h = \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} U^g f,$$

where $P_U$ denotes the orthogonal projection onto the space of $U$-invariant vectors in $\mathcal{H}$.

---

\(^3\)Recall this means that $U^{gh} = U^h U^g$ for all $g, h \in G$ where $U^g : \mathcal{H} \to \mathcal{H}$ is unitary, with $U^1 = \text{Id}_{\mathcal{H}}$. 
1.3. Spherical recurrence and Euclidean patterns

1.3.1. Euclidean patterns in positive density subsets of \( \mathbb{Z}^d \). One may see, through a simple application of Szemerédi’s Theorem, that any positive density subset of \( \mathbb{Z}^d \) contains arbitrarily long finite arithmetic progressions. However, it is easy to see that it may not contain an infinite one. Nonetheless, Magyar has shown that, for \( d \geq 5 \), the (squared) distance set of such a set contains an infinite arithmetic progression which starts at 0. A precise statement of this may be stated as follows.

**Theorem 1.3.1** (Magyar [24]). For all \( \epsilon > 0 \) and integers \( d \geq 5 \) there exists a positive integer \( q = q(\epsilon, d) > 0 \) such that the following holds: If \( B \subset \mathbb{Z}^d \) has upper Banach density

\[
d^*(B) := \lim_{L \to \infty} \max_{x \in \mathbb{Z}^d} \frac{|B \cap (x + [0, L]^d)|}{L^d} > \epsilon,
\]

then there exists a positive integer \( N_0 = N_0(B) \) such that

\[
q\mathbb{Z}_{>N_0} \subset \left\{ \|b_1 - b_2\|^2 \mid b_1, b_2 \in B \right\}.
\]

Magyar’s proof is Fourier Analytic. In [10], the author of this thesis has studied related problems using Ergodic theory. This has given rise to a shorter ergodic proof of Magyar’s result, as well as some new higher order extensions. This approach involves establishing a spherical (multiple) recurrence result for measure preserving \( \mathbb{Z}^d \) actions, from which the aforementioned combinatorial results follow upon an application of Furstenberg’s correspondence principle. Before we state the general result, let us provide an instructive example of a special case.

**Proposition 1.3.2** (Chains with prescribed gaps [10]). For all \( \epsilon > 0 \) and integers \( m > 0 \) and \( d \geq 5 \) there exists a positive integer \( q = q(\epsilon, m, d) > 0 \) such that the following holds: For all \( B \subset \mathbb{Z}^d \) with \( d^*(B) > \epsilon \) there exists a positive integer \( N_0 = N_0(B, m) \) such that for all integers \( t_1, \ldots, t_{m-1} \geq N_0 \) there exist distinct \( b_1, \ldots, b_m \in B \) such that

\[
\|b_i - b_{i+1}\|^2 = qt_i \text{ for all } i = 1, \ldots, m - 1.
\]

In other words, we may locally isometrically embed, into a subset \( B \subset \mathbb{Z}^d \) of positive density, all path graphs with edge lengths in some infinite arithmetic progression, whose common difference depends only on the upper Banach density of \( E \) and the number of vertices of the given path graph (as well as the dimension \( d \)). A continuous analogue of this result was recently obtained by Bennett, Iosevich and Taylor in [2], where they proved that a set of large enough Hausdorff dimension in Euclidean space contains long chains whose gaps can assume any value in some fixed open interval of \( \mathbb{R} \) (which depends on the length of the sought chain). More generally, the
1.3. SPHERICAL RECURRENCE AND EUCLIDEAN PATTERNS

The author has shown in [10] that this holds if one replaces path graphs with any tree. To make this statement precise, we introduce the following terminology.

Definition 1.3.3 (Edge labelled trees and local isometries). An edge-labelled tree is a tuple $\tau = (V, E, \phi, L)$ where $(V, E)$ is a finite tree (connected acyclic graph) with vertex set $V$ and edge set $E$ and $\phi : E \to L$ is a function to a set $L$. If $L \subset \mathbb{Z}_{>0}$ then a locally isometric embedding of $\tau$ into $\mathbb{Z}^d$ is an injective map $\iota : V \to \mathbb{Z}^d$ such that for each edge $e = \{v_1, v_2\} \in E$ we have that

$$\|\iota(v_1) - \iota(v_2)\|^2 = \phi(e).$$

In other words, one should think of $\phi(e)$ as being the square of the length of the edge $e$ and $\iota$ as being an embedding that preserves distances between adjacent vertices (but not necessarily the distance between non-adjacent vertices, hence $\iota$ is only a local isometry).

The main combinatorial result obtained by the author of this thesis in [10] may now be stated as follows.

Theorem 1.3.4. Let $\epsilon > 0$, $d \geq 5$ and suppose that $B \subset \mathbb{Z}^d$ with $d^*(B) > \epsilon$. Then for each positive integer $m$, there exists a positive integer $q = q(\epsilon, m, d)$ and a positive integer $N_0 = N_0(B, m)$ such that whenever $\tau = (V, E, \phi, \mathbb{Z}_{> N_0})$ is an edge-labelled tree, with $|V| = m$, then there exists a locally isometric embedding $\iota$ of $\tau$ into $\mathbb{Z}^d$ such that

$$q \cdot \iota(V) \subset B - b$$

for some $b \in B$.

In particular, all edge-labelled trees of the form $\tau = (V, E, \phi, q^2\mathbb{Z}_{> N_0})$, with $|V| = m$, may be locally isometrically embedded into the set $B$.

By considering the trees of diameter 2 (i.e., the trees of the form $(V, E)$ where $V = \{v_0, \ldots, v_{m-1}\}$ and $E = \{\{v_0, v_i\} \mid i = 1, \ldots, m - 1\}$) we recover a recent result of Lyall and Magyar on pinned distances [23]. We remark that their result for this particular family of trees is quantitatively superior to what we have stated as it turns out that, after restricting to this family of trees, the integer $q$ does not depend on $m$ (only on $\epsilon$ and $d$). See Section 3.6 below for more details. This leads to the following unresolved question.

Question 1.3.5. Does the positive integer $q$ in the conclusion of Theorem 1.3.4 only depend on $\epsilon$ and $d$ (i.e., not on $m$)?

We now finish our overview of these combinatorial results with a recent optimal spherical distribution improvement of Magyar’s Theorem due to Lyall and Magyar [23]. Define the discrete sphere $S_N = \{v \in \mathbb{Z}^d : \|v\|^2 = N\}$. 

Theorem 1.3.6 (Lyall-Magyar [23]). Let \( \epsilon > 0, d \geq 5 \) and suppose that \( B \subset \mathbb{Z}^d \) with \( d^*(B) > \epsilon \). Then there exist integers \( q = q(\epsilon, d) > 0 \) and \( N_0 = N_0(B) \) such that the following holds: Given \( N \geq N_0 \), there exists \( b \in B \) such that
\[
\frac{|B \cap (qS_N + b)|}{|S_N|} > d^*(B) - \epsilon.
\]

In other words, we may find \( \mathbb{Z}^d \) translates of spheres in \( q\mathbb{Z}^d \) with centers in \( B \), of all sufficiently large radii, which contain at least the correct proportion of points of \( B \). As we shall see, we may also recover their result with our new approach.

1.3.2. Spherical recurrence in \( \mathbb{Z}^d \) systems. Furstenberg’s correspondence principle for \( \mathbb{Z}^d \) measure preserving actions (Proposition 1.1.6) allows us to reduce Magyar’s theorem above (Theorem 1.3.1) to the following spherical recurrence result, which we will prove in Chapter 3 (more specifically, Section 3.4).

Proposition 1.3.7. Fix \( \epsilon > 0 \) and an integer \( d \geq 5 \). Then there exists a positive integer \( q = q(\epsilon, d) \) such that the following holds: For all measure preserving systems \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) and \( B \subset X \) with \( \mu(B) > \epsilon \), there exists \( N_0 \) such that for all \( N \geq N_0 \) there exists \( a \in \{ v \in \mathbb{Z}^d : \|v\|^2 = N \} \) such that
\[
\mu(B \cap T^{qa}B) > 0.
\]

More generally, we will establish the following recurrence result which implies our main combinatorial result about locally isometrically embedded trees in positive density subsets (Theorem 1.3.4).

Theorem 1.3.8. Fix \( \epsilon > 0 \) and integers \( m \geq 2 \) and \( d \geq 5 \). Then there exists a positive integer \( q = q(\epsilon, m, d) \) such that the following holds: Suppose that \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) is a measure preserving action and \( B \subset X \) with \( \mu(B) > \epsilon \). Then there exists \( N_1 \) such that for all edge-labelled trees \( \tau = (V, E, \phi, \mathbb{Z}_{> N_1}) \), with \( |V| = m \), there exists a locally isometric embedding \( \iota \) of \( \tau \) into \( \mathbb{Z}^d \) such that
\[
\mu \left( \bigcap_{v \in V} T^{q\iota(v)}B \right) > 0.
\]

We now finish with a pointwise recurrence result, whose combinatorial consequence is the recent optimal spherical distribution result of Lyall-Magyar stated in Theorem 1.3.6. Recall the notation \( S_N = \{ v \in \mathbb{Z}^d : \|v\|^2 = N \} \). We shall use the shorthand “\( P(b) \) holds for \( \mu \)-many \( b \in B \)” to mean \( \mu(\{ b \in B \mid P(b) \}) > 0 \).
1.4. Twisted recurrence and algebraic patterns

Theorem 1.3.9. Fix $\epsilon > 0$ and an integer $d \geq 5$. Then there exists a positive integer $q = q(\epsilon, d)$ such that the following holds: Let $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ be an ergodic measure preserving action and let $B \subset X$, with $\mu(B) > \epsilon$. Then there exists $N_0$ such that for all $N \geq N_0$ we have that

$$\frac{1}{|S_N|} \sum_{a \in S_N} 1_{B}(T^{qa}b) > \mu(B) - \epsilon$$

for $\mu$-many $b \in B$.

In Section 3.6, we will give the details of the rather subtle application of Furstenberg’s correspondence principle which deduces Theorem 1.3.6 from this result.

1.4. Twisted recurrence and algebraic patterns

1.4.1. Background. It is natural to ask whether Magyar’s theorem on discrete (squared) distance sets (Theorem 1.3.1) holds when one replaces the Euclidean norm with some other function. We may precisely formulate this question as follows.

Question 1.4.1. Given a function $F : \mathbb{Z}^d \rightarrow S$, where $S$ is some set (usually $\mathbb{Z}$), is it true that for each $B \subset \mathbb{Z}^d$ of positive upper Banach density, there exists a positive integer $k$ such that

$$F(B - B) \supset F(k\mathbb{Z})?$$

Note that Magyar’s theorem implies that this is true when $F(x) = \|x\|^2$ (with $S = \mathbb{Z}$ and $d \geq 5$), though we remark that Magyar’s theorem is stronger as it contains some quantitative information about $k$ (provided one is willing to exclude finitely many elements of $k\mathbb{Z}$).

Let us remark that the answer to Question 1.4.1 is negative when $F : \mathbb{Z}^d \rightarrow \mathbb{Z}$ is a non-trivial linear map, as may be seen by taking $B$ to be a suitably chosen aperiodic Bohr set (see Section 2.2 for the general definition, examples will suffice for now). For example, let $F(x_1, x_2) = x_1 + x_2$ and consider the Bohr set $B = B(\theta, \epsilon) \times B(\theta, \epsilon) \subset \mathbb{Z}^2$, where

$$B(\theta, \epsilon) = \{ x \in \mathbb{Z} \mid x\theta \in (-\epsilon, \epsilon) \pmod{1} \},$$

for some irrational $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and small enough $\epsilon > 0$. Then $d^*(B) = 4\epsilon^2 > 0$ but

$$F(B - B) = B(\theta, \epsilon) - B(\theta, \epsilon) + B(\theta, \epsilon) - B(\theta, \epsilon) = B(\theta, 4\epsilon)$$

is another Bohr set, which cannot contain a non-trivial subgroup of $\mathbb{Z}$ whenever $\theta$ is irrational and $\epsilon < \frac{1}{8}$, as $k\mathbb{Z}\theta$ is dense in $\mathbb{T}$ for all non-zero integers $k$. On the other hand, a theorem of Bogolyubov [8] (see also [31]) states that for any $B \subset \mathbb{Z}$ of positive upper Banach density, the set $B - B + B - B$ contains a Bohr set (as defined in Section 2.2).
A recent series of works, initiated by Björklund-Fish [7] and further developed by Björklund and the author of this thesis [5], answers Question 1.4.1 in the affirmative for non-negative definite integral quadratic forms (with at least 3 variables) and various other homogeneous polynomials. The techniques in those works were ergodic-theoretic and exploited the fact that such functions were preserved by a sufficiently large and algebraically structured subgroup of $\text{SL}_d(\mathbb{Z})$, to which one could apply recent measure rigidity and equidistribution results of Benoist-Quint [3], [4] and those of Bourgain-Furman-Lindenstrauss-Moses [9]. The general statement obtained in [5] may be formulated as follows.

**Theorem 1.4.2** (Twisted patterns). Let $\Gamma \leq \text{SL}_d(\mathbb{Z})$ be a non-trivial finitely generated subgroup such that the linear action $\Gamma \acts \mathbb{R}^d$ is irreducible and the Zariski closure $G \leq \text{SL}_d(\mathbb{R})$ of $\Gamma$ is Zariski-connected, a semi-simple Lie group and has no compact factors. Then for all sets $B \subset \mathbb{Z}^d$ of positive upper Banach density and positive integers $m$, there exists a positive integer $k = k(B, m)$ such that the following holds: For all $a_1, \ldots, a_m \in k\mathbb{Z}^d$ there exists $\gamma_1, \ldots, \gamma_m \in \Gamma$ and $b \in B$ such that

$$\{\gamma_1 a_1, \ldots, \gamma_m a_m\} \subset B - b.$$ 

In particular, if $F : \mathbb{Z}^d \to \mathcal{S}$ is a function ($\mathcal{S}$ is any set) preserved by $\Gamma \leq \text{SL}_d(\mathbb{Z})$ (this means that $F \circ \gamma = F$ for all $\gamma \in \Gamma$) then for all $R_1, \ldots R_m \in F(k\mathbb{Z}^d)$ there exists $b \in B$ such that

$$R_1, \ldots, R_m \in F(B - b).$$

Moreover, if $B$ is an aperiodic Bohr set (see Definition 2.2.1 in Section 2.2), then we may take $k = 1$.

Note that the $m = 1$ case was obtained by Björklund and Fish in [7] gives an analogue of Magyar’s theorem (with $F$ playing the role of $\| \cdot \|^2$). This was then extended to general $m \geq 1$ by Björklund and the author in [5]. This extension may be viewed as an analogue of the pinned distances result discussed in Section 1.3. We now explore some examples observed in the aforementioned papers (we will mainly focus on the $m = 1$ case).

**Example 1.4.3** (Non-positive-definite quadratic forms, see [5]). Fix integers $p, q \geq 1$, with $d := p + q \geq 3$ and let $Q : \mathbb{Z}^d \to \mathbb{Z}$ be an integral quadratic form of signature $p, q$ (for example, $Q : \mathbb{Z}^3 \to \mathbb{Z}$ given by $Q(x, y, z) = x^2 + y^2 - z^2$). Then the integral special orthogonal group

$$\Gamma = \text{SO}(Q)\mathbb{Z} := \{ \gamma \in \text{SL}_d(\mathbb{Z}) \mid Q \circ \gamma = \gamma \}$$

of $Q$ satisfies the hypothesis of Theorem 1.4.2. Hence we obtain the following analogue of Magyar’s theorem for $Q$: For all $B \subset \mathbb{Z}^d$, of positive upper Banach density, there exists an integer
$k \geq 1$ such that

$$Q(B - B) \supset Q(k\mathbb{Z}^d) = k^2Q(\mathbb{Z}^d).$$

**Example 1.4.4** (Characteristic polynomials of traceless integer matrices, see [7]). Let $n \geq 2$ and let $\mathfrak{sl}_n(\mathbb{Z})$ denote the integer matrices of trace zero, which is isomorphic (as an abelian group) to $\mathbb{Z}^{n^2-1}$ and hence has a notion of upper Banach density (we fix an identification of $\mathfrak{sl}_n(\mathbb{Z})$ with $\mathbb{Z}^{n^2-1}$ for the remainder of this example). It turns out that the adjoint representation

$$\text{Ad} : \text{SL}_n(\mathbb{Z}) \curvearrowright \mathfrak{sl}_n(\mathbb{Z}) \cong \mathbb{Z}^{n^2-1},$$

given by $\text{Ad}(g)A = gAg^{-1}$ for $g \in \text{SL}_n(\mathbb{Z})$ and $A \in \mathfrak{sl}_n(\mathbb{Z})$ has the desired properties; namely, $\Gamma = \text{Ad}(\text{SL}_n(\mathbb{Z})) \leq \text{SL}_d(\mathbb{Z})$ (where $d = n^2 - 1$) satisfies the hypothesis of Theorem 1.4.2. Since the adjoint representation preserves characteristic polynomials, we may consider the characteristic polynomial map

$$\text{char} : \mathfrak{sl}_n(\mathbb{Z}) \to \mathbb{Z}[t]$$

given by $\text{char}(A) = \det(tI - A)$ and conclude that for all $B \subset \mathfrak{sl}_n(\mathbb{Z})$ of positive upper Banach density, we have that $\text{char}(B - B)$ contains

$$\text{char}(k\mathfrak{sl}_n(\mathbb{Z})) = \left\{ \sum_{j=0}^{n} k^{n-j}a_jt^j \mid a_0, \ldots a_n \in \mathbb{Z} \text{ with } a_n = 1 \text{ and } a_{n-1} = 0. \right\} \subset \mathbb{Z}[t]$$

for some integer $k \geq 1$. Likewise, the determinant map $\det : \mathfrak{sl}_n(\mathbb{Z}) \to \mathbb{Z}$ is preserved by the adjoint representation and hence for all $B \subset \mathfrak{sl}_n(\mathbb{Z})$ of positive upper Banach density, we have that $\det(B - B)$ contains a non-trivial subgroup $k\mathbb{Z}$, for some integer $k \geq 1$.

**1.4.2. A self-contained approach.** As alluded to above, the proof of Theorem 1.4.2 given in [7] and [5] relies on some very deep results of Benoist-Quint obtained in [3] and [4]. In a very recent joint work with Fish [12], the author of this thesis has found a much more elementary approach (by avoiding these works of Benoist-Quint) to such extensions of Magyar’s theorem, which will also allow us to furnish some new examples that are not obtainable from the previous works [7] and [5]. As such, we will not reproduce the proofs obtained by Björklund and the author in [5] and we will instead focus on the aforementioned joint work with Fish [12]. We begin with one such general result that we are able to obtain by completely self-contained and classical means.

**Theorem 1.4.5.** Let $\Gamma \leq \text{GL}_d(\mathbb{Z})$ (with $d \geq 2$) be a subgroup such that

1. The linear action of $\Gamma$ on $\mathbb{R}^d$ is irreducible.
2. There exists a finite set $S \subset \Gamma$ of unipotent matrices which generate $\Gamma$. 
Then for all $B \subset \mathbb{Z}^d$ of positive upper Banach density and positive integers $m$, there exists a positive integer $k = k(B, m)$ such that the following holds: For all $a_1, \ldots, a_m \in k\mathbb{Z}^d$ there exists $\gamma_1, \ldots, \gamma_m \in \Gamma$ and $b \in B$ such that \( \{ \gamma_1 a_1, \ldots, \gamma_m a_m \} \subset B - b \). Moreover, if $B$ is an aperiodic Bohr set, then we may take $k = 1$.

In particular, we recover Example 1.4.4 as well as some (but not all) of the cases in Example 1.4.3. To see this, note that $\text{SL}_n(\mathbb{Z})$ is generated by a finite set of unipotents (for example, elementary matrices) and if $g \in \text{SL}_n(\mathbb{Z})$ is unipotent, then so is the endomorphism $\text{Ad}(g) : A \mapsto gAg^{-1}$ (for irreducibility, see Section 4.4). On the other hand, there are no non-trivial unipotent matrices in the group $\text{SO}(Q)(\mathbb{Z})$, where $Q(x, y, z) = x^2 + y^2 - 3z^2$. To see this, note that $Q(v) \neq 0$ for all non-zero $v \in \mathbb{Q}^3$, and hence, by Proposition 5.3.4 in [26], $\text{SO}(Q)(\mathbb{Z})$ is a cocompact lattice in $\text{SO}(Q)(\mathbb{R})$, from which it follows, by Corollary 4.4.4 in [26], that $\text{SO}(Q)(\mathbb{Z})$ contains no non-trivial unipotent elements. Let us mention however that we are still able to recover Example 1.4.3 for many interesting quadratic forms.

**Example 1.4.6.** The group $\text{SO}(Q)(\mathbb{Z})$, where $Q(x, y, z) = xy - z^2$ contains a subgroup $\Gamma$ that acts irreducibly on $\mathbb{R}^3$ and is generated by unipotents. This is clear once we write

$$Q(x, y, z) = \det \begin{pmatrix} z & -y \\ x & -z \end{pmatrix}$$

and regard $Q$ as the determinant map on $\mathfrak{sl}_2(\mathbb{Z})$. So we may take $\Gamma = \text{Ad}(\text{SL}_2(\mathbb{Z}))$, as observed in Example 1.4.4, which is generated by unipotents.

**Example 1.4.7.** Let $Q(x, y, z) = x^2 - y^2 - z^2$ and observe that

$$Q(x, y, z) = \det \begin{pmatrix} z & -(x+y) \\ x-y & -z \end{pmatrix}.$$ 

Hence we may regard $Q$ as the determinant map on the abelian subgroup

$$\Lambda = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{Z}) \mid a_{21} \equiv a_{12} \mod 2 \right\}.$$

Notice however that $\Gamma = \text{Ad}(\Gamma_0)$ preserves this abelian subgroup, where

$$\Gamma_0 = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle,$$

and acts irreducibly on $\mathfrak{sl}_2(\mathbb{R})$ (see Section 4.4). Hence Theorem 1.4.5 applies.

In fact, we may extend this example to higher dimensions as follows.
Proposition 1.4.8. Consider the quadratic form \( Q(x_1, \ldots, x_p, y_1, \ldots, y_q) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2 \) where \( p \geq 1 \) and \( q \geq 2 \). Then \( SO(Q)(\mathbb{Z}) \) contains a subgroup acting irreducibly on \( \mathbb{R}^d \) that is generated by finitely many unipotent elements. In particular, we recover the following non-definite analogue of Magyar’s theorem: If \( B \subset \mathbb{Z}^{p+q} \) has positive upper Banach density, then \( Q(B - B) \) contains a non-trivial subgroup of \( \mathbb{Z} \).

Again, we refer the reader to Section 4.4 for the details regarding the construction of these unipotent generators.

1.4.3. A generalization to non-linear actions. It turns out that our approach to Theorem 1.4.5 may be generalized to non-linear semigroups. In particular, an analogous result holds for semigroups consisting of certain polynomial, rather than linear, transformations on \( \mathbb{Z}^d \). This, in turn, may be used to obtain a Magyar-type result for certain non-homogeneous polynomials \( F : \mathbb{Z}^d \to \mathbb{Z} \). Before stating the main result, we will firstly illustrate this phenomena with the following example.

Theorem 1.4.9. Let \( P(z) \in \mathbb{Z}[z] \) be a polynomial of degree \( \geq 2 \), with \( P(0) = 0 \), and let \( F(x, y, z) = xy - P(z) \). Then for all \( B \subset \mathbb{Z}^3 \) of positive upper Banach density there exists a positive integer \( q = q(B) \) such that \( q\mathbb{Z} \subset F(B - B) \). More generally, for all positive integers \( m \) there exists a positive integer \( q = q(B, m) \) such that for all \( R_1, \ldots, R_m \in q\mathbb{Z} \) there exists \( b \in B \) such that

\[
\{R_1, \ldots, R_M\} \subset F(B - b).
\]

Moreover, if \( B \) is an aperiodic Bohr set, then we may take \( k = 1 \).

In particular, we recover Example 1.4.3 for the quadratic forms \( Q(x, y, z) = xy - dz^2 \), where \( d \in \mathbb{Z} \setminus \{0\} \). We remark that the group of linear transformations which preserve \( xy - z^3 \) spectacularly fails the hypothesis of Theorem 1.4.2, as it preserves the \( z = 0 \) plane (hence the linear action is not irreducible) and is actually a finite group. We now turn to describing our general result. To do this, we begin by generalizing the notion of unipotency to non-linear polynomial transformations.

Definition 1.4.10. If \( \Gamma \subset \{\mathbb{Z}^d \to \mathbb{Z}^d\} \) is a semigroup of mappings, then a polynomial walk in \( \Gamma \) is a map \( s : \mathbb{Z}_{\geq 0} \to \Gamma \) such that

1. There exist polynomials

\[
q_1(t_1, \ldots, t_{d+1}), \ldots, q_d(t_1, \ldots, t_{d+1}) \in \mathbb{Z}[t_1, \ldots, t_{d+1}]
\]
such that for each \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and non-negative integer \( n \) we have that
\[
s(n)x = (q_1(n, x_1, \ldots, x_d), \ldots, q_d(n, x_1, \ldots, x_d)).
\]

(2) \( s(0) = \text{Id}_{\mathbb{R}^d} \).

To see the connection to unipotency, observe that if \( \gamma \in \text{SL}_d(\mathbb{Z}) \) is unipotent then the sequence \( S(n) = \gamma^n \) is a polynomial walk in \( \text{SL}_d(\mathbb{Z}) \).

We now turn to extending the notion of an irreducible representation to non-linear actions.

**Definition 1.4.11.** We say that a set \( A \subset \mathbb{R}^d \) is **hyperplane-fleeing** if for all proper affine subspaces \( W \subset \mathbb{R}^d \) we have that \( A \not\subset W \). A sequence \( a_1, a_2, \ldots \) is defined to be **hyperplane-fleeing** if the set \( \{a_1, a_2, \ldots\} \) is hyperplane-fleeing.

For instance, a linear representation of a group \( \Gamma \) on a \( \mathbb{R}^d \), where \( d \geq 2 \), is irreducible if and only if the \( \Gamma \)-orbit of each non-zero vector is hyperplane-fleeing. To see this, note that if \( w \in \mathbb{R}^d \) is a non-zero vector such that \( \Gamma w \subset W \) for some proper affine subspace \( W \), then, as \( d \geq 2 \), we can find \( \gamma \in \Gamma \) such that \( \gamma w - w \neq 0 \) (otherwise \( \mathbb{R}w \) is a one dimensional representation). But then \( \Gamma(\gamma w - w) \subset W - W \) is contained in a proper linear subspace. We are now in a position to state our main result, which is a non-linear generalization of Theorem 1.4.5.

**Theorem 1.4.12.** Let \( \Gamma \subset \{\mathbb{Z}^d \to \mathbb{Z}^d\} \) be a semigroup of maps and suppose that there exists a finite set \( \{s_1, \ldots, s_r\} \) of polynomial walks in \( \Gamma \) such that \( \{s_j(n) \mid n \geq 0 \text{ and } j = 1, \ldots, r \} \) generates \( \Gamma \). Then for \( B \subset \mathbb{Z}^d \) of positive upper Banach density and \( m \in \mathbb{Z}_{>0} \), there exists a positive integer \( k \) such that the following holds: For all \( v_1, \ldots, v_m \in k\mathbb{Z}^d \) with each \( \Gamma v_i \) hyperplane-fleeing, there exists \( \gamma_1, \ldots, \gamma_m \in \Gamma \) such that
\[
\gamma_1 v_1, \ldots, \gamma_m v_m \in B - b \quad \text{for some } b \in B.
\]
Moreover, if \( B \) is an aperiodic Bohr set, then we may take \( k = 1 \).

To see how Theorem 1.4.9 follows, we consider the following example.

**Example 1.4.13.** Let \( P(z) \in \mathbb{Z}[z] \) be a polynomial of degree \( \geq 2 \) with \( P(0) = 0 \). The polynomial \( F(x, y, z) = xy - P(z) \) has the following symmetries which are polynomial walks:
\[
S_1(n)(x, y, z) = (x, y + H(n, x, z), z + nx)
\]

---

4By an affine subspace, we mean a translated linear subspace.
and

$$S_2(n)(x, y, z) = (x + H(n, y, z), y, z + ny)$$

where

$$H(n, x, z) = \frac{P(z + nx) - P(z)}{x} \in \mathbb{Z}[n, x, z]$$

is a polynomial. Observe that, as a polynomial in \( n \) (with coefficients in \( \mathbb{Z}[x, z] \)) the degree of \( H(n, x, z) \) is the degree of \( P \) (at least 2), with leading term of the form \( Cx^{\deg P - 1}n^{\deg P} \) for some integer \( C \neq 0 \). Let \( \Gamma \) be the semigroup generated by these polynomial walks.

**Claim:** For all \( v_0 = (x_0, y_0, z_0) \in \mathbb{Z}^3 \) with \( x_0 \neq 0 \), we have that \( \Gamma v \) is hyperplane-fleeing.

**Proof.** We may assume, without loss of generality, that \( y_0 \neq 0 \) since we may find \( n \in \mathbb{N} \) such that \( v'_0 = S_1(n)v_0 = (x_0, y'_0, z_0) \) where \( y'_0 \neq 0 \) (since, as observed above, \( H(n, x_0, z_0) \) is non-constant when \( x_0 \neq 0 \)). Suppose firstly that the sequence \( S_1(n)(v_0) \) is contained in a hyperplane \( ax + by + cz = d \). As observed earlier, \( H(n, x_0, z_0) \) has leading term \( Cx_0^{\deg P - 1}n^{\deg P} \) and hence since \( x_0 \neq 0 \), it has degree \( \deg P \geq 2 \). From this we conclude that \( b = 0 \). But since \( z_0 + nx_0 \) is non-constant, we conclude that \( c = 0 \). Thus we have shown that if \( x_0 \neq 0 \), we have that \( S_1(n)(x_0, y_0, z_0) \) can only be contained in the hyperplane \( x = x_0 \). Finally, by symmetry, we have that \( S_2(n)(v_0) \) can only be contained in the hyperplane \( y = y_0 \) (in particular, not \( x = x_0 \)). This means that \( \Gamma v_0 \) is hyperplane-fleeing. \( \square \)

Using this claim and Theorem 1.4.12, we reduce Theorem 1.4.9 to the statement that for all integers \( k \geq 1 \), the set \( F(k\mathbb{Z}^3 \setminus (\{0\} \times \mathbb{Z}^2)) \) contains \( q\mathbb{Z} \) for some integer \( q \geq 1 \). But \( F(k, ka, 0) = k^2a \) and so we may take \( q = k^2 \).

We now state one example where only a single polynomial walk is required.

**Example 1.4.14.** Let \( F(x, z) = x - P(z) \) where \( P(z) \in \mathbb{Z}[z] \) has degree at least 2 and \( P(0) = 0 \). Notice that \( F \) is preserved by the polynomial walk \( S(n)(x, z) = (x + P(z + n) - P(z), z + n) \) for \( n \in \mathbb{Z}_{\geq 0} \). Moreover, for all \( v_0 = (x_0, z_0) \in \mathbb{Z}^2 \), we have that \( S(n)v_0 \) is hyperplane-fleeing since \( x_0 + P(z_0 + n) - P(z_0) \) is non-linear in \( n \). Consequently, Theorem 1.4.12 is applicable to the group generated by this polynomial walk. This demonstrates that Theorem 1.4.9 also holds for this function \( F : \mathbb{Z}^2 \to \mathbb{Z} \); in particular, we have the following analogue of Magyar’s Theorem.

**Theorem 1.4.15.** Let \( F(x, z) = x - P(z) \) where \( P(z) \in \mathbb{Z}[z] \) is of degree \( \geq 2 \) with \( P(0) = 0 \). Then for all \( B \subset \mathbb{Z}^2 \) of positive upper Banach density, we have that \( F(B - B) \) contains a non-trivial subgroup of \( \mathbb{Z} \).
1.4.4. Ergodic formulation. As in the previous works [7] and [5], the correspondence principle of Furstenberg (Proposition 1.1.6) allows us to reduce Theorem 1.4.12 to the following ergodic-theoretic twisted multiple recurrence statement.

**Theorem 1.4.16.** Let \( \Gamma \subset \{ \mathbb{Z}^d \to \mathbb{Z}^d \} \) be a semigroup of maps and suppose that there exists a finite set \( \{ s_1, \ldots, s_r \} \) of polynomial walks in \( \Gamma \) such that \( \{ s_j(n) \mid n \geq 0 \text{ and } j = 1, \ldots, r \} \) generates \( \Gamma \). Suppose that \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) is a measure preserving system. Then for all \( B \subset X \) with \( \mu(B) > 0 \) and positive integers \( m \) there exists a positive integer \( k = k(B, m, \epsilon) > 0 \) such that the following holds: If \( v_1, \ldots, v_m \in k\mathbb{Z}^d \) are such that each orbit \( \Gamma v_i \) is hyperplane-fleeing, then there exist \( \gamma_1, \ldots, \gamma_m \in \Gamma \) such that

\[
\mu(B \cap T^{-\gamma_1 v_1} B \cap \cdots \cap T^{-\gamma_m v_m} B) > \mu(B)^{m+1} - \epsilon.
\]

Moreover, if the rational Kronecker factor of \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) is trivial (consists of only the constant functions) then we may take \( k = 1 \) (see Definition 4.2.1 for the definition of the rational Kronecker factor).

To see how we recover from this our results for aperiodic Bohr sets, one may use the correspondence principle for aperiodic Bohr sets (given in Section 2.2) and note that the corresponding measure preserving system (a Kronecker system on a connected compact abelian group) has a trivial rational Kronecker factor (see Appendix C).

1.4.5. Some remarks. The author would like to remark that his joint work with Björklund [5] on twisted multiple recurrence inspired the ergodic-theoretic approach to Magyar’s theorem and Euclidean patterns discussed in Section 1.3 and detailed in Chapter 3. The opposite is, perhaps, suggested from the ordering of the sections and chapters. However, it is worth pointing out that this ergodic approach to Magyar’s theorem involves some additional work (an ergodic measure increment argument) that allows one to obtain quantitative information about the positive integer \( q \) appearing in the theorems on spherical recurrence and Euclidean patterns (locally isometric embeddings of trees) contained therein. It would be interesting to see if something similar could be done in the case of twisted recurrence. In particular, we have the following unresolved question.

**Question 1.4.17.** Does the positive integer \( k \) appearing in the combinatorial (resp. ergodic) results presented above depend only on \( d^*(B) \) (resp. \( \mu(B) \))?
1.5. Ergodic Plünnecke inequalities

We now depart from the study of patterns in large sets and turn our attention to another central theme in Additive Combinatorics, namely, the study of sumsets. Given two subsets $A, B$ of an abelian group $(G, +)$ we may define their sumset

$$A + B = \{a + b \mid a \in A, b \in B\}$$

and for positive integers $k$, we may define the iterated sumsets

$$kA = \{a_1 + \cdots + a_k \mid a_1, \ldots, a_k \in A\}.$$ 

More generally, if $G$ is a (not necessarily abelian) group then we define the product sets $AB$ and iterated products sets $A^k$ in the analogous way for $A, B \subset G$.

1.5.1. Plünnecke inequalities for cardinality and density. Quantitative estimates of the density of sumsets may be traced back to Erdős [14]. He demonstrated that if $A \subset \mathbb{Z}_{\geq 0}$ is a basis of order $k$ (that is, $kA = \mathbb{Z}_{\geq 0}$) then

$$\sigma(A + B) \geq \sigma(B) + \frac{1}{2k} \sigma(B)(1 - \sigma(B))$$

where

$$\sigma(B) = \inf_{N \geq 1} \frac{|B \cap [1, N]|}{N}$$

is the Schnirelmann density of $B \subset \mathbb{Z}_{\geq 0}$. In particular $\sigma(A + B) > \sigma(B)$ whenever $0 < \sigma(B) < 1$.

Plünnecke [29] then significantly improved this inequality of Erdős by demonstrating that for any $A, B \subset \mathbb{Z}_{\geq 0}$ with $0 \in A$, we have that

$$\sigma(A + B) \geq \sigma(kA)^{1/k} \sigma(B)^{1 - \frac{1}{k}}.$$  \hspace{1cm} (1.1)

Plünnecke’s techniques also gave rise to general bounds on the cardinality of finite sumsets, which have played an important role in Additive Combinatorics (see [31], [35], [32] for an overview).

We begin with one of the most general formulations of Plünnecke’s inequality (for cardinality).

**Theorem 1.5.1** (Plünnecke [29]). For finite subsets $A, B$ of an abelian group define, for all $k \in \mathbb{Z}_{\geq 0}$, the magnification ratios

$$D_k = D_k(A, B) = \min_{\emptyset \neq B' \subset B} \frac{|kA + B'|}{|B'|}.$$ 

Then

$$D_j^{1/j} \geq D_k^{1/k} \quad \text{for integers } 0 < j < k.$$
1. Introduction

The \( j = 1 \) case has the following interesting and useful corollary, which is usually referred to as Plünnecke’s inequality in the literature.

**Theorem 1.5.2.** If \( A \) is a finite subset of an abelian group, then

\[
\frac{|kA|}{|A|} \leq \left( \frac{|A + A|}{|A|} \right)^k.
\]

(1.2)

More generally, if \( B \) is another finite subset of this abelian group, then

\[
|A + B| \geq |kA|^{1/k} |B|^{1-1/k}.
\]

(1.3)

Notice that the lower bound given in (1.3) is precisely the cardinality analogue of (1.1). It was later noticed by Ruzsa [31] and also by Jin [21] that this holds for the lower asymptotic density in \( \mathbb{N} \). Jin was also able to extend this to Banach densities. More precisely, he established the following Plünnecke inequalities for Banach densities in the semigroup \( (\mathbb{Z}_{\geq 0}, +) \).

**Theorem 1.5.3 (Jin [21]).** For \( A, B \subset \mathbb{Z}_{\geq 0} \) we have the estimates

\[
d^*(A + B) \geq d^*(kA)^{1/k} d^*(B)^{1-1/k}
\]

and

\[
d_*(A + B) \geq d^*(kA)^{1/k} d_*(B)^{1-1/k}.
\]

1.5.2. Plünnecke inequalities for measure preserving systems. We now turn to studying ergodic-theoretic analogues of sumsets. Given a group action \( G \curvearrowright X \) one may consider, for \( A \subset G \) and \( B \subset X \), the action set

\[
AB = \{ ab \mid a \in A, b \in B \}.
\]

Of course, a product set is nothing more than an action set for the action of \( G \) on itself by translations. We note that if \( G \curvearrowright (X, \mu) \) is an ergodic measure preserving action of a countable group \( G \) (countability ensures that action sets are measurable), then for non-null \( B \subset X \) we have that \( GB \) is a \( G \)-invariant non-null set and hence \( \mu(GB) = 1 \). Moreover, if \( H \leq G \) is a finite index subgroup then, since \( |G : H| \) translates of \( HB \) cover \( GB \), we have that

\[
\mu(HB) \geq \frac{1}{|G : H|}.
\]

More generally, one can show (see Proposition 2.5.3) that if \( G \) is amenable then

\[
\mu(AB) \geq d^*(A)
\]

(1.4)

for all \( A \subset G \). We will however be interested in finding non-trivial lower bounds for \( \mu(AB) \) for certain \( A \subset G \) which may have zero density, as was initiated in a recent joint work of Björklund
and Fish [6]. They were able to achieve this by demonstrating the following ergodic theoretic analogue of Plünnecke’s inequality.

**Theorem 1.5.4 (Björklund-Fish [6]).** Let $G \curvearrowright (X, \mu)$ be a measure preserving system, where $G$ is a countable abelian group and define, for $A \subset G$ and non-null $B \subset X$, the magnification ratio

$$c(A, B) = \inf \left\{ \frac{\mu(AB')}{\mu(B')} \mid B' \subset B \text{ and } \mu(B') > 0 \right\}.$$ 

Then

$$c(A, B) \geq c(A^k, B)^{1/k}.$$ 

By considering the action of a finite abelian group on itself by translations, one sees that this is an extension of the $j = 1$ case of the general Plünnecke inequality given in Theorem 1.5.1.

Combining this with (1.4) gives us the following non-trivial lower estimate on the measure of action sets in ergodic systems.

**Theorem 1.5.5.** If $G \curvearrowright (X, \mu)$ is an ergodic measure preserving action, where $G$ is a countable abelian group, then for all $A \subset G$ and non-null $B \subset X$ we have the estimate

$$\mu(AB) \geq d^*(A^k)^{1/k} \mu(B)^{1-1/k}.$$ 

**Proof.** As $\mu$ is ergodic, we may apply the bound (1.4) to get that

$$c(A^k, B) = \inf_{\text{Non-null } B' \subset B} \frac{\mu(A^kB')}{\mu(B')} \geq \frac{d^*(A^k)}{\mu(B)}.$$ 

By combining this with $\mu(AB) \geq c(A, B)\mu(B)$ and Theorem 1.5.4, we complete the proof. \[\square\]

Notice that this is an ergodic analogue of the original Plünnecke inequality (1.1), as well as, of course, the other aforementioned cardinality and density analogues. Moreover, this implies interesting estimates for action sets involving certain sets of zero density. For instance, we obtain that if $(X, \mu, T)$ is ergodic, then for all non-null $B \subset X$ we have that

$$\mu \left( \bigcup_{n \in \Box} T^n B \right) \geq \mu(B)^{\frac{2}{3}},$$

where $\Box = \{0, 1, 4, 9, \ldots \}$ is the set of squares, since $4\Box = \mathbb{Z}_{\geq 0}$. 

1.5.3. Some more ergodic Plünnecke inequalities. Given that Theorem 1.5.4 is the ergodic analogue of the $j = 1$ case of Theorem 1.5.1, it is interesting to ask whether the ergodic analogue holds for all integers $0 < j < k$. Indeed, this was established by the author of this thesis in a joint work with Fish [11].

**Theorem 1.5.6.** Let $G \curvearrowright (X, \mu)$ be a measure preserving action where $G$ is a countable abelian group. Then for all $A \subset G$ and non-null $B \subset X$ we have that
\[
c(A^j, B)^{\frac{1}{j}} \geq c(A^k, B)^{\frac{1}{k}}\quad \text{for all integers } 0 < j < k.
\]

By the same arguments as before we obtain the following consequence.

**Theorem 1.5.7.** If $G \curvearrowright (X, \mu)$ is an ergodic measure preserving action, where $G$ is a countable abelian group, then for all $A \subset G$ and non-null $B \subset X$ we have the estimate
\[
\mu(A^j B) \geq d^*(A^k)^{\frac{j}{k}} \mu(B)^{\frac{1}{k}}.
\]

For example, if $C = \{0, 1, 8, 27, \ldots \}$ is the set of all cubes, then for all ergodic $(X, \mu, T)$ and non-null $B \subset X$ we have that
\[
\mu\left( \bigcup_{n \in C-C} T^n B \right) \geq \mu(B)^{\frac{7}{9}}
\]
since $9C = \mathbb{Z}_{\geq 0}$.

1.5.4. Remarks on the proofs. We now give a brief summary of the strategies employed in the proofs of the Plünnecke inequalities presented above. The original proof of Plünnecke [29] was graph-theoretic and involved an application of Menger’s theorem. As such, it seems that there is no ergodic analogue of this rather complicated graph-theoretic proof. However, in no later than 2011, Petridis [28] found a remarkably simple combinatorial proof (which was not graph theoretic) of the $j = 1$ case of Theorem 1.5.1. This elegant approach of Petridis was a key ingredient in the work of Björklund-Fish [6] which established Theorem 1.5.4. An interesting discussion of this insight of Petridis continued on Gowers’ blog [20]. In particular, after a combinatorial tour de force, Christian Reiher was able to find a rather complicated extension of the technique of Petridis to give a proof of Theorem 1.5.1 in general. Unfortunately, this technique seems to not generalize to the ergodic case as it seems to rely on the fact that, for discrete probability measures, the infimum in the definition of $c(A, B)$ is attained for some $B' \subset B$. In other words, such an approach is not stable under perturbations in $B'$, which is overall a key issue that appears in the proofs of the Ergodic Plünnecke inequalities in [6] and [11]. However, in an earlier work, Petridis was able to find a simplified version of the original graph-theoretic proof of Plünnecke which avoids Menger’s
theorem. Fortunately, this technique extends naturally to the ergodic case (it is stable), as was carefully established by the author of this thesis in a joint work with Fish [11]. We will present this proof of Theorem 1.5.6 in Chapter 5.

1.5.5. Density Plünnecke inequalities for countable abelian groups. We now turn to generalizations of Jin’s Plünnecke inequalities for Banach density (Theorem 1.5.3). It turns out that one may replace finite intersections with infinite unions in the Furstenberg correspondence principle (see Section 2.5 for details) which allows one to establish results about sumsets through ergodic-theoretic means. In particular, it was noticed by Björklund and Fish that if one applies the Furstenberg correspondence principle for sumsets (Theorem 2.5.1 and Theorem 2.5.2) to Theorem 1.5.5, then one may extend Jin’s result to all countable abelian groups. By using the more general Theorem 1.5.7 instead, we get the following more general Banach density version of Plünnecke’s inequality.

Theorem 1.5.8. Suppose that $G$ is a countable abelian group and $A, B \subseteq G$. Then for integers $0 < j < k$ we have

$$d^*(A^j B) \geq d^*(A^k)^{j/k} d^*(B)^{1-j/k}$$

and

$$d_* (A^j B) \geq d^*(A^k)^{j/k} d_*(B)^{1-j/k}.$$

1.5.6. Density estimates for product sets in non-abelian groups. So far, we have only focused on estimates involving iterated sumsets in abelian groups. Recall that Theorem 1.5.2 says that if $A$ is a finite subset of an abelian group and $K$ is a constant such that $|A + A| \leq K|A|$, then $|nA| \leq K^n|A|$. Some simple examples show that this fails spectacularly in non-abelian groups. In particular, there does not exist a function $f : [1, \infty) \to [1, \infty)$ such that for any finite subset $A$ of a non-abelian group we have that

$$|A^3| \leq f \left( \frac{|A^2|}{|A|} \right) |A|.$$ 

For example, if we take $A = H \cup \{x\}$ where $H$ is a finite group such that $|HxH| = |H|^2$, then $|A^2| \leq 3|H| + 1 \leq 3|A|$ but $|A^3| \geq |HxH|^2 = (|A| - 1)^2$. On the other hand, by works of Ruzsa (see [31]) and Tao [34] we know that one has triple product estimates of the form

$$|A^n| \leq \left( \frac{|A^3|}{|A|} \right)^{O(n)} |A|.$$
and similar estimates for product sets in compact groups, where cardinality is replaced with Haar measure. It is thus natural to ask whether one can find a similar estimate for the upper Banach density in countable amenable (not necessarily abelian) groups. In an unpublished note with Fish, the author has been able to find such a bound for symmetric subsets.

**Proposition 1.5.9.** Let $A$ be a symmetric subset ($A = A^{-1}$) of a countable amenable group $G$, with positive (left) upper Banach density $d^*(A) > 0$. Then for positive integers $n > 3$ we have

\[
d^*(A^n) \leq \left(\frac{d^*(A^3)}{d^*(A)}\right)^{2n-5} d^*(A).
\]

It would be interesting to see if any sort of bound holds for arbitrary (non-symmetric) subsets of countable amenable groups.
Correspondence principles

In this section we state and prove several versions and analogues of Furstenberg’s correspondence principle, which allow us to deduce our combinatorial results from our ergodic-theoretic ones. These ideas already appear throughout the literature [17], [18], [19], [1], [6], [11]. Nonetheless, we have aimed to write an account of these in a way that is as self-contained and as uniform as possible.

2.1. Return times and density

Given a group action $G \curvearrowright X$ together with a point $x_0 \in X$ and a subset $B \subset X$, we define the corresponding set of return times to be

$$B_{x_0} := \{ g \in G \mid gx_0 \in B \}.$$

Most of the correspondence principles involve establishing relationships between the statistical properties of the set of return times and the properties of the dynamical system. This motivates the following definition.

Definition 2.1.1. A topological dynamical system is an action $G \curvearrowright X$ of a group, by homeomorphisms, on a compact metrizable space. Moreover, we say that $x_0 \in X$ is a generic point for a Borel probability measure $\mu$ along a Følner sequence $(F_N)$ in $G$ if the sequence of measures

$$\mu_N := \frac{1}{|F_N|} \sum_{g \in F_N} \delta_{gx_0}$$

converges in the weak* topology to $\mu$ (cf. Definition 4.19 in [13]). Note that this implies that $\mu$ is in the space of $G$-invariant probability measures, which we denote as $P(X)^G$ (cf. Theorem 4.1, Corollary 4.2 and the proof of Theorem 8.10 in [13]). In particular, $P(X)^G$ is a non-empty weak* compact convex set, whenever $G$ is countable amenable.

We begin with a simple relation between the open or closed sets and their set of return times, which may be seen as a precursor to the correspondence principle.
Lemma 2.1.2. Let $G \actson X$ be a topological dynamical system, and suppose that $x_0 \in X$ is a generic point for a measure $\mu \in P(X)^G$ along a Følner sequence $(F_N)$. Then for all finite $A \subset G$ we have that

$$\mu \left( \bigcap_{a \in A} aU \right) \leq d_{(F_N)} \left( \bigcap_{a \in A} aU_{x_0} \right)$$

for all open $U \subset X$ and

$$\mu \left( \bigcap_{a \in A} aV \right) \geq d_{(F_N)} \left( \bigcap_{a \in A} aV_{x_0} \right)$$

for all closed $V \subset X$.

Moreover, the same estimates hold if all intersections are replaced with unions.

Proof. Note that if $B$ is open or closed then so is $C = \bigcap_{a \in A} aB$. Moreover, we have that $C_{x_0} = \bigcap_{a \in A} aB_{x_0}$. These assertions also hold for unions in place of intersections. Hence it is enough to focus on the case where $A = \{ 1_G \}$. But for

$$\mu_N := \frac{1}{|F_N|} \sum_{g \in F_N} \delta_{gx_0}$$

we have, for all open $U \subset X$, the following standard estimate on weak* limits

$$\liminf_N \mu_N(U) \geq \mu(U).$$

The reverse estimate (where also $\liminf$ changes to $\limsup$) holds for closed sets (one can immediately see this from taking complements). But

$$\mu_N(U) = \frac{1}{|F_N|} |\{ g \in F_N \mid gx_0 \in U \}| = \frac{1}{|F_N|} |F_N \cap U_{x_0}|.$$

2.2. Correspondence principle for Bohr sets

Using the rather simple Lemma 2.1.2, we immediately obtain a correspondence principle for Bohr sets, which form an important class of sets in Additive Combinatorics.

Definition 2.2.1 (Bohr sets). We say that $B \subset \mathbb{Z}^d$ is a Bohr-set if $B = \tau^{-1}(U)$ for some

- compact-metrizable abelian group $(K, +)$ (the standard example is $\mathbb{T}^r \oplus F$, where $F$ is finite).
- homomorphism $\tau : \mathbb{Z}^d \to K$ with dense image.
- open subset $U \subset K$ with $m_K(U) = m_K(U)$, where $m_K$ is the Haar probability measure (a standard example, when $K = \mathbb{T}^r$, is a box $U = (-\epsilon, \epsilon)^r$).

Moreover, if we may take $K$ to be connected, then we say that $B$ is aperiodic.
For example, a proper finite index subgroup of $\mathbb{Z}^d$ is a Bohr set that is not aperiodic. Notice that a Bohr set $B = B(\tau, K, U)$ may be naturally realized as a set of return times as follows: We have an action of $G = \mathbb{Z}^d$ on $K$ by translations ($a \in \mathbb{Z}^d$ acts on $\theta \in K$ to give $\tau(a) + \theta$) and $B = U_{x_0}$ is the set of return times of $x_0 = 0 \in K$. Since $\tau$ has dense image, we have that $P(K)^G$ contains only the Haar probability measure and $x_0$ is generic. By setting $A = \{0\}$ and using $m_K(U) = m_K(\bar{U})$, the two inequalities in Lemma 2.1.2 become equalities and thus $m_K(U) = \bar{d}_{(F_N)}(B)$ for any Følner sequence. We also get
\[
d_{(F_N)} \left( \bigcap_{a \in A} (a + B) \right) \geq m_K \left( \bigcap_{a \in A} (\tau(a) + U) \right),
\]
which demonstrates the correspondence principle for Bohr sets.

### 2.3. Return times for ergodic measures

We have already established a correspondence principle for a very special class of set of return times, namely, Bohr sets. We now turn to generalizing this to all sets of return times. It will be helpful for us to obtain ergodicity in the correspondence principle. We begin by recalling some standard facts about the ergodic measures of a topological dynamical system.

**Lemma 2.3.1.** If $G \act X$ is a topological dynamical system with $G$ countable and amenable, then the following hold:

1. The extreme points of $P(X)^G$ are precisely the ergodic measures.
2. If $\mu \in P(X)^G$ is ergodic, then it has a generic point along some Følner sequence in $G$.

**Proof.** The characterization of ergodic measures is standard, cf. Theorem 4.4 in [13]. We now turn to the existence of generic points. For $G = \mathbb{Z}^d$, we may take the Følner sequence $[1, N]^d \cap \mathbb{Z}^d$ and use the pointwise ergodic theorem. For general amenable groups, one may use the ergodic theorem of Lindenstrauss [22]. For a more self contained proof, the mean ergodic theorem suffices as follows: Take any Følner sequence $F_N \subset G$ and apply the mean ergodic theorem to get $L^2$-convergence
\[
\frac{1}{|F_N|} \sum_{g \in F_N} f \circ g \to \int f \, d\mu \tag{2.1}
\]
for all $f \in L^2(X, \mu)$. But, for each fixed $f \in C(X)$, we may pass to a subsequence $(F_N)$ on which we have $\mu$-almost sure pointwise convergence in (2.1). Now apply this to a countable dense set of $f \in C(X)$ and use a diagonalization argument to obtain a Følner sequence on which we have almost sure pointwise convergence in (2.1) for all $f \in C(X)$. We remark that this more
elementary proof does not demonstrate (unlike those involving pointwise ergodic theorems) that the final Følner sequence does not depend on $\mu$, but we are not concerned by this as our applications do not require it.

\[ \square \]

**Proposition 2.3.2.** Let $G \curvearrowright X$ be a topological dynamical system and let $x_0 \in X$ with a dense $G$-orbit, i.e., $\overline{Gx_0} = X$. Then for all clopen $C \subset X$ and $\mu \in \mathcal{P}(X)^G$ we have that

\[ d_*(C_{x_0}) \leq \mu(C) \leq d^*(C_{x_0}). \]

Moreover, for each fixed clopen $C \subset X$, these extremes are attained by some ergodic $\mu \in \mathcal{P}_G(X)$.

**Proof.** The map $\mu \mapsto \mu(C)$ is weak$^*$ continuous and affine and hence, by Bauer’s maximal principle, the minima and maxima are attained on the extreme points of $\mathcal{P}(X)^G$, i.e., the ergodic measures. So it will suffice to prove these bounds for ergodic $\mu \in \mathcal{P}_G(X)$. By Lemma 2.3.1, there exists a generic point $x_0' \in X$ for $\mu$ along a Følner sequence $F_N \subset G$. By continuity of $1_C$, we have that

\[ \mu(C) = \lim_{N \to \infty} \sum_{g \in F_N} \frac{1}{|F_N|} 1_C(gx_0'). \]

Since the orbit of $x_0$ is dense in $X$, we may find, for each positive integer $N$, an element $h_N \in G$ such that $h_Nx_0$ is so close to $x_0'$ that

\[ 1_C(gx_0) = 1_C(gh_Nx_0) \quad \text{for all } g \in F_N. \]

Hence

\[ \mu(C) = \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} 1_C(gh_Nx_0) \]

\[ = \lim_{N \to \infty} \frac{1}{|F_Nh_N|} \sum_{g \in F_Nh_N} 1_C(gx_0) \]

\[ = \lim_{N \to \infty} \frac{1}{|F_Nh_N|} |C_{x_0} \cap F_Nh_N|. \]

But as $F_Nh_N$ is also Følner, we have that this limit is between the lower and upper Banach density of $C_{x_0}$, which proves the desired estimate. To show that these extremes are attained on ergodic measures, it is enough to show that they are attained on some not necessarily ergodic measure (again by the Bauer principle). To do this, take a Følner sequence $(F_N)$ on which the upper (resp. lower) Banach density of $C_{x_0}$ is attained. By using the weak$^*$ compactness, $x_0$ is generic for some measure $\mu$ along a subsequence of $(F_N)$. Replace $(F_N)$ by this subsequence; of course it still attains the upper (resp. lower) Banach density of $C_{x_0}$. We have that

\[ \mu(C) = \lim_{N \to \infty} \frac{1}{|F_N|} |C_{x_0} \cap F_N|. \]
and hence it is equal to the desired value. □

2.4. The Furstenberg system and a proof of the correspondence principle

Given a countable group $G$, the set $\Omega$ of all its subsets may be canonically identified with $\{0, 1\}^G$, which has the product topology. We will fix this topology on $\Omega$. Equivalently, it is the smallest topology so that all the maps $\delta_g : \Omega \to \{0, 1\}$ given by $\delta_g(B) = 1_{B(g)}$ are continuous. The group $G$ naturally acts on $\Omega$, thus giving a topological dynamical system. Given $B \subset G$, we define the corresponding Furstenberg system to be the topological dynamical system $G \actson X$ where

$$X = \overline{Gx_0} \subset \Omega$$

is the orbit closure of $x_0 = B^{-1} \in \Omega$. Now let

$$\tilde{B} = \{x \in X \mid 1_G \in x\},$$

which is a clopen set, and observe that its set of return times is precisely $\tilde{B}_{x_0} = B$. This, together with Proposition 2.3.2, finally gives us Furstenberg’s correspondence principle.

**Proposition 2.4.1.** If $G$ is a countable amenable group and $B \subset G$, then the corresponding Furstenberg system $G \actson X$ and $\tilde{B} \subset X$ satisfy the following property: There exists an ergodic $\mu \in P(X)^G$ such that $d^*(B) = \mu(\tilde{B})$ and

$$\mu \left( \bigcap_{a \in A} a\tilde{B} \right) \leq d^* \left( \bigcap_{a \in A} aB \right)$$

for finite $A \subset G$.

**Proof.** By Proposition 2.3.2, we may take ergodic $\mu \in P(X)^G$ so that $d^*(B) = \mu(\tilde{B})$. But for all finite $A \subset G$, we have that $C = \bigcap_{a \in A} a\tilde{B}$ is still clopen and $C_{x_0} = \bigcap_{a \in A} aB$, so Proposition 2.3.2 gives the desired estimate. □

2.5. Correspondence principle for product sets

We now turn to establishing a correspondence principle for product sets. The idea is to replace intersections with unions in the original correspondence principle. However, as we will be interested in products of infinite sets, we will need to invest some extra effort into studying infinite unions of translates. This turns out to be simpler for upper Banach density.
**Theorem 2.5.1.** If \( G \) is a countable amenable group and \( B \subset G \), then the corresponding Furstenberg system \( G \actson X \) and \( \tilde{B} \subset X \) satisfy the following property: There exists an ergodic \( \mu \in P(X)^G \) such that \( d^*(B) = \mu(\tilde{B}) \) and
\[
\mu(AB) \leq d^*(AB) \quad \text{for all } A \subset G.
\]

**Proof.** By Proposition 2.3.2, we may take ergodic \( \mu \in P(X)^G \) so that \( d^*(B) = \mu(\tilde{B}) \). Now let \( A_n \subset G \) be finite nested subsets such that \( A = \bigcup_{n=1}^\infty A_n \). We have that \( A_N \tilde{B} \) is still clopen and \( A_N B = (A_N \tilde{B})_{x_0} \), so Proposition 2.3.2 tells us that
\[
\mu(A_N B) \leq d^*(A_N B) \leq d^*(AB).
\]
But \( \mu(AB) = \lim_{N \to \infty} \mu(A_N B) \). \( \square \)

We have to do a bit more work to get a correspondence principle for the lower Banach density of product sets.

**Theorem 2.5.2.** Let \( B \subset G \) be a subset of a countable amenable groups and consider the corresponding Furstenberg system \( G \actson X \) and \( \tilde{B} \subset X \). Then for all \( A \subset G \) there exists an ergodic \( \mu \in P(X)^G \) such that \( d_*(B) = \mu(\tilde{B}) \) and
\[
d_*(AB) \geq \mu(\tilde{A}B).
\]

**Proof.** Of course, the first estimate holds for all \( \mu \in P(X)^G \) by Proposition 2.3.2. Now since \( \tilde{A}B \) is open, the map \( \mu \mapsto \mu(\tilde{A}B) \) is lower semicontinuous and so it has minima which are attained on the extreme points (i.e., ergodic measures) of \( P(X)^G \) by Bauer’s minimal principle. This means that it is enough to construct just one \( \mu \in P(X)^G \), not necessarily ergodic, that satisfies the bound \( d_*(AB) \geq \mu(\tilde{A}B) \). To do this, choose a Følner sequence \( F_N \) in \( G \) such that
\[
d_*(AB) = \frac{|AB \cap F_N|}{|F_N|}
\]
and replace it with a subsequence, if necessary, such that \( x_0 \) equidistributes to some \( \mu \in P(X)^G \) along \( F_N \). Again, using that \( (\tilde{A}B)_{x_0} = AB \) and the fact that \( \tilde{A}B \) is open, Lemma 2.1.2 tells us that
\[
\mu(\tilde{A}B) \leq d_*(AB),
\]
as desired. \( \square \)

We now finish with an estimate that is very useful when combined with the correspondence principle for product sets. In fact, it is the main reason why we put effort into ensuring ergodicity in the correspondence principle.
Proposition 2.5.3. Let $G$ be a countable amenable group and let $T : G \curvearrowright (X, \mu)$ be ergodic. Then for $A \subset G$ and non-null $B \subset X$ we have that

$$
\mu(AB) \geq d^*(A).
$$

Proof. Choose a Følner sequence $F_N$ in $G$ such that

$$
d^*(A) = \lim_{N \to \infty} \frac{|A \cap F_N|}{|A|}.
$$

By the Mean Ergodic Theorem, we have that

$$
\mu(AB) = \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} T^g 1_{AB}. \tag{2.2}
$$

By replacing $F_N$ with a suitable subsequence, we may assume that we have $\mu$-almost sure point-wise convergence in (2.2). In particular, as $B$ is non-null, there exists $b \in B$ such that

$$
\mu(AB) = \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} 1_{AB}(gb)
\geq \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} 1_A(g)
= d^*(A).
$$

$\square$
CHAPTER 3

Spherical recurrence

We now embark on proving the results on Euclidean patterns and spherical recurrence presented in Section 1.3. This material appears in the paper of the author [10]. Due to the delicate nature of the quantitative arguments, we will only prove our recurrence results for ergodic measure preserving systems. This is no restriction, as it is standard practice in Ergodic Ramsey Theory to reduce to the ergodic case via the ergodic decomposition (see Section 7.2.3 of [13] for details). Alternatively, if the reader wishes to avoid the ergodic decomposition and is only interested in the combinatorial consequences, then perhaps the most direct justification is the observation that ergodicity may be assumed in the Furstenberg correspondence principle, as demonstrated in Proposition 2.4.1.

3.1. Main tool from discrete harmonic analysis

We now state the main blackbox that we will use in our proofs, which is also used in Magyar’s original proof [24] as well as the recent optimal improvements and pinned generalizations of Magyar and Lyall in [23]. Fix an integer \( d \geq 5 \). For \( N \in \mathbb{N} \), we define the discrete sphere

\[
S_N = \{ x \in \mathbb{Z}^d \mid \|x\|_2^2 = N \}.
\]

For \( \eta > 0 \) and \( C > 0 \) let

\[
q_{\eta,C} = \text{lcm}\{ q \in \mathbb{Z} \mid 1 \leq q \leq C \eta^{-2} \}
\]

Magyar-Stein-Wainger gave approximations for exponential sums on a discrete sphere [25]. The following is a rather straightforward consequence of these approximations proved in Magyar’s original paper [24] (this particular formulation is stated in a recent work of Lyall and Magyar [23]).

**Theorem 3.1.1** (Exponential sum estimates on discrete spheres). There exists a constant \( C = C_d > 0 \) (depends only on \( d \geq 5 \)) such that the following is true: Given \( \eta > 0 \), an integer \( N \geq C \eta^{-4} \) and

\[
\theta \in \mathbb{R}^d \setminus \left( q_{\eta,C}^{-1} \mathbb{Z}^d + \left[ -\left( \eta N \right)^{-1/2}, \left( \eta N \right)^{-1/2} \right] \right),
\]

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then
\[ \left| \frac{1}{|S_N|} \sum_{x \in S_N} \exp (2\pi i \langle x, \theta \rangle) \right| \leq \eta. \]

We stress that this consequence of [25] has less than a one page proof in [24] and thus only takes up a small portion of that paper. As such, using this as a blackbox does not detract much from our alternative proof of Magyar’s Theorem (Theorem 1.3.1) given in Section 3.4.

**Important Notational Convention:** For brevity, let us now write \( q = q_{\eta, C} \) where \( C = C_d \) is as above (the dimension \( d \geq 5 \) will from now on be fixed for the rest of this chapter).

### 3.2. \((q, \delta)\)-Equidistributed sets.

In this section, we introduce the notion of a \((q, \delta)\)-equidistributed subset of an ergodic system, which may be of independent interest. This will enable us to employ a measure increment argument from which we will obtain good control of the integer \( q = q(\epsilon, m, d) \) appearing in Theorems 1.3.4 and 1.3.8 above. We briefly remark that a combinatorial analogue of such an increment argument (called the density increment argument) is often used in Additive Combinatorics. In fact, it is used in Magyar’s original proof of Theorem 1.3.1 as well as in the recent work of Lyall and Magyar [23]. However, the details are slightly more technical in our ergodic setting.

For the remainder of this section, let \( F_N = [1, N]^d \cap \mathbb{Z}^d \).

**Definition 3.2.1.** Let \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) be an ergodic measure preserving action. Then we say that \( B \subset X \) is \((q, \delta)\)-equidistributed if for almost all \( x \in X \) we have
\[ \lim_{N \to \infty} \frac{1}{|F_N|} \left| \{ a \in F_N \mid T^{qa} x \in B \} \right| \leq (1 + \delta) \mu(B). \]

**Definition 3.2.2.** If \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) is a measure preserving action and \( Q \) is a positive integer, then we may define a new measure preserving actions \( T^Q : \mathbb{Z}^d \curvearrowright (X, \mu) \) given by
\[ (T^Q)^a x = T^{qa} x \quad \text{for } a \in \mathbb{Z}^d \text{ and } x \in X. \]

**Definition 3.2.3** (Conditional probability and ergodic components). If \((X, \mu)\) is a probability space and \( C \subset X \) is measurable with \( \mu(C) > 0 \) then we define the conditional probability measure \( \mu(\cdot | C) \) given by \( \mu(B | C) = \frac{\mu(B \cap C)}{\mu(C)} \). We note that if \( C \) is invariant under some measure preserving action, then \( \mu(\cdot | C) \) is also preserved by this action. If \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) is ergodic and \( Q \) is a positive integer, then the action \( T^Q : \mathbb{Z}^d \curvearrowright (X, \mu) \) may not be ergodic; but it is easy to see (see Appendix A) that there exists a \( T^Q \)-invariant subset \( C \subset X \) such that the action of \( T^Q \) on \( C \) is ergodic (more precisely, \( \mu(\cdot | C) \) is \( T^Q \)-ergodic) and the translates of \( C \) disjointly cover \( X \) (there
are at most $Q^d$ distinct translates, hence $\mu(C) \geq Q^{-d})$. Note that the translates of $C$ also satisfy these properties of $C$. We call such a measure $\mu(\cdot|C)$ a $T^Q$-ergodic component of $\mu$. It follows that $\mu$ is the average of its distinct $T^Q$-ergodic components.

We may now introduce our measure increment technique, which will be used to reduce our recurrence theorems, such as Theorem 1.3.8, to ones which assume sufficient equidistribution.

**Lemma 3.2.4** (Ergodic measure increment argument). Let $\delta, \epsilon > 0$, let $q$ be a positive integer and let $T: \mathbb{Z}^d \curvearrowright (X, \mu)$ be ergodic. If $B \subset X$ with $\mu(B) > \epsilon$ then there exists a positive integer $Q \leq q^{\log(\epsilon^{-1})/\log(1+\delta)}$ and a $T^Q$-ergodic component, say $\nu$, of $\mu$ such that $\nu(B) \geq \mu(B)$ and $B$ is $(q, \delta)$-equidistributed with respect to $T^Q: \mathbb{Z}^d \curvearrowright (X, \nu)$.

To study the limits appearing in Definition 3.2.1 we make use of the well known Pointwise Ergodic Theorem.

**Proposition 3.2.5** (Pointwise ergodic theorem). Let $T: \mathbb{Z}^d \curvearrowright (X, \mu)$ be a measure preserving action. Then for all $f \in L^2(X, \mu)$ there exists $X_f \subset X$ with $\mu(X_f) = 1$ such that

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{a \in F_N} f(T^a x) \to P_T f(x)$$

for all $x \in X_f$.

**Proof of Lemma 3.2.4.** If $B$ is $(q, \delta)$ equidistributed, then we are done. Otherwise, it follows from the Pointwise ergodic theorem (applied to the action $T^q$ and the indicator function of $B$) that there exists a $T^q$-ergodic component of $\mu$, say $\nu_1$, such that $\nu_1(B) \geq (1 + \delta)\mu(B)$. Continuing in this fashion, we may produce a maximal sequence of ergodic components $\nu_1, \nu_2, \ldots, \nu_J$ of $T^q, T^{q^2}, \ldots, T^{q^J}$, respectively, such that $\nu_{j+1}(B) \geq (1 + \delta)\nu_j(B)$. Clearly we must have $\epsilon(1 + \delta)^J \leq 1$ and so this finishes the proof with $Q = q^J$. \qed

We now turn to demonstrating the key spectral properties of a $(q, \delta)$-equidistributed set.

**Definition 3.2.6** (Eigenspaces). If $T: \mathbb{Z}^d \curvearrowright (X, \mu)$ is a measure preserving action and $\chi \in \hat{\mathbb{Z}}^d$ is a character on $\mathbb{Z}^d$, then we say that $f \in L^2(X, \mu)$ is a $\chi$-eigenfunction if

$$T^a f = \chi(a) f \text{ for all } a \in \mathbb{Z}^d.$$ 

We let $\text{Eig}_T(\chi)$ denote the space of $\chi$-eigenfunctions and for $R \subset \hat{\mathbb{Z}}^d$ we let

$$\text{Eig}_T(R) = \overline{\text{Span}} \{ f \mid f \in \text{Eig}(\chi) \text{ for some } \chi \in R \}^{L^2(X, \mu)}.$$
In particular, we will be interested in the sets $R_q = \{ \chi \in \hat{\mathbb{Z}}^d | \chi^q = 1 \}$ and $R_q^* = R_q \setminus \{1\}$, where $q \in \mathbb{Z}$. Note that the spaces $\text{Eig}_T(\chi)$ are orthogonal to each other and hence $\text{Eig}_T(R_q^*)$ has an orthonormal basis consisting of $\chi$-eigenfunctions, for $\chi \in R$. Note also that ergodicity implies that each $\text{Eig}_T(\chi)$ is at most one dimensional.

Proposition 3.2.7. Let $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ be an ergodic measure preserving action and suppose that $B \subset X$ is $(q, \delta)$-equidistributed. Let $h \in L^2(X, \mu)$ be the orthogonal projection of $1_B$ onto $\text{Eig}_T(R_q^*)$. Then
\[
P_{T^q} 1_B = \mu(B) + h
\]
and
\[
\|h\|_2 \leq \sqrt{(2\delta + \delta^2)}\mu(B).
\]

Proof. Note that $\text{Eig}_T(R_q) = L^2(X, \mu)^{T^q}$. This, together with the ergodicity of $T$, shows that $h = P_{T^q} 1_B - \mu(B)$. Now the pointwise ergodic theorem, applied to the action $T^q$, combined with the $(q, \delta)$-equidistribution of $B$ immediately gives that
\[
\|h\|_2^2 = \|P_{T^q} 1_B\|_2^2 - \|\mu(B)\|_2^2 \leq (1 + \delta)^2\mu(B)^2 - \mu(B)^2 = (2\delta + \delta^2)\mu(B)^2.
\]

3.3. Spherical mean ergodic theorem

Our next result says that the ergodic averages along the discrete spheres $S_N$ of a well enough equidistributed set $B$ must almost converge (that is, are eventually very close to) in $L^2$ to the constant function $\mu(B)$.

Theorem 3.3.1 (Spherical Mean Ergodic theorem). Let $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ be an ergodic measure preserving action and suppose that $B \subset X$ is $(q_\eta, \delta)$-equidistributed. Then
\[
\limsup_{N \to \infty} \left\| \mu(B) - \frac{1}{|S_N|} \sum_{a \in S_N} T^a 1_B \right\|_2 \leq \sqrt{3\delta + \eta}.
\]

We first prove the following lemma using Bochner’s theorem. Using the notation introduced in Definition 3.2.6, let $\text{Rat} = \bigcup_{q \in \mathbb{N}} R_q$ denote the set of rational characters, let $L^2_{\text{Rat}}(X, \mu, T) = \text{Eig}_T(\text{Rat})$ denote the rational Kronecker factor and let $P_{\text{Rat}} : L^2(X, \mu) \to L^2_{\text{Rat}}(X, \mu, T)$ denote the orthogonal projection onto it.

---

1This follows from the fact that all finite dimensional representations of a finite abelian group can be decomposed into one dimensional representations.
Lemma 3.3.2. Let \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) be a measure preserving action and suppose that \( f \in L^2_{\text{Rat}}(X, \mu, T) \). Then
\[
\lim_{N \to \infty} \frac{1}{|S_N|} \sum_{a \in S_N} T^a f = 0.
\]

Proof. By Bochner’s theorem (Theorem B.1) there exists a positive finite Borel measure \( \sigma \) on \( \mathbb{T}^d \) such that
\[
\langle T^a f, f \rangle = \int \exp(2\pi i \langle u, a \rangle) d\sigma(u) \quad \text{for all } a \in \mathbb{Z}^d.
\]
Since \( f \in L^2_{\text{Rat}}(X, \mu, T) \) we have, by Lemma B.2, that \( \sigma(\mathbb{Q}^d / \mathbb{Z}^d) = 0 \) and hence
\[
\left\| \frac{1}{|S_N|} \sum_{a \in S_N} T^a f \right\|_2^2 = \int_{\Omega} \left\| \frac{1}{|S_N|} \sum_{a \in S_N} \exp(2\pi i \langle u, a \rangle) \right\|^2 d\sigma(u)
\]
where \( \Omega = \mathbb{T}^d \setminus (\mathbb{Q}^d / \mathbb{Z}^d) \). But
\[
\frac{1}{|S_N|} \sum_{a \in S_N} \exp(2\pi i \langle u, a \rangle) \to 0 \quad \text{as } N \to \infty
\]
for all \( u \in \Omega \) by Theorem 3.1.1. Now the dominated convergence theorem finally completes the proof. \( \square \)

Proof of Theorem 3.3.1. Let \( q = q_0 \). By Lemma 3.3.2, the left hand side of (3.1) remains unchanged if we replace \( \mathbb{1}_B \) with \( P_{\text{Rat}} \mathbb{1}_B \). We can write
\[
P_{\text{Rat}} \mathbb{1}_B = \mu(B) + \sum_{\chi \in R_q^*} c_{\chi} \rho_{\chi} + \sum_{\chi \in \text{Rat} \setminus R_q} c_{\chi} \rho_{\chi}
\]
where \( \rho_{\chi} \) is a \( \chi \)-eigenfunction of norm 1 and \( c_{\chi} \in \mathbb{C} \). From Proposition 3.2.7 we get that
\[
\left\| \frac{1}{|S_N|} \sum_{a \in S_N} T^a \sum_{\chi \in R_q^*} c_{\chi} \rho_{\chi} \right\|_2^2 \leq \left\| \sum_{\chi \in R_q^*} c_{\chi} \rho_{\chi} \right\|_2^2 \leq (2\delta + \delta^2) \mu(B) \leq 3\delta.
\]
(3.2)
Now by Theorem 3.1.1 we get that
\[
\limsup_{N \to \infty} \left\| \frac{1}{|S_N|} \sum_{a \in S_N} \chi(a) \right\|_2 \leq \eta \quad \text{for all } \chi \in \text{Rat} \setminus R_q.
\]
This implies that
\[
\limsup_{N \to \infty} \left\| \frac{1}{|S_N|} \sum_{a \in S_N} T^a \sum_{\chi \in \text{Rat} \setminus R_q} c_{\chi} \rho_{\chi} \right\|_2^2 = \limsup_{N \to \infty} \left\| \sum_{\chi \in \text{Rat} \setminus R_q} \left( \frac{1}{|S_N|} \sum_{a \in S_N} \chi(a) \right) c_{\chi} \rho_{\chi} \right\|_2^2
\]
\[
\leq \eta^2 \sum_{\chi \in \text{Rat} \setminus R_q} c_{\chi}^2
\]
\[ \leq \eta^2 \mu(B) \leq \eta^2. \]

Finally, combining this estimate with (3.2) and using the triangle inequality gives the desired estimate (3.1).

\[ \square \]

### 3.4. Spherical Recurrence

We are now in a position to prove Proposition 1.3.7. By Lemma 3.2.4, this reduces to the following.

**Proposition 3.4.1.** Fix \( \epsilon > 0 \) and let \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) be an ergodic measure preserving action. Suppose that \( B \subset X \) is \((q_\eta, \delta)\)-equidistributed where \( \eta < \frac{1}{2} \epsilon \) and \( \delta = \frac{1}{3} \eta^2 \). Then there exists \( N_0 \) such that for all \( N \geq N_0 \) we have that

\[ \frac{1}{|S_N|} \sum_{a \in S_N} \mu(B \cap T^a B) > \mu(B)^2 - \epsilon. \]

**Proof.** From Cauchy-Schwartz followed by the Spherical Mean Ergodic theorem (Theorem 3.3.1) we get that

\[
\left| \frac{1}{|S_N|} \sum_{a \in S_N} \mu(B \cap T^a B) - \mu(B)^2 \right| = \left| \langle 1_B, \frac{1}{|S_N|} \sum_{a \in S_N} T^a 1_B - \mu(B) \rangle \right|
\]

\[
\leq \| 1_B \|_2 \left\| \frac{1}{|S_N|} \sum_{a \in S_N} T^a 1_B - \mu(B) \right\|_2
\]

\[
< \epsilon
\]

for sufficiently large integers \( N \).

\[ \square \]

### 3.5. Locally isometric embeddings of trees

We now turn to proving our main recurrence result (Theorem 1.3.8). To this end, it will be useful to relax the notion of a locally isometric embedding, introduced in Definition 1.3.3, to the broader notion of a **locally isometric immersion**.

**Definition 3.5.1 (Locally isometric immersions).** Recall that an edge-labelled tree is a a tuple \( \tau = (V, E, \phi, L) \) where \((V, E)\) is a finite tree (connected acyclic graph) with vertex set \( V \) and edge set \( E \) and \( \phi : E \to L \) is a function to a set \( L \). If \( L \subset \mathbb{Z}_{>0} \) then a **locally isometric immersion** of \( \tau \) into \( \mathbb{Z}^d \) is a map \( \iota : V \to \mathbb{Z}^d \) such that for each edge \( e = \{v_1, v_2\} \in E \) we have that

\[ \| \iota(v_1) - \iota(v_2) \|^2 = \phi(e). \]

As per Definition 1.3.3, we say that \( \iota \) is a **locally isometric embedding** if it is injective.
By Lemma 3.2.4, the following result implies Theorem 1.3.8.

**Theorem 3.5.2.** Let $\epsilon > 0$ and suppose that $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ is an ergodic measure preserving action and $B \subset X$ is $(q_\eta, \delta)$ equidistributed with $\eta < \frac{1}{2}\epsilon$ and $\delta = \frac{1}{3}\eta^2$. Then there exists $N_1$ such that for all edge-labelled trees $\tau = (V, E, \phi, \mathbb{Z}_{\geq N_1})$, with $|V| = m$, there exists a locally isometric embedding $\iota$ of $\tau$ into $\mathbb{Z}^d$ such that

$$\mu \left( \bigcap_{v \in V} T^{\phi(v)} B \right) \geq \mu(B)^m - m\epsilon. \quad (3.4)$$

Before we embark on the proof, let us introduce the notion of a rooted edge-labelled tree.

**Definition 3.5.3.** A **rooted edge-labelled tree** is a tuple $\tau = (V, v_0, E, \phi, \mathbb{Z}_{\geq N_1})$ where $(V, E, \phi)$ is an edge-labelled tree and $v_0 \in V$ is a distinguished vertex, which we call the root of $\tau$.

It will be convenient to use the averaging notation

$$\mathbb{E}_{a \in A} f(a) = \frac{1}{|A|} \sum_{a \in A} f(a)$$

for finite sets $A$.

**Proof of Theorem 3.5.2.** Choose, by the Spherical Mean Ergodic theorem (Theorem 3.3.1), a positive integer $N_0 > 0$ such that

$$\|\mathbb{E}_{a \in S_N} T^n 1_B - \mu(B)\|_2 < \epsilon \text{ for all } N > N_0. \quad (3.3)$$

Fix a rooted edge-labelled tree $\tau = (V, v_0, E, \phi, \mathbb{Z}_{\geq N_0})$. We let

$$\mathcal{I} = \mathcal{I}(\tau) = \{ \iota : V \to \mathbb{Z}^d \mid \iota \text{ is a locally isometric immersion of } \tau \text{ with } \iota(v_0) = 0. \}.$$

We now aim to show that

$$\left\| \mathbb{E}_{a \in \mathcal{I}} \prod_{v \in V \setminus \{v_0\}} T^{\phi(v)} 1_B - \mu(B)^{m-1} \right\|_{L^2(X, \mu)} \leq (m - 1)\epsilon. \quad (3.4)$$

This may be proven by induction on $|V| = m$ as follows: The $m = 2$ case is precisely the estimate (3.3). Now suppose $m > 2$ and let $e^* = \{v_1, v^*\}$ be an edge of $\tau$ such that $v^*$ is a leaf (i.e., $e^*$ is the only edge which contains it) and $v^* \neq v_0$. Now consider the rooted edge-labelled tree $\tau' = \tau - v^*$ obtained by deleting this leaf, more precisely

$$\tau' = (V \setminus \{v^*\}, v_0, E \setminus \{e^*\}, \phi|_{E \setminus \{e^*\}}, \mathbb{Z}_{\geq N_0}),$$

and let $\mathcal{I}' = \mathcal{I}(\tau')$ be the corresponding set of immersions. We have the recursion
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\[ \mathbb{E}_{\iota \in \mathcal{I}} \prod_{v \in V \setminus \{v_0\}} T^{u(v)} \mathbf{1}_B = \mathbb{E}_{\iota \in \mathcal{I}'} \left( \left( \prod_{v \in V \setminus \{v_0, v^*\}} T^{u(v)} \mathbf{1}_B \right) \left( \prod_{a \in S_{\phi(v^*)}} T^{u(v_1) + a} \mathbf{1}_B \right) \right) \]

\[ = \mu(B) \cdot \mathbb{E}_{\iota \in \mathcal{I}'} \left( \prod_{v \in V \setminus \{v_0, v^*\}} T^{u(v)} \mathbf{1}_B \right) \]

\[ + \mathbb{E}_{\iota \in \mathcal{I}'} \left( \left( \prod_{v \in V \setminus \{v_0, v^*\}} T^{u(v)} \mathbf{1}_B \right) \left( \prod_{a \in S_{\phi(v^*)}} T^{u(v_1) + a} \mathbf{1}_B - \mu(B) \right) \right). \]

But the $L^2$-norm of second term is at most

\[ \left\| \mathbb{E}_{a \in S_{\phi(v^*)}} T^{u(v_1) + a} \mathbf{1}_B - \mu(B) \right\|_{L^2(X, \mu)} = \left\| \mathbb{E}_{a \in S_{\phi(v^*)}} T^a \mathbf{1}_B - \mu(B) \right\|_{L^2(X, \mu)} < \epsilon \]

where in the equality we only used the fact that $T^{u(v_1)}$ is an isometry that fixes constant functions.

Combining this estimate with the recursion and the inductive hypothesis finally completes the induction step, and thus establishes (3.4). Now let $\mathcal{E} = \mathcal{E}(\tau) \subset \mathcal{I}(\tau)$ be those elements of $\mathcal{I}$ which are embeddings. Note that

\[ |\mathcal{I}| = \prod_{e \in E} |S_{\phi(e)}| \]

and that

\[ |\mathcal{E}| \geq \prod_{j=1}^{|E|} (|S_{\phi(e_j)}| - j + 1), \]

where $e_1, e_2, \ldots, e_{m-1}$ is some enumeration of $E$ such that $e_1$ contains $v_0$ and the subgraph with edges $e_1, \ldots, e_i$ is connected, for all $i \leq m - 1$. This means that, as $N_0 \to \infty$, an arbitrarily large proportion of elements of $\mathcal{I}$ are embeddings. More precisely, as $|S_N| \geq \sqrt{N}$ by Lagrange’s theorem\(^2\), we have the uniform bound

\[ \frac{|\mathcal{E}|}{|\mathcal{I}|} \geq \left( 1 - \frac{m}{\sqrt{N_0}} \right)^m \to 1 \text{ as } N_0 \to \infty. \]

This means that we may choose $N_1 = N_1(B, m, \epsilon) > N_0$ (it does not depend on the tree $\tau$, only on its size) such that for all trees of the form $\tau = (V, v_0, E, \phi, Z_{>N_1})$, with $|V| = m$, we have that

\[ \left\| \mathbb{E}_{\iota \in \mathcal{E}} \prod_{v \in V \setminus \{v_0\}} T^{u(v)} \mathbf{1}_B - \mathbb{E}_{\iota \in \mathcal{I}} \prod_{v \in V \setminus \{v_0\}} T^{u(v)} \mathbf{1}_B \right\| < \epsilon, \]

\(^2\)There at least $\sqrt{N}$ solutions to $x_1^2 + \cdots + x_d^2 = N$ since for each choice of integer $x_1 \in [0, \sqrt{N}]$ there is at least one solution by Lagrange’s theorem (when $d \geq 5$).
where \( E = E(\tau) \) and \( I = I(\tau) \). Combining this with (3.4) we obtain that
\[
\left\| \mathbb{E}_{\iota \in E} \prod_{v \in V \setminus \{v_0\}} T^{(v)} \mathbb{1}_{B} - \mu(B)^{m-1} \right\|_{L^2(X, \mu)} < m\epsilon.
\]

Now Cauchy Schwartz gives (by the same argument as in the proof of Proposition 3.4.1)
\[
\left\| \mathbb{E}_{\iota \in E} \mu \left( \prod_{v \in V} T^{(v)} B \right) - \mu(B)^{m} \right\| < m\epsilon.
\]

It now immediately follows that we may choose an embedding \( \iota \in E \) satisfying the conclusion of the theorem. \( \square \)

### 3.6. Optimal Pointwise Recurrence

We now establish the pointwise recurrence result stated in Theorem 1.3.9. In what follows, we shall use the shorthand “\( P(b) \) holds for \( \mu \)-many \( b \in B \)” to mean \( \mu(\{b \in B \mid P(b)\}) > 0 \).

**Proposition 3.6.1** (Pointwise recurrence). Fix \( \epsilon > 0 \). Let \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) be an ergodic measure preserving action and suppose that \( B \subset X \), with \( \mu(B) > \epsilon \), is \((\eta, \delta)\)-equidistributed where \( \eta < \frac{1}{2} \epsilon^{2} \) and \( \delta = \frac{1}{3} \eta^{2} \). Then there exists \( N_0 \) such that for all \( N \geq N_0 \) we have that
\[
\frac{1}{|S_N|} \sum_{a \in S_N} 1_B(T^{qa} b) > \mu(B) - \epsilon \quad \text{for } \mu\text{-many } b \in B.
\]

More generally, we have the following pointwise multiple recurrence result.

**Proposition 3.6.2** (Pointwise multiple recurrence). Fix \( \epsilon > 0 \) and a positive integer \( m \). Let \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) be an ergodic measure preserving action and suppose that \( B \subset X \), with \( \mu(B) > \epsilon \), is \((\eta, \delta)\)-equidistributed where \( \eta < \frac{1}{2} \epsilon^{2} m^{-1} \) and \( \delta = \frac{1}{3} \eta^{2} \). Then there exists a positive integer \( N_0 \) such that for all \( N_0 < N_1 < \cdots < N_m \) we have that there exist \( \mu\)-many \( b \in B \) such that for all \( j \in \{1, \ldots, m\} \) we have that
\[
\frac{1}{|S_{N_j}|} \sum_{a \in S_{N_j}} 1_{T^a B}(b) > \mu(B) - \epsilon.
\]

**Remark 3.6.3.** The combinatorial consequence of this pointwise multiple recurrence result is similar to a recent optimal pinned distances result obtained by Lyall and Magyar (Theorem 2 and Theorem 4 in [23]). The difference is that they instead have the hypothesis \( \eta \ll \epsilon^{3} \), where the implied constant does not depend on \( m \), and in fact no other parameter depends on \( m \) (in particular, \( N_0 \)). It would be interesting to see whether this can be proven in the ergodic setting.
3.6. Optimal Pointwise Recurrence

**Proof.** We let

\[ U_N = \{ x \in X : | \mu(B) - \frac{1}{|S_N|} \sum_{a \in S_N} T^a 1_B(x) | \geq \epsilon \}. \]

We have that

\[
\mu(U_N) \leq \frac{1}{\epsilon} \left\| \mu(B) - \frac{1}{|S_N|} \sum_{a \in S_N} T^a 1_B \right\|_1 \\
\leq \frac{1}{\epsilon} \left\| \mu(B) - \frac{1}{|S_N|} \sum_{a \in S_N} T^a 1_B \right\|_2
\]

and so by Theorem 3.3.1 we have that

\[
\limsup_{N \to \infty} \mu(U_N) \leq \frac{1}{\epsilon} \cdot 2\eta < \frac{\epsilon}{m}.
\]

Hence for all sufficiently large \( N_1, \ldots, N_m \) we have that

\[
\mu \left( B \setminus \bigcup_{j=1}^m U_{N_j} \right) > \epsilon - m \cdot \frac{\epsilon}{m} = 0.
\]

\[ \Box \]

Let us now demonstrate how the spherical pointwise recurrence theorem (as formulated in Theorem 1.3.9) implies the result of Lyall-Magyar on *optimal spherical distribution* (Theorem 1.3.6).

**Proof of Theorem 1.3.6 from Theorem 1.3.9.** Let \( B \subset \mathbb{Z}^d \) be of positive density upper Banach density and consider the corresponding Furstenberg system \( T : \mathbb{Z}^d \curvearrowright (X, \mu) \) given in the correspondence principle (Proposition 2.4.1). Recall the point \( x_0 \in X \) and clopen set \( \tilde{B} \subset X \) described in Section 2.4. Fix \( \epsilon > 0 \) and choose, by Theorem 1.3.9, a positive integer \( q = q(\epsilon, d) \) such that for all large enough \( N \) we have

\[
\frac{1}{|S_N|} \sum_{a \in S_N} \mathbf{1}_{\tilde{B}}(T^{qa}x) > \mu(\tilde{B}) - \epsilon \quad \text{for some } x \in \tilde{B}.
\]

(3.5)

Now fix such an \( N \) and such an \( x \in \tilde{B} \). Notice that, since the orbit of \( x_0 \) is dense in \( X \) and \( \tilde{B} \) is open, we may find \( b \in \mathbb{Z}^d \) such that \( T^b x_0 \) is so close to \( x \) that

(a) \( T^b x_0 \in \tilde{B} \).

(b) \( T^{b+qa} x_0 \in \tilde{B} \) whenever \( a \in S_N \) is such that \( T^{qa} x \in \tilde{B} \).

Notice that \( T^b x_0 \in \tilde{B} \) implies that \( b \in B \) (recall that \( B = \tilde{B}_{x_0} \)). Likewise, (b) gives us that

\[
\frac{|B \cap (b + qS_N)|}{|S_N|} \geq \frac{1}{|S_N|} \sum_{a \in S_N} \mathbf{1}_{\tilde{B}}(T^{qa}b) > \mu(\tilde{B}) - \epsilon.
\]
But $\mu(\tilde{B}) = d^*(B)$, which finishes the proof.
Twisted Multiple Recurrence

In this chapter, we prove the twisted multiple recurrence result presented in Section 1.4 (Theorem 1.4.16). We also include, for the sake of completeness, proofs of some of the basic algebraic claims made in the examples of Theorem 1.4.5 given therein. This material appears in the joint work of the author and Fish [12].

4.1. Constructing hyperplane-fleeing polynomial walks

We now turn to the first part of the strategy of our proof, which is a construction of hyperplane-fleeing polynomial walks in hyperplane-fleeing orbits. We first state our result in the special case where our action is linear.

**Theorem 4.1.1.** Let $\Gamma \leq \text{GL}_d(\mathbb{Z})$, where $d \geq 2$, be a subgroup such that

1. The linear action of $\Gamma$ on $\mathbb{R}^d$ is irreducible.
2. There exists a finite set $S \subset \Gamma$ of unipotent matrices which generate $\Gamma$.

Then for all $v \in \mathbb{Z}^d \setminus \{0\}$ there exists a sequence $\gamma_1, \gamma_2, \ldots \in \Gamma$ such that

$$\gamma_n v = (p_1(n), \ldots, p_d(n))$$

for some polynomials $p_1(t), \ldots, p_d(t) \in \mathbb{Z}[t]$ such that no non-trivial linear combination of these polynomials is constant (i.e., $1, p_1(t), \ldots, p_d(t)$ are linearly independent over $\mathbb{Z}$). In other words, the sequence $\gamma_n v$ is hyperplane-fleeing.

Let us now return to the more abstract setting of non-linear actions. We begin with a simple lemma which will allow us to perform certain useful operations on polynomial walks.

**Lemma 4.1.2.** If $S, R : \mathbb{Z}_{\geq 0} \to \Gamma$ are polynomial walks then:

1. The map $S \circ R : \mathbb{Z}_{\geq 0} \to \Gamma$ given by $(S \circ R)(n) = S(n) \circ R(n)$ is a polynomial walk.
2. If $\ell$ is a positive integer then $S(\ell^n)$ is a polynomial walk.
3. For all integers $k$ and $n \geq 0$, we have that $S(kn)(k\mathbb{Z}^d) \subset k\mathbb{Z}^d$. 

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Proof of (3). By definition $S(0)0 = \text{Id} 0 = 0$. This means that each entry of $S(kn)(kx_1,\ldots,kx_d)$ is an integer polynomial, with zero constant term, in $kn,kx_1,\ldots,kx_d$ and hence a multiple of $k$ whenever $n,x_1,\ldots,x_d$ are integers. □

The non-linear generalization of Theorem 4.1.1 may now be formulated as follows.

Theorem 4.1.3. Let $\Gamma \subset \{\mathbb{Z}^d \to \mathbb{Z}^d\}$ be a semigroup of maps and suppose that there exists a finite set $\{s_1,\ldots,s_r\}$ of polynomial walks in $\Gamma$ such that $\{s_j(n) \mid n \geq 0 \text{ and } j = 1,\ldots,r\}$ generates $\Gamma$. Suppose that $v \in \mathbb{R}^d$ satisfies the property that the orbit $\Gamma v$ is not contained in any proper affine subspace of $\mathbb{R}^d$. Then there exists a polynomial walk $S : \mathbb{Z}_{\geq 0} \to \Gamma$ such that

$$S(n)v = (p_1(n),\ldots,p_d(n))$$

for some polynomials $p_1(t),\ldots,p_n(t) \in \mathbb{Z}[t]$ such that no non-trivial $\mathbb{R}$-linear combination of these polynomials is constant (i.e., $1,p_1(t),\ldots,p_d(t)$ are linearly independent over $\mathbb{R}$). In other words, the sequence $S(n)(v)$ is hyperplane-fleeing.

Proof. Use cyclic notation to define $s_N = s_{N \text{ mod } r}$ for all positive integer $N$ and let

$$A_N = \{s_N(t_N)\cdots s_1(t_1)v \mid t_1,\ldots,t_N \in \mathbb{Z}_{\geq 0}\}.$$ 

Notice that

$$\Gamma v = \bigcup_{N \geq 1} A_N$$

(this follows from $s_j(0) = \text{Id}$) and $A_n \subset A_{n+1}$. Let

$$V_N = \{L : \mathbb{R}^d \to \mathbb{R} \mid L \text{ is affine such that } L(a) = 0 \text{ for all } a \in A_N\}$$

denote the vector space of all affine maps (linear plus a constant) which annihilate $A_N$. Then $V_N \supset V_{N+1}$ and $\bigcap_{N \geq 1} V_N = \{0\}$ is trivial since $\Gamma v$ is hyperplane-fleeing. So there exists a positive integer $N$ such that $V_N = \{0\}$, which means that $A_N$ is hyperplane-fleeing. Hence the vector of polynomials

$$(p_1(t_1,\ldots,t_N),\ldots,p_d(t_1,\ldots,t_N)) = s_N(t_N)\cdots s_1(t_1)v \in (\mathbb{Z}[t_1,\ldots,t_N])^d$$

satisfies the property that no non-trivial linear combination of its entries is constant. It is easy to construct a rapidly growing sequence $e_1 < e_2 < \ldots < e_N$ of positive integers such that the polynomials

$$p_1(t^{e_1},\ldots,t^{e_N}),\ldots,p_d(t^{e_1},\ldots,t^{e_N}) \in \mathbb{Z}[t]$$
also satisfy the property that no non-trivial linear combination of them is constant (one may take 
\(e_k = R^k\) where \(R\) is the largest power of some \(t_i\) appearing in the polynomial vector above). This
means that

\[
S(n) = s_N(n^{e_N}) \cdots s_1(n^{e_1})
\]
satisfies the desired conclusion (it is a polynomial walk by Lemma 4.1.2). □

4.2. A uniform estimate of ergodic averages along polynomial walks

In this section, we study ergodic averages along hyperplane-fleeing polynomial walks. Our main
technical result is a uniform estimate of their limits, which depends only on the rational spectrum.

Definition 4.2.1. Let us recall some notation introduced in Chapter 3. We let

\[
\text{Rat} = \{ \chi \in \hat{\mathbb{Z}}^d \mid \text{The image of } \chi \text{ is finite.}\}
\]
denote the set of rational characters, where \(\hat{\mathbb{Z}}^d\) denotes the group of characters on \(\mathbb{Z}^d\) (homomorphisms from \(\mathbb{Z}^d\) to the multiplicative group of unit complex numbers). If \(T : \mathbb{Z}^d \rhd (X, \mu)\) is a measure
preserving system and \(\chi \in \hat{\mathbb{Z}}^d\) is a character then a \(\chi\)-eigenfunction of \((X, \mu, T)\) is a function
\(f \in L^2(X, \mu)\) such that \(T^v f = \chi(v) f\) for all \(v \in \mathbb{Z}^d\). Furthermore, if \(\chi \in \text{Rat}\) then we say that \(f\) is
a rational eigenfunction. We let \(L^2_{\text{Rat}}(X, \mu, T)\) denote the rational Kronecker factor, i.e., the norm
closed subspace spanned by rational eigenfunctions. We let \(P_{\text{Rat}} : L^2(X, \mu, T) \to L^2_{\text{Rat}}(X, \mu, T)\)
denote the orthogonal projection onto the rational Kronecker factor.

Proposition 4.2.2. Suppose that \(T : \mathbb{Z}^d \rhd (X, \mu)\) is a measure preserving system. Let \(f \in
L^2(X, \mu)\) be orthogonal to the rational Kronecker factor of \((X, \mu, T)\). Then for all polynomials \(p_1(n), \ldots, p_d(n) \in \mathbb{Z}[n]\) such that no non-trivial \(\mathbb{R}\)-linear combination of them is constant we have that

\[
\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{p(n)} f \right\|_2 = 0,
\]

where \(p(n) = (p_1(n), \ldots, p_d(n))\).

Proof. By Bocher’s theorem (Theorem B.1), there exists a positive Borel measure \(\sigma\) on \(\mathbb{T}^d\) such
that

\[
\langle T^v f, f \rangle = \int_{\mathbb{T}^d} e(\langle v, \theta \rangle) d\sigma(\theta) \quad \text{for all } v \in \mathbb{Z}^d.
\]

Since \(f \in L^2_{\text{Rat}}(X, \mu, T)\) we have, by Lemma B.2, that \(\sigma(\mathbb{Q}^d/\mathbb{Z}^d) = 0\) and hence

\[
\left\| \frac{1}{N} \sum_{n=1}^N T^{p(n)} f \right\|_2^2 = \int_{\Omega} \left\| \frac{1}{N} \sum_{n=1}^N e(\langle p(n), \theta \rangle) \right\|^2 d\sigma(\theta)
\]
where $\Omega = \mathbb{T}^d \setminus (\mathbb{Q}^d / \mathbb{Z}^d)$. But by Weyl’s polynomial equidistribution theorem (see Theorem 1.4 in [13]), we have that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(\langle p(n), \theta \rangle) = 0 \quad \text{for all } \theta \in \Omega$$

since $\langle p(n), \theta \rangle \notin \mathbb{R} + \mathbb{Q} [n]$, i.e., it is a polynomial in $n$ with at least one irrational non-constant term. The dominated convergence theorem now completes the proof. □

### Proposition 4.2.3

Suppose that $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ is a measure preserving system. Then for $f \in L^2(X, \mu)$ and $\epsilon > 0$ there exists a positive integer $k = k(f, \epsilon)$ such that the following holds: For all polynomials $p_1(n), \ldots, p_d(n) \in \mathbb{Z} [n]$ such that no non-trivial $\mathbb{R}$-linear combination of them is constant, we have that the limit

$$Q_p f := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{p(n)} f$$

exists in the $L^2(X, \mu)$ norm topology, where $p(n) = (p_1(n), \ldots, p_d(n))$, and if $p(\mathbb{Z}) \subset k \mathbb{Z}^d$ then

$$\|Q_p f - P_{\text{Rat}} f\|_2 < \epsilon.$$  

**Proof.** Decompose $f = h + h^\perp$ where $h$ is in the rational Kronecker factor and $h^\perp$ is orthogonal to the rational Kronecker factor. We already know that the limit $Q_p h^\perp$ exists and equals 0. Now write

$$h = \sum_{\chi \in \text{Rat}} c_\chi h_\chi$$

where $h_\chi$ is a $\chi$ eigenfunction (meaning that $T^v f = \chi(v) f$ for all $v \in \mathbb{Z}^d$) with $\|h_\chi\|_2 = 1$. Now if $\chi$ is rational then there exists a positive integer $k_\chi$ such that $\chi(v) = 1$ for all $v \in k_\chi \mathbb{Z}^d$, which means that the sequence $T^{p(n)} h_\chi$ is $k_\chi$ periodic and so we have the existence of the limit

$$Q_p h_\chi := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{p(n)} h_\chi = \frac{1}{k_\chi} \sum_{n=1}^{k_\chi} \chi(p(n)) h_\chi.$$  

The existence of the limit $Q_p f$ now immediately follows from a basic approximation argument. From this expression it also follows that if $p(\mathbb{Z}) \subset k_\chi \mathbb{Z}^d$ then

$$Q_p h_\chi = h_\chi.$$  

This means that if $k$ is a positive integer such that $p(\mathbb{Z}) \subset k \mathbb{Z}^d$, we will have

$$\|Q_p f - h\|_2 = \|Q_p h - h\|_2 \leq 2 \sqrt{\sum_{\{\chi : k \not\equiv 0 \mod k_\chi\}} c_\chi^2}.$$
But we may choose a highly divisible enough positive integer $k$ such that this upper bound is less than $\epsilon$. Notice that $k$ depends only on $\epsilon$ and the coefficients $c_\chi$ and is uniform across all such polynomial sequences $p(n)$.

4.3. Proof of the main recurrence result

We are now in a position to prove Theorem 1.4.16, which we restate verbatim for the convenience of the reader as follows.

**Theorem.** Let $\Gamma \subset \{\mathbb{Z}^d \to \mathbb{Z}^d\}$ be a semigroup of maps and suppose that there exists a finite set $\{s_1, \ldots, s_r\}$ of polynomial walks in $\Gamma$ such that $\{s_j(n) \mid n \geq 0 \text{ and } j = 1, \ldots, r\}$ generates $\Gamma$. Suppose that $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ is a measure preserving system. Then for all $B \subset X$ with $\mu(B) > 0$ and positive integers $m$ there exists a positive integer $k = k(B, m, \epsilon) > 0$ such that the following holds: If $v_1, \ldots, v_m \in k\mathbb{Z}^d$ satisfy the property that each orbit $\Gamma v_i$ is hyperplane-fleeing then there exist $\gamma_1, \ldots, \gamma_m \in \Gamma$ such that

$$\mu(B \cap T^{-\gamma_1 v_1} B \cap \cdots \cap T^{-\gamma_m v_m} B) > \mu(B)^{m+1} - \epsilon.$$ 

**Proof.** Apply Proposition 4.2.3 with $f = 1_B$ and let $k = k(f, \frac{\epsilon}{m}) > 0$ be as in the corresponding conclusion of that proposition. We apply Theorem 4.1.3 to each $v_i$ to obtain polynomial walks $R_i : \mathbb{Z}_{\geq 0} \to \Gamma$ such that each sequence $R_i(n)v_i$ is hyperplane-fleeing. Note the crucial observation that $v_i \in k\mathbb{Z}^d$ implies that the sequence $P_i(n) := R_i(kn)v_i$ satisfies $P_i(n) \in k\mathbb{Z}^d$ for all $n \in \mathbb{Z}_{\geq 0}$ (See Lemma 4.1.2) and is also hyperplane-fleeing (if a polynomial sequence enters a hyperplane infinitely many times, it is always there). Hence we may indeed apply Proposition 4.2.3 with $p(n) = P_i(n)$ to get that

$$\left\| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{P_i(n)} 1_B - P_{\text{Rat}} 1_B \right\|_2 < \frac{\epsilon}{m}. \quad (4.1)$$

Now consider the following average of correlations

$$C(N_1, \ldots, N_m) = \frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} \mu(B \cap T^{-P_1(n_1)} B \cap \cdots \cap T^{-P_m(n_m)} B)$$

$$= \frac{1}{N_1 \cdots N_m} \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} \int_X 1_B(T^{P_1(n_1)} 1_B) \cdots (T^{P_m(n_m)} 1_B) d\mu$$

and notice that

$$\lim_{N_1 \to \infty} \cdots \lim_{N_m \to \infty} C(N_1, \ldots, N_m) = \int_X 1_B(Q_{P_1} 1_B) \cdots (Q_{P_m} 1_B) d\mu$$
where \( Q_{P_i} \) is as defined in the statement of Proposition 4.2.3. But now applying the estimates (4.1) iteratively for \( i = 1, \ldots, m \) together with the fact that \( \| P_{\text{Rat}} \mathbbm{1}_B \|_\infty \leq 1 \) (this follows from the well-known fact \( P_{\text{Rat}} \) is a conditional expectation) we get that
\[
\int_X 1_B(Q_{P_1} \mathbbm{1}_B) \cdots (Q_{P_m} \mathbbm{1}_B) d\mu > \int_X 1_B(P_{\text{Rat}} \mathbbm{1}_B)^m d\mu - \epsilon.
\]
Now we use the fact that \( (P_{\text{Rat}} \mathbbm{1}_B)^m \in L^2_{\text{Rat}}(X, \mu, T) \) (since \( L^2_{\text{Rat}}(X, \mu, T) \cap L^\infty(X, \mu) \) is an algebra) and Jensen’s inequality to get
\[
\int_X 1_B(P_{\text{Rat}} \mathbbm{1}_B)^m d\mu = \langle 1_B, (P_{\text{Rat}} \mathbbm{1}_B)^m \rangle \\
\geq \langle P_{\text{Rat}} \mathbbm{1}_B, (P_{\text{Rat}} \mathbbm{1}_B)^m \rangle \\
= \langle P_{\text{Rat}} \mathbbm{1}_B, (P_{\text{Rat}} \mathbbm{1}_B)^m \rangle \\
\geq \left( \int P_{\text{Rat}} \mathbbm{1}_B d\mu \right)^{m+1} \\
= \left( \int 1_B d\mu \right)^{m+1}.
\]
This finishes the proof, as we have shown that
\[
\lim_{N_1 \to \infty} \cdots \lim_{N_m \to \infty} C(N_1, \ldots, N_m) > \mu(B)^{m+1} - \epsilon.
\]
\[ \square \]

### 4.4. Some basic algebraic facts

We now prove some elementary algebraic results that are needed to justify many of the examples of Theorem 1.4.5 stated in Section 1.4.

**Lemma 4.4.1.** Let \( \Gamma \leq G \leq \text{GL}_n(\mathbb{R}) \) be groups such that \( G \) is the Zariski closure of \( \Gamma \). Suppose that \( \rho : G \to \text{GL}_d(\mathbb{R}) \) is an irreducible representation such that \( \rho \) is a polynomial map. Then the restriction \( \rho|_\Gamma : \Gamma \to \text{GL}_d(\mathbb{R}) \) is also irreducible.

**Proof.** Suppose on the contrary that the restriction is reducible. This means that there exists a proper linear subspace \( W \leq \mathbb{R}^d \) and \( w \in W \) such that \( \rho(\Gamma)w \subset W \). Let \( \pi : \mathbb{R}^d \to \mathbb{R}^d/W \) denote the quotient map. Then \( P : G \to \text{GL}_d(\mathbb{R}) \) given by \( P(g) = \pi(\rho(g)w) \) is a polynomial in \( g \) which vanishes for all \( g \in \Gamma \). Since \( G \) is the Zariski closure of \( \Gamma \), we get that \( P \) also vanishes on \( G \) and hence \( \rho(G)w \subset W \), which contradicts the irreducibility of \( \rho \). \[ \square \]
4.4. Some basic algebraic facts

**Lemma 4.4.2.** Let \( a, b \in \mathbb{Z} \setminus \{0\} \) be non-negative integers. Then the subgroup

\[
\Gamma_0 = \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right\rangle
\]

is Zariski dense in \( \text{SL}_2(\mathbb{R}) \).

**Proof.** Let

\[
U(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]

We wish to show that the Zariski closure of \( \Gamma_0 \) contains \( U(t) \) and its transpose, for all \( t \in \mathbb{R} \), as these generate \( \text{SL}_2(\mathbb{R}) \). Now suppose that \( P : \text{SL}_2(\mathbb{R}) \to \mathbb{R} \) is a polynomial map which vanishes on all of \( \Gamma_0 \). Then, in particular, the polynomial \( R : \mathbb{R} \to \mathbb{R} \) given by \( R(x) = P(U(x)) \) vanishes on the infinite set \( a\mathbb{Z} \), and so \( R(x) \) is the zero polynomial. Hence \( P \) vanishes on \( U(t) \), for all \( t \in \mathbb{R} \). This shows that \( U(t) \) is in the Zariski closure, and a similar argument applies to its transpose. \( \square \)

**Example 4.4.3.** The adjoint representation \( \text{Ad} : \text{SL}_d(\mathbb{R}) \to \text{GL}(\text{sl}_d(\mathbb{R})) \) is a polynomial map. It is an irreducible representation and hence the above lemmata may be applied to verify the claims in Example 1.4.7.

**Lemma 4.4.4.** Suppose that \( V = W_0 \oplus W \oplus W_1 \) is a direct sum of linear spaces, with \( W \neq \{0\} \), and let \( V_0 = W_0 + W \) and \( V_1 = W + W_1 \). Let \( \Gamma_1, \Gamma_2 \leq \text{GL}(V) \) be groups such that for \( i = 0, 1 \) we have that

1. \( \Gamma_i V_i = V_i \) and the representation \( \Gamma_i \curvearrowright V_i \) is irreducible.
2. \( \Gamma_i \) acts trivially on \( W_{1-i} \) (i.e., \( \gamma v = v \) for all \( \gamma \in \Gamma_i \) and \( v \in W_{1-i} \)).

Then the action of \( \Gamma = \langle \Gamma_0 \cup \Gamma_1 \rangle \) on \( V \) is irreducible.

**Proof.** It will be more convenient to work with the group algebras \( A_i = \mathbb{F}[\Gamma_i] \) and \( A = \mathbb{F}[\Gamma] \). Let \( v \in V \) be non-zero. We wish to show that \( Av = V_0 + V_1 \). Since \( v \neq 0 \) we get that either \( A_1 v \) or \( A_2 v \) contains a vector with \( W \) component non-zero (from the irreducibility of \( \Gamma_1 \) and \( \Gamma_2 \)). Let \( v' \in Av \) be such a vector and write \( v' = w_0' + w' + w'_1 \) where \( w' \in W, w'_0 \in W_0 \) and \( w'_1 \in W_1 \). As \( w' \neq 0 \), the hypothesis on \( \Gamma_i \) gives us that \( A_i w' = V_i \) but \( A_i w_{1-i} = \{w_{1-i}\} \) for \( i = 0, 1 \). This means that

\[
A_0 v' = V_0 + w'_1 \quad \text{and} \quad A_1 v' = w'_0 + V_1.
\]

Hence \( Av \) contains \( A_0 v' + A_1 v' = V_0 + V_1 + w'_0 + w'_1 = V_0 + V_1 = V. \) \( \square \)

**Proposition 4.4.5.** Consider the quadratic form \( Q(x_1, \ldots, x_p, y_1, \ldots, y_q) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2 \) where \( p \geq 1 \) and \( q \geq 2 \). Then \( \text{SO}(Q)(\mathbb{Z}) \) contains a subgroup acting irreducibly on \( \mathbb{R}^d \) that is generated by finitely many unipotent elements.
Proof. Let $e_1, \ldots, e_p, u_1, \ldots, u_q$ denote the standard basis for $\mathbb{R}^{p+q}$. For $1 \leq a \leq p$ and $1 \leq b < c \leq q$ let $V_{a, b, c}$ be the vector space spanned by $e_a, u_b, u_c$. Then, as we have already seen in the $(p, q) = (1, 2)$ case (Example 1.4.7), we have a group $\Gamma_{a, b, c} \leq SL_d(\mathbb{Z})$, generated by a finite set of unipotents, which acts irreducibly on $V_{a, b, c}$, preserves $x_a^2 - y_a^2 - y_b^2$ and acts trivially on the complement (w.r.t. this basis) of $V_{a, b, c}$. By iteratively applying Lemma 4.4.4 to suitable combinations of the $\Gamma_{a, b, c}$ and $V_{a, b, c}$, we may construct the desired subgroup. \qed
This chapter is devoted to a proof of the general Plünnecke inequality for measure preserving actions of countable abelian groups (Theorem 1.5.6). This has appeared in the author’s joint publication with Fish [11]. Let us give a brief outline of this chapter. The main object introduced in this chapter is, what we call, a measure graph. Section 5.1 provides all the relevant definitions and basic properties. Intuitively, a measure graph is a directed edge-labelled graph equipped with a measure on the vertex set that mimics certain elementary combinatorial properties of the classical graph-theoretic notion of a matching. The aim is to prove a measure-theoretic version of the classical Plünnecke inequality for commutative graphs. The classical approach employs Menger’s theorem, which has no obvious measure theoretic analogue. However, Petridis [27] has recently found a new proof of this inequality that avoids the use of Menger’s theorem. In Section 5.2 we generalize this proof to measure graphs. This immediately implies our Ergodic Plünnecke inequality (Theorem 1.5.6) when $A$ is finite. We then, in Section 5.4, turn to extending this to the general case where $A$ is countable.

We finish this chapter with the proof of the (unpublished) result, obtained by the author and Fish, on triple product estimates in countable amenable groups for symmetric sets (Proposition 1.5.9).

5.1. Definitions

By a labelled directed graph we mean a tuple $(V, E, A)$ where $V$ and $A$ are sets and $E \subset V \times V \times A$. We regard an element $(v, w, a) \in E$ as an edge directed from $v$ to $w$ and labelled $a$. For subsets $W \subset V$ and labels $a \in A$ the $a$-image and $a$-preimage are defined, respectively, as

$$\text{Im}_a^+(W) = \{v \in V \mid (w, v, a) \in E \text{ for some } w \in W\}$$

and

$$\text{Im}_a^-(W) = \{v \in V \mid (v, w, a) \in E \text{ for some } w \in W\}.$$ 

That is, the $a$-image of $W$ consists of the vertices that may be approached to from $W$ by walking, in the direction of the orientation, along an edge labelled $a$. Moreover we define for $W \subset V$ the (pre)image $\text{Im}^\pm(W) = \bigcup_{a \in A} \text{Im}_a^\pm(W)$. For each integer $h$ we have the $h$-fold image
\[
I_m^h(W) \text{ defined recursively by } I_m^0(W) = W \text{ and } I_m^h(W) = I_m^+(I_m^{h-1}(W)) \text{ for } h > 0 \\
\text{and } I_m^h(W) = I_m^-(I_m^{h+1}(W)) \text{ for } h < 0. \text{ In other words, } I_m^h(W) \text{ consists of all end points of walks with } |h| \text{ steps that begin at } W \text{ and agree (resp. disagree) with the orientation of each edge if } h > 0 \text{ (resp. } h < 0). \text{ Define also the incoming and outgoing degrees of a vertex } v \text{ as } \\
d^-(v) = |\{(x, y, a) \in E \mid y = v\}| \text{ and } d^+(v) = |\{(x, y, a) \in E \mid x = v\}| \text{ respectively. Note that } \\
|I_m^+(\{v\})| \leq d^+(v) \text{ with strict inequality possible in case of multiple edges between two vertices (of course any two such edges would have different labels). Given an edge } e \text{ from } v \text{ to } w, \text{ we will call } v \text{ the tail, denoted } \text{tail}(e), \text{ and } w \text{ the head, denoted by } \text{head}(e). \text{ Let } E^+(v) \text{ denote the edges whose tail is } v \text{ and } E^-(v) \text{ those edges whose head is } v.
\]

**Definition 5.1.1.** A measure graph is a tuple \( \Gamma = (V, B, \mu, A, E) \) where \((V, B, \mu)\) is a finite measure space (that is, \(\mu(V) < \infty\)), \(A\) is a finite set and \((V, E, A)\) is a labelled directed graph such that

1. For each \(a \in A\) the sets 
   \[
   L_a^+(\Gamma) = L_a^+ = \{x \in V \mid \text{There exists } y \in V \text{ such that } (x, y, a) \in E\}
   \]
   and
   \[
   L_a^-(\Gamma) = L_a^- = \{x \in V \mid \text{There exists } y \in V \text{ such that } (y, x, a) \in E\}
   \]
   are measurable.
2. For \(a \in A\) and measurable \(W \subset L_a^\pm\) we have that \(I_m^\pm(W)\) is measurable and \(\mu(W) = \mu(I_m^\pm(W))\).
3. For each label \(a \in A\) and vertex \(x \in V\) there is at most one outgoing and at most one incoming \(a\)-labelled edge incident to \(x\). That is, \(|I_m^\pm(x)| \leq 1\).

**Example 5.1.2** (The \((A, Y, h)\)-orbit graph). Given an abelian group \(G\) acting on a measure space \((X, B, \mu)\) one may form for each integer \(h > 0\), finite \(A \subset G\) and \(Y \in B\) of finite measure, a measure graph whose underlying vertex set is \(\bigsqcup_{k=0}^h A^kY \times \{k\}\) with edge set 
\[
\{(x, k), (a.x, k + 1), a)\mid a \in A, x \in A^kY, k = 0, 1, \ldots, h - 1\}.
\]
The measure is the restriction of the natural product measure on \(X \times \{0, 1, \ldots, h\}\).

Given a labelled graph \(\Gamma = (V, E, A)\) and \(W \subset V\), the subgraph induced by \(W\) is the directed labelled graph \((W, E_W, A)\) where \(E_W = \{(w_1, w_2, a)\mid w_2 \in W, a \in A \text{ and } (w_1, w_2, a) \in E\}\). We say that a subgraph of \(\Gamma\) is an induced subgraph if it is induced by some subset of \(V\).
Example 5.1.3. If $\Gamma = (V, B, \mu, A, E)$ is a measure graph and $W \subset V$ is measurable then the subgraph of $\Gamma$ induced by $W$ is a measure graph (with the restricted measure, restricted $\sigma$-algebra and the same edge-label set $A$). Note that the set of vertices with an outgoing edge labelled $a \in A$ is $L_a^+ \cap W \cap Im_a^{-} (L_a^- \cap W)$ and thus is measurable as required.

Note that the $(A, Y, h)$-orbit graph defined above is a generalization of the commutative addition graph studied in classical Additive Combinatorics, see for example [31], [35]. It is also an example of what is known as a commutative, or Plünnecke, graph which may be defined as follows.

Definition 5.1.4. A layered-graph is a directed labelled graph $(V, E, A)$ together with a partition $V = V_0 \sqcup V_1 \sqcup \ldots \sqcup V_h$ such that if $e = (x, y, a) \in E$ is a directed edge then $x \in V_i$ and $y \in V_{i+1}$ for some $i \in \{0, \ldots, h - 1\}$. We call $V_k$ the $k$-th layer and we say that $(V, E, A)$ is a $h$-layered graph (we regard the partition as part of the data of a layered graph). A semi-commutative (or semi-Plünnecke) graph is a layered graph $(V, E, A)$ such that if $(x, y, a) \in E$ is an edge then there is an injection $\phi : E^+(y) \rightarrow E^+(x)$ such that $(\text{head}(\phi(e)), \text{head}(e), a) \in E$ for all $e \in E^+(y)$.

A commutative, or Plünnecke, graph $\Gamma$ is a directed layered graph such that both $\Gamma$ and the dual graph (the layered graph obtained by reversing edges and the ordering of the layers) $\Gamma^*$ are semi-commutative.

Example 5.1.5. The $(A, Y, h)$-orbit graph defined above is a commutative graph with layering $V = \bigsqcup_{j=0}^h A^j Y \times \{j\}$. To check semicommutativity, take a typical edge $((x, j), (ax, j + 1), a)$ running from $A^j Y \times \{j\}$ to $A^{j+1} Y \times \{j + 1\}$ where $0 \leq j \leq h - 1$. Then for edges

$$e = ((ax, j + 1), (a'ax, j + 2), a') \in E^+((ax, j + 1))$$

we may choose $\phi(e) = ((x, j), (a'x, j + 1), a')$ since, by commutativity of $G$, $a.(a'.x) = (a'a.x)$ and thus there is an $a$-labelled edge from $\text{head}(\phi(e))$ to $\text{head}(e)$ as required. The semi-commutativity of the dual can be similarly verified.

The following is an easy exercise in commutative graphs (see [31]).

Proposition 5.1.6. Suppose that $(V, E)$ is a $h$-layered commutative graph with layers $V = V_0 \sqcup \ldots \sqcup V_h$. Then for $S \subset V_j$ and $T \subset V_k$, where $0 \leq j < k \leq h$, we have that the channel between $S$ and $T$ (that is, the subgraph consisting of all directed paths from $S$ to $T$) is a commutative graph. We denote this subgraph $\text{ch}(S, T)$.

We will be interested in studying channels of an $(A, Y, h)$-orbit graph, it turns out these are measurable.
Lemma 5.1.7. Given an $h$-layered measure graph $\Gamma = (V, \mathcal{B}, \mu, A, E)$ with layering \( V = V_0 \sqcup \ldots \sqcup V_h \) and measurable $S \subset V_i$, $T \subset V_j$ where $0 \leq i < j \leq h$ we have that the channel $ch(S, T)$ has measurable vertex set.

Proof. Let us denote the vertex set of a subgraph $\Gamma'$ as $V(\Gamma')$. We use induction on $j - i$. The base case $j = i + 1$ holds since then $ch(S, T)$ has vertex set $V(ch(S, T)) = S \cap \text{Im}^- (T) \sqcup T \cap \text{Im}(S)$. Now suppose that $j - i > 1$. Then by the induction hypothesis we have that $ch(S, \text{Im}^- (T))$ has measurable vertex set. By the base case, $ch(\text{Im}^- (T), T)$ has measurable vertex set. Now let $U = V(ch(S, \text{Im}^- T)) \cap V(ch(\text{Im}^- T, T)) \subset V_{j-1}$. Finally we have $V(ch(S, T)) = V(ch(S, U)) \cup V(ch(U, T))$ which is measurable again by the induction hypothesis. \qed

Note that the previous Lemma and Example 5.1.3 demonstrate that the channel between two measurable sets may be naturally viewed as a measure graph (as channels are induced subgraphs).

We now turn to generalizing the notion of the number of edges in a bipartite graph.

Definition 5.1.8. Fix a 1-layered commutative measure graph $(U, \mathcal{B}, \mu, A, E)$ with layering $U = U_0 \sqcup U_1$. Define the flow of $\Gamma$ to be the quantity

$$\text{Flow}(\Gamma) = \int_{U_0} d^+(v) d\mu(v).$$

We now show that the flow behaves nicely and that $d^\pm$ is a measurable function.

Proposition 5.1.9. Under the setting of the previous definition, the map $d^+ : U \to \mathbb{R}$ is measurable and $\text{Flow}(\Gamma) = \text{Flow}(\Gamma^*)$, that is

$$\int_{U_0} d^+(v) d\mu(v) = \int_{U_1} d^-(v) d\mu(v).$$

Proof. Since (by definition of a measure graph) $|\text{Im}^\pm_a(\{v\})| \leq 1$, we may express

$$d^\pm = \sum_{a \in A} 1_{L^\pm_a}$$

and thus $d^\pm$ is measurable. Consequently we have that

$$\text{Flow}(\Gamma) = \sum_{a \in A} \mu(L^+_a) = \sum_{a \in A} \mu(\text{Im}^+_a(L^+_a)) = \sum_{a \in A} \mu(L^-_a) = \text{Flow}(\Gamma^*)$$

as required. \qed

\footnote{We always assume implicitly that each layer is measurable.}
5.2. Plünnecke’s inequality for measure graphs

Definition 5.2.1. Given a commutative measure graph $\Gamma = (V, B, \mu, A, E)$ with layering $V = V_0 \sqcup \ldots \sqcup V_h$, the magnification ratio of order $j$, where $j \in \{1, \ldots h\}$, is

$$D_j = \inf_{Y \subseteq V_0, \mu(Y) > 0} \frac{\mu(\text{Im}^j(Y))}{\mu(Y)}.$$  

Moreover, for $C > 0$, define the weight (corresponding to $C$) to be the measure on $B$ given by

$$w(S) = \sum_{j=0}^h C^{-j} \mu(S \cap V_j)$$

for $S \in B$. Furthermore, we say that $S \in B$ is a cutset if any path from $V_0$ to $V_h$ intersects $S$ and that $S$ is an $\epsilon$-minimal cutset if $S$ is a cutset such that

$$w(S) \leq m_0 + \epsilon$$

where $m_0 = \inf \{\mu(Y) \mid Y \in B \text{ is a cutset}\}$.

Lemma 5.2.2. Fix a 2-layered commutative measure graph $(U, B, \mu, A, E)$ with layering $U = U_0 \sqcup U_1 \sqcup U_2$ and $C > 0$. Then if $U_1$ is an $\epsilon$-minimal cutset (with respect to the weight corresponding to $C$), then $U_0$ is an $f(\epsilon)$-minimal cutset where

$$f(\epsilon) = \epsilon + 4|A|^2 C \epsilon + 4|A|^2 \epsilon.$$  

Proof. Let $m_0 = \inf \{w(S) \mid S \in B \text{ is a cutset}\}$. Firstly note that for measurable $S \subseteq U_1$ we have that $\text{Im}(S) \sqcup (U_1 \setminus S)$ is a cutset and thus $m_0 \leq w(\text{Im}(S)) + w(U_1 \setminus S)$. On the other hand, since $U_1$ is $\epsilon$-minimal we have that $w(S) + w(U_1 \setminus S) - \epsilon \leq m_0$ and thus $w(S) \leq w(\text{Im}(S)) + \epsilon$. A similar argument yields that $w(S) \leq w(\text{Im}^{-1}(S)) + \epsilon$. Thus

$$C \mu(S) \leq \mu(\text{Im}(S)) + C^2 \epsilon \quad (\ast)$$

and

$$C^{-1} \mu(S) \leq \mu(\text{Im}^{-1}(S)) + \epsilon \quad (\dagger)$$

for measurable $S \subseteq U_1$.

For each integer $i \geq 0$ let

$$X_i = \{u \in U_1 \mid d^- (v) = i\}$$

$$Y_i = \{u \in U_2 \mid d^- (v) = i\}$$

$$X_i' = \{u \in U_1 \mid d^+ (v) = i\}$$
The $X_i$ are measurable and partition $U_1$. Let $k = |A|$. Define now inductively $T_k = \text{Im}(X_k)$ and $T_i = \text{Im}(X_i) \setminus T_{i+1}$ for $i = k-1, k-2, \ldots, 1$. Note that the $T_i$ partition the set of vertices in $U_2$ that have at least one incoming edge. Moreover, by the definition of a commutativity we have that each vertex in $T_i$ has inwards degree at least $i$ (specifically, this is by the semicommutativity of the dual). Thus we obtain

$$
\sum_{i=1}^{k} i\mu(T_i) \leq \sum_{i=1}^{k} \text{Flow}(ch(U_1, T_i)) = \text{Flow}(ch(U_1, U_2)) \tag{5.1}
$$

where the right hand side is well defined since induced subgraphs with measurable vertex sets are measure graphs. From now on we will use the shorthand notation $\text{Flow}(U_i, U_j) := \text{Flow}(ch(U_i, U_j))$.

Moreover, since $\text{Im}(X_j \sqcup \ldots \sqcup X_k) = T_j \sqcup \ldots \sqcup T_k$ we have by ($\star$)

$$
C \sum_{i=j}^{k} \mu(X_i) \leq \sum_{i=j}^{k} \mu(T_i) + C^2 \epsilon. \tag{5.2}
$$

Adding these inequalities for $j = 1, \ldots, k$ we obtain

$$
C \cdot \text{Flow}(U_0, U_1) = C \sum_{i=1}^{k} i\mu(X_i) \leq \sum_{i=1}^{k} i\mu(T_i) + kC^2 \epsilon \tag{5.3}
$$

which implies, together with the preceding inequality, that

$$
C \cdot \text{Flow}(U_0, U_1) \leq \text{Flow}(U_1, U_2) + kC^2 \epsilon. \tag{5.4}
$$

We will now apply the same argument to the dual graph to obtain an inequality of the form

$$
\text{Flow}(U_1, U_2) \leq C \cdot \text{Flow}(U_0, U_1) + O_{k,C}(\epsilon).
$$

To do this, inductively define $T'_k = \text{Im}^{-1}(X'_k)$ and $T'_i = \text{Im}^{-1}(X'_i) \setminus T'_{i+1}$ for $i = k-1, k-2, \ldots, 1$. This time we have, by the duals of the previous arguments (with ($\dagger$) in place of ($\star$)) and the fact that flows are the same for duals (Proposition 5.1.9), that

$$
\sum_{i=1}^{k} i\mu(T'_i) \leq \text{Flow}(ch_{\Gamma^*}(U_1, U_0)) = \text{Flow}(U_0, U_1) \tag{5.5}
$$

and, for each $\ell \in \{1, 2 \ldots k\}$,

$$
C^{-1} \sum_{i=\ell}^{k} \mu(X'_i) \leq \sum_{i=\ell}^{k} \mu(T'_i) + \epsilon \tag{5.6}
$$
which, by summing as before, gives
\[ C^{-1} \text{Flow}(U_1, U_2) \leq \text{Flow}(U_0, U_1) + k\epsilon. \]

Thus \(\text{Flow}(U_1, U_2)\) is close to \(C \cdot \text{Flow}(U_0, U_1)\). This means that the inequalities above must have been close to being equalities. We will now explicitly estimate how close. Let us start with the inequality (5.6). We obtain from it, (5.4), and (5.5) that for each \(j \in \{1, \ldots, k\}\) we have
\[
C^{-1} \sum_{i=j}^{k} \mu(X_i') - \sum_{i=j}^{k} \mu(T_i') + (k - 1)\epsilon \geq C^{-1} \sum_{i=1}^{k} i\mu(X_i') - \sum_{i=1}^{k} i\mu(T_i') \\
\geq C^{-1} \text{Flow}(U_1, U_2) - \text{Flow}(U_0, U_1) \\
\geq -kC\epsilon
\]

where the first inequality is obtained by summing (5.6) for \(\ell \in \{1, 2, \ldots, k\} \setminus \{j\}\). This finally gives
\[
C^{-1} \sum_{i=j}^{k} \mu(X_i') - \sum_{i=j}^{k} \mu(T_i') \geq -kC\epsilon - (k - 1)\epsilon
\]

and so
\[
|C^{-1} \sum_{i=j}^{k} \mu(X_i') - \sum_{i=j}^{k} \mu(T_i')| \leq \max\{kC\epsilon + (k - 1)\epsilon, \epsilon\} \leq kC\epsilon + k\epsilon. \tag{5.8}
\]

Thus the triangle inequality gives
\[
|C^{-1} \mu(X_i') - \mu(T_i')| \leq 2kC\epsilon + 2k\epsilon. \tag{5.9}
\]

Now we wish to show that \(T_i'\) is approximately \(Y_i'\) (that is, they have small symmetric difference). Note that \(T_j' \sqcup \ldots \sqcup T_k' \subset Y_j' \sqcup \ldots \sqcup Y_k'\) since, as before, the definition of a commutative graph implies that each vertex in \(T_i'\) has outwards degree at least \(i\). Thus
\[
\sum_{i=j}^{k} \mu(T_i') - \sum_{i=j}^{k} \mu(Y_i') \leq 0. \tag{5.10}
\]

Combining this with (5.6) and (5.4) we have that for each \(j \in \{1, \ldots, k\}\) we have
\[
\sum_{i=j}^{k} \mu(T_i') - \sum_{i=j}^{k} \mu(Y_i') \geq \sum_{i=1}^{k} i\mu(T_i') - \sum_{i=1}^{k} i\mu(Y_i')
\]
\[ \geq C^{-1} \sum_{i=1}^{k} i\mu(X'_i) - k\epsilon - \sum_{i=1}^{k} i\mu(Y'_i) \]
\[ = C^{-1}\text{Flow}(U_1, U_2) - \text{Flow}(U_0, U_1) - k\epsilon \]
\[ \geq -kC\epsilon - k\epsilon. \]

Thus
\[ |\sum_{i=j}^{k} \mu(T'_i) - \sum_{i=j}^{k} \mu(Y'_i)| \leq kC\epsilon + k\epsilon \]
from which the triangle inequality implies
\[ |\mu(T'_i) - \mu(Y'_i)| \leq 2kC\epsilon + 2k\epsilon. \quad (5.11) \]

Combining this with (5.9) yields
\[ |C^{-1}\mu(X'_i) - \mu(Y'_i)| \leq 4kC\epsilon + 4k\epsilon. \]

Finally we get
\[ |w(U_1) - w(U_0)| = |C^{-1} \sum_{i=1}^{k} \mu(X'_i) - \sum_{i=1}^{k} \mu(Y'_i)| \leq 4k^2C\epsilon + 4k^2\epsilon \]
and so in fact \( U_0 \) is \((\epsilon + 4k^2C\epsilon + 4k^2\epsilon)\)-minimal. \( \square \)

We will now inductively apply the above lemma to construct \( \epsilon \)-minimal cutsets that lie in the union of the top and bottom layers.

**Lemma 5.2.3.** Suppose that \( \Gamma = (V, B, \mu, A, E) \) is a \( h \)-layered commutative measure graph with layering \( V = V_0 \sqcup \ldots \sqcup V_h \). Fix \( C > 0 \) and let \( w \) be the weight on \( \Gamma \) corresponding to \( C \). Then for each \( \epsilon > 0 \) there exists an \( \epsilon \)-minimal cutset \( S \in B \) such that \( S \subset X_0 \sqcup X_h \).

**Proof.** We will prove, by induction on \( j \in \{h - 1, \ldots, 1, 0\} \), that there exists an \( \epsilon \)-minimal cutset contained in \( V_0 \sqcup V_1 \sqcup \ldots \sqcup V_j \sqcup V_h \). The base case \( j = h - 1 \) is clear. Thus fix \( \delta > 0 \) and suppose that \( j \in \{1, \ldots, h - 1\} \) and \( S \subset V_0 \sqcup V_1 \sqcup \ldots \sqcup V_j \sqcup V_h \) is a \( \delta \)-minimal cutset. Let \( S_i = S \cap X_i \) for \( i \in \{1, \ldots, h\} \). Let \( U_0 \subset V_{j-1} \) be those vertices that may be approached to \( V_{j-1} \) from \( V_0 \) along a path that does not intersect \( S \). Let \( U_2 = V_{j+1} \cap \text{ch}(V_{j+1}, V_h \setminus S_h) \) be the set of vertices in \( V_{j+1} \) that may be approached to \( V_h \setminus S_h \). We know that \( U_2 \) is measurable by the measurability of channels and similarly \( U_0 \) is measurable by an application of the measurability of channels to the subgraph induced by \( \bigsqcup_{i=0}^{h} V_i \setminus S_i \). Let \( H = \text{ch}(U_0, U_2) \) and let \( U_1 \subset V_j \) by the vertices in \( H \) that lie in \( V_j \). Thus \( H \) is a 2-layered measure subgraph of \( \Gamma \) that is also commutative. Let us equip \( H \) with the
measure $C^{-j+1}\mu$ instead of $\mu$, since then the weight function on $H$ corresponding to $C$ agrees with that of $\Gamma$.

**Subclaim:** The middle layer $U_1$ is a $\delta$-minimal cutset of $H$.

To see this, firstly note that $U_1 \subset S_j$ (see Figure 1). If $U_1$ is not $\delta$-minimal, then there exists a cutset $T$ in $H$ of weight $w(T) < w(U_1) - \delta \leq w(S_j) - \delta$. But then the set $S' = S_0 \cup \ldots S_{j-1} \cup T \cup S_h$ is a cutset of $\Gamma$ of weight

$$w(S') \leq \sum_{i=0}^{j-1} w(S_i) + w(T) + w(S_h) < \sum_{i=0}^{j-1} w(S_i) + w(S_j) - \delta + w(S_h) = w(S) - \delta,$$

contradicting $S$ being $\delta$-minimal. This proves the subclaim.

![Figure 1](image)

**Figure 1.** A dotted 2-length path as shown cannot exist as this gives rise to a path from $V_0$ to $V_h$ which avoids $S$, as shown, by the definition of $U_0$ and $U_2$. Thus $U_1 \subset S_j$. One can similarly argue that the $S'$ given in the proof of the subclaim is a cutset.

Hence we get by Lemma 5.2.2 that $(S \cup U_0) \setminus S_j$ is a $(\delta + f(\delta))$-minimal cutset, where $f$ is as in the respective lemma (which we may take with the parameters of $\Gamma$, i.e: we consider $H$ as having labelling set $A$ and thus this $f$ does not depend on $H$). Taking $\delta \to 0$ finishes the induction step and hence the proof of this lemma. □
We are now ready to show that in the case \( C = D_1^{1/h} \) the bottom layer is in fact a cutset of minimal weight.

**Corollary 5.2.4.** Suppose that \( \Gamma = (X, \mathcal{B}, \mu, A, E) \) is a \( h \)-layered commutative measure graph with layering \( X = X_0 \sqcup \ldots \sqcup X_h \). Suppose that \( D_1^{1/h} > 0 \) and let \( w \) be the weight corresponding to \( C = D_1^{1/h} \). Then \( X_0 \) is a cutset of minimal weight.

**Proof.** We want to show that \( X_0 \) is \( \epsilon \)-minimal for all \( \epsilon > 0 \). Choose \( \epsilon > 0 \) and by the above lemma an \( \epsilon \)-minimal cutset \( S \subset X_0 \sqcup X_h \). Write \( S_i = X_i \cap S \). As \( S \) is a cutset we have \( \text{Im}^h(X_0 \setminus S_0) \subset S_h \) and so

\[
\mu(S_h) \geq \mu(\text{Im}^h(X_0 \setminus S_0)) \geq D_h \mu(X_0 \setminus S_0) = C_h \mu(X_0) - C_h \mu(S_0).
\]

Thus

\[
w(S) = \mu(S_0) + C^{-h} \mu(S_h) \geq \mu(X_0) = w(X_0)
\]

and so \( w(X_0) \) is \( \epsilon \)-minimal. \( \square \)

We may now finally prove the Plünnecke inequality for measure graphs.

**Theorem 5.2.5 (Plünnecke inequality for measure graphs).** Suppose that \( \Gamma = (X, \mathcal{B}, \mu, A, E) \) is a \( h \)-layered commutative measure graph with layering \( X = X_0 \sqcup \ldots \sqcup X_h \). Then for \( j \in \{1, \ldots, h\} \) we have

\[
D_j^h \geq D_j^i.
\]

**Proof.** If \( D_h = 0 \) then we are done. If \( D_h > 0 \) then we may set \( C = D_1^{1/h} \) and apply the above corollary as follows. For each non-null measurable \( Z \subset X_0 \) we have that \( (X_0 \setminus Z) \sqcup \text{Im}^j(Z) \) is a cutset and thus by minimality of \( X_0 \) we have

\[
\mu(X_0) = w(X_0) \leq w((X_0 \setminus Z) \sqcup \text{Im}^j(Z)) = \mu(X_0) - \mu(Z) + D_h^{-j/h} \mu(\text{Im}^j(Z))
\]

and so

\[
D_h^{-j/h} \leq \frac{\mu(\text{Im}^j(Z))}{\mu(Z)}
\]

which completes the proof as \( Z \subset X_0 \) was arbitrary. \( \square \)
5.3. Applications to measure preserving systems

Let us recall the magnification ratios for measure preserving systems introduced by Björklund and Fish in [6], which were defined in Theorem 1.5.4.

**Definition 5.3.1.** Suppose that $G$ is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. Define for $A \subset G$ and $B \in \mathcal{B}$ of positive finite measure the magnification ratio

$$c(A, B) = \inf \left\{ \frac{\mu(AB')}{\mu(B')} \mid B' \subset B, \mu(B') > 0 \right\}.$$ 

Moreover, for $\delta > 0$ we may define the $\delta$-heavy magnification ratio

$$c_\delta(A, B) = \inf \left\{ \frac{\mu(AB')}{\mu(B')} \mid B' \subset B, \mu(B') \geq \delta \cdot \mu(B) \right\}.$$ 

Furthermore, if $E \subset X$ is measurable then we may define the restricted magnification ratio

$$c(A, B, E) = \inf \left\{ \frac{\mu(AB' \setminus E)}{\mu(B')} \mid B' \subset B, \mu(B') > 0 \right\}.$$ 

By applying the Plünnecke inequality for measure graphs to the case of orbit graphs we obtain Theorem 1.5.6 in the case where $A$ is finite, i.e., we have the following.

**Theorem 5.3.2.** Suppose that $G$ is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. Then for $A \subset G$ finite and measurable $B \in \mathcal{B}$ of positive finite measure, we have

$$c(A^j, B)^{1/j} \geq c(A^k, B)^{1/k}$$

for positive integers $j < k$.

We may also obtain the ergodic analogue of a classical restricted addition result.

**Theorem 5.3.3.** Suppose that $G$ is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. For finite $A \subset G$, measurable $B \subset X$ of positive finite measure and measurable $E \subset X$ we have

$$c(A^j, B, A^j A^{j-1} E)^{1/j} \geq c(A^k, B, A^k A^{k-1} E)^{1/k}$$

for positive integer $j < k$.

**Proof.** Consider the subgraph of the $(A, B, k)$-orbit graph induced by the subset

$$B \times \{0\} \sqcup \bigcup_{j=1}^k (A^j B \setminus A^{j-1} E) \times \{j\}.$$ 

One may check that this subgraph is indeed commutative (see [31]).
5.4. Countable set of translates

The inequalities established in Section 5.3 required the set of translates \( A \subset G \) to be finite. We now turn to extending Theorem 5.3.2 to the case where \( A \) is countable. We use the techniques developed by Björklund and Fish in [6].

The following proposition is analogous to Proposition 2.2 in [6].

**Proposition 5.4.1.** Suppose that \( G \) is an abelian group acting on a probability space \((X, \mathcal{B}, \mu)\) and fix a finite \( A \subset G \) and non-null \( B \in \mathcal{B} \) together with a \( 0 < \delta < 1 \) and positive integers \( j \leq k \). If \( B' \subset B \) is measurable and satisfies

\[
\left( \frac{\mu(A^k B')}{\mu(B')} \right)^{1/k} \leq (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}.
\]  

(5.12)

Then \( \mu(B') \geq \delta \cdot \mu(B) \) or there exists \( B' \subset B'' \subset B \) such that \( \mu(B'' \setminus B) > 0 \) and \( B'' \) satisfies (5.12).

**Proof.** Firstly we note that if the hypothesis holds for \( B' = B_1 \) and \( B' = B_2 \) with \( B_1 \) and \( B_2 \) disjoint, then it holds for \( B_1 \sqcup B_2 \) since the hypothesis may be rewritten as the inequality

\[
\mu(A^kB') \leq (1 - \delta)^{-k/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{k/j} \mu(B').
\]

By Theorem 5.3.2 we know that there exists non-null measurable \( B' \subset B \) such that (5.12) is satisfied. Suppose that \( \mu(B') < \delta \cdot \mu(B) \), thus we wish to construct a strictly larger \( B'' \supset B \) that satisfies (5.12) and is contained in \( B \). Set \( B_0 = B \setminus B' \). We have that

\[
\frac{\mu(B_0)}{\mu(B)}(1 - \delta)^{-1} > 1
\]

and thus there exists \( B'_0 \subset B_0 \) such that

\[
\left( \frac{\mu(A^kB'_0)}{\mu(B'_0)} \right)^{1/k} \leq \left( \frac{\mu(B_0)}{\mu(B)}(1 - \delta)^{-1} \right)^{1/j} \left( \frac{\mu(A^j B_0)}{\mu(B_0)} \right)^{1/j}
\]

\[
= (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B_0)}{\mu(B)} \right)^{1/j}
\]

\[
\leq (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}
\]

and thus we may set \( B'' = B' \sqcup B'_0 \). \( \square \)
We will now apply the above lemma to construct a set $B' \subset B$ such that $\mu(B') \geq \delta \cdot \mu(B)$ and (5.12) holds. The idea is to choose a set $B' \subset B$ that satisfies (5.12) and that is maximal in the sense that $B$ does not contain any measurable $B'' \supset B'$ of strictly larger measure that satisfies (5.12). Such a set would have to necessarily satisfy $\mu(B') \geq \delta \cdot \mu(B)$. The existence of such a maximal $B'$ follows from the continuity of measure together with the following easy lemma on monotone classes.

**Lemma 5.4.2.** Suppose that $(X, \mathcal{B}, \mu)$ is a finite measure space and $\mathcal{M} \subset \mathcal{B}$ is non-empty and closed under countable nested unions (that is, if $M_i \in \mathcal{M}$ with $M_i \subset M_{i+1}$ then $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$). Then there exists $M \in \mathcal{M}$ such that $\mu(M) = \mu(M')$ for all $M' \in \mathcal{M}$ with $M \subset M'$.

**Proof.** For $M \in \mathcal{M}$ let $s(M) = \sup\{\mu(M') | M' \in \mathcal{M}, M \subset M'\}$. Choose $M_1 \in \mathcal{M}$. Now inductively choose $M_{n+1} \in \mathcal{M}$ such that $M_n \subset M_{n+1}$ and $\mu(M_{n+1}) \geq \frac{\mu(M_n) + s(M_n)}{2}$. Let $M = \bigcup_{n=1}^{\infty} M_n$. We claim that $\mu(M) = s(M)$. To see this, note that $\mu(M_n) \to \mu(M)$ and $s(M_n) \geq s(M)$. Thus

$$
\mu(M) = \lim_{n \to \infty} \mu(M_{n+1}) \geq \limsup_{n \to \infty} \frac{\mu(M_n) + s(M_n)}{2} \geq \limsup_{n \to \infty} \frac{\mu(M_n) + s(M)}{2} = \frac{\mu(M) + s(M)}{2}
$$

and thus $\mu(M) \geq s(M)$ as required.

If we set $\mathcal{M} = \{B' \subset B \mid B' \text{ satisfies (5.12)}\}$ then we see that $\mathcal{M}$ is non-empty by Theorem 5.3.2 and is closed under countable nested unions by the continuity of measure. Thus by the discussion above we obtain a $B' \subset B$ such that $\mu(B') \geq \delta \cdot \mu(B)$ and (5.12) holds. Consequently we have shown

**Lemma 5.4.3.** Suppose that $G$ is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a finite $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. Then

$$
c_{\delta}(A^k, B)^{1/k} \leq (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}.
$$

We may now obtain our first result about the case where $A \subset G$ is not necessarily finite.

**Lemma 5.4.4.** Suppose that $G$ is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. Then

$$
\sup\{c_{\delta}(A', B)^{1/k} | A' \subset A^k, \ A' \text{ is finite}\} \leq (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}.
$$
Proof. If $A' \subset A^k$ is finite then one may choose a finite $A_0 \subset A$ such that $A' \subset A_0^k$. Consequently
\[c_\delta(A', B)^{1/k} \leq c_\delta(A_0^k, B)^{1/k} \leq (1 - \delta)^{-1/j} \left( \frac{\mu(A_0^j B)}{\mu(B)} \right)^{1/j} \leq (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}\]
and so as $A'$ was arbitrary this completes the proof.

The next non-trivial result due to Björklund and Fish allows us to extend the Plünnecke inequalities for a finite set of translates (Theorem 5.3.2) to the case of an infinite set of translates.

**Theorem 5.4.5** (Proposition 4.1 of [6]). Suppose that $G$ is a countable group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$. Then
\[c(A, B) \leq \sup \{c_\delta(A', B) \mid A' \subset A, \ A' \text{ is finite} \}.

**Theorem 5.4.6.** (Plünnecke inequalities for an infinite set of translates) Suppose that $G$ is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. Then
\[c(A^k, B)^{1/k} \leq c(A^j, B)^{1/j} \leq \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}.

Proof. By the previous two results we obtain for each $\delta > 0$ the inequalities
\[c(A^k, B)^{1/k} \leq \sup \{c_\delta(A', B)^{1/k} \mid A' \subset A^k, \ A' \text{ is finite} \} \leq (1 - \delta)^{-1/j} \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}.

Taking $\delta \to 0$ gives
\[c(A^k, B)^{1/k} \leq \left( \frac{\mu(A^j B)}{\mu(B)} \right)^{1/j}.

Now applying this to non-null $B_i \subset B$ such that
\[\frac{\mu(A^j B_i)}{\mu(B_i)} \to c(A^j, B)\]
gives
\[c(A^k, B)^{1/k} \leq c(A^k, B_i)^{1/k} \leq \left( \frac{\mu(A^j B_i)}{\mu(B_i)} \right)^{1/j} \to c(A^j, B)^{1/j}\]
as desired. \qed
5.5. Product set estimates in countable non-abelian amenable groups

We now turn to proving Proposition 1.5.9. We start with the following ergodic analogue of the classical Ruzsa triangle inequality (see [35] and [31]).

**Lemma 5.5.1.** Suppose that $G \acts (X, \mu)$ is a measure preserving action. If $A, F \subset G$ are finite and $B \subset X$, then

$$|A|\mu(FB) \leq |FA^{-1}|\mu(AB).$$

**Proof.** We may partition

$$FB = \bigsqcup_{f \in F} fB_f$$

where $B_f \subset X$ is measurable and $B = \bigsqcup_{f \in F} B_f$. Now consider the mapping

$$\psi : A \times FB \to FA^{-1} \times AB$$

given by

$$\psi(a, fb) = (fa^{-1}, ab)$$

for $f \in F$ and $b \in B_f$. It is easy to see that $\psi$ is an injection. Now equip $G \times X$ with the product measure $\nu = |\cdot| \times \mu$, where $|\cdot|$ is the cardinality measure on $G$. We see that $\psi$ maps the rectangle $\{a\} \times fB_f$ onto the rectangle $\{fa^{-1}\} \times aB_f$. Both of these rectangles have $\nu$-measure $\mu(B_f)$, thus the image of $\psi$ has the same $\nu$-measure as its domain. But this image is a subset of $FA^{-1} \times AB$. \qed

Recall our convention that all Banach densities and Følner sequences are left.

**Lemma 5.5.2.** Let $G$ be a countable amenable group acting on a probability space $(X, B, \mu)$. Suppose that $A, F \subset G$ are not necessarily finite sets and $B \subset X$. Then

$$d^*(A)\mu(FB) \leq d^*(FA)\mu(A^{-1}B).$$

**Proof.** Note that it is sufficient to show this in the case that $F$ is finite, thus we assume this in the proof. Let $I_n \subset G$ be a (left) Følner sequence which attains the (left) upper Banach density of $A$, that is

$$\frac{|A \cap I_n|}{|I_n|} \to d^*(A).$$

By applying the previous lemma to the finite set $(A \cap I_n)^{-1}$ we obtain that

$$|A \cap I_n|\mu(FB) \leq |F(A \cap I_n)|\mu((A \cap I_n)^{-1}B) \leq |FA \cap FI_n|\mu(A^{-1}B).$$
as $F(A \cap I_n) \subset FA \cap FI_n$ and $(A \cap I_n)^{-1}B \subset A^{-1}B$. Thus
\[
\frac{|A \cap I_n|}{|I_n|} \mu(FB) \leq \frac{|FA \cap FI_n|}{|I_n|} \mu(AB).
\]
(5.13)
Thus to prove the first inequality it is enough to show that
\[
\left| \frac{|FA \cap FI_n|}{|I_n|} - \frac{|FA \cap I_n|}{|I_n|} \right| \rightarrow 0
\]
as then we can take the lim sup of (5.13). To see this note that
\[
\left| \frac{|FA \cap FI_n|}{|I_n|} - \frac{|FA \cap I_n|}{|I_n|} \right| \leq \frac{1}{|I_n|} |FI_n \triangle I_n| \rightarrow 0
\]
since $F$ is finite and $I_n$ is a (left) Følner sequence.

We are now in a position to prove Proposition 1.5.9, which we restate for the convenience of the reader as follows.

**Proposition 1.5.9.** Let $G$ be a countable amenable group and let $A \subset G$ be symmetric ($A = A^{-1}$). Then for integers $n > 3$, we have that
\[
d^*(A)^{2n-6} d^*(A^n) \leq d^*(A^3)^{2n-5}.
\]
In other words, if $d^*(A) > 0$ and $d^*(A^3) \leq K d^*(A)$ then
\[
d^*(A^n) \leq K^{2n-5} d^*(A).
\]

**Proof.** Suppose that $d^*(A) > 0$ (the result is trivial otherwise). By the correspondence principle for product sets, there exists an ergodic action $G \curvearrowright (X, B, \mu)$ on some probability space, together with a measurable set $\tilde{A} \subset X$ such that $\mu(A) = d^*(A) > 0$ and $\mu(A^{n-2} \tilde{A}) \leq d^*(A^{n-1})$. Since $\mu(\tilde{A}) > 0$, Proposition 2.5.3 tells us that
\[
d^*(A^n) \leq \mu(A^n \tilde{A}).
\]
Altogether, these estimates along with the previous lemma (with $F = A^2$ and $B = A^{n-2} \tilde{A}$) yield
\[
d^*(A) d^*(A^n) \leq d^*(A) \mu(A^n \tilde{A})
\]
\[
\leq d^*(A^3) \mu(A^{n-1} \tilde{A})
\]
\[
\leq K d^*(A) \mu(A^{n-1} \tilde{A})
\]
\[
\leq K d^*(A^3) \mu(A^{n-2} \tilde{A})
\]
\[
\leq K^2 d^*(A) d^*(A^{n-1}).
\]
Thus we have shown that

\[ d^r(A^n) \leq K^2 d^r(A^{n-1}) , \]

which can be applied recursively to obtain the desired result. \[\Box\]
Appendix A

Sets invariant under finite index subgroups

For the sake of completeness, we provide a simple proof of the claims made in Definition 3.2.3 regarding the $T^Q$-invariant sets.

Lemma A.1. Let $(X, \mathcal{A}, \mu)$ be a probability space such that $\mathcal{A}$ does not have sets of arbitrarily small positive measure, i.e.,

$$r := \inf \{ \mu(A) \mid A \in \mathcal{A}, \mu(A) > 0 \} > 0.$$ 

Then the $\sigma$-algebra $\mathcal{A}$ is finite (modulo null sets, as always).

Proof. Let $r$ be as in the statement of the Lemma. Take $A_0 \in \mathcal{A}$ with $\mu(A_0) < \frac{3r}{2}$. If there exists an $\mathcal{A}$-measurable $A' \subset A_0$ with $0 < \mu(A') < \mu(A_0)$ then $\mu(A') \geq r$ and so $\mu(A_0 \setminus A') \leq \frac{r}{2}$, contradicting the definition of $r$. Hence it follows that $A_0$ is an atom of $\mathcal{A}$. By considering the space $X \setminus A_0$, we may continue in this way to obtain a partition of $X$ into a finite union of atoms of $\mathcal{A}$. □

Proposition A.2. Let $G \acts (X, \mu)$ be an ergodic measure preserving action of a group $G$. Then for finite index normal subgroups $H \triangleleft G$ there exists a $H$-invariant $C \subset X$ such that $H \acts (C, \mu(\cdot | C))$ is ergodic and a finitely many $g_1, \ldots, g_n \in G$ such that the collection of sets $\{g_iC \mid i = 1, \ldots, n\}$ is a partition of $X$.

Proof. Let $A = \{ A \subset X \mid hA = A \text{ for all } h \in H \}$ be the $\sigma$-algebra of $H$ invariant sets. Notice that since $H$ is normal in $G$, we have that $\mathcal{A}$ is $G$-invariant (i.e., $gA \in \mathcal{A}$ for all $g \in G$ and $A \in \mathcal{A}$). Furthermore, as $H$ fixes each $A \in \mathcal{A}$, there is a natural action of $G/H$ on $\mathcal{A}$. From the ergodicity of $G$, it follows that for non-null $A \in \mathcal{A}$ we have that

$$X = \bigcup_{g \in G} gA = \bigcup_{u \in G/H} uA$$

and so $\mu(A) \geq \frac{1}{|G/H|} > 0$. Applying Lemma A.1 above we may take an atom $C \in \mathcal{A}$ of positive measure. Since $C$ does not contain any non-trivial element of $\mathcal{A}$, it follows that the action of $H$ on $C$ is ergodic, as desired. Any translate of $C$ is also an atom of $\mathcal{A}$, hence the distinct translates of $C$ are disjoint. Moreover, they cover $X$ by the ergodicity of the action of $G$. □
APPENDIX B

Atoms of the spectral measure

We recall the spectral theorem of Bochner.

**Theorem B.1** (Bochner’s Theorem 4.18 in [16]). Let $U : G \curvearrowright \mathcal{H}$ be a unitary representation of a locally compact abelian group $G$. Then for all $f \in \mathcal{H}$ there exists a unique positive Borel measure $\sigma = \sigma_{f, U}$ on the dual $\hat{G}$, called the spectral measure of $f$, such that

$$\langle U^g f, f \rangle = \int_{\hat{G}} \chi(g) d\sigma(\chi) \quad \text{for all } g \in G.$$  

We now prove the following simple but useful relation between the atoms of the spectral measure and the eigenfunctions. Recall that if $U : G \curvearrowright \mathcal{H}$ is a unitary representation of an abelian group and $\chi \in \hat{G}$, then a $\chi$-eigenfunction is an element $f \in \mathcal{H}$ such that $U^g f = \chi(g) f$ for all $g \in G$.

**Lemma B.2.** Let $U : G \curvearrowright \mathcal{H}$ be a unitary representation of a discrete countable abelian group $G$ on a Hilbert space and suppose that $f \in \mathcal{H}$ and let $\sigma$ be the spectral measure of $f$. Then

$$\sigma(\{\chi_0\}) = \langle P_{\chi_0} f, f \rangle$$

for characters $\chi_0 \in \hat{G}$, where $P_{\chi_0}$ denotes the orthogonal projection onto the closed subspace of all $\chi_0$-eigenvectors.

**Proof.** Let $F_1, F_2, \ldots$ be a Følner sequence in $G$. Then Lemma B.3 tells us that for each character $\chi \in \hat{G}$ we have that

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{a \in F_n} \chi(a) \to \mathbf{1}_{\{1\}}(\chi)$$

and hence for each character $\chi_0$ we have that

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{a \in F_n} \chi_0^{-1} \chi(a) \to \mathbf{1}_{\{\chi_0\}}(\chi)$$

Thus by Lebesgue’s dominated convergence theorem we obtain that

$$\sigma_f(\{\chi_0\}) = \langle \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{a \in F_n} \chi_0^{-1}(a) U^a f, f \rangle.$$  

\footnote{Where 1 denotes the trivial character.}
But Von Neumann’s mean ergodic theorem applied to the unitary representation $V^a = \chi_0^{-1}(a)U^a$ shows that
\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{a \in F_n} \chi_0^{-1}(a)U^a \sum_{a \in F_n} \chi_0^{-1}(a)U^a f = P_{HV} f
\]
where $P_{HV} f$ denotes the orthogonal projection onto the space of $V$-invariant vectors, which are precisely the $\chi_0$-eigenvectors. \hfill \square

**Lemma B.3.** Let $F_1, F_2, \ldots$ be a Følner sequence in a countable amenable group $G$ and suppose that $\chi \in \hat{G}$ is a non-trivial character. Then
\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{a \in F_n} \chi(a) = 0
\]

**Proof.** Since $\chi$ is non-trivial, there exists $b \in G$ such that $\chi(b) \neq 1$. Now observe that
\[
\left| \frac{1}{|F_n|} \sum_{a \in F_n} \chi(a) - \frac{1}{|F_n|} \sum_{a \in bF_n} \chi(a) \right| \leq \frac{|F_n \Delta bF_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty,
\]
by the definition of a Følner sequence. But this difference may be written as
\[
\frac{1}{|F_n|} \sum_{a \in F_n} \chi(a) - \frac{1}{|F_n|} \sum_{a \in bF_n} \chi(a) = \frac{1}{|F_n|} \sum_{a \in F_n} \chi(a) - \chi(b) \frac{1}{|F_n|} \sum_{a \in F_n} \chi(a)
\]
\[
= (1 - \chi(b)) \frac{1}{|F_n|} \sum_{a \in F_n} \chi(a).
\]
Since $1 - \chi(b) \neq 0$, we obtain the desired convergence. \hfill \square
Spectrum of Kronecker systems

Given a compact metrizable abelian group \((K, +)\) and a homomorphism \(\tau : \mathbb{Z}^d \to K\) with dense image, we may define the Kronecker system \(T : \mathbb{Z}^d \curvearrowright (K, m_K)\) given by \(T^a x = \tau(a) + x\).

**Lemma C.1.** If \(K\) is connected, then any Kronecker system \(T : \mathbb{Z}^d \curvearrowright (K, m_K)\) has a trivial rational Kronecker factor (no non-constant rational eigenfunctions).

**Proof.** Suppose that \(f \in L^2(X, m_K)\) is a rational eigenfunction. Thus its \(T\) orbit \(\{T^a f \mid a \in \mathbb{Z}^d\}\) is finite. Consider the Fourier decomposition

\[
f = \sum_{\rho \in \hat{K}} c_{\rho} \rho.
\]

Now suppose that \(\rho \in \hat{K}\) such that \(c_{\rho} \neq 0\). Since the orbit of \(f\) is finite, then so must be the orbit of \(\rho\), which means that the set

\[
\{\rho(\tau(a)) \mid a \in \mathbb{Z}^d\}
\]

is finite. But as \(\rho\) is continuous and the image of \(\tau\) is dense, we have that \(\rho(K)\) is finite. As \(K\) is connected, this must mean that \(\rho\) is constant. In summary, \(c_{\rho} = 0\) for all non-trivial characters \(\rho \in \hat{K}\) and so \(f\) is constant. \(\square\)
References


