Appendix
Rotations in 3-D

A.1 Coordinate Frames

Rotations, like vectors, are described relative to a three-dimensional (3-d) coordinate frame that has three base axes (x, y, and z) pointing in mutually orthogonal directions from the origin of the coordinate frame. The x-, y-, and z-axes of the coordinate frame are referenced to the head, and correspond to the naso-occipital, inter-aural, and dorso-ventral axes depicted in figure A1.1. Rotations about the x-, y-, and z-axes constitute torsional, vertical, and horizontal rotations, respectively. In the scientific literature, torsional, vertical, and horizontal rotations of the head are often referred to as being roll, pitch, and yaw rotations, respectively.

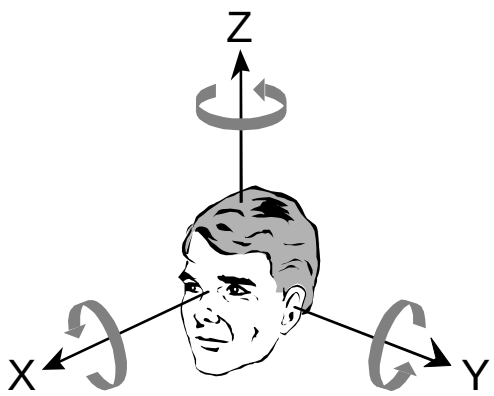

Figure A1.1 The 3-d head-fixed coordinate frame used to describe eye and head rotations is illustrated. The arrows around the three base axes indicate the directions of positive rotations (from Aw et al. 1996a).

In the current series of studies, rotations were described as being positive or negative using the right-hand rule, which states that if the thumb points along the axis of rotation, the curled fingers of the right hand indicate the direction of rotation. The

rotations were described from the subject's point of view. Hence, positive rotations about the x-, y-, and z-axes are clockwise, downward, and leftward rotations, respectively (see figure A1.1).

There are three coordinate planes for a given 3-d coordinate frame. For the head-fixed coordinate frame shown in figure A1.1, the three coordinate planes are the roll (ZY), pitch (ZX), and yaw (YX) planes. The coordinate planes, illustrated in figure A1.2, allowed for clear presentation of 3-d eye and head rotation data (see sections 7.10 and 7.11).

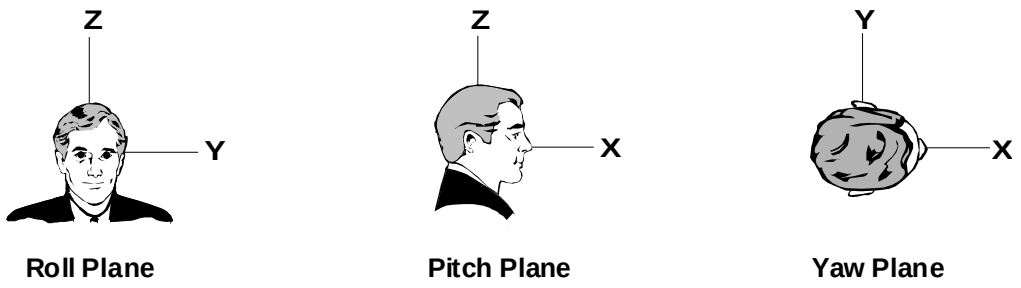

Figure A1.2 The roll, pitch, and yaw planes of the head-fixed coordinate system are illustrated (from Aw et al. 1996a).

Rotations of the eye and head were recorded relative to space using the magnetic search coil technique (see section 5.2) in the current series of studies. The rotations were therefore recorded relative to a *space-fixed* coordinate frame. Since the extra-ocular muscles and the peripheral vestibular apparatus are fixed within the head, a *head-fixed* coordinate frame was chosen to study the recorded rotations (see chapter 7).

A.2 Non-commutativity

Mathematicians initially sought to describe rotations in 3-d space using three sequential rotations about the three base axes (see Haslwanter 1995). When the order of the sequential base rotations is altered, however, the final orientation of the object is different (see figure A1.3). Since the order in which rotations are carried out is important

in determining the final orientation of an object, rotations are said to be *non-commutative* (MacKenzie 1997).

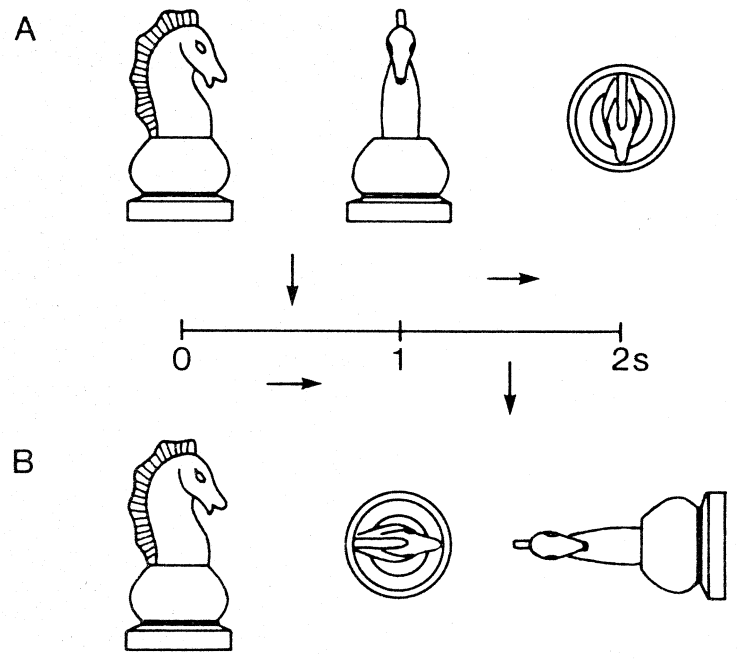

Figure A1.3 A. The first knight is initially rotated 90° about a vertical axis, and is then rotated 90° about a horizontal axis. **B.** The second knight undergoes the same rotations, but in the reverse order. The final orientation of the second knight is different from that of the first knight, due to the non-commutativity of rotations (from Tweed and Vilis 1987).

A.3 Rotation Matrices

Both Fick (1854) and Helmholtz (1867) developed ways of describing 3-d rotations, using sequential rotations about each of the three base axes. The Fick sequence involves a rotation about the z-axis first, followed by a rotation about the y-axis, and then a rotation about the x-axis; the three components of the Fick sequence are called Fick angles. The Helmholtz sequence differs in that the first two rotations are performed in the reverse order; the components of the Helmholtz sequence are called Helmholtz angles. Since the order of the sequential rotations in the Fick and Helmholtz sequences is different, the Fick and Helmholtz angles are not equal to one another for a given rotation.

Hence, the Fick and Helmholtz angles are not representative of the true components of the 3-d rotation.

The Fick angles and Helmholtz angles may be utilized to accurately calculate the rotation matrix corresponding to the rotation (see section 7.2). In the experiments presented in this thesis, Fick angles were calculated from the raw data and then used to calculate the corresponding rotation matrices.

A.4 Representations of 3-d Rotations

Unfortunately, rotation matrices are difficult to interpret, although they are truly representative of rotations in 3-d space. Hamilton (1899) developed an elegant way of expressing rotations in 3-d space using quaternions. Quaternions (q) are two-part operators with a scalar component (q_0) and a vector component ($\mathbf{q}$):

$$q = q_0 + \mathbf{q}$$

For a rotation of α degrees about an axis in space, the vector component of the quaternion points in the direction of the position axis and its magnitude is equal to $\sin(\alpha/2)$. The scalar component of the quaternion is equal to $\cos(\alpha/2)$. The three components of the vector part of the quaternion may be used to represent the three components of the rotation in 3-d space. The quaternions describing a series of rotations can be derived from the rotation matrices (Tweed et al. 1990).

Alternatively, rotations in 3-d space may be represented using rotation vectors (Haustein 1989). The rotation vector (r) representing a rotation can be calculated from the quaternion (q) using the following equation:

$$r = \frac{q}{q_0}$$

The rotation vector points in the direction of the position axis and its magnitude is equal to $\tan(\alpha/2)$. Rotation vectors can also be calculated from the Fick angles representing the rotation (see section 7.5).

Rotations in 3-d space may also be expressed in the axis-angle format (Schnabolk and Raphan 1994; Raphan 1997, 1998), where the 3-d orientation of the position axis (*n*) and the angle about this axis that the object rotates (Φ) is calculated. Due to its simplicity, the axis-angle format was chosen to represent angular position in this thesis. The steps required to convert rotation matrix data into axis-angle format are summarized in section 7.5.

A.5 Angular Velocity

The instantaneous angular velocity of a rotating object can be described with a velocity vector. The velocity vector points in the direction of the velocity axis and its magnitude is equal to the instantaneous angular velocity (in degrees per second). However, the velocity vector cannot be obtained simply by calculating the time derivative of the quaternion (or rotation vector). Instantaneous angular velocity is dependent on the time derivative of the quaternion (or rotation vector) *and* the instantaneous quaternion (or rotation vector) representing the rotation (Tweed and Vilis 1987; see section 7.7). Hence, the position and velocity axes often **do not** correspond.

The spatial characteristics of the angular velocity vectors (relative to the head-fixed coordinate frame) can be appreciated by plotting one component of the velocity vector against another. When the horizontal component of the velocity vector is plotted against the torsional component, the spatial position of the vectors can be viewed in the pitch plane (from the right side of the head; see figure A1.2). When the horizontal component of the velocity vector is plotted against the vertical component, the vectors are viewed in the roll plane (from the front of the head; see figure A1.2). Finally, when the vertical component of the velocity vector is plotted against the torsional component, the vectors are viewed in the yaw plane (from the top of the head; see figure A1.2).