A q-discrete Analogue of the Third Painlevé Equation and its Linear Problem

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This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

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CONTENTS

Abstract

In this thesis we investigate the rational and Riccati type special solutions for particular parameter values of a q-discrete analogue of the third Painlevé equation, with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2 + A_1)^{(1)}$. The general solutions of this equation are highly transcendental in nature. We work closely with an associated system of discrete linear equations, which we refer to as 'the linear problem.'We demonstrate that the linear problem can be solved both in terms of q-Gamma functions and series expansions for different parameter values of our discrete Painlevé equation. By developing a Schlesinger transformation, which transforms a series expansion of the linear problem for one parameter value to another series expansion of the linear problem for another parameter value, we are able to develop determinantal representations of the rational and Riccati type special solutions. These determinantal forms appear different to those discovered previously by Kajiwara [**33**, **34**]. This technique has only been used to develop the determinantal forms of two other continuous and discrete Painlevé equations and hence the results presented here further indicate its potential.

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CHAPTER 1

An Introduction to the Painlevé Equations

1.1. An Introduction to Special Solutions

Let us begin by introducing the following equation, which is referred to as the second Painlevé equation, or P_{II} for short:

$$\frac{d^2 f}{dt^2} = 2f^3 + tf - \alpha \tag{1.1}$$

where $f = f(t)$ and α is a parameter. This second-order nonlinear differential equation belongs to a set of six equations called the Painlevé equations. For general parameter values, the solutions to these equations are all highly transcendental and cannot be expressed in terms of other special functions [**12**, **15**, **56**].

Despite this, we can observe that for particular values of α, (1.1) permits either Hypergeometric solutions, or solutions that are rational in t. We refer to solutions for specific parameter values, as 'special solutions'and denote them as $f_\alpha(t)$.

Example 1.1. *For $\alpha = 0$ (1.1) simplifies to*

$$\frac{d^2 f_0}{dt^2} = 2f_0^3 + tf_0. \tag{1.2}$$

We can observe that $f_0 = 0$ is a trivial rational solution.

Example 1.2. *For $\alpha = \frac{1}{2}$, it can be checked by substitution that (1.1) is satisfied by the following Riccati equation*

$$f'_{\frac{1}{2}} = -f_{\frac{1}{2}}^2 - \frac{t}{2}. \tag{1.3}$$

Now, (1.3) can be linearised by $f_{\frac{1}{2}} = \frac{y'(t)}{y(t)}$ to give the Airy equation

$$y'' = -\frac{t}{2}y, \tag{1.4}$$

such that we find

$$f_{\frac{1}{2}} = \frac{A'((-2)^{-\frac{1}{3}}t)}{A((-2)^{-\frac{1}{3}}t)}, \tag{1.5}$$

where A is used to denote a solution to the Airy equation.

In fact, for each $\alpha \in \mathbb{N}$ and each half integer α, there exists a rational and Hypergeometric special solution respectively, resulting in two countably infinite sets of special solutions. These sets of special solutions can be calculated using a Bäcklund transformation: a rational transformation that relates a solution of an equation for one parameter value, to a solution for another value. Below is a Bäcklund transformation for (1.1), where $f_\alpha(t)$ denotes the solution for the particular value of α.

$$f_{\alpha+1}(t) = -f_\alpha(t) - \frac{(\alpha + 1/2)}{f'_\alpha(t) - f_\alpha^2(t) - t/2}. \tag{1.6}$$

Note that the solution for parameter $\alpha + 1$ only relies on the solution for the previous parameter $f_\alpha(t)$, such that we can generate all the rational special solutions by beginning with $f_0 = 0$ or the Hypergeometric special solutions by beginning with $f_{\frac{1}{2}}$. Below are a few examples of the simpler rational special solutions.

$$f_1(t) = \frac{1}{t} \tag{1.7}$$

$$f_2(t) = \frac{2\left(t^3 - 2\right)}{t\left(t^3 + 4\right)} \tag{1.8}$$

$$f_3(t) = \frac{3t^2\left(t^6 + 8t^3 + 160\right)}{\left(t^3 + 4\right)\left(t^6 + 20t^3 - 80\right)} \tag{1.9}$$

$$f_4(t) = \frac{4\left(t^{15} + 50t^{12} + 1000t^9 - 22400t^6 - 112000t^3 - 224000\right)}{t\left(t^6 + 20t^3 - 80\right)\left(t^9 + 60t^6 + 11200\right)}. \tag{1.10}$$

Not only have we demonstrated that in spite of the insoluble nature of (1.1) there are two hierarchies of special solutions, by factorising the above rational solutions we can see certain polynomials in the denominator are repeated, suggesting that there is some deeper structure to be uncovered.

This structure was discovered by Flaschka and Newell in [**10**], where the repeated polynomials are represented as determinants of square matrices whose entries are polynomial in t. The following two theorems are both originally from [**10**].

Theorem 1.1. *The rational special solutions of P_{II} are given by $f_k(t)$ for $\alpha = k$ where $k \in \mathbb{N}$ such that*

$$f_k(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)} \tag{1.11}$$

where $\tau_k(t)$ is a degree $k(k+1)/2$ polynomial, represented as a $k \times k$ determinant as follows

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-k+2} & T_{-k+3} & \dots & T_1 \end{vmatrix}, \tag{1.12}$$

where the entries $T_k(t)$ are degree k Laguerre polynomials that satisfy two recurrence relations:

$$jT_j = 4iT_{j-3} + itT_{j-1} \tag{1.13}$$

$$\frac{dT_j}{dt} = iT_{j-1} \tag{1.14}$$

and $T_0(t) = 1$, $T_j = 0$ for $j < 0$.

Theorem 1.2. *The Hypergeometric special solutions of P_{II} are given by $\alpha = k + \frac{1}{2}$ for $k \in \mathbb{N}$ such that*

$$f_{k+\frac{1}{2}}(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)} \tag{1.15}$$

where $\tau_k(t)$ is a determinant whose entries are functions of $A((-2)^{-\frac{1}{3}}t)$,

$$\tau_k(t) = \begin{vmatrix} A & \frac{d}{dt}A & \dots & \frac{d^{k-1}}{dt^{k-1}}A \\ \frac{d}{dt}A & \frac{d^2}{dt^2}A & \dots & \frac{d^k}{dt^k}A \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{k-1}}{dt^{k-1}}A & \frac{d^k}{dt^k}A & \dots & \frac{d^{2k-2}}{dt^{2k-2}}A \end{vmatrix}. \tag{1.16}$$

Additionally the following equation is satisfied

$$\frac{d^2 A((-2)^{-\frac{1}{3}})t}{dt^2} = -\frac{t}{2}A((-2)^{-\frac{1}{3}}t). \tag{1.17}$$

Revealing the determinant structure of the special solutions provides a simpler way of calculating solutions as α is iterated over the Bäcklund transformation. Substituting a rational solution into the Bäcklund transformation, which itself is a rational transformation, results in increasingly difficult expressions to simplify. On the other hand, observe that both (1.12) and (1.16) increase from a $k \times k$ determinant to a $(k+1) \times (k+1)$ determinant as $k \to k+1$. The difference in calculation is simply an additional row or column expansion. It is for this reason that the determinantal forms of the special solutions of the Painlevé equations, are objects of continued investigation and the subject of this thesis.

1.2. Outline of the Thesis

The main objective of this thesis is to develop the determinantal forms of the rational and Riccati type special solutions of an asymmetric (coupled), discrete analogue of the third Painlevé equation, given by:

$$g(qt) = \frac{a}{g(t)f(t)} \frac{1 + tf(t)}{t + f(t)} \tag{1.18a}$$

$$f(qt) = \frac{a}{f(t)g(qt)} \frac{1 + btg(qt)}{bt + g(qt)} \tag{1.18b}$$

where $t = t_0 q^n$ for $n \in \mathbb{N}$ and both a and b are parameters. We refer to an equation with this type of discretisation as a 'q-discrete equation,' and the above pair of equations (1.18) we call qP_{III}. In the remainder of this introductory Chapter we introduce both the Painlevé equations, their discrete analogues, and a system of linear equations called 'the linear problem', the latter being a system of linear equations, closely associated with the Painlevé equations and invaluable in reaching our goal. We will then discuss the motivation behind these equations and outline the relevant known results. We finally conclude by showing our main results, before reviewing some relevant known results in more depth in Chapter 2.

In Chapter 2 we review the work of Flaschka and Newell [**10**] on P_{II} (1.1), which we summarised in Section 1.1, in greater detail. After outlining the relationship between the linear problem and P_{II} itself, we demonstrate how studying the linear problem enables the construction of the determinantal form solutions. Also in Chapter 2, we similarly review Joshi and Shi's extension of Flaschka and Newell's work in [**10**] to a discrete analogue of P_{II} [**32**], called qP_{II}, given by

$$g(x/q)g(qx) = \frac{\alpha x^2(g(x) + x^2)}{g(x)(g(x) - 1)} \tag{1.19}$$

where α is a parameter. We provide a review of the two methods, in preparation for our investigation of (1.18).

In Chapter 3 we present our new results regarding qP_{III}. We first introduce the relationship between (1.18) and its linear problem. We then demonstrate how the simplest rational and Riccati type special solutions simplify the associated linear problem. Using these results we then construct a Schlesinger transformation of the linear problem which is an essential tool for developing the determinantal forms for both the rational and Riccati type special solutions. We also provide examples of some of the simpler determinants for simpler special solutions.

In Chapter 4 we provide a summary and conclusion based on our new results presented in Chapter 3. We then conclude the thesis by discussing the differences and similarities between our results on qP_{III} and those derived in [**33**, **34**] by Kajiwara et. al. The latter of which presents a different determinantal representation for the rational and Riccati type solutions than in this thesis.

1.3. The Painlevé Equations

Differential equations have been invaluable in mathematical modelling for hundreds of years. Yet even in the 18th century it was entirely understood that a model may be insoluble. For some, this caused a shift in focus to classification and understanding the extent to which a system could be analysed. In particular the fields of Lagrangian and Hamiltonian dynamics made great advancements in understanding just how much information could be accessed from these kinds of systems. These fields provided early definitions for when a system could be considered 'sufficiently well behaved'or Integrable.

It is in this spirit, that in the early 1900s, Painlevé, with additions by Gambier and Fuchs, set out to classify all second order linear differential equations of the form

$$f''(x) = R(x; f, f'(x)) \tag{1.20}$$

for rational R, that possessed what is called The Painlevé Property [**12**, **15**, **56**].

Definition 1.1 (The Painlevé Property)**.** *An equation possesses the Painlevé property if all the moveable singularities of all its solutions are poles.*

This is another form of integrability. Painlevé was able to categorise these differential equations into 50 canonical forms, 44 of which could be solved in terms of previously known functions. However, six of these 50 equations were found to be inexpressible in terms of previously known functions and hence their solutions in fact defined new functions, called 'Painlevé Transcendants'; much like the Airy function $A(x)$ is defined as a solution to the equation:

$$f''(x) = xf(x). \tag{1.21}$$

At the end of Painlevé, Fuch and Gambier's research, these six equations were identified to have the following form, with $f = f(t)$ and parameters

$\alpha,\ \beta,\ \gamma,\ \delta$. We refer to the six Painlevé equations as P_I to P_{VI}:

$$f'' = 6f^2 + t \tag{P_I}$$

$$f'' = 2f^3 + tf + \alpha \tag{P_{II}}$$

$$tff'' = t(f')^2 - ff' + \delta t + \beta t + \alpha f^3 + \gamma t f^4 \tag{P_{III}}$$

$$ff'' = \frac{1}{2}(f')^2 + \beta + 2(t^2 - \alpha)f^2 + 4tf^3 + \frac{3}{2}f^4 \tag{P_{IV}}$$

$$\begin{aligned} f'' = \left(\frac{1}{2f} + \frac{1}{f-1}\right)(f')^2 - \frac{f'}{t} + \frac{(f-1)^2}{t^2}\left(\alpha f + \frac{\beta}{f}\right) \\ + \gamma\frac{f}{t} + \delta\frac{f(f+1)}{f-1} \end{aligned} \tag{P_V}$$

$$\begin{aligned} f'' = \frac{1}{2}\left(\frac{1}{f} + \frac{1}{f-1} + \frac{1}{f-t}\right)(f')^2 - \left(\frac{1}{t} + \frac{1}{1-t} + \frac{1}{f-t}\right)f' \\ + \frac{f(f-1)(f-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{f^2} + \gamma\frac{t-1}{(f-1)^2} + \delta\frac{t(t-1)}{(f-t)^2}\right). \end{aligned} \tag{P_{VI}}$$

The Painlevé equations have been found in many fields of physics, including statistical mechanics models [**15**], optics [**16**] and quantum field theory [**22**]. These natural occurrences of the Painlevé equations have helped fuel interest in the area since their discovery.

In addition to the above applications, the Painlevé transcendents have been shown to be closely related to other important partial differential equations (PDEs). For example, it has been demonstrated that the Painlevé equations can be derived by similarity reductions from the non-linear Schrödinger equation [**5**], the Korteweg-de Vries equation [**4**] and the modified Korteweg-de Vries equation [**49**], which are all completely integrable.

1.4. The Discrete Painlevé Equations

We now turn our attention to the discrete Painlevé equations, for which more information can be in the following reviews [**9**, **18**, **39**]. The discrete Painlevé equations are defined by Grammaticos, Ramani and Hietarinta in [**18**, **61**] as integrable, second order, non-autonomous mappings which, at the continuum limit go over to one of the continuous Painlevé equations. It is in this sense that the discrete Painlevé equations are discretisations of the

continuous Painlevé equations. As there are many discrete analogues for each of the continuous Painlevé equations, further information is required to classify the discrete Painlevé equations.

In 2001, Sakai classified the rational surfaces obtained via a nine-point blow up of the complex projective plane and demonstrated that the discrete Painlevé equations arise as transformations that act on these surfaces [**63**]. In this paper Sakai discovered not only the previously known discrete Painlevé equations, but also a new class of discrete mappings that exist on an elliptic curve. These discoveries refined how we refer to and distinguish the discrete Painlevé equations. For example equations (1.18), collectively are a discrete analogue of P_{III} and yet so is the following, the first discrete analogue of P_{III} to be discovered [**61**]:

$$f(qt)f(q^{-1}t) = \frac{(f(t)-at)(f(t)-bt)}{(1-cf(t))(1-df(t))} \tag{1.22}$$

where $a,\ b,\ c,\ d$ are parameters. The asymmetric analogue given by $(1.18a)$ and $(1.18b)$, which we study in this thesis, is of type $A_5^{(1)}$ whereas (1.22) is of type $A_3^{(1)}$. Nishioka has recently proved the transcendence of q-Painlevé equations of type $A_6^{(1)}$ and $A_7^{(1)}$ in [**52**,**53**], whereas previously the transcendence of discrete Painlevé equations had been open to speculation.

Laguerre was the first to examine non-autonomous mappings, in his work on orthogonal polynomials [**45**], however the first confirmed discrete Painlevé equation was discovered in 1939 by Shohat [**64**], who was also studying orthogonal polynomials and produced the first discrete analogue of P_1 (although this was not realised at the time).

The early 1990s marked a sudden rush of interest into the discrete Painlevé equations when further discrete analogues of P_I appeared in applied journals. First a discrete analogue of P_I was developed in the investigation of the partition function in a 2D model of quantum gravity [**6**,**11**]. Periwal and Shevitz were then the first mathematicians to discover a discrete analogue of P_{II}, again with a physical context [**59**] and as an example of the variation in methodology, Nijhoff and Papageorgiou were able to derive the exact same discrete analogue using a different technique [**50**]. More recently, the discrete Painlevé equations have continued to appear in a physical context, such as in the following theoretical physics papers [**26**, **55**]. However, in the early 90s there was still no systematic method for discovering discrete Painlevé equations.

The development of singularity confinement introduced in [**19**] revolutionised the way people were able to hunt down discrete Painlevé equations.

As Grammaticos describes the derivation of the third discrete Painlevé equation: 'It was the first transcendent that was not discovered 'accidentally'in some physical application, but instead, was derived 'on request" [**17**]. The discrete analogues of the third through fifth Painlevé equations promptly followed [**61**] and eventually the sixth discrete Painlevé equation was found in 1996 [**28**].

While singularity confinement was for a time hailed as a sufficient condition for integrability [**19**, **61**], until Hietarinta and Viallet disproved this theory in [**25**] by demonstrating, amongst other facts, that the mapping

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2} \tag{1.23}$$

has a confined singularity which iterates as $\{0, \infty, \infty, 0\}$, while also displaying chaotic dynamics. This meant that the only reliable surviving method for identifying integrable maps was to use *algebraic entropy* [**2**]. The existence of an associated linear problem is also considered an indication of an equations integrability, whether continuous or otherwise and allows us to investigate the discrete Painlevé equations in greater depth.

1.5. The Associated Linear Problems

Both the continuous and discrete Painlevé equations have been shown to be closely related to an associated linear problem (also referred to as a Lax pair in the literature). In the continuous case, each Painlevé equation arises as a compatibility condition for two linear systems. One of these systems is a 2×2 matrix differential equation (often called a spectral equation) whose monodromy data is preserved by the other linear system [**13**,**27**]. The linear problem is therefore also referred to as the 'iso-monodromy deformation problem'.

We now continue our use of P_{II} and [**10**] as an introductory example. Note that we refer to (1.24) and (1.26) collectively as the linear problem:

Example 1.3. *The spectral equation for the linear problem of P_{II} is*

$$\frac{d}{dx}\Psi(x,t) = \Psi_x(x,t) = M(x,t)\Psi(x,t) \tag{1.24}$$

where

$$M(x,t) = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} -i\left(2f^2+t\right) & 2if' \\ -2if' & i\left(2f^2+t\right) \end{pmatrix} + \begin{pmatrix} 0 & 4f \\ 4f & 0 \end{pmatrix} x + \begin{pmatrix} -4i & 0 \\ 0 & 4i \end{pmatrix} x^2 \tag{1.25}$$

with monodromy variable x, and the deformation half, or the 'deformation equation'is given by

$$\frac{d}{dt}\Psi(x,t) = \Psi_t(x,t) = N(x,t)\Psi(x,t) \tag{1.26}$$

where

$$N(x,t) = \begin{pmatrix} -ix & f \\ f & ix \end{pmatrix} \tag{1.27}$$

with deformation variable (also called the Painlevé variable) t. We now set the compatibility condition, that the mixed derivatives be equal:

$$\Psi_{xt} = \Psi_{tx} \implies M_t - N_x = NM - MN \tag{1.28}$$

which in turn gives $f'' = 2f^3 + tf - \alpha$, or P_{II}.

The discrete case is identical in principle, except that it is a discrete linear system being deformed. For examples see (2.46) and (2.47), and (3.2) and (3.3). Early linear problems for the discrete Painlevé equations were comprised of 3×3 or even 4×4 linear systems, in comparison to the above 2×2 example, the development of which can be found in [**7**, **51**, **57**, **58**]. Since this early research, efforts have been directed at developing 2×2 linear problems.

This was shown to be a realistic goal when 2×2 linear problems for discrete analogues of P_I and P_{II} were discovered by Joshi et. al. in [**29**] and for a q-discrete analogue of P_{VI} by Jimbo and Sakai in [**28**]. Further progress was made by Murata when he discovered 2×2 linear problems for q-discrete P_I through P_V [**48**] and classified them in terms of the previously mentioned rational surfaces in [**63**].

One of the applications of a linear problem, for a discrete or continuous equation, is that it allows us to investigate the special solutions of the associated equation.

1.6. Special Solutions

As explained previously, the general solutions of the Painlevé equations are highly transcendental. However, there do exist hierarchies of exact solutions of P_{II} through P_{VI} (recall that P_I has no parameters to set!) for certain parameter values. These solutions fall into both rational and Hypergeometric categories.

The rational solutions can be found through what are referred to as special polynomials, such that P_{II} has Yablonskii-Vorob'ev polynomials [**68**, **69**] (these are the repeating polynomials we observed in the opening section), P_{IV} has Okamoto polynomials [**54**] and P_{III}, P_V and P_{VI}

have Umemura polynomials [**14**]. Additionally the Hypergeometric solutions of P_{II} through P_{VI} have been categorised respectively into, Airy [**1**], Bessel [**46**], Parabolic Cylinder [**21**], Whittaker [**20**] and Gauss Hypergeometic [**47**].

The above special polynomials can be expressed as determinants of Laguerre (for example those of P_{II}) and Hermite orthogonal polynomials. Okamoto and Kajiwara have derived the determinantal forms for several of the Painlevé equations as can be seen in [**35**, **40**, **41**], and Flaschka and Newell produced determinantal representations for the special solutions of P_{II} in [**10**] using an alternate method. In [**10**], it is demonstrated that one can obtain the determinantal structure of the special solutions of P_{II} through an examination of the associated linear problem.

Similarly, discrete Painlevé equations, with the exception of dP_I, also possess both rational and q-Hypergeometric special solutions. The q - Hypergeometric solutions of the discrete P_{II} through P_{VI} have been identified as discrete analogues of Airy [**60**], Bessel, Parabolic Cylinder [**65**], Confluent Hypergeometric [**66**] and Gauss hypergeometric functions [**28**] respectively. Kajiwara has shown that one discrete analogue of P_{III} also has Riccati type special solutions [**34**], in addition to q-Hypergeometric solutions.

Over the past two decades, Kajiwara et. al. developed the determinantal forms for both the rational and hypergeometric solutions of various discrete analogues of $P_{II}-P_V$ [**23**,**24**,**33**,**34**,**36–38**,**42–44**]. The determinantal form of the special solutions for a discrete analogue of P_{VI} have also been found on multiple occasions, first by Sakai in [**62**] and secondly by Tsuda [**67**]. After this period of investigation, the method of [**10**] was extended to q-discrete Painlevé equations, when Shi and Joshi produced the determinantal forms of the rational and q-Hypergeometric special solutions for a discrete analogue of P_{II}, with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2 + A_1)^{(1)}$ [**31**, **32**].

1.7. New Results: The Determinantal Form of qP_{III} Via Its Linear Problem

In this thesis, we use a similar approach to [**10**], and [**31**, **32**] to develop the determinantal forms of both the rational and Riccati type special solutions of (1.18a,1.18b) via an analysis of its associated linear problem. We now provide a summary of our new results, which can be found in Chapter 3.

The linear problem (3.2,3.3) associated with equations (1.18), have the form

$$\widehat{\Psi}(x,t) = \Psi(qx,t) = A(x,t)\Psi(x,t) \tag{1.29}$$

$$\overline{\Psi}(x,t) = \Psi(x,qt) = B(x,t)\Psi(x,t) \tag{1.30}$$

where $A(x,t)$ and $B(x,t)$ are 2×2 matrices. A Schlesinger transformation is a transformation, for a parameter k, such that

$$\Psi^{(k+1)}(x,t) = L_k(x,t)\Psi^{(k)}(x,t) \tag{1.31}$$

where $\Psi^{(k)}(x,t)$ denotes $\Psi(x,t)$ evaluated for some parameter k. For our particular Schlesinger transformation, which we believe to be a new result, k corresponds to the values for which equations (1.18) have special solutions, with $a = a_k = q^{2k}$. We find

$$L_k(x,t) = \begin{pmatrix} p_k(t)x & -1 \\ 1 & 0 \end{pmatrix} \tag{1.32}$$

where

$$p_k(t) = -\frac{btq^k\left(q^{2k+1}-1\right)}{g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)}. \tag{1.33}$$

Studying how the Schlesinger transformation acts on the series expansions for sequential rational and Riccati type special solutions allows us to develop the determinantal form of these special solutions. These determinantal forms, which are our primary results, are given in the following theorems. Note that as qP_{III} (1.18) possesses the symmetry $f_{-k} = \frac{1}{f_k}$ and $g_{-k} = \frac{1}{g_k}$, we only need to explicitly calculate the case for positive integer k and positive half integer k in the following.

Theorem 1.3. *The asymmetric qP_{III} (3.71) with $a = a_k = q^{2k}$ for integer k has a hierarchy of rational special solutions given by*

$$f_k(t) = \frac{q\left(ba_k+\Pi_k(t)\right)\left(qa_k-\Pi_k(t)\Pi_{k+1}(t)\right)}{bt\Pi_k(t)\Pi_{k+1}(t)-bq^2ta_k^2} \tag{1.34}$$

and

$$g_k(t) = \frac{\Pi_k(t)\left(bt\Pi_k(t)\Pi_{k+1}(t)-bq^2ta_k^2\right)}{q\left(ba_k+\Pi_k(t)\right)\left(qa_k-\Pi_k(t)\Pi_{k+1}(t)\right)} \tag{1.35}$$

where we have defined $\Pi_k(t) := f_k(t)g_k(t)$. We find that

$$\Pi_k(t) = q^{2k}\frac{\tau_k(qt)\tau_{k-1}(t)}{b\tau_k(t)\tau_{k-1}(qt)}. \tag{1.36}$$

We also have

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & T_2 & T_3 \\ 0 & \dots & \dots & T_0 & T_1 \end{vmatrix} \tag{1.37}$$

and $T_j(t)$ is determined by

$$T_j(t) = \frac{q^{j-1}\left((bt+q+t)T_{j-2}(t) - it(bq+bt+q)T_{j-1}(t) + iT_{j-3}(t)\right)}{bt^2\left(q^j-1\right)} \tag{1.38}$$

with t-evolution governed by

$$T_j(t) = \frac{bt^2T_j(qt) - i(b+1)tT_{j-1}(t) + T_{j-2}(t)}{bt^2} \tag{1.39}$$

and $T_0(qt) = bT_0(t)$, $T_j(t) = 0,\ j < 0$.

Theorem 1.4. *The asymmetric qP_{III} (1.18a,1.18b) with $a = a_k = q^{2k}$ for half integer k where $k = n - \frac{1}{2}$ for integer n, has a hierarchy of Riccati type special solutions. $f_{n+\frac{1}{2}}(t)$ satisfies the following quadratic equation:*

$$\begin{aligned} &tf_{n+\frac{1}{2}}(t)^2\Pi_{n+\frac{1}{2}}(qt)\left(a_{n+\frac{1}{2}} + b\Pi_{n+\frac{1}{2}}(t)\right) + \\ &f_{n+\frac{1}{2}}(t)\left(-a_{n+\frac{1}{2}}\left(bt^2a_{n+\frac{1}{2}} + \Pi_{n+\frac{1}{2}}(t)\right) + a_{n+\frac{1}{2}}\Pi_{n+\frac{1}{2}}(qt) + bt^2\Pi_{n+\frac{1}{2}}(t)\Pi_{n+\frac{1}{2}}(qt)\right) \\ &- ta_{n+\frac{1}{2}}\left(ba_{n+\frac{1}{2}} + \Pi_{n+\frac{1}{2}}(t)\right) = 0 \end{aligned} \tag{1.40}$$

where we have defined $\Pi_{n+\frac{1}{2}}(t) := f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)$. We find that

$$\Pi_{n+\frac{1}{2}}(t) = q^{2n+1}\frac{M_{n+1}(t)\left(t_{-1,n}(qt)\Lambda_{n+2}(qt) - \tau_{n+1}(qt)a_3^{(n-1/2)}(qt)\right)}{bM_{n+1}(qt)\left(t_{-1,n}(t)\Lambda_{n+2}(t) - \tau_{n+1}(t)a_3^{(n-1/2)}(t)\right)} \tag{1.41}$$

for odd n and

$$\Pi_{n+\frac{1}{2}}(t) = q^{2n+1}\frac{t_{0,n}(qt)\Lambda_{n+1}(qt)\tau_n(t)}{bt_{0,n}(t)\Lambda_{n+1}(t)\tau_n(qt)} \tag{1.42}$$

for even n, such that

$$g_{n+\frac{1}{2}}(t) = \frac{\Pi_{n+\frac{1}{2}}(t)}{f_{n+\frac{1}{2}}(t)}. \tag{1.43}$$

We have the following, where $\bar{c}_n^{(\frac{1}{2})} = c_n^{(\frac{1}{2})}(qt)$

$$\Lambda_n(t) = \begin{vmatrix} c_{n-1}^{(\frac{1}{2})} & \bar{c}_{n+1}^{(\frac{1}{2})} & c_{n+1}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ c_{n-3}^{(\frac{1}{2})} & \bar{c}_{n-1}^{(\frac{1}{2})} & c_{n-1}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ & & & & & & \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix}, \tag{1.44}$$

$$\tau_n(t) = \begin{vmatrix} \bar{c}_n^{(\frac{1}{2})} & c_n^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ & & & & & \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix}, \tag{1.45}$$

$$M_n(t) = \begin{vmatrix} \bar{c}_{n+2}^{(\frac{1}{2})} & c_{n+2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n}^{(\frac{1}{2})} & c_{2n}^{(\frac{1}{2})} \\ \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ \bar{c}_{n-4}^{(\frac{1}{2})} & c_{n-4}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-6}^{(\frac{1}{2})} & c_{2n-6}^{(\frac{1}{2})} \\ \bar{c}_{n-6}^{(\frac{1}{2})} & c_{n-6}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-8}^{(\frac{1}{2})} & c_{2n-8}^{(\frac{1}{2})} \\ & & & & & \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix}, \tag{1.46}$$

where after the 2nd row, all subsequent rows' subscripts decrease by 2 and

$$t_{-1}(t) = i^{n-1}\frac{q^{1/2}}{bt^2}. \tag{1.47}$$

$a_3^{(n)}(t)$ *can be expressed in terms of* $c_n^{(\frac{1}{2})}(t)$ *through applications of*

$$q^{-j+1}\sqrt{a}bt^2g(t)\left(aq^j-1\right)a_j^{(n+\frac{1}{2})}(t)=$$
$$\sqrt{a}a_{j-2}^{(n+\frac{1}{2})}(t)(at+f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)(bt+qg_{n+\frac{1}{2}}(t)))$$
$$-g(t)\left(tg_{n+\frac{1}{2}}(t)c_{j-1}^{(n+\frac{1}{2})}(t)(abq+btf_{n+\frac{1}{2}}(t)+qf_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t))-ac_{j-3}^{(n+\frac{1}{2})}(t)\right) \tag{1.48a}$$

$$q^{-j+1}bt^2f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)^2\left(q^j-1\right)c_j^{(n+\frac{1}{2})}(t)=$$
$$a(btg_{n+\frac{1}{2}}(t)+q)\left(\sqrt{a}ta_{j-1}^{(n+\frac{1}{2})}(t)+g_{n+\frac{1}{2}}(t)c_{j-2}^{(n+\frac{1}{2})}(t)\right)$$
$$+f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)\left(\sqrt{a}bqta_{j-1}^{(n+\frac{1}{2})}(t)-\sqrt{a}g_{n+\frac{1}{2}}(t)a_{j-3}^{(n+\frac{1}{2})}+tg_{n+\frac{1}{2}}(t)^2c_{j-2}^{(n+\frac{1}{2})}(t)\right) \tag{1.48b}$$

and $a_1^{(n+\frac{1}{2})}(t)=\mu c_0^{(n+\frac{3}{2})}(t)$. *We also have*

$$t_{0,n}(t)=\frac{(bt^2-q^{\frac{1}{2}}\sum_{j=1}^{(n/2)}p_{(4j-3)/2}(t))i^n}{bt^2} \tag{1.49}$$

with $t_{0,0}(t)=1$ *and*

$$p_k(t)=-\frac{btq^k\left(q^{2k+1}-1\right)}{g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)}. \tag{1.50}$$

$c_n^{(\frac{1}{2})}$ *satisfies both*

$$c_n^{(\frac{1}{2})}(t)=-\frac{q^n\left(t^2\left(\left(b^2+1\right)c_{n-2}^{(\frac{1}{2})}(t)+c_{n-2}^{(\frac{1}{2})}(qt)\right)+c_{n-4}^{(\frac{1}{2})}(t)\right)}{b^2t^4\left(q^n-1\right)} \tag{1.51}$$

$$b^2q^2t^2c_n^{(\frac{1}{2})}(t)=\frac{-q^2c_{n-4}^{(\frac{1}{2})}(t)}{t^2}+q^2t^2\left(\left(b^2\left(t^2-1\right)-1\right)c_n^{(\frac{1}{2})}(qt)-c_n^{(\frac{1}{2})}\left(q^2t\right)\right)$$
$$+\left(-b^2-1\right)q^2c_{n-2}^{(\frac{1}{2})}(t)-\left(q^2+1\right)c_{n-2}^{(\frac{1}{2})}(qt) \tag{1.52}$$

with

$$c_0^{(\frac{1}{2})}\left(q^2t\right)+\left(1-b^2\left(t^2-1\right)\right)c_0^{(\frac{1}{2})}(qt)+b^2c_0^{(\frac{1}{2})}(t)=0 \tag{1.53}$$

and $c_j^{(\frac{1}{2})}=0$, $j<0$.

The above results demonstrate that we are able to write all of the rational solutions of qP_{III} (1.18a,1.18b) in terms of the polynomials $T_k(t)$ which can be calculated through the recurrence relations (1.38,1.39) and

similarly the Riccati type solutions can be expressed entirely in terms of $c_n^{(\frac{1}{2})}(t)$, which are calculated through the recurrence relations (1.51,1.52). This concludes the summary of our new results, all of which can be found in detail in Chapter 3, as well as this introductory Chapter. We now move on to providing a more in depth review of the key papers related to our work, [**10**] and [**32**].

CHAPTER 2

The Second Painlevé Equation and a q-discrete Analogue

In this Chapter we provide partial reviews of [**10**] and [**32**], both of which are directly relevant to our investigation. In our review of [**10**], we introduce some of the techniques for investigating the associated linear problem of a Painlevé equation, such as how to expand around the system's singularities. We then highlight how the determinantal form of the rational solutions emerges from these series expansions.

In our review of [**32**] we discuss how Joshi and Shi were able to extend the methods of [**10**] to a discrete analogue of P_{II}. We demonstrate that despite working on a discrete equation there exist analogous approaches to developing the determinantal form solutions. It is possible to find series expansions around the singularities of its associated linear problem and for the simplest q-Hypergeometric solution, we can solve the linear problem in terms of a series of discrete Airy functions. In this section we also introduce the concept of a Schlesinger transformation: a transformation that acts on the linear problem, with vector solutions $\Phi^{(k)}(x,t)$, for some parameter k, such that

$$\Phi_1^{(k+1)}(x,t) = L_k(x,t)\Phi_1^{(k)}(x,t) \tag{2.1}$$

for some matrix $L_k(x,t)$. Observing the action of the Schlesinger transformation on one of the series expansion for sequential special solutions of (2.58), allows us to produce the determinantal form for the q-Hypergeometric solutions.

2.1. The Rational Special Solutions of the Second Painlevé Equation

2.1.1. The Linear Problem of P_{II}. We now present the linear problem of P_{II}, which is used in [**10**]. Below is the spectral half of the linear problem, or the 'spectral equation'

$$\frac{d}{dx}\Psi(x,t) = \Psi_x(x,t) = M(x,t)\Psi(x,t) \tag{2.2}$$

where

$$M(x,t) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} -i\left(2f(t)^2 + t\right) & 2if'(t) \\ -2if'(t) & i\left(2f(t)^2 + t\right) \end{pmatrix} + \begin{pmatrix} 0 & 4f(t) \\ 4f(t) & 0 \end{pmatrix} x + \begin{pmatrix} -4i & 0 \\ 0 & 4i \end{pmatrix} x^2 \quad (2.3)$$

with parameter a and the deformation half, or the 'deformation equation '

$$\frac{d}{dt}\Psi(x,t) = \Psi_t(x,t) = N(x,t)\Psi(x,t) \quad (2.4)$$

where

$$N(x,t) = \begin{pmatrix} -ix & f(t) \\ f(t) & ix \end{pmatrix}. \quad (2.5)$$

We now set the compatibility condition, that the mixed derivatives be equal

$$\Psi_{xt} = \Psi_{tx} \implies M_t - N_x = NM - MN \quad (2.6)$$

which in turn gives

$$f''(t) = 2f(t)^3 + tf(t) - a, \quad (2.7)$$

or P_{II}. We refer to x as the spectral variable and t as the Painlevé variable due to P_{II}'s dependence on the latter. We now focus our attention on the spectral half of the linear problem which is polynomial in x, whereas focusing on the deformation system would be fruitless due to its dependence on the Painlevé transcendent $f(t)$.

We have purposely represented the matrix $M(x,t)$ as a series in x to highlight the singularities of the spectral equation. We see $x = 0$ is a regular singularity and $x = \infty$ is an irregular singularity.

2.1.2. A Series Expansion of the Linear Problem around $x = 0$. As we have a regular singularity at $x = 0$ we can solve the spectral system with a Frobenius series expansion.

Proposition 2.1. *There exists a fundamental solution matrix $\Phi(x,t)$ of the linear systems (2.2,2.4) around $x = 0$, given by $\Phi(x,t) = \{\phi_1, \phi_2\}$, where*

$$\phi_1(x,t) = x^a \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(t)x^{2j} \\ c_{2j+1}(t)x^{2j+1} \end{pmatrix} \quad (2.8)$$

$$\phi_2(x,t) = x^{-a} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} b_{2j+1}(t)x^{2j+1} \\ d_{2j}(t)x^{2j} \end{pmatrix} \quad (2.9)$$

such that

$$\Phi(x,t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(t)x^{2j} & b_{2j+1}(t)x^{2j+1} \\ c_{2j+1}(t)x^{2j+1} & d_{2j}(t)x^{2j} \end{pmatrix} \begin{pmatrix} x^a & 0 \\ 0 & x^{-a} \end{pmatrix}. \tag{2.10}$$

The coefficients $a_j,\ b_j,\ c_j,\ d_j$ *satisfy the following recurrences:*

$$\begin{aligned} ja_j(t) = -i\left(2f(t)^2 + t + 2f'(t)\right) c_{j-1}(t) \\ + 4f(t)a_{j-2}(t) - 4ic_{j-3}(t) \end{aligned} \tag{2.11a}$$

$$\begin{aligned} (j-2a)b_j(t) = -i\left(2f'(t) + 2f(t)^2 + t\right) d_{j-1}(t) \\ + 4f(t)b_{j-2}(t) - 4id_{j-3}(t) \end{aligned} \tag{2.11b}$$

$$\begin{aligned} (j+2a)c_j(t) = i\left(2f'(t) - 2f(t)^2 - t\right) a_{j-1}(t) \\ - 4f(t)c_{j-2}(t) - 4ia_{j-3}(t) \end{aligned} \tag{2.11c}$$

$$\begin{aligned} jd_j(t) = i\left(2f(t)^2 - t - 2f'(t)\right) b_{j-1}(t) \\ - 4f(t)d_{j-2}(t) - 4ib_{j-3}(t) \end{aligned} \tag{2.11d}$$

where $a_{odd} = b_{odd} = c_{even} = d_{even} = 0$. *We also find*

$$f(t) = \frac{a_0'}{a_0} = -\frac{d_0'}{d_0}. \tag{2.12}$$

Proof. We begin by diagonalising the spectral half of the linear problem (2.2) by the following transformation

$$\Phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Phi_1$$

such that we obtain the new linear problem

$$\Phi_{1x} = A(x,t)\Phi \tag{2.13}$$

$$\Phi_{1t} = \begin{pmatrix} f & -ix \\ -ix & -f \end{pmatrix} \Phi \tag{2.14}$$

where

$$\begin{aligned} A(x,t) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \frac{1}{x} + \begin{pmatrix} 0 & -i(2f^2 + t + 2f') \\ i(2f^2 + t + 2f') & 0 \end{pmatrix} \\ + \begin{pmatrix} 4f & 0 \\ 0 & -4f \end{pmatrix} x + \begin{pmatrix} 0 & -4i \\ -4i & 0 \end{pmatrix} x^2. \end{aligned}$$

Let the fundamental solution matrix $\Phi_1(x,t)$ be given by

$$\Phi_1(x,t) = \sum_{j=0}^{\infty} \begin{pmatrix} a_j(t) & b_j(t) \\ c_j(t) & d_j(t) \end{pmatrix} \begin{pmatrix} x^{l_1} & 0 \\ 0 & x^{l_2} \end{pmatrix} \tag{2.15}$$

for constant $l_1,\ l_2$. Substituting (2.15) into (2.13) and equating coefficients of x, we find $l_1 = a$ and $l_2 = -a$, as well as the recurrence relations (2.11).

We also find $c_0(t) = b_0(t) = 0$ such that, from (2.11) $a_{odd} = d_{odd} = c_{even} = b_{even} = 0$. Substituting (2.15) into (2.14) and equating powers of x then gives the t evolution of the coefficients (2.12). □

2.1.3. A Series Expansion of the Linear Problem Around $x = \infty$. We find that $x = \infty$ is an irregular singular point of rank 3. To expand around a singularity of this nature we can apply the WKB method to obtain a series expansions of the form

$$\psi = e^{\sum_{j=1}^{3} \omega_j x^j} \sum_{j=0}^{\infty} \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} \frac{1}{x^{j+l}} \tag{2.16}$$

where $\omega_j = \omega_j(t)$ and l is constant.

Proposition 2.2. *There exists a fundamental solution matrix $\Psi(x,t)$ of the linear systems (2.2,2.4) around $x = \infty$, given by $\Psi(x,t) = \{\psi_1, \psi_2\}$ where*

$$\psi_1(x,t) = e^{-i(\frac{4}{3}x^3+tx)} \sum_{j=0}^{\infty} \begin{pmatrix} \alpha_j(t) \\ \gamma_j(t) \end{pmatrix} \frac{1}{x^j} \tag{2.17}$$

and

$$\psi_2(x,t) = e^{i(\frac{4}{3}x^3+tx)} \sum_{j=0}^{\infty} \begin{pmatrix} \beta_j(t) \\ \delta_j(t) \end{pmatrix} \frac{1}{x^j} \tag{2.18}$$

such that

$$\Psi(x,t) = \sum_{j=0}^{\infty} \begin{pmatrix} \alpha_j(t) & \beta_j(t) \\ \gamma_j(t) & \delta_j(t) \end{pmatrix} \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} & 0 \\ 0 & e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix} \frac{1}{x^j}. \tag{2.19}$$

The expansion coefficients satisfy the following recurrence relations

$$4f\gamma_{n-1} = -2if^2\alpha_{n-2} - 2if'\gamma_{n-2} - (n-3)\alpha_{n-3} - a\gamma_{n-3} \tag{2.20a}$$

$$\begin{aligned} 8i\gamma_n = -4f\alpha_{n-1} + 2if'\alpha_{n-2} - (2if^2 + 2it)\gamma_{n-2} \\ - (n-3)\gamma_{n-3} - a\alpha_{n-3} \end{aligned} \tag{2.20b}$$

$$4f\beta_{n-1} = -2if^2\delta_{n-2} + 2if'\beta_{n-2} - (n-3)\delta_{n-3} - a\beta_{n-3} \tag{2.20c}$$

$$\begin{aligned} -8i\beta_n = -4f\delta_{n-1} - 2if'\delta_{k-2} + (2if^2 + 2it)\beta_{n-2} \\ - (n-3)\beta_{n-3} - a\delta_{n-3} \end{aligned} \tag{2.20d}$$

with $\beta_0 = \gamma_0 = 0$ and constant α_0 and δ_0. Additionally we find

$$\gamma_1 = \frac{if(t)}{2} \qquad \beta_1 = -\frac{if(t)}{2} \qquad \alpha_{1t} = -\frac{if(t)^2}{2} \qquad \delta_{1t} = \frac{if(t)^2}{2}. \tag{2.21}$$

Proof. We begin by substituting

$$\Psi(x,t) = e^{\sum_{j=1}^{3}\omega_j x^j} \sum_{j=0}^{\infty} \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} \frac{1}{x^{j+l}} \tag{2.22}$$

into both the spectral half of the linear problem (2.2) and the deformation half (2.4). Equating powers of x gives us $\omega_1 = \pm\frac{4}{3}i$, $\omega_2 = 0$ from the former and $\omega_3 = \pm it$ from the latter. Additionally we can set $l = 0$ such that we have

$$\Psi(x,t) = \sum_{j=0}^{\infty} \begin{pmatrix} \alpha_j(t) & \beta_j(t) \\ \gamma_j(t) & \delta_j(t) \end{pmatrix} \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} & 0 \\ 0 & e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix} \frac{1}{x^j}.$$

Once we have fully determined the exponential components, further equating powers of x from the spectral equation gives us the recurrence relations (2.20) and similarly from the deformation equation the t evolutions (2.21), with

$$\alpha_0' = \delta_0' = 0$$

such that $\alpha_0 = \delta_0 = 1$ is sufficient. □

These series expansions are essential to the development of the determinantal forms of the special solutions.

2.1.4. A Series Expansion For The Simplest Rational Solution Around $x = 0$. In this section we demonstrate how a simple rational special solution can simplify the associated linear problem. First we outline that there exists a hierarchy of special solutions.

Proposition 2.3. *It can be checked by substitution that the simplest rational solution of P_{II} is given by $f_0 = 0$. The Bäcklund transformation of P_{II} is given by*

$$f_{a+1}(t) = -f_\alpha(t) - \frac{(a+1/2)}{f_a'(t) - f_a^2(t) - t/2} \tag{2.23}$$

such that for each integer a there exists a rational special solution.

Additionally we can observe that P_{II} possesses a symmetry such that if $f_a(t)$ is a solution then $f_{-a}(t) = -f_a(t)$. This implies that we need only explicitly calculate $f_a(t)$ for positive a, as we then obtain those for $-a$ with trivial effort.

Proposition 2.4. *For $f_0 = 0$ the linear problem of P_{II}, denoted as $\Psi^{(0)}(x,t)$ can be solved exactly, where a solution is given by*

$$\psi_1^{(0)}(x,t) = \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} \\ e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} (-1)^j T_j(t) x^j \\ T_j(t) x^j \end{pmatrix} \tag{2.24}$$

where $T_j(t)$ is defined by the two recurrence relations

$$jT_j = 4iT_{j-3} + itT_{j-1} \tag{2.25}$$

$$\frac{dT_j}{dt} = iT_{j-1} \tag{2.26}$$

with $T_0(t) = 1$, $T_j(t) = 0$, $j < 0$.

Proof. We now observe the effect of setting this special solution on the linear problem. We obtain the diagonal systems

$$\Psi_x = \begin{pmatrix} -i(4x^2+t) & 0 \\ 0 & i(4x^2+t) \end{pmatrix} \Psi$$

$$\Psi_t = \begin{pmatrix} -ix & 0 \\ 0 & ix \end{pmatrix} \Psi$$

which gives the exact solution

$$\psi_1(x,t) = \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} \\ e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix}.$$

We now define

$$\sum_{j=0}^{\infty} T_j(t)x^j = e^{i(\frac{4}{3}x^3+tx)}. \tag{2.27}$$

Differentiating (2.27) with respect to x gives

$$\sum_{j=0}^{\infty} jT_j(t)x^{j-1} = (4ix^2+it)e^{i(\frac{4}{3}x^3+tx)}$$

$$\implies jT_j = 4iT_{j-3} + itT_{j-1},$$

whereas differentiating (2.27) with respect to t gives

$$\sum_{j=0}^{\infty} \frac{dT_j}{dt}x^j = ixe^{i(\frac{4}{3}x^3+tx)}$$

$$\implies \frac{dT_j}{dt} = iT_{j-1}.$$

□

2.1.5. The Determinantal Form Of The Rational Solutions of P_{II}. In this section we demonstrate how to develop the determinantal form of the rational solutions of P_{II}. In [**10**] the determinantal forms emerge by utilising the relationship between the general expansions of the linear problem around both $x = 0$ and $x = \infty$ [**10**].

Theorem 2.1. *The rational special solutions of P_{II} are given by $f_k(t)$ for $a = k$ where $k \in \mathbb{Z}$ such that*

$$f_k(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)} \tag{2.28}$$

where $\tau_k(t)$ is an order $k(k+1)/2$ polynomial, represented as a $k \times k$ determinant as follows

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-k+2} & T_{-k+3} & \dots & T_1 \end{vmatrix} \tag{2.29}$$

where the entries $T_k(t)$ are order k Laguerre polynomials that satisfy two recurrence relations:

$$jT_j = 4iT_{j-3} + itT_{j-1} \tag{2.30}$$

$$\frac{dT_j}{dt} = iT_{j-1} \tag{2.31}$$

and $T_0(t) = 1$, $T_j = 0$ for $j < 0$.

Proof. The key ingredients in this construction are the series expansions around $x = 0$ and $x = \infty$. As we are looking at the rational solutions of P_{II} we set $a = k$ for integer k and we denote the series expansions around $x = 0$ and $x = \infty$ for rational solutions as $\Phi^{(k)} = \left\{\phi_1^{(k)}, \phi_2^{(k)}\right\}$ and $\Psi^{(k)} = \left\{\psi_1^{(k)}, \psi_2^{(k)}\right\}$. Specifically we have

$$\phi_1^{(k)}(x,t) = x^k \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(k)}(t)x^{2j} \\ c_{2j+1}^{(k)}(t)x^{2j+1} \end{pmatrix}$$

and

$$\phi_2^{(k)}(x,t) = x^{-k} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} b_{2j+1}^{(k)}(t)x^{2j+1} \\ d_{2j}^{(k)}(t)x^{2j} \end{pmatrix}$$

for the solutions around $x = 0$. However for the solutions at $x = \infty$, one can observe how the series component, multiplying the exponential component, changes for each rational solution f_k. The series expansion actually terminates depending on the integer k such that we have

$$\psi_1^{(k)}(x,t) = e^{-i(\frac{4}{3}x^3 + tx)} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha_1^{(k)} \\ \gamma_1^{(k)} \end{pmatrix} \frac{1}{x} + \dots + \begin{pmatrix} \alpha_k^{(k)} \\ \gamma_k^{(k)} \end{pmatrix} \frac{1}{x^k} \right\}$$

and

$$\psi_2^{(k)}(x,t) = e^{i(\frac{4}{3}x^3+tx)}\left\{\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}\beta_1^{(k)}\\ \delta_1^{(k)}\end{pmatrix}\frac{1}{x} + \cdots + \begin{pmatrix}\beta_k^{(k)}\\ \delta_k^{(k)}\end{pmatrix}\frac{1}{x^k}\right\}.$$

Or, as we discovered from the simplest rational special solution at $x = 0$, that the exponential component can be expressed as a series

$$\psi_1^{(k)}(x,t) = \sum_{j=0}^{\infty}(-1)^j T_j x^j \left\{\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}a_1^{(k)}\\ c_1^{(k)}\end{pmatrix}\frac{1}{x} + \cdots + \begin{pmatrix}a_k^{(k)}\\ c_k^{(k)}\end{pmatrix}\frac{1}{x^k}\right\} \tag{2.32}$$

and

$$\psi_2^{(k)}(x,t) = \sum_{j=0}^{\infty} T_j x^j \left\{\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}b_1^{(k)}\\ d_1^{(k)}\end{pmatrix}\frac{1}{x} + \cdots + \begin{pmatrix}b_k^{(k)}\\ d_k^{(k)}\end{pmatrix}\frac{1}{x^k}\right\}. \tag{2.33}$$

We also utilise the following relation: $\beta_1 = -i\frac{f_k(t)}{2}$.

As Ψ and Φ are both solutions of (2.2,2.4) we can write $\psi_2^{(k)} = A\phi_1^{(k)} + B\phi_2^{(k)}$ for constants A and B. Or

$$\sum_{j=0}^{\infty} T_j x^j \left\{\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}\beta^{(k)}\\ \delta_1^{(k)}\end{pmatrix}\frac{1}{x} + \cdots + \begin{pmatrix}\beta_k^{(k)}\\ \delta_k^{(k)}\end{pmatrix}\frac{1}{x^k}\right\} \tag{2.34}$$

$$= Ax^k\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left\{\begin{pmatrix}a_0^{(k)}\\ 0\end{pmatrix} + \begin{pmatrix}0\\ c_1^{(k)}\end{pmatrix}x + \dots\right\}$$

$$+ Bx^{-k}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left\{\begin{pmatrix}0\\ d_0^{(k)}\end{pmatrix} + \begin{pmatrix}b_1^{(k)}\\ 0\end{pmatrix}x + \dots\right\}. \tag{2.35}$$

Our intention is to equate the coefficients of x and we note that on the right hand side (2.35), the x^k component of the first series expansion renders it inconsequential. We therefore only expand the second series expansion on the RHS. We expand up to order $2k-1$, for even k, such that we subsequently obtain $2k$ equations. The case for odd k is proved similarly and so

we only demonstrate the even case.

$$\sum_{j=0}^{\infty} T_j x^j \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \beta^{(k)} \\ \delta_1^{(k)} \end{pmatrix} \frac{1}{x} + \cdots + \begin{pmatrix} \beta_k^{(k)} \\ \delta_k^{(k)} \end{pmatrix} \frac{1}{x^k} \right\}$$
$$= Ax^k \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left\{ \begin{pmatrix} a_0^{(k)} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_1^{(k)} \end{pmatrix} x + \dots \right\}$$
$$Bx^{-k} \Bigg(d_0^{(k)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + b_1^{(k)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + d_2^{(k)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} x^2 +$$
$$b_3^{(k)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^3 + \cdots + d_{k-2}^{(k)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} x^{k-2} + b_{k-1}^{(k)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^{k-1}$$
$$+ \cdots + d_{2k-2}^{(k)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} x^{2k-2} + b_{2k-1}^{(k)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x^{2k-1} \dots \Bigg).$$

Now that we are ready to equate coefficients, we follow the notation of [**10**] and define the following $\xi_j = \beta_j^{(k)} + \delta_j^{(k)}$ and $\eta_j = \beta_j^{(k)} - \delta_j^{(k)}$. We first demonstrate our process of equating coefficients, on the lowest order term x^{-k}.

$$\frac{1}{x^k} : T_0 \begin{pmatrix} \beta_k^{(k)} \\ \delta_k^{(k)} \end{pmatrix} = Bd_0^{(k)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.36}$$

such that we have $T_0 \beta_k^{(k)} = Bd_0^{(k)}$ and $T_0 \delta_k^{(k)} = -Bd_0^{(k)}$. Adding the two equations we find

$$T_0 \xi_k = 0. \tag{2.37}$$

We continue this procress such that we obtain the following, note the alternating inclusion of ξ and η:

$$\begin{aligned}
\frac{1}{x^{k-1}} &: T_1 \eta_k + T_0 \eta_{k-1} = 0 \\
&\vdots : \vdots \\
\frac{1}{x^2} &: T_{k-2} \xi_k + T_{k-3} \xi_{k-1} + \cdots + T_0 \eta_2 = 0 \\
\frac{1}{x} &: T_{k-1} \eta_k + T_{k-2} \eta_{k-1} + \cdots + T_0 \eta_1 = 0 \\
&\vdots : \vdots \\
x^{k-2} &: T_{2k-2} \xi_k + T_{2k-3} \xi_{k-1} + \cdots + T_{k-1} \xi_1 = -T_{k-2} \xi_0 = -T_{k-2} \\
x^{k-1} &: T_{2k-1} \eta_k + T_{2k-2} \eta_{k-1} + \cdots + T_k \eta_1 = -T_{k-1} \eta_0 = T_{k-1}
\end{aligned}$$

where in the latter two equations $\xi_0 = 1$ and $\eta_0 = -1$ as $\beta_0^{(k)} = 0$ and $\delta_0^{(k)} = 1$. We now split these $2k$ equations into two sets of $k \times k$ matrix

systems. One involving $\xi_1, \xi_2, \ldots, \xi_k$:

$$\begin{pmatrix} T_{k-1} & T_k & \ldots & T_{2k-4} & T_{2k-3} & T_{2k-2} \\ T_{k-3} & T_{k-2} & \ldots & T_{2k-6} & T_{2k-5} & T_{2k-4} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_1 & T_2 & \ldots & T_{k-2} & T_{k-1} & T_k \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & T_0 & T_1 & T_2 \\ 0 & \ldots & & \ldots & 0 & T_0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \\ \\ \vdots \\ \\ \xi_k \end{pmatrix} = - \begin{pmatrix} T_{k-2} \\ T_{k-4} \\ \vdots \\ T_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.38}$$

and one involving $\eta_1, \eta_2, \ldots, \eta_k$:

$$\begin{pmatrix} T_k & T_{k+1} & \ldots & T_{2k-3} & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \ldots & T_{2k-5} & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_2 & T_3 & \ldots & T_{k-1} & T_k & T_{k+1} \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & T_1 & T_2 & T_3 \\ 0 & \ldots & & \ldots & T_0 & T_1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \\ \\ \vdots \\ \\ \eta_k \end{pmatrix} = \begin{pmatrix} T_{k-1} \\ T_{k-3} \\ \vdots \\ T_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.39}$$

Recall that if we have a matrix system given by

$$\sum_{j=1}^{n} C_j x_j = B,$$

where

$$B = (b_1, b_2, \ldots, b_n)^T$$
$$A = |C_1 C_2 \ldots C_j \ldots C_n|$$

and $(C_1 C_2 \ldots C_j \ldots C_n)$ are column vectors with jth column vector C_j, then Cramer's Rule tells us x_j is calculated by

$$x_j = \frac{1}{A} |C_1 C_2 \ldots B \ldots C_n|.$$

This allows us to calculate ξ_1:

$$\xi_1 = \frac{1}{A} |B C_2 \ldots\ldots C_k|.$$

Using the relation on T_j (2.26) we observe

$$\frac{dC_j}{dt} = iC_{j-1}$$
$$\implies B = i\frac{dC_1}{dt}$$

such that we have

$$\xi_1 = \frac{1}{A}|i\frac{dC_1}{dt}C_2 \dots C_j \dots C_k|. \tag{2.40}$$

We can further simplify (2.40) with the following

Proposition 2.5.

$$|\frac{dC_1}{dt}\boldsymbol{C_2 \dots C_j \dots C_k}| = \frac{dA}{dt}. \tag{2.41}$$

Proof.

$$\begin{aligned}
\frac{dA}{dt} &= \sum_{j=1}^{k} |C_1 C_2 \dots \frac{dC_j}{dt} \dots C_k| \\
&= |\frac{dC_1}{dt}C_2 \dots C_j \dots C_k| + \dots + |C_1 \dots C_j \dots \frac{dC_k}{dt}| \\
&= |\frac{dC_1}{dt}C_2 \dots C_j \dots C_k| + \dots + i|C_1 C_2 \dots C_j \dots C_{k-1}C_{k-1}| \\
&= |\frac{dC_1}{dt}C_2 \dots C_j \dots C_k|
\end{aligned}$$

as a determinant with two or more linearly dependent columns is equal to 0. □

We now define $\tau_k(t)$ to be the determinant of the matrix in (2.39).

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-k+2} & T_{-k+3} & \dots & T_1 \end{vmatrix}$$

where the T_k are the Laguerre polynomial satisfying (2.25,2.26) and $T_k = 0$ for negative integer k. Noting that $A = T_0\tau_{k-1} = 1 \times \tau_{k-1} = \tau_{k-1}$, we have

$$\xi_1 = i\frac{1}{A}\frac{dA}{dt} = i\frac{d}{dt}\ln A = i\frac{d}{dt}\ln \tau_{k-1}. \tag{2.42}$$

After following a similar method we obtain

$$\eta_1 = -i\frac{d}{dt}\ln \tau_k. \tag{2.43}$$

We now make use of one of the properties of our series expansion around $x = \infty$, namely $\beta_1^{(k)} = -i\frac{f_k(t)}{2}$. We also recall our initial definitions of η_1 and ξ_1, that is $\eta_1 = \beta_1^{(k)} - \delta_1^{(k)}$ and $\xi_1 = \beta_1^{(k)} + \delta_1^{(k)}$ such that

$$\xi_1 + \eta_1 = 2\beta_1^{(k)} = -if_k. \tag{2.44}$$

Substituting relations (2.42,2.43) into (2.44) finally gives us

$$-if_k = i\frac{d}{dt}(\ln \tau_{k-1} - \ln \tau_k)$$
$$f_k = \frac{d}{dt}\ln\left(\frac{\tau_k}{\tau_{k-1}}\right).$$

It can be checked by calculation that these solutions are identical to those produced by the Bäcklund transformation of P_{II} (1.6). □

In this section we have reviewed Flaschka and Newell's work on P_{II} and its linear problem, with particular focus on the rational solutions. The method outlined above has not been used in other papers which also use the linear problem, i.e. those of [**31**, **32**] or this thesis. While Hypergeometric determinantal form solutions do exist as outlined in Chapter 1, we encourage the reader to view the source material [**10**].

We now move on to Joshi and Shi's work on qP_{II} [**32**], which extends the developments made in [**10**] to the q-discrete world.

2.2. The q-Hypergeometric Special Solutions of a q-discrete Analogue

We now provide a review of [**32**] on developing the determinantal form of the q-Hypergeometric special solutions of a q-discrete analogue of the second q-discrete Painlevé equation, with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2 + A_1)^{(1)}$:

$$g(x/q)g(qx) = \frac{\alpha x^2(g(x) + x^2)}{g(x)(g(x) - 1)} \tag{2.45}$$

where α is a parameter. We refer to (2.45) as qP_{II} for short. Despite continuous and discrete systems being fundamentally different, there are similarities between the methods.

2.2.1. The Linear Problem of qP_{II}. We begin by presenting the linear problem of (2.45). As in the continuous case we have a spectral equation

$$\widehat{\Psi}(v, x) = \Psi(v/q^2, x) = A(v, x)\Psi(v, x) \tag{2.46}$$

and a deformation equation

$$\overline{\Psi}(v, x) = \Psi(v, x/q) = B(v, x)\Psi(v, x) \tag{2.47}$$

where we have

$$A(v,x) = M_0(x) + M_1(x)v + M_2(x)v^2 + M_3(x)v^3 \tag{2.48}$$

$$= \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} 0 & m_1(x) \\ m_2(x) & 0 \end{pmatrix} v + \begin{pmatrix} n_1(x) & 0 \\ 0 & n_2(x) \end{pmatrix} v^2 + \begin{pmatrix} 0 & f_1(x) \\ f_2(x) & 0 \end{pmatrix} v^3 \tag{2.49}$$

and

$$B(v,x) = \begin{pmatrix} -i\sqrt{\alpha}\frac{qx}{g(qx)} & i\frac{q^2x}{\sqrt{\alpha}}v \\ i\frac{x}{\sqrt{\alpha}}v & i\frac{g(qx)}{\sqrt{\alpha}xq} \end{pmatrix}. \tag{2.50}$$

We have parameters e_1, e_2 and $\alpha = \frac{e_1}{e_2}$ as well as numerous functions of x and $g(x)$. Namely: $m_1(x)$, $m_2(x)$, $n_1(x)$, $n_2(x)$, $f_1(x)$ and $f_2(x)$:

$$m_1(x) = \frac{e_2 g(qx)(-e_1x^2q^2 - e_2g(x) + e_2g(x)g(qx))}{e_1x^2q} \tag{2.51}$$

$$m_2(x) = -\frac{x^2\left(e_1g(qx) + e_1q^2x^2 - e_2q^2g(x)g(qx)\right)}{qg(x)g(qx)^2} \tag{2.52}$$

$$n_1(x) = \frac{e_2\left(e_1x^2 + e_2g(x)g(qx)^2 - e_2g(x)g(qx)\right)}{e_1g(qx)} \tag{2.53}$$

$$n_2(x) = -\frac{e_2\left(-e_1q^2x^2g(qx) - e_1q^4x^4 + e_2g(x)g(qx)^2\right)}{e_1q^2g(x)g(qx)} \tag{2.54}$$

$$f_1(x) = -\frac{e_2^2qx^2}{e_1} \tag{2.55}$$

$$f_2(x) = -\frac{e_2^2x^2}{e_1q}. \tag{2.56}$$

Just as the continuous Painlevé equations arise from their linear problem via a compatibility condition, so do their discrete counterparts. The compatibility condition for the discrete case is the demand that a shift in either the x or t direction followed by a shift in the other, is equivalent to if the shifts' order were reversed. Formally, we demand $\overline{\widehat{\Psi}} = \overline{\widehat{\Psi}}$, such that

$$A(v,x/q)B(v,x) - B(v/q^2,x)A(v,x) = 0. \tag{2.57}$$

From this condition we obtain qP_{II} (2.45):

$$g(x/q)g(qx) = \frac{\alpha x^2(g(x)+x^2)}{g(x)(g(x)-1)}. \tag{2.58}$$

Now that we have demonstrated the connection between qP_{II} (2.45) and its linear problem, we can begin to construct series expansions.

2.2.2. A Series Expansion of the Linear Problem around $v = 0$. We now introduce a small slice of the theory of q-equations developed by Carmichael [**8**] and Birkhoff [**3**] which will allow us to produce series expansion of the linear problem. Carmichael and Birkhoff studied $n \times n$ q-linear systems such that

$$Y(qx) = (A_0 + A_1 x + A_2 x^2 + \cdots + A_\mu x^\mu)Y(x), \tag{2.59}$$

where A_0 has eigenvalues given by q^{θ_j} and A_μ has eigenvalues given by q^{ρ_j} for $j \in \{1, \ldots, n\}$, $i \neq j$, non-integer $\theta_i - \theta_j$ and non-integer $\rho_i - \rho_j$. The combination of these constraints in the eigenvalues is referred to as the 'non-resonant'case of the coefficient matrix of the system. Carmichael found that for the non-resonant case, there exist two sets of fundamental solutions, one around a singularity at $x = 0$ and the other set can be found around $x = \infty$.

Theorem 2.2. *Carmichael's Theorem For the $n \times n$ q-discrete linear system*

$$Y(qx) = (A_0 + A_1 x + A_2 x^2 + \cdots + A_\mu x^\mu)Y(x) \tag{2.60}$$

where A_0 has eigenvalues given by q^{θ_j} and $A_\mu x^\mu$ has eigenvalues given by q^{ρ_j} for $j \in \{1, \ldots, n\}$, $i \neq j$, non-integer $\theta_i - \theta_j$ and non-integer $\rho_i - \rho_j$, the system has fundamental matrix solutions $Y_0(x)$ and $Y_\infty(x)$ such that

$$\begin{cases} Y_0(x) = (x^{\theta_j}\epsilon_{ij}(x))_{1\leq i,j\leq n} \\ Y_\infty(x) = q^{\frac{\mu}{2}(t^2-t)}(x^{\rho_j}\delta_{ij}(x))_{1\leq i,j\leq n} \end{cases} \tag{2.61}$$

where $t = \frac{\ln x}{\ln q}$ and both $(\epsilon_{ij})_{1\leq i,j\leq n}$ and $(\delta_{ij})_{1\leq i,j\leq n}$ are $n \times n$ matrices of analytic functions expandable in x either around $x = 0$ or $\frac{1}{x}$ around $x = \infty$.

This theorem is crucial to expanding the linear problem of qP_{II} (2.46,2.47). For further results on q-linear difference equations we encourage the readers to view [**3**, **8**]. We are now ready to produce a series expansion around $v = 0$.

Proposition 2.6. *There exists a fundamental solution matrix $\Phi(v, x) = \{\phi_1(v, x), \phi_2(v, x)\}$ of the linear systems (2.46), (2.47) around $v = 0$ given by*

$$\Phi(v, x) = \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)v^{2j} & b_{2j+1}(x)v^{2j+1} \\ c_{2j+1}(x)v^{2j+1} & d_{2j}(x)v^{2j} \end{pmatrix} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix} \tag{2.62}$$

where $t = -\frac{1}{2}\frac{\ln v}{\ln q}$.

The expansion coefficients satisfy the following recurrence relations:

$$e_1 a_j(1/q^{2j} - 1) = m_1 c_{j-1} + n_1 a_{j-2} + f_1 c_{j-3} \tag{2.63a}$$

$$c_j(e_1/q^{2j} - e_2) = m_2 a_{j-1} + n_2 c_{j-2} + f_2 a_{j-3} \tag{2.63b}$$

$$e_2 d_j(1/q^{2j} - 1) = m_2 b_{j-1} + n_2 d_{j-2} + f_2 b_{j-3} \tag{2.63c}$$

$$b_j(e_2/q^{2j} - e_1) = m_1 d_{j-1} + n_1 b_{j-2} + f_1 d_{j-3}. \tag{2.63d}$$

We also find that

$$g(x) = -ix\sqrt{\alpha}\frac{a_0(x/q)}{a_0(x/q^2)}. \tag{2.64}$$

Proof. Let the fundamental solution matrix of (2.46) and (2.47) be given by

$$\Phi(v, x) = \sum_{j=0}^{\infty} \begin{pmatrix} a_j(x) & b_j(x) \\ c_j(x) & d_j(x) \end{pmatrix} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix} v^j. \tag{2.65}$$

Substituting (2.65) into (2.46) and equating coefficients of v gives the recurrence relations (2.63), in addition to $b_0 = c_0 = 0$ which implies that $a_{odd} = c_{even} = d_{odd} = b_{even} = 0$. We then substitute (2.65) into the deformation equation (2.47) and again equate coefficients of v to observe the x-evolution of the coefficients. In particular we find

$$a_0(x/q) = -i\sqrt{\alpha}\frac{qx}{g(qx)}a_0(x).$$

This can be rearranged to give a relationship between solutions of qP_{II} and the expansion coefficients $a_0(x/q)$ and $a_0(x/q^2)$

$$g(x) = -ix\sqrt{\alpha}\frac{a_0(x/q)}{a_0(x/q^2)}.$$

□

Relation (2.64) provides a link between the linear problem and the solutions of (2.45) which we build upon to develop the determinantal form for the q-Hypergeometric special solutions. We will now demonstrate how the simplest q-Hypergeometric special solution simplifies the series expansion.

2.2.3. q-Hypergeometric Special Solutions of qP_{II}. We first demonstrate that qP_{II} (2.45) has a hierarchy of q-Hypergeometric special solutions.

Proposition 2.7. *qP_{II} possesses a hierarchy of q-Hypergeometric solutions for $\alpha = q^{-2k}$ and half integer k. A q-Hypergeometric special solution can be found when $k = \frac{1}{2}$ such that we have*

$$g_{\frac{1}{2}}(x) = -\frac{ix}{q}\frac{a_0(x/q)}{a_0(x/q^2)} \tag{2.66}$$

where $a_0(x)$ satisfies a q-discrete analogue of the Airy equation

$$a_0(x/q^2) - \frac{i}{qx}a_0(x/q) + \frac{a_0(x)}{q} = 0. \tag{2.67}$$

Proof. Observe the recurrence relation on b_j (2.63d) for when $j = 1$ and $e_1 = \frac{1}{q^{2k}}$, $e_2 = q^{2k}$ and $k = \frac{1}{2}$ such that $\alpha = \frac{e_1}{e_2} = q^{-2k}$:

$$b_1 \times 0 = m_1(x)d_0.$$

Now we recall that $d_0 \neq 0$ such that this equation can only be consistent if $m_1(x) = 0$. If we refer to the solution of qP_{II} for $k = \frac{1}{2}$ as $g_{\frac{1}{2}}(x)$ we have

$$\begin{aligned} -x^2 - g_{\frac{1}{2}}(x) + g_{\frac{1}{2}}(x)g_{\frac{1}{2}}(qx) = 0 \\ \implies g_{\frac{1}{2}}(qx) = \frac{x^2 + g_{\frac{1}{2}}(x)}{g_{\frac{1}{2}}(x)} \end{aligned} \tag{2.68}$$

which is a q-discrete analogue of the Riccati equation. This is analogous to the continuous case where the Hypergeometric solution was defined by a Ricatti equation [**10**]. Now recall that (2.64) relates $g(x)$ to the expansion coefficients. Substituting (2.64) into (2.68) gives us

$$a_0(x/q^2) - \frac{i}{qx}a_0(x/q) + \frac{a_0(x)}{q} = 0$$

which is a q-discrete analogue of the Airy equation. This is again analagous to the continuous case where the simplest Hypergeometric solution of P_{II} was solvable in terms of the Airy equation [**10**].

The Bäcklund transformation is given by:

$$g_{k+1}(x) = \frac{(\alpha x^2 - g_k(x/q)g_k(x)q^2 + \alpha g_k(x))x^2}{q^2 g_k(x)(g_k(x)(g_k(x/q) - 1) - x^2)} \tag{2.69}$$

where $g_k(x)$ is a solution of (2.45) for the associated value of k, $e_1 = \frac{1}{q^{2k}}$, $e_2 = q^{2k}$ and $\alpha = \frac{1}{q^{4k}}$, such that for each half integer k, qP_{II} has a q-Hypergeometric solution. □

As qP_{II} possesses a symmetry such that if $g_k(x)$ is a solution of (2.45), then $g_{-k}(x) = -x^2/g_k(x)$, we only need to explicitly calculate the special solutions for positive k.

2.2.4. A Series Expansion For The Simplest q-Hypergeometric Solution Around $v = 0$. We denote the series expansion for the linear problem, for a particular rational special solution $g_k(x)$ with parameter k as $\Phi^{(k)}(v, x)$.

Proposition 2.8. *There exists a solution $\phi_1^{(\frac{1}{2})}(v,x)$ of the linear systems (2.46, 2.47) around $v=0$ for $k=\frac{1}{2}$.*

$$\phi^{(\frac{1}{2})}(v,x) = v^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ -\frac{n_1(x)}{f_1(x)v} & \frac{1}{qf_1(x)v} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^2q^4 & ixq^2 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)v^{2j} \\ a_{2j}(x/q)v^{2j} \end{pmatrix}$$

$$= v^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^3(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^3v} \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)v^{2j} \\ a_{2j}(x/q)v^{2j} \end{pmatrix} \tag{2.70}$$

where $a_{2j}(x)$ is defined by the following 2 recurrence relations:

$$a_{2j}(x/q) = ixq^2a_{2j}(x) - \frac{i}{xq^2}(1/q^{4j+4}-1)a_{2j+2}(x) \tag{2.71}$$

$$a_{2j}(x/q^2) = -\frac{1}{q}a_{2j}(x) - x^2q^3a_{2j-2}(x) + \frac{i}{xq}a_{2j}(x/q). \tag{2.72}$$

From (2.72) we obtain the condition on a_0

$$a_0(x/q^2) - \frac{i}{xq}a_0(x/q) + \frac{1}{q}a_0(x) = 0.$$

Proof. We begin by recalling our general solution (2.62) and take the first linearly independent solution

$$\phi_1^{(\frac{1}{2})}(v,x) = e_1^t \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}v^{2j} \\ c_{2j+1}v^{2j+1} \end{pmatrix}.$$

For $k=\frac{1}{2}$ we have $e_1=\frac{1}{q}$ and so $e_1^t = v^{\frac{1}{2}}$. This inspires us to make our first transformation $\Psi(v,x) = v^{\frac{1}{2}}\Psi_1(v,x)$ producing the following

$$\widehat{\Psi}_1(v,x) = qA(v,x)\Psi_1(v,x) = A_1(v,x)\Psi_1(v,x) \tag{2.73}$$

$$\widehat{\Psi}_1(v,x) = B(v,x)\Psi_1(v,x). \tag{2.74}$$

We let $\begin{pmatrix} U(v,x) \\ V(v,x) \end{pmatrix}$ be a solution to the transformed linear problem (2.73,2.74) such that we have

$$\begin{pmatrix} U(v/q^2,x) \\ V(v/q^2,x) \end{pmatrix} = \begin{pmatrix} 1+qn_1(x)v^2 & qf_1(x)v^3 \\ qm_2(x)v + qf_2(x)v^3 & q^2+qn_2(x)v^3 \end{pmatrix} \begin{pmatrix} U(v,x) \\ V(v,x) \end{pmatrix} \tag{2.75}$$

$$\begin{pmatrix} U(v,x/q) \\ V(v,x/q) \end{pmatrix} = \begin{pmatrix} -\frac{ix}{1+\frac{x^2}{g_{\frac{1}{2}}(x)}} & iq^3xv \\ ixqv & \frac{ix}{1+\frac{x^2}{g_{\frac{1}{2}}(x)}} \end{pmatrix} \begin{pmatrix} U(v,x) \\ V(v,x) \end{pmatrix}. \tag{2.76}$$

We continue transforming our solution, based on a rearrangement of the top row of (2.75).

$$V_1 = \frac{\widehat{U} - U}{v^2} = qn_1(x)U + vqf_1(x)V. \tag{2.77}$$

This transformation is inspired by the fact it resembles the following

$$\lim_{q\to 1} \frac{y(qx) - y(x)}{x^2(q-1)} \to \frac{1}{x}\frac{dy}{dx}$$

which is analogous to the Hypergeometric case of P_{II} [**10**], which we have not covered here. Now this transformation gives us

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{n_1(x)}{f_1(x)v} & \frac{1}{qf_1(x)v} \end{pmatrix} \begin{pmatrix} U \\ V_1 \end{pmatrix} = T(v,x) \begin{pmatrix} U \\ V_1 \end{pmatrix} \tag{2.78}$$

such that the new linear problem is given by

$$\begin{pmatrix} \widehat{U} \\ \widehat{V}_1 \end{pmatrix} = \widehat{T}^{-1} A_1(v,x) T \begin{pmatrix} U \\ V_1 \end{pmatrix} = A_2(v,x) \begin{pmatrix} U \\ V_1 \end{pmatrix}$$
$$\begin{pmatrix} U \\ V_1 \end{pmatrix} = \overline{T}^{-1} B(v,x) T \begin{pmatrix} U \\ V_1 \end{pmatrix} = B_2(v,x) \begin{pmatrix} U \\ V_1 \end{pmatrix}$$

where we have

$$A_2(v,x) = \begin{pmatrix} 1 & v^2 \\ -x^2q^2(-1-q^2+x^2q^4)v^2 + x^4q^6v^4 & 1 + (-1+x^2q^2+x^2q^4)v^2 \end{pmatrix} \tag{2.79}$$

$$B_2(v,x) = \begin{pmatrix} ixq^2 & -\frac{i}{xq^2} \\ ix(-1-q^2+x^2q^4) - ix^3q^4v^2 & -\frac{i(x^2q^2-1)}{xq^2} \end{pmatrix}. \tag{2.80}$$

Transformation (2.78) has clearly simplified the linear problem, such that (2.79,2.80) do not depend on $g_{\frac{1}{2}}$. One final transformation is required. We now make use of the top row of equation (2.80) and define

$$V_2 = \overline{U} = ixq^2U - \frac{i}{xq^2}\left(\frac{\widehat{U} - U}{v^2}\right) \tag{2.81}$$

such that $V_2 = \sum_{j=0}^{\infty} a_{2j}(x/q)v^{2j}$. This gives us

$$\begin{pmatrix} U \\ V_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^2q^4 & ixq^2 \end{pmatrix} \begin{pmatrix} U \\ V_2 \end{pmatrix} = S(v,x) \begin{pmatrix} U \\ V_2 \end{pmatrix}$$
$$= S(v,x) \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)v^{2j} \\ a_{2j}(x/q)v^{2j} \end{pmatrix}$$

and we obtain the final system

$$\begin{pmatrix}\widehat{U}\\ \widehat{V}_2\end{pmatrix} = \widehat{S}^{-1}A_1(v,x)S\begin{pmatrix}U\\ V_2\end{pmatrix} = \begin{pmatrix}1+x^2q^4v^2 & ixq^2v^2\\ -ixv^2-ix^3q^4v^4 & 1+(x^2q^2-1)v^2\end{pmatrix}\begin{pmatrix}U\\ V_2\end{pmatrix}$$

$$\begin{pmatrix}\overline{U}\\ \overline{V}_2\end{pmatrix} = \overline{S}^{-1}A_1(v,x)S\begin{pmatrix}U\\ V_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{1}{q}-x^2q^3v^2 & \frac{i}{xq}\end{pmatrix}\begin{pmatrix}U\\ V_2\end{pmatrix}. \tag{2.82}$$

Recalling the definitions $U(v,x)=\sum_{j=0}^{\infty}a_{2j}(x)$ and $V_2(v,x)=\sum_{j=0}^{\infty}a_{2j}(x/q)$, we can now find conditions on a_{2j}: From (2.81) we obtain (2.71) whereas from (2.82) we obtain (2.72). We therefore have

$$\begin{aligned}\phi_1^{(\frac{1}{2})}(v,x) &= v^{\frac{1}{2}}\begin{pmatrix}U\\ V\end{pmatrix}\\ &= v^{\frac{1}{2}}T(v,x)\begin{pmatrix}U\\ V_1\end{pmatrix}\\ &= v^{\frac{1}{2}}T(v,x)S(v,x)\begin{pmatrix}U\\ V_2\end{pmatrix}\\ &= v^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ -\frac{n_1(x)}{f_1(x)v} & \frac{1}{qf_1(x)v}\end{pmatrix}\begin{pmatrix}1 & 0\\ x^2q^4 & ixq^2\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}(x)v^{2j}\\ a_{2j}(x/q)v^{2j}\end{pmatrix}.\end{aligned}$$

□

2.2.5. The Schlesinger Transformation of qP_{II}. We now develop the Schlesinger transformation for the associated linear problem. We denote the linear problem for when $e_1=\frac{1}{q^{2k}}$, $e_2=q^{2k}$, $\alpha=\frac{1}{q^{4k}}$ and $g_k(x)$ satisfies qP_{II}

$$g_k(x/q)g_k(qx) = \frac{\alpha x^2(g_k(x)+x^2)}{g_k(x)(g_k(x)-1)} \tag{2.83}$$

as follows:

$$\widehat{\Psi}^{(k)}(v,x) = A_k(v,x)\Psi^{(k)}(v,x) \tag{2.84}$$

$$\overline{\Psi}^{(k)}(v,x) = B_k(v,x)\Psi^{(k)}(v,x). \tag{2.85}$$

To develop the Schlesinger transformation, we first define $\begin{pmatrix}U^{(k)}(v,x)\\ V^{(k)}(v,x)\end{pmatrix}$ as $\phi_1^{(k)}(v,x)$, the first column of the fundamental solution matrix of (2.84,2.85) and by observing that e_1^t for $e_1=\frac{1}{q^{2k}}$ and $t=-\frac{1}{2}\frac{\ln v}{\ln q}$ simplifies to v^k we note

$$\begin{pmatrix}U^{(k)}(v,x)\\ V^{(k)}(v,x)\end{pmatrix} \sim v^k\begin{pmatrix}a_0^{(k)}(x)\\ c_1^{(k)}(x)v\end{pmatrix} \tag{2.86}$$

around $v = 0$. Similarly when $e_1 = \frac{1}{q^{2(k+1)}}$, $e_2 = q^{2(k+1)}$ and $\alpha = \frac{1}{q^{4(k+1)}}$, we have

$$\begin{pmatrix} U^{(k+1)}(v,x) \\ V^{(k+1)}(v,x) \end{pmatrix} \sim v^{k+1} \begin{pmatrix} a_0^{(k+1)}(x) \\ c_1^{(k+1)}(x)v \end{pmatrix}. \tag{2.87}$$

We also notice from (2.86) and (2.87) that $V^{(k)}(v,x) \sim U^{(k+1)}(v,x)$.

Proposition 2.9. *For some constant μ*

$$\mu a_0^{(k+1)}(x) = c_1^{(k)}(x). \tag{2.88}$$

Proof. This is equivalent to proving $\frac{a_0^{(k+1)}(x/q)}{a_0^{(k+1)}(x/q)} = \frac{c_1^{(k)}(x/q)}{c_1^{(k)}(x/q^2)}$. We then recall (2.64) which relates $g(x)$ to $a_0(x)$

$$g_k(x) = -\frac{ix}{q^{2k}} \frac{a_0^{(k)}(x/q)}{a_0^{(k)}(x/q^2)}.$$

Shifting $k \to k+1$ and rearranging we obtain

$$\frac{a_0^{(k+1)}(x/q)}{a_0^{(k+1)}(x/q^2)} = \frac{iq^{2k+2} g_{k+1}(x)}{x}. \tag{2.89}$$

We also make use of the recurrence relation on $c_1^{(k)}(x)$ (2.63b). This allows us to write $\frac{c_1^{(k)}(x/q)}{c_1^{(k)}(x/q^2)}$ in terms of $g_k(x)$.

$$\begin{aligned} \frac{c_1^{(k)}(x/q)}{c_1^{(k)}(x/q^2)} &= \frac{m_2(x/q)a^{(k)}(x/q)}{m_2(x/q^2)a^{(k)}(x/q^2)} \\ &= -\frac{q^4 g_k(x/q^2) g_k(x/q)(-x^2 + g_k(x)(-1 + q^{2+4k} g_k(x/q)))a_0^{(k)}(x/q)}{g_k(x)^2(x^2 + (q^2 - q^{4+4k} g_k(x/q^2))g_k(x/q))a_0^{(k)}(x/q^2)}. \end{aligned} \tag{2.90}$$

Applying the Bäcklund transformation (2.69) to (2.89) and the definition of qP_{II} (2.83) to (2.90) to eliminate $g_k(x/q^2)$ we see

$$\frac{a_0^{(k+1)}(x/q)}{a_0^{(k+1)}(x/q)} = \frac{c_1^{(k)}(x/q)}{c_1^{(k)}(x/q^2)}.$$

□

This inspires the following proposition

Proposition 2.10.

$$U^{(k+1)}(v,x) = V^{(k)}(q^2 v, x). \tag{2.91}$$

Proof. We now write the spectral equation (2.84) as follows

$$\begin{pmatrix} \widehat{U}^{(k)} \\ \widehat{V}^{(k)} \end{pmatrix} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{pmatrix} \begin{pmatrix} U^{(k)} \\ V^{(k)} \end{pmatrix} \tag{2.92}$$

where

$$A_{11}^{(k)}(v,x) = \frac{v^2 q^{4k}\left(q^{2k} g_k(x) g_k(qx)^2 - q^{2k} g_k(x) g_k(qx) + x^2 q^{-2k}\right)}{g_k(qx)} + q^{-2k}$$

$$A_{12}^{(k)}(v,x) = \frac{v q^{4k-1} g_k(qx)\left(-q^{2k} g_k(x) + q^{2k} g_k(x) g_k(qx) - x^2 q^{2-2k}\right)}{x^2} - v^3 x^2 q^{6k+1}$$

$$A_{21}^{(k)}(v,x) = -\frac{v x^2\left(q^{-2k} g_k(qx) - q^{2k+2} g_k(x) g_k(qx) + x^2 q^{2-2k}\right)}{q g_k(x) g_k(qx)^2} - v^3 x^2 q^{6k-1}$$

$$A_{22}^{(k)}(v,x) = q^{2k} - \frac{v^2 q^{4k-2}\left(-x^2 q^{2-2k} g_k(qx) + q^{2k} g_k(x) g_k(qx)^2 - x^4 q^{4-2k}\right)}{g_k(x) g_k(qx)}.$$

By writing the spectral equation in this form we can decouple the system to two second order q-difference equations:

$$\widehat{\widehat{U}}^{(k)} = (A_{22}^{(k)} \frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}} + \widehat{A}_{11}^{(k)})\widehat{U}^{(k)} - \frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}} |A_k| U^{(k)} \tag{2.93}$$

$$\widehat{\widehat{V}}^{(k)} = (A_{11}^{(k)} \frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}} + \widehat{A}_{22}^{(k)})\widehat{V}^{(k)} - \frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}} |A_k| V^{(k)} \tag{2.94}$$

where $|A_k|$ is the determinant of A_k. As we are particularly interested in $U^{(k+1)}$ we also have

$$\widehat{\widehat{U}}^{(k+1)} = (A_{22}^{(k+1)} \frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}} + \widehat{A}_{11}^{(k+1)})\widehat{U}^{(k+1)} - \frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}} |A_{k+1}| U^{(k+1)}. \tag{2.95}$$

We wish to prove that (2.94) is equivalent to (2.95) evaluated at $v \to v/q^2$. By using the Bäcklund transformation (2.69) and the definition of qP_{II} (2.83) we see that

$$|A_k| = |\widehat{A}_{k+1}|$$

$$\frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}}(v,x) = \frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}}(v/q^2, x)$$

$$(A_{11}^{(k)} \frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}} + \widehat{A}_{22}^{(k)})(v,x) = (A_{22}^{(k+1)} \frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}} + \widehat{A}_{11}^{(k+1)})(v/q^2, x)$$

such that $U^{(k+1)}(v/q^2, x)$ and $V^{(k)}(v,x)$ satisfy the same equation. We also know the asymptotic behaviour of $U^{(k+1)}(v/q^2, x)$ to be

$$a_0^{(k+1)}(x) \frac{v^{k+1}}{q^{2k+2}} \tag{2.96}$$

while the asymptotic behaviour of $V^{(k)}(v,x)$ is given by

$$c_1^{(k)}(x)v^{k+1} = \mu a_0^{(k+1)}(x)v^{k+1}. \tag{2.97}$$

Setting $\mu = \frac{1}{q^{2k+2}}$ such that (2.96,2.97) agree, that is

$$c_1^{(k)}(x) = \frac{a_0^{(k+1)}(x)}{q^{2k+2}}$$

finally gives us $U^{(k+1)}(v/q^2,x) = V^{(k)}(v,x)$ or $U^{(k+1)}(v,x) = V^{(k)}(q^2v,x)$. □

Proposition 2.11. *The Schlesinger transformation for the linear problem (2.84,2.85) of* qP_{II} (2.83) *is defined by*

$$\Psi^{(k+1)}(v,x) = L_k(v,x)\Psi^{(k)}(v,x). \tag{2.98}$$

where

$$L_k(v,x) = \begin{pmatrix} \frac{-A_{21}^{(k)}(q^2v)}{|A_k(q^2v)|} & \frac{A_{11}^{(k)}(q^2v)}{A_k(q^2v)} \\ \frac{A_{11}^{(k+1)}(v)A_{21}^{(k)}(q^2v)}{A_{12}^{(k+1)}(v)|A_k(q^2v)|} & \frac{1}{A_{12}^{(k+1)}(v)}(1-\frac{A_{11}^{(k+1)}(v)}{|A_k(q^2v)|}A_{11}^{(k)}(q^2v)) \end{pmatrix}. \tag{2.99}$$

Proof. We need a relation such that we can express $U^{(k+1)}(v,x)$ in terms of $U^{(k)}(v,x)$ and $V^{(k)}(v,x)$. Inverting and rearranging (2.92) we find

$$U^{(k+1)}(v) = V^{(k)}(q^2v) = -\frac{1}{|A_k(q^2v)|}(A_{21}^{(k)}(q^2v)U^{(k)} - A_{11}^{(k)}(q^2v)V^{(k)}).$$

Similar rearrangements on (2.92) for $k \to k+1$ give us

$$V^{(k+1)}(v) = \frac{A_{11}^{(k+1)}(v)A_{21}^{(k)}(q^2v)}{A_{12}^{(k+1)}(v)|A_k(q^2v)|}U^{(k)} + \frac{1}{A_{12}^{(k+1)}(v)}(1-\frac{A_{11}^{(k+1)}(v)}{|A_k(q^2v)|}A_{11}^{(k)}(q^2v))V^{(k)}$$

such that the Schlesinger transformation is given by

$$L_k(v,x) = \begin{pmatrix} \frac{-A_{21}^{(k)}(q^2v)}{|A_k(q^2v)|} & \frac{A_{11}^{(k)}(q^2v)}{A_k(q^2v)} \\ \frac{A_{11}^{(k+1)}(v)A_{21}^{(k)}(q^2v)}{A_{12}^{(k+1)}(v)|A_k(q^2v)|} & \frac{1}{A_{12}^{(k+1)}(v)}(1-\frac{A_{11}^{(k+1)}(v)}{|A_k(q^2v)|}A_{11}^{(k)}(q^2v)) \end{pmatrix}.$$

□

As is fully explained in [**31**] this Schlesinger transformation is not suitable for developing the determinantal form solutions. However it can be simplified as follows

Proposition 2.12. *Let*

$$F^{(k)}(v,x) = \phi_1^{(k)}(v/q^{2k},x) \tag{2.100}$$

$$= \frac{v^k}{q^{2k^2}} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(k)}(x)\frac{v^{2j}}{q^{4kj}} \\ c_{2j+1}^{(k)}(x)\frac{v^{2j+1}}{q^{2k(2j+1)}} \end{pmatrix}. \tag{2.101}$$

This vector function has a Schlesinger transformation given by

$$F^{(k+1)}(v,x) = \Lambda_k(v,x)F^{(k)}(v,x) \tag{2.102}$$

where

$$\Lambda_k(v,x) = A_{k+1}(v/q^{2k},x)L_k(v/q^{2k},x) = \begin{pmatrix} 0 & 1 \\ \frac{1}{q^2} & \frac{\rho_k(x)}{v} \end{pmatrix} \tag{2.103}$$

and

$$\rho_k(x) = \frac{q^{2k}(e_1 - e_2q^2)g_k(x)g_k(qx)^2}{x^2q^3(e_1x^2q^2 + (e_1 - e_2q^2g_k(x))g_k(qx))}. \tag{2.104}$$

Proof. From the Spectral equation (2.84) we have

$$\begin{aligned} \widehat{\phi}_1^{(k+1)}(v,x) &= A_{k+1}(v,x)\phi_1^{(k+1)}(v,x) \\ &= A_{k+1}(v,x)L_k(v,x)\phi_1^{(k)}(v,x), \end{aligned}$$

which, after we let $v \to \frac{v}{q^{2k}}$ allows us to involve $F^{(k)}(v,x)$:

$$\begin{aligned} \widehat{\phi}_1^{(k+1)}(v/q^{2k},x) &= A_{k+1}(v/q^{2k},x)L_k(v/q^{2k},x)\phi_1^{(k)}(v/q^{2k},x) \\ F^{(k+1)}(v,x) &= A_{k+1}(v/q^{2k},x)L_k(v/q^{2k},x)F^{(k)}(v,x). \end{aligned}$$

After defining

$$\Lambda_k(v,x) = A_{k+1}(v/q^{2k},x)L_k(v/q^{2k},x)$$

and simplifying the RHS with the Bäcklund transformation (2.69) and qP_{II} (2.83) we obtain

$$\Lambda_k(v,x) = \begin{pmatrix} 0 & 1 \\ \frac{1}{q^2} & \frac{\rho_k(x)}{v} \end{pmatrix}$$

where $\rho_k(x)$ is as required. □

2.2.6. The Determinantal Form Of The q-Hypergeometric Special Solutions of qP_{II}. We will now observe how the Schlesinger transformation acts upon series expansions of the linear problem, for sequential q-Hypergeometric special solutions. The simplest case being $k = \frac{1}{2}$ with successive solutions given by half integer k.

Due to the nature of the Schlesinger transformation we can write all $F^{(k)}(v,x)$, for half integer k, in terms of $F^{(\frac{1}{2})}(v,x)$:

$$F^{(k+1)}(v,x) = \Lambda_k F^{(k)}(v,x) \tag{2.105}$$
$$= \Lambda_k \Lambda_{k-1} F^{(k-1)}(v,x) \tag{2.106}$$
$$= \Lambda_k \Lambda_{k-1} \dots \Lambda_{\frac{3}{2}} \Lambda_{\frac{1}{2}} F^{(\frac{1}{2})}(v,x). \tag{2.107}$$

Or, recalling the definition of $F^{(\frac{1}{2})}(v,x)$:

$$F^{(k+1)}(v,x) = \Lambda_k \dots \Lambda_{\frac{1}{2}} \frac{v^{\frac{1}{2}}}{q^{\frac{1}{2}}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v} \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(\frac{1}{2})} \frac{v^{2j}}{q^{2j}} \\ \bar{a}_{2j}^{(\frac{1}{2})} \frac{v^{2j}}{q^{2j}} \end{pmatrix}, \tag{2.108}$$

where $\bar{a}_{2j}^{(\frac{1}{2})} = a_{2j}^{(\frac{1}{2})}(x/q)$. Setting $k = n - \frac{1}{2}$ such that $k+1 = n + \frac{1}{2}$, we now observe the behaviour of the Schlesinger transformation on the RHS for a few values of n, starting from $n = 1$, corresponding to $F^{(\frac{3}{2})}$:

$$n = 1 : \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v} \end{pmatrix} = \begin{pmatrix} \frac{s_1}{v} & \frac{t_1}{v} \\ s_0 + \frac{s_2}{v^2} & \frac{t_2}{v^2} \end{pmatrix} \tag{2.109}$$

$$n = 2 : \Lambda_{\frac{3}{2}} \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v} \end{pmatrix} = \begin{pmatrix} s_0 + \frac{s_2}{v^2} & \frac{t_2}{v^2} \\ \frac{s_1}{v} + \frac{s_3}{v^3} & \frac{t_1}{v} + \frac{t_3}{v^3} \end{pmatrix} \tag{2.110}$$

$$n = 3 : \Lambda_{\frac{5}{2}} \Lambda_{\frac{3}{2}} \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v} \end{pmatrix} = \begin{pmatrix} \frac{s_1}{v} + \frac{s_3}{v^3} & \frac{t_1}{v} + \frac{t_3}{v^3} \\ s_0 + \frac{s_2}{v^2} + \frac{s_4}{v^4} & \frac{t_2}{v^2} + \frac{t_4}{v^4} \end{pmatrix} \tag{2.111}$$

where we have named the coefficients of v^{-j} in the first column $s_j = s_j(x)$ and the coefficients of v^{-j} in the second column to be $t_j = t_j(x)$: such that for even integer n we have

$$\Lambda_{n-\frac{1}{2}} \dots \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v} \end{pmatrix} = \begin{pmatrix} s_0 + \frac{s_2}{v^2} + \frac{s_4}{v^4} + \dots + \frac{s_n}{v^n} & \frac{t_2}{v^2} + \dots + \frac{t_n}{v_n} \\ \frac{s_1}{v} + \frac{s_3}{v^3} + \dots + \frac{s_{n+1}}{v^{n+1}} & \frac{t_1}{v} + \frac{t_3}{v^3} + \dots + \frac{t_{n+1}}{v^{n+1}} \end{pmatrix} \tag{2.112}$$

and for odd integer n we have

$$\Lambda_{n-\frac{1}{2}} \dots \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v} \end{pmatrix} = \begin{pmatrix} \frac{s_1}{v} + \frac{s_3}{v^3} + \dots + \frac{s_n}{v^n} & \frac{t_1}{v} + \dots + \frac{t_n}{v_n} \\ s_0 + \frac{s_2}{v^2} + \dots + \frac{s_{n+1}}{v^{n+1}} & \frac{t_2}{v^2} + \dots + \frac{t_{n+1}}{v^{n+1}} \end{pmatrix}. \quad (2.113)$$

We also find that $s_0 = \frac{1}{q^2}$ and $t_1 = -\frac{i}{xq^2}$.

Theorem 2.3. *The q-discrete second Painlevé equation qP_{II} (2.45) with $\alpha = \frac{1}{q^{4k}}$ for $k = n - \frac{1}{2}$ with $n \in \mathbb{Z}$ has a hierarchy of q-Hypergeometric special solutions given by*

$$g_{n+\frac{1}{2}}(x) = -\frac{ix}{q^{2n+1}} \frac{\tau_{n+1}(x/q)\tau_n(x/q^2)}{\tau_{n+1}(x/q^2)\tau_n(x/q)} \quad (2.114)$$

for even n and

$$g_{n+\frac{1}{2}}(x) = -\frac{ix}{q^{2n+2}} \frac{\tau_{n+1}(x/q)\tau_n(x/q^2)}{\tau_{n+1}(x/q^2)\tau_n(x/q)} \quad (2.115)$$

for odd n. With the following, where $\bar{a}_n^{(\frac{1}{2})} = a_n^{(\frac{1}{2})}(x/q)$

$$\tau_n(x) = \begin{vmatrix} \frac{\bar{a}_n^{(\frac{1}{2})}}{q^n} & \frac{a_n^{(\frac{1}{2})}}{q^n} & & \dots & \frac{\bar{a}_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} & \frac{a_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} \\ \frac{\bar{a}_{n-2}^{(\frac{1}{2})}}{q^{n-2}} & \frac{a_{n-2}^{(\frac{1}{2})}}{q^{n-2}} & & \dots & \frac{\bar{a}_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} & \frac{a_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} & \frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} & \frac{a_2^{(\frac{1}{2})}}{q^2} \\ 0 & 0 & \dots & 0 & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} \end{vmatrix} \quad (2.116)$$

for even n and

$$\tau_n(x) = \begin{vmatrix} \frac{a_{n-1}^{(\frac{1}{2})}}{q^{n-1}} & \frac{\bar{a}_{n+1}^{(\frac{1}{2})}}{q^{n+1}} & \frac{a_{n+1}^{(\frac{1}{2})}}{q^{n+1}} & & \dots & \frac{\bar{a}_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} & \frac{a_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} \\ \frac{a_{n-3}^{(\frac{1}{2})}}{q^{n-3}} & \frac{\bar{a}_{n-1}^{(\frac{1}{2})}}{q^{n-1}} & \frac{a_{n-1}^{(\frac{1}{2})}}{q^{n-1}} & & \dots & \frac{\bar{a}_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} & \frac{a_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} \\ \vdots & & & \ddots & & & \vdots \\ 0 & \dots & 0 & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} & \frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} & \frac{a_2^{(\frac{1}{2})}}{q^2} \\ 0 & 0 & 0 & \dots & & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} \end{vmatrix} \quad (2.117)$$

for odd n.

Also, $a_{2j}(x)$ satisfies

$$\bar{a}_{2j}^{(\frac{1}{2})} = ixq^2 a_{2j}^{(\frac{1}{2})} - \frac{i}{xq^2}(1/q^{4j+4} - 1)a_{2j+2}^{(\frac{1}{2})} \tag{2.118}$$

$$\bar{\bar{a}}_{2j} = -\frac{1}{q}a_{2j}^{(\frac{1}{2})} - x^2q^3 a_{2j-2}^{(\frac{1}{2})} + \frac{i}{xq}\bar{a}_{2j}^{(\frac{1}{2})}. \tag{2.119}$$

Proof. Recall that $F^{(k+1)}(v,x)$ for $k = n - \frac{1}{2}$ with $n \in \mathbb{Z}$ is given by

$$F^{(n+\frac{1}{2})}(v,x) = \Lambda_{n-\frac{1}{2}}\Lambda_{n-\frac{3}{2}}\dots\Lambda_{\frac{1}{2}}\frac{v^{\frac{1}{2}}}{q^{\frac{1}{2}}}\begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{q^2(x^2+g_{\frac{1}{2}}(x))v} & -\frac{i}{xq^2v}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix} a_{2j}^{(\frac{1}{2})}\frac{v^{2j}}{q^{2j}} \\ \bar{a}_{2j}^{(\frac{1}{2})}\frac{v^{2j}}{q^{2j}}\end{pmatrix}.$$

Now if we consider the even n case, we know from studying the Schlesinger transformation we can write the RHS as:

$$\frac{v^{\frac{1}{2}}}{q^{\frac{1}{2}}}\begin{pmatrix} s_0 + \frac{s_2}{v^2} + \frac{s_4}{v^4} + \dots + \frac{s_n}{v^n} & \frac{t_2}{v^2} + \dots + \frac{t_n}{v_n} \\ \frac{s_1}{v} + \frac{s_3}{v^3} + \dots + \frac{s_{n+1}}{v^{n+1}} & \frac{t_1}{v} + \frac{t_3}{v^3} + \dots + \frac{t_{n+1}}{v^{n+1}}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix} a_{2j}^{(\frac{1}{2})}\frac{v^{2j}}{q^{2j}} \\ \bar{a}_{2j}^{(\frac{1}{2})}\frac{v^{2j}}{q^{2j}}\end{pmatrix}. \tag{2.120}$$

We also know the asymptotic behaviour of $F^{(n+\frac{1}{2})}(v,x)$ as $v \to 0$

$$\begin{aligned} F^{(n+\frac{1}{2})}(v,x) &= \phi^{(n+\frac{1}{2})}(v/q^{2n+1},x) \\ &\sim \frac{v^{(n+\frac{1}{2})}}{q^{(2n+1)(n+1/2)}}\begin{pmatrix} a_0^{(n+\frac{1}{2})} \\ c_1^{(n+\frac{1}{2})}\frac{v}{q^{2n+1}}\end{pmatrix}. \end{aligned} \tag{2.121}$$

We now equate equations (2.120) and (2.121) such that $F^{(n+\frac{1}{2})}(v,x)$ has consistent asymptotic behaviour. From the top row we equate $n+1$ equations by equating $n+1$ orders of v, $(v^{-n}, v^{-n+2}, \dots, v^{n-2}, v^n)$:

$$\begin{aligned}
\frac{1}{v^n} &: s_n a_0^{(\frac{1}{2})} + t_n\bar{a}_0^{(\frac{1}{2})} = 0 \\
\frac{1}{v^{n-2}} &: s_n\frac{a_2^{(\frac{1}{2})}}{q^2} + t_n\frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} + s_{n-2}a_0^{(\frac{1}{2})} + t_{n-2}\bar{a}_0^{(\frac{1}{2})} = 0 \\
&\vdots : \vdots \\
v^{n-2} &: s_n\frac{a_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} + t_n\frac{\bar{a}_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} + \dots + t_2\frac{\bar{a}_n^{(\frac{1}{2})}}{q^n} + s_0\frac{a_{n-2}^{(\frac{1}{2})}}{q^{n-2}} = 0 \\
v^n &: s_n\frac{a_{2n}^{(\frac{1}{2})}}{q^{2n}} + t_n\frac{\bar{a}_{2n}^{(\frac{1}{2})}}{q^{2n}} + \dots + t_2\frac{\bar{a}_{n+2}^{(\frac{1}{2})}}{q^{n+2}} + s_0\frac{a_n^{(\frac{1}{2})}}{q^n} = \frac{a_0^{(n+\frac{1}{2})}}{q^{(2n+1)(n+1/2)}}.
\end{aligned}$$

From the bottom we obtain $n+2$ equations by equating coefficients of $(v^{-n-1}, v^{-n+1}, \ldots, v^{n+1})$

$$
\begin{aligned}
\frac{1}{v^{n+1}} &: s_{n+1}a_0^{(\frac{1}{2})} + t_{n+1}\bar{a}_0^{(\frac{1}{2})} = 0 \\
\frac{1}{v^{n-1}} &: s_{n+1}\frac{a_2^{(\frac{1}{2})}}{q^2} + t_{n+1}\frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} + s_{n-1}a_0^{(\frac{1}{2})} + t_{n-1}\bar{a}_0^{(\frac{1}{2})} = 0 \\
&\vdots : \vdots \\
v^{n-1} &: s_{n+1}\frac{a_{2n}^{(\frac{1}{2})}}{q^{2n}} + t_{n+1}\frac{\bar{a}_{2n}^{(\frac{1}{2})}}{q^{2n}} + \cdots + s_1\frac{a_n^{(\frac{1}{2})}}{q^n} + t_1\frac{\bar{a}_n^{(\frac{1}{2})}}{q^n} = 0 \\
v^{n+1} &: s_{n+1}\frac{a_{2n+2}^{(\frac{1}{2})}}{q^{2n+2}} + t_{n+1}\frac{\bar{a}_{2n+2}^{(\frac{1}{2})}}{q^{2n+2}} + \cdots + s_1\frac{a_{n+2}^{(\frac{1}{2})}}{q^{n+2}} + t_1\frac{\bar{a}_{n+2}^{(\frac{1}{2})}}{q^{n+2}} = \frac{c_1^{(n+\frac{1}{2})}}{q^{(2n+1)(n+\frac{3}{2})}}.
\end{aligned}
$$

We now rewrite the above two systems as an $(n+1)\times(n+1)$ matrix system and an $(n+2)\times(n+2)$ matrix system respectively:

$$
\begin{pmatrix}
\frac{a_n^{(\frac{1}{2})}}{q^n} & \frac{\bar{a}_{n+2}^{(\frac{1}{2})}}{q^{n+2}} & \frac{a_{n+2}^{(\frac{1}{2})}}{q^{n+2}} & & \cdots & \frac{\bar{a}_{2n}^{(\frac{1}{2})}}{q^{2n}} & \frac{a_{2n}^{(\frac{1}{2})}}{q^{2n}} \\
\frac{a_{n-2}^{(\frac{1}{2})}}{q^{n-2}} & \frac{\bar{a}_n^{(\frac{1}{2})}}{q^n} & \frac{a_n^{(\frac{1}{2})}}{q^n} & & \cdots & \frac{\bar{a}_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} & \frac{a_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} \\
\vdots & & & \ddots & & & \vdots \\
0 & \ldots & 0 & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} & \frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} & \frac{a_2^{(\frac{1}{2})}}{q^2} \\
0 & 0 & 0 & \ldots & & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})}
\end{pmatrix}
\begin{pmatrix} s_0 \\ t_2 \\ \vdots \\ t_{n+1} \\ s_{n+1} \end{pmatrix}
=
\begin{pmatrix} \frac{a_0^{(n+\frac{1}{2})}}{q^{(2n+1)(n+1/2)}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\tag{2.122}
$$

$$
\begin{pmatrix}
\frac{\bar{a}_{n+2}^{(\frac{1}{2})}}{q^{n+2}} & \frac{a_{n+2}^{(\frac{1}{2})}}{q^{n+2}} & & \cdots & \frac{\bar{a}_{2n+2}^{(\frac{1}{2})}}{q^{2n+2}} & \frac{a_{2n+2}^{(\frac{1}{2})}}{q^{2n+2}} \\
\frac{\bar{a}_n^{(\frac{1}{2})}}{q^n} & \frac{a_n^{(\frac{1}{2})}}{q^n} & & \cdots & \frac{\bar{a}_{2n}^{(\frac{1}{2})}}{q^{2n}} & \frac{a_{2n}^{(\frac{1}{2})}}{q^{2n}} \\
\vdots & & \ddots & & & \vdots \\
0 & \ldots & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} & \frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} & \frac{a_2^{(\frac{1}{2})}}{q^2} \\
0 & 0 & \ldots & 0 & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})}
\end{pmatrix}
\begin{pmatrix} t_1 \\ s_1 \\ \vdots \\ t_{n+1} \\ s_{n+1} \end{pmatrix}
=
\begin{pmatrix} \frac{c_1^{(n+\frac{1}{2})}}{q^{(2n+1)(n+\frac{3}{2})}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\tag{2.123}
$$

We now define

$$\tau_n(x) = \begin{vmatrix} \frac{\bar{a}_n^{(\frac{1}{2})}}{q^n} & \frac{a_n^{(\frac{1}{2})}}{q^n} & & \cdots & \frac{\bar{a}_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} & \frac{a_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} \\ \frac{\bar{a}_{n-2}^{(\frac{1}{2})}}{q^{n-2}} & \frac{a_{n-2}^{(\frac{1}{2})}}{q^{n-2}} & & \cdots & \frac{\bar{a}_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} & \frac{a_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} & \frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} & \frac{a_2^{(\frac{1}{2})}}{q^2} \\ 0 & 0 & \cdots & 0 & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} \end{vmatrix}$$

for even n and

$$\tau_n(x) = \begin{vmatrix} \frac{a_{n-1}^{(\frac{1}{2})}}{q^{n-1}} & \frac{\bar{a}_{n+1}^{(\frac{1}{2})}}{q^{n+1}} & \frac{a_{n+1}^{(\frac{1}{2})}}{q^{n+1}} & & \cdots & \frac{\bar{a}_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} & \frac{a_{2n-2}^{(\frac{1}{2})}}{q^{2n-2}} \\ \frac{a_{n-3}^{(\frac{1}{2})}}{q^{n-3}} & \frac{\bar{a}_{n-1}^{(\frac{1}{2})}}{q^{n-1}} & \frac{a_{n-1}^{(\frac{1}{2})}}{q^{n-1}} & & \cdots & \frac{\bar{a}_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} & \frac{a_{2n-4}^{(\frac{1}{2})}}{q^{2n-4}} \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} & \frac{\bar{a}_2^{(\frac{1}{2})}}{q^2} & \frac{a_2^{(\frac{1}{2})}}{q^2} \\ 0 & 0 & 0 & \cdots & & \bar{a}_0^{(\frac{1}{2})} & a_0^{(\frac{1}{2})} \end{vmatrix}$$

for odd n. Recall that s_0 is a constant and note that in $\frac{a_0^{(n+\frac{1}{2})}}{q^{(2n+1)(n+1/2)}}$ only $a_0^{(n+\frac{1}{2})}$ is dependent on x. We can therefore apply Cramer's rule to (2.122) to obtain

$$a_0^{(n+\frac{1}{2})}(x) = \mu_n \frac{\tau_{n+1}(x)}{\tau_n(x)}$$

where μ_n represents the scalar terms. Now recall (2.64)

$$g_{n+\frac{1}{2}}(x) = -\frac{ix}{q^{2n+1}} \frac{a_0^{(n+\frac{1}{2})}(x/q)}{a_0^{(n+\frac{1}{2})}(x/q^2)}.$$

Therefore, for even n we have

$$g_{n+\frac{1}{2}}(x) = -\frac{ix}{q^{2n+1}} \frac{\tau_{n+1}(x/q)\tau_n(x/q^2)}{\tau_{n+1}(x/q^2)\tau_n(x/q)}.$$

We now require an expression for $g_{n+\frac{1}{2}}(x)$ for odd n and recall that $t_1 = -\frac{i}{q^2 x}$. Applying Cramer's rule on (2.123), similarly to the even case, we

obtain

$$c_1^{(n+\frac{1}{2})}(x) = \frac{\lambda_n}{x}\frac{\tau_{n+2}(x)}{\tau_{n+1}(x)}$$

where λ_n represents the scalar terms. Recalling $c_1^{(n+\frac{1}{2})}(x) = a_0^{(n+1+\frac{1}{2})}(x)$ we can now write

$$a_0^{(n+1+\frac{1}{2})}(x) = \frac{\lambda_n}{x}\frac{\tau_{n+2}(x)}{\tau_{n+1}(x)}. \tag{2.124}$$

If n is even, then $n+1$ is odd. Therefore, for odd n we can write the following

$$a_0^{(n+\frac{1}{2})}(x) = \frac{\delta_n}{x}\frac{\tau_{n+1}(x)}{\tau_n(x)},$$

where δ_n represents the scalar terms, such that for odd n we have

$$g_{n+\frac{1}{2}}(x) = -\frac{ix}{q^{2n+2}}\frac{\tau_{n+1}(x/q)\tau_n(x/q^2)}{\tau_{n+1}(x/q^2)\tau_n(x/q)}.$$

□

2.3. Conclusion

In this Chapter we have given a partial review of the works of Flaschka and Newell [**10**], and Joshi and Shi [**32**] on utilising the associated linear problems for both continuous and discrete Painlevé equations to develop determinantal form solutions for rational and (q-)Hypergeometric special solutions. There are some differences between the continuous and discrete case, for example one can can view in [**31**, **32**] that the Schlesinger transformation is essential in both the rational and q-Hypergeometric solutions of qP_{II}, whereas in the continuous case it is only needed for the Hypergeometric case [**10**]. The methodologies also had clear similarities, such as the series expansions around the singularities of the linear problem being crucial to the process.

It is interesting to consider that the differences between the rational and (q-)Hypergeometric solutions for both P_{II} and qP_{III} seem to suggest that there may not be a single method for developing the determinantal form solutions, but rather, each case must be treated as a new problem, with the methods and tools such as the series expansions, Bäcklund transformations and Schlesinger transformations being part of a common repertoire, to be selected as necessary. In the next Chapter we develop the determinantal form solutions for a q-discrete analogue of the third Painlevé equation.

CHAPTER 3

A q-Discrete Analogue of the Third Painlevé Equation

In this Chapter we present our original results on the determinantal representations for both the rational and Riccati type special solutions of a q-discrete analogue of P_{III} (qP_{III}) with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2 + A_1)^{(1)}$, given by:

$$g(qt) = \frac{a}{g(t)f(t)} \frac{1 + tf(t)}{t + f(t)} \tag{3.1a}$$

$$f(qt) = \frac{a}{f(t)g(qt)} \frac{1 + btg(qt)}{bt + g(qt)} \tag{3.1b}$$

for parameters a and b. While sharing the same rational surface and affine Weyl group as qP_{II} (2.58), this is a system of equations as opposed to a scalar equation like (2.58). Additionally, we have two parameters a and b which influences how we find the special solutions.

In this Chapter, we apply what we have discussed in our review of previous literature. Using Carmichael's theorem we can produce series expansions around a singularity of the linear problem (3.2,3.3) and by utilising the Bäcklund transformation of qP_{III} (3.15), we are then able to develop the Schlesinger transformation for the associated linear problem (3.82), which we believe to be a new result. Examining how the Schlesinger transformation acts on the series expansions of the linear problem, when 3.1 is solved by simple rational or Riccati type special solutions, then allows us to develop the determinantal forms for the rational and Riccati type special solutions, which are our main results.

3.1. An Associated Linear Problem of qP_{III}

We now present the associated linear problem for qP_{III}, which we use for our investigations. We have the spectral equation

$$\widehat{\Psi}(x,t) = \Psi(qx,t) = A(x,t)\Psi(x,t) \tag{3.2}$$

and the deformation equation

$$\overline{\Psi}(x,t) = \Psi(x,qt) = B(x,t)\Psi(x,t), \tag{3.3}$$

where the spectral matrix $A(x,t)$ is defined as follows

$$A(x,t) = A_0(x,t) + \frac{A_1(x,t)}{x} + \frac{A_2(x,t)}{x^2} + \frac{A_3(x,t)}{x^3} \tag{3.4}$$

with

$$A_0(x,t) = \begin{pmatrix} b\sqrt{a} & 0 \\ 0 & \frac{b}{\sqrt{a}} \end{pmatrix} \tag{3.5a}$$

$$A_1(x,t) = \begin{pmatrix} 0 & \frac{abqg(t)+btf(t)g(t)+qf(t)g(t)^2}{aqtx} \\ \frac{-abtg(t)-aq-bqf(t)g(t)}{qtxf(t)g(t)^2} & 0 \end{pmatrix} \tag{3.5b}$$

$$A_2(x,t) = \begin{pmatrix} \frac{-at-btf(t)g(t)-qf(t)g(t)^2}{\sqrt{a}qt^2x^2g(t)} & 0 \\ 0 & \frac{-abtg(t)-aq-tf(t)g(t)^2}{\sqrt{a}qt^2x^2f(t)g(t)} \end{pmatrix} \tag{3.5c}$$

$$A_3(x,t) = \begin{pmatrix} 0 & -\frac{1}{qt^2x^3} \\ \frac{1}{qt^2x^3} & 0 \end{pmatrix} \tag{3.5d}$$

and the deformation matrix $B(x,t)$ is defined as

$$B(x,t) = B_0(x,t) + \frac{B_1(x,t)}{x} + \frac{B_2(x,t)}{x^2} \tag{3.6}$$

with

$$B_0(x,t) = \begin{pmatrix} \frac{ab}{f(t)g(t)} & 0 \\ 0 & \frac{bf(t)g(t)}{a} \end{pmatrix} \tag{3.7a}$$

$$B_1(x,t) = \begin{pmatrix} 0 & \frac{ab+f(t)g(t)}{\sqrt{a}txf(t)} \\ \frac{-a-bf(t)g(t)}{\sqrt{a}txg(t)} & 0 \end{pmatrix} \tag{3.7b}$$

$$B_2(x,t) = \begin{pmatrix} -\frac{1}{t^2x^2} & 0 \\ 0 & -\frac{1}{t^2x^2} \end{pmatrix} \tag{3.7c}$$

where a and b are parameters. (3.2,3.3) are produced from a gauge transformation of the linear problem found in [**30**]. We find the compatibility condition of (3.2,3.3) $\overline{\widehat{\Psi}} = \widehat{\overline{\Psi}}$ gives us

$$A(x,qt)B(x,t) - B(qx,t)A(x,t) = 0, \tag{3.8}$$

which is satisfied by (3.1).

3.2. A General Expansion around $x = \infty$

Proposition 3.1. *There exists a fundamental solution matrix $\Phi(x,t)$ of the linear systems (3.2,3.3) around $x = \infty$, given by*

$$\Phi(x,t) = \{\phi_1(x,t), \phi_2(x,t)\} \tag{3.9}$$

where

$$\phi_1(x,t) = e_1^{\lambda} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j+1}(t)x^{-(2j+1)} \\ c_{2j}(t)x^{-2j} \end{pmatrix} \tag{3.10}$$

$$\phi_2(x,t) = e_2^{\lambda} \sum_{j=0}^{\infty} \begin{pmatrix} b_{2j}(t)x^{-2j} \\ d_{2j+1}(t)x^{-(2j+1)} \end{pmatrix} \tag{3.11}$$

with $e_1 = \frac{b}{\sqrt{a}}$, $e_2 = b\sqrt{a}$ and $\lambda = \frac{\ln x}{\ln q}$. The expansion coefficients satisfy the following recurrence relations:

$$\begin{aligned} q^{-n+1}\sqrt{a}bt^2 g(t)\left(aq^n - 1\right)a_n &= \sqrt{a}a_{n-2}(at + f(t)g(t)(bt + qg(t))) \\ &\quad - g(t)\left(tg(t)c_{n-1}(abq + btf(t) + qf(t)g(t)) - ac_{n-3}\right) \end{aligned} \tag{3.12a}$$

$$\begin{aligned} q^{-n+1}a^{3/2}bt^2 g(t)\left(q^n - 1\right)b_n &= \sqrt{a}b_{n-2}(at + f(t)g(t)(bt + qg(t))) \\ &\quad - g(t)\left(tg(t)d_{n-1}(abq + btf(t) + qf(t)g(t)) - ad_{n-3}\right) \end{aligned} \tag{3.12b}$$

$$\begin{aligned} q^{-n+1}bt^2 f(t)g(t)^2\left(q^n - 1\right)c_n &= a(btg(t) + q)\left(\sqrt{a}ta_{n-1} + g(t)c_{n-2}\right) \\ &\quad + f(t)g(t)\left(\sqrt{a}bqta_{n-1} - \sqrt{a}g(t)a_{n-3} + tg(t)^2 c_{n-2}\right) \end{aligned} \tag{3.12c}$$

$$\begin{aligned} q^{-n+1}bt^2 f(t)g(t)^2\left(q^n - a\right)d_n &= a(btg(t) + q)\left(\sqrt{a}tb_{n-1} + g(t)d_{n-2}\right) \\ &\quad + f(t)g(t)\left(\sqrt{a}bqtb_{n-1} - \sqrt{a}g(t)b_{n-3} + tg(t)^2 d_{n-2}\right). \end{aligned} \tag{3.12d}$$

We also find that

$$\frac{c_n(qt)}{c_n(t)} = \frac{bf(t)g(t)}{a}. \tag{3.13}$$

Proof. By Carmichael's Theorem (3.2,3.3) has a solution around $x = \infty$ given by

$$\Phi(x,t) = \sum_{j=0}^{\infty} \begin{pmatrix} a_j(t) & b_j(t) \\ c_j(t) & d_j(t) \end{pmatrix} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix} \frac{1}{x^j}. \tag{3.14}$$

After substituting (3.14) into (3.2) and equating powers of x we find that $a_{even}(t) = d_{even}(t) = b_{odd}(t) = c_{odd}(t) = 0$ along with recurrence relations (3.12).

Substituting (3.14) into (3.3) and similarly equating powers of x, gives the recurrence relations for the t-evolution of the coefficients, in particular

$$\frac{c_n(qt)}{c_n(t)} = \frac{bf(t)g(t)}{a}.$$

□

Equation (3.13) provides a connection between the solutions of (3.1) and the leading order expansion coefficient of the linear problem's solution around $x = \infty$. We continue to build this connection throughout the Chapter.

3.3. Special Solutions of qP_{III}

Our expansion (3.9) was general, for any values of $g(t)$, $f(t)$ and the associated parameters a and b. We now turn our attention to the linear problem (3.2,3.3) when (3.1) has special solutions with $a = q^{2k}$ for parameter k.

The Bäcklund transformation of qP_{III} (3.1) [**30**], is given by

$$g_{k+1}(t) = \frac{qa_k\left(f_k(t)\left(bt + qg_k(t)\right) + b\right)}{g_k(t)\left(bqa_k + f_k(t)\left(bt + qg_k(t)\right)\right)} \tag{3.15a}$$

$$f_{k+1}(t) = \frac{q\left(a_k\left(btf_k(t) + b\right) + f_k(t)g_k(t)\right)}{f_k(t)\left(f_k(t)\left(bt + qg_k(t)\right) + b\right)} \tag{3.15b}$$

where $a = a_k = q^{2k}$, and $f_k(t)$ and $g_k(t)$ solve (3.1) for the associated value of k.

qP_{III} (3.1) possesses a symmetry such that if $f_k(t)$ and $g_k(t)$ comprise a solution for some k, then $g_{-k}(t) = \frac{1}{g_k(t)}$ and $f_{-k}(t) = \frac{1}{f_k(t)}$ comprise a solution for $-k$. Therefore we can focus our attention on positive k.

Proposition 3.2. *The simplest rational solution of* qP_{III} (3.1) *is found when* $k = 0$ *such that*

$$f_0(t) = g_0(t) = 1. \tag{3.16}$$

This can be checked by substituting the above values into (3.1). Due to the structure of the Bäcklund transformation (3.15), each integer k corresponds to a rational special solution, such that for integer k, (3.1) possess a hierarchy of rational solutions.

Proposition 3.3. qP_{III} (3.1) *possesses a hierarchy of Riccati type solutions for half integer k. A Riccati type special solution can be found when $k = \frac{1}{2}$ such that we obtain discrete Riccati type solutions, given by*

$$f_{\frac{1}{2}}(t) = -\frac{q}{bg_{\frac{1}{2}}(t)} - t \tag{3.17a}$$

$$g_{\frac{1}{2}}(qt) = -\frac{b\left(b\left(t^2-1\right)g_{\frac{1}{2}}(t) + qt\right)}{btg_{\frac{1}{2}}(t) + q} \tag{3.17b}$$

or

$$f_{\frac{1}{2}}(t) = \frac{tF(qt)}{\mu qF(t) - F(qt)} \tag{3.18a}$$

$$g_{\frac{1}{2}}(t) = -\frac{\mu qF(t) - F(qt)}{b\mu tF(t)} \tag{3.18b}$$

for some scalar μ. Where $F(t)$ satisfies the linear difference equation

$$F\left(q^2t\right) + \mu q\left(b^2\left(t^2-1\right)-1\right)F(qt) + b^2\mu^2q^2F(t) = 0. \tag{3.19}$$

Proof. Recall (3.12a), the recurrence relation for $a_n(t)$. Let us look at when $n = 1$ for $a = q^{2k}$ which gives

$$\begin{aligned} bt^2a_1(t)g_k(t)q^k\left(q^{2k+1}-1\right) = \\ - tc_0(t)g_k(t)^2\left(btf_k(t) + bq^{2k+1} + qf_k(t)g_k(t)\right). \end{aligned} \tag{3.20}$$

We observe that for $k = -\frac{1}{2}$ (3.20) simplifies to

$$0 = -tc_0(t)g_{-\frac{1}{2}}(t)^2(btf_{-\frac{1}{2}}(t) + b + qf_{-\frac{1}{2}}(t)g_{-\frac{1}{2}}(t)).$$

As we know that in general, $c_0(t) \neq 0$, for consistency we set $(btf_{-\frac{1}{2}}(t) + b + qf_{-\frac{1}{2}}(t)g_{-\frac{1}{2}}(t)) = 0$ and rearrange to make $f_{-\frac{1}{2}}(t)$ the subject:

$$f_{-\frac{1}{2}}(t) = -\frac{b}{bt + qg_{-\frac{1}{2}}(t)}. \tag{3.21}$$

Substituting (3.21) into (3.1a) we can then solve for $g_{-\frac{1}{2}}(qt)$ which gives

$$g_{-\frac{1}{2}}(qt) = -\frac{bt + qg_{-\frac{1}{2}}(t)}{b\left(b\left(t^2-1\right) + qtg_{-\frac{1}{2}}(t)\right)}. \tag{3.22}$$

We now attempt to linearise (3.22) by setting $g_{-\frac{1}{2}} = \frac{F(t)}{G(t)}$ for some functions $F(t)$ and $G(t)$. We then obtain

$$\frac{F(qt)}{G(qt)} = -\frac{btG(t) + qF(t)}{b\left(b\left(t^2-1\right)G(t) + qtF(t)\right)}. \tag{3.23}$$

We can split (3.23) into two relations for some constant μ:

$$F(qt) = \mu(btG(t) + qF(t)) \tag{3.24}$$

$$G(qt) = b\mu\left(b\left(t^2-1\right)G(t) + qtF(t)\right). \tag{3.25}$$

We then decouple (3.24) and (3.25) to produce a 2nd order linear difference equation entirely in terms of $F(t)$:

$$F\left(q^2t\right) + \mu q\left(b^2\left(t^2-1\right)-1\right)F(qt) + b^2\mu^2q^2F(t) = 0.$$

We now rearrange (3.24) in terms of $G(t)$

$$G(t) = \frac{F(qt) - \mu qF(t)}{b\mu t}$$

such that if $F(t)$ satisfies (3.19), we have by our choice of $g_{-\frac{1}{2}}$

$$g_{-\frac{1}{2}}(t) = -\frac{b\mu tF(t)}{\mu qF(t) - F(qt)}$$

and recalling (3.21) we have

$$f_{-\frac{1}{2}}(t) = \frac{\mu qF(t) - F(qt)}{tF(qt)}.$$

We then calculate $f_{\frac{1}{2}}(t)$ (3.18a) and $g_{\frac{1}{2}}(t)$ (3.18b) with (3.17a) and (3.17b) through the symmetry $f_k = \frac{1}{f_{-k}}$ and $g_k = \frac{1}{g_{-k}}$.

Due to the structure of the Bäcklund transformation (3.15), each half integer k therefore corresponds to a Riccati type special solution. □

The equation for $F(t)$, (3.19) is equivalent to equation (3.11) in [**34**] which is used by Kajiwara et. al. to develop the determinantal form of what they refer to as the Riccati type solutions.

3.3.1. The Linear Problem When qP_{III} is Satisified By Its Simplest Rational Special Solution. We will now observe the series expansion for the linear problem when (3.1) is satisfied by Proposition (3.2). In this section we use the q-Gamma function $\Gamma_q(1-x)$ which is a solution to the following equation

$$y(qx) = \Gamma_q(1-x)y(x). \tag{3.26}$$

The q-Gamma function has both an infinite product expansion and an infinite series expansion:

$$\Gamma_q(1-x) = \frac{1}{(x;q)_\infty} \tag{3.27}$$

where $(x;q)_\infty = (1-x)(1-xq)(1-xq^2)\dots$. and

$$\Gamma_q(1-x) = \sum_{j=0}^{\infty}\frac{x^j}{(q;q)_j}. \tag{3.28}$$

Proposition 3.4. *There exists a solution $\phi_1^{(0)}(x,t)$ of (3.2,3.3) around $x = \infty$ for when $k = 0$ and* (3.1) *is solved by $f_0(t) = g_0(t) = 1$ given by*

$$\phi_1^{(0)}(x,t) = \frac{b^\lambda}{2}\begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}\begin{pmatrix} \Gamma_{\frac{1}{q}}(1-\frac{i}{qx})\Gamma_{\frac{1}{q}}(1-\frac{i}{tx})\Gamma_{\frac{1}{q}}(1-\frac{i}{btx}) \\ \Gamma_{\frac{1}{q}}(1+\frac{i}{qx})\Gamma_{\frac{1}{q}}(1+\frac{i}{tx})\Gamma_{\frac{1}{q}}(1+\frac{i}{btx}) \end{pmatrix} \tag{3.29}$$

$$= b^\lambda \sum_{j=0}^{\infty}\begin{pmatrix} T_{2j+1}(t)x^{-(2j+1)} \\ iT_{2j}(t)x^{-2j} \end{pmatrix}. \tag{3.30}$$

$$\tag{3.31}$$

where $\lambda = \frac{\ln x}{\ln q}$ and $T_j(t)$ is determined by

$$T_j(t) = \frac{q^{j-1}\left((bt+q+t)T_{j-2}(t) - it(bq+bt+q)T_{j-1}(t) + iT_{j-3}(t)\right)}{bt^2\left(q^j-1\right)} \tag{3.32}$$

with t-evolution governed by

$$T_j(t) = \frac{bt^2T_j(qt) - i(b+1)tT_{j-1}(t) + T_{j-2}(t)}{bt^2} \tag{3.33}$$

for $T_0(qt) = bT_0(t)$ and $T_j(t) = 0$ for $j < 0$.

Proof. When (3.1) is satisfied by Proposition (3.2), the linear problem (3.2,3.3) simplifies to

$$\widehat{\Psi}(x,t) = A_0(x,t)\Psi(x,t) \tag{3.34}$$

$$\overline{\Psi}(x,t) = B_0(x,t)\Psi(x,t) \tag{3.35}$$

where $A_0(x,t)$ and $B_0(x,t)$ are the simplified versions of $A(x,t)$ and $B(x,t)$, such that

$$A_0(x,t) = \begin{pmatrix} -\frac{-bqt^2x^2+q+bt+t}{qt^2x^2} & \frac{bt^2x^2+(b+1)qtx^2-1}{qt^2x^3} \\ -\frac{bt^2x^2+(b+1)qtx^2-1}{qt^2x^3} & -\frac{-bqt^2x^2+q+bt+t}{qt^2x^2} \end{pmatrix}$$

$$B_0(x,t) = \begin{pmatrix} b-\frac{1}{t^2x^2} & \frac{b+1}{tx} \\ -\frac{b+1}{tx} & b-\frac{1}{t^2x^2} \end{pmatrix}.$$

We denote the solution matrix of (3.34,3.35) as $\Psi(x,t) = \left\{\phi_1^{(0)}, \phi_2^{(0)}\right\}$. We now diagonalise both $A_0(x,t)$ and $B_0(x,t)$ with the constant matrix $C = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}$ by letting $\Psi(x,t) = C\Psi_1(x,t)$ such that we obtain

$$\widehat{\Psi}_1 = C^{-1}A_0(x,t)C\Psi_1 \tag{3.36}$$

$$\overline{\Psi}_1 = C^{-1}B_0(x,t)C\Psi_1 \tag{3.37}$$

where

$$C^{-1}A_0(x,t)C = \begin{pmatrix} b\left(1-\frac{i}{qx}\right)\left(1-\frac{i}{tx}\right)\left(1-\frac{i}{btx}\right) & 0 \\ 0 & b\left(1+\frac{i}{qx}\right)\left(1+\frac{i}{tx}\right)\left(1+\frac{i}{btx}\right) \end{pmatrix} \tag{3.38}$$

$$C^{-1}B_0(x,t)C = \begin{pmatrix} b\left(1-\frac{i}{tx}\right)\left(1-\frac{i}{btx}\right) & 0 \\ 0 & b\left(1+\frac{i}{tx}\right)\left(1+\frac{i}{btx}\right) \end{pmatrix}. \tag{3.39}$$

Let the first column of the solution matrix of systems (3.36,3.37) be given by $\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, such that

$$\phi_1^{(0)}(x,t) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} \tag{3.40}$$

where

$$u(qx,t) = b\left(1-\frac{i}{qx}\right)\left(1-\frac{i}{tx}\right)\left(1-\frac{i}{btx}\right)u(x,t) \tag{3.41}$$

$$u(x,qt) = b\left(1-\frac{i}{tx}\right)\left(1-\frac{i}{btx}\right)u(x,t) \tag{3.42}$$

$$v(qx,t) = b\left(1+\frac{i}{tx}\right)\left(1+\frac{i}{tx}\right)\left(1+\frac{i}{btx}\right)v(x,t) \tag{3.43}$$

$$v(x,qt) = b\left(1+\frac{i}{tx}\right)\left(1+\frac{i}{btx}\right)v(x,t). \tag{3.44}$$

Equation (3.41) is solved by

$$u(x,t) = b^{\lambda}u_1(x,t)u_2(x,t)u_3(x,t)$$

where $\lambda = \frac{\ln x}{\ln q}$ and

$$u_1(qx,t) = \left(1-\frac{i}{qx}\right)u_1(x,t) \tag{3.45}$$

$$u_2(qx,t) = \left(1-\frac{i}{tx}\right)u_2(x,t) \tag{3.46}$$

$$u_3(qx,t) = \left(1-\frac{i}{btx}\right)u_3(x,t) \tag{3.47}$$

can all be solved in terms of the q-Gamma function $\Gamma_q(1-x)$, such that

$$u_1(x,t) = \Gamma_{\frac{1}{q}}\left(1 - \frac{i}{qx}\right) \tag{3.48}$$

$$u_2(x,t) = \Gamma_{\frac{1}{q}}\left(1 - \frac{i}{tx}\right) \tag{3.49}$$

$$u_3(x,t) = \Gamma_{\frac{1}{q}}\left(1 - \frac{i}{btx}\right) \tag{3.50}$$

and so

$$u(x,t) = b^{\lambda}\Gamma_{\frac{1}{q}}\left(1 - \frac{i}{qx}\right)\Gamma_{\frac{1}{q}}\left(1 - \frac{i}{tx}\right)\Gamma_{\frac{1}{q}}\left(1 - \frac{i}{btx}\right). \tag{3.51}$$

Recalling $\Gamma_q(1-x) = 1/(x;q)_\infty$, where

$$(x;q)_\infty = (1-x)(1-xq)(1-xq^2)\dots.$$

one can check by substitution that (3.51) satisfies both (3.41,3.42). Similarly we can show

$$v(x,t) = b^{\lambda}\Gamma_{\frac{1}{q}}\left(1 + \frac{i}{qx}\right)\Gamma_{\frac{1}{q}}\left(1 + \frac{i}{tx}\right)\Gamma_{\frac{1}{q}}\left(1 + \frac{i}{btx}\right)$$

which satisfies (3.43,3.44). Therefore

$$\phi_1^{(0)}(x,t) = \frac{b^{\lambda}}{2}\begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}\begin{pmatrix} \Gamma_{\frac{1}{q}}(1-\frac{i}{qx})\Gamma_{\frac{1}{q}}(1-\frac{i}{tx})\Gamma_{\frac{1}{q}}(1-\frac{i}{btx}) \\ \Gamma_{\frac{1}{q}}(1+\frac{i}{qx})\Gamma_{\frac{1}{q}}(1+\frac{i}{tx})\Gamma_{\frac{1}{q}}(1+\frac{i}{btx}) \end{pmatrix}.$$

Recall that $\Gamma_q(1-x)$ also has a series expansion given by

$$\Gamma_q(1-x) = \sum_{j=0}^{\infty}\frac{x^j}{(q;q)_j}$$

such that

$$\begin{aligned} u(x,t) &= \sum_{j=0}^{\infty}\left(\frac{i}{qx}\right)^j\frac{1}{(1/q;1/q)_j}\sum_{j=0}^{\infty}\left(\frac{i}{tx}\right)^j\frac{1}{(1/q;1/q)_j}\sum_{j=0}^{\infty}\left(\frac{i}{btx}\right)^j\frac{1}{(1/q;1/q)_j} \\ &= 1 + \frac{i(bq+bt+q)}{b(q-1)tx} + \dots \end{aligned}$$

and

$$\begin{aligned} v(x,t) &= \sum_{j=0}^{\infty}\left(\frac{-i}{qx}\right)^j\frac{1}{(1/q;1/q)_j}\sum_{j=0}^{\infty}\left(\frac{-i}{tx}\right)^j\frac{1}{(1/q;1/q)_j}\sum_{j=0}^{\infty}\left(\frac{-i}{btx}\right)^j\frac{1}{(1/q;1/q)_j} \\ &= 1 - \frac{i(bq+bt+q)}{b(q-1)tx} + \dots. \end{aligned}$$

We then define

$$u(x,t) = b^{\lambda}\sum_{j=0}^{\infty}(-1)^j T_j(t)/x^j$$
$$v(x,t) = b^{\lambda}\sum_{j=0}^{\infty}T_j(t)/x^j \tag{3.52}$$

where $T_0(qt) = bT_0(t)$ and by substituting (3.52) into (3.43,3.44) and equating powers of x, we find the recurrence relations which define the remaining $T_j(t)$

$$T_j(t) = \frac{q^{j-1}\left((bt+q+t)T_{j-2}(t) - it(bq+bt+q)T_{j-1}(t) + iT_{j-3}(t)\right)}{bt^2\left(q^j-1\right)}$$

with t-evolution governed by

$$T_j(t) = \frac{bt^2T_j(qt) - i(b+1)tT_{j-1}(t) + T_{j-2}(t)}{bt^2}$$

for $T_0(qt) = bT_0(t)$ and $T_j(t) = 0$ for $j < 0$. Therefore

$$\begin{aligned}\phi_1^{(0)}(x,t) &= \frac{b^{\lambda}}{2}\begin{pmatrix}-1 & 1\\ i & i\end{pmatrix}\begin{pmatrix}\Gamma_{\frac{1}{q}}(1-\frac{i}{qx})\Gamma_{\frac{1}{q}}(1-\frac{i}{tx})\Gamma_{\frac{1}{q}}(1-\frac{i}{btx})\\ \Gamma_{\frac{1}{q}}(1+\frac{i}{qx})\Gamma_{\frac{1}{q}}(1+\frac{i}{tx})\Gamma_{\frac{1}{q}}(1+\frac{i}{btx})\end{pmatrix}\\ &= \frac{b^{\lambda}}{2}\begin{pmatrix}-1 & 1\\ i & i\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}(-1)^jT_j(t)x^{-j}\\ T_j(t)x^{-j}\end{pmatrix}\\ &= b^{\lambda}\sum_{j=0}^{\infty}\begin{pmatrix}T_{2j+1}(t)x^{-(2j+1)}\\ iT_{2j}(t)x^{-2j}\end{pmatrix}.\end{aligned}$$

□

3.3.2. The Linear Problem When qP_{III} is Satisified By Its Simplest Riccati type Special Solution.

Proposition 3.5. *There exists a solution $\phi_1^{(\frac{1}{2})}(x,t)$ of (3.2,3.3) around $x = \infty$ for when $k = \frac{1}{2}$ and* (3.1) *is solved by*

$$f_{\frac{1}{2}}(t) = -\frac{q}{bg_{\frac{1}{2}}(t)} - t \qquad g_{\frac{1}{2}}(qt) = -\frac{b\left(b\left(t^2-1\right)g_{\frac{1}{2}}(t) + qt\right)}{btg_{\frac{1}{2}}(t) + q}$$

given by

$$\phi_1^{(\frac{1}{2})}(x,t) = b^{\lambda}x^{-\frac{1}{2}}\begin{pmatrix}\frac{\sqrt{q}x}{b} & \frac{bx^2g_{\frac{1}{2}}(t)t^3+qx^2t^2+q}{b\sqrt{q}t^2x}\\ 0 & 1\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}\frac{c_{2j}(qt)}{x^{2j}}\\ \frac{c_{2j}(t)}{x^{2j}}\end{pmatrix}. \tag{3.53}$$

Where $\lambda = \frac{\ln x}{\ln q}$ *and* c_n *satisfies both*

$$c_n(t) = -\frac{q^n\left(t^2\left(\left(b^2+1\right)c_{n-2}(t)+c_{n-2}(qt)\right)+c_{n-4}(t)\right)}{b^2t^4\left(q^n-1\right)} \tag{3.54}$$

$$b^2q^2t^2c_n(t) = \frac{-q^2c_{n-4}(t)}{t^2} + q^2t^2\left(\left(b^2\left(t^2-1\right)-1\right)c_n(qt) - c_n\left(q^2t\right)\right)$$
$$+\left(-b^2-1\right)q^2c_{n-2}(t) - \left(q^2+1\right)c_{n-2}(qt) \tag{3.55}$$

with

$$c_0\left(q^2t\right) + \left(1-b^2\left(t^2-1\right)\right)c_0(qt) + b^2c_0(t) = 0 \tag{3.56}$$

and $c_j = 0$, $j < 0$. *We also find that*

$$f_{\frac{1}{2}}(t) = -\frac{tc_0(qt)}{c_0(qt)+c_0(t)} \tag{3.57a}$$

$$g_{\frac{1}{2}}(t) = -\frac{q\left(c_0(qt)+c_0(t)\right)}{btc_0(t)}. \tag{3.57b}$$

Proof. Recall that the first column of the general expansion around $x = \infty$ is

$$\phi_1(x,t) = e_1^\lambda \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j+1}(t)x^{-(2j+1)} \\ c_{2j}(t)x^{-2j} \end{pmatrix}.$$

For the simplest Riccati type solution when (3.1) is satisfied by Proposition (3.3) we have $e_1 = bq^{-1/2}$, which gives us $e_1^\lambda = b^\lambda x^{-\frac{1}{2}}$ such that

$$\phi_1(x,t) = b^\lambda x^{-\frac{1}{2}} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j+1}(t)x^{-(2j+1)} \\ c_{2j}(t)x^{-2j} \end{pmatrix}.$$

We will therefore begin our transformation by

$$\Psi(x,t) = b^\lambda x^{-\frac{1}{2}}\Psi_1(x,t)$$

such that we have

$$\widehat{\Psi}_1(x,t) = \Psi_1(qx,t) = A1(x,t)\Psi_1(x,t) \tag{3.58}$$

$$\overline{\Psi}_1(x,t) = \Psi_1(x,qt) = B(x,t)\Psi_1(x,t) \tag{3.59}$$

where

$$A1(x,t) = \frac{q^{\frac{1}{2}}A(x,t)}{b}$$
$$= \begin{pmatrix} \frac{q}{b^2t^2x^2} + q + \frac{g_{\frac{1}{2}}(t)}{btx^2} + \frac{1}{x^2q} & \frac{-bqt^2g_{\frac{1}{2}}(t)^2x^2 - t\left(b^2t^2-\left(b^2-1\right)q^2\right)g_{\frac{1}{2}}(t)x^2 - bq\left(t^2x^2+1\right)}{b^2q^{3/2}t^2x^3} \\ \frac{1}{b\sqrt{q}t^2x^3} & -\frac{g_{\frac{1}{2}}(t)}{bqtx^2} + \frac{1}{t^2x^2} + 1 \end{pmatrix}$$

and

$$B(x,t)=\begin{pmatrix} -\frac{b^2qt^2x^2+q+btg_{\frac{1}{2}}(t)}{t^2x^2\left(q+btg_{\frac{1}{2}}(t)\right)} & \frac{g_{\frac{1}{2}}(t)\left(-qb^2+tg_{\frac{1}{2}}(t)b+q\right)}{\sqrt{q}tx\left(q+btg_{\frac{1}{2}}(t)\right)} \\ \frac{b}{\sqrt{q}x} & -\frac{btg_{\frac{1}{2}}(t)}{q}-\frac{1}{t^2x^2}-1 \end{pmatrix}.$$

Letting $\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ be a solution of (3.58,3.59) we have

$$\begin{pmatrix} \widehat{u}(x,t) \\ \widehat{v}(x,t) \end{pmatrix} = A1(x,t)\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} \tag{3.60}$$

$$\begin{pmatrix} \bar{u}(x,t) \\ \bar{v}(x,t) \end{pmatrix} = B(x,t)\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}. \tag{3.61}$$

The bottom row of (3.60) gives us

$$\widehat{v}-v=\frac{1}{b\sqrt{q}t^2x^3}u+\left(-\frac{g_{\frac{1}{2}}(t)}{bqtx^2}+\frac{1}{t^2x^2}\right)v. \tag{3.62}$$

To simplify (3.60,3.61) we define a new variable

$$u_1=\widehat{v}-v=\frac{1}{b\sqrt{q}t^2x^3}u+\left(-\frac{g_{\frac{1}{2}}(t)}{bqtx^2}+\frac{1}{t^2x^2}\right)v \tag{3.63}$$

such that

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = T(x,t)\begin{pmatrix} u_1(x,t) \\ v(x,t) \end{pmatrix}$$

where

$$T(x,t)=\begin{pmatrix} b\sqrt{q}t^2x^3 & \frac{x\left(tg_{\frac{1}{2}}(t)-bq\right)}{\sqrt{q}} \\ 0 & 1 \end{pmatrix}.$$

This transformation produces the following system

$$\begin{pmatrix} \widehat{u}_1 \\ \widehat{v} \end{pmatrix} = A2(x,t)\begin{pmatrix} u_1 \\ v \end{pmatrix} \tag{3.64}$$

$$\begin{pmatrix} \bar{u}_1 \\ \bar{v} \end{pmatrix} = B2(x,t)\begin{pmatrix} u_1 \\ v \end{pmatrix} \tag{3.65}$$

where

$$A2(x,t) = \widehat{T}^{-1}(x,t).A1(x,t).T(x,t) =$$
$$\begin{pmatrix} \frac{\left(\left(t^2x^2+1\right)b^2+1\right)q^2+b^2t^2}{b^2q^4t^2x^2} & -\frac{q^2x^2+\left(b^2+1\right)t^2x^2+1}{b^2q^4t^4x^6} \\ 1 & 1 \end{pmatrix}$$
$$B2(x,t) = \overline{T}^{-1}(x,t).B(x,t).T(x,t) = \begin{pmatrix} \frac{b^2t^4x^2-1}{q^2t^2x^2} & -\frac{\left(\left(b^2+1\right)t^2+1\right)x^2+1}{q^2t^2x^4} \\ b^2t^2x^2 & -b^2-\frac{1}{t^2x^2}-1 \end{pmatrix}.$$

Observe that $A2(x,t)$ and $B2(x,t)$ are independent of $g_{\frac{1}{2}}$. The bottom row of (3.65) produces

$$\bar{v} = b^2t^2x^2u_1 + \left(-1-b^2-\frac{1}{t^2x^2}\right)v.$$

We then define

$$u_2 = \bar{v} = b^2t^2x^2u_1 + \left(-1-b^2-\frac{1}{t^2x^2}\right)v \tag{3.66}$$

such that

$$u_2 = \sum_{j=0}^{\infty} \frac{c_0(qt)}{x^{2j}}$$

and

$$\begin{pmatrix} u_1 \\ v \end{pmatrix} = S(x,t)\begin{pmatrix} u_2 \\ v \end{pmatrix},$$

where

$$S(x,t) = \begin{pmatrix} \frac{1}{b^2t^2x^2} & \frac{b^2t^2x^2+t^2x^2+1}{b^2t^4x^4} \\ 0 & 1 \end{pmatrix}.$$

This produces the following:

$$\begin{pmatrix} \widehat{u}_2 \\ \widehat{v} \end{pmatrix} = A3(x,t)\begin{pmatrix} u_2 \\ v \end{pmatrix} \tag{3.67}$$
$$\begin{pmatrix} \bar{u}_2 \\ \bar{v} \end{pmatrix} = B3(x,t)\begin{pmatrix} u_2 \\ v \end{pmatrix} \tag{3.68}$$

where

$$A3(x,t) = \widehat{S}^{-1}(x,t).A2(x,t).S(x,t) =$$
$$\begin{pmatrix} 1+\frac{1}{x^2q^2}-\frac{1}{b^2t^4x^4q^2} & -\frac{\left(t^2x^2+1\right)\left(b^2t^2x^2+1\right)}{b^2q^2t^6x^6} \\ \frac{1}{b^2t^2x^2} & \frac{\left(t^2x^2+1\right)\left(b^2t^2x^2+1\right)}{b^2t^4x^4} \end{pmatrix}$$
$$B3(x,t) = \bar{S}^{-1}(x,t).B2(x,t).S(x,t) =$$
$$\begin{pmatrix} \left(t^2-1\right)b^2-\frac{q^2+1}{q^2t^2x^2}-1 & -\frac{\left(t^2x^2+1\right)\left(b^2t^2x^2+1\right)}{t^4x^4} \\ 1 & 0 \end{pmatrix}.$$

Both $A3(x,t)$ and $B3(x,t)$ are independent of $g_{\frac{1}{2}}$. We now have:

$$\begin{aligned} \phi_1(x,t) &= b^\lambda x^{-\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= b^\lambda x^{-\frac{1}{2}} T(x,t) \begin{pmatrix} u_1 \\ v \end{pmatrix} \\ &= b^\lambda x^{-\frac{1}{2}} T(x,t) S(x,t) \begin{pmatrix} u_2 \\ v \end{pmatrix} \\ &= b^\lambda x^{-\frac{1}{2}} T(x,t) S(x,t) \sum_{j=0}^{\infty} \begin{pmatrix} \frac{c_{2j}(qt)}{x^{(2j)}} \\ \frac{c_{2j}(t)}{x^{(2j)}} \end{pmatrix}. \end{aligned}$$

Substituting $v(x,t) = \sum_{j=0}^{\infty} \frac{c_{2j}(t)}{x^{-(2j)}}$ into equation (3.66), using our definition $u_1 = \widehat{v} - v$ and equating powers of x gives us the recurrence relation (3.54). Similarly substituting $v(x,t) = \sum_{j=0}^{\infty} \frac{c_{2j}(t)}{x^{-(2j)}}$ into (3.68) and equating powers of x in the bottom row gives the t-evolution of $c_n(t)$ (3.55), in particular

$$c_0\left(q^2t\right) + \left(1 - b^2\left(t^2-1\right)\right)c_0(qt) + b^2c_0(t) = 0. \tag{3.69}$$

We notice that (3.69) is equivalent to (3.19) when $\mu = -1/q$ and we find via (3.18a,3.18b)

$$\begin{aligned} f_{\frac{1}{2}}(t) &= -\frac{tc_0(qt)}{c_0(qt)+c_0(t)} \\ g_{\frac{1}{2}}(t) &= -\frac{q\left(c_0(qt)+c_0(t)\right)}{btc_0(t)}. \end{aligned}$$

□

3.4. The Schlesinger Transformation of qP_{III}

In this section we will be looking at the linear problem of qP_{III} when $a = a_k = q^{2k}$ and $f_k(t)$ and $g_k(t)$ are solutions of

$$g_k(qt) = \frac{a_k}{g_k(t)f_k(t)}\frac{1+tf_k(t)}{t+f_k(t)} \tag{3.71a}$$

$$f_k(qt) = \frac{a_k}{f_k(t)g_k(qt)}\frac{1+btg_k(qt)}{bt+g_k(qt)}. \tag{3.71b}$$

That is

$$\widehat{\Psi}^{(k)}(x,t) = \Psi^{(k)}(qx,t) = A_k(x,t)\Psi^{(k)}(x,t) \tag{3.72a}$$

$$\bar{\Psi}^{(k)}(x,t) = \Psi^{(k)}(x,qt) = B_k(x,t)\Psi^{(k)}(x,t). \tag{3.72b}$$

We now define $\phi_1^{(k)}(x,t) = \begin{pmatrix} u^{(k)}(x,t) \\ v^{(k)}(x,t) \end{pmatrix}$ as the first column of the fundamental solution matrix of (3.71a,3.71b) given by proposition (3.9) with $a = a_k = q^{2k}$ and $f_k(t)$ and $g_k(t)$ are solutions of (3.71). The asymptotic behaviour around $x = \infty$ is given by

$$\begin{pmatrix} u^{(k)}(x,t) \\ v^{(k)}(x,t) \end{pmatrix} \sim b^{\lambda}x^{-k}\begin{pmatrix} a_1^{(k)}(t)/x \\ c_0^{(k)}(t) \end{pmatrix}, \tag{3.73}$$

where $\lambda = \frac{\ln x}{\ln q}$. This shows that $u^{(k)}(x,t)$ and $v^{(k+1)}(x,t)$ have the same leading order around $x = \infty$. This motivates us to try to relate $a_1^{(k)}(t)$ to $c_0^{(k+1)}(t)$.

Proposition 3.6.

$$a_1^{(k)}(t) = \mu c_0^{(k+1)}(t) \tag{3.74}$$

for some constant μ.

Proof. It is equivalent to prove the following statement

$$\frac{a_1^{(k)}(qt)}{a_1^{(k)}(t)} = \frac{c_0^{(k+1)}(qt)}{c_0^{(k+1)}(t)} \tag{3.75}$$

which has solution $a_1^{(k)}(t) = \mu c_0^{(k+1)}(t)$, for some constant μ. We can use (3.12a) to calculate the LHS of (3.75), in particular we have

$$a_1^{(k)}(t) = -\frac{q^{-k}c_0^{(k)}(t)g_k(t)\left(btf_k(t)+qf_k(t)g_k(t)+bq^{2k+1}\right)}{bt\left(q^{2k+1}-1\right)}. \tag{3.76}$$

Using the definition of of qP_{III} (3.71) to simplify (3.76), we arrive at

$$\frac{a_1^{(k)}(qt)}{a_1^{(k)}(t)} = \frac{b\left(f_k(t)\left(g_k(t)+btq^{2k}\right)+bq^{2k}\right)}{f_k(t)g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)}.$$

To calculate the RHS we have (3.13), that is

$$\frac{c_0^{(k+1)}(qt)}{c_0^{(k+1)}(t)} = bf_{k+1}(t)g_{k+1}(t)q^{-2(k+1)}. \tag{3.77}$$

Again, using the definition of qP_{III} (3.71) in addition to the Bäcklund transformation (3.15) we arrive at

$$\frac{c_0^{(k+1)}(qt)}{c_0^{(k+1)}(t)} = \frac{b\left(f_k(t)\left(g_k(t)+btq^{2k}\right)+bq^{2k}\right)}{f_k(t)g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)}$$

and therefore

$$\frac{a_1^{(k)}(qt)}{a_1^{(k)}(t)} = \frac{c_0^{(k+1)}(qt)}{c_0^{(k+1)}(t)}.$$

□

This proof motivates us to further investigate the relationship between $v^{(k+1)}(x,t)$ and $u^{(k)}(x,t)$.

Proposition 3.7.

$$v^{(k+1)}(x,t) = u^{(k)}(x,t). \tag{3.78}$$

Proof. Observe that in the linear problem, $v^{(k)}(x,t)$ and $u^{(k)}(x,t)$ are coupled such that we can write (3.72a) as

$$\begin{pmatrix}\widehat{u}^{(k)}(x,t)\\ \widehat{v}^{(k)}(x,t)\end{pmatrix} = A_k(x,t)\begin{pmatrix}u^{(k)}(x,t)\\ v^{(k)}(x,t)\end{pmatrix} \tag{3.79}$$

$$= \begin{pmatrix}A_{11}^{(k)} & A_{12}^{(k)}\\ A_{21}^{(k)} & A_{22}^{(k)}\end{pmatrix}\begin{pmatrix}u^{(k)}(x,t)\\ v^{(k)}(x,t)\end{pmatrix} \tag{3.80}$$

where we have

$$\begin{aligned}
A_{11}^{(k)} &= bq^k - \frac{q^{-k-1}\left(btf_k(t)g_k(t)+qf_k(t)g_k(t)^2+tq^{2k}\right)}{t^2x^2g_k(t)}\\
A_{12}^{(k)} &= \frac{q^{-2k-1}g_k(t)\left(btf_k(t)+qf_k(t)g_k(t)+bq^{2k+1}\right)}{tx} - \frac{1}{qt^2x^3}\\
A_{21}^{(k)} &= \frac{1}{qt^2x^3} - \frac{bqf_k(t)g_k(t)+btq^{2k}g_k(t)+q^{2k+1}}{qtxf_k(t)g_k(t)^2}\\
A_{22}^{(k)} &= bq^{-k} - \frac{q^{-k-1}\left(btq^{2k}g_k(t)+tf_k(t)g_k(t)^2+q^{2k+1}\right)}{t^2x^2f_k(t)g_k(t)}.
\end{aligned}$$

In order to prove our proposition we will decouple (3.79). This gives us the following two equations:

$$\begin{aligned}\widehat{\widehat{u}}^{(k)} &= (A_{22}^{(k)}\frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}} + \widehat{A}_{11}^{(k)})\widehat{u}^{(k)} - \frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}}|A_k|u^{(k)} \\ \widehat{\widehat{v}}^{(k)} &= (A_{11}^{(k)}\frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}} + \widehat{A}_{22}^{(k)})\widehat{v}^{(k)} - \frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}}|A_k|v^{(k)} \end{aligned} \tag{3.81}$$

where $|A_k|$ is the determinant of A_k. As we are interested in $v^{(k+1)}(x,t)$ we will shift (3.81) forward in k, giving

$$\widehat{\widehat{v}}^{(k+1)} = (A_{11}^{(k+1)}\frac{\widehat{A}_{21}^{(k+1)}}{A_{21}^{(k+1)}} + \widehat{A}_{22}^{(k+1)})\widehat{v}^{(k+1)} - \frac{\widehat{A}_{21}^{(k+1)}}{A_{21}^{(k+1)}}|A_{k+1}|v^{(k+1)}.$$

If we can prove that $v^{(k+1)}(x,t)$ and $u^{(k)}(x,t)$ satisfy the same equation, then the claim is proved. Firstly, as

$$|A_k| = \frac{(q^2x^2+1)\,(t^2x^2+1)\,(b^2t^2x^2+1)}{q^2t^4x^6}$$

is independent of k, we see $|A_{k+1}| = |A_k|$. Now by using the definition of qP_{III} (3.71) and the Bäcklund transformation (3.15) we can show that

$$(A_{22}^{(k)}\frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}} + \widehat{A}_{11}^{(k)}) = (A_{11}^{(k+1)}\frac{\widehat{A}_{21}^{(k+1)}}{A_{21}^{(k+1)}} + \widehat{A}_{22}^{(k+1)})$$

along with

$$\frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}} = \frac{\widehat{A}_{21}^{(k+1)}}{A_{21}^{(k+1)}}$$

and therefore $v^{(k+1)}(x,t)$ and $u^{(k)}(x,t)$ satisfy the same equation. Recall that $a_1^{(k)}(t) = \mu c_0^{(k+1)}(t)$. We set $\mu = 1$, such that $v^{(k+1)}(x,t) = u^{(k)}(x,t)$. □

This is the key to developing the Schlesinger transformation. It tell us how to calculate $v^{(k+1)}(x,t)$ in terms of $u^{(k)}(x,t)$ and allows us to calculate $u^{(k+1)}(x,t)$ in terms of $v^{(k)}(x,t)$ and $u^{(k)}(x,t)$ and hence complete the derivation of the Schlesinger transformation.

Proposition 3.8. *The Schlesinger transformation L_k is given by*

$$\Psi^{(k+1)}(x,t) = L_k(x,t)\Psi^{(k)}(x,t) \tag{3.82}$$

where

$$L_k(x,t) = \begin{pmatrix} \frac{(A_{11}^{(k)} - A_{22}^{(k+1)})}{A_{21}^{(k+1)}} & \frac{A_{12}^{(k)}}{A_{21}^{(k+1)}} \\ 1 & 0 \end{pmatrix} \tag{3.83}$$

$$= \begin{pmatrix} p_k(t)x & -1 \\ 1 & 0 \end{pmatrix} \tag{3.84}$$

where

$$p_k(t) = -\frac{btq^k\left(q^{2k+1}-1\right)}{g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)} \tag{3.85}$$

such that for rational special solutions

$$\phi_1^{(k+1)}(x,t) = L_k\phi_1^{(k)}(x,t) \tag{3.86}$$

$$= L_kL_{k-1}L_{k-2}\dots L_1L_0\phi_1^{(0)}(x,t) \tag{3.87}$$

$$= b^\lambda L_kL_{k-1}L_{k-2}\dots L_1L_0\sum_{j=0}^{\infty}\begin{pmatrix} T_{2j+1}(t)x^{-(2j+1)} \\ iT_{2j}(t)x^{-2j}\end{pmatrix} \tag{3.88}$$

where $\lambda = \frac{\ln x}{\ln q}$ *for integer* k*, and for Riccati type solutions with* $k = n - \frac{1}{2}$ *and integer* n

$$\phi_1^{(n+\frac{1}{2})}(x,t) = L_{n-\frac{1}{2}}\phi_1^{(n-\frac{1}{2})}(x,t) \tag{3.89}$$

$$= L_{n-\frac{1}{2}}L_{n-\frac{3}{2}}L_{n-\frac{5}{2}}\dots L_{\frac{1}{2}}\phi_1^{(\frac{1}{2})}(x,t) \tag{3.90}$$

$$= b^\lambda x^{-\frac{1}{2}}L_{n-\frac{1}{2}}L_{n-\frac{3}{2}}L_{n-\frac{5}{2}}\dots L_{\frac{1}{2}}T(x,t)S(x,t)\sum_{j=0}^{\infty}\begin{pmatrix} \frac{c_{2j}^{(\frac{1}{2})}(qt)}{x^{2j}} \\ \frac{c_{2j}^{(\frac{1}{2})}(t)}{x^{2j}}\end{pmatrix}. \tag{3.91}$$

Proof. We will begin by inverting (3.79), producing

$$u^{(k)}(x,t) = \frac{A_{22}^{(k)}u^{(k)}(qx,t) - A_{12}^{(k)}v^{(k)}(qx,t)}{|A_k|}$$

$$v^{(k)}(x,t) = \frac{A_{11}^{(k)}v^{(k)}(qx,t) - A_{21}^{(k)}u^{(k)}(qx,t)}{|A_k|}. \tag{3.92}$$

We will now work with (3.92) and shift $k \to k+1$

$$v^{(k+1)}(x,t) = \frac{A_{11}^{(k+1)}v^{(k+1)}(qx,t) - A_{21}^{(k+1)}u^{(k+1)}(qx,t)}{|A_{k+1}|},$$

or, via (3.78)

$$u^{(k)}(x,t) = \frac{A_{11}^{(k+1)}u^{(k)}(qx,t) - A_{21}^{(k+1)}u^{(k+1)}(qx,t)}{|A_{k+1}|}. \tag{3.93}$$

We now have no $v^{(k+1)}(x,t)$ terms and so it is a case of writing $u^{(k+1)}(x,t)$ in terms of $u^{(k)}(x,t)$ and $v^{(k)}(x,t)$. Finally by utilising (3.79) and a final application of (3.78) we can produce the desired terms such that we find

$$u^{(k+1)}(x,t) = \frac{(A_{11}^{(k)} - A_{22}^{(k+1)})}{A_{21}^{(k+1)}} u^{(k)}(x,t) + \frac{A_{12}^{(k)}}{A_{21}^{(k+1)}} v^{(k)}(x,t)$$

such that

$$L_k(x,t) = \begin{pmatrix} \frac{(A_{11}^{(k)} - A_{22}^{(k+1)})}{A_{21}^{(k+1)}} & \frac{A_{12}^{(k)}}{A_{21}^{(k+1)}} \\ 1 & 0 \end{pmatrix}$$

or after applying the Bäcklund transformation (3.15):

$$L_k(x,t) = \begin{pmatrix} -\frac{btq^k\left(q^{2k+1}-1\right)x}{g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)} & -1 \\ 1 & 0 \end{pmatrix}.$$

After multiple applications of the Schlesinger transform, we can, for the rational special solutions, write:

$$\begin{aligned} \phi_1^{(k+1)}(x,t) &= L_k L_{k-1} L_{k-2} \dots L_1 L_0 \phi_1^{(0)}(x,t) \\ &= b^\lambda L_k L_{k-1} L_{k-2} \dots L_1 L_0 \sum_{j=0}^{\infty} \begin{pmatrix} T_{2j+1}(t) x^{-(2j+1)} \\ i T_{2j}(t) x^{-2j} \end{pmatrix} \end{aligned}$$

where $\lambda = \frac{\ln x}{\ln q}$. Transforming $k \to n - \frac{1}{2}$ for integer n then gives the following for the Riccati type solutions:

$$\begin{aligned} \phi_1^{(n+\frac{1}{2})}(x,t) &= L_{n-\frac{1}{2}} \phi_1^{(n-\frac{1}{2})}(x,t) \\ &= L_{n-\frac{1}{2}} L_{n-\frac{3}{2}} L_{n-\frac{5}{2}} \dots L_{\frac{1}{2}} \phi_1^{(\frac{1}{2})}(x,t) \\ &= b^\lambda x^{-\frac{1}{2}} L_{n-\frac{1}{2}} L_{n-\frac{3}{2}} \dots L_{\frac{1}{2}} T(x,t) S(x,t) \sum_{j=0}^{\infty} \begin{pmatrix} \frac{c_{2j}^{(\frac{1}{2})}(qt)}{x^{2j}} \\ \frac{c_{2j}^{(\frac{1}{2})}(t)}{x^{2j}} \end{pmatrix}. \end{aligned}$$

□

3.5. The Determinantal Form for the Rational Special Solutions of qP_{III}

Theorem 3.1. *The asymmetric qP_{III} (3.71) with $a = a_k = q^{2k}$ for integer k has a hierarchy of rational special solutions given by*

$$f_k(t) = \frac{q\left(ba_k + \Pi_k(t)\right)\left(qa_k - \Pi_k(t)\Pi_{k+1}(t)\right)}{bt\Pi_k(t)\Pi_{k+1}(t) - bq^2 t a_k^2} \tag{3.94}$$

and

$$g_k(t) = \frac{\Pi_k(t)\left(bt\Pi_k(t)\Pi_{k+1}(t) - bq^2ta_k^2\right)}{q\left(ba_k + \Pi_k(t)\right)\left(qa_k - \Pi_k(t)\Pi_{k+1}(t)\right)} \tag{3.95}$$

where we have defined $\Pi_k(t) := f_k(t)g_k(t)$. *We find that*

$$\Pi_k(t) = q^{2k}\frac{\tau_k(qt)\tau_{k-1}(t)}{b\tau_k(t)\tau_{k-1}(qt)}. \tag{3.96}$$

We also have

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & T_2 & T_3 \\ 0 & \dots & \dots & T_0 & T_1 \end{vmatrix} \tag{3.97}$$

and $T_j(t)$ *is determined by*

$$T_j(t) = \frac{q^{j-1}\left((bt+q+t)T_{j-2}(t) - it(bq+bt+q)T_{j-1}(t) + iT_{j-3}(t)\right)}{bt^2\left(q^j-1\right)} \tag{3.98}$$

with t*-evolution governed by*

$$T_j(t) = \frac{bt^2T_j(qt) - i(b+1)tT_{j-1}(t) + T_{j-2}(t)}{bt^2} \tag{3.99}$$

and $T_0(qt) = bT_0(t)$, $T_j(t) = 0,\ j < 0$.

Proof. Recall

$$\phi_1^{(k+1)} = L_kL_{k-1}\dots L_1L_0\phi_1^{(0)}(x,t).$$

We observe that

$$L_1L_0 = \begin{pmatrix} s_1x^2 + s_{-1} & t_1x \\ s_0x & t_0 \end{pmatrix}$$
$$L_2L_1L_0 = \begin{pmatrix} s_2x^3 + s_0x & t_2x^2 + t_0 \\ s_1x^2 + s_{-1} & t_1x \end{pmatrix}$$
$$L_3L_2L_1L_0 = \begin{pmatrix} s_3x^4 + s_1x^2 + s_{-1} & t_3x^3 + t_1x \\ s_2x^3 + s_0x & t_2x^2 + t_0 \end{pmatrix}$$

where we have labelled the coefficients of x^j in the first column $s_{j-1}(t)$ and those in the second column $t_j(t)$. Due to the structure of the Schlesinger

equation, we find s_1 and t_0 are always scalars. If k is odd we therefore have

$$\phi_1^{(k+1)} = b^\lambda \begin{pmatrix} s_k x^{k+1} + \cdots + s_1 x^2 + s_{-1} & t_k x^k + \cdots + t_3 x^3 + t_1 x \\ s_{k-1} x^k + \cdots + s_2 x^3 + s_0 x & t_{k-1} x^{k-1} + \cdots + t_2 x^2 + t_0 \end{pmatrix} \times \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j+1}(t) x^{-(2j+1)} \\ c_{2j}(t) x^{-(2j)} \end{pmatrix}. \quad (3.100)$$

We also know from (3.73) that around $x = \infty$

$$\phi_1^{(k+1)} \sim b^\lambda \begin{pmatrix} a_1^{(k+1)}(t)/x^{k+2} \\ c_0^{(k+1)}(t)/x^{k+1} \end{pmatrix} \quad (3.101)$$

where $\lambda = \frac{\ln x}{\ln q}$. Equating (3.100) and (3.101) we see that from the top row we get $k+2$ equations:

$$\begin{aligned} x^k &: s_k a_1 + t_k c_0 = 0 \\ x^{k-2} &: s_k a_3 + t_k c_2 + s_{k-2} a_1 + t_{k-2} c_0 = 0 \\ &\vdots : \vdots \\ \frac{1}{x^k} &: s_k a_{2k+1} + t_k c_{2k} + \cdots + s_1 a_{k+2} + t_1 c_{k+1} + s_{-1} a_k = 0 \\ \frac{1}{x^{k+2}} &: s_k a_{2k+3} + t_k c_{2k+2} + \cdots + s_1 a_{k+4} + t_1 c_{k+3} + s_{-1} a_{k+2} = a_1^{(k+1)}. \end{aligned}$$

Now, from the bottom row we obtain $k+1$ equations:

$$\begin{aligned} x^{k-1} &: s_{k-1} a_1 + t_{k-1} c_0 = 0 \\ x^{k-3} &: s_{k-1} a_3 + t_{k-1} c_2 + s_{k-3} a_1 + t_{k-3} c_0 = 0 \\ &\vdots : \vdots \\ \frac{1}{x^{k-1}} &: s_{k-1} a_{2k-1} + t_{k-1} c_{2k-2} + \cdots + s_0 a_k + t_0 c_{k-1} = 0 \\ \frac{1}{x^{k+1}} &: s_{k-1} a_{2k+1} + t_{k-1} c_{2k} + \cdots + s_0 a_{k+2} + t_0 c_{k+1} = c_0^{(k+1)}. \end{aligned}$$

We will now write the above equations in matrix form. Beginning with the equations derived from the top row, we obtain a $(k+2) \times (k+2)$ matrix

form,

$$\begin{pmatrix} a_{k+2} & c_{k+3} & \dots & c_{2k+2} & a_{2k+3} \\ a_k & c_{k+1} & \dots & c_{2k} & a_{2k+1} \\ & & \ddots & & \\ 0 & & \dots & c_0 & a_1 \end{pmatrix} \begin{pmatrix} s_{-1} \\ t_1 \\ s_1 \\ \vdots \\ t_k \\ s_k \end{pmatrix} = \begin{pmatrix} a_1^{(k+1)} \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix} \tag{3.102}$$

and from the bottom entry we get a $(k+1) \times (k+1)$ matrix form

$$\begin{pmatrix} c_{k+1} & a_{k+2} & \dots & c_{2k} & a_{2k+1} \\ c_{k-1} & a_k & \dots & c_{2k-2} & a_{2k-1} \\ & & \ddots & & \\ 0 & & \dots & c_0 & a_1 \end{pmatrix} \begin{pmatrix} t_0 \\ s_0 \\ t_2 \\ \vdots \\ t_{k-1} \\ s_{k-1} \end{pmatrix} = \begin{pmatrix} c_0^{(k+1)} \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix}. \tag{3.103}$$

As k is an odd integer $k+1$ is even. Therefore we define

$$\sigma_{k+1}(t) = \begin{vmatrix} c_{k+1} & a_{k+2} & \dots & c_{2k} & a_{2k+1} \\ c_{k-1} & a_k & \dots & c_{2k-2} & a_{2k-1} \\ & & \ddots & & \\ 0 & & \dots & c_0 & a_1 \end{vmatrix}$$

and

$$\sigma_{k+2}(t) = \begin{vmatrix} a_{k+2} & c_{k+3} & \dots & c_{2k+2} & a_{2k+3} \\ a_k & c_{k+1} & \dots & c_{2k} & a_{2k+1} \\ & & \ddots & & \\ 0 & & \dots & c_0 & a_1 \end{vmatrix}.$$

We now use Cramer's Rule: Recall that if we have a matrix system given by

$$\sum_{j=1}^{n} C_j x_j = B,$$

where

$$B = (b_1, b_2, \dots, b_n)^T$$
$$A = |C_1 C_2 \dots C_j \dots C_n|$$

then x_j is calculated by

$$x_j = \frac{1}{A}|C_1 C_2 \dots B \dots C_n|.$$

Applying Cramer's Rule to (3.102) lets us calculate s_{-1}:

$$s_{-1} = \frac{a_1^{(k+1)}(t)\sigma_{k+1}(t)}{\sigma_{k+2}(t)}$$
$$\implies a_1^{(k+1)}(t) = \frac{s_{-1}\sigma_{k+2}(t)}{\sigma_{k+1}(t)}.$$

However as we know $a_1^{(k+1)}(t) = c_0^{(k+2)}(t)$,

$$c_0^{(k+2)}(t) = \frac{s_{-1}\sigma_{k+2}(t)}{\sigma_{k+1}(t)}.$$

Similarly, we apply Cramer's rule to (3.103) to obtain

$$t_0 = \frac{c_0^{(k+1)}(t)\sigma_k(t)}{\sigma_{k+1}(t)}$$
$$\implies c_0^{(k+1)}(t) = \frac{t_0\sigma_{k+1}(t)}{\sigma_k(t)}.$$

We now have expressions for $c_0^{(k)}(t)$ for both odd and even integer k. Recall that

$$\phi^{(0)}(x,t) = b^\lambda \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j+1}(t)x^{-(2j+1)} \\ c_{2j}(t)x^{-2j} \end{pmatrix} = b^\lambda \sum_{j=0}^{\infty} \begin{pmatrix} T_{2j+1}(t)x^{-(2j+1)} \\ iT_{2j}(t)x^{-2j} \end{pmatrix}$$

such that $a_{2j+1}(t) = T_{2j+1}(t)$, $c_{2j}(t) = iT_{2j}(t)$ for $j \in \mathbb{N}$. We now define the following determinant of a $k \times k$ matrix for integer k

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & T_2 & T_3 \\ 0 & \dots & \dots & T_0 & T_1 \end{vmatrix}$$

and note that τ_k is proportional to σ_k. Additionally, as we found that s_{-1} and t_0 were always constants in t we can say

$$c_0^{(k)}(t) = \mu_k \frac{\tau_k(t)}{\tau_{k-1}(t)} \tag{3.104}$$

where μ_k represents the scalar term. Finally, using (3.77) we have

$$\frac{c_0^{(k)}(qt)}{c_0^{(k)}(t)} = bf_k(t)g_k(t)q^{-2k}$$
$$\implies f_k(t)g_k(t) = q^{2k}\frac{\tau_k(qt)\tau_{k-1}(t)}{b\tau_{k-1}(qt)\tau_k(t)}. \tag{3.105}$$

We now define $\Pi_k(t) := f_k(t)g_k(t)$ for simplicity in the following manipulations. Recall the Bäcklund transformation of qP_{III}, given by (3.15a) and (3.15b). After multiplying (3.15a) and (3.15b) we obtain

$$\Pi_{k+1}(t) = \frac{q^2 a_k\left(a_k\left(btf_k(t)+b\right)+\Pi_k(t)\right)}{\Pi_k(t)\left(bqa_k+btf_k(t)+q\Pi_k(t)\right)}. \tag{3.106}$$

Equation (3.106) can be rearranged to give

$$f_k(t) = \frac{q\left(ba_k+\Pi_k(t)\right)\left(qa_k-\Pi_k(t)\Pi_{k+1}(t)\right)}{bt\Pi_k(t)\Pi_{k+1}(t)-bq^2ta_k^2}$$

By our definition of $\Pi_k(t)$ we can find $g_k(t)$ via $g_k(t) = \frac{\Pi_k(t)}{f_k(t)}$ to be

$$g_k(t) = \frac{\Pi_k(t)\left(bt\Pi_k(t)\Pi_{k+1}(t)-bq^2ta_k^2\right)}{q\left(ba_k+\Pi_k(t)\right)\left(qa_k-\Pi_k(t)\Pi_{k+1}(t)\right)}.$$

□

We find the first few special solutions to be

$$f_0(t) = 1 \qquad g_0(t) = 1 \tag{3.107}$$
$$f_1(t) = \frac{q(bt+b+1)}{bt+b+q} \qquad g_1(t) = \frac{q(bt+b+q)}{bq+bt+q} \tag{3.108}$$

$$f_2(t) = \frac{b^3(t+1)\left(q^2t+qt^2+q+t\right)+b^2\left(q^2+q+1\right)(t+1)^2+b\left(q^2+q+1\right)(t+1)+q}{b^3(t+1)\left(q^2t+qt^2+q+t\right)+b^2\left(q^2+q+1\right)q(t+1)^2+b\left(q^2+q+1\right)q^2(t+1)+q^4} \times \frac{q^2(bt+b+q)}{(bt+b+1)} \tag{3.109}$$

$$g_2(t) = \frac{b^3(t+1)\left(q^2t+qt^2+q+t\right)+b^2\left(q^2+q+1\right)q(t+1)^2+b\left(q^2+q+1\right)q^2(t+1)+q^4}{b^3(t+1)\left(q^3+q^2t+qt+t^2\right)+b^2\left(q^2+q+1\right)(q+t)^2+bq\left(q^2+q+1\right)(q+t)+q^3} \times \frac{q(bq+bt+q)}{(bt+b+q)}. \tag{3.110}$$

3.6. The Determinantal Form for the Riccati type Special Solutions of qP_{III}

Theorem 3.2. *The asymmetric qP_{III} (1.18a,1.18b) with $a = a_k = q^{2k}$ for half integer k where $k = n - \frac{1}{2}$ for integer n, has a hierarchy of Riccati type special solutions. $f_{n+\frac{1}{2}}(t)$ satisfies the following quadratic equation:*

$$\begin{aligned}
&tf_{n+\frac{1}{2}}(t)^2\Pi_{n+\frac{1}{2}}(qt)\left(a_{n+\frac{1}{2}} + b\Pi_{n+\frac{1}{2}}(t)\right) + \\
&f_{n+\frac{1}{2}}(t)\left(-a_{n+\frac{1}{2}}\left(bt^2a_{n+\frac{1}{2}} + \Pi_{n+\frac{1}{2}}(t)\right) + a_{n+\frac{1}{2}}\Pi_{n+\frac{1}{2}}(qt) + bt^2\Pi_{n+\frac{1}{2}}(t)\Pi_{n+\frac{1}{2}}(qt)\right) \\
&- ta_{n+\frac{1}{2}}\left(ba_{n+\frac{1}{2}} + \Pi_{n+\frac{1}{2}}(t)\right) = 0
\end{aligned} \tag{3.111}$$

where we have defined $\Pi_{n+\frac{1}{2}}(t) := f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)$. We find that

$$\Pi_{n+\frac{1}{2}}(t) = q^{2n+1}\frac{M_{n+1}(t)\left(t_{-1,n}(qt)\Lambda_{n+2}(qt) - \tau_{n+1}(qt)a_3^{(n-1/2)}(qt)\right)}{bM_{n+1}(qt)\left(t_{-1,n}(t)\Lambda_{n+2}(t) - \tau_{n+1}(t)a_3^{(n-1/2)}(t)\right)} \tag{3.112}$$

for odd n and

$$\Pi_{n+\frac{1}{2}}(t) = q^{2n+1}\frac{t_{0,n}(qt)\Lambda_{n+1}(qt)\tau_n(t)}{bt_{0,n}(t)\Lambda_{n+1}(t)\tau_n(qt)} \tag{3.113}$$

for even n, such that

$$g_{n+\frac{1}{2}}(t) = \frac{\Pi_{n+\frac{1}{2}}(t)}{f_{n+\frac{1}{2}}(t)}. \tag{3.114}$$

We have the following, where $\bar{c}_n^{(\frac{1}{2})} = c_n^{(\frac{1}{2})}(qt)$

$$\Lambda_n(t) = \begin{vmatrix} c_{n-1}^{(\frac{1}{2})} & \bar{c}_{n+1}^{(\frac{1}{2})} & c_{n+1}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ c_{n-3}^{(\frac{1}{2})} & \bar{c}_{n-1}^{(\frac{1}{2})} & c_{n-1}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ & & & & & & \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix}, \tag{3.115}$$

$$\tau_n(t) = \begin{vmatrix} \bar{c}_n^{(\frac{1}{2})} & c_n^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ & & & & & \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix}, \tag{3.116}$$

$$M_n(t) = \begin{vmatrix} \bar{c}_{n+2}^{(\frac{1}{2})} & c_{n+2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n}^{(\frac{1}{2})} & c_{2n}^{(\frac{1}{2})} \\ \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ \bar{c}_{n-4}^{(\frac{1}{2})} & c_{n-4}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-6}^{(\frac{1}{2})} & c_{2n-6}^{(\frac{1}{2})} \\ \bar{c}_{n-6}^{(\frac{1}{2})} & c_{n-6}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-8}^{(\frac{1}{2})} & c_{2n-8}^{(\frac{1}{2})} \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix}, \tag{3.117}$$

where after the 2nd row, all subsequent rows' subscripts decrease by 2 and

$$t_{-1}(t) = i^{n-1}\frac{q^{1/2}}{bt^2}. \tag{3.118}$$

$a_3^{(n)}(t)$ *can be expressed in terms of* $c_n^{(\frac{1}{2})}(t)$ *through applications of*

$$\begin{aligned} q^{-j+1}\sqrt{a}bt^2 g(t)\left(aq^j-1\right)a_j^{(n+\frac{1}{2})}(t) = \\ \sqrt{a}a_{j-2}^{(n+\frac{1}{2})}(t)(at + f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)(bt+qg_{n+\frac{1}{2}}(t))) \\ - g(t)\left(tg_{n+\frac{1}{2}}(t)c_{j-1}^{(n+\frac{1}{2})}(t)(abq + btf_{n+\frac{1}{2}}(t) + qf_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)) - ac_{j-3}^{(n+\frac{1}{2})}(t)\right) \end{aligned} \tag{3.119a}$$

$$\begin{aligned} q^{-j+1}bt^2 f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)^2\left(q^j-1\right)c_j^{(n+\frac{1}{2})}(t) = \\ a(btg_{n+\frac{1}{2}}(t)+q)\left(\sqrt{a}ta_{j-1}^{(n+\frac{1}{2})}(t) + g_{n+\frac{1}{2}}(t)c_{j-2}^{(n+\frac{1}{2})}(t)\right) \\ + f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)\left(\sqrt{a}bqta_{j-1}^{(n+\frac{1}{2})}(t) - \sqrt{a}g_{n+\frac{1}{2}}(t)a_{j-3}^{(n+\frac{1}{2})} + tg_{n+\frac{1}{2}}(t)^2c_{j-2}^{(n+\frac{1}{2})}(t)\right) \end{aligned} \tag{3.119b}$$

and $a_1^{(n+\frac{1}{2})}(t) = \mu c_0^{(n+\frac{3}{2})}(t)$. *We also have*

$$t_{0,n}(t) = \frac{(bt^2 - q^{\frac{1}{2}}\sum_{j=1}^{(n/2)} p_{(4j-3)/2}(t))i^n}{bt^2} \tag{3.120}$$

with $t_{0,0}(t) = 1$ *and*

$$p_k(t) = -\frac{btq^k\left(q^{2k+1}-1\right)}{g_k(t)\left(f_k(t)(qg_k(t)+bt) + bq^{2k+1}\right)}. \tag{3.121}$$

$c_n^{(\frac{1}{2})}$ *satisfies both*

$$c_n^{(\frac{1}{2})}(t) = -\frac{q^n\left(t^2\left(\left(b^2+1\right)c_{n-2}^{(\frac{1}{2})}(t)+c_{n-2}^{(\frac{1}{2})}(qt)\right)+c_{n-4}^{(\frac{1}{2})}(t)\right)}{b^2t^4\left(q^n-1\right)} \tag{3.122}$$

$$b^2q^2t^2c_n^{(\frac{1}{2})}(t) = \frac{-q^2c_{n-4}^{(\frac{1}{2})}(t)}{t^2}+q^2t^2\left(\left(b^2\left(t^2-1\right)-1\right)c_n^{(\frac{1}{2})}(qt)-c_n^{(\frac{1}{2})}\left(q^2t\right)\right)$$
$$+\left(-b^2-1\right)q^2c_{n-2}^{(\frac{1}{2})}(t)-\left(q^2+1\right)c_{n-2}^{(\frac{1}{2})}(qt) \tag{3.123}$$

with

$$c_0^{(\frac{1}{2})}\left(q^2t\right)+\left(1-b^2\left(t^2-1\right)\right)c_0^{(\frac{1}{2})}(qt)+b^2c_0^{(\frac{1}{2})}(t)=0 \tag{3.124}$$

and $c_j^{(\frac{1}{2})}=0$, $j<0$.

Proof. Recall

$$\phi_1^{(n+\frac{1}{2})}(x,t) = b^\lambda x^{-\frac{1}{2}}L_{n-\frac{1}{2}}L_{n-\frac{3}{2}}\dots L_{\frac{1}{2}}TS\sum_{j=0}^{\infty}\begin{pmatrix}\frac{c_{2j}^{(\frac{1}{2})}(qt)}{x^{2j}}\\ \frac{c_{2j}^{(\frac{1}{2})}(t)}{x^{2j}}\end{pmatrix}.$$

where $\lambda=\frac{\ln x}{\ln q}$. We observe that

$$L_{\frac{1}{2}}TS=\begin{pmatrix}s_2x^2 & t_0+t_2x^2\\ s_1x & \frac{t_{-1}}{x}+t_1x\end{pmatrix}$$

$$L_{\frac{3}{2}}L_{\frac{1}{2}}TS=\begin{pmatrix}s_1x+s_3x^3 & \frac{t_{-1}}{x}+t_1x+t_3x^3\\ s_2x^2 & t_0+t_2x^2\end{pmatrix}$$

$$L_{\frac{5}{2}}L_{\frac{3}{2}}L_{\frac{1}{2}}TS=\begin{pmatrix}s_2x^2+s_4x^4 & t_0+t_2x^2+t_4x^4\\ s_1x+s_3x^3 & \frac{t_{-1}}{x}+t_1x+t_3x^3\end{pmatrix}$$

$$L_{\frac{7}{2}}L_{\frac{5}{2}}L_{\frac{3}{2}}L_{\frac{1}{2}}TS=\begin{pmatrix}s_1x+s_3x^3+s_5x^5 & \frac{t_{-1}}{x}+t_1x+t_3x^3+t_5x^5\\ s_2x^2+s_4x^4 & t_0+t_2x^2+t_4x^4\end{pmatrix}$$

where we have labelled the coefficients of x^j in the first column $s_j(t)$ and those in the second column $t_j(t)$. However, due to the structure of the Schlesinger equation, we find for odd n

$$t_{-1}(t)=t_{-1,n}(t)=\frac{q^{\frac{1}{2}}}{bt^2}i^{n-1}$$

and for even n

$$t_0(t)=t_{0,n}(t)=\frac{(bt^2-q^{\frac{1}{2}}\sum_{j=1}^{(n/2)}p_{(4j-3)/2}(t))i^n}{bt^2}$$

with $t_{0,0}(t) = 1$. If n is even we have

$$\phi_1^{(n+\frac{1}{2})} =$$

$$b^\lambda x^{-\frac{1}{2}} \begin{pmatrix} s_{n+1}x^{n+1} + \cdots + s_3x^3 + s_1x & t_{n+1}x^{n+1} + \cdots + t_1x + \frac{t_{-1}}{x} \\ s_nx^n + \cdots + s_4x^4 + s_2x^2 & t_nx^n + \cdots + t_2x^2 + t_0 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} \frac{\bar{c}_{2j}^{(\frac{1}{2})}}{x^{2j}} \\ \frac{c_{2j}^{(\frac{1}{2})}}{x^{2j}} \end{pmatrix} \tag{3.126}$$

where we write $c_{2j}^{(\frac{1}{2})}(qt) = \bar{c}_{2j}^{(\frac{1}{2})}$ for simplicity. We also know from (3.73)

$$\phi_1^{(n+\frac{1}{2})} \sim b^\lambda x^{-(n+\frac{1}{2})} \begin{pmatrix} a_1^{(n+\frac{1}{2})}(t)/x \\ c_0^{(n+\frac{1}{2})}(t) \end{pmatrix}. \tag{3.127}$$

Equating (3.126) and (3.127) we see the following. First, from the top row we get $n+3$ equations:

$$\begin{aligned}
x^{n+1} &: t_{n+1}c_0^{(\frac{1}{2})} + s_{n+1}\bar{c}_0^{(\frac{1}{2})} = 0 \\
x^{n-1} &: t_{n+1}c_2^{(\frac{1}{2})} + s_{n+1}\bar{c}_2^{(\frac{1}{2})} + t_{n-1}c_0^{(\frac{1}{2})} + s_{n-1}\bar{c}_0^{(\frac{1}{2})} = 0 \\
&\vdots : \vdots \\
x^{-n+1} &: t_{n+1}c_{2n}^{(\frac{1}{2})} + s_{n+1}\bar{c}_{2n}^{(\frac{1}{2})} + \cdots + t_1c_n^{(\frac{1}{2})} + s_1\bar{c}_n^{(\frac{1}{2})} + t_{-1}c_{n-2}^{(\frac{1}{2})} = 0 \\
x^{-n-1} &: t_{n+1}c_{2n+2}^{(\frac{1}{2})} + s_{n+1}\bar{c}_{2n+2}^{(\frac{1}{2})} + \cdots + t_1c_{n+2}^{(\frac{1}{2})} + s_1\bar{c}_{n+2}^{(\frac{1}{2})} + t_{-1}c_n^{(\frac{1}{2})} = a_1^{(n+\frac{1}{2})} \\
x^{-n-3} &: t_{n+1}c_{2n+4}^{(\frac{1}{2})} + s_{n+1}\bar{c}_{2n+4}^{(\frac{1}{2})} + \cdots + t_1c_{n+4}^{(\frac{1}{2})} + s_1\bar{c}_{n+4}^{(\frac{1}{2})} + t_{-1}c_{n+2}^{(\frac{1}{2})} = a_3^{(n+\frac{1}{2})}.
\end{aligned}$$

Now, from the bottom row we obtain $n+1$ equations:

$$\begin{aligned}
x^{n} &: t_nc_0^{(\frac{1}{2})} + s_n\bar{c}_0^{(\frac{1}{2})} = 0 \\
x^{n-2} &: t_nc_2^{(\frac{1}{2})} + s_n\bar{c}_2^{(-\frac{1}{2})} + t_{n-2}c_2^{(\frac{1}{2})} + s_{n-2}\bar{c}_0^{(\frac{1}{2})} = 0 \\
&\vdots : \vdots \\
x^{-n+1} &: t_nc_{2n-2}^{(\frac{1}{2})} + s_n\bar{c}_{2n-2}^{(\frac{1}{2})} + \cdots + t_2c_n^{(\frac{1}{2})} + s_2\bar{c}_n^{(\frac{1}{2})} + t_0c_{n-2}^{(\frac{1}{2})} = 0 \\
x^{-n} &: t_nc_{2n}^{(\frac{1}{2})} + s_n\bar{c}_{2n}^{(\frac{1}{2})} + \cdots + t_2c_{n+2}^{(\frac{1}{2})} + s_2\bar{c}_{n+2}^{(\frac{1}{2})} + t_0c_n^{(\frac{1}{2})} = c_0^{(n+\frac{1}{2})}.
\end{aligned}$$

The equations derived from the top row give an $(n+3)\times(n+3)$ matrix form.

$$\begin{pmatrix} c_{n+2}^{(\frac{1}{2})} & \bar{c}_{n+4}^{(\frac{1}{2})} & c_{n+4}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n+4}^{(\frac{1}{2})} & c_{2n+4}^{(\frac{1}{2})} \\ c_{n}^{(\frac{1}{2})} & \bar{c}_{n+2}^{(\frac{1}{2})} & c_{n+2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n+2}^{(\frac{1}{2})} & c_{2n+2}^{(\frac{1}{2})} \\ c_{n-2}^{(\frac{1}{2})} & \bar{c}_{n}^{(\frac{1}{2})} & c_{n}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n}^{(\frac{1}{2})} & c_{2n}^{(\frac{1}{2})} \\ c_{n-4}^{(\frac{1}{2})} & \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & & \bar{c}_{0}^{(\frac{1}{2})} & c_{0}^{(\frac{1}{2})} \end{pmatrix} \begin{pmatrix} t_{-1} \\ s_1 \\ t_1 \\ \vdots \\ s_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} a_3^{(n+\frac{1}{2})} \\ a_1^{(n+\frac{1}{2})} \\ \vdots \\ 0 \end{pmatrix} \tag{3.128}$$

and from the bottom row we also obtain an $(n+1)\times(n+1)$ matrix form

$$\begin{pmatrix} c_{n}^{(\frac{1}{2})} & \bar{c}_{n+2}^{(\frac{1}{2})} & c_{n+2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n}^{(\frac{1}{2})} & c_{2n}^{(\frac{1}{2})} \\ c_{n-2}^{(\frac{1}{2})} & \bar{c}_{n}^{(\frac{1}{2})} & c_{n}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & & \bar{c}_{0}^{(\frac{1}{2})} & c_{0}^{(\frac{1}{2})} \end{pmatrix} \begin{pmatrix} t_0 \\ s_2 \\ t_2 \\ \vdots \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} c_0^{(n+\frac{1}{2})} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.129}$$

We then define the following determinants:

$$\Lambda_n(t) = \begin{vmatrix} c_{n-1}^{(\frac{1}{2})} & \bar{c}_{n+1}^{(\frac{1}{2})} & c_{n+1}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ c_{n-3}^{(\frac{1}{2})} & \bar{c}_{n-1}^{(\frac{1}{2})} & c_{n-1}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & & \bar{c}_{0}^{(\frac{1}{2})} & c_{0}^{(\frac{1}{2})} \end{vmatrix},$$

$$\tau_n(t) = \begin{vmatrix} \bar{c}_{n}^{(\frac{1}{2})} & c_{n}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-2}^{(\frac{1}{2})} & c_{2n-2}^{(\frac{1}{2})} \\ \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \bar{c}_{0}^{(\frac{1}{2})} & c_{0}^{(\frac{1}{2})} \end{vmatrix}$$

and

$$M_n(t) = \begin{vmatrix} \bar{c}_{n+2}^{(\frac{1}{2})} & c_{n+2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n}^{(\frac{1}{2})} & c_{2n}^{(\frac{1}{2})} \\ \bar{c}_{n-2}^{(\frac{1}{2})} & c_{n-2}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-4}^{(\frac{1}{2})} & c_{2n-4}^{(\frac{1}{2})} \\ \bar{c}_{n-4}^{(\frac{1}{2})} & c_{n-4}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-6}^{(\frac{1}{2})} & c_{2n-6}^{(\frac{1}{2})} \\ \bar{c}_{n-6}^{(\frac{1}{2})} & c_{n-6}^{(\frac{1}{2})} & & \dots & \bar{c}_{2n-8}^{(\frac{1}{2})} & c_{2n-8}^{(\frac{1}{2})} \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & & \bar{c}_0^{(\frac{1}{2})} & c_0^{(\frac{1}{2})} \end{vmatrix},$$

where after the 2nd row, all subsequent rows' subscripts decrease by 2.

Applying Cramer's rule to (3.128) we find

$$t_{-1}(t) = \frac{a_3^{(n+\frac{1}{2})}(t)\tau_{n+2}(t) - M_{n+2}(t)a_1^{(n+\frac{1}{2})}(t)}{\Lambda_{n+3}(t)}$$

or after rearranging:

$$a_1^{(n+\frac{1}{2})}(t) = \frac{t_{-1}(t)\Lambda_{n+3}(t) - a_3^{(n+\frac{1}{2})}(t)\tau_{n+2}(t)}{-M_{n+2}(t)}. \tag{3.130}$$

As $c_0^{(k+1)}(t) = a_1^{(k)}(t)$ we can write (3.130) as

$$c_1^{(n+1+\frac{1}{2})}(t) = \frac{t_{-1}(t)\Lambda_{n+3}(t) - a_3^{(n+\frac{1}{2})}(t)\tau_{n+2}(t)}{-M_{n+2}(t)}.$$

For even n, $n+1$ is odd and so we can now write, for odd n

$$c_1^{(n+\frac{1}{2})}(t) = \frac{t_{-1}(t)\Lambda_{n+2}(t) - a_3^{(n-\frac{1}{2})}(t)\tau_{n+1}(t)}{-M_{n+1}(t)}.$$

Also, recall once more

$$\frac{c_0^{(n+\frac{1}{2})}(qt)}{c_0^{(n+\frac{1}{2})}(t)} = bf_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t)q^{-2n-1}$$

such that for even n we obtain

$$f_{n+\frac{1}{2}}(t)g_{n+\frac{1}{2}}(t) = q^{2n+1}\frac{M_{n+1}(t)\left(t_{-1,n}(qt)\Lambda_{n+2}(qt) - \tau_{n+1}(qt)a_3^{(n-\frac{1}{2})}(qt)\right)}{bM_{n+1}(qt)\left(t_{-1,n}(t)\Lambda_{n+2}(t) - \tau_{n+1}(t)a_3^{(n-\frac{1}{2})}(t)\right)}. \tag{3.131}$$

$a_3^{(n-\frac{1}{2})}(t)$ can be expressed in terms of $c_0^{(\frac{1}{2})}(t)$ through applications of (3.12a), (3.12c) and $c_0^{(k+1)}(t) = a_1^{(k)}(t)$.

We now apply Cramer's rule to (3.129), producing

$$c_0^{(n+\frac{1}{2})} = t_0 \frac{\Lambda_{n+1}}{\tau_n}$$

such that for even n we have

$$f_{n+\frac{1}{2}}(t) g_{n+\frac{1}{2}}(t) = q^{2n+1} \frac{t_{0,n}(qt)\Lambda_{n+1}(qt)\tau_n(t)}{bt_{0,n}(t)\Lambda_{n+1}(t)\tau_n(qt)}.$$

We now define $\Pi_{n+\frac{1}{2}}(t) := f_{n+\frac{1}{2}}(t) g_{n+\frac{1}{2}}(t)$ for simplicity in the following manipulations. We now write (3.71a) with $\Pi_{n+\frac{1}{2}}(t)$ where possible and multiply (3.71b) on the left and right by $g_{n+\frac{1}{2}}(qt)$, to give

$$g_{n+\frac{1}{2}}(qt) = \frac{a_{n+\frac{1}{2}}(tf_{n+\frac{1}{2}}(t)+1)}{(f_{n+\frac{1}{2}}(t)+t)\Pi_{n+\frac{1}{2}}(t)} \tag{3.132}$$

$$\Pi_{n+\frac{1}{2}}(qt) = \frac{a_{n+\frac{1}{2}}(btg_{n+\frac{1}{2}}(qt)+1)}{f_{n+\frac{1}{2}}(t)(g_{n+\frac{1}{2}}(qt)+bt)}. \tag{3.133}$$

Substituting (3.132) into (3.133) then gives us the following condition on $f_{n+\frac{1}{2}}(t)$

$$\begin{aligned}
&tf_{n+\frac{1}{2}}(t)^2 \Pi_{n+\frac{1}{2}}(qt)\left(a_{n+\frac{1}{2}} + b\Pi_{n+\frac{1}{2}}(t)\right) + \\
&f_{n+\frac{1}{2}}(t)\left(-a_{n+\frac{1}{2}}\left(bt^2 a_{n+\frac{1}{2}} + \Pi_{n+\frac{1}{2}}(t)\right) + a_{n+\frac{1}{2}}\Pi_{n+\frac{1}{2}}(qt) + bt^2\Pi_{n+\frac{1}{2}}(t)\Pi_{n+\frac{1}{2}}(qt)\right) \\
&- ta_{n+\frac{1}{2}}\left(ba_{n+\frac{1}{2}} + \Pi_{n+\frac{1}{2}}(t)\right) = 0
\end{aligned}$$

which is quadratic in $f_{n+\frac{1}{2}}(t)$. We can then find $g_{n+\frac{1}{2}}(t)$ via $g_{n+\frac{1}{2}}(t) = \frac{\Pi_{n+\frac{1}{2}}(t)}{f_{n+\frac{1}{2}}(t)}$. □

We find the first couple of special solutions to be

$$f_{\frac{1}{2}}(t) = -\frac{tc_0(qt)}{c_0(qt)+c_0(t)} \qquad g_{\frac{1}{2}} = -\frac{q\left(c_0(qt)+c_0(t)\right)}{btc_0(t)} \tag{3.134}$$

$$f_{\frac{3}{2}}(t) = -\frac{q^2\left(c_0(qt)+c_0(t)\right)\left(\left(1-b^2\left(t^2-1\right)\right)c_0(t)c_0(qt)+b^2c_0(t)^2+c_0(qt)^2\right)}{tc_0(qt)\left(c_0(t)\left(q^2-b^2\left(t^2-1\right)\right)c_0(qt)+b^2c_0(t)^2+q^2c_0(qt)^2\right)} \tag{3.135}$$

$$g_{\frac{3}{2}}(t) = \frac{bqtc_0(t)\left(c_0(t)\left(b^2\left(t^2-1\right)-q^2\right)c_0(qt)-b^2c_0(t)^2-q^2c_0(qt)^2\right)}{\left(c_0(qt)+c_0(t)\right)\left(c_0(t)\left(\left(b^2+1\right)q^2-b^2t^2\right)c_0(qt)+b^2q^2c_0(t)^2+q^2c_0(qt)^2\right)}. \tag{3.136}$$

3.7. Conclusion

In this Chapter we presented our original results on developing the determinantal forms of both the rational and Riccati type special solutions for the asymmetric qP_{III} (3.71) with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2+A_1)^{(1)}$. Using the linear problem (3.2,3.3) presented in [**30**], we developed a series expansion around one of the singularities of the system $x = \infty$. By developing the Schlesinger transformation (3.83) for qP_{III}, we were able to represent all the solutions of the linear problem for when qP_{III} had rational or Riccati type solutions in terms of the simplest series expansion for the respective hierarchies. This allowed us to obtain the determinantal forms for both the rational and Riccati type special solutions of qP_{III}.

In the following conclusion to this thesis, we further expand on the results given here, and consider the implications of these results on future study.

CHAPTER 4

Conclusion

This thesis aimed to develop the determinantal forms of the rational and Riccati type special solutions of qP_{III} (1.18) with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2+A_1)^{(1)}$. This is the third Painlevé equation, continuous or discrete, to possess determinantal forms for their special solutions developed via its associated linear problem, the first two being P_{II} [**10**] and a discrete analogue of P_{II} with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2+A_1)^{(1)}$ [**31**, **32**].

In Chapter 2 we reviewed [**10**] and [**32**] to demonstrate how the linear problem of P_{II} and qP_{II} respectively could be used to develop the determinantal forms of the rational solutions for P_{II} and the q-Hypergeometric solutions for qP_{II}. In both cases it was essential to develop series expansions of the linear problem around the singularities of the system. While in [**10**] relating the two series expansions together was enough to develop the determinantal form, in [**32**] it was necessary to develop a Schlesinger transformation and observe how it acted on the series expansions of the linear problem for sequential q-Hypergeometric special solutions.

Chapter 3 contained our investigations into qP_{III}. After producing a series expansion around a singularity of the linear problem, $x=\infty$, we examined this expansion for the simplest rational and Riccati type special solutions of qP_{III}, we obtained

$$\begin{aligned}\phi_1^{(0)}(x,t) &= \frac{b^\lambda}{2}\begin{pmatrix}-1 & 1\\ i & i\end{pmatrix}\begin{pmatrix}\Gamma_{\frac{1}{q}}(1-\frac{i}{qx})\Gamma_{\frac{1}{q}}(1-\frac{i}{tx})\Gamma_{\frac{1}{q}}(1-\frac{i}{btx})\\ \Gamma_{\frac{1}{q}}(1+\frac{i}{qx})\Gamma_{\frac{1}{q}}(1+\frac{i}{tx})\Gamma_{\frac{1}{q}}(1+\frac{i}{btx})\end{pmatrix}\\ &= b^\lambda\sum_{j=0}^{\infty}\begin{pmatrix}T_{2j+1}(t)x^{-(2j+1)}\\ iT_{2j}(t)x^{-2j}\end{pmatrix}\end{aligned}$$

where $\lambda=\frac{\ln x}{\ln q}$ and

$$\phi_1^{(\frac{1}{2})}(x,t) = b^\lambda x^{-\frac{1}{2}}\begin{pmatrix}\frac{\sqrt{q}x}{b} & \frac{bx^2g_{\frac{1}{2}}(t)t^3+qx^2t^2+q}{b\sqrt{q}t^2x}\\ 0 & 1\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}\frac{c_{2j}(qt)}{x^{2j}}\\ \frac{c_{2j}(t)}{x^{2j}}\end{pmatrix}$$

respectively. We were then able to construct a Schlesinger transformation, such that

$$\Psi^{(k+1)}(x,t) = L_k(x,t)\Psi^{(k)}(x,t)$$

where

$$L_k(x,t) = \begin{pmatrix} -\frac{btq^k\left(q^{2k+1}-1\right)x}{g_k(t)\left(f_k(t)(qg_k(t)+bt)+bq^{2k+1}\right)} & -1 \\ 1 & 0 \end{pmatrix}.$$

This allowed us to express $\phi_1^{(k+1)}$ in terms of the series expansions for the simplest rational and Riccati type solutions, such that

$$\phi_1^{(k+1)}(x,t) = L_k L_{k-1} \dots L_0 \phi_1^{(0)}(x,t)$$

for integer k and

$$\phi_1^{(k+1)}(x,t) = L_k L_{k-1} \dots L_{\frac{1}{2}} \phi_1^{(\frac{1}{2})}(x,t)$$

for half integer k. This Schlesinger transformation was of a suitable form to develop the determinantal forms for the rational and Riccati type special solutions of qP_{III}, despite the latter being unusually convoluted. These determinants depended exclusively on the series coefficients of the simplest rational or Riccati type solutions respectively. For example in the rational case we have

$$\tau_k(x) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & T_2 & T_3 \\ 0 & \dots & \dots & T_0 & T_1 \end{vmatrix}.$$

It is these results, given by Theorems (3.1 and (3.2), that make qP_{III} (1.18) the third Painlevé equation, continuous or discrete, to possess a determinantal representation for its special solutions, derived from its associated linear problem.

However, different determinantal form solutions have previously been discovered by Kajiwara in [**33**,**34**], by an alternate method. The determinant structures in [**33**, **34**] are different to ours and currently general relations between these structures and the determinants of this thesis are not known. However, as we would expect, we can still observe for at least one case, that the solutions obtained from these different structures are identical. Looking at the rational solutions, Kajiwara, denoting the solutions as $f_0(z;y,q^N)$ and $f_1(z;y,q^N)$ for parameters y and N (f_0 and f_1 are just the function values, like our f and g), finds the first non-trivial rational solution, in his

notation to be:

$$f_0(z;y,q) = \frac{qz+q+y}{y+z+1} \tag{4.1}$$

$$f_1(z;y,q) = \frac{q(y+z+1)}{q+y+z}. \tag{4.2}$$

If we rename z to be t, and then send $y \to q$ we obtain

$$f_0(t) = \frac{q(t+2)}{q+t+1} \tag{4.3}$$

$$f_1(t) = \frac{q(q+t+1)}{2q+t} \tag{4.4}$$

which is precisely what we obtain for our $f_1(t)$ and $g_1(t)$ via either the Bäcklund transformation or determinantal form for $b = 1$. This would correspond to the parameter $\nu = 0$ in [**33**].

One question that needs to be asked is why is it P_{II} in addition to qP_{II} and qP_{III}, the latter both with rational surface $A_5^{(1)}$ and affine Weyl group $(A_2{+}A_1)^{(1)}$, that have determinantal forms for their special solutions, developed via the associated linear problem? Are all three cases completely coincidental with no unifying approach for all the Painlevé equations? Should we conversely expect the determinantal forms for special solutions of all the Painlevé equations to be developed from their linear problem? Or perhaps this shared property of these three equations is an emergent property of their affine Weyl group $(A_2 + A_1)^{(1)}$, with P_{II} then inheriting the properties as it is simply the continuum limit of qP_{II}?

As outlined in Chapter 1, the determinantal forms of many special solutions have been found, primarily through methods different to our own, however the methods highlighted in this thesis continue to demonstrate their potential as a powerful tool for analysing the Painlevé and discrete Painlevé equations. Further pursuit of the linear problem as a tool for studying these, or perhaps higher order equations, may further indicate its potential and shed some light on these questions.

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