All Spun Out

Limits of aerial techniques when performing somersaults

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BE in Mechanical Engineering (Biomedical) (Hons. I and Medal)

A thesis submitted in fulfilment of the requirements of the degree of Doctor of Philosophy

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Submitted December 2015
Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the University or other institute of higher learning, except where due acknowledgement has been made in the text.

Joanne Mikl

30 December 2015
Abstract

The somersault is a key skill in gymnastics and diving. Almost all the rotation required must be performed while the athlete is airborne; whilst airborne the athlete’s angular momentum is constant. The postures chosen, and any postural change that occurs while airborne, will determine the rotation achieved by the athlete.

Equations are derived that describe the possible rotational states in terms of the somersault and twist rotation, thereby determining which rotational states are possible and useful for performing somersaults and twisting somersaults. Equations describing the results of idealised postural changes intended to initiate twist in a somersault are also derived.

Inertial property data was both collated from the literature and estimated from measurements of current athletes. The data thus represented a range of ‘possible athletes’ which were applied to the derived equations of motion to predict which skills are achievable using different postures and actions.

Recommendations were made as to the ‘best’ twist initiation actions and postures to use for different somersault skills. For twisting somersaults it was shown that previously published aerial techniques, and slight variations of these, are inadequate to allow the majority of athletes to achieve the highest numbers of twists per somersault observed in current international competition. It was concluded that contact twist, or aerial techniques yet to be mathematically described, must be used.

Based on the predictions of skills achievable it was clear that some athletes have a natural advantage over others. Further, it was found that the postures or techniques which were the ‘best’ for one athlete were not necessarily the ‘best’ for all athletes. Differences in predicted skill achievement and which posture or technique was most suitable varied with the gender and squad (related to age and years of training) but these categories did not explain all of the variation.
Acknowledgements

Completing a PhD is a long journey. I would like to thank all those who aided me on my journey.

Thanks to my family for understanding my choice to return to university and then supporting and encouraging me throughout the journey.

Thanks to Holger Dullin for being my first contact with the “Bodies in space” project. After discussing initial project ideas, I made the choice to start my PhD in the area of biomechanics and in particular researching the somersault. Thanks also to Holger for the various discussions along the way.

I acknowledge the support I received through the funding provided through ARC Linkage Project grant LP100200245.

Thanks to the NSW Institute of Sport for allowing me to invite their athletes to participate in my studies and providing access to facilities in which to conduct my studies.

Thanks to David Rye for becoming my principal supervisor from the second semester of my PhD. David was a great help when it came to navigating the university systems, as well as the numerous “curve balls” life threw me during my candidature.

Thanks to Peter Sinclair for always being willing to assist in any way he could; in particular, for assisting with measurement set ups, providing spreadsheets to aid smoothing of data, and commenting on drafts of my work from a sport science perspective.

Thanks to Cherie Walker for always being happy to lend a helping hand, and being someone who understood the project’s ups and downs, and who was ever ready to have a laugh about it all.
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## Glossary

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<tr>
<td>Abduction (of a segment)</td>
<td>Movement of the body segment away from the midline of the body or body part to which it is attached</td>
</tr>
<tr>
<td>Adduction (of a segment)</td>
<td>Movement of the body segment toward the midline of the body or body part to which it is attached</td>
</tr>
<tr>
<td>Allometric equation</td>
<td>An allometric equation is one in the form $Y=aX^b$, where $a$ and $b$ are scaling factors (Zatsiorsky, 2002).</td>
</tr>
<tr>
<td>Anatomical position</td>
<td>The posture in which a person is standing upright with feet and palms facing forwards.</td>
</tr>
<tr>
<td>Angular momentum $H$</td>
<td>Angular momentum of a differential piece of mass about a point is defined as the cross product of the vector from that point to a piece of mass and its linear momentum (mass times velocity). The angular momentum of the body is then the ‘sum’ of all these differential masses over the body. i.e. $H = \int_{body} r \times v, dm$ Angular momentum is constant when there no external forces or moments act on the body.</td>
</tr>
<tr>
<td>Anterior</td>
<td>Toward the front of the body</td>
</tr>
<tr>
<td>Arch</td>
<td>Posture such that the hips are hyper-extended: the angle between the front of the body and the front of the thighs exceeds $180^\circ$.</td>
</tr>
<tr>
<td>Axes and planes of the body</td>
<td>There are three axes (Longitudinal, Transverse, and Medial) and three planes (Frontal, Sagittal, and Transverse) of the body. These</td>
</tr>
</tbody>
</table>
are illustrated in the sketch below.

From Dyson (1973). Figure 73

Awarded (in terms of judging) The skill performed was identified by the judges and the corresponding difficulty value added to the athlete’s score.

Body link A straight line that extends from one joint to the next through a body segment. The link is an “idealised” segment (Zatsiorsky, 2002; Dempster, 1955). Bones form a rigid support for a segment of the body but they are not links themselves. The bones may be shorter or longer than the link depending on the position of the points of rotation (Leveau, 1992).

Body segment A part of the body that is assumed to be rigid for the purposes of modelling. Segments are connected to each other by joints. The complexity of the model determines which segments are used. For example, one model may consider the arm as one segment, while a more detailed model may consider the arm as consisting of three segments: the upper arm, lower arm, and hand.

Categorical The data that can be divided into categories.

Code of points The document which specifies the judging criteria and competition rules, lists all the skills defined in the sport by apparatus, and
specifies the difficulty ranking or the points awarded for performing each skill.

Common Language Effect size = CLES

The “probability that a score sampled at random from one distribution will be greater than a score sampled from another” e.g. males and females differ in height by an effect size of 2. This translates to a CLES of 0.92, or 92 out of 100 blind dates will mean that the male is taller than the female (Coe, 2002). The dominance statistic, d, (Cliff, 1993) relates to the CLES since it is the difference in CLES for each order of the pair. So for the 100 blind dates d is 0.92−0.08 = 0.84. CLES = (d+1)/2

Constraint

A constraint is something that limits or prevents certain types of motion. Constraints may be classed as: Positional, also known as geometrical or holonomic or Kinematic, also known as non-holonomic (Josephs & Huston, 2002)

Degree of difficulty

The difficulty score given to a particular skill. The higher the score, the harder the skill is considered to be.

Depth (of a pike)

A widely used description of the amount of forward flexion in a piked posture. A deep pike is one with a large amount of forward flexion, while a shallow pike has only a small amount of forward flexion. The words deep and shallow are not precise mathematical concepts.

Digitization (of footage)

In a sports science context digitization of footage refers to extracting image coordinates of points of interest from an image.

Direction cosines

The magnitude of a vector in a particular direction may be determined by the cosine of the angle between the vector and the specified direction. When the directions of interest are the x, y, and z axes then the magnitude of the components of a vector along each of the axes as determined by the cosines of the angles between the vector and the axes are known as the direction cosines. The angles between each of the axes and the vector can specify the position of the vector.

Distal

Farther from the point of attachment to the torso. Distal does not have a clear usage with the torso; however in this thesis it will be
taken to mean farther away from the reference segment (the pelvis).

**Dominance (d)** The dominance statistic, d, is the proportion of all possible pairs created between two sample groups, A & B, with the A item greater than the B item minus the reverse situation. It is a non-parametric and distribution free measure of effect size (Cliff, 1993). It has a value between -1 (all B is > A) and 1 (all A is > B). When equal numbers of pairs have A > B then d is 0.

**Easier** In this thesis a posture will be considered easier than another posture if, for the same athlete and in the same time, to achieve the same somersault rotation will require less angular momentum.

**Effect size** The effect size is the number of standard deviations difference between the means of two sample groups. The choice of which standard deviation to use depends on the sample groups. For more information see Coe (2002). It is the same as a Z-score of a standard normal distribution. The effect size thus only has meaning when comparing groups from normal distributions.

**Extension** An action that increases the angle between two segments of the body. It can be viewed as straightening the body. It is the opposite of flexion.

**Flexion** Action that decreases the angle between two segments of the body.

**Iliocristale** The point on the iliac crest aligned with the midline of the torso.

**Inertial frame** A frame that is either stationary or translating with constant velocity. Hence Newton’s laws of motion apply.

If the frame chosen is not inertial then forces to account for the effect of an accelerating frame need to be applied to the body.

**Inward somersault** A forward somersault performed after a jump backwards. Divers typically stand at the end of the board or platform and jump backwards while throwing into a front somersault which rotates towards the board.

**Kick-out** When the athlete changes shape quickly and decisively from a tuck.
or pike. It is used for aesthetic reasons or to quickly slow rotation and avoid over-rotation when practicing the entry to a multiple somersault skill when not actually wishing to complete the multiple rotations. It is also used to refer to a technique to initiate twist where part of the motion involves the athlete extending from a tuck or pike.

**Lateral**
Away from the midline of the body

**Lateral rotation**
Turning away from the midline of the body

**Layout**
For the purposes of awarding the difficulty assigned to a layout, the posture required is generally one where the knee-hip-shoulder angle (measured from the front of the body) is greater than 135 degrees, and the legs are straight. Even though the legs are expected to also be straight, bent legs while maintaining the knee-hip-shoulder angle would attract a deduction rather than the skill not being called a layout. *Note:* The judging criteria for each sport will specify how long the layout posture must be held for the skill to be deemed a layout.

**Learning progressions**
Exercises or skills used as intermediate steps to learning a skill. Learning progressions may break the skill down into components and teach each separately and then gradually combine them. They may also be other similar, but easier skills.

**Longitudinal axis**
An axis of symmetry closest to the head-to-toe axis for the whole body or the local segment axis aligned with the whole body longitudinal axis when standing in the anatomical position.

**Longitudinal principal axis**
The principal axis closest to the longitudinal axis of the body. In this context closest means the principal axis with the largest component in the direction of the body axis.

**Medial**
Towards the midline of the body

**Medial axis**
The axis pointing forwards. Anterior-posterior axis is also used except that the positive direction is now backwards.

**Medial principal**
The principal axis closest to the medial axis of the body. In this context closest means the principal axis with the largest component
axis in the direction of the body axis.

Noise An unwanted disturbance to a signal or observation. It makes the signal harder to distinguish.

Open (posture) An open posture is one where there is a moderate amount of space between body segments and the overall moment of inertia is large. E.g. an open pike is a posture where the hip angle is larger than that desired in a pike, but it is not large enough to be called a layout.

Open out Movement from a tuck or pike posture into a layout posture.

Over-rotated The athlete has completed more somersault than necessary, often causing them to step or fall in the horizontal direction of travel.

Orientation Orientation describes the angular measure of a body from a specific position. To determine orientation, a reference frame for the body whose orientation is to be described, and a fixed world frame that the orientation will be determined with respect to, need to be defined. The angles between corresponding axes in these two frames describe the orientation. This is true of a rigid body and a non-rigid body.

P-value The probability of the null hypothesis being true.

Pancaking A term used to describe an increase in tilt of the longitudinal axis of the body off a vertical axis in a twisting somersault as the somersault inverts.

Peaking A term used to describe positioning of posture changes or timing the full extensions for the greatest aesthetic effect. For example, completing as much rotation as possible centred around the apex of a somersault’s trajectory through changing one’s shape, or having a strong decisive ‘kick-out’ of a tuck somersault.

Pike Posture such that the knee-hip-shoulder angle is less than 135 degrees and the legs are straight.

Note: In twisting pike somersaults the pike position is shown in the non-twisting phases, while a layout position is generally used in
the twisting phases.

**Posture**  
A description of the relative position of the various body segments with respect to each other. In models of rigid segments connected by rotating joints the posture is fully defined by the joint angles.

Tuck, puck, pike, and layout are general terms that describe specific postures of interest observed during somersaults.

Any person can hold the same posture, but their shape and so inertial properties in that posture differ, since their body proportions differ.

**Posterior**  
Towards the back of the body.

**Practically significant**  
An observation is practically significant if it makes a difference which is deemed to be important, in the application to which it is applied. Some differences may be statistically significant, yet not practically significant. Practical significance is subjective, although may be linked to other outcomes to justify one's choice of what is practically significant. E.g. a value less than 5° may be practically insignificant to one person due to belief about what angles could be perceived by the judge’s eye, yet to another it is practically significant as it changes the ‘feel’ for the athlete and the way forces travel through the body.

**Principal Axes or directions**  
Cartesian axes where the tensor of inertia with respect to these axes is a diagonal matrix. If the body rotates about one of these axes it will continue to do so indefinitely with an angular velocity equal to the angular momentum divided by the moment of inertia for that axis: the rotation can be represented by planar rotation.

**Proximal**  
Closer to point of attachment to the torso. Proximal does not have a clear usage with the torso; however, in this thesis it will be taken to mean closer to the reference segment (the pelvis).

**Puck**  
A posture between a tuck and a pike; it is allowed in trampolining gymnastics for multiple somersaults with twist. It is a compromise posture between having a small moment of inertia about the longitudinal or transverse axis. The posture is given as a picture
only in the CoP TRAMP (2009)

Quasi-  Is not but appears to be, and behaves sufficiently like for modelling, the item after the hyphenation. E.g. Quasi-rigid has been used to describe the behaviour of the body when an athlete holds a posture and behaves similarly to a rigid body.

Radiale  The proximal and lateral border of the radius

Reference segment  A segment of the body whose motion and orientation is of particular interest. The positions and orientations of the rest of the body’s segments are defined relative to this segment. In this thesis the reference segment used is the pelvis.

Reverse somersault  A backward somersault performed after a forward jump: the athlete rotates backwards while they travel forwards.

Rigid body  A body that does not change shape or deform regardless of the application of forces, or the deformation is sufficiently small that it may be neglected. Mathematically the lack of deformation may be defined as there being a constant length between any two points and a constant cross product of the vectors from the origin of the rigid body’s frame and those two points.

Salto  A somersault. Salto is a very common short form used in the code of points and is a common expression.

Scaled orthography  The camera model where a single magnification is applied. This model works under the assumption that the scene pictured is in one plane. Such an assumption is acceptable if the scene depth is small compared to the average distance from the camera. When the coordinates are given as normalized coordinates (all still in one plane) the method is known as orthographic projection.

Score  The number representing the combination of difficulty points (as specified in the code or points or difficulty tables) and deductions taken (for errors as specified in the sport’s judging criteria). The score is used to rank the athlete’s performances.

Short (in reference to a)  A landing that has insufficient somersault rotation and/or height. If the athlete is short of rotation they will be leaning away from the
landing) horizontal direction of travel: falling backwards out of a front somersault or forwards out of a back somersault. If they are short due to lack of height they will probably need to squat deeply on landing.

Somersault  A rotation about a horizontal axis, during which the body becomes inverted. In a non-twisting somersault the rotation will be about the transverse axis of the body. In a twisting somersault there will be a small angle between the horizontal and the transverse axis of the body.

Statistically significant  When it is improbable that an observation was the result of random chance or random variation in the collected data. A probability- (or p-) value should be quoted which gives the probability of making the observation or one more extreme under the null hypothesis.
A p-value less than 0.05 is commonly taken to mean that a result is statistically significant. However, depending on the consequences of an error the desired p-value may differ.

Steady  Has constant velocity (linear or angular). Continues with the same motion unless disturbed. For a motion to be steady it does not necessarily mean the absence of forces or moments. It is the motion observed that means it is named steady.

Stylion  Bony protrusion on the thumb side of the wrist

Superiority (Probability of)  The probability that a random variable from one group is greater than a random variable from the other group. It is the same as the common language effect size (CLES)

Technique  A term used in sports to describe the way in which segments of the body were moved with respect to each other, and the position of the body with respect to the apparatus that allows the athlete to perform a skill. That is, an athlete’s technique includes the actions performed and the postures used. Good technique achieves the skill whilst also being aesthetically pleasing.

Transverse axis  The axis pointing from a body’s right to a body’s left.

Transverse  The principal axis closest to the transverse axis of the body. In this
<table>
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<th><strong>Glossary</strong></th>
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<tbody>
<tr>
<td><strong>principal axis context closest</strong></td>
</tr>
<tr>
<td><strong>Throw (for a somersault or twist)</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Tight</strong></td>
</tr>
<tr>
<td><strong>Tilt</strong></td>
</tr>
<tr>
<td><strong>Trend</strong></td>
</tr>
<tr>
<td><strong>Trunk lateral flexion</strong></td>
</tr>
<tr>
<td><strong>Tuck</strong></td>
</tr>
</tbody>
</table>

*Note:* The sport will determine how long the tuck position must be seen in a tuck twisting somersault. It is common in diving and artistic gymnastics to show the tuck in the non-twisting phases, and use a layout position in the twisting phases. In trampoline gymnastics the tuck, or puck (for multiple somersaults with twist),
is to be shown throughout.

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twist</td>
<td>A rotation about the longitudinal axis of the body in a somersault.</td>
</tr>
<tr>
<td>Twisting Somersault</td>
<td>A somersault skill where the body also spins about its longitudinal axis.</td>
</tr>
<tr>
<td>Under- (rotated or twisted)</td>
<td>Being under-rotated means the athlete has completed an insufficient amount of somersault rotation.</td>
</tr>
<tr>
<td></td>
<td>Being under-twisted means the athlete has completed insufficient twist.</td>
</tr>
</tbody>
</table>
# Nomenclature

<table>
<thead>
<tr>
<th>Symbol/Acronym</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1U1D</td>
<td>Sport specific posture, where the body is laid-out and one arm is raised while the other is lowered. Defined in Section 3.9.3.</td>
</tr>
<tr>
<td>1U1DB</td>
<td>Sport specific posture, where the arms are allowed to flex at the elbow to ~90° in a 1U1D posture. Defined in Section 3.9.5.</td>
</tr>
<tr>
<td>1U1DB LF</td>
<td>Sport specific posture, where the body laterally flexes in a 1U1DB posture. Defined in Section 3.9.6.</td>
</tr>
<tr>
<td>A</td>
<td>Sport specific posture, where there is backward flexion through the torso. Defined in Section 3.9.8.</td>
</tr>
<tr>
<td>a, b, c, d, e</td>
<td>Constants that are combinations of the inertial properties of the body, and initial orientation angles to aid evaluation of elliptical integrals. Defined in Section 4.2.4.</td>
</tr>
<tr>
<td>A, B, C, D, E, F, G, L, N</td>
<td>Constants that are combinations of the inertial properties of the body for the three segment planar model in Equations (4-32) and (4-33), in Section 4.3.2.</td>
</tr>
<tr>
<td>Angle_disp17</td>
<td>Computer programme used to determine Υ. See Section A.6</td>
</tr>
<tr>
<td>BT</td>
<td>Sport specific posture, showing differences to T when performing a back somersault. Defined in Section 3.9.16.</td>
</tr>
<tr>
<td>CT</td>
<td>Sport specific posture, similar to TT, but allowing some separation of the legs. Defined in Section 3.9.19.</td>
</tr>
<tr>
<td>CLES</td>
<td>The Common Language Effect Size, as described in Appendix B.6.2.</td>
</tr>
<tr>
<td>D_i</td>
<td>The vector between the centre of gravity of the body as a whole and the centre of gravity of segment i. Used in equations in Section 4.3</td>
</tr>
<tr>
<td>EP</td>
<td>Sport specific posture used on entry to forward somersaults. Defined in Section 3.9.12.</td>
</tr>
</tbody>
</table>
**Nomenclature**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT</td>
<td>Sport specific posture, showing differences to T when performing a forward somersault. Defined in Section 3.9.16.</td>
</tr>
<tr>
<td>Glo</td>
<td>The global inertial frame defined in Section 4.1.4.</td>
</tr>
<tr>
<td>H</td>
<td>Magnitude of the angular momentum vector.</td>
</tr>
<tr>
<td>HVLV</td>
<td>Sport specific posture, where the body is laid out and the arms are in a High V and Low V position. Defined in Section 3.9.4.</td>
</tr>
<tr>
<td>ICG17</td>
<td>A programme used to determine the inertial properties of a group of segments of the body as a whole. See Section 3.8</td>
</tr>
<tr>
<td>I&lt;sub&gt;xx&lt;/sub&gt;, I&lt;sub&gt;yy&lt;/sub&gt;, I&lt;sub&gt;zz&lt;/sub&gt;</td>
<td>The principal moments of inertia about the medial, transverse, and longitudinal principal axes of the body respectively.</td>
</tr>
<tr>
<td>JL</td>
<td>Sport specific posture that meets the minimum requirement to be awarded a layout when performing a somersault. Defined in Section 3.9.11.</td>
</tr>
<tr>
<td>L</td>
<td>Sport specific posture, similar to the anatomical position but with the palms facing the body. Defined in Section 3.9.7</td>
</tr>
<tr>
<td>LAP</td>
<td>Sport specific posture, where the body is in L and the arms are laterally outstretched. Defined in Section 3.9.2</td>
</tr>
<tr>
<td>LAU</td>
<td>Sport specific posture, where the body is in L and the arms are raised. Defined in Section 3.9.1</td>
</tr>
<tr>
<td>LHF</td>
<td>Sport specific posture, where there is lateral flexion through the torso. Defined in Section 3.9.9.</td>
</tr>
<tr>
<td>N&lt;sub&gt;0&lt;/sub&gt;</td>
<td>The amount of somersault required for one cycle of $\phi$. This is for a half twist when there is continuous twist.</td>
</tr>
<tr>
<td>OP</td>
<td>Sport specific posture between JL and P. Defined in Section 3.9.13.</td>
</tr>
<tr>
<td>P</td>
<td>Sport specific posture representing the posture called pike. Defined in Section 3.9.14.</td>
</tr>
<tr>
<td>p-value</td>
<td>The probability that the null hypothesis is correct when applying a statistical test.</td>
</tr>
<tr>
<td>Pri</td>
<td>The principal frame of reference which has axes aligned with the principal directions. It is defined in Section 4.1.3.</td>
</tr>
<tr>
<td>Pu</td>
<td>Sport specific posture, where there is some flexion at the hips and</td>
</tr>
</tbody>
</table>
Nomenclature

knees in a 1U1D posture. Defined in Section 3.9.10.

$R^2$ The coefficient of determination

Ref The reference frame defined in Section 4.1.2

T Sport specific posture representing the posture called tuck. Defined in Section 3.9.15.

TP Sport specific posture representing the greatest hip flexion, with straight legs which are not separated. Defined in Section 3.9.17.

TT Sport specific posture representing the greatest hip and knee flexion without allowing separation of the legs. Defined in Section 3.9.18.

$V_1, V_{21}, V_{23}, V_3$ Vectors from the centre of gravity of a segment to the joint with another segment in the 3 segment planar model of Section 4.3.2.

$\alpha_i$ Equivalent angle or rotation of a segment from its reference position with respect to the reference frame. Used in equations in Section 4.3

$\theta$ Angle representing the somersault completed, defined in Section 4.1.4

$\tau$ The relative difficulty of a pure somersault. $\tau = I_{yy}/I_{yy.L}$. See Section 5.2.

$\Upsilon$ Angular displacement of the reference frame due to both posture change and the angular momentum possessed by the body. Used in equations in Section 4.3

$\phi$ Angle between the longitudinal axis of the body and the angular momentum vector, defined in Section 4.1.4

$\phi_{Crit}$ Values of $\phi_o$, that separate continuous and oscillating twist. The subscripts 1 or 2 are added when referring specifically to the $\phi_{Crit}$ value for when the transverse or the medial moment of inertia is the intermediate-valued moment of inertia respectively. Defined in Section 4.2.3.

$\phi_{o1}, \phi_{Crit2}$

$\phi_o$ Initial value of $\phi$.

$\psi$ Angle representing the twist completed, defined in Section 4.1.4

$\psi_o$ Initial value of twist. Due to sporting expectations it should be small.
1.1 Biomechanics and the somersault

In both diving and gymnastics the pure somersault and twisting somersault are essential skills. Thus, researching and understanding the somersault, pure or twisting, is a useful focus area for performance enhancement in these sports. In competitions gymnasts and divers are awarded a score that is a combination of the quality of the performance, known as the execution score, and the perceived difficulty, known as the difficulty score.

For somersault-based skills, the difficulty score increases with the number of twists and somersaults performed. A biomechanist will assist an athlete to maximise their difficulty score by suggesting techniques, or technique adjustments, which comply with the sporting rules. Technique options include the postures chosen and the actions performed. It is then the coach’s role to apply these suggestions in the most appropriate fashion for their specific athletes and sport.

The somersault is an airborne skill. The rotation performed while airborne defines the skill performed. Actions performed while airborne may be used to increase the speed of the somersault rotation, introduce twist, increase the speed of twist, or remove twist. The take-off determines the angular momentum once airborne that will allow the rotation, and the landing requires that the athlete is able to reduce their angular momentum to zero in a controlled fashion. Vertical linear momentum does not alter the rotation, but is essential for enabling the athlete to leave the ground. When considering its effect on the rotation that may be achieved it is sufficient to specify a flight time. Horizontal linear momentum will also be present, as either a secondary effect of vertical linear momentum or angular momentum generation, or to allow the athlete to avoid striking the equipment. When defining skills and maximising the difficulty score the aerial phase is the logical place to start.
Different athletes will have different inertial properties, and these properties will alter the effectiveness of a technique. In turn this means that some athletes will find certain skills easier or harder to perform; as a result they may be forced to generate additional angular momentum or use a different technique so as to achieve a desired skill. When making suggestions about techniques, or in some cases when choosing athletes for elite programmes, it is important to consider the variation in inertial properties. Different inertial properties will have the greatest effect during the airborne phase, since the athlete is no longer in contact with the ground and so they rely on using changes in their body posture to alter their rotation.

As a result of differing inertial properties which alter the effectiveness of a technique, it is not possible to make a single statement about which skills can be achieved by using a specific technique. Instead it is helpful to find reasonable upper bounds to the number of twists and somersaults that may be achieved by any athlete and by the majority of the athlete population when using different techniques. Further, since inertial properties measured will always be estimates, and inevitably only represent a sample of the athlete population, using bounds helps to draw conclusions based on what is reasonable to expect rather than stating an absolute.

An athlete has little control over their inertial properties and because divers and gymnasts perform wearing only swimsuits or leotards, both of which have negligible mass, the effect of inertial properties is limited to the segments of the human body; there are no extra pieces of equipment that need to be considered. Even if using the same aerial technique different athletes will be able to achieve different amounts of twist and somersault. The consequences of these differences would support or challenge the notion of an ‘ideal body type’.

The aim of this thesis is thus to

*Establish limits of the number of twists and somersaults resulting from different technique choices, and how these vary due to differences in athlete inertial properties, during the aerial phase of the somersault considering the rules governing the sports of diving and gymnastics.*

In addressing the aim of this thesis the following are of practical interest to coaches:

1. Results relating to the proportions of athletes that can achieve a selection of skills when using each known technique may be used to suggest a priority order of techniques to employ for the best chance of achieving a particular skill.
2. Results relating to the spread of skill achievement and any variations to the effectiveness of techniques may be used to better match techniques to the athlete.
3. Results may be used to support, or challenge the notions of an ‘ideal body type’ in terms of inertial properties. If the notion is supported, coaches may better identify those athletes with a natural advantage.

4. The estimates of the limits to the number of twists that may be performed would indicate the most difficult skill that may be performed using only acceptable aerial techniques.

1.2 Principal contributions

The principal contributions of this thesis fall into two main areas: 1) the derivation of mathematical equations allowing the somersault and twisting somersault to be modelled, and 2) evaluating these equations over a broad range of inertial properties, reflecting the gymnastic and diving populations, for the purpose of investigating aerial techniques.

The mathematical equations were derived and interpreted considering anatomy, and sporting rules; however, these equations are not limited to such an environment but are broadly applicable to any multi-segment model. The techniques investigated are specific to the sports of gymnastics and diving. The inertial properties are a collation of previously published data and measurements of current athletes; thus the conclusions drawn are specific to this athlete population.

The contributions are summarised as follows.

1. Equations were derived for the quasi-rigid phase of the somersault thus allowing skills to be defined and the required angular momentum to be determined. From these equations rotation was classified by twist behaviour. Postures that allow sufficient twist to achieve a skill were identified. (See Section 4.2 and a summary in Section 7.1)

Skills are defined by the number of somersaults completed and the number of twists completed within these somersaults. As a result it is necessary to evaluate the twist completed as a function of somersault, not time. This extends previous work that derived equations for twist with respect to time. When determining the angular momentum required for any somersault skill time becomes important. As a result equations for somersault achieved were derived with respect to time. This is another extension on previous work which gave an average somersault rate for a given angular momentum.

2. The error as a result of assuming that the medial and transverse moments of inertia are equal has been evaluated in light of the predictions regarding skills that may be
achieved. As a result, situations under which the assumption is appropriate, are discussed. (Sections 4.2.5, 5.3.6 and a summary in 7.1.1)

The assumption that the medial and transverse moments of inertia are equal has been made in the previous literature without providing an assessment of the resulting error or any indication of when the assumption is appropriate. This shortfall is addressed in this thesis by considering a range of inertial properties and postures that comply with sporting rules.

3. Equations for the reorientation of the body as a result of postural change, both when the body possesses zero and non-zero angular momentum, were derived for a model of the body comprised of rigid segments (Section 4.3).

4. A broad range of inertial properties, intended to reflect the current diver and gymnast population, were compiled (Chapter 3). It was shown that the data available in the literature was inadequate to explore what rotational behaviours could be observed (Section 7.4.1).

There are no previously published inertial property data sets specific to the diving and gymnastic population. Hence it was necessary to estimate inertial properties from measurements of current divers and gymnasts. The sample included males and females, adults and children. This inertial property data was used to explore aerial techniques and is provided in Appendix A.5.

5. Suggestions of the ‘best’ postures and actions to employ for various somersault skills were made. Comparisons of postures and actions were made in Chapter 5 and Chapter 6, then summarised in Sections 7.2 and 7.3.

The techniques considered the ‘best’ were those that allowed the greatest achievement while considering sporting constraints.

6. Adjustments to known techniques that increase the number of twists that may be performed are presented; the known techniques are those expected to be acceptable in gymnastics and diving (Chapter 6).

7. Evidence is presented supporting the notion that some athletes have a more advantageous set of inertial properties than others (presented throughout Chapter 5 and Chapter 6, with a brief summary in Section 7.4).

Variations in skill achievement, when different inertial properties are substituted into equations describing any particular technique, show that some inertial property combinations are more or less advantageous. This variation is discussed and the
categories based on characteristics that may be used to distinguish those with a natural advantage are appraised.

1.3 Publications resulting from this work

The following papers relating to this thesis have been published by the author:

**Journal articles**


This paper focuses on the mathematical equations derived in Section 4.2 for the quasi-rigid phase of the somersault. The paper uses one example athlete to illustrate the behaviour of the derived equations. This athlete is not the same example athlete as used in this thesis when presenting a single example; the data set used in this thesis was an estimate from measurements of a current athlete. Some discussion regarding postures and the assumption of equality of the transverse and medial moments of inertia, which is discussed in Section 5.2.9, is also presented in this journal paper.

**Conference papers**


This paper estimated the inertial properties of one individual using a selection of different published methods. The estimates were used to calculate parameters of interest when performing somersaults. Comparing the calculated parameters revealed that each method would mean slightly different conclusions, in regard to technique choice, would be drawn depending on the method used.


This paper presented a methodology for reconstructing 3-D world coordinates from two camera views. This methodology used free software and a checkerboard that is moved through the field of view of the cameras. By determining lengths and angles from the 3-D world coordinates of a known object moved in the field of view of the camera an accuracy assessment was completed.

This paper presents parameters used to compare postures and athletes. Initial comparisons are made and sub-populations identified. The content is mostly covered in Section 5.2, which also presents further comparisons of specific postures.


The content of this paper is covered in Sections 6.1.1, 6.1.2 and 6.5.1. The paper focuses on the tilt produced by the idealized arm actions FullS, DiverS, Drop, and Raise. Chapter 6 explores these and other idealized actions, as well as seeking to translate the tilt produced into skills that may be achieved.


The contents of this paper are addressed in Section 6.5.1. The paper discusses the use of different postures to boost tilt production and what instruction would assist the athlete to move the arms in the correct plane for each posture, in order to attain the boost in tilt.

### 1.4 Outline of the thesis

Following this introductory chapter, *Chapter 2* introduces the reader to the somersault in its sporting context, providing a general background to the terminology and the sports of diving and gymnastics, as well as reviewing the previous literature.

*Chapter 3* presents the mathematical derivations and equations while the athlete is holding a posture and when they are changing posture. These equations will be used later to evaluate skill achievement, using various postures and actions (techniques).

*Chapter 4* reviews the methods for estimating inertial properties and then presents the methodology that was used to collate a representative sample of inertial properties for use in this thesis.

*Chapter 5* introduces a selection of postures held when somersaulting and then evaluates them by calculating the number of somersaults and twists they would allow to be performed under any given set of initial conditions. The inertial properties collated in Chapter 4 are used to estimate the spread in achievement and changes to a hierarchy of postures that could be expected across different athletes.

*Chapter 6* reviews known twist initiation actions and then investigates adjustments to these actions to boost the number of twists that may be achieved. Again the inertial properties
from Chapter 4 are used to estimate the spread in achievement and changes to a hierarchy of postures that could be expected across different athletes.

Chapter 7 summarises the conclusions of the previous chapters. It also seeks to rephrase the conclusions in such a way that they may easily be interpreted by coaches.

Appendix A supplements Chapter 3. It provides greater detail on the methods of estimating inertial properties and describes the computer programme ICG17 used to calculate inertial properties of the body as a whole or groups of segments as a unit.

Appendix B presents a summary of mathematical equations and mechanics concepts, written in the forms, and with notation, used in this thesis. Solutions to integrals that appear multiple times in the thesis are also provided.

Appendix C presents the detailed methodology for the collection and processing of video footage of somersault performances.

Appendix D to G provides additional discussions of references, explanations, and alternative methods which supplement the arguments presented in the body of the thesis.

Appendix H presents the full mathematical working that was not detailed in the chapters.
Chapter 2

Background

This chapter introduces the somersault and twisting somersault in its sporting context. A review of the literature, presenting the mechanics of the airborne phase of a somersault, and the techniques athletes employ to perform a twisting somersault is then conducted.

2.1 The somersault and its sporting context

Based on sporting rules and judging criteria, to perform the skill known as the somersault, an athlete must leave the take-off surface, rotate about an axis parallel to the ground and perpendicular to the horizontal direction of travel, and then land after all the desired rotation is complete. No intermediate landing is allowed: all the rotation must be completed while airborne. In a twisting somersault the athlete completes a somersault while also adding a rotation about the longitudinal (approximately head-to-toe) axis of the body, known as twist. Each twisting somersault skill is described by the number of somersaults and the number of twists completed.

The international governing body for gymnastics is the Fédération Internationale de Gymnastique (FIG) and for diving is Fédération Internationale de Natation (FINA). Since the FIG governs women’s artistic gymnastics (WAG), men’s artistic gymnastics (MAG) and trampolining gymnastics (TRAMP), to avoid confusion each reference to a code of points (CoP) will use the title abbreviations CoP WAG, CoP MAG, and CoP TRAMP respectively, rather than an author as is traditional in referencing documents.

Almost all of the skills in diving and trampolining are somersaults, while in gymnastics and tumbling, the somersault is one of the required skills. Each somersault skill recognised by each sport is listed with an illustration in the ‘code of points’ or ‘table of degree of difficulty’ published by the international governing body. In general, adding an additional somersault rotation—or an additional half somersault rotation in diving, adding an extra half twist to a somersault, or changing the posture used constitutes a
different skill. In most competitions an athlete will need to perform more than one somersault skill.

2.1.1 Naming of somersault skills

Somersault skills are named according to the number of twists, if any; the number of somersaults, if more than one; the combination of the direction of the rotation and horizontal translation of the somersault; and the posture used. For example, a full twisting double back tuck, has one twist completed within two somersaults, both of which are rotating backwards while the athlete is travelling backwards, and it is executed in a tuck posture. The number of twists may also be given at the end following the word ‘with’. For example, a full twisting double back may also be called a double back somersault with full twist. There are slight variations to this naming system as well as a number of abbreviations and special names specific to each sport and geographic region; these names will not be used except when quoting the works of other authors, in which case the name, according to the system presented here will be given in parentheses. To aid the reader unfamiliar with somersaults, each aspect of the naming is described below.

Posture

The posture of a somersault is defined by the ankle-knee-hip (“knee”) angle and knee-hip-shoulder (“hip”) angle. There are three classes of posture: tuck, where the knees and hips are flexed; pike where there is no flexion at the knee and there is flexion at the hips; and layout where there is no flexion at the knees and the hip angle is greater than 135°. Figure 2-1 gives examples of tuck, pike, and layout positions; the arm and head positions are optional. The specified posture needs to be clearly shown during the somersault, although it does not need to be held for the entirety of the somersault.

Figure 2-1: The tuck (a), pike (b) and layout (c) postures.
Directions of somersault rotation and horizontal travel

In gymnastics and diving there are four somersault types depending on the directions of rotation and horizontal travel\(^1\). The horizontal direction of travel may not always be large, but is present (especially when dismounting to avoid accidentally striking the take-off surface) and does effect the naming of skills, particularly in diving. The four somersault types are forward, backward, inward and reverse, which are illustrated in in Figure 2-2 to Figure 2-5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2-2}
\caption{A forward 1½ somersault pike.} \\
\textit{After O’Brien (2003).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2-3}
\caption{A backward 1½ somersault pike.} \\
\textit{After O’Brien (2003).}
\end{figure}

\(^1\) There are only four combinations possible with two horizontal directions of travel (forward and backward) and two directions of rotation (forward and backward). The diving terminology has been used and all four combinations are performed in diving. In gymnastics the reverse is often called a gainer, although this can also refer to take-off from one rather than two feet when rotating backwards. Dismounts from the bars in gymnastics correspond to the inwards and reverse somersaults are referred to as forward and backward flyaways; however the Code of Points does provide a detailed description.
Number of somersaults

The number of somersaults is the number of times that the athlete rotates about an axis parallel to the ground and perpendicular to the horizontal direction of travel.

Figure 2-2 to Figure 2-5 illustrate 1½ somersaults. The number of somersaults is generally given as a mixed numeral although a double somersault is given in words.

Gymnasts typically land on their feet\(^2\) and so somersaults are named in whole somersault increments. In diving, the athlete may enter the water feet first or head first; feet-first entries are common in training and competitions for young athletes, while the

\(^2\) Although landing on the feet is most common, there are some skills defined in CoP MAG as three-quarter somersaults, where the last quarter rotation back to the feet is performed on the ground; that is, the athlete rolls.
head-first entry is almost exclusively performed in any major competition. As a result, in diving, the somersault skills are named in half somersault increments.

**Number of twists**

The number of twists is the number of rotations about the longitudinal, approximately head-to-toe, axis performed during a somersault. Skills are named by half twist increments\(^3\). Figure 2-6 illustrates a full twisting 1½ somersault.

When writing the number of twists fractions are typically given, although for whole numbers of twists the words full, double, triple and so on are used.

![Figure 2-6: A forward full twisting 1½ somersault.](From FINA (2009).)

**2.1.2 Judging**

The winner of a diving, gymnastic, or trampolining competition is the athlete with the highest score as awarded by the panel of judges. The score comprises two sub-scores: one for difficulty and the other for execution. Each skill has a difficulty value assigned by the code of points or table of degree of difficulty published by the governing body. The difficulty value is intended to reward athletes for performing skills that are perceived to be more challenging. The execution component is out of ten, and is the result of a judge subtracting deductions for errors they observe during the performance according to their interpretation of the judging criteria, which are published by the governing bodies alongside the code of points or table of degree of difficulty. To maximise their score an athlete needs to balance difficulty against their ability to execute the skill.

\(^3\) There is a gymnastics skill known as a side-somersault, which may be thought of as a somersault with ¼ twist. It is, however, not typically used in the same context as any of the other somersaults and is closer to an aerial cartwheel than a traditional somersault, and so will not be considered.
To be awarded the difficulty value for a somersault skill, the athlete must complete the required number of somersaults and/or twist rotations and show the specified posture at some point during the execution of the skill. For a whole number of somersaults the skill is awarded if the feet touch the landing surface or water before any other part of the body. When an odd number of half somersaults are performed in diving, the athlete is awarded the difficulty value if the head or hands touch the water first. In diving it is also necessary that the diver has performed the dive they nominated to receive a score (FINA, 2009). For a twisting somersault, completion definitions vary slightly between sports: in women’s artistic gymnastics the amount of twist completed is determined by the direction that the front foot is facing on landing, or the direction that the athlete travels in the following connected skill (CoP WAG, 2013). In men’s gymnastics and diving the twist is awarded when it is within 90° of the required amount; which body segment to use is not stated and it appears to be up to the judge to decide (CoP MAG, 2013; FINA, 2009). The trampolining judging criteria do not state specific completion criteria; rather, it specifies deductions for not achieving the ‘exact’ amount of twist. The less twist completed the greater the deduction, to a maximum where presumably a lesser skill is awarded (CoP TRAMP, 2013).

Once the difficulty value of a somersault is awarded the judge will take deductions for the errors that they observed during the performance. The judging criteria are guides that are intend to create a fair judging system. They are not, however, precise criteria in a mathematical sense. This means there is scope for personal preference amongst judges. For example, an athlete may choose to confine their twist to a small portion of the somersault, to create contrast within the skill, or they may spread the twist over the whole somersault and use a lower rate of twist to create a perception of ‘ease’. Which is ‘better’ is a matter of personal preference. Further, what the judge ‘sees’ can vary according to their physical position and orientation with respect to the performance, their familiarity with the skill and its common errors, the speed of the movement and what the athlete did immediately before and after. There is no video replay used by the judges: they must make a judgement about the performance based on a single observation of the somersault, which is generally completed in less than a second. Inevitably there is some level of subjectivity and personal preference which affects the deductions taken.

**2.1.3 Phasing of a somersault**

To provide the biomechanist with a framework for the analysis of any somersault, the somersault can be divided into phases, with each phase showing distinctly different characteristics. The phases may differ due to the athlete’s intention for that phase, for example maximising somersault rotation or twist rotation, or differ due to the actions
made by the athlete or the forces and moments applied. The end of one phase sets the
initial conditions for the next and so it is important to consider the compatibility of the
phases before the results of studying each phase independently is translated into
recommendations for a performance of the entire skill.

In addition to providing a framework for the biomechanist, distinct phases are expected
as part of any good performance. The notion of “peaking” (George, 1980) explains the
expectation in terms of aesthetics. Further, lack of distinct phases would incur
deductions for “insufficient exactness” (CoP WAG, 2013).

All somersaults will have a take-off phase, during which all the required linear and
angular momentum needs to be generated, an airborne phase where gravity is the only
external force of any appreciable magnitude acting on the body, and a landing phase
where the athlete lands or enters the water.

If the athlete is performing a twisting somersault the twist may be initiated during the
take-off phase, or the airborne phase, or may be spread across both phases, depending
on the technique(s) used. If there is no twist the athlete moves directly into the posture
required by the skill they are performing. In both cases the athlete will hold the posture
they have assumed for a portion of the skill while airborne, and so may be considered as
a quasi-rigid body. The existence of a quasi-rigid phase is a common assumption based
on the general experience of previous authors (Batterman, 1968; Frohlich, 1979;
Yeadon, 1993a). Furthermore, the more highly ranked subjects have been observed
holding postures during which the bulk of the twist is performed. In contrast those given
a lower ranking were observed showing continual movement of body segments
(Sanders, 1995; Sanders, 1999). It follows that a quasi-rigid phase is highly desirable. In
skills involving multiple somersaults, an athlete may show a series of quasi-rigid
phases. The phase in which the athlete prepares for landing will involve the slowing or
removing of any twist present and the movement into a posture suitable for the start of
the landing phase.

2.1.4 Institutes of sports and talent identification

In Australia there is a national institute of sport, the Australian Institute of Sport (AIS),
as well as state-based institutes of sport. The New South Wales Institute of Sport
(NSWIS) is based at Sydney Olympic Park. The institutes provide facilities, coaches
and support staff to their athletes, with the intention of assisting their progression
through the sport to ideally represent Australia at international competitions.
Athletes may be invited to join an institute of sport if they are deemed to have talent according to an institute’s specific talent identification criteria. The criteria used vary both between institutes and between sports within the same institute.

Talent identification often involves judging the athlete’s performance of a series of basic skills and taking body measurements. Body measurements are often taken since it is generally believed that there “are optimal size and proportions underpinning the ideal performance” (Stewart & Sutton, 2012, p. 155). When analysing somersaults in terms of dynamics, “size and proportions” translates to the inertial properties of the various body segments. Even though the inertial properties may suggest that one athlete has a natural advantage over another at the time of measurement, as the athlete grows these will change. When using body measurements to identify talent it is thus important to consider what the athlete’s likely proportions will be when at competitive age (Stewart & Sutton, 2012).

This thesis analyses the airborne phase of somersaults in general, analysing different postures and techniques, as well as exploring the effects of different inertial properties. The conclusions regarding the effect of inertial properties may be used to guide talent identification for future competitors, or to assist matching techniques to those athletes already competing. Section 2.5 reviews the current understanding of how an athlete’s inertial properties influence the performance of somersaults. The effects of inertial properties will be a key discussion point in subsequent chapters.

It is also of interest to know if a reduced number of body measurements, a ratio or an index strongly correlates with desirable inertial properties. If a correlation could be found this would allow quicker identification of athletes with advantageous inertial properties. The following common biomechanical indices and ratios will be considered: BMI, Rohrer, Ponderal and Androgyny indices and V-ratio, sitting height-, arm- and leg-length to stature ratios (Park et al., 2007; Stewart & Sutton, 2012; Bradshaw & Rossignol, 2004).

### 2.2 Historical explanations of the mechanics of the somersault

A somersault has been defined in Section 2.1 as a rotation about a horizontal axis parallel to the ground and perpendicular to the horizontal direction of travel. Defining a somersault in this way means that the definition of the somersault aspect of rotation is consistent across all types of somersaults. In a pure somersault this horizontal axis will be aligned with the transverse axis of the body, which will also be parallel to a principal axis direction. In a twisting somersault the transverse and horizontal axes do not align.
and the athlete displays continuous rotation about their longitudinal axis. Although a pure somersault is by nature a non-twisting somersault, there exist other somersaults which are non-twisting, but are also not pure. Mathematical descriptions of the different types of somersault are given in Section 4.2. Which type of somersault is displayed depends on the relative values of the principal moments of inertia and their orientation with respect to the angular momentum vector.

Since during a pure somersault the athlete is required to rotate about their transverse axis, which is also a principal axis, the mechanics can be reduced to a planar analysis with the transverse axis perpendicular to the plane. There has been no historical controversy or difficulty describing the mechanics of the pure somersault.

The twisting somersault is more complex than the pure somersault, and it took considerable time before suitable dynamic equations were written. When watching a twisting somersault without the advantage of slow motion videography, it is easy to conclude that the athlete rotates simultaneously about their longitudinal and transverse axes. When the athlete is in a layout position these axes are approximately parallel to the principal axes and so the athlete would appear to violate the law of the conservation of angular momentum (Frohlich, 1979). Further, if an athlete leaves the ground performing a pure somersault, there has been controversy surrounding if, and how, they may initiate twist after becoming airborne.

An athlete may perform a twisting somersault using what is termed contact twist, which means the athlete generates angular momentum about both the somersault axis and the twist axis prior to take-off (Aaron, 1970; Boone, 1974; Frohlich, 1979; George, 1980; Yeadon, 1993b). This technique was initially the only one taught (Aaron, 1970; Boone, 1974) and hence has also been termed classic twist (Biesterfeldt, 1974). When the contact twist technique is used the longitudinal axis of the athlete will tip from vertical until the half-somersault position and then return by the full somersault position. Boone (1974) related this tipping to the vector addition of the angular momentum vectors of the somersault and twist components, and stated that the tipping off vertical increases when twist angular momentum increases or somersault angular momentum decreases. George (1980) observed this tipping behaviour and termed it “pancaking”. Yeadon (1984) has described pancaking mathematically, and integrated it with his mathematical description of a twisting somersault. This is the description of the twisting somersault accepted to date; it is described in Section 2.3.

Following interest in how a cat turns over to land on its feet when held upside down and dropped, the concept of “self-rotation” was proposed as the means by which twist is produced in a somersault (McDonald, 1961; Boone, 1974): just as a cat with zero
angular momentum could turn over using body posture changes, an athlete could twist about their longitudinal axis even when there was no component of angular momentum about this axis. Such twist would occur only while the athlete is changing shape. The limit to the number of twists that could be performed was believed to be limited to the speed at which the athlete could perform the required shape-changes to self-rotate about their longitudinal axis relative to the time available (McDonald, 1961). For example, elite diver Brian Phelps (McDonald, 1961) took on average 0.3 seconds to turn a half twist (600°/sec), while subjects in (Bartee & Dowell, 1982) could only rotate 246° about their longitudinal axis when jumping from the 7.5 metre tower (an average of \(\approx 200°/sec\)). The theory of twist being a self-rotation is no longer held, having been superseded by the theory of tilt-twist, since tilt-twist can produce continuous twist. Nevertheless the concept of self-rotation is used to suggest actions to introduce tilt to a pure somersault and thereby initiate twist via the tilt-twist theory of the twisting somersault.

The theory of tilt-twist was proposed in the late 1960s (Batterman, 1968). Tilt-twist requires the longitudinal axis of the body to tilt towards the angular momentum vector during a pure somersault so that continuous twist is produced (Batterman, 1968; Rackham, 1970; Frohlich, 1979). By tilting the longitudinal axis towards the angular momentum vector some of the somersault rotation is “transferred” to twist (Rackham, 1970). Rackham (1970) states that

> “only a small amount of the original somersaulting angular momentum is transferred into twist angular momentum, but as the moment of inertia (resistance) about the twist axis is small, the rate of twist is quite large”

and so by eye the tilt may not even be noticeable. Rackham’s phrasing can cause confusion since the angular momentum vector has a fixed magnitude and direction with respect to an inertial frame of reference. From the surrounding context and figures it appears that Rackham believes that when twisting, the somersault is no longer about the transverse axis of the body and so the somersault no longer requires all the angular momentum the body possesses and this allows the twist to occur. Even though Rackham does not present a clear mathematical formulation, his description did aid the development of the theory of tilt-twist.

Frohlich (1979) expanded on Rackham’s description by likening the twisting somersault to “a rigid symmetrical top experiencing force-free motion”. Frohlich makes the assumption that the athlete’s transverse and medial moments of inertia are equal. This appears to be a reasonable assumption, since the moments of inertia in the example twist position he specified are within 4% of each other. However, assessing the error of such an assumption, and determining whether the errors are sufficiently small, has not
been the focus of any published paper. Frohlich’s description matches the case of “steady precession with zero moment” (Meriam & Kraige, 1998, p. 587), where twist is equivalent to spin and the somersault is equivalent to precession. Frohlich, however, made an error when writing the mathematical equation for the ratio of the twist and somersault angular velocities; this was identified by Yeadon (1993a). Yeadon (1993a) then refined and extended the mathematics of tilt-twist, as well as relating it to contact twist. Even though somersault and twist can be equated to precession and spin when the transverse and medial moments of inertia are equal, this can readily cause confusion, since the words spin and precession are most commonly used when precession results from the application of an external moment to a symmetrical object spinning about its axis of symmetry. Further, spin and precession do not consider anatomy, but the definitions of twist and somersault depend on anatomy.

Self-rotation about the medial axis of the body was identified, at the same time as the tilt-twist theory was proposed, as a means of introducing tilt into a pure somersault so that continuous twist could be initiated. Asymmetric rotation of the arms in a frontal plane (Batterman, 1968; Rackham, 1970; Frohlich, 1979) and torsion between the trunk and legs when extending from a pike (Rackham, 1970) were identified as actions that will produce self-rotation about the medial axis.

2.3 Current understanding of the mechanics of the somersault

Assuming that air resistance forces are negligible, once an athlete is airborne their angular momentum is fixed in magnitude and direction with respect to a fixed world frame. The angular velocity of the athlete need not be constant and depends on the principal moments of inertia and their directions at any instant in time. Thus, an athlete may alter their angular velocity within limits that are governed by the postures they can physically move between or hold.

A quasi-rigid phase of a twisting somersault, during which a large portion of the rotation is completed, has been assumed by previous authors (Yeadon, 1993a), and hence rigid body mechanics has formed the basis of describing the rotation during a twisting somersault. The rotation observed will depend on the values of the principal moments of inertia and the directions of the principal axes with respect to the angular momentum vector. The values of the principal moments of inertia in turn depend on the posture held by the athlete. The directions of the principal axes relative to the angular momentum vector depend on the posture held and the initial orientation of the anatomical axes with respect to the angular momentum vector.
For a pure somersault there is only somersault rotation, and the transverse principal axis of the body is parallel to the somersault axis. Changing posture to alter the transverse principal moment of inertia would cause the somersault angular velocity to increase or decrease proportionally. Assuming that the posture changes do not introduce any tilt of the longitudinal axis towards the angular momentum vector, then the only change would be to the somersault angular velocity.

A twisting somersault possesses both somersault rotation and twist rotation. Yeadon (1984; 1993a) presents equations to describe the rotational behaviour of a twisting somersault when the angular momentum vector is parallel to the ground and perpendicular to the horizontal direction of travel, and the principal moments of inertia in order of decreasing magnitude are transverse, medial, and longitudinal. Yeadon also considers the special cases when the transverse and medial moments of inertia are equal and when the medial and longitudinal moments of inertia are equal. Further, Yeadon stated that by introducing a phase shift of 90° in twist the same equations may be used when the order of the principal moments of inertia is medial, transverse, and longitudinal in decreasing order of magnitude. No comment is made regarding the situation when the longitudinal moment of inertia is the maximum-valued or the intermediate-valued moment of inertia; no justification is given for not considering these situations.

The somersault is a rotation about the angular momentum vector and twist occurs when the longitudinal principal axis is not perpendicular to the angular momentum vector. Yeadon (1984; 1993a) identifies two modes of twisting: one where there is continuous twist and the other where the twist angle oscillates about zero and never reaches a half twist. The relative magnitudes of the moments of inertia and the initial conditions determine which is observed. The magnitude of tilt of the longitudinal principal axis away from a position perpendicular to the angular momentum vector determines the rate of twist, and hence the number of twists that may be performed during a somersault. Yeadon’s (1984; 1993a) formulation is mathematically sound and provides the currently accepted description of the twisting somersault in the quasi-rigid phase. Yeadon derives an average twist-to-somersault ratio which, although it allows an estimate to be made, does not allow accurate determination of the amount of somersault required for a particular amount of twist. Equations describing twist rotation as a function of somersault rotation, rather than as a function of time or a time-average, are derived in Section 4.2 of this thesis. Although Yeadon (1984; 1993a) starts with analytic equations there is very little discussion regarding these equations and the examples presented were the result of a numerical computer simulation. Two example postures, to illustrate the two twisting modes, are given. There is, thus, room for further analytic investigation.
and investigation of the behaviour of a broader range of postures including identifying any constraints on initial conditions.

Using contact twist means the athlete leaves the ground with somersault and twist rotation. This means that the athlete’s angular momentum vector is not parallel to the ground. Instead it is inclined to the horizontal at an angle equal to the inverse tangent of the ratio of the components of twist and somersault angular momentum. The athlete would experience the same rotation relative to the angular momentum vector; however, when the athlete is in a posture that will display continuous twist, an observer would see the longitudinal axis tilt towards an axis parallel to the ground as the somersault increases to a half somersault and then the longitudinal axis would return to being vertical (Yeadon, 1984). This is the phenomenon that George (1980) termed “pancaking”. As a means of producing a twisting somersault the inclination of the angular momentum vector is limited to less than 45°. At 45° the longitudinal axis would move from a vertical to a horizontal position and then return. Such a rotation could not be called a somersault since an observer cannot see any rotation about a horizontal axis (Yeadon, 1984).

Using tilt twist means an athlete initiates twist after becoming airborne. Yeadon (1993c) affirms the proposals of previous authors for using asymmetric rotation of the arms in a frontal plane (Batterman, 1968; Rackham, 1970; Frohlich, 1979), and torsion between the trunk and legs when extending from a pike (Rackham, 1970). Yeadon (1993c) also adds lateral flexion as a possible technique. To eliminate twist generated by any of these actions the athlete may reverse the action after an even number of half twists. At an odd number of half twists, since the angular momentum vector is now on the opposite side of the body, it is necessary to repeat the initiating action to remove the tilt; performing the reverse action would only increase the tilt. It is possible to repeat actions at even numbers of half twists to increase the tilt in a stepwise fashion, although it is important to remember that the tilt would also need to be removed in a stepwise fashion.

### 2.4 Techniques used by athletes to perform a somersault

An understanding of the mechanics of the twisting somersault provides an explanation as to why some techniques are more successful than others and allows techniques to be compared without the need for an athlete to perform the technique. As a result, certain techniques may be recommended above others. The actual performance of skills and the techniques used has also been an area of research. By observing athletes, techniques of interest for mechanical investigation, potential constraints following changes in technique observed with differing apparatus or take-off conditions, and techniques
preferred by judges may be identified. Previous observational studies of athletes performing twisting somersaults are reviewed below.

### 2.4.1 Techniques observed and resulting contributions to twist

Yeadon has conducted a number of observational studies of athletes performing twisting somersaults during training and competition (Yeadon, 1987; Yeadon, 1989; Yeadon, et al., 1990; Yeadon, 1994; Yeadon, 1997; Yeadon & Kerwin, 1999). By using numerical computer simulations Yeadon (1993d) has divided an athlete’s technique into the contributions to tilt made by various observed actions. The actions considered were asymmetrical arm actions, chest rotation, lateral hip flexion, symmetrical actions, and contact twist. Based on observations and simulations it was concluded that even in situations where contact twist was used—and could justifiably be assumed by sight alone to be the only cause of the twist—there were still significant contributions made to the twist by aerial actions. Further, aerial techniques tended to dominate over contact techniques. As a result, Yeadon strongly advocates the use of aerial techniques since they allow the same somersault entry for twisting and non-twisting somersaults and can be removed before landing (Yeadon, et al., 1990). Although contributions to tilt by various actions are classified, and suggested learning progressions given (Yeadon, 1997b), there is no discussion regarding why some aerial actions contribute more than others: for example, are they more effective actions, longer actions, or preferred actions? Further no indication is given as to whether the athletes have or have not reached the theoretical limit of each action, and hence which actions an athlete could reasonably focus on to increase the number of twists they may complete in a somersault.

### 2.4.2 Better or worse performances

In work to determine what constitutes a better executed performance, Sanders (1995; 1999) and Yeadon & Kerwin (1999) compared performances of athletes with different rankings, as given by accredited judges.

Sanders (1995) observed eleven trampolinists of varying ability. It was found that the less-skilled trampolinists tended to rush their actions, twisting earlier and with more vigour, and used greater amounts of tilt (related to twist speed) than the more-skilled performers. In addition, the more-skilled performers tended to only use necessary actions and showed symmetry of the amount of twist completed about the apex of the somersault trajectory, enhancing the impression of being un rushed.
Sanders (1999) observed ten nationally-competitive divers on the 3 metre springboard. The more skilled divers started with less hip flexion, as a result achieving more height off the springboard. They then extended sooner and as a result started twisting sooner. They then completed the twist earlier, using what Sanders termed an “aesthetically pleasing” tight pike posture and extended for a clean entry in a time that was considered neither too “long” nor too “short”.

Yeadon (1999) compared techniques used in a full twisting back somersaults in the gymnastics compulsory floor routine at the 1996 Olympic Games in Atlanta. He divided the athletes into two groups: “Group A comprised the nine highest-scoring competitors (9.70 - 9.85) while Group B comprised the nine lowest scoring competitors (8.61 - 9.26)”]. The conclusion was that “no evidence of a difference in technique between the highest and lowest scoring competitors” existed. As a result it was proposed that all the athletes were elite and so the differences in score were due to “technical” rather than “mechanical” reasons. Presumably Yeadon uses the phrase “technical reasons” to mean reasons such as poor posture and steps on landing, and “mechanical reasons” to mean the twist technique used. The fact that scores are awarded for the whole routine, which includes skills additional to the full twisting back somersault being analysed, was neglected: differences in score are not necessarily due to the full-twisting back somersault.

### 2.4.3 Changes in technique with increasing amounts of twist

A very short paper by Wiley (1964) compared a plain back somersault, a full twisting back somersault and a double twisting back somersault following a round-off back-flip. No description was given of the take-off surface, but considering the year of publication it was probably unsprung. Wiley concluded that the twisting motion was established before the performer left the mat, and in the double twist the first twist was completed before a half somersault was completed. From the table of summarised parameters it appears that when twisting the athletes gained less height but had higher average angular velocities about the somersault axis. Perhaps, to perform the twist the change in the actions affected the generation of vertical linear momentum thereby reducing the height and forcing the angular rotation to increase in order to still complete the somersault. No joint angles were given to describe the actions or postures used.

Even though the focus of the paper by Sanders (1995) was on the effect of athletic ability on technique, the same athletes performing a “Barani” (1/2 twist forward somersault) and a “Rudi” (1½ twist forward somersault) were observed. The amount of arm asymmetry, the amount and speed of chest rotation at take-off, the speed of hip
extension from the entry pike position, and the amount of lateral hip flexion used was observed to differ between the two twisting somersault skills. Sanders’ attributed the changes to the need to generate more twist in the Rudi somersault. No attempt to relate the changes in technique to the mechanics of twist initiation was made.

Well considered progressions for learning a twisting somersault should, through a series of defined and manageable steps, add to and adjust an athlete’s technique so that they may perform an increasing number of twists within a somersault. Yeadon (1999; 2001) presented progressions for learning a twisting somersault that were “based on the results of computer simulations and make a single change in technique from one stage to the next” (Yeadon, 1999). The learning progressions given were for learning a forward and a backward twisting somersault. The progression steps included the use of asymmetrical arms and asymmetrical hip actions. Minimal explanation was given as to why these particular techniques were recommended or what criteria were used with the computer simulation. Although these progressions may enable an athlete to eventually perform more twists in a somersault, it is unclear if they are the most effective techniques and if they are suitable for all reasonable athlete inertial properties. Furthermore, no reason is given as to why, in one case (Yeadon, 1999), asymmetrical hips is suggested for forward somersaults and asymmetrical arms for backward somersaults, while in a later paper (Yeadon, 2001) both asymmetrical arms and asymmetrical hips are suggested for both forward and backward somersaults.

Although there exists some understanding that technique would change when increasing the numbers of twists performed, there has been no research focussed on classifying, or analysing these changes in technique.

### 2.4.4 Different apparatus or sport

Al-Haroun’s (1980) PhD thesis compared the same athletes performing full twists on three different gymnastic apparatus. Differences in the distance travelled, flight time, as well as velocities of various body landmarks were observed. Joint angles and body orientation during the flight phase were not given, meaning that it is difficult to relate Al-Haroun’s work to current understanding. Al-Haroun commented that each performer seemed to elect to use different techniques; as a consequence it was concluded that it is unreasonable to use average velocities across athletes on any apparatus and instead comparisons between apparatus should be made separately for each athlete.

In his many studies Yeadon observed performances across different sports and apparatus. These performances were made by different athletes and so, unfortunately, any differences in technique cannot directly be related to the change in apparatus or sport.
Inevitably, changing apparatus or sport would require an athlete to adjust their technique due to different initial and end conditions, different time constraints, and different expectations. It is not, however, known what adjustments athletes actually make.

### 2.5 Inertial properties and performance

Inertial properties of a rigid body are constants within the equations of motion, yet their values can alter not only the magnitude but also the type of motion observed.

Inertial properties will differ between athletes and so different athletes will experience different motions, even though they may perform the same actions, use the same postures, and have the same initial conditions on take-off. The differences in motion between athletes may or may not be practically significant. Yeadon (1984) and Vieten & Riehle (1992) concluded that for the performance of somersaults, variations in inertial properties across athletes were practically insignificant.

Yeadon (1984) applied the inertial properties of three adult elite trampolinists to his computer simulation model for twisting somersaults and concluded from the outputs that inertial properties had little effect. Although this may be true for the three adult elite trampolinists that Yeadon measured, the sample size is too small and too narrow to assume that differences in inertial properties between athletes will not have an impact on their performance.

Vieten & Riehle (1992) concluded that the inertial properties were of “marginal importance” since the “relative inertia tensor, shows only insignificant changes for all individuals” [6 top trampolinists and 18 semi-trained adults]. The “relative inertia tensor” used was essentially the principal transverse moment of inertia in one of seven predefined postures divided by the principal transverse moment of inertia in a layout posture. Comparing this “relative inertia tensor” across athletes would determine if, when performing a layout at a fixed angular velocity and then changing to one of the other predefined postures, any particular athlete would experience a greater increase in somersault angular velocity. Vieten & Riehle (1992) quote standard deviations of their measured “relative inertia tensors” but do not indicate what magnitude difference equates to what difference in achievement in terms of the number of somersaults that could be completed. As a result it is not clear why they concluded that the values of the standard deviations quoted were not practically significant.

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4 It is not strictly a tensor but a ratio of the moments of inertia.
In addition to different athletes having different inertial properties, the estimation method used to determine these inertial properties will produce different data sets and so would result in different motions being predicted in any simulations. Kwon (1993) estimated the inertial properties of “three collegiate male gymnasts” using ten different inertial property estimation methods. These estimates were input into a simulation of a full twisting double back somersault dismount from the high bar, using techniques extracted from actual performances. Kwon compared the results of the simulation with actual filmed performances, to determine the simulation accuracy. The different estimation methods produced significantly different inertial properties for all segments (p < 0.05). These differences then affected the calculated magnitude of the average airborne angular momentum and the accuracy of the simulations. Kwon calculated the relative simulation errors as the simulation error divided by the maximum angle ranges. The mean relative simulation errors he observed were 2.9±0.7% for somersault, and 17.2±7.3% for twist. It was thus concluded that “all estimation methods provided equally accurate somersault simulation results”, but that there were significant differences between methods of estimating inertial properties and the accuracy of the twist simulation. It was suggested that the twist angle was more susceptible to error than the somersault angle since the twist moment of inertia was smaller, and so the same magnitude of error in the inertial properties would have a larger effect. This reason, however, gives no consideration to the cause of the error: Would the same magnitude of error in fact be likely?

Due to the greater accuracy in twist simulations, Kwon (1993) recommends using the estimation method based on Yeadon, stepwise regression equations from cadavers, and a method based on Hanavan, in that order, for any “complex airborne movements”. Even though Kwon (1993), recommends some estimation errors over others it is acknowledged that the results are not broadly applicable since only three athletes were studied, all of whom were male college students of similar age (21 or 22 years), height (172.7-180.0cm), and mass (63.6-74.4kg). In addition only one manoeuvre was observed. Kwon did consider the effect of perturbations of the inertial properties, although each perturbation of an inertial property was considered to be independent, but not how the perturbations may occur as a result of the production of an inertial property estimate. As a result this analysis does not address where the errors lie in any estimation method, nor suggest what may be done to improve the accuracy of any of the methods. It is thus impossible to predict with any confidence, the situations in which the recommended methods would be the most accurate methods beyond the three athletes studied.

From Kwon’s work it is clear that applying inertial property estimates based on different methods alters the results achieved. Without knowing what the results should
be, it is impossible to determine which inertial property estimate is the most accurate for any particular performance. The most accurate inertial properties for one skill are not necessarily the most accurate for all skills (Kwon, 1993). It is thus necessary to choose a focus: either to explore the mechanical equations with reasonable estimates, or to replicate a specific individual’s performance. The focus of this thesis is to understand and explore the mechanical equations, and assess the effect of inertial properties on these equations with the intention of increasing understanding of the mechanics of a twisting somersault and recommending certain postures or techniques; it is not to replicate a specific athlete’s performance. Thus, the inertial property data used should reflect a region of inertial properties in which gymnasts and divers are expected to lie; they do not need to be accurate estimates of any particular individual. This is the rationale behind the inertial property data set collation process in Chapter 3. All conclusions are to be treated as reasonable expectations for athletes in general, not accurate predictions of any individual’s performance.
Chapter 3

Body model and representative inertial properties

In order to mathematically investigate the possible rotations an athlete may achieve, it is necessary to select a model for the body, along with appropriate inertial properties, and then derive appropriate equations of motion. This Chapter defines the model of the body used and collates inertial properties representative of current divers and gymnasts. Chapter 4 then derives the equations of motion for both when the athlete holds a quasi-rigid posture and when there is a postural change.

The inertial properties collated in this chapter are intended to be representative of the athlete population of interest, not necessarily accurate values for specific individuals. These inertial properties are then used in Chapter 5 and Chapter 6 to compare postures and techniques and predict what skills could reasonably be expected to be achievable. Further, differences in achievement are used to discuss whether or not some athletes possess a natural advantage due to their inertial properties.

3.1 Multi-segment representation of the human body

The rigidity of the skeleton means that postural change may be assumed to occur only by altering the joint angles. The human body thus lends itself to being modelled as a set of rigid segments connected by joints. Treating the body as a multi-rigid-segment system is common in biomechanics (Miller, 1970; Pike, 1980; Van Gheluwe, 1981; Baughman, 1983; Hatze, 1980; Yeadon, 1984; Huston, 2009). Even when using the skeleton to determine the segments used, each segment is not a truly rigid body due to the presence of soft tissues, muscle movement, and the nature of breathing where the chest expands and contracts. Nevertheless Hatze (1980) concludes that the error in inertial property estimation of segments as a result of the segments not being truly rigid is less than 6%, and so Hatze deemed the rigidity assumption to be acceptable. King, Kong & Yeadon (2009) considered the effect of soft-tissue, by adding wobbling masses to rigid segments and simulating forward dives. They concluded that the wobbling masses had no significant effect on their
model predictions. This thesis is concerned with gross movement, as was the case in the aforementioned studies; thus modelling the human body as a multi-rigid-segment body is considered an acceptable approach.

The rigid segments of the body model are connected by joints. Rather than modelling each joint individually, joints are modelled as spherical joints with constraints applied to the allowed orientation angles. These constraints reflect different types of joints and reasonable physiological limits to the range of motion at that joint.

### 3.1.1 Segment numbers and allowed joints

The skeleton may be used to guide the division of segments. Motion may only occur at joints and so it is reasonable to define each segment as the tissue between two joints. It is not necessary that there be movement at every joint. If no movement is allowed at a joint then the body part on either side of the joint may be considered as one segment. For example, if there is no movement allowed at the wrist or elbow, then the arm may be considered as one segment; however if movement is allowed at the elbow then the upper and lower arm need to be treated as separate segments.

When investigating somersaults previous authors have used four (Miller, 1970) segments to model pike and layout somersaults, five (Pike, 1980) to model full twisting layout somersaults, six (Van Gheluwe, 1981) to model twists in back somersaults allowing flexion through the torso, eleven (Yeadon, 1984) to model twisting somersaults in a variety of postures and using a variety of twist initiation actions, and 15 (Kwon, 1993) when modelling a full twisting double back somersault dismount from high bar. In other areas of literature 15 (Baughman, 1983) and 17 (Huston, 2009) segments have been used; the additional segments increased the number of joints and so allowed greater diversity of movement to be modelled.

When investigating the mechanics of techniques used in somersaults, motion at the wrists, ankle, and neck may be ignored. Movement at the wrists and ankles will be small as athletes’ are seeking clean lines during performance. Rotation of the hand about the longitudinal axis of the forearm, although not strictly occurring at the wrist joint, will also be ignored. Motion of the head will occur predominantly so that the athlete can stabilise the head and use visual cues, and is not the focus of any technique. It is, thus, reasonable to combine the hand and lower arm, the foot and lower leg, and the head, neck and chest. Since the spine is made up of a large number of vertebrae it is difficult to know how many segments are required to represent the torso. It is clear that athletes do flex through the torso, as may be seen in a number of sport-specific postures defined in Section 3.9, and so a one-segment torso is insufficient. Previous authors, who have included flexion in the torso, have used two (Hanavan, 1964; Van Gheluwe, 1981) and three (Baughman, 1983;
Body model and inertial properties

Yeadon, 1984; Huston, 2009) segments. Considering the inertial property data that could be collated the torso can be divided into at most three segments: pelvis, abdomen, and chest. It would thus be reasonable to use an eleven segment model similar to Yeadon (1984): pelvis, abdomen, chest-neck-head, two upper arms, two lower arms combined with hands, two upper legs, two lower legs combined with feet. When discussing techniques and actions no more than these eleven segments are used. However, since inertial property data sets and estimates often give the hands, feet, neck and head separately it is necessary to start with a model which can accept these inputs. Thus the body model used consists of 17 segments and be reduced to fewer segments as required. The segments are illustrated and numbered in Figure 3-1, and Table 3-1 lists the numbers against a description of the segments.

![Figure 3-1: The seventeen-segment model of the body. Joints are in black text, segments are numbered in blue. After (Anthropology Research Project, 1988)](image)

<table>
<thead>
<tr>
<th>Number</th>
<th>Description</th>
<th>Number</th>
<th>Description</th>
<th>Number</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pelvis</td>
<td>6</td>
<td>Left Upper arm</td>
<td>12</td>
<td>Right upper arm</td>
</tr>
<tr>
<td>2</td>
<td>Abdomen</td>
<td>7</td>
<td>Left lower arm</td>
<td>13</td>
<td>Right lower arm</td>
</tr>
<tr>
<td>3</td>
<td>Chest</td>
<td>8</td>
<td>Left hand</td>
<td>14</td>
<td>Right hand</td>
</tr>
<tr>
<td>4</td>
<td>Neck</td>
<td>9</td>
<td>Left upper leg</td>
<td>15</td>
<td>Right upper leg</td>
</tr>
<tr>
<td>5</td>
<td>Head</td>
<td>10</td>
<td>Left lower leg</td>
<td>16</td>
<td>Right lower leg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>Left foot</td>
<td>17</td>
<td>Right foot</td>
</tr>
</tbody>
</table>
The methods that are used to collate inertial properties (Section 3.5) agree on the anatomical features used to specify the joints between segments, except for the joints of the torso: the abdomen-chest was sometimes at the level of the xiphoid process (Hanavan, 1964) and at other times at the level of the tenth rib or bottom of the ribcage (Baughman, 1983; Yeadon, 1984). The pelvis-abdomen was sometimes at the level of the iliac crest (Baughman, 1983) and sometimes at the umbilicus (Yeadon, 1984). These differences reflect differences in where these authors believed the joints should be, and the ease of identifying anatomical features that divide the segments. Even though the torso segment divisions differ slightly, these are still the only places at which the model of the torso is allowed to flex, and so the way in which the segments connect is unaltered. Thus, if the pelvis segmentation is at the umbilicus and not the iliac crest, the estimation method would be considered to produce an athlete with a slightly larger pelvis, and smaller abdomen. In Chapter 5 when joint angles are measured, the top of the iliac crest is used to divide the pelvis and abdomen, since it is easier than the umbilicus to identify when observing performances. These joint angles are used to define postures and techniques. Differences in predictions using different methods of estimating inertial properties thus include differences in the values of the inertial properties and the slight differences in joint position.

### 3.1.2 Inertial properties required for each segment

Twisting somersaults are three-dimensional skills and so it is necessary that the inertial properties for each segment cover all three dimensions. Described below are the inertial properties required of a segment: the mass, the inertia tensor or the principal moments of inertia and their directions, the location of the centre of gravity, and the origin.

The **mass** is constant and independent of any frame of reference.

The **inertia tensor** of a segment is the inertia tensor about the segment’s centre of gravity, and aligned with the local frame (Section 4.1.1). This matches inertial property data available.

The **centre of gravity** vector of a segment gives the coordinates of the centre of gravity of that segment with respect to the local frame. Since each segment is rigid this vector is a constant with respect to the local frame.

The **origin** vector of a segment is the vector from the origin of the more proximal segment to the origin of the current segment; it is known with respect to the local frame of the proximal segment. There are seventeen origins, although the origin of the local frame for the pelvis is (0, 0, 0). The origin vector is independent of the joint angle and is thus an
inertial constant since it is a vector between two fixed points of a rigid segment. Figure 3-2 illustrates the origin vector.

![Figure 3-2: The relationships between the origin vector and the local frames](image)

In this picture the origin vector is the vector from the origin of the local frame of segment 1 to the origin of the local frame of segment 2. It is known with respect to the local frame of segment 1.

### 3.2 Choosing a representative sample

Somersaults and twisting somersaults in diving and gymnastics are explored in this thesis. Therefore, inertial properties collated should reflect current divers and gymnasts, rather than a member of the general public. The inertial properties do not need to be accurate representations of specific people, but simply to reflect reasonable variation within the diving and gymnastic population; this is because the inertial properties are used to explore what expectations are generally reasonable, rather than what a specific athlete may achieve. Mathematically speaking choosing a representative sample limits the inertial property parameters in the equations of motion from all real numbers to a region where divers and gymnasts inertial properties are expected to be found. This focuses the exploration of the nature of these equations to allow a physical interpretation in a sporting context to be made.

It is insufficient to apply one set of inertial properties as an ‘average’ and expect all other athletes to be a scaled version of this average. It is insufficient because, in the words of Daniels, (1952): “it is virtually impossible to find an ‘average man’… because of the great variability of bodily dimensions which is characteristic of all men.” Daniels went on to seek an approximately average man (within 30% of the mean) in the U.S Air Force population and found no such man.

It is clear that between athletes differences in body proportions and inertial properties exist. Significant differences have been reported between genders (Johnson, 1976; Hall & Depauw, 1982; Finch, 1985; Zatsiorsky, 2002; Ma, et al., 2011; Nikolova & Toshev, 2007), between elite female gymnasts and a control sample (Sands, et al., 2003), between age groups (Jensen, 1989; Hall & Depauw, 1982; Zatsiorsky, 2002), between stomatotypes (Dempster, 1955; Jensen, 1978; Finch, 1985; Park, et al., 1999), and between ethnicities (2002).
Body model and inertial properties

(Cheng, et al., 2000; Shan & Bohn, 2003; Nikolova, 2010a; Nikolova, 2010b; Ma, et al., 2011). Differences in body proportions and inertial properties affect technique choice; for example, Shan & Bohn, (2003), hypothesise that due to their proportionately longer legs Caucasians would “prefer to amplify stride length” while Asians would opt to increase stride frequency when seeking to increase running speed. In order to appreciate the practical significance of the variation between athletes it is important to collate inertial property data sets from a broad range of gymnasts and divers.

No inertial property data sets providing the inertial data as listed in Section 3.1.1 and specifically representing divers or gymnasts were available in the literature. Thus, the inertial properties of current divers and gymnasts were estimated. Recruitment was by invitation through the NSW Institute of Sport to their diving and gymnastic squads and associated divers and gymnasts. No exclusion criteria were set so that the broadest range of athletes were included.

Fifteen inertial property data sets providing the data required in Section 3.1.1 which represented various percentiles or ‘average’ people (Anthropology Research Project, 1988; Nikolova & Toshev, 2007; Huston, 2009; Nikolova, 2010a; Nikolova, 2010b) were found. In the same way that Daniels (1952) could not find an average man, these cannot be considered to be real people, even though they are based on real people. They are presented in the literature with the suggestion they can be used when conducting theoretical biomechanical explorations or used in ergonomic design. It is for this reason they are included in the data sets used to explore the effect of inertial properties on rotational behaviour.

3.3 Short review of methods available for determining inertial properties

There are numerous ways of estimating inertial properties. A short review is provided in this section. The methods may generally be classed as direct, motion based, geometric scaling, averages and regression equations, or geometric models. There is not a clear boundary between each classification as refinements of one classification can merge into another; however the classifications provide a helpful structure when examining the methods available.

3.3.1 Direct

Methods classed as “Direct” have an inertial related property as the first output of the measurement procedure. The inertial property is determined from measurements taken on the actual body or body-segment or a casting of it. If a casting is used, a density adjustment
is required (Drillis, et al., 1964; Plagenhoef, 1983). The results obtained from direct measurements are specific to the person/segment measured.

Direct measurements that relate to the definition of inertial properties are the balance/suspension method (Contini, et al., 1963; Zatsiorsky, 2002), the rotating platform method (Page, 1969; Griffiths, 2006) and the method of using the effective line of action of the weight (Contini, et al., 1963). The balance/suspension method balances the segment on a knife edge or suspends it to find the balance point. The centre of gravity is defined as the point about which all the mass is balanced and so this method enables the centre of gravity to be found. The rotating platform method determines the moments of inertia by using the definition of the moment of inertia as the constant of proportionality between the applied moment and the angular acceleration, when the axis of rotation is a principal axis (Page, 1969; Griffiths, 2006). The effective line of action may be calculated from statics once the reaction force and distribution required to support the segment are known (Contini, et al., 1963). These three methods are limited to determining inertial properties of cadaver segments, castings, or the whole body.

Inertial properties are determined by the shape and density of a segment; by measuring shape and density or applying an assumed density, the inertial properties may be determined using their integral definitions. Gamma scanning (Zatsiorsky, 2002), CT scans (Brown, 1987) and duel energy x-ray absorptiometry (DXA) (Durkin, et al., 2002) may be used to determine both shape and density using the attenuation of the gamma or x-rays. Some knowledge of the tissue to be scanned is required, so that the process can be calibrated. MRI may be used to determine shape and tissue type to which an assumed density of that tissue may be applied (Brown, 1987). Due to the non-linear relationship between signal strength and mass density, it is more common to assign an average density to each tissue type (Brown, 1987). Shape and volume are calculated from the number of pixels in each image of a certain tissue type multiplied by the difference in height between where the images were taken (Cheng, et al., 2000). All inertial properties may be determined from each scan using their integral definitions. Anatomical landmarks, or bony landmarks, in the images are used to define the ends of segments (Durkin, et al., 2002; Zatsiorsky, 2002). Gamma scanning, CT scans and DXA all expose the athlete to radiation while MRI does not. The largest errors in the estimates are for the longitudinal moments of inertia (Brown, 1987; Zatsiorsky, et al., 1990). The DXA scan can only determine inertial properties in the plane scanned. Overlap of segments may preclude its use for determining these inertial properties (Durkin, et al., 2002). The major limitation of all four methods is the expense (Pearsall & Reid, 1994). They are also time consuming for the athlete, since they must attend a location with the required equipment.
Photographgrammetry (Contini, et al., 1963; Herron, et al., 1976; Tong, et al., 2012) determines the shape of the body using a series of photographs and/or images from a depth camera. An assumed density is then applied. All inertial properties may be determined from the scan using their integral definitions. Photographgrammetry will be less accurate than Gamma, CT, DXA, or MRI since it provides no information regarding tissue type or density, but relies on using an average density based on what is available in the literature. This literature data for density is derived from cadaver data (Appendix A.1.1 for some discussion on cadaver data). The method of immersion (Contini, et al., 1963; Dempster & Gaughran, 1967) allows volumes of segments to be determined, but not their shapes. As a result, the mass of a segment may be estimated by applying a constant density, but the moments of inertia cannot be estimated. Due to assuming a constant density the centre of gravity is located at the centre of volume.

Inertial properties are constants within static and dynamic force equations. By observing specific predefined static positions or dynamic actions that move only one segment or one fixed combination of segments, and then measuring forces, velocities and/or accelerations, the inertial properties may be determined. Such methods include the reaction change (Drillis, et al., 1964), quick release (Drillis, et al., 1964), pendulum (DuBois & Santschi, 1962; Drillis, et al., 1964; Chandler, et al., 1975), and torsional pendulum (Drillis, et al., 1964; Tichonov, 1975; Klose, et al., 1993). The reaction change method (Drillis, et al., 1964) can determine the mass and/or the location of the centre of gravity by comparing the difference in reaction forces on a support when the body is in two different postures. The quick release method (Drillis, et al., 1964) can determine the moment of inertia of a limb segment about its proximal joint by measuring the instantaneous acceleration of the limb following the removal of a known force against which the muscles had been contracting isometrically. The pendulum method swings the segment of interest like a pendulum with and without the addition of known weights, and measures the period of oscillation in order to determine the effective suspension point, centre of gravity, and moment of inertia (Drillis, et al., 1964) by comparing the period of oscillation between three cases: body segment alone, body segment with a known weight at a known location, and body segment with a second known weight at a known location. The pendulum method may be easily applied to cadaver segments and castings. It may only be applied to limb segments of a living person if it can be assumed that any resisting forces are negligible and the axis of rotation is known. The torsional pendulum method (Drillis, et al., 1964; Tichonov, 1975; Klose, et al., 1993) measures torsional oscillations to determine the moment of inertia of the whole body about a central axis. By comparing the moment of inertia and the location of the centre of gravity when the body is in different postures, which differ only by the position of one segment, the difference in the segment’s moment of inertia and centre of gravity location for the two positions may be determined (Tichonov, 1975). This method is
typically limited to limb segments (Drillis, et al., 1964; Tichonov, 1975). Further, it is important to realise that the moment of inertia determined is about the central axis and so includes the moment of inertia about the centre of gravity and the centre of gravity’s position with respect to the central axis.

**3.3.2 Motion based**

The inertial properties are parameters that are applied to segments within a model of the human body to personalise the model to an individual. If the model is known to reflect reality, then rather than estimating the inertial properties from measurements of the athlete, they can be taken to be the values that allow the personalised model to simulate observed motion of the specific athlete. The accuracy of the inertial properties depends on how well the number and connection of segments reflects reality, the accuracy of the kinetic and kinematic data collected, and the accuracy of the numerical solvers used to determine the inertial properties.

The simplest motion based method involves only two segments. Kane (1973) explains how the moment of inertia of two segments can be obtained from the momentum equations of motion when an athlete is placed on a frictionless turntable such that the joint between the two segments is placed on the turntable axis, and one segment is fixed to the turntable. Measuring the angular displacement of each segment as a result of one segment moving relative to the other allows the ratio of the moments of inertia of the two segments about the joint axis to be calculated. Independently measuring the combined moment of inertia when the segments are held rigid allows the actual values of the moment of inertia of each segment about the common joint to be calculated.

It is not necessary to know all the inertial properties independently. Instead it is sufficient to determine the barycentric parameters, which are the minimum inertial-property related parameters required for the specific model of interest. Lu & Ma (2012) defines and demonstrates how barycentric parameters may be determined.

As the model complexity increases so does the number of barycentric parameters, and hence the computational cost of solving the equations of motion simultaneously. It is also necessary that the movements observed excite all degrees of freedom in different ways to ensure that there are sufficient data points to determine the inertial properties (Lu & Ma, 2012). For movements performed in contact with the ground such as in Venture et al. (2008) it is necessary to measure the ground reaction forces and the joint angles of the body simultaneously. Lu & Ma (2012) recommends using impulse-momentum equations since the internal forces, including energy dissipating forces, do not need to be known; they found that the impulse-momentum equations were less sensitive to measurement errors than the equations using acceleration as an input.
The presence of measurement error in the kinematic and kinetic inputs means that the
equations cannot be solved analytically. Least-squares or another optimisation technique
would be required (Lu & Ma, 2012). The major limitations of motion based methods are,
that it may not be possible to excite all degrees of freedom, and the computational cost of
processing sufficient data to determine all the barycentric parameters is high.

### 3.3.3 Geometric scaling

Geometric scaling assumes that an athlete has the same proportions as another athlete
whose inertial properties are known. The scaling factor is determined by a single or several
geometric measurements on the current athlete. The most basic geometric scaling assumes
isometry, or equality of measure, and so a single length measurement is used to alter the
length scale; density is assumed to be the same. A scaling factor may also be applied
segment by segment, using the length of the segment to determine the scaling factor. A
density scaling factor may be applied based on total body mass. For example Hinrich
(1985) assumes that every segment is a circular cylinder and that all segments are
proportional to the standing height. He assumes that the body has constant density, which
is proportional to the total body mass. In the derivation of the transverse moment of inertia,
Hinrich assumes that the radius of a segment is small compared to its length.

Forwood, et al. (1985) sought to determine the accuracy and hence the value of geometric
scaling as a method for estimating transverse moments of inertia. Two scaling methods
were considered: scaling using body mass and standing height as used by Hinrich (1985)
and scaling using body mass and segment length. For each subject Forwood, et al. (1985)
applied both scaling methods to the mean moment of inertia of the group, and to the
subject in the group whose height-to-body-mass ratio was closest to the subject of interest.
The estimates from each scaling method were compared to each subject’s actual moment
of inertia. It was found that only 16-43% of the estimated moments of inertia were within
5% of the subject’s actual moment of inertia; the moments of inertia of the taller and
heavier subject were estimated less accurately, being “markedly underestimated”. No
consistent pattern of error was found across the segments in either of the scaling methods.

Further, Zatsiorsky (2002) observed that the mass of individual body segments as a
percentage of total body mass varies with the total body mass. In particular he identifies
that the relative mass of the abdomen increases while the relative mass of the head, feet,
and hands decrease with increasing total body mass. This is almost certainly the result of
the tendency for body fat to be stored in the abdominal region (Zatsiorsky, 2002).

The findings of Forwood, et al. (1985) and Zatsiorsky (2002) indicate that scaling is not an
appropriate means of estimating inertial properties.
3.3.4 Averages and regression equations

Regression equations are simply curves that are deemed to best fit observed data (Zatsiorsky, 2002). Observed data could be from cadaver studies, (Clauser, et al., 1969), studies on living subjects, (Plagenhoef, 1983; Contini, 1972; Young, et al., 1983; Zatsiorsky, 2002) or studies using models and anthropometric measurements (Woolley, 1972).

There are two fundamental assumptions underlying the use of regression equations. Firstly, the inertial property of interest is a function of only the independent variables considered, or the effect of other variables is negligible; secondly, the person whose inertial properties are being estimated is similar to the people whose data went into forming the regression equations; it can thus be expected that the current person’s inertial properties lie on the regression line. The independent variables used should reflect some aspect of the inertial property, and since using regression equations reduces equipment and time they should be easy to measure. Geometric scaling (Section 3.3.3) may be thought of as a regression equation that must pass through the origin.

Regression equations may take a number of forms, although linear or allometric equations are typically sought. The input parameters may also vary. The most common input parameters are body mass and standing height, although some regression equations use combinations of segment length, segment width, a body fat indicator, circumferences and/or diameters (McConville, et al., 1980; Zatsiorsky, 2002). In a multi-input regression equation inputs are chosen based on the measurements available and the strength of each possible input’s correlation with the inertial property. Increasing the number of inputs would increase the accuracy of the regression equation; however, the improvements achieved with each additional input progressively decreases. Three was the maximum reported in McConville, et al. (1980) and a maximum of four, for some segments, was reported in Zatsiorsky (2002). It is also important that regression equations for moments of inertia are only for the principal moments of inertia, so that the assumed principal directions are mutually perpendicular (Chandler, et al., 1975; McConville, et al., 1980). Appendix A.2 provides a summary of common regression equations or averages. Not all of these can be used in this thesis since not all of them allow all inertial properties to be estimated.

Ultimately, without knowing an individual’s inertial properties and comparing them to the calculated inertial properties from regression equations, the error in the estimation is not known. There is no simple test to determine if an individual is sufficiently similar to the sample group on which the equations are based. Checks to ensure that the values would be physically possible, and checks on the reasonableness of the data, are thus important.
Nevertheless regression equations are useful when seeking inertial properties at the start of the design process when only general information about the people involved is available.

3.3.5 Geometric models

Geometric models use shapes which can be defined by relatively simple analytical equations to represent segments of the body. Uniform density within a segment is assumed. The inertial properties are then determined mathematically by the geometric shape and the assumed density.

To determine the parameters of the required geometric shapes it is necessary to take anthropometric measurements of an athlete or to apply previously published representative measurements. For example, if the upper arm is to be modelled as a cylinder then the length and circumference (or diameter) of the upper arm are the required anthropometric measurements. The density used may be a whole body average calculated by dividing the mass of the person by the sum of the volumes of the geometric shapes representing that person, or chosen based on published values.

The accuracy of geometric models is limited by the choice of the shapes and the knowledge of suitable density values: body segments are not simple geometric models, and variations in tissue type mean that the density of a segment is not uniform. Applying an average density to the whole segment affects the estimates of mass, moments of inertia and the location of the centre of gravity. More discussion of the uniform density assumption is given in Appendix A.1.

Many geometric models have been proposed in the literature. Distinctly different models have been presented by Hanavan (1964), Hatze (1979), Baughman (1983), Yeadon (1984), and Nikolova & Toshev (2002). The geometric methods that are used in this thesis are discussed individually in Section 3.5.6. The method presented by Hatze (1979) is not used since the required perimeters could not be obtained from the photographs that could be taken considering the constraints of estimating inertial properties of current athletes (section 3.5.5).

3.4 Constraints when selecting a method

Which inertial properties are required, the nature of the people whose inertial properties are being estimated and the equipment available to measure these inertial properties or the required anthropometric measurements restricts the methods that can be used.

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5 The choice of shape alters mass distribution of the body segment. The difference between the shape used in the model and reality means that there will be some errors in all the inertial properties: mass, location of the centre of gravity, and the magnitude and direction of the principal moments of inertia. It is typical to choose simple shapes where the principal moments of inertia are aligned with anatomical axes of the segment.
The inertial properties required were given in Section 3.1.2: they were mass, centre of gravity, origin, and tensor of inertia. These inertial properties need to be determined for all segments, with the exception that the hand and lower arm, the foot and lower leg, and/or the head, neck and chest could be combined. In addition athletes are living people and so the estimation method cannot be invasive. Since balance/suspension, immersion, and torsional pendulum methods do not estimate all inertial properties independently and the quick release and pendulum methods are limited to limb segments on living people, these methods are not suitable. Gamma scanning, CT scans, dual energy x-ray absorptiometry (DXA) and MRI are prohibitively expensive. Geometric scaling, as identified in Section 3.3.3, is not appropriate and further there is no representative data for athletes participating in gymnastics or diving that may be scaled. Regression equations are limited to sources that provide regression equations for all inertial properties or those that could be complemented by anthropometric measurements.

The number of athletes, especially those in elite programmes, is quite limited. To encourage participation the time required of the athlete could be no longer than 15 minutes and should not interfere with training schedules. Measurements were to be taken just before or after a session at the training location. Even though invitations were sent to the Australian Institute of Sport (AIS) gymnastics programme and Diving Australia, as well as NSWIS, only NSWIS allowed invitations to be given to their athletes. The AIS gymnastics programme representative did not give a reason for rejecting the invitation and Diving Australia indicated that participation was too burdensome on their athletes.

To remain within a 15 minute time limit and since any equipment required needed to be portable, it was only possible to measure height and weight and take a few other anthropometric measurements, or to mark key points and take a photograph that could be later scaled, or take a video for a motion-based method.

Motion-based methods require a motion capture system and, if the motion is on the ground, force plates to measure ground reaction forces. The available budget precluded portable force plates. Motion performed during a flight phase would eliminate the need for a force plate. During the flight phase it would be necessary to excite all the required degrees of freedom in the minimum number of jumps and ensure a safe and comfortable landing. To achieve a true representative sample it is necessary that athletes across a broad range of ages and abilities are included, but in doing so, there is no guarantee that all these athletes could safely perform the required actions. Thus motion-based methods were not suitable. Applying estimates of inertial properties to a simulation and matching this to observations of an athlete performing a sequence of skills within their capabilities can, however, be used as a check on the accuracy of the inertial property estimates. This is done in Section 3.10.
It was necessary to accept these constraints to maximise participation by the target group. The methods of estimating inertial properties used were regression equations⁶ and geometric models. The athlete’s height and weight was measured and then a photograph taken which was later scaled to obtain any additional length, width or depth measurements. In addition to fitting within the time constraints the use of photographs allowed any unusual measurement to be checked and corrected without requiring more time of the athlete. This helped to reduce errors in recording the anthropometric measurements. The height of the athlete, as extracted from the photographs and that measured directly was compared to check on the accuracy of the scaling of the measurements extracted from the photograph. Agreement within ±2% was considered acceptable, and only those photographs were used.

### 3.5 Methodology for estimating inertial properties of current athletes

This section describes the measurement process used to estimate inertial properties of current athletes. The work was approved by the University of Sydney Human Research Ethics Committee (HREC) protocol number 14974 on the 29th of June 2012.

#### 3.5.1 Athletes that participated

Thirty-four current gymnasts and divers volunteered to participate. Table 3-2 shows the “squad” (related to age and years of training) and gender distribution, while Figure 3-3 illustrates the spread of heights and total body mass (commonly referred to as an athlete’s “weight”).

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>12-or-under</td>
<td>7</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>Teen (13-16)</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>Senior</td>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Master</td>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>16</td>
<td>18</td>
<td>34</td>
</tr>
</tbody>
</table>

⁶ Section A-2 gives a summary of regression equations readily available in the literature. Only those methods that produce all the required inertial properties and whose inputs can be obtained from a photograph were used.
To assist in comparing the current athletes to the previous literature, the subjects’ heights and weights were compared to the eleven subjects from Sanders (1995) and the three from Kwon (1993) in Figure 3-3. The current athletes were also compared to the subjects used to develop the regression equations that are used in Section 3.5.6 (Finch, 1985; Zatsiorsky, 2002; Shan & Bohn, 2003; Ma, et al., 2011). Figure 3-5 and Figure 3-6 add boxes whose sides are the mean plus and minus two standard deviations (Finch, 1985; Zatsiorsky, 2002; Ma, et al., 2011) or points connected by lines for the minimum, mean, and maximum data available (Shan & Bohn, 2003).
Figure 3-5: Male athletes in the current study compared to those in previous literature. The boxes show the region that is the mean plus and minus two standard deviations (Finch, 1985; Zatsiorsky, 2002; Ma, et al., 2011) and the points connected by lines are for the minimum, mean, and maximum data available (Shan & Bohn, 2003).

Figure 3-6: Female athletes in the current study compared to those in previous literature. Boxes and lines are as for Figure 3-5 but for female athletes.

The subjects in Sanders (1995) and Kwon (1993) fell within the scatter pattern of the athletes included in the current study. It is thus reasonable to consider the current study athletes to be a part of the same population of athletes in gymnastics or diving.

Subjects in Finch (Finch, 1985), Zatsiorsky (Zatsiorsky, 2002), Shan & Bohn (Shan & Bohn, 2003) and Ma et al. (Ma, et al., 2011) are similar to the heavier and taller athletes in the current study. This is to be expected since the literature sources focussed on adults, while the current study includes children. Children constitute a large proportion of the gymnast and diver population and so it is important to include them.
3.5.2 Total body mass and stature

The athletes’ total body mass was determined using a set of portable scales which read kilograms to one decimal place. Even though Marfell-Jones et al (2006) recommends measuring stretched standing stature to provide the greatest consistency in measurement, the athlete’s standing stature was measured, so that it would be directly comparable to the stature measurements taken from the photograph. The maximum difference in stretched standing stature or standing stature is expected to be 1%. The athlete’s standing stature was measured using Procedure 3-1.

Procedure 3-1: Measuring the athlete’s height.

1. Ask the athlete to stand with the tape measure aligned with the midline of their body and with their back to the wall with feet, buttocks and shoulder blades against the wall. Ask the athlete to have the feet slightly apart in a comfortable standing position so that the measurement will be the same as in the photograph where slight leg separation is needed to distinguish the legs.
2. Check head is level: The Frankfort plane is parallel to the ground as illustrated to the right (Marfell-Jones, et al., 2006).
3. Ask the athlete to breathe in without tilting their head. Lower a headboard to the vertex of the skull, read, and record the athlete’s height.

3.5.3 Markers to identify required levels

When extracting measurements from photographs it is important that the anatomical features defining the measurement levels are clear in both the front and the side photographs (Section 3.5.4). Some features such as eye level are obvious in both the front and side views, while other features, for example the level of the trochanter, are not clear in both views. To assist in identifying the correct levels, before the photographs were taken, the following anatomical features were marked with coloured strapping tape on the lateral side of the body so that they would be visible in both photographs:

- Mid-calf (greatest muscle bulk)
- Mid-thigh (greatest muscle bulk)
- Leg at groin/crotch
- Trochanter on both legs
- Below the gluteal flap
- Buttocks (greatest muscle bulk)
- Iliac crest
• Level of the umbilicus
• 10th rib
• Xiphoid process
• Level of the nipple
• C7 vertebra
• Acromion
• Mid-forearm

3.5.4 Front and side photographs

It was necessary to take two photographs so that measurements of length, width and depth could be extracted. To scale the photograph a third photograph of a checkerboard (11-by-11 squares each with side length 10 cm) is required. Procedure 3-2 gives the steps for taking the necessary photographs.

Procedure 3-2: Taking the photographs from which anthropometric measurements would be extracted.

1. Draw a cross on the floor with one stroke aligned with the approximate optical axis of the camera.
2. Set the camera to take a portrait shot and zoom so that the athlete will be in the field of view and the umbilicus is approximately at the centre of the image.
3. Ensure the camera is set to “photograph” at the maximum resolution.
4. Place the checkerboard (11-by-11 squares each with side length 10 cm) along the stroke of the cross perpendicular to the optical axis of the camera. Take a photograph of the checkerboard.
5. Ask each athlete to stand with their longitudinal axis on the centre of the cross and their frontal plane aligned with the line perpendicular to the camera’s optical axis. This will mean that the scaling achieved by the checkerboard is correct for the frontal plane.
6. The Frankfort plane should be parallel to the ground. Ask the athlete to breathe in just prior to the photograph being taken. Take the front photograph.
7. Ask each athlete to then stand with the length of the foot closest to the camera on the line perpendicular to the camera’s optical axis. This will mean that the scaling achieved by the checkerboard is correct for the plane through the centre of the leg closest to the camera. The arm will be slightly closer to the camera and so will tend to be slightly overestimated. Torso measurements, which are widest at the centre of the body, will be slightly underestimated as they will be behind the line.
8. Again check the Frankfort plane is parallel to the ground and ask the athlete to breathe in just prior the photograph being taken. Take the side photograph.

There will be a small error in scaling dimensions from a photograph due to the three-dimensionality of the athlete. This out-of-plane effect was estimated to result in approximately 2.5% error in the case of the largest participant, and considerably less for the majority of the participants. As a consequence these small errors were neglected.
3.5.5 Extracting measurements from the photographs

The photographs were digitised using the programme Tracker (Brown, 2012). Using a photograph and Tracker means that the key points may be reviewed and if any unusual measurements are identified the position of the key points on the photograph may be checked, in case they were erroneously placed. The key points and axis system should be defined for the first photograph processed. Then they may be imported for each new photograph and moved to the required locations. In Tracker the image was scaled using the “calibration stick” tool, aligning it with the length of the checkerboard, and entering the real world length of 110 cm. After this scaling process, lengths, widths, and depths may be determined by using the coordinates of the key points. Distortion of the image was assumed negligible, and this was confirmed by checking other lengths along the checkerboard. Procedure 3-3 lists the steps required, and Figure 3-7 illustrates how anthropometric measurements may be extracted.

1. Copy the checkerboard photograph, front, and side view photographs into the appropriate folder for the specific athlete. Rename the checkerboard photograph as <AthleteID>1, the front view as <AthleteID>2, and the side view as <AthleteID>3.
2. Import the photograph sequence into Tracker.
3. Create a calibration stick and align it with a horizontal checkerboard line. This will define the scale and any tilt of the axes. Enter the length of the calibration stick in cm. If there is no glare the full checkerboard may be used and its length will be 110 cm.
4. Place the Tracker reference frame on the centre of the cross marked on the floor and align the x-axis with the line on the floor. The origin at the centre of the cross means the y coordinate of the vertex gives standing stature.
5. Align the reference y-axis with the vertex or the x-axis with the line on the floor to account for any tilt of the image. It is assumed that the athlete is standing perpendicular to the floor.
6. Import the markers using File>>>Import>>>Video. Select file, choose import, select all features except the video and click ok.
7. Move each marker to the appropriate point on the current athlete. The same marker is used for the front and side views corresponding to the same anatomical feature. The location in the front view will be recorded in the Tracker tables as the second time step, while the side view will be recorded as the third time step. (The checkerboard is the first time step). Symmetry of the body is assumed and so measurements of only one arm and one leg are required.
8. Save the tracker file, and copy data for each marker into Excel for calculating lengths, widths and depths.

Procedure 3-3: Extracting the coordinates of key points from the photographs.
3.5.6 Estimating inertial properties

Since, as stated in Section 3.1.1, an eleven segment model is all that is required, any method of estimating inertial properties or data from literature which gives the inertial properties for these eleven segments, is sufficient.

Nineteen methods of estimating inertial properties were used to produce initial estimates. These estimates were checked and “unreasonable” ones rejected (section 3.5.7). These methods were compatible with the procedure presented above using a photograph of the athlete: they required only slight or no adjustments to be made to the extracted measurements. The nineteen methods include fifteen published methods and four methods which are slight modifications of these. The methods are

- Finch (1985), who measured Canadian female college students
  1. Average percentages for females, Somatotype unknown
  2. Average percentages for females, Endomorph
  3. Average percentages for females, Mesomorph
  4. Average percentages for females, Ectomorphs

*Note:* Methods based on Finch (1985) are only applied to the female athletes.
Body model and inertial properties

- Zatsiorsky (2002), who measured Russian athletes
  5. Linear regression
  6. Non-linear regression
  7. Average percentages
- Shan & Bohn, (2003), who measured on young adults
  8. Germans, linear regression equations
  9. Chinese, linear regression equations
- Ma et al (2011), who measured Korean adults,
  10. Linear regression equations
- Hanavan (1964) geometric based methods
  11. HanavanBP: the excess mass is evenly proportioned across the segments.
  12. Using mass distribution as presented in Woolley (1972)
  13. HanavanY, using densities from Dempster (1955)
  14. Hanavan3, using densities from Dempster (1955) and adding a third torso segment
- GOBD
  15. Basic geometric model
  16. GOBD-Y, using densities from Dempster (1955)
- Yeadon
  17. As presented in Yeadon (1984)
  18. Reduced Yeadon, eliminating mid-limb measurements
- Nikolova & Toshev
  19. As presented in Nikolova & Toshev (2007)

Each of these methods is described briefly in Sections 3.5.6.1 to 3.5.6.8. Additional descriptions and equations presented in Appendix A.3 help to clarify the equations used in these models.

Since front and side photographs were used, widths, depths, and lengths, but not circumferences, may be measured. As a result, when a circumference was required, it was estimated from width and depth measurements. When the intention of measuring the circumference was to determine width, depth or a radius, then the actual width or depth was used, or half the average of the width and depth when a radius was required. When the actual circumference is required, as in the case of Zatsiorsky’s (2002) non-linear regression equations, the circumference was calculated as the average of the width and depth of the segment multiplied by π. This matches Zatsiorsky’s (2002) assumption that the segments are cylinders with an adjustment factor applied.
The regression equations from Finch (1985), Shan & Bohn (2003), and Ma et al (2011) did not include a regression equations for the transverse locations of the hip and shoulder. Measurements from the photographs were used instead. The regression equations for length were then used to check if the inertial property data sets produced were reasonable (Section 3.5.7).

### 3.5.6.1 Finch (1985)

Finch (1985) presents regression equations based on fifteen Canadian female college students selected to represent the three somatotypes. Even though Finch (1985) uses only two torso segments, the model was included since it focused on females, of which there are few suitable studies, and included three somatotypes. The in-text equations were used rather than the final tabular presentations, following Finch’s suggestion that the in-text equations should be used by the “more conservative reader”. Finch (1985) includes regression equations for the length of segments using the total body height as an input.

### 3.5.6.2 Zatsiorsky (2002)

Zatsiorsky (2002) presents average percentages, and linear and non-linear regression equations based on a sample of 100 male and 15 female athletes. Standing height and total body mass are required as inputs, and for the non-linear regression equations segment circumference is also required. Zatsiorsky’s equations were for mass, moment of inertia, and the location of the centre of gravity. All lengths were extracted from the photographs.

### 3.5.6.3 Shan & Bohn (2003)

Shan & Bohn (2003) present linear regression equations requiring the inputs of total body mass, stature, and ethnicity (German or Chinese). The regression equations were based on 50 young Chinese and 50 young Germans. In the study “Young” was defined as 20–38 years old. Current athlete’s ethnicity was not collected, and so both methods were included.

### 3.5.6.4 Ma, et al. (2011)

Ma, et al. (2011) presented linear regression equations requiring the inputs of total body mass and stature. The regression equations were based on 40 male and 40 female subjects selected from 3-D body scan data from the “SizeKorean” database. The sample chosen was intended to reflect the wider Korean population.

### 3.5.6.5 Hanavan (1964)

Whitsett (1963) and Hanavan (1964) propose quite similar geometric models of the body. The model shapes differ only in the treatment of the feet, and the number of torso
segments. Hanavan’s (1964) method was used since it is the most recent, has two rather than one torso segment, and the full paper describing the model could be obtained.

Hanavan (1964) compared the calculated inertial properties of his model with measurements made when subjects held a series of set postures. Half of the predicted centre of gravity estimates were within 0.5 inches of the measured values and difference between predicted and measured values of the moments of inertia, were within 10% for the medial and transverse moments of inertia, and within 20% for the longitudinal moment of inertia; the predicted moment of inertia values for the hands and feet showed the greatest deviation from those measured. Hanavan (1964) concluded that the accuracy obtained was sufficient for the human body. Figure 3-8 illustrates Hanavan’s model, showing the geometric shapes used and divisions between segments.

To finalise the inertial properties an estimation of segment mass or density is required. Hanavan (1964) elected to estimate the segment mass by applying the regression equations, derived from cadaver studies, from Barter (1957). Barter does not separate the head and torso, or the upper and lower torso and so Hanavan (1964) splits these using average proportions and density information. Barter’s regression equations use total body mass as the input. However, the sum of the estimated segments may not equal the total
Body mass, due to the nature of the regression equations. As a result the remaining mass needs to be appropriately distributed amongst the segments. Hanavan simply stated that the extra mass should be distributed proportionately. Due to the uncertainty of the way in which to proportion the remaining mass two methods were considered: the first, call it HanavanBP, distributes the remaining mass in proportion to the mass predicted by the regression equations. The second is the approach taken by Woolley (1972) where the mass input into the regression equations is adjusted so that the sum of the regression equations gives the total mass.

Miller and Morrison (1975) suggest using multi-step regression equations from Clauser et al. (1969). Miller and Morrison (1975) suspect that these regression equations would be more accurate than Barter’s since they were based on a larger number of cadavers, were closer to living norms in terms of height and weight, and are multi-step regression equations. They could not, however, show the superiority of either method, since both produced reasonable estimated values of inertial properties of their subjects when compared with other studies. These multi-step regression equations also require skin fold measurements, which cannot be taken from a photograph and so this approach was not used.

Miller and Morrison (1975) found that when applying Hanavan’s model to a group of 30 highly-skilled athletes, the densities of the segments differed from densities provided by Dempster (1955); they were generally larger, especially for the leg densities. It is thus reasonable to also consider applying densities from Dempster (1955) to Hanavan’s geometric model. This method will be called HanavanY.

Hanavan’s model only has two torso segments. To allow for motions requiring three torso segments the lower torso was split into two elliptical cylinders at the level of the tenth rib. The umbilicus level will be used to define the cross-section of the segment. The density used will be taken from Dempster (1955). This method will be called Hanavan3.

### 3.5.6.6 GOBD (Baughman, 1983)

The GOBD model described by Baughman (1983) was intended to provide inertial property data that could be used with a previously defined articulated body model used when modelling automobile crashes and aircraft cockpit ejections. The model used to generate the inertial property data consisted of 15 segments of the body, all of which are “contact ellipsoids” or truncated right elliptical cones. The dimensions of each solid are derived from anthropometric measurements.

The density applied is equal for all segments and is the average density determined by dividing the total body mass by the volume of the calculated segments. The method
GOBD-Y will then use the geometry of the GOBD model but use densities from Dempster (1955).

### 3.5.6.7 Yeadon (1984)

Yeadon (1984) developed a geometric model for use with athletes performing twisting somersaults. The geometric model uses 30 sub-segments which are then combined into the segments used to model motion.

Yeadon (1984) checked the accuracy of the inertial properties by simulating observed twisting somersaults. It was found that the difference between measured total body mass and the model estimate of total body mass was less than 3%. In addition the difference in somersault and twist achieved in the simulated replications of actual performances were less than 0.05 and 0.14 revolutions respectively or 18° and 50° respectively.

Figure 3-9, Figure 3-10, and Figure 3-11 illustrate Yeadon’s sub-segments and give their shape and the measurements that are required at each level: P for perimeter, W for width, D for depth. Each segment’s height is determined as the distance between the levels defining its end points. The numbering is the same as Yeadon used, with S prefixing the segment numbers and L prefixing the numbers for the levels of measurement. Yeadon does not use a segment numbered 9 or 10. The mass of the feet and hands was determined from the assumed conical shape rather than by immersion as done by Yeadon (1984). In Yeadon (1990b), a model similar to the 1984 model, is presented, but uses 4 sub-segments to represent each foot and 3 sub-segments to model the hands. Since the hands and feet would not be the most significant segments the extra detail is deemed unnecessary.
Body model and inertial properties

Figure 3-9: The torso and head sub-segments in Yeadon’s model: sub-segments 1 to 8. After Yeadon (1984), figure 39.

Segment 8:
Semi-ellipsoid of revolution

Segments 6-7: Truncated right circular cones

Segments 1-5 are stadium solids defined by stadium shapes determined at levels 0 to 5.

Figure 3-10: The arm sub-segments in Yeadon’s model: sub-segments 11 to 15. After Yeadon (1984) figure 40.

Segment 15:
right circular cone

Length from wrist to knuckle III needs to be determined (Double this to get H15)
Figure 3-11: The leg sub-segments in Yeadon’s model: sub-segments 21 to 26. Segments 27 to 32 for the other leg are identical. After Yeadon (1984) figure 40.

Table 3-3 shows how the sub-segments from Yeadon’s model are combined to give the 17 segments of the model defined in Section 3.1.1. The axes of sub-segments that are combined are assumed to be aligned, and the final local axis for the combined segment has this same alignment. Yeadon applies density values extracted from Dempster (1955). These densities are also given in Table 3-3.

Table 3-3: Mapping Yeadon’s inertial segments to the 17-segment model

<table>
<thead>
<tr>
<th>Segment in 17 segment model</th>
<th>Sub-segments from Yeadon’s model</th>
<th>Density from Dempster (1955)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.01</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.01</td>
</tr>
<tr>
<td>3</td>
<td>3, 4, 5</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Yeadon’s segments 3 &amp; 4)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Yeadon’s segment 5)</td>
</tr>
<tr>
<td>4 combined with 5:</td>
<td>6, 7, 8</td>
<td>1.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 &amp; 12</td>
<td>11 &amp; 12</td>
<td>1.07</td>
</tr>
<tr>
<td>7 &amp; 13</td>
<td>13 &amp; 14</td>
<td>1.13</td>
</tr>
<tr>
<td>8 &amp; 14</td>
<td>15</td>
<td>1.16</td>
</tr>
<tr>
<td>9 &amp; 15</td>
<td>21, 22 &amp; 23</td>
<td>1.05</td>
</tr>
<tr>
<td>10 &amp; 16</td>
<td>24 &amp; 25</td>
<td>1.09</td>
</tr>
<tr>
<td>11 &amp; 17</td>
<td>26</td>
<td>1.10</td>
</tr>
</tbody>
</table>
Yeadon’s approach is distinct from other methods by its use of stadium solids for the torso and because it also takes measurements at the broadest part of the limbs in addition to the joints. An intermediary estimate “Reduced Yeadon”, which eliminates the additional measurement at the broadest part of the limbs, was also used. Reduced Yeadon combines Yeadon’s sub-segments 3 & 4, 6 & 7, 11 & 12, and 13 & 14.

**3.5.6.8 Nikolova & Toshev (2007)**

Nikolova and Toshev (2007) present a 16-segment model of the body and estimate the body segment parameters of the average Bulgarian male and female based on the anthropometric data of 5,290 Bulgarians. Nikolova (Nikolova, 2010a; Nikolova, 2010b) extends the model by adding right elliptical stadium solids for the limbs. Since longitudinal rotation about the longitudinal axis of any of the limb segments would be ignored in this thesis, the addition of right elliptical stadium solids for the limbs is unnecessary. Hence, the original model of Nikolova and Toshev (2007) is used. Table 3-4 lists the shapes applied to the 16 segments and Figure 3-12 illustrates the 16-segment model with the required dimensions labelled. Nikolova et al (2002) was used to supplement understanding of how the anthropometric measurements and the geometric shapes matched. Density values quoted differ from those given by Dempster (1955) by less than 0.02 kg/L and so it appears Nikolova intended to use the data from Dempster but when converting units there may have been a rounding error.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head and neck</td>
<td>Ellipsoid</td>
</tr>
<tr>
<td>Upper torso</td>
<td>Truncated reverted right elliptical cone.</td>
</tr>
<tr>
<td></td>
<td>The acromial, mid-sternum breadth and xiphoid</td>
</tr>
<tr>
<td></td>
<td>level measurements are used.</td>
</tr>
<tr>
<td>Middle torso</td>
<td>Elliptical cylinder</td>
</tr>
<tr>
<td>Lower torso</td>
<td>Elliptical cylinder and reverted elliptical</td>
</tr>
<tr>
<td></td>
<td>cone</td>
</tr>
<tr>
<td>Upper arm</td>
<td>Truncated cone</td>
</tr>
<tr>
<td>Lower arm</td>
<td>Truncated cone</td>
</tr>
<tr>
<td>Hand</td>
<td>Sphere</td>
</tr>
<tr>
<td>Thigh</td>
<td>Truncated cone</td>
</tr>
<tr>
<td>Shank</td>
<td>Truncated cone</td>
</tr>
<tr>
<td>Foot</td>
<td>Truncated cone</td>
</tr>
</tbody>
</table>

Table 3-4: The shapes of the 16 segments from Nikolova and Toshev (2007)
3.5.7 Rejecting unreasonable estimates

To ensure that clearly unreasonable estimates of inertial properties do not affect the theoretical exploration in this thesis, it is necessary to set some criteria for what is unreasonable and reject estimates accordingly. Estimates were rejected if one or more of the following occurred:

- A segment was predicted to have negative mass, negative moment of inertia, or principal moments of inertia that disobey the triangle inequality (that the sum of any two is greater than the third). This will be referred to as “Negative” for brevity.
- The absolute value of the difference in actual mass and predicted total mass of the athlete is greater than 20%. This will be referred to as “Body weight” for brevity.
The predicted lengths from the regression equations differed from measured lengths by more than 20%. This will be referred to as “Lengths” for brevity. Figure 3-13 is a stacked column graph showing the percentage of all the inertial property estimates that were rejected for one of the above reasons.

Without a method of accurately measuring the inertial properties, rather than estimating them, the underlying cause for each rejection is unknown, but will be related to the difference between the current athlete group and the group on which the method was designed. For example, as seen in Figure 3-5 and Figure 3-6, estimations for some of the current athletes would require extrapolation of regression equations beyond the original data set. Alternatively, body segment shapes may differ between groups or the density chosen is no longer suitable. Since it is not known which aspect has caused the method to be rejected it is not possible to suggest corrections to the estimate. For example if “Body weight” was the reason for rejection, this could be due to an under- or over-estimate of the shape of the segments or the densities used. Adjusting just the densities may produce the total body mass as required, but if the shape was erroneous then the moments of inertia of the segments will still be incorrect; it is impossible to know if this is the case.

Hanavan-BP and GOBD had the least number of rejections, with only one rejection each: Hanavan-BP had one under-12-female with the hands having negative mass and GOBD had the abdomen of one female master disobey the triangle inequality for the principal moments of inertia.
Each of the original sample groups used by Finch (1985), Shan & Bohn (2003), and Ma et al. (2011) to develop regression equation methods, were not athletes. All the rejections of these methods included the reason “Lengths”. The particular segments posing a problem varied.

The Zatsiorsky methods were rejected mostly for the reason “Negative”. The tallest and heaviest females in the current study were not quite as tall or heavy as the average in Zatsiorsky’s sample. The majority of the males in the current study were also below the mean height and mass of Zatsiorsky’s sample. It is apparent that the athletes in Zatsiorsky’s sample are from a different sub-population of athletes to those in the current study. The high number of rejections is thus not surprising. Estimates of the centre of gravity location for males and females differed when using Zatsiorsky’s linear regression equations. For males the regression equations predicted the location of the centre of gravity as a distance from the proximal end, while for females it was predicted as a percentage of segment length. As a result, it was possible that when using the regression equations for males, if the athlete’s segment of interest was shorter than that in Zatsiorsky’s sample the centre of gravity could be predicted as outside the segment. This is a systematic problem and did result in impossible values for the inertial properties of some of the males.

Nikolova had “Negative” and “Body weight” as reasons for rejection. The chest and pelvis were the segments which predominantly caused the rejection. Nikolova uses the acromial and mid-sternum measurements when determining mass, but uses the mid-sternum and xiphoid process measurements when determining moments of inertia. It is not known why the anatomical features used were changed; if the relationship between the anatomical features differs from whatever was assumed this could be one cause of the rejections.

Considering the Hanavan-based methods it is clear that the method of mass distribution used has caused the majority of the rejections. This can be seen by the jump in rejections due to “Body weight” from Hanavan-BP (~0%) and Woolley (~0%) to HanavanY (~56%). The change in density between GOBD and GOBD-Y does increase the number of rejections, but the increase is less than for the Hanavan-based methods. The smaller increase in the number of rejections could be due to the average density in GOBD being closer to the densities used in GOBD-Y. It thus appears that for the segment shapes of Hanavan’s model, the densities of Dempster (1955) are not the most suitable densities to apply.

The Yeadon methods have rejections due to “Negative” and “Body weight”. Since the number of rejections due to “Body weight” is less for Reduced Yeadon than Yeadon this suggests that the extra mid-segment measurements in Yeadon lead to an over-estimate of shape. However, it is also possible that the density used is not suitable, and the change in
shape from Yeadon to Reduced Yeadon cancels some of the error. The “negative” rejections are all due to the abdomen disobeying the triangle inequality. Since the moments of inertia are determined by geometry, such a rejection can only occur if the measurements are not compatible with the assumed shape.

Figure 3-14 is a scatter plot showing the number of rejections for each of the 34 athletes, with the athlete’s squad and gender indicated by colour. Finch was excluded from the counts so that gender could be compared and since almost all the Finch (1985) data sets were rejected. Visually it can be seen that the 12-or-under athletes had more rejected data sets than the other squads. Gender does not in general appear to make a significant difference, although it may if a particular method is considered.

Of 582 estimates of inertial properties, 357 were rejected, leaving 225 inertial data sets. All 34 athletes were represented in these remaining 225 data sets; thus these data sets still reflect the same squads and genders.

![Figure 3-14: Number of rejections by squad and gender (Finch excluded)](image)

### 3.6 Inertial property data sets from the literature

Fifteen inertial property data sets may be constructed from the literature: three from The Anthropology Research Project (1988), six from Huston (2009), four from Nikolova & Toshev (2007), one from Nikolova (2010a) and one from Nikolova (2010b).

The Anthropology Research Project (1988) presented inertial properties of hypothetical U.S. male aviators based on adjusting for changes in average height and mass and adapting data from previous studies. The data produced was stated as representing the 3rd, 50th and 95th percentiles of male aviators.

Huston (2009) presents inertial properties of segments said to represent the 5th, 50th, and 95th percentile male and female segments. These are based on some geometric simplifications of the body, but Huston believes they are still useful for a “large class of simulations”. Segments from the same percentile group were combined to give six data
sets: female 5th percentile, female 50th percentile, female 95th percentile, male 5th percentile, male 50th percentile, and male 95th percentile.

Nikolova & Toshev (2007) and Nikolova (2010a; 2010b) presented inertial property data for an average Bulgarian male and an average Bulgarian female. The average was determined by using a mathematical model and average anthropometric measurements based on an anthropometric survey of 5,290 Bulgarians. The four data sets from Nikolova & Toshev (2007) represent the male and female data sets using the tabulated un-adjusted and then adjusted segment parameters with Nikolova & Toshev’s model of each segment. The “adjusted” dimensions were derived by adjusting the dimensions of the assumed geometric shapes of each segment so that they better reproduce masses of segments as predicted by regression equations already published by Zatsiorsky (2002) and Shan & Bohn (2003). In this thesis the longitudinal rotation of the arm was not modelled or observed and so the transverse and medial moments of inertia of the arm from Nikolova (2010a) and Nikolova (2010b) was set as equal and to their average value.

3.7 Using collated inertial property data sets in theoretical modelling

The intention of collating inertial properties was limit the range of inertial properties applied to the equations of motions from the set of all real numbers to a region where the inertial properties of gymnasts and divers are expected to lie. The intention was not to name a specific individual and optimise a technique for them. Thus, the inertial properties do not need to be accurate representations of specific athletes, but merely to reflect the region of ‘possible’ athletes.

With this in mind, all 225 estimates of athletes that were not rejected in Section 3.5.6 and the 15 data sets from the literature would be considered as ‘possible’ athletes. These ‘possible’ athletes reflect both variability in the population sampled and variability due to applying different inertial property models. This gives a total of 240 inertial property data sets; these may be found in Appendix A.5. From this point forward these will simply be referred to as the ‘athletes’, with the single quotation marks retained to signify that this label is used for the purpose of brevity and should not be taken to imply that each data set represents a specific living individual.

Some of the data sets combine segments and so do not allow the independent movement of all 17 segments. These estimates will simply be considered to be particular athletes with a range of motion restriction between those segments. In situations where an action requires three torso segments, estimates based on Finch (1985) or Hanavan (1964) with two rather than three torso segments, are not used, while for postures the definitions of the postures
was modified for these data sets (Section 4.7). None of the techniques explored in this thesis involve movement at the wrists, ankles, or neck and so inertial property data sets with restrictions at these joints require no special treatment.

The 225 reasonable inertial property data sets were based on 34 athletes with the number of rejections per athlete varying as shown in Figure 3-14. This means that these data sets are not entirely independent; however, due to the difference in estimation methods they cover different parts of the region where the inertial properties of gymnasts and divers are expected to lie.

Estimates based on the same athlete have the same classification, such as gender, and the same biomechanic indices, such as BMI. Testing if a classification or a biomechanic index identifies those data sets with favourable inertial properties is not unreasonable since the difference in estimation methods can be considered to reflect smaller variations between possible ‘athletes’. If the data sets produce quite different outcomes, then it is reasonable to say that the classification or index is unable to identify ‘athletes’ with favourable inertial properties. However, if it is found that a classification or index is able to identify favourable inertial properties then care must be taken when interpreting statistical probabilities, since each data set is not entirely independent of the others, as is typically assumed in most statistical tests. Such situations should be treated as helping to identify classifications of indices that would be worthwhile for further investigation. Such investigation would require greater numbers of athletes and greater confidence in the accuracy of the estimated inertial properties.

### 3.8 Inertial properties for a specific posture or group of segments (ICG17)

All 17 segments of the body model (Section 3.1) were not required to model the rotation of an athlete in the remainder of this thesis. As a result it was necessary to be able to determine the inertial properties for groups of segments or the body as a whole. A Matlab programme “ICG17” was written for this purpose.

The programme ICG17 takes the inertial properties of the 17 segments of the body defined in Section 3.1.1, and a specified posture, to determine the inertial properties of the body as a whole in that posture. If the inertial properties of a group of segments, rather than the whole body, are required, then the inertial properties of the segments not in the group are set to zero. A full description of the programme including the required inputs and outputs, and their form as well as the mathematical logic used is provided in Appendix A.4.
3.8.1 Inertial properties for a group of segments

When the inertial properties of a group of segments held in a fixed posture, so that they act as a single rigid segment, are required, ICG17 may be used with the inputs for the inertial properties of segments not in the desired group set to zero. The rotation matrices for the group of segments of interest must describe the posture of those segments, and the remaining rotation matrices should be set to the identity matrix. If the group of segments needs to be used elsewhere and treated as one rigid segment then the inertia tensor of the new segment is Principal_I, and the centre of gravity vector will be the transpose of the Principal_directions matrix times the location of the proximal joint of the new segment. Figure 3-15 gives a flow chart of the process of grouping segments.

Table 3-5 describes each of the grouped segments that are required in Chapter 6.

Figure 3-15: Creating a grouped segment
**Table 3-5: Grouped segments required in Chapter 6**

<table>
<thead>
<tr>
<th>Grouped segment</th>
<th>Segments with non-zero inertial properties or non-identity rotation matrices</th>
<th>ICG17 outputs that are used later</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Straight arms:</strong> The arms are kept separate. There is no flexion at the elbows or wrists. Rotation about the longitudinal axis is ignored and so the medial and transverse moments of inertia are assumed to be equal. This grouped segment is required in Sections 6.1 and 6.2.1.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mass &amp; Moment of inertia</td>
<td>Origins</td>
</tr>
<tr>
<td></td>
<td>6-8</td>
<td>7-8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| **Lower arms and hands:** This combination is required when exploring actions allowing flexion at the elbow, but not allowing flexion at the wrists. As with straight arms the medial and transverse moments of inertia are assumed to be equal, with a value equal their average. This grouped segment is required in Section 6.2.2. | | |
| | Mass & Moment of inertia | Origins | Centre of Gravities | Rotation matrices | Frontal plane models | Sagittal plane models |
| | 7-8 | 8 | 7-8 | All identity | - Mass | Both arms move together: |
| | | | | | - (I_{xx} + I_{yy})/2, | - 2*Mass |
| | | | | | - Distance along the z-axis to the shoulder, Joints(3,7) | - (I_{xx} + I_{yy}) |
| | | | | | | - Distance along the z-axis to the shoulder, Joints(3,7) |
### Inertial properties

<table>
<thead>
<tr>
<th>Grouped segment</th>
<th>Segments with non-zero inertial properties or non-identity rotation matrices</th>
<th>ICG17 outputs that are used later</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Body without arms in layout:</strong> When seeking to explore the actions of the arms, the body is the ‘unmoving’ segment that the arms act against. The body is in the reference position. This grouped segment is required in Sections 6.1 and 6.2</td>
<td><strong>Mass &amp; Moment of inertia</strong></td>
<td><strong>Frontal plane models</strong></td>
</tr>
<tr>
<td></td>
<td><strong>Origins</strong></td>
<td><strong>Centre of Gravities</strong></td>
</tr>
<tr>
<td></td>
<td><strong>1-5, 9-11, 15-17</strong></td>
<td><strong>1-5, 9-11, 15-17</strong></td>
</tr>
<tr>
<td></td>
<td><strong>Vertical distance to the shoulder:</strong> JRP'*Joints(:,6) &amp; JRP'*Joints(:,12)</td>
<td></td>
</tr>
</tbody>
</table>
### Inertial properties

#### Segments with non-zero inertial properties or non-identity rotation matrices

<table>
<thead>
<tr>
<th>Grouped segment</th>
<th>Segments with non-zero inertial properties or non-identity rotation matrices</th>
<th>ICG17 outputs that are used later</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Body without arms in EP:</strong> <strong>EP</strong> is defined in Section 3.9.12 and this grouped segment is used in Section 6.5.1.</td>
<td><strong>Mass &amp; Moment of inertia</strong></td>
<td><strong>Frontal plane models</strong></td>
</tr>
<tr>
<td><strong>Mass</strong></td>
<td><strong>Moment</strong></td>
<td><strong>Sagittal plane models</strong></td>
</tr>
<tr>
<td>1-5, 9-11, 15-17</td>
<td>1-5, 9-11, 11, 16-17</td>
<td>1-5, 9-11, 15-17</td>
</tr>
</tbody>
</table>
| JRP*Joints(:,6) | JRP*Joints(:,12) | R1 = \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30) \\
0 & 1 & 0 \\
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] |
| R2 = \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30) \\
0 & 1 & 0 \\
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] | R8=R14 = \[
\begin{bmatrix}
\cos(-20) & 0 & \sin(-20) \\
0 & 1 & 0 \\
-\sin(-20) & 0 & \cos(-20)
\end{bmatrix}
\] |
| - Mass | - I\textsubscript{xx} | - Not required in Chapter 6. |
| - The vectors to the each shoulder with respect to the principal axes: JRP*Joints(:,6) and JRP*Joints(:,12). These will be in the principal x-z plane. | - The angle between the principal x-axis and the pelvis’ x-axis: \(\cos^{-1}(\text{JR}P(1,1))\) |
**Inertial properties**

<table>
<thead>
<tr>
<th>Grouped segment</th>
<th>Segments with non-zero inertial properties or non-identity rotation matrices</th>
<th>ICG17 outputs that are used later</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Body without arms in A:</strong> A is defined in Section 3.9.8, and this grouped segment is used in Section 6.5.1.</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Mass &amp; Moment of inertia</strong></td>
<td><strong>Origins</strong></td>
<td><strong>Centre of Gravities</strong></td>
</tr>
<tr>
<td></td>
<td>1-5, 9-11, 15-17</td>
<td>1-5, 9-11, 15-17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Grouped segment

<table>
<thead>
<tr>
<th>Segments with non-zero inertial properties or non-identity rotation matrices</th>
<th>ICG17 outputs that are used later</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mass &amp; Moment of inertia</strong></td>
<td><strong>Origins</strong></td>
</tr>
</tbody>
</table>
| Body less the right arm: This is the 'unmoving' segment when exploring the action of a single arm, which is allowed to flex at the elbow. This grouped segment is required in Section 6.2.2. | 1-11,15-17 | 2-11,15-17 | 1-11,15-17 | All identity | - Mass  
- $I_{xx}$  
- The vector to the right shoulder with respect to the principal axes: $\text{JRPP}^\text{Joints(:,12)}$. This will be in the principal x-z plane. | Not required in Chapter 6. |
| Pelvis and straight legs: This will be one segment when exploring lateral flexion, which occurs at the pelvis-abdomen joint and the abdomen-chest joint. This grouped segment is required in Section 6.4. | 1, 9-11, 15-17 | 9-11, 15-17 | 1, 9-11, 15-17 | All identity | - Mass  
- $I_{xx}$  
- Distance along the z-axis to the pelvis-abdomen joint from the centre of gravity: $\text{JRPP}^\text{Joints(:,2)}$ | Not required in Chapter 6. |
<table>
<thead>
<tr>
<th>Grouped segment</th>
<th>Segments with non-zero inertial properties or non-identity rotation matrices</th>
<th>ICG17 outputs that are used later</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mass &amp; Moment of inertia</td>
<td>Origins</td>
</tr>
<tr>
<td>Chest, head, neck and arms in reference position: This is another one of the segments when exploring lateral flexion, which occurs at the pelvis-abdomen joint and the abdomen-chest joint. This grouped segment is required in Section 6.4.1.</td>
<td>All identity</td>
<td>- Mass</td>
</tr>
</tbody>
</table>
3.9 Key somersault postures

Postures are defined by the joint angles or the rotation matrices between segments of the body. Any change in a joint angle would produce a new posture in a mathematical sense; however, sporting rules and expectations restrict the postures that should be used by athletes. The postures identified here as significant were those presented in the literature (Biesterfeldt, 1975; Frohlich, 1979; Yeaton, 1984), those mentioned or described in the sporting rules (FINA, 2009; CoP MAG, 2013; CoP WAG, 2013; CoP TRAMP, 2013), and those identified in discussions with accredited gymnastic and diving judges and/or coaches. Since sporting rules do change, the reader should confirm the current rules for their sport before applying the conclusions of this thesis. The joint angles used to define each posture were selected after considering joint angle values measured from photographs of a current athlete asked to hold each of the postures, angles extracted from photographs given by Biesterfeldt (1975), observations by Sanders (1995), and the range of motion of various joints presented by Dempster (1955) and Laubach (1979).

Twenty postures were selected and are described under their own sub-headings (Sections 3.9.1 to 3.9.19). A short description, including whether the posture is more common in twisting or non-twisting somersaults, is given along with a photograph of an athlete holding the posture and a stick figure representation of the same posture. The stick figure representation was produced by the programme ICG17 (Section 3.8): the red lines are the body links connecting the joints, the blue dots represent the location of the centre of gravity of each segment, and the black cross is the overall centre of gravity of the body. For each posture the rotation matrices required to define the posture are given. The numbering of the rotation matrices is as required when using the programme ICG17. How the numbering of the rotation matrices matches the 17-segment model defined in Section 3.1 is given in Appendix A.4.5. Any flexion at the neck, wrists, and ankles was ignored, as specified in Section 3.1. The feet were assumed to have toes pointed (plantar flexed) when the athlete is airborne. This means that for some postures the feet and head will be in different positions in the photograph and stick figure.

The abbreviation given in the sub-heading will be used throughout the remainder of this thesis to indicate the specific posture as defined here and not a general description of the posture. For example the word “tuck” used generally covers any posture with flexion at the knees and hips, while posture “T” will specify the posture defined in Section 3.9.15 below. Since some inertial property data sets only have two torso segments, it is necessary to make some adjustments to the posture and these are specified in Section 3.9.20.

When any of these postures is held the whole body inertial properties are calculated using ICG17.m; the inputs are the inertial properties of all seventeen segments of the body model.
and the rotation matrices defining the desired posture. When the inertial properties are known to include segments that are already combinations of some of the seventeen segments, these may be entered as the inertial properties for the most proximal segment then the inertial properties of the other segments in the combination are set to zero. The rotation matrix between combined segments should be entered as the identity matrix.

### 3.9.1 Layout with arms raised (LAU)

A layout with arms raised (LAU) is often observed as the athlete becomes airborne in a backward rotating somersault. It follows from the throw of the arms that is used to generate backwards rotation.

**Figure 3-16: Posture LAU**

<p>| Table 3-6: Rotation matrices defining posture LAU |
|---|---|---|---|---|</p>
<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
\cos(10) & 0 & \sin(10) \\
0 & 1 & 0 \\
-\sin(10) & 0 & \cos(10)
\end{bmatrix}
\] | 5 | \[
\begin{bmatrix}
\cos(-140) & 0 & \sin(-140) \\
0 & 1 & 0 \\
-\sin(-140) & 0 & \cos(-140)
\end{bmatrix}
\] | 11 \[=R5\] |
| 2 | \[
\begin{bmatrix}
\cos(-35) & 0 & \sin(-35) \\
0 & 1 & 0 \\
-\sin(-35) & 0 & \cos(-35)
\end{bmatrix}
\] | 6 | \[
\begin{bmatrix}
\cos(-20) & 0 & \sin(-20) \\
0 & 1 & 0 \\
-\sin(-20) & 0 & \cos(-20)
\end{bmatrix}
\] | 12 \[=R6\] |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | \[
\begin{bmatrix}
\cos(-10) & 0 & \sin(-10) \\
0 & 1 & 0 \\
-\sin(-10) & 0 & \cos(-10)
\end{bmatrix}
\] | 14 \[=R8\] |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
### 3.9.2 Layout with arms laterally outstretched (LAP)

A layout posture with the arms laterally outstretched (LAP) is a common pre-twist posture for forward somersaults with twist in diving (Frohlich, 1979). This posture is also used as part of “checking”, or slowing a twist in preparation for landing. Further, LAP was a posture specifically discussed in Section 4.2.6.1 as part of a strategy to increase twist in a somersault. This strategy will be discussed further in Section 5.3.5.2.

![Figure 3-17: Posture LAP](image)

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Identity</td>
<td>5</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; \cos(90) &amp; -\sin(90) \ 0 &amp; \sin(90) &amp; \cos(90) \end{bmatrix}$</td>
<td>11</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; \cos(-90) &amp; -\sin(-90) \ 0 &amp; \sin(-90) &amp; \cos(-90) \end{bmatrix}$</td>
</tr>
<tr>
<td>2</td>
<td>Identity</td>
<td>6</td>
<td>Identity</td>
<td>12</td>
<td>Identity</td>
</tr>
<tr>
<td>3</td>
<td>Identity</td>
<td>7</td>
<td>Identity</td>
<td>13</td>
<td>Identity</td>
</tr>
<tr>
<td>4</td>
<td>Identity</td>
<td>8</td>
<td>Identity</td>
<td>14</td>
<td>Identity</td>
</tr>
<tr>
<td></td>
<td>Identity</td>
<td>9</td>
<td>Identity</td>
<td>15</td>
<td>Identity</td>
</tr>
<tr>
<td></td>
<td>Identity</td>
<td>10</td>
<td>Identity</td>
<td>16</td>
<td>Identity</td>
</tr>
</tbody>
</table>
3.9.3 Layout with one arm raised and the other lowered (1U1D)

This posture is the natural end position when using any of the idealised straight arm aerial twist initiation techniques; these techniques are analysed in Section 6.1.

![Figure 3-18: Posture 1U1D](image)

**Table 3-8: Rotation matrices for posture 1U1D**

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>cos(−140) 0 sin(−140)</td>
</tr>
<tr>
<td></td>
<td>Identity</td>
<td>5</td>
<td>Identity</td>
<td>11</td>
<td>0 1 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>− sin(−140) 0 cos(−140)</td>
</tr>
<tr>
<td>1</td>
<td>Identity</td>
<td>6</td>
<td>Identity</td>
<td>12</td>
<td>Identity</td>
</tr>
<tr>
<td>2</td>
<td>cos(−35) 0 sin(−35)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>− sin(−35) 0 cos(−35)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Identity</td>
<td>7</td>
<td>Identity</td>
<td>13</td>
<td>Identity</td>
</tr>
<tr>
<td>4</td>
<td>Identity</td>
<td>8</td>
<td>Identity</td>
<td>14</td>
<td>Identity</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Identity</td>
<td>10</td>
<td>Identity</td>
<td>15</td>
<td>Identity</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Identity</td>
<td>16</td>
<td>Identity</td>
<td>16</td>
<td>Identity</td>
</tr>
</tbody>
</table>
3.9.4 Layout with one arm in a high “V” and the other in a low “V” position (HVLV)

This posture was observed on some occasions when gymnasts and divers performed twisting somersault skills that were well within their capability.

![Figure 3-19: Posture HVLV](image)

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | Identity | 5 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(35) & -\sin(35) \\
0 & \sin(35) & \cos(35)
\end{bmatrix}
\] | 11 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-155) & -\sin(-155) \\
0 & \sin(-155) & \cos(-155)
\end{bmatrix}
\] |
| 2 | \[
\begin{bmatrix}
\cos(-35) & 0 & \sin(-35) \\
0 & 1 & 0 \\
-\sin(-35) & 0 & \cos(-35)
\end{bmatrix}
\] | 6 | Identity | 12 | Identity |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | Identity | 14 | Identity |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
3.9.5 1U1D with both arms bent (1U1DB)

This posture was observed when divers performed forward somersaults with twist and it was reported as a common twist posture by Frohlich (1979).

Figure 3-20: Posture 1U1DB

Table 3-10: Rotation matrices for posture 1U1DB

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | Identity | 5 | Identity | 11 | \[
\begin{bmatrix}
\cos(-140) & 0 & \sin(-140) \\
0 & 1 & 0 \\
-\sin(-140) & 0 & \cos(-140)
\end{bmatrix}
\] |
| 2 | \[
\begin{bmatrix}
\cos(-35) & 0 & \sin(-35) \\
0 & 1 & 0 \\
-\sin(-35) & 0 & \cos(-35)
\end{bmatrix}
\] | 6 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-90) & -\sin(-90) \\
0 & \sin(-90) & \cos(-90)
\end{bmatrix}
\] | 12 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(90) & -\sin(90) \\
0 & \sin(90) & \cos(90)
\end{bmatrix}
\] |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | Identity | 14 | Identity |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
3.9.6 1U1DB combined with a laterally flexed posture (1U1DBLF)

When raising or lowering one arm the athlete may continue to stretch the raised arm over
the head or the lowered arm across the body, resulting in a posture with lateral flexion at
the abdomen-chest joint.

Figure 3-21: Posture 1U1DBLF

Table 3-11: Rotation matrices for posture 1U1DBLF

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | Identity | 5 | Identity | 11 | \[
\begin{bmatrix}
\cos(-14\degree) & 0 & \sin(-14\degree) \\
0 & 1 & 0 \\
-\sin(-14\degree) & 0 & \cos(-14\degree)
\end{bmatrix}
\times
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(5\degree) & -\sin(5\degree) \\
0 & \sin(5\degree) & \cos(5\degree)
\end{bmatrix}
\] |
| 2 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-30\degree) & -\sin(-30\degree) \\
0 & \sin(-30\degree) & \cos(-30\degree)
\end{bmatrix}
\] | 6 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(90\degree) & -\sin(90\degree) \\
0 & \sin(90\degree) & \cos(90\degree)
\end{bmatrix}
\times
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-90\degree) & -\sin(-90\degree) \\
0 & \sin(-90\degree) & \cos(-90\degree)
\end{bmatrix}
\] |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | \[
\begin{bmatrix}
\cos(-10\degree) & 0 & \sin(-10\degree) \\
0 & 1 & 0 \\
-\sin(-10\degree) & 0 & \cos(-10\degree)
\end{bmatrix}
\] | 14 | \$R8$ |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
3.9.7 Layout (L)

Layout is one of the postures that may be used to classify a skill. Ideally the body is straight, as if standing, but with toes pointed. This posture has also been used to “normalise” parameters and so allow comparisons across different athletes (Section 2.5 and Section 5.2).

Figure 3-22: Posture L

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Identity</td>
<td>5</td>
<td>Identity</td>
<td>11</td>
<td>Identity</td>
</tr>
<tr>
<td>2</td>
<td>Identity</td>
<td>6</td>
<td>Identity</td>
<td>12</td>
<td>Identity</td>
</tr>
<tr>
<td>3</td>
<td>Identity</td>
<td>7</td>
<td>Identity</td>
<td>13</td>
<td>Identity</td>
</tr>
<tr>
<td>4</td>
<td>Identity</td>
<td>8</td>
<td>Identity</td>
<td>14</td>
<td>Identity</td>
</tr>
<tr>
<td>9</td>
<td>Identity</td>
<td>15</td>
<td>Identity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Identity</td>
<td>16</td>
<td>Identity</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 3.9.8 Arch (A)

An arch posture is often seen in double layout backward somersaults, presumably following the strong ‘throw’ of the chest backwards to aid rotation, and since the nature of the shape is expected to reduce the moment of inertia about the somersault axis, will still be classified as a layout for purposes of determining the difficulty score.

![Figure 3-23: Posture A](image)

**Table 3-13: Rotation matrices for posture A**

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
\cos(-10) & 0 & \sin(-10) \\
0 & 1 & 0 \\
-\sin(-10) & 0 & \cos(-10)
\end{bmatrix}
\] | 5 | \[
\begin{bmatrix}
\cos(35) & 0 & \sin(35) \\
0 & 1 & 0 \\
-\sin(35) & 0 & \cos(35)
\end{bmatrix}
\] | 11 | =R5 |
| 2 | \[
\begin{bmatrix}
\cos(-20) & 0 & \sin(-20) \\
0 & 1 & 0 \\
-\sin(-20) & 0 & \cos(-20)
\end{bmatrix}
\] | 6 | \[
\begin{bmatrix}
\cos(-20) & 0 & \sin(-20) \\
0 & 1 & 0 \\
-\sin(-20) & 0 & \cos(-20)
\end{bmatrix}
\] | 12 | =R6 |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | \[
\begin{bmatrix}
\cos 10 & 0 & \sin 10 \\
0 & 1 & 0 \\
-\sin 10 & 0 & \cos 10
\end{bmatrix}
\] | 14 | =R14 |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
### 3.9.9 Lateral hip flexion (LHF)

Lateral hip flexion has been previously observed by Van Gheluwe (1981), Yeadon (1984), and Sanders (1995). The posture is a flexion where the hips appear to be pushed to one side. The three authors do not give the same description of lateral hip flexion. Van Gheluwe (1981) applies all the flexion at the hips. Yeadon (1984) describes the position using the angle between the longitudinal axis of the thorax and the midline of the upper leg; this would suggest that Yeadon uses a two segment representation of the torso, but the angle for the lateral hip flexion is not actually at a single joint. Sanders (1995) does not specify any joints but defines lateral hip flexion as the ankle-hip-shoulder angle which is projected onto the plane containing the hips and shoulders. Due to the differing definitions, measurements were taken from a current athlete. The flexion occurred predominantly through the torso, at the pelvis-abdomen and abdomen-chest joints, rather than at the hips.

![Figure 3-24: Posture LHF](image)

It should be noted that although the athlete shows some torsion of the chest and places the arms to the side. This torsion was not included in the mathematical model of this posture so as to isolate the lateral flexion.

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 \cos(-20) & -\sin(-20) & 0 \\
0 \sin(-20) & \cos(-20) & 0
\end{bmatrix}
\] | 5 | Identity | 11 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 \cos(40) & -\sin(40) & 0 \\
0 \sin(40) & \cos(40) & 0
\end{bmatrix}
\] |
| 2 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 \cos(-20) & -\sin(-20) & 0 \\
0 \sin(-20) & \cos(-20) & 0
\end{bmatrix}
\] | 6 | \[
\begin{bmatrix}
\cos(-150) & 0 & \sin(-150) \\
0 & 1 & 0 \\
-\sin(-150) & 0 & \cos(-150)
\end{bmatrix}
\] | 12 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 \cos(120) & -\sin(120) & 0 \\
0 \sin(120) & \cos(120) & 0
\end{bmatrix}
\] |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | Identity | 14 | Identity |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
### 3.9.10 Puck (Pu)

In the CoP TRAMP (2009) the puck could be used without deduction when performing multiple twists in multiple somersaults. Illustrations rather than joint angles were given showing acceptable hip and knee flexion. Considering its use in twisting somersaults the arms will be placed one up and one down with both bent.

![Figure 3-25: Posture Pu](image)

Table 3-15: Rotation matrices for posture Pu

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1  | \[
\begin{bmatrix}
\cos(20) & 0 & \sin(20) \\
0 & 1 & 0 \\
-\sin(20) & 0 & \cos(20)
\end{bmatrix}
\] | 5  | Identity                                    | 11 | \[
\begin{bmatrix}
\cos(140) & 0 & \sin(140) \\
0 & 1 & 0 \\
-\sin(140) & 0 & \cos(140)
\end{bmatrix}
\] |
| 2  | \[
\begin{bmatrix}
\cos(-25) & 0 & \sin(-25) \\
0 & 1 & 0 \\
-\sin(-25) & 0 & \cos(-25)
\end{bmatrix}
\] | 6  | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(90) & -\sin(90) \\
0 & \sin(90) & \cos(90)
\end{bmatrix}
\] | 12 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-90) & -\sin(-90) \\
0 & \sin(-90) & \cos(-90)
\end{bmatrix}
\] |
| 3  | Identity                                    | 7  | Identity                                    | 13 | Identity                                    |
| 4  | Identity                                    | 8  | \[
\begin{bmatrix}
\cos(-45) & 0 & \sin(-45) \\
0 & 1 & 0 \\
-\sin(-45) & 0 & \cos(-45)
\end{bmatrix}
\] | 14 | =R8                                        |
| 9  | \[
\begin{bmatrix}
\cos(65) & 0 & \sin(65) \\
0 & 1 & 0 \\
-\sin(65) & 0 & \cos(65)
\end{bmatrix}
\] | 15 | =R9                                        |
| 10 | Identity                                    | 16 | Identity                                    |
3.9.11 Just layout (JL)

To be classified as layout in gymnastics the knee-hip-shoulder angle needs to be at least 135°. This posture is of interest since it may provide some rotational advantage over using the ideal layout, posture L, while still being awarded the difficulty of a layout.

Figure 3-26: Posture JL

Table 3-16: Rotation matrices for posture JL

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30) \\
0 & 1 & 0 \\
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] | 5 | \[
\begin{bmatrix}
\cos(-30) & 0 & \sin(-30) \\
0 & 1 & 0 \\
-\sin(-30) & 0 & \cos(-30)
\end{bmatrix}
\] | 11 | R5 |
| 2 | \[
\begin{bmatrix}
\cos(20) & 0 & \sin(20) \\
0 & 1 & 0 \\
-\sin(20) & 0 & \cos(20)
\end{bmatrix}
\] | 6 | Identity | 12 | Identity |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | \[
\begin{bmatrix}
\cos(-10) & 0 & \sin(-10) \\
0 & 1 & 0 \\
-\sin(-10) & 0 & \cos(-10)
\end{bmatrix}
\] | 14 | R8 |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
3.9.12 Entry pike (EP)

The entry pike posture may be seen at the beginning of a forward somersault with twist. Twist is often observed straight after an athlete has extended their body from this piked posture into a layout posture. The posture has also been observed after the twist has ended in preparation for landing.

![Figure 3-27: Posture EP](image)

### Table 3-17: Rotation matrices for posture EP

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30) \\
0 & 1 & 0 \\
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] | 5 | \[
\begin{bmatrix}
\cos(-40) & 0 & \sin(-40) \\
0 & 1 & 0 \\
-\sin(-40) & 0 & \cos(-40)
\end{bmatrix}
\] | 11 | \[
= R_5
\] |
| 2 | \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30) \\
0 & 1 & 0 \\
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] | 6 | Identity | 12 | Identity |
| 3 | Identity | 7 | Identity | 13 | Identity |
| 4 | Identity | 8 | \[
\begin{bmatrix}
\cos(-20) & 0 & \sin(-20) \\
0 & 1 & 0 \\
-\sin(-20) & 0 & \cos(-20)
\end{bmatrix}
\] | 14 | \[
= R_8
\] |
| 9 | Identity | 15 | Identity |
| 10 | Identity | 16 | Identity |
3.9.13 Open pike (OP)

To be awarded a pike the legs must be straight and the knee-hip-shoulder angle less than 90°. An open pike is a position which only just meets the criteria for being a pike. The open pike has been observed slowing a somersault in order to avoid over-rotation, and also as a developmental step towards a layout somersault.

![Figure 3-28: Posture OP](image-url)

Table 3-18: Rotation matrices for posture OP

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
\cos(35) & 0 & \sin(35) \\
0 & 1 & 0 \\
-\sin(35) & 0 & \cos(35)
\end{bmatrix}
\] | 5 | \[
\begin{bmatrix}
\cos(-60) & 0 & \sin(-60) \\
0 & 1 & 0 \\
-\sin(-60) & 0 & \cos(-60)
\end{bmatrix}
\] | 11 | $=R5$
| 2 | \[
\begin{bmatrix}
\cos(35) & 0 & \sin(35) \\
0 & 1 & 0 \\
-\sin(35) & 0 & \cos(35)
\end{bmatrix}
\] | 6 | Identity | 12 | Identity
| 3 | Identity | 7 | Identity | 13 | Identity
| 4 | Identity | 8 | \[
\begin{bmatrix}
\cos(-40) & 0 & \sin(-40) \\
0 & 1 & 0 \\
-\sin(-40) & 0 & \cos(-40)
\end{bmatrix}
\] | 14 | $=R8$
| 9 | Identity | 15 | Identity
| 10 | Identity | 16 | Identity
### 3.9.14 Pike (P)

Posture “P” has been chosen to reflect an aesthetically pleasing posture with the ankle-hip-shoulder angle less than 90°.

![Figure 3-29: Posture P](image)

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\begin{bmatrix} \cos(45) &amp; 0 &amp; \sin(45) \ 0 &amp; 1 &amp; 0 \ -\sin(45) &amp; 0 &amp; \cos(45) \end{bmatrix}$</td>
<td>5</td>
<td>$\begin{bmatrix} \cos(-80) &amp; 0 &amp; \sin(-80) \ 0 &amp; 1 &amp; 0 \ -\sin(-80) &amp; 0 &amp; \cos(-80) \end{bmatrix}$</td>
<td>11</td>
<td>$= R5$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{bmatrix} \cos(10) &amp; 0 &amp; \sin(10) \ 0 &amp; 1 &amp; 0 \ -\sin(10) &amp; 0 &amp; \cos(10) \end{bmatrix}$</td>
<td>6</td>
<td>$\begin{bmatrix} \cos(-70) &amp; 0 &amp; \sin(-70) \ 0 &amp; 1 &amp; 0 \ -\sin(-70) &amp; 0 &amp; \cos(-70) \end{bmatrix}$</td>
<td>12</td>
<td>$= R12$</td>
</tr>
<tr>
<td>3</td>
<td>Identity</td>
<td>7</td>
<td>Identity</td>
<td>13</td>
<td>Identity</td>
</tr>
<tr>
<td>4</td>
<td>Identity</td>
<td>8</td>
<td>$\begin{bmatrix} \cos(-95) &amp; 0 &amp; \sin(-95) \ 0 &amp; 1 &amp; 0 \ -\sin(-95) &amp; 0 &amp; \cos(-95) \end{bmatrix}$</td>
<td>14</td>
<td>$= R8$</td>
</tr>
<tr>
<td>9</td>
<td>Identity</td>
<td>15</td>
<td>Identity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Identity</td>
<td>16</td>
<td>Identity</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 3.9.15 Tuck (T)

The tuck posture is one where the hips and knees are flexed, with the hip and knee angles each less than $90^\circ$. A few versions of tuck are possible. This one was chosen as an aesthetically pleasing tuck. Tighter positions will be TT and CT defined in Sections 3.9.18 and 3.9.19 respectively.

![Figure 3-30: Posture T](image-url)

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1  | \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30) \\
0 & 1 & 0 \\
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] | 5  | \[
\begin{bmatrix}
\cos(-60) & 0 & \sin(-60) \\
0 & 1 & 0 \\
-\sin(-60) & 0 & \cos(-60)
\end{bmatrix}
\] | 11 | =R5    |
| 2  | \[
\begin{bmatrix}
\cos(20) & 0 & \sin(20) \\
0 & 1 & 0 \\
-\sin(20) & 0 & \cos(20)
\end{bmatrix}
\] | 6  | \[
\begin{bmatrix}
\cos(-20) & 0 & \sin(-20) \\
0 & 1 & 0 \\
-\sin(-20) & 0 & \cos(-20)
\end{bmatrix}
\] | 12 | =R6    |
| 3  | Identity | 7  | Identity | 13 | Identity |
| 4  | Identity | 8  | \[
\begin{bmatrix}
\cos(-100) & 0 & \sin(-100) \\
0 & 1 & 0 \\
-\sin(-100) & 0 & \cos(-100)
\end{bmatrix}
\] | 14 | =R8    |
|    |         | 9  | \[
\begin{bmatrix}
\cos(110) & 0 & \sin(110) \\
0 & 1 & 0 \\
-\sin(110) & 0 & \cos(110)
\end{bmatrix}
\] | 15 | =R9    |
|    |         | 10 | Identity | 16 | Identity |
3.9.16 Back tuck (BT) and Front tuck (FT)

From observations of back and front tuck somersaults it was apparent that the curvature of the torso differed. Presumably this was due to a different initiation of the somersault and the sighting requirements for the landing. In a back tuck the torso was observed to be straighter and the hip angle greater, while in a front tuck the torso was more curved and the hip angle was smaller. Figure 3-31 and Figure 3-32 present key frames when performing a back tuck somersault and a front tuck somersault.
Table 3-21 gives the rotation matrices that are used to define BT and FT and Figure 3-33 then gives the corresponding stick figure representations.

Figure 3-31: Key frames showing the posture used during a back tuck somersault
(a) tuck posture just reached (b) mid-flight, (c) just prior to opening out

Figure 3-32 Key frames showing the posture used during a front tuck somersault
(a) j tuck posture just reached (b) mid-flight, (c) just prior to opening out
### Table 3-21: Rotation matrices for BT and FT

<table>
<thead>
<tr>
<th>R</th>
<th>BT</th>
<th>FT</th>
</tr>
</thead>
</table>
| 1 | \[
\begin{bmatrix}
\cos(40) & 0 & \sin(40)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(40) & 0 & \cos(40)
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(50) & 0 & \sin(50)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(50) & 0 & \cos(50)
\end{bmatrix}
\] |
| 2 | \[
\begin{bmatrix}
\cos(30) & 0 & \sin(30)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(30) & 0 & \cos(30)
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(40) & 0 & \sin(40)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(40) & 0 & \cos(40)
\end{bmatrix}
\] |
| 3 | Identity | Identity |
| 4 | Identity | Identity |
| 5 & 11 | \[
\begin{bmatrix}
\cos(-60) & 0 & \sin(-60)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(-60) & 0 & \cos(-60)
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(-70) & 0 & \sin(-70)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(-70) & 0 & \cos(-70)
\end{bmatrix}
\] |
| 6 & 12 | \[
\begin{bmatrix}
\cos(-35) & 0 & \sin(-35)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(-35) & 0 & \cos(-35)
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(-30) & 0 & \sin(-30)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(-30) & 0 & \cos(-30)
\end{bmatrix}
\] |
| 7 & 13 | Identity | Identity |
| 8 & 14 | \[
\begin{bmatrix}
\cos(-90) & 0 & \sin(-90)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(-90) & 0 & \cos(-90)
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(-70) & 0 & \sin(-70)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(-70) & 0 & \cos(-70)
\end{bmatrix}
\] |
| 9 & 15 | \[
\begin{bmatrix}
\cos(140) & 0 & \sin(140)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(140) & 0 & \cos(140)
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(140) & 0 & \sin(140)
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\sin(140) & 0 & \cos(140)
\end{bmatrix}
\] |

Figure 3-33: BT (a) and FT (b)
3.9.17 Tight pike (TP)

This posture was chosen to represent the tightest pike reasonably achievable. It is expected to be the piked posture with the lowest moment of inertia about the somersault axis.

![Figure 3-34: Posture TP](image)

**Table 3-22: Rotation matrices for posture TP**

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1   | \[
\begin{bmatrix}
\cos(45) & 0 & \sin(45) \\
0 & 1 & 0 \\
-\sin(45) & 0 & \cos(45)
\end{bmatrix}
\] | 5   | \[
\begin{bmatrix}
\cos(-50) & 0 & \sin(-50) \\
0 & 1 & 0 \\
-\sin(-50) & 0 & \cos(-50)
\end{bmatrix}
\] | 11  | \[
\begin{bmatrix}
\cos(-50) & 0 & \sin(-50) \\
0 & 1 & 0 \\
-\sin(-50) & 0 & \cos(-50)
\end{bmatrix}
\] |
| 2   | \[
\begin{bmatrix}
\cos(5) & 0 & \sin(5) \\
0 & 1 & 0 \\
-\sin(5) & 0 & \cos(5)
\end{bmatrix}
\] | 6   | \[
\begin{bmatrix}
\cos(-120) & 0 & \sin(-120) \\
0 & 1 & 0 \\
-\sin(-120) & 0 & \cos(-120)
\end{bmatrix}
\] | 12  | \[
\begin{bmatrix}
\cos(-120) & 0 & \sin(-120) \\
0 & 1 & 0 \\
-\sin(-120) & 0 & \cos(-120)
\end{bmatrix}
\] |
| 3   | Identity                    | 7   | Identity                    | 13  | Identity                    |
| 4   | Identity                    | 8   | \[
\begin{bmatrix}
\cos(-115) & 0 & \sin(-115) \\
0 & 1 & 0 \\
-\sin(-115) & 0 & \cos(-115)
\end{bmatrix}
\] | 14  | \[
\begin{bmatrix}
\cos(-115) & 0 & \sin(-115) \\
0 & 1 & 0 \\
-\sin(-115) & 0 & \cos(-115)
\end{bmatrix}
\] |
| 9   | Identity                    | 15  | Identity                    |
| 10  | Identity                    | 16  | Identity                    |
### 3.9.18 Tight tuck (TT)

This posture is intended to represent the tightest tuck position reasonably achievable without allowing the legs to separate.

![Figure 3-35: TT](image)

#### Table 3-23: Rotation matrices for TT

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1  | \[
|    | \begin{bmatrix}
|    | \cos(30) & 0 & \sin(30) \\
|    | 0 & 1 & 0 \\
|    | -\sin(30) & 0 & \cos(30)
|    \end{bmatrix}
|    | \]                                                                 | 5  | \[
|    | \begin{bmatrix}
|    | \cos(-30) & 0 & \sin(-30) \\
|    | 0 & 1 & 0 \\
|    | -\sin(-30) & 0 & \cos(-30)
|    \end{bmatrix}
|    | \times \]                                                              | 11 | \[
|    | \begin{bmatrix}
|    | \cos(-30) & 0 & \sin(-30) \\
|    | 0 & 1 & 0 \\
|    | -\sin(-30) & 0 & \cos(-30)
|    \end{bmatrix}
|    | \]                                                                 |
| 2  | \[
|    | \begin{bmatrix}
|    | \cos(30) & 0 & \sin(30) \\
|    | 0 & 1 & 0 \\
|    | -\sin(30) & 0 & \cos(30)
|    \end{bmatrix}
|    | \]                                                                 | 6  | \[
|    | \begin{bmatrix}
|    | \cos(-110) & 0 & \sin(-110) \\
|    | 0 & 1 & 0 \\
|    | -\sin(-110) & 0 & \cos(-110)
|    \end{bmatrix}
|    | \times \]                                                              | 11 | \[
|    | \begin{bmatrix}
|    | \cos(-110) & 0 & \sin(-110) \\
|    | 0 & 1 & 0 \\
|    | -\sin(-110) & 0 & \cos(-110)
|    \end{bmatrix}
|    | \]                                                                 |
| 3  | Identity                                                               | 7  | Identity                                                               | 13 | Identity                                                               |
| 4  | Identity                                                               | 8  | \[
|    | \begin{bmatrix}
|    | \cos(-120) & 0 & \sin(-120) \\
|    | 0 & 1 & 0 \\
|    | -\sin(-120) & 0 & \cos(-120)
|    \end{bmatrix}
|    | \]                                                                 |
| 9  | \[
|    | \begin{bmatrix}
|    | \cos(30) & 0 & \sin(30) \\
|    | 0 & 1 & 0 \\
|    | -\sin(30) & 0 & \cos(30)
|    \end{bmatrix}
|    | \]                                                                 | 15 | \[
|    | \begin{bmatrix}
|    | \cos(30) & 0 & \sin(30) \\
|    | 0 & 1 & 0 \\
|    | -\sin(30) & 0 & \cos(30)
|    \end{bmatrix}
|    | \]                                                                 |
| 10 | Identity                                                               | 16 | Identity                                                               |
3.9.19 Cowboy tuck (CT)

Allowing the legs to separate means the athlete can pull the legs into the plane of their torso. Such a posture is often referred to as a cowboy tuck. It is expected to allow faster rotation than TT. In the past allowing the legs to separate has been considered an error, although more recently, as athletes are pushing the bounds of the number of somersaults they can complete, its use appears to have increased and become more accepted.

Table 3.24: Rotation matrices for CT

<table>
<thead>
<tr>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
<th>R</th>
<th>Matrix</th>
</tr>
</thead>
</table>
| 1  | \[
\begin{bmatrix}
\cos(-10) & 0 & \sin(-10) \\
0 & 1 & 0 \\
-\sin(-10) & 0 & \cos(-10)
\end{bmatrix}
\]                      | 5  | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos 30 & -\sin 30 \\
0 & \sin 30 & \cos 30
\end{bmatrix}
\]                      | 11 | \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-30) & -\sin(-30) \\
0 & \sin(-30) & \cos(-30)
\end{bmatrix}
\]                      |
| 2  | \[
\begin{bmatrix}
\cos(50) & 0 & \sin(50) \\
0 & 1 & 0 \\
-\sin(50) & 0 & \cos(50)
\end{bmatrix}
\]                      | 6  | \[
\begin{bmatrix}
\cos(-100) & 0 & \sin(-100) \\
0 & 1 & 0 \\
-\sin(-100) & 0 & \cos(-100)
\end{bmatrix}
\]                      | 12 | =R6                                       |
| 3  | Identity                                   | 7  | Identity                                   | 13 | Identity                                   |
| 4  | Identity                                   | 8  | \[
\begin{bmatrix}
\cos(-150) & 0 & \sin(-150) \\
0 & 1 & 0 \\
-\sin(-150) & 0 & \cos(-150)
\end{bmatrix}
\]                      | 14 | \[
\begin{bmatrix}
\cos(-150) & 0 & \sin(-150) \\
0 & 1 & 0 \\
-\sin(-150) & 0 & \cos(-150)
\end{bmatrix}
\]                      |
| 9  | \[
\begin{bmatrix}
\cos(120) & 0 & \sin(120) \\
0 & 1 & 0 \\
-\sin(120) & 0 & \cos(120)
\end{bmatrix}
\]                      | 10 | Identity                                   | 15 | =R9                                       |
### 3.9.20 Adjustments for only two torso segments

Some inertial property data sets collated in Chapter 3 had only two torso segments. The “lower torso” comprised the pelvis and the abdomen segments. Any flexion at the pelvis-abdomen joint cannot be described by such a model, but must be accounted for by adjusting the lower torso-upper torso and hip angles. The pelvis-abdomen rotation matrix must be set to the identity matrix and then the rotation matrices of the abdomen-chest and the pelvis-upper leg (hip) must be adjusted. The exact angles that give the same hip and shoulder position as when using a three torso segment depend on the lengths of the segments. So the rotation matrices used may be specified once for each posture and the angles of flexion between the lower and upper torso will be taken as the sum of the flexion at the pelvis-abdomen joint and the abdomen-chest joint, then the rotation matrix $R_8$ will be replaced with the values given in Table 3-25.

<table>
<thead>
<tr>
<th>Posture</th>
<th>$R_8$ (Flexion at Hips)</th>
<th>Posture</th>
<th>$R_8$ (Flexion at Hips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAU</td>
<td>$\begin{bmatrix} \cos(-5) &amp; 0 &amp; \sin(-5) \ 0 &amp; 1 &amp; 0 \ -\sin(-5) &amp; 0 &amp; \cos(-5) \end{bmatrix}$</td>
<td>JL</td>
<td>$\begin{bmatrix} \cos(-35) &amp; 0 &amp; \sin(-35) \ 0 &amp; 1 &amp; 0 \ -\sin(-35) &amp; 0 &amp; \cos(-35) \end{bmatrix}$</td>
</tr>
<tr>
<td>LAP</td>
<td>No change</td>
<td>EP</td>
<td>$\begin{bmatrix} \cos(-15) &amp; 0 &amp; \sin(-15) \ 0 &amp; 1 &amp; 0 \ -\sin(-15) &amp; 0 &amp; \cos(-15) \end{bmatrix}$</td>
</tr>
<tr>
<td>1U1D</td>
<td>No change</td>
<td>OP</td>
<td>$\begin{bmatrix} \cos(-50) &amp; 0 &amp; \sin(-50) \ 0 &amp; 1 &amp; 0 \ -\sin(-50) &amp; 0 &amp; \cos(-50) \end{bmatrix}$</td>
</tr>
<tr>
<td>HVLV</td>
<td>No change</td>
<td>P</td>
<td>$\begin{bmatrix} \cos(-115) &amp; 0 &amp; \sin(-115) \ 0 &amp; 1 &amp; 0 \ -\sin(-115) &amp; 0 &amp; \cos(-115) \end{bmatrix}$</td>
</tr>
<tr>
<td>1U1DB</td>
<td>No change</td>
<td>T</td>
<td>$\begin{bmatrix} \cos(-120) &amp; 0 &amp; \sin(-120) \ 0 &amp; 1 &amp; 0 \ -\sin(-120) &amp; 0 &amp; \cos(-120) \end{bmatrix}$</td>
</tr>
<tr>
<td>1U1DB LF</td>
<td>$\begin{bmatrix} \cos(-10) &amp; 0 &amp; \sin(-10) \ 0 &amp; 1 &amp; 0 \ -\sin(-10) &amp; 0 &amp; \cos(-10) \end{bmatrix}$</td>
<td>BT</td>
<td>$\begin{bmatrix} \cos(-120) &amp; 0 &amp; \sin(-120) \ 0 &amp; 1 &amp; 0 \ -\sin(-120) &amp; 0 &amp; \cos(-120) \end{bmatrix}$</td>
</tr>
<tr>
<td>L</td>
<td>No change</td>
<td>TP</td>
<td>$\begin{bmatrix} \cos(-140) &amp; 0 &amp; \sin(-140) \ 0 &amp; 1 &amp; 0 \ -\sin(-140) &amp; 0 &amp; \cos(-140) \end{bmatrix}$</td>
</tr>
</tbody>
</table>
Inertial properties

3.10 The accuracy of inertial property estimates

The intention of estimating the inertial properties of current athletes was to generate reasonable inertial property data sets; rather than to determine the inertial properties of a particular individual athlete. However, it is prudent to be aware of the level of accuracy of the inertial property estimation methods used.

The complete set of the actual inertial properties of any athlete is not known: it is impossible to directly measure all the inertial properties of each segment of a living person. Thus, it is not possible to compare estimated inertial properties with actual inertial properties. The accuracy of inertial properties is thus typically assessed by the accuracy of simulations intended to replicate an observed skill. In this situation, errors due to digitisation, the extraction of the joint angles, the number of segments chosen and how they are connected, and errors resulting from the fact that only estimated inertial properties are applied, are inseparable.

To make a partial assessment of the accuracy of the inertial properties across several skills, one athlete (female, 58.8kg, 163.3cm) volunteered to perform a tuck, pike and layout, and double-tuck forward somersault. The inertial property data set produced by each method in Section 3.5.6, for that athlete, was applied to a simulation of the skills. The angular momentum was a constant input, and the joint angles input matched those observed. The output was the somersault angle. See Appendix C for details of the process. The angular momentum chosen ensured that the simulation of each somersault skill completed the required amount of somersault rotation. The magnitude of the difference in the somersault angle between the simulation and the filmed performance, at any point during the skill, was determined: the maximum magnitudes are given in Table 3-26.
Table 3-26: Maximum magnitude difference between simulation and observation (degrees).

<table>
<thead>
<tr>
<th>Model (Ordered Alphabetically)</th>
<th>Somersault skill</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tuck</td>
</tr>
<tr>
<td>Finch, All</td>
<td>36</td>
</tr>
<tr>
<td>Finch, Ectomorph</td>
<td>36</td>
</tr>
<tr>
<td>Finch, Endomorph</td>
<td>36</td>
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<tr>
<td>Finch, Mesomorph</td>
<td>36</td>
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<tr>
<td>GOBD</td>
<td>17</td>
</tr>
<tr>
<td>GOBD-Y</td>
<td>18</td>
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<tr>
<td>Hanavan - BP</td>
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<td>Hanavan3</td>
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<td>HanavanY</td>
<td>40</td>
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<td>Ma, Kwon, et al.</td>
<td>26</td>
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<tr>
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<td>20</td>
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<tr>
<td>Reduced Yeadon</td>
<td>23</td>
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<td>Shan &amp; Bohn - Chinese</td>
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<td>Shan &amp; Bohn - German</td>
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<td>Woolley</td>
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<td>Zatsiorsky Average percentage</td>
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<td>43</td>
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<tr>
<td>Zatsiorsky non-linear regression</td>
<td>43</td>
</tr>
<tr>
<td>Mean</td>
<td>31</td>
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</tbody>
</table>

Hanavan3 produced the minimum error for the tuck and pike somersaults, has a reasonably small error in the layout and is in the lower half for error in the double tuck. It is thus a reasonable method to use when comparing the different skills: it is the method that will be used in Section 5.2.2 when comparing these pure somersaults, and is used for many examples throughout this thesis. It is not surprising that Hanavan3 has much less error than HanavanY, since the extra torso segment allows a better description of the curvature of the torso, which was apparent to the eye when watching the somersault skills in slow motion.

It is apparent from Table 3-26 that different methods of estimating inertial properties produce different magnitudes of error and that the error also depends on the skill performed. The same method is not the most accurate for all four skills, and in some cases there can be quite a large increase in error between the skills. It is thus unreasonable to choose one method of estimating inertial properties and expect it to be the ‘best’ method for all the applications. For this particular athlete GOBD, GOBD-Y, Hanavan-BP, Hanavan3, Nikolova, Woolley, and Zatsiorsky average percentage were the methods deemed reasonable using the criteria in Section 3.5.7 above. There does not, however, appear to be a relationship between being accepted as reasonable and the accuracy of the
simulation. This emphasizes the point that when applying these estimates to the equations of motion, the results represent inertial properties that are within reasonable bounds, rather than specific athletes.

The layout somersault displayed the least error for all estimation methods. This is presumably because the layout somersault displayed the least change in posture during the course of the skill, and so the scaling of the magnitude of angular momentum would have been more effective at accounting for the differences between the inertial property estimation methods.

Comparing the angular displacement when $H = 0$ also shows considerable difference between the estimation methods; even in the layout posture, which had only small postural changes due to movement of the arms and some torso curvature.

It is quite clear from these observations and simulations of pure somersaults that some estimation methods are more accurate for some skills than others. It is unnecessary to consider twisting somersaults as well to appreciate this fact. Let us then continue through the next chapters, remembering our intention is to explore the possible rotational behaviours of ‘athletes’ by applying inertial properties that reflect athletes of various ages, and both genders, but are not meant to replicate specific athletes.
Chapter 4

Mathematical descriptions of the somersault

This chapter derives the mathematical equations required to analytically describe the airborne phase of a somersault using the model of the body presented in Section 3.1. It starts by defining the frames of reference and orientation angles which will allow somersault, and twist to be described.

For situations where an athlete displays a quasi-rigid phase, rigid body equations of motion are used to describe the rotation of the athlete. The body posture held and the body’s orientation with respect to the angular momentum vector, affects whether or not twist is produced, and the skills that can be achieved. All combinations of equality or inequality of the whole body moments of inertia will be considered. Instantaneous changes of shape between postures will also be discussed as a means of altering the rotation observed.

Twist is initiated in a pure somersault following re-orientation of the body with respect to the angular momentum vector. This re-orientation is achieved by moving segments of the body relative to each other. Movement of the segments relative to each other introduces no external force or moment and so the conservation of angular momentum will be used to write an equation for the re-orientation as a function of the joint angles and velocities. A reduced form applicable to planar two- and three- segment bodies will then be given, and methods of analytic exploration will be discussed. Finally the logic underlying the computer programme, Angle_disp17.m, which allows the 3-D total angular displacement to be solved numerically, will be presented.
4.1 Frames of reference

In order to mathematically describe actions, body posture, and whole body orientation, it is necessary to define frames of reference.

Posture is defined by the relative positions and orientations of each segment with respect to the other segments. To mathematically describe posture “local” frames of reference, frames 1 to 17, for each segment will be defined in Section 4.1.1. Posture is fully defined by the rotation matrices, which are set by the joint angles, between these local frames. Sixteen rotation matrices are required to define the posture of the 17 segment representation of the body. Actions allow movement of the segments by altering joint angles and hence the rotation matrices between segments.

How orientation of the body as a whole should be described is not obvious, since the body can assume many different postures, all of which have the segments orientated differently with respect to the other segments. Further, depending on the application it may be of interest to know the orientation of specific segments or the orientation of the principal axes of the body. As a result, to describe the orientation of the body three frames will be used; frame Glo, defined in Section 4.1.4, so the orientation may be related to the ‘world’; frame Ref, defined in Section 4.1.2 which is used to specify the orientation of the body, using a reference segment, and frame Pri, defined in Section 4.1.3, which describes the orientation of the principal axes. Both frame Ref and frame Pri may be considered as ‘body’ frames, since they move with the body.

If the body is held in a fixed posture then it may be treated as a single rigid body and so the principal axes then become convenient axes about which to describe the rotation; this is done in Section 4.2. When the body is changing posture, it is more convenient to observe the change of a key segment, the reference segment; this is what will be done in Section 4.3.

The focus of this chapter is to formulate the mathematics for defining the pure and twisting somersault, and the actions required for airborne initiation of twist; thus, the orientation will be described in terms of somersault angle $\theta$, and twist angle, $\psi$. A third independent angle is required to fully define orientation (Paul, 1981; Khatib & Kolarov, 2006); the angle between the somersault axis and the principal longitudinal axis of the body, $\phi$, will be used. These angles and the rotation matrices they produce are defined in Section 4.1.5. Appendix B.1.1 describes methods by which a rotation matrix may be created.

Let us now clarify notation that will be used in equations. Prepended superscripts will be used to specify the frame of reference for any vector, or axis direction. For example, the x-axis in frame 1 will be written $^1x$, and the angular momentum with respect to frame Pri is
Mathematical descriptors of the somersault

$^pH$. Symbols of vectors will be in bold face; and when written as a 3X1 column vector each row corresponds to the x, y and z- components within the same frame of reference. The unit vectors in the x-, y-, and z-axis directions will be $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The appended superscript “T” will indicate transpose. For example, $^{\text{Glo}}\mathbf{H}=[0, H, 0]^T$, states that the angular momentum vector with respect to frame Glo, is in the direction of the $^{\text{Glo}}\mathbf{y}$-axis and has magnitude H. To specify the “to” and “from” frames of a rotation matrix, a prepended superscript will be used for the “to” frame and an appended subscript for the “from” frame. That is, $^{\text{to}}R_{\text{from}}$. Using this notation if a point $\mathbf{P}$ is known with respect to frame 2, its coordinates may be multiplied by the rotation matrix from frame 2 to frame 1, to give the coordinates of $\mathbf{P}$ with respect to frame 1. That is $^{\text{1}}\mathbf{P} = ^{\text{1}}R^{\text{2}}_{\text{2}}\mathbf{P}$.

For those unfamiliar with anatomical terminology, let us momentarily digress to introduce the terms medial, transverse, longitudinal, frontal and sagittal. The traditional way of defining these terms is illustrated in Figure 4-1.

![Figure 4-1: Anatomical axes and planes of the body.](From Dyson (1973), Figure 73.)

The medial, transverse, and longitudinal axes are related to anatomy; they are not principal axes, although are generally close to principal axes. These terms should be treated as general descriptions and not terms with precise mathematical definitions. When defining frames of reference in this thesis the terms medial, transverse, and longitudinal, will be used to describe the axis closest in orientation to its namesake in Figure 4-1 when the athlete is standing upright.
4.1.1 Local frames of reference (1-17)

The local frame of each segment will have its origin at the more proximal joint; that is the joint closer to the pelvis; the local frame of the pelvis will have its origin at its centre of gravity. Each local frame of reference will be orientated such that a line connecting the proximal and distal joint of each segment, known as the body-link, is the z-axis. When the athlete is standing upright, as in Figure 4-1, the positive z-direction is approximately aligned with the longitudinal axis and so will be referred to as the longitudinal axis of the segment. The x and y axes will complete a right-handed orthogonal frame with the x-axis aligned as closely as possible with the medial axis when the athlete is standing upright, as in Figure 4-1; the y-axis will then be closely aligned with the transverse axis when the athlete is standing upright. The x- and y-axes will be referred to as the medial and transverse axes of the segment respectively. For almost all the inertial property data sets collated in Chapter 3 the local frame will also align with the principal axes for that segment. This is essentially the same system of defining local frames as presented in Huston (2009). The local frames will be numbered the same as the segments are numbered, see Section 3.1.1.

The inertial properties of a segment (Section 3.1.2) will be constant in the local frame, since the segment is rigid and the local frame is attached to the segment. It is thus convenient to define the inertial properties of a segment in terms of the local frame of that segment.

4.1.2 The reference frame (Ref)

When describing orientation of the body as it changes posture it is helpful to define a “reference frame”. The choice of reference frame is arbitrary; however, it is helpful to choose a reference frame that simplifies the mathematical description of orientation and has some practical meaning. Smith and Kane (1967) and Kwon (1993) chose to use the local frame of one segment as the reference frame. Yeadon (1984) took a different approach and defined the reference frame by the location of the knee, hip, and shoulder joint centres. The justification given was that by using more than one segment to define the reference frame it would better represent the whole body orientation for a sporting application; the example given was a pike jump where the reference frame defined above rotates very little. This example was intended to affirm the choice of a reference frame since a pike jump is not a somersault. However, if the athlete performed a tuck jump, which is also not a somersault, such a reference frame would rotate in a different way.

Choosing a reference frame whose origin is at the centre of gravity of the body as a whole and is aligned with one segment’s local frame will simplify calculations while still allowing all the necessary rotations to be described. This is the approach that will be used.
Frame Ref will be aligned with frame 1 (the local frame of the pelvis) and have its origin at the centre of gravity of the body as a whole.

If, for any reason, it is desirable to change the reference frame the new orientation may be determined from the orientation of the current reference frame and the posture. If the new reference segment is to be segment 2 instead of 1, then the new orientation will be the vector sum of the current orientation, \( \gamma_1 \), and the orientation of segment 2 with respect to segment 1, \( \gamma_2 \); the sum is multiplied by the rotation matrix between segment 1 and 2, \( R_{12} \), so that the orientation is not only for segment 2 but also is with respect to segment 2’s local frame, \( \gamma_2 \). Written as an equation this is, \( \gamma_2 = R_{12} (\gamma_1 + \gamma_2) \).

### 4.1.3 The principal axes (Frame Pri)

The principal axes of the body as a whole are the axes for which the tensor of inertia of the body as a whole is a diagonal matrix. The principal moments of inertia are the moments of inertia about the principal axes. The orientation of the axes depends on the inertial properties of the segment and the posture used.

The origin of frame Pri will be the centre of gravity of the body as a whole. The x-axis will be the principal axis closest to the medial axis of the pelvis. The y-axis will be the principal axis closest to the transverse axis of the pelvis, and the z-axis will be the principal axis closest to the longitudinal axis of the pelvis. The ‘closest’ principal axis is the one with the largest component in the direction of a particular anatomical axis of the pelvis. The positive direction of the \( \text{Pri}_x \)-axis, is back-to-front in a forward somersault, and front-to-back in a backward somersault. The positive direction of the \( \text{Pri}_y \)-axis is right-to-left in a forward somersault, and left-to-right in a backward somersault. The positive direction of the \( \text{Pri}_z \)-axis is from the feet to the head. Frame Pri is thus a right-handed orthogonal frame. These axis directions are chosen so that at the start of a somersault the frame Pri will be aligned with the frame Glo (Section 4.1.4).

### 4.1.4 The global inertial frame (frame Glo)

Frame Glo is a non-rotating frame with its origin at the centre of gravity of the body as a whole. Since gravity is the only external force acting on the body, frame Glo, may be treated mathematically as an inertial frame (Smith & Kane, 1967): it is in an inertial frame that Newton’s laws of motion, in particular the conservation of angular momentum, apply.

The focus of this thesis is on how an athlete may rotate and so what skills may be completed. The rotation of frame Ref or frame Pri with respect to frame Glo will describe the body’s orientation. The translation of the body is independent of the body posture or
orientation and so a time constraint, the flight time, is sufficient to specify the maximum
time available to complete the required rotation.

Frame Glo will be orientated in order to aid the description of a somersault as seen by an
observer. The z-axis will be vertical. The y-axis will be the horizontal axis about which a
somersault occurs; in backward rotating somersaults this will be to the athlete’s right and
in a forward rotating somersault this will be to the athlete’s left. The x-axis will be in the
horizontal direction of travel for forward and backward somersaults; the negative x-axis is
the horizontal direction of travel for reverse and inwards somersaults. Figure 4-2 illustrates
frame Glo, along with an approximation of frame Pri, when an athlete is performing a pure
forward somersault.

![Frame Glo and Pri](image)

**Figure 4-2: Frames of reference Glo (black) and Pri (red).**
After O’Brien (2003). A pure somersault is illustrated and so Glo_y and Pri_y are coincident and
directed into the page. Should the athlete twist, Glo_y and Pri_y will no longer be coincident; Glo_y
will remain into the page but Pri_y will rotate as the athlete twists.

### 4.1.5 Orientation angles (θ, ψ, φ) defining orientation in
terms of somersault and twist angles.

The orientation angles θ, ψ, and φ,\(^7\) may be used to describe the angular displacement of
the principal axes of the body when it is moving as a unit, or the reorientation of the
reference frame, Ref, following postural changes, or actions. So that the equations may be
written for either situation let the term *Body frame* stand for either the principal frame, Pri,
or the reference frame, Ref, with the interpretation occurring when the equations are
actually used. For those readers unfamiliar with rotation matrices Appendix B.1.1 may be
helpful.

\(^7\) These symbols do not match the usage by Yeadon (1984). These symbols are commonly used for angles
and hence are used here to describe angles. They are fully defined here and this definition used throughout
this thesis.
The y-axis of the global inertial frame ($G^y_{\text{Glo}}$) is the axis about which the somersault occurs. The somersault angle ($\theta$) will thus be the angular displacement about $G^y_{\text{Glo}}$. The rotation matrix relating the position of the athlete after a somersault rotation of magnitude $\theta$ will thus be

$$G^y_{\text{Glo}} R_{\text{Body}, \theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

For twist to occur in a somersault, it is necessary that the longitudinal axis of the body is not perpendicular to the angular momentum vector (Section 2.3). In a pure somersault the angular momentum vector will be aligned with $G^y_{\text{Glo}}$. The angular momentum vector is the only quantity that is fixed in magnitude and orientation since there are no external moments applied. It is therefore desirable to measure the third angle from the angular momentum vector. In addition, for aesthetic reasons, it is of interest to know the angle between the longitudinal axis of the body and $G^y_{\text{Glo}}$. Thus let the third orientation angle be $\phi$ and let it be the angle between the somersault axis $G^y_{\text{Glo}}$ and the longitudinal axis of the body, $\text{Body}_z$.

A change in $\phi$ will be a rotation about the x-axis of a frame, say $\text{St}$, which tilts and somersaults with the body but does not twist. The orientation defined by $\theta$ and $\phi$ will be two successive rotations through YX Euler angles and so the rotation matrix relating the position of the body is now

$$G^y_{\text{Glo}} R_{\text{Body}, \theta \phi} = G^y_{\text{St}} R_{\text{St}} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi - \pi/2) & -\sin(\phi - \pi/2) \\ 0 & \sin(\phi - \pi/2) & \cos(\phi - \pi/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \phi & \cos \phi \\ 0 & -\cos \phi & \sin \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta \cos \phi & \sin \theta \sin \phi \\ 0 & \sin \phi & \cos \phi \\ -\sin \theta & -\cos \theta \cos \phi & \cos \theta \sin \phi \end{bmatrix}$$

Twist is the rotation about the longitudinal axis of the body, $\text{Body}_z$. Let $\psi$ be the twist angle which is the angular displacement about the $\text{Body}_z$-axis. If $\psi$ is increasing this is a left-twist, while if $\psi$ is decreasing this is a right-twist. Adding twist means we have three successive rotations through YXZ Euler angles
Since $\theta$, $\phi$, and $\psi$ are independent variables all orientations of the body frame with respect to frame Glo may be described.

### 4.2 Rotation in the quasi-rigid phase

The quasi-rigid phase is where an athlete holds a posture, and thereby rotates as a single unit. In such a phase the rigid body equations of motion may be used. As was mentioned in Section 2.1.3, having a quasi-rigid phase is in line with sporting expectations of an ideal performance, and has been observed and assumed in the previous literature. The amount of rotation completed during the quasi-rigid phase will vary, and there may even be more than one quasi-rigid phase, depending on the skill being performed. This section will derive the equations of motion, using the conservation of angular momentum, for the quasi-rigid phase in terms of the angles $\theta$, $\phi$, $\psi$ defined in Section 4.1.5 with frame Pri as the body frame; hence the motion can be directly related to a somersault, with or without twist.

#### 4.2.1 Initial conditions and posture parameters

The type of motion observed—determined by how $\theta$, $\phi$, $\psi$ change—depends on the posture used and the initial conditions.

The posture used determines the values of the principal moments of inertia; these are constant throughout the quasi-rigid phase, although their value will differ depending on the posture used. This means that the moments of inertia are parameters in the equations of motion. The values and orders of the moments of inertia for a selection of key sporting body postures, are given in Section 5.1.

The initial conditions will be the initial values of the orientation angles, $\theta_0$, $\phi_0$, and $\psi_0$, at the start of the quasi-rigid phase and the magnitude and direction of the angular momentum vector, which is constant while the athlete is airborne. The actions performed prior to the instant of take-off set the value and direction of the angular momentum vector, while the initial orientation angles are determined by the orientation on take-off and any subsequent reorientation that occurs as the athlete transitions from the take-off posture to the quasi-rigid posture. To assist in solving the equations of motion, while still allowing all skills in
Mathematical descriptors of the somersault

Diving and gymnastics to be described mathematically, the initial conditions will be restricted.

The orientation of the body with respect to the angular momentum vector is what determines the rotation observed. Since \( \theta \), and \( \phi \), were defined with respect to frame Glo it is important to consider the direction of the angular momentum vector with respect to frame Glo before writing the equations of motion.

If an athlete becomes airborne performing a pure somersault—aerial twist initiation techniques may occur after take-off—then the angular momentum vector would be parallel the \( \text{Glo}y \)-axis; that is \( \text{Glo} \mathbf{H} = [0, H, 0]^T \). Any component in the \( \text{Glo}x \)-axis direction is clearly erroneous and so may be disregarded.

Contact twist techniques introduce a component of angular momentum in the \( \text{Glo}z \)-axis direction. This means that the angular momentum vector would be inclined to the \( \text{Glo}y \)-axis by an angle (say \( \beta \)) equal to the inverse tan of the ratio of the component in the \( \text{Glo}z \) direction and the component in the \( \text{Glo}y \) direction. Provided that \( \beta \) is less than \( \pi/4 \) the athlete would still appear to somersault as their head would drop below the \( \text{Glo}y \)-axis (Yeadon, 1984), however, “pancaking” (George, 1980; Yeadon, 1984; Yeadon & Kerwin, 1999) would be observed. Figure 4-3 illustrates “pancaking” when \( I_{xx} = I_{yy} \) for the posture held.

![Figure 4-3: Inclination of the angular momentum vector](image)

**Figure 4-3: Inclination of the angular momentum vector**

a) When the angular momentum vector is parallel to the \( \text{Glo}y \)-axis b) when the angular momentum vector is inclined to the \( \text{Glo}y \)-axis. In both cases the black outline is for a whole number of somersaults and the blue outline an odd number of half somersaults. Note the greater angle between the \( \text{Glo}z \)-axis and the negative \( \text{Glo}z \)-axis at the half-somersault position. This increase is what is known as “pancaking”. When \( I_{xx} \neq I_{yy} \) then additional changes in \( \phi \) would also be present (Section 4.2.3), but these are equally applied to a) and b) relative to the angular momentum vector.

Somersault skills are defined according to the number of half-somersaults and half-twists performed. When \( \beta \) is less than \( \pi/4 \), a half-rotation about the angular momentum vector
may still be considered as a half-somersault. Pancaking does not alter the skill definition; it only affects the aesthetics of the skill. Thus, it is sufficient to analyse the situation when \( \text{GloH} = [0, H, 0]^T \) as though the athlete entered the air performing a pure somersault.

The \( \text{Glo} \) y-axis (Section 4.1.4) was defined as the axis about which somersault occurred; thus, regardless of the direction of rotation of the somersault, \( \dot{\theta} \) will always be positive. The direction of rotation is then interpreted as forward or backward rotation depending on the direction the athlete is facing. Ultimately, the purpose of solving the equations of motion is to determine what somersault skills may be performed. It is thus sufficient to set \( \theta_0 \) to zero, determine the motion in the quasi-rigid phase and then add any somersault rotation that occurred prior to the quasi-rigid phase to determine the total somersault completed.

Since \( \text{GloH} = [0, H, 0]^T \) then \( \phi \) is the angle between the angular momentum vector and the \( \text{Pri}z \)-axis. A non-zero twist rate is expected when \( \phi \neq \pi/2 \) or \( 3\pi/2 \), since a component of angular momentum is in the direction of the \( \text{Pri}z \)-axis. Somersault and twist would become indistinguishable if \( \phi = 0 \) or \( \pi \), as the athlete would be rotating about the \( \text{Glo} \) y-axis and the \( \text{Pri} \)z-axis. If \( \phi_o \) is in the second quadrant the twist will start in the opposite direction to when \( \phi_o \) was in the first quadrant but otherwise the twist behaviour will be the same. If \( \phi_o \) is in the third or fourth quadrant this would be equivalent to the athlete starting upside-down. As a result it is sufficient to explore when \( 0 < \phi_o \leq \pi/2 \). Then to describe all rotational cases it is only necessary to analyse the motion when \( 0 < \phi \leq \pi/2 \), since symmetry of the behaviour may be used to describe the full motion.

Ideally an athlete starts with zero initial twist; that is, \( \psi_o = 0 \). In reality, some unintended twist, in the direction of twist once airborne could be present. This should be ‘small’ or the athlete would incur deductions, for ‘cheating’ the twist. Since \( \phi_o \) has been restricted to the first quadrant, a positive twist rate is initially expected. The allowed initial twist would then be positive. What constitutes small will be explored in Section 4.2.3 in light of the different rotational cases identified there.

In the quasi-rigid phase the orientation of the anatomical axes of each segment and the principal axes do not change. Thus, to swap between frame \( \text{Pri} \) and frame \( \text{Ref} \), or any of the local frames, a constant angle determined from the posture, should be added.

Based on the above discussion the equations of motion will be derived and solved for a twisting somersault when, i) \( \text{GloH} = H[0, 1, 0]^T \), where \( H \) is a positive scalar; ii) \( \theta_0 = 0 \); iii) \( 0 < \phi_o \leq \pi/2 \); iv) \( \phi \neq \{0, \pi\} \), since somersault and twist would then be indistinguishable; and v) \( \psi_o \geq 0 \) and small. The equations given in Sections 4.2.2 to 4.2.5 are the key equations, demonstrating the mathematical process, and giving final results: the full working is given in Appendix H by equation number.
4.2.2 Equations of motion

To solve for $\theta$, $\phi$, $\psi$ it is necessary to resolve the athlete’s rigid-body angular velocity, $\omega$, into components $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$. Thus, the total angular velocity may be written as

$$\omega_{\text{Total}} = \dot{\theta}^{\text{Glo}} \mathbf{j} + \dot{\phi}^{\text{St}} \mathbf{i} + \dot{\psi}^{\text{Pri}} \mathbf{k}$$

Writing it with respect to frame Pri

$$\omega_{\text{Total}}^{\text{Pri}} = \dot{\theta}^{\text{Pri}} R_{\text{St}} S \mathbf{j} + \dot{\phi}^{\text{Pri}} R_{\text{Glo}}^{\text{Glo}} \mathbf{j} + \dot{\psi}^{\text{Pri}} \mathbf{k}$$

The rotation matrices $R_{\text{Glo}}$ and $R_{\text{St}}$ are the inverse rotation matrices of $R_{\text{Body}}$ and $R_{\text{Body}}$ from Section 4.1.5 respectively. Thus,

$$\omega_{\text{Total}}^{\text{Pri}} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} c\theta & s\theta & c\phi & s\psi & -s\theta & c\phi & s\psi \\ -s\theta & c\theta & s\phi & c\psi & s\theta & c\phi & c\psi \\ s\theta & s\phi & c\psi \ \ c\phi & c\theta & s\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $s$ and $c$ denote the sine and cosine operators.

Frame Pri is aligned with the principal axes of the body, which means that the tensor of inertia of the body with respect to Pri is a diagonal matrix. That is

$$I_{\text{Pri}} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

The angular momentum with respect to frame Pri is thus,

$$H_{\text{Pri}} = I_{\text{Pri}} \omega_{\text{Total}}^{\text{Pri}}$$

Since Pri is a rotating frame, the angular momentum with respect to Pri changes over time. It will, however, be constant in Glo, since it is an inertial frame. Further, from Section 4.2.1 the angular momentum vector was set to being in the Glo y-axis direction. Thus,

$$0 \quad H \quad 0 \quad = H_{\text{Glo}} = R_{\text{Pri}}^{\text{Glo}} H_{\text{Pri}}$$

Expanding and equating components of angular momentum we have:
0 = \dot{\theta}\left(\sin\phi\sin\psi\cos\theta\cos\psi\left(I_{xx} - I_{yy}\right) - \left(\sin\theta\cos\phi\sin\phi\right)\left(I_{xx} \sin^2\psi + I_{yy} \cos^2\psi - I_{zz}\right)\right)
+ \dot{\phi}\left(-\sin\theta\cos\phi\sin\psi\cos\psi\left(I_{xx} - I_{yy}\right) + \left(\cos\theta\right)\left(I_{xx} \cos^2\psi + I_{yy} \sin^2\psi\right)\right)
+ \dot{\psi}\left(I_{zz} \sin\theta \sin\phi\right)

H = \dot{\theta}\left(\sin^2\phi\left(I_{xx} \sin^2\psi + I_{yy} \cos^2\psi\right) + I_{zz} \cos^2\phi\right)
+ \dot{\phi}\cos\psi\sin\phi\sin\psi\left(I_{xx} - I_{yy}\right)
+ \dot{\psi}\left(I_{zz} \cos\phi\right)

0 = -\dot{\theta}\left(\sin\phi\sin\psi\sin\theta\cos\psi\left(I_{xx} - I_{yy}\right) + \left(\sin\phi\cos\theta\cos\phi\right)\left(I_{xx} \sin^2\psi + I_{yy} \cos^2\psi - I_{zz}\right)\right)
- \dot{\phi}\left(\cos\phi\cos\theta\cos\phi\sin\psi\left(I_{xx} - I_{yy}\right) + \left(\sin\theta\right)\left(I_{xx} \cos^2\psi + I_{yy} \sin^2\psi\right)\right)
+ \dot{\psi}\left(I_{zz} \cos\theta \sin\phi\right)

These three equations must hold simultaneously. By observing common factors in the expansion (Appendix H) it is not necessary to use the principal of the conservation of energy as well as the principal of the conservation of momentum, as previous authors have done (Synge & Griffith, 1959; Yeadon, 1993a). In Appendix D it is shown that the equations of motion do still satisfy the principle of the conservation of energy. The equations of motion may then be simplified to

\frac{\dot{\phi}}{\theta} = -\sin\phi\left(\frac{\sin\psi\cos\psi\left(I_{xx} - I_{yy}\right)}{I_{xx} - \left(I_{xx} - I_{yy}\right) \sin^2\psi}\right)

(4-1)

\frac{\dot{\psi}}{\theta} = \left(\frac{\cos\phi}{I_{zz}}\right)\left[\frac{I_{xx} I_{yy}}{I_{xx} - \left(I_{xx} - I_{yy}\right) \sin^2\psi} - I_{zz}\right]

(4-2)

\dot{\theta} = H\left(\frac{I_{xx} - \left(I_{xx} - I_{yy}\right) \sin^2\psi}{I_{xx} I_{yy}}\right)

(4-3)

From Equation (4-2) it is clear that an athlete wishing to increase the rate of twist within a somersault should decrease the angle \(\phi\). This is as expected since it increases the component of the angular momentum vector in the direction of the athlete’s longitudinal axis. Another immediately apparent strategy is to reduce \(I_{zz}\). It is, however, important to realise that all three moments of inertia appear in Equation (4-2) and affect the ratio of the twist angular velocity \(\dot{\psi}\) to the somersault angular velocity \(\dot{\theta}\); thus, the strategy should be to reduce \(I_{zz}\) relative to \(I_{yy}\) and \(I_{xx}\).

The moments of inertia of the body and H are positive, and so from Equation (4-3), the somersault angular velocity is always positive, oscillating between \(H/I_{yy}\) and \(H/I_{xx}\) as twist continues to increase. Adding twist only alters the somersault angular velocity as a result of \(I_{yy}\) and \(I_{xx}\) being unequal, not because there is any ‘extra’ rotation. The tilt means that
Mathematical descriptors of the somersault

the somersault is no longer about the transverse axis of the body and so one may view the situation as the tilt allowing the angular momentum to be shared between the somersault and the twist. The somersault angular velocity will oscillate between \( \frac{H}{I_{yy}} \) and \( \frac{H}{I_{xx}} \) depending on the value of \( \psi \). It is important to realise that while \( I_{xx} \neq I_{yy} \) the rate of change of \( \psi \) is not constant and in fact depends on the value of \( \phi \); this means that the way in which the somersault angular velocity oscillates between \( \frac{H}{I_{yy}} \) and \( \frac{H}{I_{xx}} \) depends on \( \phi \) and so based on Equation (4-3) it is not possible to say whether it is better to confine the twist to a small portion of the somersault or spread it over a larger portion in order to reduce the angular momentum required to achieve the skill. The change in angular momentum required when adding twist to a somersault is explored in Sections 4.2.7 and 5.3.4. If \( I_{xx} \) and \( I_{yy} \) are equal then \( \dot{\phi} \) is constant and the athlete somersaults at a steady rate and so the presence of twist or the portion of somersault used for twist does not alter the amount of somersault that may be completed.

When contemplating which postures to use in the quasi-rigid phase, it is of interest to know which twisting somersault skills could be achieved and what additional angular momentum would be required above a pure somersault to achieve these twisting somersault skills. Skills are awarded based on the twist completed within a somersault, not the twist completed with respect to time. Thus Equations (4-1) and (4-2) will be solved simultaneously to establish the relationship between \( \phi \) and \( \psi \) (Section 4.2.3), thereby determining the rotational states possible; based on this relationship the amount of somersault \( \theta \) required to achieve specific amounts of twist will be determined (Section 4.2.4). The angular momentum required will ultimately depend on the duration of the pure somersault and twisting phases when the skill is actually performed; however, it is still of interest to know the relative difficulty during the twisting phase due to the presence of twist and this is discussed in Section 4.2.7.

It is possible to determine \( \phi \) and \( \psi \) as functions of time. Yeadon (1993a) took this approach. Working and solutions for \( \phi \) and \( \psi \) with respect to time are given in Appendix D for all combinations of which moment of inertia is the maximum, minimum, and intermediate-valued moment of inertia. Those cases, which match cases presented with equations by Yeadon (1993a) are also identified. Equations of \( \phi \) with respect to time will be required in Section 4.2.7 when determining relative difficulty.

### 4.2.3 The relationship between \( \Phi \) and \( \psi \)

Solving Equation (4-1) and Equation (4-2) simultaneously, then integrating will give the relationship between \( \phi \) and \( \psi \). The form of the final relationship will depend on which moment of inertia is the intermediate-valued moment of inertia and on the initial conditions. This section will describe twelve cases. The trivial case where \( I_{xx} = I_{yy} = I_{zz} \) is
not included since rotation about the angular momentum vector is the obvious result. The combination of the initial orientation angles, \( \phi_o \) and \( \psi_o \), and which moment of inertia is the intermediate-valued moment of inertia separates the cases. Section 4.2.4 will then determine the somersault required for a period of oscillation of \( \phi \), for those cases of sporting interest.

Regardless of the order of the moments of inertia, if \( \phi_o = \pi/2 \) and \( \psi_o = 0 \), as would be expected for the start of a pure somersault, then \( \dot{\psi} = 0 \) and \( \dot{\phi} = 0 \), and so a pure somersault will be produced and without disturbance will continue. Let this be case 1.

Excluding the case when \( I_{xx} = I_{yy} = I_{zz} \), and case 1 where \( \phi_o = \pi/2 \), means that Equation (4-1) may be divided by Equation (4-2). This will allow the derivation of the equations for the relationship between \( \phi \) and \( \psi \) for the remaining eleven cases:

\[
\frac{\dot{\phi}}{\dot{\psi}} = -\sin \phi \frac{I_{zz}(I_{xx} - I_{yy}) \sin \psi \cos \psi}{\cos \phi \left[ I_{ax}(I_{yy} - I_{zz}) + I_{cz}(I_{xx} - I_{yy}) \sin^2 \psi \right]}
\]

Then, separating variables

\[
\frac{\cos \phi \, \dot{\phi}}{\sin \phi \, \dot{\psi}} = \frac{-I_{zz}(I_{xx} - I_{yy}) \sin \psi \cos \psi}{\left[ I_{ax}(I_{yy} - I_{zz}) + I_{cz}(I_{xx} - I_{yy}) \sin^2 \psi \right]}
\]

Both sides of this equation are in the form of the derivative of the natural logarithm. Since \( 0 < \phi < \pi \) the denominator on the left-hand side of the equation is positive. The denominator on the right-hand side may be positive or negative, and so both situations need to be considered. The full working is given in Appendix H. Conveniently, it is possible to write an equation that combines both situations:

\[
\sin \phi = \sin \phi_o \sqrt{\frac{I_{ax}(I_{yy} - I_{zz}) + I_{cz}(I_{xx} - I_{yy}) \sin^2 \psi_o}{I_{ax}(I_{yy} - I_{zz}) + I_{cz}(I_{xx} - I_{yy}) \sin^2 \psi}}
\]

(4-4)

In Equation (4-4), \( \psi \) appears only within the \( \sin^2 \psi \) term and so its initial direction is not clear. From Equation (4-2) it can be seen that, with \( \phi_o \) and \( \psi_o \) in the first quadrant, the initial twist direction is to the left (\( \dot{\psi} > 0 \)) when \( I_{yy} > I_{zz} \). Table 5-1 shows that, for the vast majority of postures, \( I_{yy} > I_{zz} \).

Equation (4-4) contains a square root. For the relationship between \( \psi \) and \( \phi \) to remain real, the value under the square root must not be negative for any angles of \( \phi \) and \( \psi \); meaning that the numerator and denominator must have the same sign. The common form of the numerator and denominator means that initially they will be the same sign. Since the numerator is constant then they will remain the same sign provided that the denominator
never crosses zero. Thus when the denominator equals zero, this is a bound of the allowed values of \( \psi \). That is
\[
I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi \neq 0
\]
Which when re-arranged gives
\[
\sin^2 \psi \neq -\frac{I_{xx}(I_{yy} - I_{zz})}{I_{zz}(I_{xx} - I_{yy})}
\]
(4-5)

Equation (4-5) will always be true when \( I_{yy} \) is the intermediate-valued moment of inertia, since the right hand side of Equation (4-5) is less than zero. Equation (4-5) will also be true when \( I_{xx} \) is the intermediate-valued moment of inertia, since the right hand side of Equation (4-5) is greater than one. However, when \( I_{zz} \) is the intermediate-valued moment of inertia the right hand side of Equation (4-5) is less than one. This limits the value of \( \psi \) and so also values of \( \psi_o \) that could be considered small. Let,
\[
\psi_{Small} = \sin^{-1} \left( -\frac{I_{xx}(I_{yy} - I_{zz})}{I_{zz}(I_{xx} - I_{yy})} \right)
\]
(4-6)

The value of \( \psi_{Small} \) as seen from Equation (4-6), decreases as \( I_{yy} \) and \( I_{zz} \) approach each other, and as \( I_{yy} \) diverges from \( I_{xx} \). If \( \psi_o \) were to equal \( \psi_{Small} \), Equation (4-4) could not be used to describe the relationship between \( \psi \) and \( \phi \). Returning to Equation (4-2), the initial twist to somersault angular velocity would be:
\[
\left( \frac{\psi}{\theta} \right)_{initial} = \left( \frac{\cos \phi}{I_{cz}} \right) \left[ \frac{I_{xx}I_{yy} - I_{zz}(I_{xx} - I_{yy})}{I_{xx}(I_{xx} - I_{yy}) - I_{zz}(I_{xx} - I_{yy})} \right] = 0
\]
Thus \( \psi_o \) will remain unchanged.

Returning to Equation (4-1), \( \frac{\phi}{\theta} \) will be positive when \( I_{yy} > I_{xx} \) and negative when \( I_{xx} > I_{yy} \). Thus \( \phi \) will progressively increase to \( \pi/2 \), for \( I_{yy} > I_{xx} \) and decrease to zero for \( I_{xx} > I_{yy} \). This situation produces a non-twisting, but not a pure, somersault. Since it occurs when \( \psi_o \) is on the bounds of being small it will be the limiting situation that is approached when \( I_{zz} \) is the intermediate-valued moment of inertia and \( \psi_o \) increases. Let this be Case 2.

The only restriction on \( \phi_o \) is that it does not equal zero. Yet its value with respect to \( \psi_o \) and the moments of inertia will determine whether continuous or oscillating twist is observed. From Equation (4-2), \( \phi = \pi/2 \) is a turning point for \( \psi \); this condition may be used to determine if continuous or oscillating twist would be observed. If \( \phi \) never reaches \( \pi/2 \) then
continuous twist would occur; if $\phi = \pi/2$ but $\psi \neq 0$ or $\pi/2$, then the twist reverses direction and the nature of the trigonometric functions in the equations of motion means that oscillating twist would be produced; if $\phi = \pi/2$ and $\psi = 0$ or $\pi/2$ simultaneously then no further twist occurs. Substituting $\phi = \pi/2$ and $\psi = 0$ or $\pi/2$, into Equation (4-4) gives equations for the initial conditions where no further twist occurs, and hence the initial conditions that form the transition between continuous and oscillating twist being displayed:

$$\sin^2 \phi_o \frac{I_{zx}(I_{yy} - I_{zz}) + I_{xx}(I_{zz} - I_{yy})}{I_{xx}(I_{yy} - I_{zz})} \sin^2 \psi_o = 1$$

$$\sin^2 \phi_o \frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy})}{I_{xx}(I_{yy} - I_{zz})} \sin^2 \psi_o = 1$$

Athletes intentionally change $\phi_o$ to achieve twist and so the value of $\phi_o$ will be deemed the critical initial condition. Rearranging the above two equations gives the critical values of $\phi_o$ that separate continuous and oscillating twist as

$$\phi_{Crit1} = \sin^{-1} \left( \frac{I_{xx}(I_{yy} - I_{zz})}{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy})} \sin^2 \psi_o \right)$$

(4-7 a)

$$\phi_{Crit2} = \sin^{-1} \left( \frac{I_{yy}(I_{xx} - I_{zz})}{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy})} \sin^2 \psi_o \right)$$

(4-7 b)

The critical value separating continuous twist and twist that oscillates about $\psi = \pi/2$ is $\phi_{Crit1}$. When $\psi_o = 0$, then $\phi_{Crit1} = \pi/2$. The critical value separating continuous twist and twist that oscillates about $\psi = 0$ is $\phi_{Crit2}$.

When $\psi_o = 0$, then $\phi_{Crit2} = \sin^{-1} \sqrt{I_{yy}(I_{xx} - I_{zz})/I_{xx}(I_{yy} - I_{zz})}$; this is the same as the equation given by Yeadon (1993a) as being “the singular solution separating the twisting and wobbling [oscillating twist] modes”.

When $I_{yy}$ is the intermediate-valued moment of inertia then $\phi_{Crit2}$ is undefined, and thus the twist may not oscillate about $\psi = 0$. However, $\phi_{Crit1}$ is defined and gives the value of $\phi_o$ that represents the separation between continuous twist and twist oscillating about $\psi = \pi/2$. Let the continuous twist case be Case 3; the case where twist oscillates about $\psi = \pi/2$ be case 4; and when $\phi = \phi_{Crit1}$, so that twist that stops at $\pi/2$, be Case 5. If $\psi_o = 0$, only case 3 will be observed since $\phi_{Crit1} = 90^\circ$.

To illustrate the relationship between $\psi$ and $\phi$ in Case 3 and Case 4, Figure 4-4 and Figure 4-5 are $\psi$-$\phi$ phase plots for an example athlete, holding the postures 1U1DB, L, and Pu, which all have $I_{yy}$ as the intermediate-valued moment of inertia and are common twist
postures. The difference in initial conditions determines whether the same postures may display behaviours classified as case 3 or case 4.

Figure 4-4: Example of Case 3.
Relationship between $\phi$ and twist angle $\psi$ for the athlete inertial properties in Section 3.10 in postures 1U1DB, L, and Pu, with initial conditions $\psi_o = 0$ and $\phi_o = 80^\circ$.

Figure 4-5: Example of Case 4.
Relationship between $\phi$ and twist angle $\psi$ for the athlete inertial properties in Section 3.10 in postures 1U1DB, L, and Pu, with initial conditions $\psi_o = 3^\circ$ and $\phi_o = 90^\circ$. Oscillating twist occurs and so the value of $\phi$ and $\psi$ trace around the curves in an anti-clockwise direction.

Case 3 will show oscillations in $\phi$: the maximum value of $\phi$ occurs when $\psi = 0$, and the minimum value occurs when $\psi = \pi/2$; the magnitude of the oscillation is thus

$$\sin^{-1}\left(\sin\phi \sqrt{\frac{I_{yy}(I_{zz} - I_{xx}) + I_{zz}(I_{xx} - I_{yy})\sin^2\psi}{I_{xx}(I_{zz} - I_{yy})}}\right) - \sin^{-1}\left(\sin\phi \sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{yy}(I_{zz} - I_{xx})\sin^2\psi}{I_{yy}(I_{xx} - I_{zz})}}\right)$$

where if $\psi_o = 0$ this reduces to
\[ \phi_o - \sin^{-1}\left(\frac{I_{xx}(I_{yy} - I_{zz})}{I_{yy}(I_{xx} - I_{zz})}\right) \]

Thus, the magnitude of the oscillation of \( \phi \) is greater for postures where any of the following are true individually or in combination: \( I_{xx} \) diverges from \( I_{yy} \); \( I_{zz} \) approaches \( I_{yy} \); and \( \phi_o \) approaches 90°. A small oscillation in \( \phi \) in Case 3 indicates that when performing a twisting somersault in any of these postures the oscillation in \( \phi \) is not an aesthetic concern.

When \( I_{zz} \) is the intermediate-valued moment of inertia, \( \psi_{o,small} \) has a value less than \( \pi/2 \). This means that the twist will oscillate about \( \psi = 0 \) and there are no initial conditions that will produce continuous twist. This is confirmed by the fact that \( \phi_{crit\ 1} \) is undefined, so \( \phi \) and \( \psi \) can never reach \( \pi/2 \) at the same time, and the fact that \( \phi_{crit\ 2} \) is \( \pi/2 \) and so there will be no oscillations about \( \psi = \pi/2 \). Let this be Case 6.

To illustrate the general relationship between \( \psi \) and \( \phi \) in Case 6, Figure 4-6 illustrates the \( \psi-\phi \) phase plane for an example athlete holding posture P, which has \( I_{zz} \) as the intermediate-valued moment of inertia.

![Figure 4-6: Example of Case 6.](image)

Relationship between \( \phi \) and twist angle \( \psi \) for the athlete inertial properties in Section 3.10 in posture P, with initial conditions \( \psi_o = 0 \) and \( \phi_o = 80° \).

The values between which \( \psi \) oscillates may be determined from Equation (4-4) by substituting \( \phi = \pi/2 \) and choosing the values for which \( \psi \) is in the first or fourth quadrant. The first quadrant bound is,

\[ \psi = \sin^{-1}\left(\frac{I_{xx}(I_{yy} - I_{zz})}{I_{yy}(I_{xx} - I_{zz})}\cos^2 \phi_o + \sin^2 \phi_o \sin^2 \phi_o \right) \]

When \( \psi_o = 0 \) this reduces to
When \( I_{xx} \) is the intermediate-valued moment of inertia, \( \phi_{\text{Crit}1} \) is \( \pi/2 \) when \( \psi_o = 0 \) and undefined otherwise, while \( \phi_{\text{Crit}2} \) will have a value less than \( \pi/2 \). This means that when \( I_{xx} \) is the intermediate-valued moment of inertia, it is possible to observe either continuous or oscillating twist. Let Case 7 be when continuous twist is displayed; let Case 8 be when twist that oscillates about \( \psi = 0 \) is displayed; and let Case 9 be the transition between Case 7 and Case 8 where twist stops at \( \pi/2 \).

For Case 7 the magnitude of the oscillation in \( \phi \) has the same equation as when \( I_{yy} \) is the intermediate-valued moment of inertia; however, the minimum value of \( \phi \) is now at \( \psi = 0 \), and the maximum at \( \psi = \pi/2 \).

For Case 8 the oscillation in \( \psi \) is about \( \psi = 0 \), and the magnitude of these oscillations will have the same equation as when \( I_{zz} \) is the intermediate-valued moment of inertia, although continuous twist can be displayed when \( I_{xx} \) is the intermediate-valued moment of inertia, and so these oscillations can be any size up to \( \pi/2 \).

To illustrate the relationship between \( \psi \) and \( \phi \) in Cases 7, 8, and 9, Figure 4-7, Figure 4-8, and Figure 4-9 are \( \psi-\phi \) phase plots for an example athlete for three different values of \( \phi_o \), holding the posture T, which has \( I_{xx} \) as the intermediate-valued moment of inertia. It is the difference in the value of \( \phi_o \) which determines the case observed.

![Figure 4-7: Example of Case 7.](image)

Relationship between \( \phi \) and twist angle \( \psi \) for the athlete inertial properties in Section 3.10 in posture T, with initial conditions \( \psi_o = 0 \) and \( \phi_o = 50^\circ \).

Considering Figure 4-7 compared to Figure 4-4, although it is possible for this athlete to twist in posture T, a much smaller value of \( \phi_o \) is required than for postures 1U1DB, L, or Pu; Posture T is thus not a practically useful twisting posture. Further, even when
continuous twist was produced the larger oscillations in $\phi$ would potentially detract from the performance.

![Figure 4-8: Example of Case 8.](image)

*Figure 4-8: Example of Case 8.*

Relationship between $\phi$ and twist angle $\psi$ for the athlete inertial properties in Section 3.10 in posture T, with initial conditions $\psi_o = 0$ and $\phi_o = 80^\circ$.

Considering Figure 4-8 compared to Figure 4-6, the larger oscillations in twist angle $\psi$ when holding posture T, compared to posture P, means that posture T is a less effective posture for appearing to cease continuous twist. The oscillations in twist angle $\psi$ in Figure 4-8 are expected to be clearly visible.

![Figure 4-9: Example of Case 9.](image)

*Figure 4-9: Example of Case 9.*

Relationship between $\phi$ and twist angle $\psi$ for the athlete inertial properties in Section 3.10 in posture T, with initial conditions $\psi_o = 0$ and $\phi_o \approx 68.4^\circ$. The twist stops at $90^\circ$ because $\psi$ and $\phi$ reach $90^\circ$ at the same time.

When any of the moments of inertia are equal, cases between those already numbered will be seen. When $I_{xx} = I_{yy}$ the rotational behaviours will be between Case 3 and Case 7, since
as $I_{xx}$ approaches $I_{yy}$ the oscillations in $\phi$ decrease, and $\phi_{Crit1}$ and $\phi_{Crit2}$ approach $\pi/2$. Letting $I_{xx} = I_{yy} = I_a$ and substituting this into Equation (4-1) and Equation (4-2) gives

$$\frac{\dot{\phi}}{\theta} = 0$$

and

$$\frac{\dot{\psi}}{\theta} = \frac{(I_a - I_z) \cos \phi}{I_z}.$$  \hspace{1cm} (4-8)

Since $\phi$ is constant, a steady twisting somersault is produced when $\phi_o < \pi/2$. The value of $\psi_o$ has no effect on the rotational state. Let this be Case 10. Equation (4-8) is equivalent to “steady precession with zero moment” (Meriam & Kraige, 1998, p. 587), and the equation given by Yeadon (1993 b)\(^8\) for his “rod mode” when $I_{xx} = I_{yy}$.

It is often assumed that $I_{zz}$ is the minimum-valued moment of inertia (Yeadon, 1993 b), When considering a range of posture and inertial property data sets this would be generally, although not always, true (Table 5-1). Nevertheless in Equation (4-8) there is no restriction on whether $I_{zz}$ is the minimum- or maximum-valued moment of inertia, the only difference being the direction of the twist: a left twist ($\psi > 0$) occurs when $I_{zz}$ is the minimum-valued moment of inertia and a right twist ($\psi < 0$) occurs when $I_{zz}$ is the maximum-valued moment of inertia.

When $I_{xx} = I_{zz}$, the transition between when $I_{zz}$ is the intermediate-valued moment of inertia and when $I_{xx}$ is the intermediate moment of inertia is represented. Let this be Case 11. Applying $I_{xx} = I_{zz}$ to Equation (4-4) gives

$$\sin \phi = \sin \phi_o \frac{\cos \psi}{\cos \psi_o}.$$  \hspace{1cm} (4-4)

This means that the relationship between $\phi$ and $\psi$ is set by the initial conditions only. Twist will oscillate about $\psi = 0$: with the bounds as

$$\psi = \pm \cos^{-1}(\sin \phi_o \cos \psi_o).$$

Figure 4-10 is a $\phi$-$\psi$ phase plot for the situation when $I_{xx} = I_{zz}$.

\(^8\) Yeadon has sine rather than cosine because he uses the angle of tilt of the longitudinal axis measured from the vertical when the angular momentum vector is horizontal rather than the angle between the angular momentum vector and the longitudinal axis that is used here: The two angles are complementary.
When $I_{yy} = I_{zz}$, the transition between when $I_{zz}$ is the intermediate-valued moment of inertia and when $I_{yy}$ is the intermediate moment of inertia is represented. Let this be Case 12.

As $I_{zz}$ approaches $I_{yy}$, the value of $\psi_{oSmall}$ decreases to zero. Large oscillations in both $\phi$ and $\psi$ are apparent, provided that $\psi_{o} \neq 0$. If $\psi_{o} = 0$, then a non-twisting somersault will be produced regardless of the value of $\phi_{o}$, because while $\psi = 0$, even though anatomically the athlete would be seen to have tilted, mechanically there has been no change to the moments of inertia with respect to the angular momentum vector.

Applying $I_{yy} = I_{zz}$ to Equation (4-4) gives

$$\sin \phi = \sin \phi_{o} \frac{\sin \psi_{o}}{\sin \psi}.$$

The angle $\psi$ will always have the same sign as $\psi_{o}$. Figure 4-11 illustrates the relationship between $\psi$ and $\phi$ for two values of $\psi_{o}$.

**Figure 4-10:** Example of Case 11.
Relationship between $\phi$ and twist angle $\psi$ when $\psi_{o} = 0$ and $\phi_{o} = 80^\circ$.

**Figure 4-11:** Example of Case 12.
Relationship between $\phi$ and twist angle $\psi$ when $\psi_{o} = 0$ and $\phi_{o} = 80^\circ$. 
Considering all the cases, it is expected that postures where $I_{yy}$ is the intermediate-valued moment of inertia would be the most useful during the twisting phase of a twisting somersault since it is expected to display continuous twist for a value of $\phi_o$ closest to $\pi/2$. Postures where $I_{zz}$ is the intermediate-valued moment of inertia are expected to be the most useful for preventing twist (Section 4.2.6.2) since they would always show oscillations in twist. This is only true in part, since as will be shown in Section 4.2.4, the relative magnitudes of the three moments of inertia will determine the amount of somersault required for any amount of twist.

It is possible to transition between cases by adjusting the initial conditions and/or the moments of inertia. Figure 4-13 is a diagrammatic representation of the process. Changing the initial conditions effectively shifts the curves representing the relationship between $\psi$ and $\phi$ up or down; continuous twist curves will eventually meet when $\phi = \pi/2$ and so break into oscillating twist curves or vice versa. Changing the moments of inertia increases or decreases the magnitude of the oscillations in $\phi$, again allowing continuous twist curves to eventually meet when $\phi = \pi/2$ and so break into oscillating twist curves or vice versa.

**The effect of initial twist**

The relationship between $\psi$ and $\phi$ for the continuous twist Cases 3 and 7 differs in the direction in which the curves move after $\psi = 0$. In Case 3 the value of $\phi$ is a maximum when $\psi = 0$, while in Case 7 $\phi$ is a minimum when $\psi = 0$.

![Figure 4-12: The effect of initial twist on the curve relating $\psi$ and $\phi$ for continuous twist.](image)

(a) The curves for Case 3 shift upwards as $\psi_o$ increases and so decreases twist rate; (b) The curves for Case 7 shift downwards as $\psi_o$ increases and so increases twist rate.
If an athlete is able to achieve the same value of $\phi_o$ regardless of whether or not any twist is present, then the existence of initial twist, $\psi_o \neq 0$, in Case 3 will shift the curve upwards and ultimately reduce the twist rate. Figure 4-12 a) illustrates. Thus, if using a posture with $I_{yy}$ as the intermediate-valued moment of inertia and wishing for the highest twist rate possible, an athlete must seek to avoid any initial twist. In contrast, for Case 7, the presence of initial twist will shift the curve downwards and increase the twist rate. Figure 4-12 b) illustrates. Thus, if using a posture with $I_{xx}$ as the intermediate-valued moment of inertia and seeking the highest twist rate possible, an athlete would benefit from having some initial twist, although it may aesthetically detract from performance.

A note about stability

When performing a pure somersault, Case 1, the athlete is rotating about their principal transverse axis, $\text{Pri}_y$. If there was a disturbance resulting in a small decrease in $\phi$ from $\pi/2$, or a small increase in $\psi$ from 0, or both, then twist would be observed. If $I_{yy}$ was the intermediate-valued moment of inertia then Case 3, 4, or 5 may result; these involve large changes in the twist even for small changes of $\phi$ from $\pi/2$ or $\psi$ from 0. As a result of these large oscillations for small orientating “disturbances”, rotating about the principal axis corresponding to the intermediate-valued moment of inertia may be considered unstable in a theoretical sense. Ultimately the number of somersaults required for appreciable twist to be observed determines if the instability is of practical significance. If $I_{xx}$ was the intermediate-valued moment of inertia, a small disturbance produces oscillations in twist about $\psi = 0$, that is Case 8. What constitutes small is determined by $\phi_{\text{Crit}2}$; if the disturbance is larger than $\phi_{\text{Crit}2}$ continuous twist will occur. If $I_{zz}$ is the intermediate-valued moment of inertia, only oscillating twist may ever be observed; provided the twist disturbance is small as determined by $\psi_{\text{small}}$. The result is an oscillation about $\psi = 0$; that is case 6.

If assessing whether or not a particular posture is more stable than another, it is important to know the relative magnitudes of inertia and the size of the expected disturbance. Nevertheless $I_{xz}$ as the intermediate-valued moment of inertia may be considered more stable than when $I_{xx}$ or $I_{yy}$ is the intermediate-valued moment of inertia, since regardless of the disturbance size, only oscillating twist will be observed.

---

Depending on the direction of the disturbance and whether $I_{xz}$ is the maximum- or minimum-valued moment of inertia, Case 5, will either cause the twist to return to zero or go to 180° and then stop. For example, if $I_{xz}$ is the minimum-valued moment of inertia, and $\phi$ remains at 90°, then if the disturbance caused $\psi_o$ to be positive it will continue to increase to 180° and then stop; however, if the disturbance caused $\psi_o$ to be negative it will return to 0° and then stop. It is most likely that the disturbance will produce Cases 3 or 4 rather than 5 and so it is reasonable to consider rotation about the principal axis corresponding to the intermediate-valued moment of inertia as ‘unstable’.
Mathematical descriptors of the somersault

Figure 4-13: The cases and their relationships.
The arrows show the changes to the initial conditions and the moments of inertia required to move from one case to another. The symbol “→” represents “approaches”. 
4.2.4 Somersault required to complete set amounts of twist

Using the appropriate relationship between $\phi$ and $\psi$ found in Section 4.2.3, $\psi$ may be eliminated from Equation (4-1) and the resulting equation solved using elliptical integrals to give an expression for $\phi$ as a function of $\theta$. Substituting the result into Equation (4-4) will give $\psi$ as a function of $\theta$. This must be done for each case from Section 4.2.3. The nature of the relationship between $\psi$ and $\phi$ means that, if oscillating twist is produced, $\psi$ will have the same period as $\phi$ and if continuous twist is produced then the twist angle $\psi$ will increase by $\pi/2$ with each period of $\phi$. Since skills are defined in increments of a half twist ($\psi$ increases by $\pi/2$) then it is sufficient to determine the fraction of a somersault required for one period of $\phi$; let this be $N_\theta$. Multiplying $N_\theta$ by the number of half twists to be performed will give the amount of somersault required for any particular skill; if this number is less than the number of somersaults allowed then the skill may be achieved.

To simplify the appearance of equations and to aid identification of parts of the integral as elliptical integrals, it is convenient to define the following constants:

\[
\begin{align*}
    a &= I_{xx}(I_{yy} - I_{zz}) \cos^2 \phi_o - I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o \\
    b &= I_{zz}(I_{yy} - I_{zz}) \\
    c &= I_{yy}(I_{xx} - I_{zz}) - \sin^2 \phi_o \left(I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi_o \right) \\
    d &= I_{zz}(I_{xx} - I_{yy}) + I_{xx}(I_{yy} - I_{zz}) \sin^2 \psi_o \\
    f &= I_{zz}(I_{xx} - I_{yy}) + I_{xx}(I_{yy} - I_{zz}) = I_{yy}(I_{xx} - I_{zz}) \\
    g &= \sin^2 \phi_o \left(I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi_o \right) = f - c = b - a
\end{align*}
\]

Re-writing (4-4) using these constants gives

\[
\sin^2 \phi = \frac{g}{b + (f - b) \sin^2 \psi}
\]

Which may be re-arranged to give

\[
\sin^2 \psi = \frac{g - b \sin^2 \phi}{(f - b) \sin^2 \phi}
\]

(4-10)

Substituting this into Equation (4-1) gives

\[
\frac{\dot{\phi}}{\theta} = -\sin \phi \frac{\sqrt{(g - b \sin^2 \phi)(f \sin^2 \phi - g)}}{I_{xx}I_{yy} \sin^2 \phi - g}
\]

Inverting and integrating gives

\[
\theta = \int_{\phi_0}^{\phi} \frac{g - I_{xx}I_{yy} \sin^2 \phi}{\sin \phi \sqrt{(g - b \sin^2 \phi)(f \sin^2 \phi - g)}} \, d\phi
\]
Since $\sin \phi \neq 0$ then

$$\theta = g \int_{\phi_0}^{\phi} \frac{1}{\sin \phi \sqrt{(g-b \sin^2 \phi) f \sin^2 \phi - g}} \, d\phi - I_{xx} I_{yy} \int_{\phi_0}^{\phi} \frac{\sin \phi}{\sqrt{(g-b \sin^2 \phi) f \sin^2 \phi - g}} \, d\phi$$

(4-11)

Let $u = \cos \phi$; thus $du/d\phi = -\sin \phi$ and $d\phi = -du/\sqrt{1-u^2}$. This is the same substitution used by Yeadon (1993a) when seeking an equation for $\phi$ with respect to time. It allows the equation to be written in a form that can be more easily identified with Elliptic integrals of the first and third kind. Following this substitution, Equation (4-11) becomes

$$\theta = g \int_{c_0}^{c} \frac{1}{\sqrt{1-u^2} \sqrt{1-u^2} \sqrt{1-u^2} \sqrt{1-u^2} \sqrt{1-u^2}} \, d\phi$$

(4-12)

The first integral in Equation (4-12) may be written as an elliptical integral of the third kind and the second integral as an elliptical integral of the first kind (Appendix B.3.4).

It is only necessary to evaluate Equation (4-12) for the cases of sporting interest. It is highly unlikely that an athlete would be able to achieve the exact initial conditions for the transition cases, and so rather than evaluating Equation (4-12) for the transition cases, these will simply be regarded as limiting situations that occur as $\phi_o$ approaches either $\phi_{crit1}$ or $\phi_{crit2}$. In Section 5.1, equality of moments of inertia was found to occur only for one set of inertial properties, and one posture; thus it is reasonable to ignore the equality situations. This leaves cases 3, 4, 7, and 8. Case 10 is also evaluated so that the consequences of the assumption that $I_{xx} = I_{yy}$, can be specifically explored in Section 4.2.5. The full working is presented in Appendix H.

**Case 3**

$$\theta = \left( \frac{f}{\sqrt{bc}} \right) \left( \frac{bc - af}{b(c - f)} \right) \sin^{-1} \left( \frac{\sqrt{bc - af}}{bc - af} \right) \frac{1}{bc} \left[ \frac{1}{bc} \right] \frac{1}{bc}$$

(4-13)

**Case 4**

$$\theta = \left( \frac{f}{\sqrt{bc - af}} \right) \left( \frac{c}{c - d} \right) \sin^{-1} \left( \frac{\sqrt{bc - af}}{bc - af} \right) \frac{bc}{bc - af}$$

(4-14)
Mathematical descriptors of the somersault

Case 6

\[ \theta = \frac{-e}{\sqrt{-bc}} \left[ \prod \left( \frac{a}{b}, \sin^{-1}\left( \frac{b}{\sqrt{a}} \cos \phi \right) \frac{af}{bc} \right) \right]_{\phi}^{\psi} + \frac{l_{xx}}{\sqrt{-bc}} \left[ F \left( \sin^{-1}\left( \frac{b}{\sqrt{a}} \cos \phi \right) \frac{af}{bc} \right) \right]_{\phi}^{\psi} \]

(4-15)

Case 7

\[ \theta = \frac{b}{\sqrt{af}} \left[ \prod \left( \frac{a}{b}, \sin^{-1}\left( \frac{f(a-b)\cos^2 \phi}{af - bc} \right) \frac{af}{bc} \right) \right]_{\phi}^{\psi} + \frac{l_{xx}}{\sqrt{af}} \left[ F \left( \sin^{-1}\left( \frac{f(a-b)\cos^2 \phi}{af - bc} \right) \frac{af}{bc} \right) \right]_{\phi}^{\psi} \]

(4-16)

Case 8

\[ \theta = \frac{b}{\sqrt{af - bc}} \left[ \prod \left( \frac{a}{b}, \sin^{-1}\left( \frac{1-b}{a} \cos \phi \right) \frac{af}{af - bc} \right) \right]_{\phi}^{\psi} + \frac{l_{xx}}{\sqrt{af - bc}} \left[ F \left( \sin^{-1}\left( \frac{1-b}{a} \cos \phi \right) \frac{af}{af - bc} \right) \right]_{\phi}^{\psi} \]

(4-17)

Case 10 \((I_{xx} = I_{yy})\)

Throughout the motion \(\phi\) has a constant value (Section 4.2.3) and so Equation (4-8) when integrated with respect to \(\theta\) becomes,

\[ \psi = \frac{I_{yy} - I_{zz}}{I_{zz}} \cos \phi \theta + \psi_o \]

Thus the somersault required for a particular amount of twist is

\[ \theta = \frac{I_{zz}}{\left( I_{yy} - I_{zz} \right) \cos \phi} (\psi - \psi_o) \]

(4-18)

When integrating Equation (4-12) it was assumed that \(\phi\) and \(\psi\) were both in the first quadrant. Based on the initial conditions set in Section 4.2.1, this will be true and so a bound to the integral of \(\phi_o\) is acceptable. The other bound should then be less than or equal to the next extrema value of \(\phi\) or \(\pi/2\): this is when \(\psi\) or \(\phi\) moves into the next quadrant respectively. If the integral for a larger region is required, symmetry should be used to relate it back to first quadrant values.

To find the angle of somersault required for one cycle of \(\phi\) then Equation (4-13) through to Equation (4-18) may be evaluated with bounds chosen to be the extrema of \(\phi\), as may be determined from Equation (4-4) in Section 4.2.3. To determine the fraction of somersault required Equations (4-13) to (4-18) must be divided by \(2\pi\). This value is \(N_{\theta}\). The extrema of \(\phi\) and \(\psi\) and \(N_{\theta}\) for each case are presented in Table 4-1, and the evaluation is shown in Appendix H.
Table 4-1: Extrema of $\phi$ and $\psi$, and $N_\theta$ for each case.

When $\Pi$ or $F$ are in the form $\Pi(n; \frac{\pi}{2}, k^2)$ or $F\left(\frac{\pi}{2}, k^2\right)$, these are known as the complete elliptical integrals of the third and first kind respectively.

<table>
<thead>
<tr>
<th>Case</th>
<th>Conditions on constants</th>
<th>Max and min value of $\phi$</th>
<th>Max and min value of $\psi$</th>
<th>$N_\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$I_{yy}$ is the intermediate and $g &lt; b$; hence $a &gt; 0$</td>
<td>Max: $\phi = \sin^{-1}\left(\frac{g}{b}\right) = \cos^{-1}\left(\frac{a}{b}\right)$</td>
<td>Continuous twist</td>
<td>$-\frac{f}{\pi \sqrt{bc}} \Pi\left(\frac{bc - af}{b(c - f)}, \frac{\pi}{2}, 1 - \frac{af}{bc}\right)$ $+ \frac{I_{xy} I_{yy}}{\pi \sqrt{bc}} F\left(\frac{\pi}{2}, 1 - \frac{af}{bc}\right)$</td>
</tr>
<tr>
<td>4</td>
<td>$I_{yy}$ is the intermediate and $g &lt; b$; hence $a &gt; 0$</td>
<td>Max: $\phi = \pi - \sin^{-1}\left(\frac{g}{b}\right)$</td>
<td>Max: $\psi = \pi - \sin^{-1}\sqrt{-\frac{a}{f - b}}$</td>
<td>$-\frac{2f}{\pi \sqrt{bc - af}} \Pi\left(\frac{c}{c - f}, \frac{\pi}{2}, \frac{bc - af}{bc - af}\right)$ $+ \frac{2I_{xy} I_{yy}}{\pi \sqrt{bc - af}} F\left(\frac{\pi}{2}, \frac{bc}{2}, \frac{bc - af}{bc - af}\right)$</td>
</tr>
<tr>
<td>6</td>
<td>$I_{x}$ is the intermediate and $g &lt; 0$</td>
<td>Max: $\phi = \pi - \sin^{-1}\left(\frac{g}{b}\right)$</td>
<td>Max: $\psi = \sin^{-1}\sqrt{-\frac{a}{g - b}}$</td>
<td>$-\frac{2g}{\pi \sqrt{-bc}} \Pi\left(\frac{a}{b}, \frac{\pi}{2}, \frac{af}{bc}\right)$ $+ \frac{2I_{xy} I_{yy}}{\pi \sqrt{-bc}} F\left(\frac{\pi}{2}, \frac{af}{2}, \frac{bc}{bc}\right)$</td>
</tr>
<tr>
<td>7</td>
<td>$I_{xx}$ is the intermediate and $g &lt; f$; hence $c &gt; 0$</td>
<td>Max: $\phi = \sin^{-1}\sqrt{-\frac{a}{f - b}}$</td>
<td>Continuous twist</td>
<td>$-\frac{b}{\pi \sqrt{af}} \Pi\left(\frac{af - bc}{f(a - b)}, \frac{\pi}{2}, \frac{1 - bc}{af}\right)$ $+ \frac{I_{xy} I_{yy}}{\pi \sqrt{af}} F\left(\frac{\pi}{2}, 1 - \frac{bc}{af}\right)$</td>
</tr>
<tr>
<td>Page</td>
<td>Conditions</td>
<td>Equations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>------------</td>
<td>-----------</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 8    | $I_{xx}$ is the intermediate and $g > d$; hence $c < 0$ | Max: $\phi = \pi - \sin^{-1}\left(\frac{g}{\sqrt{b}}\right)$
Min: $\phi = \sin^{-1}\left(\frac{g}{\sqrt{b}}\right)$
Oscillation will be about $\psi = 0$
Max: $\psi = \sin^{-1}\left(-\frac{a}{\sqrt{f - b}}\right)$
Min: $\psi = -\sin^{-1}\left(-\frac{a}{\sqrt{f - b}}\right)$
Provided $\phi_o \neq 0$ then continuous twist occurs. |
| 10   | $I_{xx} = I_{yy}$ Hence, $b = f$ and $a = c$ | $\phi$ remains at a constant value. |
|      | $\phi$ does not oscillate but the somersault required for a half-twist is $\frac{I_{zz}}{2(I_{yy} - I_{zz})}\cos\phi$. |
To illustrate the relationship between $N_\theta$ and the initial conditions for cases 3, 6, 7 and 8, all of which have use in the performance of twisting somersaults (case 4 is covered in the discussion of stability in Section 5.2.10), Figure 4-14, Figure 4-15, and Figure 4-16 plot $N_\theta$ as a function of $\phi_o$, when $\psi_o = 0$, for postures L, P and T each of which have a different intermediate-valued moment of inertia.

**Figure 4-14: Example of $N_\theta$ as a function of $\phi_o$ when $\psi_o = 0$, Case 3**

The inertial properties are for the athlete identified in Section 3.10 holding posture L.

The shape of Figure 4-14 is as expected, with $N_\theta$ decreasing as $\phi_o$ decreases. As the longitudinal axis approaches the angular momentum vector the twist rate increases and hence the somersault required, $N_\theta$, decreases; as $\phi_o$ approaches $\pi/2$, the somersault required approaches infinity since when $\phi_o = \pi/2$ ($\psi_o = 0$) a pure somersault occurs. Based on Figure 4-14 this athlete must start with $\phi_o \sim 86.5^\circ$ ($\psi_o = 0$) to achieve a full twist in a single somersault, ($N_\theta = \frac{1}{2}$), using posture L.

**Figure 4-15: Example of somersault required for one period of oscillation of $\phi$ in Case 6**

The inertial properties are for the athlete identified in Section 3.10 holding posture P.
From Figure 4-15, when in the posture P, this athlete would experience oscillations in twist, regardless of the value of $\phi_o$. As $\phi_o$ decreases, both $N_\theta$ and the amplitude of the oscillations increase. This characteristic makes postures with $I_{zz}$ as the intermediate-valued moment of inertia, such as posture P, useful for preventing continuous twist: larger oscillations also have a larger value of $N_\theta$ and so the impact of larger oscillations would not be seen as early. If this athlete had $\phi_o \sim 86.5^\circ$ ($\psi_o = 0$) which would produce a full twist in a single somersault when in posture L, then in posture P they would have a maximum twist of $\sim 1^\circ$ and this maximum would be seen in 0.37 of a somersault; that is $N_\theta \sim 1.5$ and the oscillation is between $\pm 1^\circ$.

Figure 4-16: Example of $N_\theta$ as a function of $\phi_o$ when $\psi_o = 0$ in Cases 7 & 8
The inertial properties are for the athlete identified in Section 3.10 holding posture T. Posture T transitions from displaying oscillating twist, for high values of $\phi_o$, to displaying continuous twist. Asymptotic behaviour is seen between these situations at $\phi_o \sim 68.4^\circ$ since on the transition $\phi$ and $\psi$ reach 90° at the same time and the twist stops.

Figure 4-16 shows a discontinuity at $\phi_o \sim 68.4^\circ$: this is the transition value where posture T moves from displaying continuous twist to oscillating twist. $N_\theta$ decreases as $\phi_o$ decreases irrespective of whether continuous or oscillating twist is displayed. This would suggest that postures with $I_{xx}$ as the intermediate-valued moment of inertia would be less effective postures for preventing twist since, as the oscillations grow, the decreasing value of $N_\theta$ would mean that the oscillations are more readily observed. However, in the case of posture T vs. posture P the values of the moments of inertia mean that for values of $\phi_o$ close to $\pi/2$ $N_\theta$ is so large that the oscillations in posture T would hardly be seen. For example, if this athlete had $\phi_o \sim 86.5^\circ$ ($\psi_o = 0$) which would produce a full twist in a single somersault when in posture L, then in posture T they would have a maximum twist of $\sim 9^\circ$ and this maximum would be seen after 46 somersaults. As a result in the 0.37 somersaults in which posture P displayed $\sim 1^\circ$ of twist, posture T displays $< 0.1^\circ$ of twist.
The scale on the y-axis of Figure 4-12, indicates that posture T requires a much larger $N_\theta$ when displaying continuous twist, than posture L; this is not just the result of $I_{xx}$ being the intermediate-valued moment of inertia, but also because a much larger value of $I_{zz}$, as compared to $I_{xx}$ and $I_{yy}$, occurs in posture T than when in posture L.

When seeking to perform twisting somersaults, questions of interest related to the quasi-rigid phase are which posture and what initial orientation would minimise the somersault required to achieve a specified number of twists. In Section 3.9, key somersault postures are defined and then in Section 5.3 $N_\theta$ is evaluated under various initial conditions to compare these postures. Figure 4-17 and Figure 4-18 present $N_\theta$ for a range of ratios of the moments of inertia, under two different initial conditions. The domain reflects the ratios of the common twist postures (Figure 5-4): $I_{yy}/I_{zz}$ from 3 to 30 and $I_{xx}/I_{yy}$ from 0.9 to 1.3. The white area is where oscillating twist occurs.

![Figure 4-17: $N_\theta$ when $\phi_o = 80^\circ$ and $\psi_o = 0$.](image)

It is clear that from the colour pattern of Figure 4-17 and Figure 4-18, a higher value of $I_{yy}/I_{zz}$ and $I_{xx}/I_{yy}$ reduces $N_\theta$. A high value of $I_{yy}/I_{zz}$ reduces $N_\theta$ since from Equation (4-2) a small value of $I_{zz}$ will increase the rate of twist within a somersault. A higher value of $I_{xx}/I_{yy}$ will show continuous twist for larger values of $\phi_o$, and then a greater decrease in $\phi$ as it oscillates, thereby increasing the rate of twist in the somersault. Alternatively one may think about the situation as due to the higher $I_{xx}$ value slowing the somersault down, as may be seen by inspecting Equation (4-3), and hence more twist is completed within the somersault. It is important to remember that it is the ratios of the moments of inertia that determine $N_\theta$ and not the actual values of the moments of inertia.
Figure 4-18: $N_\theta$ when $\phi_o = 70^\circ$ and $\psi_o = 0$.

The pattern of the plots in Figure 4-17 and Figure 4-18 changes as $\phi_o$ decreases, with the curves moving towards vertical lines. This indicates that the value of $I_{yy}/I_{zz}$ becomes the critical inertial property parameter as $\phi_o$ decreases. The postures that are anatomically possible and acceptable within a sporting environment restrict the ratios of the moments of inertia that an athlete may obtain. As a result it may not be possible to increase $I_{yy}/I_{zz}$ and $I_{xx}/I_{yy}$ at the same time. Sport specific postures will be compared in Section 5.3.2.

When performing a twisting somersault it is ideal that an athlete start without any twist: that is with $\psi_o = 0$. To determine the value of $\phi_o$ required to achieve any particular skill then the equations for $N_\theta$ should be rearranged to make $\phi_o$ the subject. For Case 10, if the fraction of a somersault available to complete a half-twist is $S_{Aval}$, then the value of $\phi_o$ required would be

$$
S_{Aval} = \frac{I_{zz}}{2(I_o - I_{zz})\cos \phi_o} \cos^{-1}\left(\frac{I_{zz}}{2(I_o - I_{zz})S_{Aval}}\right)
$$

(4-19)

For Cases 3 and 7, the inverse of each complete elliptical integral depends on the elliptical parameters $n$ and $k$, which are both functions of $\phi_o$. Thus, to determine the value of $\phi_o$, for a particular value of $N_\theta$ a numerical approach is required. The roots could be found by first applying the values of $I_{xx}$ and $I_{yy}$ for the posture of interest to Equation (4-19) for Case 10 to give upper and lower values for $\phi_o$ between which the root should lie; which is the upper and which is the lower depends on the case under consideration. Then the root between
4.2.5 Using the assumption that $I_{xx} = I_{yy}$ to simplify descriptions of the twisting somersault

Assuming that $I_{xx} = I_{yy}$ greatly simplifies the equations that describe the twisting somersault, allowing the relatively simple expression, given in Equation (4-18), to describe the twist-to-somersault ratio. This assumption has previously been used (Frohlich, 1979; Yeadon, 1993a), although no investigations have been conducted to determine how similar $I_{xx}$ and $I_{yy}$ need to be for the assumption to be reasonable. The assumption of equality would be reasonable if the errors due to making the assumption are small, in a practical sense. For the twisting somersault the errors of interest are those in the magnitude of oscillation of $\phi$, and the value of $N_\theta$.

Before considering the error associated with assuming $I_{xx} = I_{yy}$, it is essential to ensure that the posture will in fact display continuous twist. Thus, only when either $I_{yy}$ is the intermediate-valued moment of inertia and $\phi_o < \phi_{\text{crit}1}$ or $I_{xx}$ is the intermediate-valued moment of inertia and $\phi_o < \phi_{\text{crit}2}$ would it be reasonable to consider assuming $I_{xx} = I_{yy}$.

Once it is known that continuous twist will occur, there will always be a point when $\psi = 0$, so to specify the relationship between $\psi$ and $\phi$ and so it is sufficient to specify the values of $I_{xx}/I_{yy}$, $I_{yy}/I_{zz}$, and $\phi_o$ for $\psi_o = 0$.

When $I_{xx} = I_{yy}$, the range of $\phi$ is zero. The error in the magnitude of the oscillation of $\phi$ is then quite simply the magnitude of oscillation of $\phi$. From Cases 3 and 7 in Table 4-1 the range of $\phi$ is

$$\left|\cos^{-1}\left(\sqrt{ab}\right) - \cos^{-1}\left(\sqrt{c/f}\right)\right| = \phi - \cos^{-1}\left(1 - \sin^2\phi_o\left[\frac{I_{xx}(1-I_{xx}/I_{yy})}{I_{yy}(I_{xx}/I_{yy}-I_{zz}/I_{yy})}\right]\right)$$

when $\psi_o = 0$.

This range will approach zero when any of the following are true, alone or in combination: $I_{xx}/I_{yy} \to 1$; $I_{yy}/I_{zz} \to \infty$; or $\phi_o \to 0$. Figure 4-19 and Figure 4-20 illustrate the range of $\phi$ when $\phi_o \to \pi/2$ and when $\phi_o = 80^\circ$. 
Mathematical descriptors of the somersault

Figure 4-19: Limit of the magnitude of oscillation of $\phi$ as $\phi_o$ approaches 90°.
Only when $I_{yy}$ is the intermediate-valued moment of inertia will the magnitude of oscillation have a real value. Hence $I_{xx}/I_{yy}$ is only shown from the value of one.

Figure 4-20: Range of $\phi$, for the case when $\phi_o = 80^\circ$.
The coloured region is where continuous twist is observed.

The coloured region is larger in Figure 4-20 since the smaller value of $\phi_o$ means that continuous twist is observed for more situations when $I_{xx}$ is the intermediate-valued moment of inertia. It is clear from both these plots that the range in $\phi$ is less when $I_{xx}/I_{yy}$ is closer to one and also for larger values of $I_{yy}/I_{zz}$. The plot fans out as $I_{yy}/I_{zz}$ increases, showing that if a certain range of $\phi$ is deemed acceptable, for larger values of $I_{yy}/I_{zz}$, an acceptable value of $I_{xx}/I_{yy}$ also may be progressively broader. For example, $I_{xx}/I_{yy} = 1.01$ and $I_{yy}/I_{zz} = 5$; $I_{xx}/I_{yy} = 1.023$ and $I_{yy}/I_{zz} = 10$; and $I_{xx}/I_{yy} = 1.049$ and $I_{yy}/I_{zz} = 20$ all have the same error. When $\phi_o = 80^\circ$ this error is $0.39^\circ$ and when $\phi_o = 90^\circ$ this error is $2.85^\circ$. These differences emphasize the importance of considering what the expected values of $I_{xx}/I_{yy}$, $I_{yy}/I_{zz}$ and $\phi_o$ are, before deciding whether or not the assumption of equality is reasonable.
Also of interest is the error in the value of $N_\theta$; that is $N_\theta(error) = N_\theta(\text{Assumption}) - N_\theta(\text{Actual})$. Figure 4-21 plots $N_\theta(error)$ as a function of $\phi_o$ for the example athlete used thus far, holding the posture L. From Figure 4-21 the error can clearly be seen to decrease as $\phi_o$ decreases. Thus as the number of twists performed increases $N_\theta(error)$ decreases and the assumption that $I_{xx} = I_{yy}$ becomes more reasonable. For a full twist in a somersault, which from Figure 4-14 required $\phi_o \sim 86.5^\circ$ ($\psi_o=0$), then from Figure 4-21, $N_\theta(error) = 0.06$ of a somersault, which is ~21°; this is not a small error. However, for a triple twist when $\phi_o \sim 83.9^\circ$ ($\psi_o=0$) $N_\theta(error) = 0.01$ of a somersault (~5°), which is a small error.

![Figure 4-21: Example $N_\theta(error)$ as a function of $\phi_o$ when $\psi_o = 0$ in Case 3](image)

The inertial properties are for the athlete identified in Section 3.10 holding posture L. $N_\theta(error) = N_\theta(\text{Assumption}) - N_\theta(\text{Actual})$.

When discussing which postures will show a greater or lesser value of $N_\theta(error)$, two possible general scenarios are of interest: the value of $N_\theta(error)$ for set values of $\phi_o$, and the value of $N_\theta(error)$ when the value of $\phi_o$ is set to the value that means $N_\theta(\text{Assumption})$ equals one divided by the number of half twists to be completed.

Figure 4-22 plots $N_\theta(error)$ for $\phi_o = 80^\circ$. Figure 4-23 and Figure 4-24 plot the error in the number of somersaults required for an athlete to complete a 1/2 or 4/1 twists within a somersault; that is when $\phi_o$ is set so that $N_\theta(\text{Assumption})$ equals 1 or 1/8 respectively. As for previous plots, $3 \leq I_{yy}/I_{zz} \leq 30$ and $0.9 \leq I_{xx}/I_{yy} \leq 1.3$. The white areas are where oscillating twist is observed. The colour scale was set so that the pattern of values could be most easily seen and regarding errors greater than 0.3 to be very large errors in practical terms (deductions start at ~0.03 of a somersault); the end of the scale values thus need to be interpreted as the error is greater than or equal to that value.
Figure 4-22: \( N_\theta(\text{error}) \) when \( \phi_o = 80^\circ \), \( \psi_o = 0 \)

The fan pattern in Figure 4-22 confirms that as \( I_{yy}/I_{zz} \) increases, then \( I_{xx}/I_{yy} \) can be further from one and the error still small. When \( I_{xx}/I_{yy} = 1 \), the error is zero, since \( I_{xx} \) and \( I_{yy} \) are in fact equal. When \( I_{xx}/I_{yy} > 1 \) the error is positive while when \( I_{xx}/I_{yy} < 1 \) the error is negative; this means that the assumption of equality has overestimated the number of somersaults required when \( I_{yy} \) is the intermediate-valued moment of inertia, and underestimated the number of somersaults required when \( I_{xx} \) is the intermediate-valued moment of inertia. In practical terms it is better to overestimate the somersault required, since the error would mean the athlete finishes the twist early, while an underestimation would result in the athlete not completing the required twists prior to landing.

As can be seen from Figure 4-23 and Figure 4-24, when \( \phi_o \) is set so that \( N_\theta(\text{Assumption}) = 1 \) and 1/8 respectively, then, when \( I_{xx}/I_{yy} > 1 \) the assumption that \( I_{xx} = I_{yy} \) still overestimates the value of \( N_\theta \) required, and when \( I_{xx}/I_{yy} < 1 \) it still underestimates the value of \( N_\theta \) required. In Figure 4-23 the moderately sized white region is where the oscillating twist occurs; this is because the predicted value of \( \phi_o < \phi_{\text{Crit2}} \).
Mathematical descriptors of the somersault

Figure 4-23: Error in the number of somersaults, when $\phi_o$ is set so that $N_\theta(\text{Assumption}) = 1$.
Error in the number of somersaults required for $\frac{1}{2}$ twist = $N_\theta(\text{error}) = 1 - N_\theta(\text{Actual})$. $I_{xx}$ and $I_{yy}$ were set to the average of the actual values. A value of 0 occurs when $I_{xx}$ and $I_{yy}$ are in fact equal.

Figure 4-24: Error in the number of somersaults, when $\phi_o$ is set such that $N_\theta(\text{Assumption}) = 1/8$.
Error in the number of somersaults required = $8N_\theta(\text{error}) = 1 - 8N_\theta(\text{Actual})$. $I_{xx}$ and $I_{yy}$ were set to the average of the actual values. A value of 0 occurs when $I_{xx}$ and $I_{yy}$ are in fact equal.
Decreasing $\phi_o$ from the value where $N_{\theta(Assumption)} = 1$ to where $N_{\theta(Assumption)} = 1/8$ greatly reduces the error. This can be seen quite clearly by the necessary change in scale from Figure 4-23 to Figure 4-24 so that the variation over the domain is still perceivable.

The pattern of error is quite different in Figure 4-22 compared to Figure 4-24 and Figure 4-23. It is thus clear that the way in which the assumption $I_{xx} = I_{yy}$ is applied is a critical component of the magnitude of the error, and hence whether or not the error would be acceptable. When $\phi_o$ is set to a value predicted under the assumption that $I_{xx} = I_{yy}$, as in Figure 4-24 and Figure 4-23, the magnitude of the error increases as $I_{yy}/I_{zz}$ increases; this is the opposite to when $\phi_o$ was set to a fixed value. The reason for this is that for larger values of $I_{yy}/I_{zz}$ the predicted value of $\phi_o$ is closer to $\pi/2$ and this contributes to the increases in error as seen in the shape of Figure 4-21.

It is thus not possible to give a simple answer to when it is reasonable to assume $I_{xx} = I_{yy}$: the error depends on the ratios $I_{xx}/I_{yy}$ and $I_{yy}/I_{zz}$, the values of $\phi_o$ and $\psi_o$, and whether a prediction of the amount of somersault is required or a prediction of the initial conditions is required. It is also important to know what magnitude of error would be of practical concern, as well as if it is an over- or under-estimation.

In Section 5.3.5 the assumption of equality will be considered with reference to sport-specific postures along with the consequences of the error; as a result any further conclusions, as to when assuming $I_{xx} = I_{yy}$ is reasonable, will be deferred until then. Additional plots of the error when assuming $I_{xx} = I_{yy}$ are given in Appendix F for the interested reader.

### 4.2.6 Altering the value of the moments of inertia and its effect on twist

In different phases of a skill an athlete may elect to hold a different posture. As a result they will rotate differently in each phase. This section (4.2) has focussed on mathematically describing rigid-body motion in terms of $\theta$, $\psi$, and $\phi$. Let us now discuss what effects may be observed due to instantaneous changes in the moments of inertia. The effect of changing posture is considered generally; actual inertial property data set values in specific postures, and hence the expected practical results of posture changes, will be considered in Chapter 5.

#### 4.2.6.1 Altering the speed of the twist

As identified in Section 4.2.2, reducing $I_{zz}$ relative to $I_{yy}$ and $I_{xx}$, for any value of $\phi < \pi/2$, will increase the twist speed within the somersault.
Changing the moments of inertia will also change the way in which $\phi$ varies and so the timing of any postural change is significant. Yeadon (1993a) proposed a method of instantaneously swapping between continuous twist postures with higher and lower amplitudes of their oscillation in $\phi$ every quarter-twist as a means of progressively shifting the value of $\phi$ downwards. Moving between posture LAP and posture L was the specific example given by Yeadon (1993a). The proposal could be extended to any two postures that display continuous twist for the given initial conditions. Figure 4-25, Figure 4-26 and Figure 4-27 illustrate the three possible situations: 1) both postures have $I_{yy}$ as the intermediate-valued moment of inertia; 2) both have $I_{xx}$ as the intermediate-valued moment of inertia; and 3) one posture has $I_{yy}$ as the intermediate-valued moment of inertia and the other has $I_{xx}$ as the intermediate-valued moment of inertia. In all cases the reduction in $\phi$ is the absolute value of the difference in the curves representing the relationship between $\psi$ and $\phi$ at $\pi/2$.

Figure 4-25: Changing between two postures with $I_{yy}$ as the intermediate-valued moment of inertia at the quarter-twist position. The oscillation in $\phi$ of the starting posture is the red trace and oscillation in $\phi$ of the second posture is the blue trace. Both have $I_{yy}$ as the intermediate moment of inertia.

Figure 4-26: Changing between two postures with $I_{xx}$ as the intermediate-valued moment of inertia at the quarter-twist position. The oscillation in $\phi$ of the starting posture is the red trace and oscillation in $\phi$ of the second posture is the blue trace. Both have $I_{xx}$ as the intermediate moment of inertia.
Figure 4-27: Changing, at the quarter-twist position, between two postures both with continuous twist and one with $I_{yy}$ as the intermediate-valued moment of inertia and the other with $I_{xx}$ as the intermediate-valued moment of inertia.

The oscillation in $\phi$ of the starting posture is the red trace and oscillation in $\phi$ of the second posture is the green trace. The first posture has $I_{yy}$ as the intermediate moment of inertia and the second has $I_{xx}$ as the intermediate moment of inertia.

Whether or not this proposal is of practical use will depend on the magnitude of the difference in oscillation of $\phi$ between the two postures, the value of $\phi_o$, and whether or not an athlete could reasonably move between these two postures with sufficient speed, and if the change was acceptable aesthetically. This proposal is assessed for moving between posture L and posture LAP in Section 5.3.5.2 by applying athlete inertial properties from Chapter 4.

4.2.6.2 Moving between continuous and oscillating twist

Whether a posture displays continuous or oscillating twist depends on the moments of inertia and the initial conditions ($\phi_o$ and $\psi_o$). Appropriate selection of postures means that swapping between postures can be used to change the rotation from continuous to oscillating twist or vice versa. For example, swapping between the posture L and posture P at small values of $\psi$, which corresponds to swapping between Cases 3 and 6, would change the twist rotation from continuous to oscillating twist. In a sporting context moving into a posture that displays oscillating rather than continuous twist may be used to cease continuous twist and then complete the somersault rotation. The magnitude of the oscillations in the twist will determine whether the approach would be aesthetically pleasing or not. This strategy will be considered in Section 5.3.3 with the key somersault postures from Section 3.9.
4.2.6.3 Entering a posture which displays oscillating twist about $\psi = 0$, wait until $\phi = \pi/2$ then ‘open’ for entry

A posture that displays oscillating twist about $\psi = 0$ will have $\phi$ oscillate about $\pi/2$. If the athlete “opens out” at $\phi = \pi/2$, Yeadon (1984; 1993b) suggested that the twist will have been removed or be small and so insignificant.

When twist is oscillating about $\psi = 0$ then, when $\phi = \pi/2$, $\psi$ is at its maximum value: it is not zero. Opening out means that for the new posture $\phi_o = \pi/2$ and $\psi_o \neq 0$. As a result twist will occur. When $I_{zz}$ or $I_{xx}$ are the intermediate-valued moments of inertia in the open posture then oscillations in twist about $\psi = 0$ will be observed and will be small; however, when $I_{yy}$ is the intermediate-valued moment of inertia oscillations about $\psi = \pi/2$ will be observed, that is case 4 (Section 4.2.3). This strategy does not remove twist, but by choosing the postures well it will mean that twist is slow, and not of practical concern, when opening out for entry. Whether or not any restart of twist is of practical concern for any of the key somersault postures, as defined in Section 3.9, is considered in Section 5.3.3.1.

4.2.6.4 Entering a posture that oscillates in $\phi$ (and $\psi$) to correct alignment for a vertical entry

When contact twist is used pancaking is observed (see Figure 4-3). Yeadon (1997b) proposed that in diving, where the athlete traditionally enters the water after an odd number of half somersaults, a pike may be used to change from a twisting somersault to one that oscillates in $\phi$ and $\psi$, and the athlete may then wait until the oscillation allows for a vertical entry before opening out and entering the water. Figure 4-28 illustrates the proposal.

![Figure 4-28: Using an oscillating posture to achieve a vertical entry](image)

The dashed outline indicates the position at an odd number of half somersaults, just prior to entering the pike. The solid line shows the athlete orientation when the oscillation reaches a position allowing a vertical entry.
This proposal relies on the amount of oscillation in $\phi$, that is $\pi - 2\phi_o$ (from $\phi_o$ to $\pi - \phi_o$) being greater than the deviation from the vertical entry position, that is $\pi/2 - \phi_o - \beta$.

Oscillations in $\phi$ are accompanied by oscillations in $\psi$; so that the skill would be aesthetically pleasing the oscillation in $\psi$ must not be “excessive” (Yeadon, 1997b).

It is essential that there be enough somersault remaining for sufficient oscillation to occur in the piked position and allow the desired vertical entry. Further, there should not be too much extra somersault remaining, lest when opening into the entry position an observable amount of twist occurs before entry. Fortunately, when the twist restarts it will be slower than the original twist since, unless $\phi$ has reached the other extreme of its oscillation, it will be closer to $\pi/2$. If desiring to investigate the skills for which this strategy would be suitable it is necessary to know values of $\beta$. Unfortunately suggested values of $\beta$ could not be found in the literature, and the equipment required to measure forces and calculate $\beta$ was not available. As a result this strategy will not be further considered in Chapter 5.

### 4.2.7 Relative difficulty of a twisting vs. a pure somersault

Let us define relative difficulty as the ratio of the number of somersaults that may be completed in the same time and with the same angular momentum, in two different situations.

The amount of somersault that may be completed may be determined by integrating Equation (4-3) with respect to time.

\[
\dot{\theta} = H \left( \frac{I_{xx} - (I_{xx} - I_{yy})\sin^2 \psi}{I_{xx} I_{yy}} \right)
\]

\[
[\theta]_{End}^{Start} = \int_{Start}^{End} H \left( \frac{I_{xx} - (I_{xx} - I_{yy})\sin^2 \psi}{I_{xx} I_{yy}} \right) dt
\]

\[
[\theta]_{End}^{Start} = H \left( \frac{I_{xx} \left[ I_{yy}^{End} - (I_{xx} - I_{yy}) \int_{Start}^{End} \sin^2 \psi dt \right]}{I_{xx} I_{yy}} \right)
\]

In twisting somersaults the function for $\psi$ with respect to time will depend on the Case (Section 4.2.3) while a pure somersault will have $\psi = 0$ at all times.

Let us define the relative difficulty of a twisting somersault as the ratio of the number of pure somersaults that may be completed to the number of twisting somersaults that may be completed in the same time with the same angular momentum.
Substituting Equation (4-10) into Equation (4-20) gives

\[
\begin{align*}
\left[\theta_{\text{Pure}}\right]_{\text{End}} &= \left[\theta_{\text{Twisting}}\right]_{\text{End}} - \frac{H}{I_{yy}} \left[\nu\right]_{\text{End}} \\
&= \frac{I_{xx} \left[v\right]_{\text{End}} - (I_{xx} - I_{yy}) \int_{\text{Start}}^{\text{End}} \sin^2 \psi dt}{I_{xx} I_{yy}} \\
&= \frac{I_{xx} \left[v\right]_{\text{End}} - (I_{xx} - I_{yy}) \int_{\text{Start}}^{\text{End}} \frac{g - b \sin^2 \phi}{f - b \sin^2 \phi} dt}{I_{xx} I_{yy}} \\
&= \frac{I_{xx} I_{zz} \left[v\right]_{\text{End}} - \int_{\text{Start}}^{\text{End}} \frac{g - b \sin^2 \phi}{\sin^2 \phi} dt}{I_{xx} I_{yy}} \\
&= \frac{(I_{xx} I_{zz} + b) \left[v\right]_{\text{End}} - \int_{\text{Start}}^{\text{End}} \frac{1}{\sin^2 \phi} dt}{I_{xx} I_{yy}}
\end{align*}
\]

Equation (4-21) describes the situation when \(I_{xx} \neq I_{yy}\). If \(I_{xx} = I_{yy}\) the relative difficulty would simply be one.

To complete the integral it is necessary to know \(\phi\) as a function of time. Appendix D, gives equations for \(\phi\) as a function of time for each of the cases in Section 4.2.3. Case 3 and Case 7 allow twisting somersaults to be performed and so the solution to the integral

\[
\int_{\text{Start}}^{\text{End}} dt / \sin^2 \phi
\]

for these cases from Appendix D will be substituted into Equation (4-21). This gives,
Case 3:

\[
\begin{align*}
\theta_{\text{Pure Start}}^\text{End} & = I_{ax} I_{zz}^\text{End} \\
\theta_{\text{Twisting Start}}^\text{End} & = \frac{I_{ax} I_{zz}}{(I_{ax} I_{zz} + b) I_{zz}} + \frac{Hf \sqrt{bc}}{I_{ax} I_{zz}^\text{End}} \\
& \times \left[ \Pi \left( \frac{af - bc}{bg} \right) \sin^{-1} \left( \frac{sn \left( Dn^{-1} \left( \frac{I_{ax} I_{zz}}{I_{ax} I_{zz}^\text{End}} \right) \right)}{k} \right) \right] \end{align*}
\]

where \( k = \sqrt{1 - \frac{af}{bc}} \)

\( (4-22) \)

Case 7:

\[
\begin{align*}
\theta_{\text{Pure Start}}^\text{End} & = I_{ax} I_{zz}^\text{End} \\
\theta_{\text{Twisting Start}}^\text{End} & = \frac{I_{ax} I_{zz}}{(I_{ax} I_{zz} + b) I_{zz}} + \frac{Hf \sqrt{af}}{I_{ax} I_{zz}^\text{End}} \\
& \times \left[ \Pi \left( \frac{bc - af}{fg} \right) \sin^{-1} \left( \frac{sn \left( Dn^{-1} \left( \frac{I_{ax} I_{zz}}{I_{ax} I_{zz}^\text{End}} \right) \right)}{k} \right) \right] \end{align*}
\]

where \( k = \sqrt{1 - \frac{bc}{af}} \)

\( (4-23) \)

When performing twisting somersault skills multiples of a half twist must be performed; a half twist will be performed in the same time as one period of \( \phi \). Due to the symmetry about the quarter-twist position, the integrals may simply be evaluated over a quarter-twist: the start time will be when \( \phi = \phi_o \) and the end time when \( \phi_o \) is at its maximum or minimum value, which from Table 4-1 would be \( \phi = \cos^{-1} \left( \frac{c}{f} \right) \) for both Cases 3 and 7. For

continuously twisting somersaults \( \psi \) can equal zero and so it is sufficient to use situations when \( \psi_o = 0 \) to evaluate the relative difficulty of a twisting somersault. As a result the relative difficulty for a continuously twisting somersault (full working in Appendix F) will be

Case 3: Where \( \psi_o = 0 \)

\[
\begin{align*}
\frac{I_{ax} I_{zz}}{(I_{ax} I_{zz} + b) F \left( \frac{\pi}{2} \right) - f \left( \frac{1 - \frac{f}{c} \cos^2 \phi_o}{g} \right)} & = f \left( \frac{1 - \frac{f}{c} \cos^2 \phi_o}{g} \right)
\end{align*}
\]

\( (4-24) \)
Case 7 Where $\psi_o = 0$

$$I_{xx} I_{zz} F\left(\frac{\pi}{2}, 1 - \frac{bc}{af}\right)$$

$$\frac{(I_{xx} I_{zz} + b) F\left(\frac{\pi}{2}, 1 - \frac{bc}{af}\right) - b \Pi\left(\frac{f - b}{f}, \frac{\pi}{2}, \sqrt{1 - \frac{bc}{af}}\right)}{(I_{xx} I_{zz} + b) F\left(\frac{\pi}{2}, 1 - \frac{bc}{af}\right) - b \Pi\left(\frac{f - b}{f}, \frac{\pi}{2}, \sqrt{1 - \frac{bc}{af}}\right)}$$

(4-25)

Figure 4-29 and Figure 4-30 plot the relative difficulty as a function of $\phi_o$ when $\psi_o = 0$ for the athlete from Section 3.10 holding the postures L and T respectively for the purposes of illustrating the shape of the functions in Equations (4-24) and (4-25); these curves do not necessarily correspond to an observed skill. For posture L, $I_{xx}/I_{yy} \sim 1.03$; this forms an absolute upper bound to the relative difficulty. As can be seen in Figure 4-29 the relative difficulty never reaches this upper bound, instead it plateaus at approximately 1.015. For this athlete and posture the relative difficulty of a twisting somersault is only slightly greater than one (a pure somersault) and would be of no practical significance, since the slight additional rotation could be easily achieved during the exit phase of the twist and in the preparation for landing. For posture T, $I_{xx}/I_{yy} \sim 0.82$, which means that a twisting somersault is easier than a pure somersault. A relative difficulty of 0.82 is an absolute lower bound. The shape of the curves in Figure 4-29 and Figure 4-30 means that the relative difficulty increases as $\phi_o$ decreases but this does plateau. As a result increasing the rate of twist by decreasing $\phi_o$ would initially require greater angular momentum but further increasing the rate of twist has little effect on the angular momentum required. The plateau shape is to be expected since as $\phi_o$ decreases, the oscillations in $\phi$ decrease, and the speed of the twist becomes more steady (Section 4.2.5): the relative difficulty of a twisting vs. a pure somersault in the same posture will thus approach the average of $I_{xx}/I_{yy}$ and one. The average of $I_{xx}/I_{yy}$ and one, and the smaller of $I_{xx}/I_{yy}$ and one will be the two bounds of relative difficulty, and these may be used to estimate the relative difficulty; alternatively if the lowest value of $\phi_o$ is known, then this can be used instead as an estimate of the maximum relative difficulty (this is what is done in Section 5.3.4).
Figure 4-29: Example of relative difficulty as a function of $\phi_o$ ($\psi_o = 0$) when holding the posture L for the athlete from Section 3.10.

Figure 4-30: Example of relative difficulty as a function of $\phi_o$ ($\psi_o = 0$) when continuously twisting and holding the posture T using the athlete from Section 3.10. The curve stops at 68.4° where posture T transitions from displaying continuous twist to displaying oscillating twist. The equation for the relative difficulty of a continuous twist is no longer valid once $\phi_o > 68.4°$.

When performing twisting somersaults it is the number of twists within the somersault that determines the skill that may be achieved; this is the result of both the posture and the angle $\phi$. It would be desirable to choose a posture which allows the required number of twists to be completed and minimises the relative difficulty. Twist initiation techniques
may restrict the value of $\phi$ and so in order to still achieve the desired number of twists, it is perhaps necessary to use a posture with greater relative difficulty. As a result there are multiple factors to consider when suggesting the most suitable postures to use when twisting. Relative difficulty will be discussed in more detail in Section 5.3.4, for the key somersault postures defined in Section 3.9 and the inertial property data sets collated in Chapter 3.

As was suggested in Section 4.2.6.2, changing posture may be used to cease continuous twist. If this strategy is used there will be oscillations in twist. The oscillations in twist will alter the relative difficulty and thereby affect the completion of the skill. However, in contrast to continuous twist where it is known that the athlete will be performing multiples of a half twist, it is not known how many oscillations the athlete will perform. The number will depend on the remaining somersault, which depends on the portion of the somersault already used when they were continuously twisting. Further, when using such a strategy the posture of choice will be one that minimises oscillation in twist, thereby meaning any changes in relative difficulty between a pure somersault and one displaying oscillating twist will be small. As a result the relative difficulty of Cases 6 and 8 will not be explored.

**4.3 Re-orientation with posture change**

After take-off an athlete must move into the desired posture for the quasi-rigid phase. This includes performing any aerial twist initiation techniques. Since the human body is not a rigid body, it is possible to re-orientate the body through postural changes.

The following derivations follow a similar approach to Dullin (2011) and Dullin & Tong (2015) using the conservation of angular momentum. Previous authors have used the conservation of angular momentum to determine or describe reorientation of a body following postural change; however, they do so for a specific number of segments only (Miller, 1970; Frohlich, 1979; Pike, 1980; Van Gheluwe, 1981), when the centre of gravity of each segment lies on the longitudinal axis (Smith & Kane, 1967), with different final frames of reference and different definitions of the orientation angles\(^{10}\) (Kwon, 1993), or they neglect some terms during the derivation\(^{11}\) (Yeadon, 1984). Dullin (2011) also identified the time independence of the resulting equations for planar situations.

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\(^{10}\) Inspecting Figure C.2. and the equations leading to Equation C.20b given by Kwon (1993) it appears that in the figure the orientation angles are Euler angles, while the equations are written for rotations about the fixed frame of reference.

\(^{11}\) See Appendix G.3 for a comparison between Yeadon’s equations and those derived in this thesis.
4.3.1 Equations for angular displacement

The angular momentum of a group of rigid body segments is

\[ H = \sum m_i \left( r_{Gi} \times \dot{r}_{Gi} \right) + I_i \omega_i \]

where \( I_i \) is the moment of inertia about segment \( i \)’s centre of gravity, \( r_{Gi} \) is the displacement vector from the origin of the frame in which the angular momentum is being determined and the centre of gravity of segment \( i \), and \( m_i \) is the mass of segment \( i \).

Frame Ref will be used to determine the re-orientation due to postural change. Frame Glo is the inertial frame in which the conservation of angular momentum holds. Let \( \dot{\gamma} \) be the angular velocity of frame Ref with respect to frame Glo. Let \( \dot{\alpha} \) be the angular velocity of each segment with respect to frame Ref, and \( \dot{D}_i \) be the vector between the origin of frame Ref and the centre of gravity of segment \( i \) with respect to frame Ref. Using the frames Glo and Ref, and the local frames then the variables in Equation (4-26) are,

\[
\begin{align*}
\omega_i & = \dot{\gamma} + \text{Glo} R_{\text{Ref}} \dot{\alpha}_i \\
\dot{r}_{Gi} & = \text{Glo} R_{\text{Ref}} D_i \\
\dot{r}_{Gi} & = \text{Glo} \dot{R}_{\text{Ref}} D_i + \text{Glo} R_{\text{Ref}} \dot{D}_i \\
\text{Glo} I_i & = \text{Glo} R_{\text{Ref}} R_{\text{Local},i} I_i \left[ R_{\text{Local},i} \right]^{-1} \left[ \text{Glo} R_{\text{Ref}} \right]^{-1}
\end{align*}
\]

Thus, for the 17-segment model Equation (4-26), written with respect to Frame Glo, will be

\[
\text{Glo} H = \sum_{i=1}^{17} m_i \left( \text{Glo} R_{\text{Ref}} D_i \times \left( \text{Glo} \dot{R}_{\text{Ref}} D_i + \text{Glo} R_{\text{Ref}} \dot{D}_i \right) \right) + \text{Glo} R_{\text{Ref}} I_{\text{Overall}} \left[ \text{Glo} R_{\text{Ref}} \right]^{-1} \dot{\gamma} + \text{Glo} R_{\text{Ref}} \sum_{i=1}^{17} \left( m_i D_i \times \dot{D}_i + \text{Ref} I_i \dot{\alpha}_i \right)
\]

This equation may be simplified (full working in Appendix H) by using the nature of the derivative of a rotation matrix (Appendix B.1.2), the parallel axis theorem (Appendix B.2.3), and letting \( I_{\text{Overall}} \) be the moment of inertia for the body as a whole in each posture used. Thus, the simplified form is

\[
\text{Glo} H = \text{Glo} R_{\text{Ref}} I_{\text{Overall}} \left[ \text{Glo} R_{\text{Ref}} \right]^{-1} \dot{\gamma} + \text{Glo} R_{\text{Ref}} \sum_{i=1}^{17} \left( m_i D_i \times \dot{D}_i + \text{Ref} I_i \dot{\alpha}_i \right)
\]

The angular momentum is the sum of two parts. The first is due to the posture held and the orientation with respect to the angular momentum vector; when there is no posture change, as in the quasi-rigid phase, this will be the only part. The second part is due to postural change, having a non-zero value only when there are segments moving relative to other
segments, and hence $\mathbf{D}_i$ and $\mathbf{a}_i$ are non-zero. The way in which the body changes shape may be thought of as determining which segments possess a greater or lesser share of the overall angular momentum at any one time. The two parts are not independent, since posture change affects both parts and each part will contribute to changing $^{\text{Glo}}R_{\text{Ref}}$, which in turn alters how each part contributes to the total angular momentum.

If the motion is in two dimensions the frames Glo and Ref, and each local frame will have one axis aligned. Then $^{\text{Glo}}\mathbf{H}$, $\mathbf{a}_i$ and $\dot{\gamma}$ will be about the same axis, and so may be considered as scalars. The tensor of inertia will reduce to a single moment of inertia about the common axis. Equation (4-27) then reduces to

$$^{\text{Glo}}\mathbf{H} = I_{\text{Overall}} \dot{\gamma} + \sum_{i=1}^{17} \left( m_i \mathbf{D}_i \times \dot{\mathbf{D}}_i + I_i \dot{a}_i \right)$$

(4-28)

The first part of Equation (4-28) is the familiar expression for the angular momentum of a rigid body in two dimensions, and the second part may be considered as due to postural change.

In two dimensions $\dot{\gamma}$ is the angular velocity about an axis perpendicular to the plane in which the motion occurs. The value $\gamma$ then also has a clear physical meaning as a rotation about the axis perpendicular to the plane in which the motion occurs. It is thus possible to determine $\gamma$ from Equation (4-28), by rearranging to make $\dot{\gamma}$ the subject and then integrating. This gives

$$\gamma = ^{\text{Glo}}\mathbf{H} \int_{\text{Start}}^{\text{End}} \frac{1}{I_{\text{Overall}}} dt - \int_{\text{Start}}^{\text{End}} \sum_{i=1}^{17} \left( m_i \mathbf{D}_i \times \dot{\mathbf{D}}_i + I_i \dot{a}_i \right) dt$$

(4-29)

The two integrals in Equation (4-29) may be evaluated independently, if the posture at all points in time is known. Further, the second integral is independent of time, since time may be eliminated and the integral evaluated as a function of the $a_i$ values; this means that when $\mathbf{H} = [0, 0, 0]^T$ then the reorientation depends only on the postural change, not the speed of that change, and the integral may be evaluated as a work integral.

In three dimensions $\dot{\gamma}$ is a vector which varies in magnitude and direction; it cannot be integrated in the same way as the two dimensional situation. To determine the angular displacement of frame Ref with respect to frame Glo from Equation (4-27) it is necessary to incrementally add the effect of $\dot{\gamma}$ to determine the change in $^{\text{Glo}}R_{\text{Ref}}$. The change in $^{\text{Glo}}R_{\text{Ref}}$ for an increment of time is
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\[
G_{\text{Re}} f_{\text{next}}^{\text{Glo}} = R(\gamma \Delta t)^{\text{Glo}} G_{\text{Re}} f_{\text{previous}}^{\text{Glo}}
\]

(4-30)

Where,

\[
\dot{\gamma} = \left[ G_{\text{Ref}}^{\text{Glo}} I_{\text{Overall}}^{\text{Ref}} G_{\text{Ref}}^{\text{Glo}} \right]^{-1} \left[ G_{\text{Ref}}^{\text{Glo}} H \right]
\]

\[
- \left[ G_{\text{Ref}}^{\text{Glo}} I_{\text{Overall}}^{\text{Ref}} G_{\text{Ref}}^{\text{Glo}} \right]^{-1} G_{\text{Ref}}^{\text{Glo}} \sum_{i=1}^{17} \left( m_i \bar{D}_i \times \bar{D}_i^{\text{Ref}} + I_{\text{Ref}}^{\text{Ref}} \bar{q}_i \right)
\]

and \( R(\gamma \Delta t) \) stands for the rotation matrix constructed using the equivalent angle-axis method (Appendix B.1.1).

From \( G_{\text{Ref}}^{\text{Glo}} \), the angles \( \theta, \phi, \) and \( \psi \), as defined in Section 4.1.5, may be extracted at each time increment. These angles may be extracted from \( G_{\text{Ref}}^{\text{Glo}} \) by equating it to \( G_{\text{Body}, \text{Req}}^{\text{Glo}} \) from Section 4.1.5, and assuming that, during the postural change, \( \theta \) and \( \psi \) change by less than \( \pi/2 \), and \( \phi \) is in the range \( 0 \) to \( \pi \). Thus,

- \( \phi = \cos^{-1} (G_{\text{Ref}}^{\text{Glo}}(2,3)) \)
  - Tilt = \( (\pi-\phi) \) when \( H = 0 \) so the re-orientation is only due to postural change, this is helpful when visualising the response to an action.

- Somersault
  - \( \theta = \sin^{-1} (G_{\text{Ref}}^{\text{Glo}}(1,3) / \sin \phi) \)
  - Check if \( \theta \) is in the first or fourth quadrant by checking that \( G_{\text{Ref}}^{\text{Glo}}(3,3) > 0 \). Otherwise the assumption that the reorientation due to the postural change was small is false.

- Twist
  - \( \psi = \sin^{-1} (G_{\text{Ref}}^{\text{Glo}}(2,1) / \sin \phi) \)
  - Check if \( \psi \) is the first or fourth quadrant by checking that \( G_{\text{Ref}}^{\text{Glo}}(2,2) > 0 \). Otherwise the assumption that the reorientation due to the postural change was small is false.

Equations (4-30) and (4-29) will be evaluated numerically when \( H \neq 0 \), 3-D actions are performed, or the number of segments moving is greater than three. A Matlab programme, Angle_disp_17.m was written for this purpose. See Appendix A.6.

As with the planar situation when \( H = [0, 0, 0]^T \) the angular displacement that occurs depends on the postural change and not the speed of the change. This occurs because when \( H = [0, 0, 0]^T \) the only component of Equation (4-30) that remains is the component that depends on posture change; \( I_{\text{Overall}}^{\text{Ref}} \) and \( D_i \) are all defined by the posture and so are functions of \( \alpha_i \), and \( D_i \) is a function of \( \dot{\alpha}_i \). Thus, under these circumstances time can be eliminated from the equation of angular displacement for each increment in \( \gamma \).
### 4.3.2 Planar 2 or 3 segment models when $H = [0, 0, 0]^T$

When segments of the seventeen segment model of the body, presented in Section 3.1, move as one unit enabling the body to be reduced to two or three grouped segments, the angular displacement will be a function of one or two variables respectively. Thus, the angular displacement may be easily graphed as a line or a surface, and analytic methods used to explore the effect of different actions on the angular displacement.

The angular displacement achieved using a three segment planar model and zero angular momentum from Equation (4-29) will be

$$\gamma_{H=0} = - \int_{\text{Start}}^{\text{End}} \sum_{i=1}^{3} \left( m_i \mathbf{D}_i \times \mathbf{D}_i + I_{\text{Overall}} \alpha_i \right) dt$$

(4-31)

The segments will simply be labelled 1, 2, and 3, and will be taken to be connected in a chain in that order. It is important not to confuse these numbers with the segment numbers for the seventeen segments used to model the body in Section 3.1.

To maintain the generality of the equations derived below, so that they may be applied to any set of two or three grouped segments where any of the three segments may be the reference segment, let us place each grouped segment’s local frame at its centre of gravity and use vectors ($\mathbf{V}_1$, $\mathbf{V}_{21}$, $\mathbf{V}_{23}$, $\mathbf{V}_3$) from the centre of gravity of each grouped segment to the applicable joints. These vectors are easily determined for a group of segments using ICG17 for grouped segments (Section 3.8). Figure 4-31 illustrates the vectors $\mathbf{V}_1$, $\mathbf{V}_{21}$, $\mathbf{V}_{23}$, and $\mathbf{V}_3$.

---

**Figure 4-31: Vectors $\mathbf{V}_1$, $\mathbf{V}_{21}$, $\mathbf{V}_{23}$, $\mathbf{V}_3$**

$\mathbf{V}_1$ is the vector from the 1st segment’s Centre of Gravity to the joint between segment 1 and 2, in terms of the 1st segment’s local frame. $\mathbf{V}_{21}$ is the vector from segments 2’s Centre of Gravity to the joint between segment 1 and 2 in terms of segment 2’s local frame. $\mathbf{V}_{23}$ is the vector from segment 2’s Centre of Gravity to the joint between segments 2 and 3, in terms of the 2nd segment’s local frame. $\mathbf{V}_3$ is the vector from the 3rd segment’s Centre of Gravity to the joint between segment 2 and 3, in terms of the 3rd segment’s local frame.
In Equation (4-31) $m_i$ and $I_i$ are the mass and moment of inertia of each grouped segment, and $\alpha_i$ is the angle of rotation of the grouped segment’s local frame from the local frame of the reference segment. If the middle segment is the reference segment then $\alpha_i$ is directly related to the joint angle between the two segments. In this case $\alpha_1$ and $\alpha_2$ are reasonable to use as the input variables for determining angular displacement.

The value of each $D_i$ may be written as a function of all the $\alpha_i$’s using the definition of the overall centre of gravity, and the values of $V_1$, $V_{21}$, $V_{23}$, $V_3$, $m_1$, $m_2$, and $m_3$. $I_{\text{overall}}$ may also be written as a function of the $\alpha_i$’s using the definition of the overall moment of inertia and the values of each $D_i$ and each $I_i$. When both $D_i$ and $I_{\text{overall}}$ are substituted into Equation (4-31) and segment 2 is the reference segment (Appendix H for full working) then,

$$y_{i_{\text{ref}}=2} = \alpha_{i_{\text{ref}}} \left( I_1 + A + C \sin \alpha_i - D \cos \alpha_i + G \sin(\alpha_i - \alpha_i) + L \cos(\alpha_i - \alpha_i) \right) d\alpha_i$$

$$- \alpha_{i_{\text{ref}}} \left( I_3 + B + E \sin \alpha_i - F \cos \alpha_i + G \sin(\alpha_i - \alpha_i) + L \cos(\alpha_i - \alpha_i) \right) d\alpha_i$$

where,

$$A = \frac{m_1(m_2 + m_3)}{(m_1 + m_2 + m_3)} \left( V_{x_2}^2 + V_{y_2}^2 \right)$$

$$B = \frac{m_2(m_1 + m_3)}{(m_1 + m_2 + m_3)} \left( V_{x_2}^2 + V_{y_2}^2 \right)$$

$$C = \frac{m_1(m_2 + m_3)}{(m_1 + m_2 + m_3)} \left( V_{x_2} V_{y_2} - V_{x_2} V_{x_1} \right) + \frac{-m_1 m_3}{(m_1 + m_2 + m_3)} \left( V_{y_2} V_{x_2} - V_{x_1} V_{y_1} \right)$$

$$D = \frac{m_2(m_1 + m_3)}{(m_1 + m_2 + m_3)} \left( V_{x_2} V_{x_2} - V_{x_2} V_{y_2} \right) + \frac{-m_1 m_3}{(m_1 + m_2 + m_3)} \left( V_{y_2} V_{x_2} + V_{y_2} V_{y_2} \right)$$

$$E = \frac{m_1(m_2 + m_3)}{(m_1 + m_2 + m_3)} \left( V_{x_2} V_{y_2} - V_{x_2} V_{x_1} \right) + \frac{-m_1 m_3}{(m_1 + m_2 + m_3)} \left( V_{y_2} V_{x_2} - V_{x_2} V_{y_2} \right)$$

$$F = \frac{m_2(m_1 + m_3)}{(m_1 + m_2 + m_3)} \left( V_{x_2} V_{x_2} + V_{x_2} V_{y_2} \right) + \frac{-m_1 m_3}{(m_1 + m_2 + m_3)} \left( V_{y_2} V_{x_2} + V_{y_2} V_{y_2} \right)$$

$$G = \frac{-m_1 m_3}{(m_1 + m_2 + m_3)} \left( V_{x_1} V_{x_2} - V_{x_1} V_{y_2} \right)$$

$$L = \frac{-m_1 m_3}{(m_1 + m_2 + m_3)} \left( V_{x_1} V_{y_2} + V_{y_2} V_{y_2} \right)$$
\[ N = I_1 + I_2 + I_3 + \left( \frac{m_2 + m_3}{m_2 + m_3 + m_4} \right) V_{2x}^2 + V_{2y}^2 + V_{3y}^2 + V_{0y}^2 + \left( \frac{m_2 + m_3}{m_2 + m_3 + m_4} \right) V_{2x} V_{3x} + V_{2y} V_{3y} \]

(4-32)

Once \( \gamma_H=0, \text{ref}=2 \) is known it is easy to determine \( \gamma_H=0, \text{ref}=1 \), by adding the change in the angle of rotation between the first segment and the middle segment, \( [\alpha_1 - \alpha_2] = \alpha_1 \) since the second segment is the reference segment. It is desirable to use joint angles rather than angles of rotation with respect to the reference segment, that is use \( \alpha_{2/1} \) and \( \alpha_{3/2} \).

The joint angle between the 1\textsuperscript{st} and 2\textsuperscript{nd} segment, \( \alpha_{2/1} \), is \((-\alpha_1)\) in the equation for \( \gamma_H=0, \text{ref}=2 \) and the joint angle between the 2\textsuperscript{nd} and 3\textsuperscript{rd} segment, \( \alpha_{3/2} \), is \((-\alpha_3)\) in the equation for \( \gamma_H=0, \text{ref}=2 \).

The angular displacement when the first segment is the reference segment is thus,

\[
\gamma_H=0, \text{ref}=1 = \gamma_H=0, \text{ref}=2 + \left[ \alpha_{1/2} \right]_{\text{Start}}^{\text{End}}
\]

\[
= \int_{\alpha_{1/2}^{\text{Start}}}^{\alpha_{1/2}^{\text{End}}} \left( \frac{I_1 + A - C \sin \alpha_{2/1} - D \cos \alpha_{2/1} - G \sin(\alpha_{2/1} + \alpha_{3/2}) + L \cos(\alpha_{2/1} + \alpha_{3/2})}{N - 2G \sin(\alpha_{2/1} + \alpha_{3/2}) + 2L \cos(\alpha_{2/1} + \alpha_{3/2})} \right) d\alpha_{2/1}
\]

\[
- \int_{\alpha_{1/2}^{\text{Start}}}^{\alpha_{1/2}^{\text{End}}} \left( \frac{I_3 + B \sin \alpha_{3/2} - F \cos \alpha_{3/2} - G \sin(\alpha_{3/1} + \alpha_{3/2}) + L \cos(\alpha_{3/1} + \alpha_{3/2})}{N - 2G \sin(\alpha_{3/1} + \alpha_{3/2}) + 2L \cos(\alpha_{3/1} + \alpha_{3/2})} \right) d\alpha_{3/2}
\]

\[-[\alpha_{2/1}]_{\text{Start}}^{\text{End}} \]

(4-33)

In the situation where there are only two grouped segments rather than three \( \alpha_3 \) is zero.

Applying this to Equation (4-32) gives

\[
\gamma_H=0, \text{ref}=2 = - \int_{\alpha_{1/2}^{\text{Start}}}^{\alpha_{1/2}^{\text{End}}} \left( \frac{I_1 + A + C \sin \alpha_{1} - D \cos \alpha_{1} + G \sin \alpha_{1} + L \cos \alpha_{1}}{N + 2G \sin \alpha_{1} + 2L \cos \alpha_{1} + 2C \sin \alpha_{1} - 2D \cos \alpha_{1}} \right) d\alpha_{1}
\]

\[- \int_{\alpha_{1/2}^{\text{Start}}}^{\alpha_{1/2}^{\text{End}}} \left( \frac{I_1 + A}{N + 2G \sin \alpha_{1} + 2L \cos \alpha_{1} + 2C \sin \alpha_{1} - 2D \cos \alpha_{1}} \right) d\alpha_{1}
\]

Further since segment 3 no longer exists all its inertial properties are zero and hence the constants B, E, F, G and L are all zero and so

\[
\gamma_H=0, \text{ref}=2 = - \int_{\alpha_{1/2}^{\text{Start}}}^{\alpha_{1/2}^{\text{End}}} \left( \frac{I_1 + A + C \sin \alpha_{1} - D \cos \alpha_{1}}{N + 2C \sin \alpha_{1} - 2D \cos \alpha_{1}} \right) d\alpha_{1}
\]

(4-34)
When the centre of gravity is on the longitudinal axis of each segment $C = 0$, and Equation (4-34) is then equivalent to the equation 3.2.8 presented by Smith and Kane (1967) for their two body planar model. See Appendix G.1 for the solution to the integral, matched to the solution presented by Smith and Kane (1967).

Using the auxiliary angle method, the integrand, in Equation (4-34) may be rewritten as

$$\frac{(I_1 + A + C \sin \alpha_i - D \cos \alpha_i)}{(N + 2C \sin \alpha_i - 2D \cos \alpha_i)} = \left( \frac{I_1 + A - \sqrt{C^2 + D^2} \sin(\alpha_i + \mu)}{N - 2\sqrt{C^2 + D^2} \sin(\alpha_i + \mu)} \right)$$

$$= \frac{1}{2} + \frac{I_1 + A - N}{2(N - 2\sqrt{C^2 + D^2} \sin(\alpha_i + \mu))}$$

where $\sin(\mu) = \frac{-D}{\sqrt{C^2 + D^2}}$ and $\cos(\mu) = \frac{C}{\sqrt{C^2 + D^2}}$

A sketch of the general shape of this function is given in Figure 4-32. The area under the curve gives the magnitude of $\gamma$ and the direction of $\gamma$ is opposite to the direction of $\alpha_1$. The $\alpha_1$-axis has not been shown on the sketch since it is not known if the curve will or will not cross this axis, until the inertial constants are given specific values.

If the curve crosses the $\alpha_1$-axis then the angular displacement will be in the same direction as the rotation of segment 1. Smith and Kane (1967) did identify this possibility. When exploring the motion of two rods they found it was possible for these two rods joined by a hinge to rotate in the same direction although the system possessed zero angular momentum (Smith & Kane, 1967).
The oscillating shape of the graph means that the angular displacement achieved for a specific change in $\alpha_1$ is not constant. The angular displacement depends on both the magnitude of the change in $\alpha_1$ and the value of $\alpha_1$. This is an important consideration when the change in $\alpha_1$ is limited in some way. If the magnitude of the change in $\alpha_1$ is limited then for any given magnitude of change in $\alpha_1$ the maximum angular displacement will be achieved if the change in $\alpha_1$ is centred around $(\pi/2-\mu)$ if the curve is mostly positive, or $(3\pi/2-\mu)$, if the curve is mostly negative. The magnitude and phase of the oscillation depend on the inertial properties of the segments and how different the two segments are. If the two segments are equal $C = 0$, $A = D$, and $N = 2(I_1+A)$, there will be no oscillation in the graph but rather a constant value of $1/2$, giving the expected result that each segment moves half the distance of the joint angle change.

When there are three segments the plot will be a surface. Slices parallel to either axis will have the same general shape as the two segment situation. The actual inertial properties will determine how the relative maxima and minima of the slices combine to produce maximum(s), minimum(s), or saddle(s) of the surface. Equations (4-33) and (4-34) are in the form of two-dimensional work integrals i.e. $\int_c F \cdot dr = \int_c F_1 dx + F_2 dy$ (Appendix B.3.5).

These integrals may thus be solved via parameterisation, or using Green’s theorem.

Parameterisation requires that the change in the joint angles ($\alpha_1$ or $\alpha_2/1$ and $\alpha_3$) is written with respect to a single parameter. The integrand for $\gamma$ is then integrated with respect to this parameter.

Green’s theorem may be used to convert a cyclic integral into a double integral. Green’s theorem (Appendix B.3.6), written using $\alpha_1$ and $\alpha_3$ rather than $x$ and $y$ states

$$\int_c F_1(\alpha_1, \alpha_3) d\alpha_1 + F_2(\alpha_1, \alpha_3) d\alpha_3 = \int_K \left( \frac{\partial F_2(\alpha_1, \alpha_3)}{\partial \alpha_1} - \frac{\partial F_1(\alpha_1, \alpha_3)}{\partial \alpha_3} \right) d\alpha_1 d\alpha_3$$

For convenience $\frac{\partial F_2(\alpha_1, \alpha_3)}{\partial \alpha_1} - \frac{\partial F_1(\alpha_1, \alpha_3)}{\partial \alpha_3}$ will be referred to as Green’s function.

Green’s theorem applied to Equation (4-32) gives

$$\gamma_{\beta=\alpha_1, \alpha_2} = \int_K \frac{\cos(\alpha_1 - \alpha_3)[2G(N-I_1-B-I_4-A)+2CF-2ED]}{+\sin(\alpha_1 - \alpha_3)[2L(I_1+B-I_4+A-N)+2DF+2EC]} - \frac{\cos \alpha_3[-2C(I_1+B)-2FG+2EL]}{+\sin \alpha_3[-2D(I_1+B)+2GE+2FL]} + \frac{\cos \alpha_3[2E(I_1+A)-2GD+CL]}{+\sin \alpha_3[2F(I_1+A)-2LD-2CG]} d\alpha_1 d\alpha_3$$

(4-35)
and Green’s theorem applied to Equation (4-33) gives

\[
\int \left( \left( \cos(\alpha_{\text{z}_1} + \alpha_{\text{z}_2})\right) - 2G(I_1 + B + I_i + A - N) + 2FC - 2ED \right) \\
+ \sin(\alpha_{\text{z}_1} + \alpha_{\text{z}_2})\left( -2L(I_1 + B + I_i + A - N) - 2CE - 2FD \right) \\
+ \left[-2DL + 2GC + 2F(I_i + A)\right] \sin(\alpha_{\text{z}_2}) \\
+ \left[-2GD - 2CL + 2E(I_i + A)\right] \cos(\alpha_{\text{z}_2}) \\
+ \left[-2FL - 2GE + 2D(I_i + B)\right] \sin(\alpha_{\text{z}_1}) \\
+ \left[-2GF + 2EL - 2C(I_i + B)\right] \cos(\alpha_{\text{z}_1}) \right) \, \, d\alpha_{\text{z}_2} d^2\alpha_i
\]

(4-36)

The cyclic integral in Green’s theorem may be split into two work integrals starting and ending at the same point. The area enclosed by these two paths may then be used to compare the two integrals. This is how Green’s theorem will be used in Section 6.2 to explore adjustments to the idealized asymmetrical arm actions in order to increase the angular displacement produced by these actions.

If the direct route between a start and end posture is the basic action under consideration, then actions that, along with the direct route, enclose a net positive area on the left (or net negative area on the right) will result in a greater angular displacement than the direct route. The maximum angular displacement that can be achieved between a fixed start and end posture will be produced when the action follows the zero contour of Green’s function. Figure 4-33 illustrates this concept; it is a similar philosophy to the Green-Miele method (Mastroeni, 2001) which sought the maximum of a cyclic work integral.

The zero-contour may be found by setting the numerator of Equation (4-35) or Equation (4-36) to zero and solving. Alternatively, when seeking to generally describe or find a
straight-line path close to the zero-contour, such as in Section 6.2, a plot of Green’s function will be sufficient.

If the start and end postures are the same posture in an anatomical sense—the joints angles are $2\pi$—then the start and end could be exchanged and the actions performed in the reverse direction. In this case it is necessary to compare the maximums when travelling in either direction to determine the absolute maximum between those two points.

### 4.3.3 Adjusting planar actions for when $H \neq 0$

When actions in the frontal plane are performed with the intention of producing tilt, which reduces $\phi$, and hence initiates twist; unless the actions can be performed instantaneously twist will start before the action is completed. As the athlete starts to twist the angular momentum vector will move relative to the athlete; it will appear to trace a cone with a semi-vertex angle of $\phi$. Figure 4-34 illustrates.

![Figure 4-34: Perceptions of the angular momentum vector](image)

If the actions are restricted to the frontal plane then, as the action continues in the frontal plane, the tilt it produces will not only change $\phi$, but will also change $\theta$. At zero twist, the action will alter $\phi$; as the twist increases the angular displacement produced by the action will contribute to changing $\phi$ and $\theta$; at a quarter-twist the angular displacement will now only alter $\theta$; as the twist continues the angular displacement now alters $\phi$ and $\theta$ in the opposite directions. Pike (1980) identifies this effect and presents tables describing how $\phi$ (she calls this “rolling”) and somersault rotation are affected in each twist quadrant in two situations: the wrapping and unwrapping phases of a twisting somersault, when the arms only move in the frontal plane. How the angular displacement due to actions in the frontal plane translates into increasing or decreasing $\phi$ and/or $\theta$ is shown in Figure 4-35 according to the quadrant that the angular momentum vector head is in with respect to the $\text{Ref}^x$-axis.
Mathematical descriptors of the somersault

Figure 4-35: Describing angular displacement about $Ref_x$-axis in terms of $\theta$ and $\phi$ as $\psi$ increases

This is for a backward somersault. For a forward somersault the same logic applies but the planar action must cause an angular displacement about the negative $Ref_x$-axis.

To a close approximation, the maximum change in $\phi$ achievable by an action restricted to the frontal plane will be when the action is maximised in the plane and the action is performed instantaneously. Since an athlete cannot move instantaneously they should move as fast as possible and use as slow a somersault as possible: A slow somersault, achieved by increasing flight time, means that the twist rate produced will be slower and hence the action will progress further for any amount of twist completed. The end result is a greater change in $\phi$, producing a greater twist-to-somersault ratio, even though both twist and somersault are slower with respect to time.

If an athlete is able to adjust their planar action they should do so in order to continue in the plane defined by the $Pri_z$-axis and the angular momentum vector which is parallel to the $Glo_y$-axis. The change in $\phi$ will then be closer to the instantaneous situation. The plane defined by the $Pri_z$-axis of the body frame and $Glo_y$-axis, is also the y-z plane of a frame, frame St (Section 4.1.5), which is a rotation by $\theta$ and $\phi$ from frame Glo. To decrease $\phi$ the angular displacement should be about the negative x-axis of frame St. As the athlete twists, the $St_x$-axis moves in a circle, with respect to the athlete, at the same rate as the twist, but in the opposite direction. Thus the adjusted actions, may be described in terms of the body frame by

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12 As the body starts twisting as identified in Section 4.2.3 there will be an oscillation in $\phi$. As the action continues the overall inertial properties of the body mean that the amplitude of this oscillation changes and can have a net effect on the effective value of $\phi$, as described in Section 4.2.6.1. However, if the action is not performed instantaneously the fact that the tilt produced no longer wholly contributes to a change in $\phi$ will dominate over this effect.
In practice, rather than seeking to describe the adjusted action in the body frame it may be more helpful for the athlete to combine postural descriptions with external references. For example, rather than describing how to adjust the motion of the arm in the body frame as it is dropped, an instruction could be: start dropping one arm laterally, and continue dropping it so the fingers continue to point at the same wall while your body twists into the arm.

Adjusting the action means that the arm is now moving both up/down with respect to the body, and also left/right. The up/down component will alter $\phi$, which was the reason for performing an adjustment. The left/right will add some twist, since the arm is now moving in the opposite direction to the twist relative to the body. The additional twist will be in the desired direction of twist but will be quite small.

If the athlete’s reference segment has its medial and transverse moments of inertia equal and the joints of moving segments are on the longitudinal axis then, at any instant, the adjusted action may be treated as a planar action for the purposes of determining the angular displacement. If the reference segment does not have the medial and transverse movements of inertia equal, then as the athlete twists, the angular displacement produced will differ since the moments of inertia in the plane of action are changing. Further the products of inertia will no longer be zero so a planar reduction is no longer suitable. As a result the adjusted action may produce more or less change in $\phi$ depending on the actual inertial properties. Nevertheless if an athlete can only move at a set speed then adjusting the action will produce a greater change than not making the adjustment.

In some cases it may be that, as the twist occurs, the change of axis about which the body is required to rotate in order to alter $\phi$, may mean that some actions are no longer anatomically possible, while others may become anatomically possible. For example moving an arm lateral to the body in the frontal plane may be initially performed, but as the athlete twists, the adjusted action may require the arm to be moved in front of the body, and then across the body. The arm will not remain in the frontal plane when across the body, thus this adjustment is limited. However, at the same time the other arm will become available to be moved instead.

Just as the planar actions used to alter $\phi$ need to be adjusted to account for twist, actions which intend to increase $\phi$ back to $\pi/2$, need also to be adjusted to account for twist. Actions that caused a rotation about the negative $St_x$-axis decreased $\phi$, and so actions that cause a rotation about the positive $St_x$-axis will increase $\phi$. The $St_x$-axis is aligned with the
Ref $x$-axis only after an even number of half-twists are completed. If an odd number of half-twists have been completed the angular momentum vector is on the opposite side of the athlete. This means that it is necessary to repeat an action to undo the original change in $\phi$ and hence remove the twist. Thus only for an even number of half-twists will reversing the action used to reduce $\phi$, increase $\phi$ back to $\pi/2$.

If the actions cannot be performed instantaneously, then they must start before the desired number of twists is reached, and adjustments made accordingly. For example, dropping the arm laterally initially in the frontal plane, will be adjusted to progressively move in front of the body; when close to a full twist has been completed the arm will need to be raised starting behind the body and as the twist continues is raised laterally in the frontal plane.

In summary, to maximise the change in $\phi$ due to the angular displacement an athlete should perform the actions as quickly as possible, using the slowest somersault possible, and as far as anatomically possible adjust the action so that it is completed in the y-z plane in frame $St$ as defined in Section 4.1.5.
The quasi-rigid phase

Chapter 5

The quasi-rigid phase

As explained in Section 2.1.3 there is typically a quasi-rigid phase, where at least for more skilled athletes, a considerable portion of the desired rotation occurs. By understanding the rotational behaviour during the quasi-rigid phase it is possible to predict upper limits to the numbers of twists and somersaults that may be achieved for reasonable initial conditions. Such an understanding may also be used to suggest what initial conditions would be desirable for this quasi-rigid phase when performing a twisting somersault and hence the re-orientation that is desirable during the twist initiation phase.

To determine which of the rotational cases from Section 4.2 are likely to be observed and to aid the assessment of which postures are most suitable for use in somersaults, the inertial properties of the various sport specific postures defined in Section 3.9 were applied in this chapter to the equations governing the rotational cases.

This chapter begins by presenting the inertial properties of the sport-specific postures from Section 3.9 which were determined using ICG17.m (as described in Section 3.8) and all 240 of the inertial property data sets specified in Section 3.6.

Postures are evaluated in terms of their relative difficulty for pure and twisting somersaults, and the number of somersaults that would be required to achieve each half-increment of twist or the twist that occurs if the posture displays oscillating twist. As a result suggestions may be made regarding which postures would aid the performance of various somersault skills. Differences in achievement, for different inertial property data sets when using the same posture, are used to discuss whether or not some ‘athletes’ have a natural advantage over others due to their inertial properties. The statistical tests used are described in Sections B.4 to B.6; the tests are only named in this chapter with the stated p-value and effect size statistic. For brevity the Kruskal-Wallis H-test will simply be referred to as the H-test and the Mann-Whitney U-test will be simply referred to as the U-test.
5.1 The moments of inertia in somersault postures

5.1.1 Order of the moments of inertia

In Section 4.2 it was found that, which moment of inertia was the intermediate-valued moment of inertia, determined which of the rotational cases (Figure 4-13) would be observed. It was also found that from Equation (4-2) with $\phi_o$ and $\psi_o$ restricted to the first quadrant, when $I_{yy} > I_{zz}$ the initial twist direction is to the left ($\psi > 0$). As a result to determine what rotational case and direction of twist may be observed for any particular posture it is necessary to know the order, in terms of magnitude, of the moments of inertia of a body posture. Table 5-1 gives the percentage of the inertial property data sets from Chapter 3 that have each possible order of the principal moments of inertia, in terms of their magnitude, for each of the postures defined in Section 3.9.

Based on Table 5-1 it is reasonable to focus on the situations when $I_{xx} \neq I_{yy} \neq I_{zz}$. There was one inertial property data set and posture combination that displayed equality of any of the moments of inertia to the level of significance that the inertial property data for each input segment was known: this was for FT, which had $I_{xx} = I_{zz}$.

The percentages given in Table 5-1 indicate that it is important to analyse situations when $I_{xx} > I_{yy} > I_{zz}$ which corresponds to case 3, when $I_{yy} > I_{zz} > I_{xx}$ which corresponds to case 6, and when $I_{yy} > I_{xx} > I_{zz}$ which corresponds with cases 7&8, depending on the initial conditions. Yeadon (1993a) gave two example postures, which are similar to posture L and posture OP, and both of which have $I_{yy} > I_{xx} > I_{zz}$. These examples are not sufficient to provide coverage of all the motions that are expected to be observed. As can be seen from Table 5-1 the examples provided by Yeadon do not cover all the orders of the moments of inertia that are observed across a broad range of postures.

A general pattern may be seen in Table 5-1, with the postures that have extended hips and knees tending to have $I_{xx} > I_{yy} > I_{zz}$. These are the postures typically used when twisting. Adding flexion at the hips means $I_{yy}$ becomes larger than $I_{xx}$. With further flexion at the hips and some at the knees $I_{yy}$ tends to be the maximum moment while it is mixed as to whether $I_{xx}$ or $I_{zz}$ is larger; this is because they become quite similar and so a slight variation in one will result in the order of the moments of inertia changing. The postures defined in Section 3.9 are postures that are expected to be held for a portion of a somersault, rather than used during a transition to other postures. An athlete could adopt any number of postures between these. For the purposes of understanding how the order of the moments of inertia change as flexion is added at the hips and knees Figure 5-1 plots the values of the moments of inertia when the athlete from Section 3.10, smoothly transitions from posture L to posture P to posture T. Figure 5-2 then plots the somersault angle.
between frame Pri and frame Ref. The discontinuities in Figure 5-1 are where $I_{xx}$ and $I_{zz}$ swap, and match with when the somersault angle between frame Pri and frame Ref reaches 45°. For this athlete, the postures L, P and T are away from the discontinuities, and so may be considered good example postures for when a different moment of inertia is the intermediate-valued moment of inertia.

![Figure 5-1: Example of the variation in the moments of inertia when smoothly transitioning from posture L to posture P to posture T.](image)

The x-axis shows hip flexion from L (at 0°) to P (at 95°, marked by the black line), followed by 110° of knee flexion to T (at 205°) with hip flexion constant at 95°. All segments move at a steady coordinated rate so that when the hip and knee angles reach those required for posture P or posture T all segments are in the correct positions. The inertial property data used was of the athlete from Section 3.10.
The quasi-rigid phase

Figure 5-2: Example of the variation in the angle between the z-axis of frame Pri and the z-axis of frame Ref.

This angle is only in the direction of somersault since, due to symmetry of the postures, tilt and twist will be zero throughout. The discontinuities are where the angle reaches ±45°, and so the naming of the x and z axis of frame Pri swaps. These correspond to the change in the colour of the plots of $I_{xx}$ and $I_{zz}$ in Figure 5-1.
Table 5-1: The percentage of inertial property data sets with each combination of the order of the moments of inertia in terms of their magnitude. Percentages are only given when they are greater than zero; otherwise the cell is left blank. The percentages are given to the nearest percent; a zero in the table indicates that the percentage is positive but less than 1%.

<table>
<thead>
<tr>
<th>Moment of inertia</th>
<th>Percentage of ‘athlete’ with this order of the moments of inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LA</td>
</tr>
<tr>
<td>xx &gt; yy</td>
<td>100</td>
</tr>
<tr>
<td>yy &gt; xx</td>
<td>10</td>
</tr>
<tr>
<td>xx=yy = xx</td>
<td>83</td>
</tr>
<tr>
<td>xx&gt;yy = zz</td>
<td></td>
</tr>
<tr>
<td>yy = zz</td>
<td></td>
</tr>
<tr>
<td>xx = yy=zz</td>
<td>1</td>
</tr>
<tr>
<td>yy &lt; zz</td>
<td></td>
</tr>
<tr>
<td>xx&gt;yy &lt; zz</td>
<td></td>
</tr>
<tr>
<td>xx=yy &lt; zz</td>
<td></td>
</tr>
<tr>
<td>xx&lt;yy &lt; zz</td>
<td></td>
</tr>
</tbody>
</table>
5.1.2 Ratios of the principal moments of inertia

As was shown in Section 4.2 it is the relative magnitudes of the three moments of inertia, rather than the actual values themselves, that along with the initial conditions determine the number of somersaults required per half twist or per oscillation in twist \((N_0)\) and the magnitude of twist oscillation about \(\psi = 0\) is observed.

To aid visualization of the ratios of the moments of inertia, postures may be plotted on a scatter plot where the axes are the ratios \(I_{yy}/I_{zz}\) and \(I_{xx}/I_{yy}\). Figure 5-3 indicates the order of the moments of inertia in each region bounded by \(I_{xx} = I_{yy}, I_{yy} = I_{zz}\) or \(I_{xx} = I_{zz}\). Figure 5-4 and Figure 5-5 then give scatter plots for the common twist postures and those that are not typically used when twisting respectively.

![Figure 5-3: Scatter plot regions](image)
The order, in terms of magnitude, of the three moments of inertia in each region bounded by the lines \(I_{yy} = I_{zz}, I_{xx} = I_{yy}, \) and \(I_{xx} = I_{zz}\) is indicated. Understanding what each region represents will aid in interpreting Figure 5-4 and Figure 5-5.

![Figure 5-4: Ratios of the moments of inertia for common twist postures](image)
The quasi-rigid phase

Figure 5-5: Ratios of the moments of inertia for postures not usually used when twisting

The postures with minimal hip flexion, which are typically used when twisting (Figure 5-4), are scattered around $I_{xx} = I_{yy}$, with the scatter being narrower at larger values of $I_{yy}/I_{zz}$. This scatter pattern suggests that it is worth investigating whether the assumption, $I_{xx} = I_{yy}$, would be reasonable for these postures. The postures with deeper hip flexion (Figure 5-5) show scatter across different regions, and so a diverse range of rotational behaviour is expected. Some of these postures will even show continuous twist, although the small $I_{yy}/I_{zz}$ ratio indicates that the twist will be slow.

5.1.3 Initial orientation angles due to posture

The equations derived in Section 4.2 relied on the fact that $\theta$, $\phi$, and $\psi$ were all defined with respect to frame Pri; that is a coordinate frame whose axes are parallel to the principal directions for the posture held. When discussing the reorientation that occurs due to an action, the equations in Section 4.3 used frame Ref, since it was easier to identify with respect to anatomical features. It is thus important to determine the orientation angles between frame Pri and frame Ref when in each of the key somersault postures defined in Section 3.9 so that these angles may be included in the initial conditions when determining somersault skills that may be performed. The orientation angles between frame Pri and frame Ref were determined for each specific posture by using ICG17 (Appendix A.4).

Figure 5-6, Figure 5-8, and Figure 5-7 give box plots for the angle of tilt (a rotation about the $Ref_x$), the somersault angle (a rotation about $Ref_y$), and the twist angle (a rotation about $Ref_z$) of the frame Pri with respect to frame Ref (Section 4.1). Obviously the maximum angle between correspondingly named axes of frame Pri and frame Ref is $45^\circ$. This is because the axes of frame Pri were named by the closest axes of frame Ref.
The quasi-rigid phase

Figure 5-6: Tilt angle between frame Pri and frame Ref for the key somersault postures
The key somersault postures were defined in Section 3.9.

In Chapter 6 the angle of tilt between frame Pri and frame Ref will be added to the tilt due to any action performed before determining the twisting somersaults that may be achieved since it will be assumed that the longitudinal axis of the pelvis was vertical when the athlete left the ground.

Figure 5-7: Twist angle between frame Pri and frame Ref
The key somersault postures were defined in Section 3.9.
The twist angle between frame Pri and frame Ref is one means by which some initial twist, $\psi_0$, (Section 4.2.1) may be present, even when the athlete does not ‘cheat’ the twist when considering anatomical features.

![Figure 5-8: Somersault angle, $\theta$ between frame Pri and frame Ref](image)

The key somersault postures were defined in Section 3.9.

The somersault angle makes no difference to the skills that may be achieved or the rotational case (Figure 4-13) that may be observed. What it does mean is that the twist is not about an axis parallel to the longitudinal axis of the pelvis, but rather an axis inclined to the longitudinal axis of the pelvis. For the postures LAU through to EP, which will later be found (Section 5.3.1) to display predominantly continuous twist, the z-axis of frame Pri will still have a head-to-toe feel and so the difference between frame Pri and frame Ref is probably not observable when watching performances with the naked eye. The postures where the somersault angle between frame Pri and frame Ref is large, OP to CT, will be seen to be those that display predominantly oscillating twist. There will still be a feel of moving to the left or right; but twist and tilt motions will be expected to blend when observed with the naked eye.

### 5.2 The pure somersault

The pure somersault occurs when $\phi = \pi/2$ and $\psi = 0$. The athlete is rotating about their principal transverse axis and $\theta$ is the only angle that changes.

Relative difficulty is the ratio of the number of somersaults that could be completed in the same time and with the same angular momentum for two different situations. In Section 4.2.7 the equation for the relative difficulty of a pure vs. a twisting somersault
was derived. When performing a pure somersault \( \phi = \pi/2 \). There is no twist \((\psi = 0)\) and the motion can be reduced to a planar motion. The angular momentum required is

\[
H_{\text{Pure}} = I_{yy} \left[ \theta_{\text{End}}^{\text{End}} - \theta_{\text{Start}}^{\text{Start}} \right].
\]

The ratio of the number of somersaults in two different postures that may be completed when angular momentum and time are the same will thus be

\[
n_2 = \frac{I_{yy} \cdot 1}{I_{yy} \cdot 2}.
\]

The layout posture, L, will be used as posture 2 to facilitate comparisons between ‘athletes’ and postures. Any posture could be used; however, the layout posture is commonly used on entry and exit, and the layout has previously been used for the purpose of the ‘normalizing’ of parameters of interest (Vieten & Riehle, 1992), and thereby facilitate comparisons between ‘athletes’. Thus let the relative difficulty for a pure somersault with respect to a layout, \( \tau \), be

\[
\tau = \frac{I_{yy} \cdot 1}{I_{yy} \cdot L}.
\]

If it is assumed that the layout somersault is equally difficult for all ‘athletes’ then the \( \tau \) may be used to compare postures and ‘athletes’. It would be reasonable for sport governing bodies to use \( \tau \) as one of the factors in determining difficulty score. Alternatively, for a given set of rules the value of \( \tau \) for different postures may be used to choose the easiest posture for an athlete to achieve a required difficulty score. Any difference in \( \tau \) of a particular posture between ‘athletes’ indicates that some ‘athletes’ have a natural advantage. It is important to realise that \( \tau \) is only applicable for the quasi-rigid portion of the somersault. When performing an actual skill the entry and exit phases must also be considered.

### 5.2.1 The variation of \( \tau \) across ‘athletes’ and postures

Table 5-2 gives the median values \( \tau = I_{yy}/I_{yy,L} \) and Figure 5-9 gives box plots for the value of \( \tau \) for different ‘athletes’ across the postures. The postures were ordered by the median rank, then if the median ranks of two postures are equal then these two postures were ranked in order of the mean rank. The median relative difficulties of postures TT and CT suggest they should be in the reverse order; however, for the majority of ‘athletes’ posture CT has a lower relative difficulty than posture TT.

<table>
<thead>
<tr>
<th>LAU</th>
<th>LAP</th>
<th>1U</th>
<th>1I</th>
<th>1U</th>
<th>1I</th>
<th>L</th>
<th>A</th>
<th>LHF</th>
<th>Pu</th>
<th>JL</th>
<th>EP</th>
<th>OP</th>
<th>P</th>
<th>T</th>
<th>BT</th>
<th>TP</th>
<th>FT</th>
<th>TT</th>
<th>CT</th>
</tr>
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<tbody>
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<td></td>
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</tbody>
</table>
The quasi-rigid phase

Figure 5-9: Box plots showing the range of relative somersault difficulty, $\tau$, over all ‘athletes’. Posture L is the reference posture and so $\tau$ for posture L has the value of one for all ‘athletes’.

The order of the postures does reflect the general expectation that $\tau$ is lower for the more ‘compact’ postures that are associated with lower difficulty scores. The box plots show overlapping values of $\tau$ between postures. It is thus not clear if one posture is always better than another. Figure 5-10 illustrates the spread in the ranks assigned to each posture, when the postures are ranked by the value of $\tau$ for each ‘athlete’.

Figure 5-10: Box plots showing the ranks assigned to each posture when they are ranked by the value of the relative somersault difficulty, $\tau$, for each ‘athlete’.
The purple line shows the median rank. If part of the box plot is below the purple line then for those ‘athletes’ that posture is ranked higher than the median rank. If part of the box plot is above the purple line then for those ‘athletes’ that posture is ranked lower than the median rank.

From Figure 5-10 it can be seen that the ranks are generally not symmetric about the median rank. In a number of cases there are adjacent postures with the lower quartile to median box (green) of one posture overlapping the median to upper quartile box (blue) of the other. This indicates that for many ‘athletes’ the rank order of these two postures is swapped. Where there are larger overlaps in the spread of the ranks, some non-adjacent postures may have their ranks swapped for some athletes. There are also some postures which overlap in ranks and yet do not overlap in relative difficulty value spread (e.g. LAP & L, A &L); this indicates that the ranks of these postures vary not because some ‘athletes’ have their ranks swapped but that the ranks of other postures slot between, thus changing their rank with respect to each other.

The only posture that was assigned the same rank for all ‘athletes’ was posture LAU. It had the greatest value of \( \tau \) for all ‘athletes’. Thus, even though there is a spread in \( \tau \) that does overlap with other postures, the overlap is due to some ‘athletes’ having a \( \tau \) value in LAU overlapping with higher values of \( \tau \) in other postures of a different ‘athlete’. Posture LAU has the greatest value of \( \tau \) for all ‘athletes’ and so when using this posture each ‘athlete’ will be rotating at the slowest possible speed for the angular momentum they possess. Thus, posture LAU is probably a more suitable reference posture than posture L for determining relative difficulty, and thereby comparing ‘athletes’.

Overlaps of ranks indicate that the ranking is not constant across ‘athletes’. In fact no ‘athlete’ ranked the postures in the same order as the median ranks. There were 115 different posture orders observed with the 240 ‘athletes’. Ten ‘athletes’ was the largest number of ‘athletes’ that ranked the postures the same. It is thus not reasonable to suggest even a few likely alternative orders of postures, or match these to a few sub-populations.

The spread of \( \tau \) for each posture shows that there are differences between ‘athletes’. Those with lower \( \tau \) values in a particular posture would have a natural advantage for that posture. When the ‘athletes’ are ordered by their value of \( \tau \) for each of the postures, the same ‘athletes’ do not have the same ranking. This means that different ‘athletes’ would have a natural advantage in different postures.

Biesterfeldt (1975) and Frohlich (1979) present \( I_{yy} \) or ratios including \( I_{yy} \), for a few of the postures described in Section 3.9 based on one individual each. From these values \( \tau \) may be determined.

Table 5-3 gives the values of \( \tau \), and Figure 5-11 superimposes these on Figure 5-9.
Biesterfeldt’s and Frohlich’s ‘athletes’ are within the range of the current data. They are approximately the median for posture TP, yet tend to the extremes otherwise. As a result, conclusions from the current study may be considered to supplement and expand upon existing knowledge rather than representing the same group of athletes or being a distinctly separate group. The variation of the position of Biesterfeldt’s and Frohlich’s athletes compared to the box plots from the current study highlights the fact that any one individual is not necessarily in the same percentile group for all postures. This is why considering ranks and not just values, is important when making comparisons.
5.2.2 Equivalence

It is not immediately clear what difference in $\tau$ is practically significant. The difference needs to be translated into something that describes the consequences of the difference for an athlete. Let the term “equivalent somersaults” refer to the ratio of $\tau$ between ‘athletes’, postures, or a combination of both. For example, if one athlete has a value of $\tau = 0.5$ in posture T, and another athlete has $\tau = 0.4$ then the equivalent somersaults is $\tau_1/\tau_2 = 0.4/0.5 = 0.8$, which means that the first athlete would complete 0.8 of a somersault in posture T for every somersault the second athlete completes, in the same time, and with angular momentum specific to each athlete, so that a layout was of equal difficulty.

Figure 5-9 illustrated the variation in $\tau$ across ‘athletes’ for each posture. Figure 5-12 now illustrates the spread of equivalent somersaults if each ‘athlete’ possessed the median value of $\tau$ for that posture.

![Figure 5-12: Equivalent somersaults compared to the median](image)

The value of the equivalent somersaults was calculated by dividing the value of $\tau$ for each ‘athlete’ by the median value of $\tau$ for the posture of interest. Posture L is assumed to be equally difficult and so the “equivalent somersaults” for all ‘athletes’ is one.

In Figure 5-12 it can be seen that postures with greater knee and/or hip flexion tend to have a greater range in the value of equivalent somersaults. This is to be expected since these postures differ the most from posture L, which was used as the reference posture when defining $\tau$, and so was assumed to be of equal difficulty across ‘athletes’. What is of particular interest is how great the range becomes. Even though a tenth may sound small, a tenth of a somersault is practically significant: it is $36^\circ$ of over- or under-rotation. This amount of over- or under-rotation would result in deductions and make the landing/entry much more difficult. The majority of the postures have a spread of
greater than a tenth of a somersault. It is thus clear that some ‘athletes’ have a practically significant advantage over other ‘athletes’ due to their inertial properties.

This conclusion is the opposite to that reached by Vieten & Riehle (1992) in their study of “rotational ability”. Vieten & Riehle calculated the relative difficulty of seven postures: layout, layout with arms raised, tuck, pike, layout with arms laterally perpendicular to the body, lateral flexion, and layout with arms in front. Six of these correspond with postures used in the current study. By considering the difference in the value of τ Vieten & Riehle (1992) concluded that there were “insignificant changes for all individuals”. Vieten & Riehle only considered the magnitude of τ and not equivalent somersaults. Their group consisted only of adults, whose inertial properties were estimated using a Hanavan method (presumably Hanavan-BP). The Hanavan-BP “Senior” and “Master” inertial property data sets in the current study show similar standard deviations in the range of τ (0.017 for “Senior” and 0.036 for “Master”) to Vieten & Riehle’s trampolinists (0.024 for “top trampolinists” and 0.031 for “semi-trained adults”). These seem small, yet the range in the value of equivalent somersaults calculated by comparing each ‘athlete’ to the median value of τ in each posture, for just these ‘athletes’ is still practically significant: for example, for the Senior and Master groups combined, the range of equivalent somersaults in posture LAU was 0.10, in posture LAP was 0.05, in posture P was 0.07 and in posture T was 0.21.

5.2.3 Complete and multiple somersaults

The values of τ for each posture are only applicable for the quasi-rigid phase of the somersault. When performing the full skill of a somersault, the ‘athlete’ would only hold the specified posture for part of the somersault, and there will be an ‘entry’ and ‘exit’ phase where an athlete moves into and then out of the specified posture. The relative difficulty of a complete somersault depends on all three phases. Some examples of actual performances of pure somersaults, where entry and exit phases may be seen, are given in Sections 5.2.8 and 5.2.9.

For multiple somersaults τ, as determined in Section 5.2.1, needs to be multiplied by the number of somersaults to give the relative difficulty during the quasi-rigid phase. As an example, Table 5-4 gives a conservative estimate of relative difficulties of complete somersaults; this estimate is based on holding posture L for a quarter somersault entry and a quarter somersault exit phase, and an instantaneous change between posture L and the specified posture which is held for the remaining half somersault. In reality there will be a transition from posture L to the specified posture which will have a smaller relative difficulty than posture L. Based on observations of athletes and allowing some
rotation during take-off and landing, the degrees of rotation for a single, double, triple, and quadruple somersault will be 350°, 700°, 1060°, 1420° respectively.

Table 5-4: Median relative difficulties for complete somersaults

<table>
<thead>
<tr>
<th>Posture</th>
<th>L</th>
<th>A</th>
<th>LHF</th>
<th>Pu</th>
<th>JL</th>
<th>EP</th>
<th>OP</th>
<th>P</th>
<th>T</th>
<th>BT</th>
<th>TP</th>
<th>FT</th>
<th>TT</th>
<th>CT</th>
</tr>
</thead>
<tbody>
<tr>
<td>From Table 5-2</td>
<td>1.00</td>
<td>0.96</td>
<td>0.94</td>
<td>0.92</td>
<td>0.88</td>
<td>0.82</td>
<td>0.70</td>
<td>0.49</td>
<td>0.46</td>
<td>0.41</td>
<td>0.41</td>
<td>0.34</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>Single (350°)</td>
<td>1.00</td>
<td>0.98</td>
<td>0.97</td>
<td>0.96</td>
<td>0.94</td>
<td>0.91</td>
<td>0.85</td>
<td>0.75</td>
<td>0.74</td>
<td>0.72</td>
<td>0.71</td>
<td>0.68</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>Double (700°)</td>
<td>2.00</td>
<td>1.93</td>
<td>1.91</td>
<td>1.89</td>
<td>1.82</td>
<td>1.74</td>
<td>1.55</td>
<td>1.24</td>
<td>1.19</td>
<td>1.13</td>
<td>1.13</td>
<td>1.02</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>Triple (1060°)</td>
<td>3.00</td>
<td>2.89</td>
<td>2.84</td>
<td>2.81</td>
<td>2.69</td>
<td>2.56</td>
<td>2.24</td>
<td>1.73</td>
<td>1.65</td>
<td>1.54</td>
<td>1.54</td>
<td>1.35</td>
<td>1.38</td>
<td></td>
</tr>
<tr>
<td>Quadruple (1420°)</td>
<td>4.00</td>
<td>3.84</td>
<td>3.78</td>
<td>3.73</td>
<td>3.57</td>
<td>3.38</td>
<td>3.24</td>
<td>2.94</td>
<td>2.22</td>
<td>2.11</td>
<td>1.95</td>
<td>1.95</td>
<td>1.69</td>
<td>1.73</td>
</tr>
</tbody>
</table>

A comparison between entries in Table 5-4 could be used by governing bodies to guide assignment of difficulty scores. This table is only a guide since other factors also affect difficulty; for example, limitations or non-linear relationships of the generation of momentum, joint torques to hold posture, flexibility to attain a posture, what skills may precede the somersault, and the ease of sighting the landing.

5.2.4 Specific posture reflections and comparisons

For the purposes of aiding coaching decisions relating to the postures chosen once airborne, this section will make a few posture comparisons. The number of “equivalent somersaults” of one posture compared to the other will be determined for all the inertial properties collated in Chapter 3 (referred to as ‘athletes’ for brevity), and the result converted to degrees of over- or under-rotation, and then plotted. The x-axis thus gives the degrees over- or under-rotated if the ‘athlete’ had sufficient angular momentum to perform a single somersault in one posture but used the other. The captions indicate the postures. The y-axis then gives the proportion of ‘athletes’ that achieve more (red) or less (black) than that amount of rotation.

Tighter postures, such as using posture TT or CT rather than postures T, or TP rather than P, have lower \( \tau \) values for any particular ‘athlete’. By tightening the posture an ‘athlete’ would gain additional rotation. This is as expected and only becomes clear with a specific posture comparison since there is some overlap in the value of \( \tau \) (Figure 5-9) and the ranks given to the postures (Figure 5-10) due to the underlying variation between different ‘athletes’.

Postures FT and BT are both tuck postures but differ in the curvature of the spine. Both had the same median relative difficulty; however, from Figure 5-9 they do not have the
The quasi-rigid phase

same spread. Posture BT was ranked as easier, although there was moderate overlap in the ranks assigned. Figure 5-13 shows the over- or under-rotation of BT vs. FT. Half the ‘athletes’ found FT easier and the other half found BT easier and the proportion of over- or under-rotation was similar in either case; thus it is not possible to make a general statement about which is easier.

![Figure 5-13: Proportion of ‘athletes’ and the over- or under-rotation predicted when an ‘athlete’ uses BT instead of FT or vice versa.](image)

(Black) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture BT but used posture FT instead. (Red) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture FT but used BT instead.

Posture CT is used in multiple somersaults because it is believed that it allows faster rotation even though it is an unaesthetic posture. CT was ranked as the easiest posture in Table 5-2, although there was considerable overlap in ranks, with posture TT and slight overlap with the ranks of postures P, T, BT, TP, and FT. Figure 5-14 illustrates the over- or under-rotation of CT vs. TT. Posture CT is easier for the majority of ‘athletes’; yet approximately 38% were predicted to find posture TT easier and so for these ‘athletes’ CT would result in unnecessary deductions. The over- or under-rotation is up to 40° in either case. If all ‘athletes’ could perform posture CT, and wished to choose the more aesthetically pleasing posture TT then, for 50% of the ‘athletes’, the loss, if any, would be less than 5° per somersault and for 90% of ‘athletes’ the loss is less than 25°. The choice between posture CT and posture TT is not clear cut, considering the deduction associated with posture CT. It is advisable to start with posture TT.
Figure 5-14: Proportion of ‘athletes’ and the over- or under-rotation predicted when an ‘athlete’ uses posture CT instead of posture TT or vice versa.
(Black) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture CT but used posture TT instead. (Red) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture TT but used posture CT instead.

A pike is generally awarded a greater difficulty score than a tuck. Table 5-2 lists posture P as more difficult than posture T, and posture TP as more difficult than posture TT; however, there is considerable overlap in the relative difficulties shown in Figure 5-9, and from Figure 5-10 a clear overlap in the ranks of posture P and posture T and a slight overlap in the ranks of postures TP and TT may be seen.

Figure 5-15: Proportion of ‘athletes’ and the over- or under-rotation predicted when an ‘athlete’ uses P instead of T or vice versa.
(Black) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture T but used posture P instead. (Red) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture P but used posture T instead.
The quasi-rigid phase

Figure 5-16: Proportion of ‘athletes’ and the over- or under-rotation predicted when an ‘athlete’ uses TT instead of TP or vice versa.

(Black) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture TT but used posture TP instead. (Red) The ‘athlete’ possesses enough angular momentum to perform a somersault in posture TP but used TT instead.

Posture T is predicted to be easier than posture P for 60% of the ‘athletes’ (Figure 5-15), while TT is easier than TP for 80% of the ‘athletes’ (Figure 5-16). These proportions support the conclusion that piked postures are harder than tuck. However, some ‘athletes’ would have a natural advantage, gaining extra points for a posture that is easier in angular momentum terms for them to perform. The change in the proportions between postures P and T, and TP and TT indicate that flexing at the knees to reduce the value of $I_{yy}$ and so increase rotational speed is more advantageous when accompanied by greater hip flexion.

Considering postures T and TP as a pair, 78% of the ‘athletes’ are predicted to over-rotate in posture TP if they possessed sufficient angular momentum to perform a somersault in posture T. Thus, from an angular momentum perspective when wishing to transition from a tucked posture to a piked posture, a logical learning progression would be to start with learning a somersault in posture TT, and then to progressively open out from posture TT into posture T, since it requires greater angular momentum but is a more familiar posture than a piked posture. Once able to perform a somersault in posture T, transitioning to posture TP would be easy. Then, with additional training to generate greater angular momentum they can progress to using posture P. Of course the strength required to actually hold each of the postures T, TT, P and PT may affect what is actually ‘easier’ for the ‘athlete’, and hence the exact progression used.

Considering only the symmetric postures used for pure somersaults the layout difficulty would be awarded for postures JL, A, L, LAP or LAU. The relative difficulties $\tau$ of each of the postures JL, A, L and LAP (Figure 5-9) do not overlap. Thus the order of difficulty is the same for all ‘athletes’: posture LAP is the hardest, followed by postures
L, A, and JL. This is not surprising since, posture A has more flexion through the torso than posture L, and posture JL has more flexion through the torso than posture A. To maximise somersault posture JL would be the best choice, although it may be deemed less aesthetically pleasing than posture L.

The layout posture is not common in competitive diving. Single layouts are very common in gymnastics. Double layout back somersaults are common in gymnastics, but double front layout somersaults are not. It is interesting to note that posture A is very common in double back layout somersaults. Even though posture A is harder than posture JL, there may be other reasons for using posture A over posture JL in back layouts, such as the sighting requirements or the follow through from the take-off.

Postures LAU and LAP have no torso flexion, but the position of the arms places the mass of the arms at the shoulder level or above and hence further from the centre of gravity than in posture L. This is why these postures have a higher value of τ. If an ‘athlete’ only requires a difficulty of a layout to be awarded then using postures LAU or LAP rather than posture L would make the skill unnecessarily difficult to achieve. However, postures LAU and LAP are of use in a “flying” somersault, where the intention is to have a slow rotation for the first ¼ to ½ somersault, and then more quickly rotate the rest of the somersault. A greater angular velocity difference between the two phases would be achieved by using posture LAU or posture LAP for the first ¼ to ½ somersault rather than posture L, thus making the skill more aesthetically pleasing.

5.2.5 Gender differences

In Sections 5.2.1 and 5.2.2 the spread in τ means that some ‘athletes’ have a natural advantage when performing pure somersaults. It is of interest to know if gender affects the choices of posture to use or the expected achievement.

The full range of τ—the maximum value of τ minus the minimum value of τ for a specific ‘athlete’—was found to be significantly different for males and females under the U-test (p=0.0013). The females tended to have a higher full range of τ than the males, with a CLES of 61.3% and a dominance statistic of 0.225. Figure 5-17 shows the full range of τ, from the most to the least difficult posture, against the proportion of ‘athletes’. The male and female curves do not cross, indicating that a higher proportion of females have at least each full range of τ.
The quasi-rigid phase

For the range of $\tau$, from posture L to the posture with the minimum value of $\tau$, females tended to have even higher ranges than the males, with a CLES of 66.7% and a dominance statistic of 0.333. For the range of $\tau$, from posture L to the posture with the maximum value of $\tau$, males and females were not significantly different under the U-test ($p=0.16$).

Considering the value of $\tau$ for each posture, significant differences, using the U-test, were found between the genders for 1U1DB ($p = 0.09$), Pu ($p << 0.001$), OP ($p = 0.03$), P ($p = 0.01$), T ($p << 0.001$), BT ($p << 0.001$), TP ($p << 0.04$), FT ($p << 0.001$), TT ($p << 0.001$), and CT ($p << 0.001$). In all of these postures, except 1U1DB, females had a lower value of $\tau$. Table 5-5 gives the CLES percentages for each of these postures. The rotational advantage that females have over males depends on the specific male and female; however, to illustrate the potential practical significance, the median value of $\tau$ for these postures is given for each gender, and the extra rotation the median female would achieve over the median male is also given.

Based on the amount of over-rotation for the median female compared to the median male, shown in Table 5-5, even though all the postures listed showed a statistical difference, it is in the tuck positions (postures T, BT, FT, TT, and CT) that the females have a clear practically significant advantage. A coach may then expect that when performing drills to build strength for generating angular momentum, females would not need to be able to complete as many layouts as males before they are ready to transition to multiple tuck somersaults. A female would also experience a greater
change in angular velocity from layout to tuck. This is an interesting finding since traditionally it is males who push the bounds of difficulty in terms of the number of somersaults achieved. Males must have a different advantage over females; this could simply be a greater ability to generate angular momentum or generate greater vertical linear momentum giving greater flight times, thereby allowing them to achieve larger numbers of somersaults.

**Table 5-5: CLES for \( \tau \) by posture where male and females were different.**

For posture P the medians are the same; the statistical difference is due to the distribution around the median. The over-rotation in the last row is given per somersault for the median female if she had the same angular momentum as the median male.

<table>
<thead>
<tr>
<th>Posture</th>
<th>1U1DB</th>
<th>Pu</th>
<th>OP</th>
<th>P</th>
<th>T</th>
<th>BT</th>
<th>TP</th>
<th>FT</th>
<th>TT</th>
<th>CT</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLES %</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Females have a lower value of ( \tau ))</td>
<td>45.0</td>
<td>63.2</td>
<td>56.8</td>
<td>58.5</td>
<td>69.0</td>
<td>66.3</td>
<td>56.6</td>
<td>66.1</td>
<td>68.0</td>
<td>65.4</td>
</tr>
<tr>
<td>Median female ( \tau )</td>
<td>1.05</td>
<td>0.91</td>
<td>0.69</td>
<td>0.49</td>
<td>0.44</td>
<td>0.39</td>
<td>0.41</td>
<td>0.39</td>
<td>0.31</td>
<td>0.32</td>
</tr>
<tr>
<td>Median male ( \tau )</td>
<td>1.04</td>
<td>0.92</td>
<td>0.70</td>
<td>0.49</td>
<td>0.49</td>
<td>0.45</td>
<td>0.42</td>
<td>0.44</td>
<td>0.37</td>
<td>0.38</td>
</tr>
<tr>
<td>Degrees of over-rotation of median female vs. median male.</td>
<td>-3.5</td>
<td>4.0</td>
<td>5.2</td>
<td>0.0</td>
<td>40.9</td>
<td>55.4</td>
<td>8.8</td>
<td>46.2</td>
<td>69.7</td>
<td>67.5</td>
</tr>
</tbody>
</table>

The genders may not only differ in the value of \( \tau \) for a posture, but the relationship between postures may also be affected. Table 5-6 compares those pairs of postures discussed in Section 5.2.3, where one posture was not clearly easier than the other.

**Table 5-6: CLES for comparing postures where male and females were different**

<table>
<thead>
<tr>
<th>Posture comparison</th>
<th>p-value for genders under the U-test.</th>
<th>Dominance statistic and CLES</th>
<th>Proportions within a gender</th>
</tr>
</thead>
<tbody>
<tr>
<td>BT vs. FT</td>
<td>Highly significant ( p &lt;&lt; 0.001 )</td>
<td>( d = 0.312 ) \nCLES = 66% with more males, than females, finding FT easier than BT.</td>
<td>42% of females found FT easier and 60% of males found FT easier</td>
</tr>
<tr>
<td>CT vs. TT</td>
<td>Not Significant ( p = 0.421 )</td>
<td>( d = 0.015 ) \nCLES = 50.7%</td>
<td>65% of females found CT easier and 59% of males found CT easier</td>
</tr>
<tr>
<td>P vs. T</td>
<td>Highly significant ( p &lt;&lt; 0.001 )</td>
<td>( d = 0.406 ) \nCLES = 70.3% with more females than males finding T easier than P.</td>
<td>74% of females found T easier and 42% of males found T easier</td>
</tr>
<tr>
<td>TP vs. TT</td>
<td>Highly significant ( p &lt;&lt; 0.001 )</td>
<td>( d = 0.394 ) \nCLES = 69.7% With more females finding TT easier than TP.</td>
<td>89% of females found TT easier and 72% of males found TT easier</td>
</tr>
</tbody>
</table>
It is not surprising that a higher proportion of females than males found tuck easier than pike, since there was a greater difference between the values of $\tau$ for tuck and pike between genders. A coach should thus expect females to have more difficulty than males when transitioning from tuck to pike due to the greater increase in $\tau$ compared to males. It may also be advisable for females, depending on the actual sport and scoring difference, to pursue multiple tuck somersaults rather than pike somersaults.

For each gender, the proportion of that gender who were predicted to find posture FT or BT easier than the other is no longer one half, as was found in Section 5.2.3. Females tended to find posture BT easier and the males tended to find posture FT easier. Thus, gender may be used to help guide whether or not a coach should encourage a straighter or more curved torso.

### 5.2.6 Differences between squads

In Section 5.2.5 gender was considered as one characteristic that may describe some of the ‘athletes’ with a natural advantage. The other categorical data that was collected about the athlete was their squad, which is related to age and years of training. Let us similarly consider if squad is a category that may be used to describe ‘athletes’ with a natural advantage.

In terms of the range of $\tau$ the squads were found to be from different populations under the H-test ($p = 0.0025$). Figure 5-18 illustrates the proportion of ‘athletes’ with at least the range of $\tau$ given on the x-axis. As can be seen from this plot the 12-or-under, Master and the literature data stand out as different, while the Teen and Senior squads are quite similar. The U-test between each squad and the combinations of the other squads confirms that the 12-or-under and other squads are distinctly different from the other groups ($p<<0.01$). The 12-or-under ‘athletes’ have a much lower range of $\tau$ while the literature data has a much larger range of $\tau$. The Master group shows a tendency to be different from the other squads ($p=0.06$).

Considering the range of $\tau$ from posture L to the posture with the minimum value of $\tau$ and the range of $\tau$ from posture L to the posture with the maximum value of $\tau$, the squads are still not from the same homogeneous population ($p<<0.01$ for both cases using the H-test). In both cases the 12-or-under squad still shows a clearly smaller range of $\tau$. The literature data again shows a very narrow spread of the range of $\tau$; the range of $\tau$ is high in value compared to the other squads for the range from posture L to the posture with the minimum value of $\tau$, but is close to the middle for the range from posture L to the posture with the maximum value of $\tau$. 
The quasi-rigid phase

Figure 5-18: Proportions of ‘athletes’ from each squad having at least the given full range of \( \tau \).

The full range of \( \tau \) is the maximum value of \( \tau \) minus the minimum value of \( \tau \) for a specific ‘athlete’.

Since the 12-or-under ‘athletes’ have a lower range of \( \tau \), a coach should expect that their 12-or-under ‘athletes’ would gain less from changing posture once airborne. They should also expect that these ‘athletes’ would take longer to achieve their first somersault, but then transition between the postures more quickly than older ‘athletes’.

The Master squad was almost exclusively female (1 male and 5 females). Since females tended to have higher values of \( \tau \) than males, and the path is similar to the female line in Figure 5-17 it is possible that the gender, not the squad, made this group distinct.

The literature data showed a very narrow spread in the range of \( \tau \), from 0.85-0.96, which is a high value of \( \tau \) compared to the other squads. This small spread suggests that these data sets were of similarly proportioned ‘athletes’. This could be a result of how the data sets were collected, in that the data sets are averages not actual people, or similarities in the original people sampled; the reason cannot be known. However, using only the literature data sets limits understanding of what could be achievable and this supports the decision in Chapter 3 to take measurements of current ‘athletes’ to collate a broader range of inertial properties.

Considering the value of \( \tau \) for each posture under the H-test, the squads were found to be significantly different for all postures except A and Pu and EP. The 12-or-under squad and the literature data stood out as different. The 12-or-under squad tended to have higher values of \( \tau \); the postures harder than layout were minimally harder for them,
while those postures easier than layout were considerably more difficult for them. This
difference causes the low range of $\tau$ for the 12-or-under ‘athletes’. The literature data
tended to have lower values of $\tau$, most notably in posture P or easier postures. The
Teens, Senior, and Master squads were not statistically the same, but which squad was
more similar to another varied with the posture. Rather than examining every posture it
is of more interest to return to the specific posture comparisons from Section 5.2.4.

Table 5-7 gives the proportion of ‘athletes’ that find one posture easier than the other.

<table>
<thead>
<tr>
<th>Posture comparison</th>
<th>Proportion of squad for which posture statement is true (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12 or under</td>
</tr>
<tr>
<td>FT is easier than BT</td>
<td>58</td>
</tr>
<tr>
<td>CT is easier than TT</td>
<td>54</td>
</tr>
<tr>
<td>T is easier than P</td>
<td>40</td>
</tr>
<tr>
<td>TT is easier than TP</td>
<td>65</td>
</tr>
</tbody>
</table>

It is clear that the literature data is a separate sub-population and the decision regarding
which posture is easier is straightforward. As when considering the range of $\tau$ it is clear
that the data collated from literature only provides limited understanding of what is
achievable.

It is of particular interest to observe the much lower proportion of 12-or-under ‘athletes’
who find posture T easier than posture P, and posture TT easier than posture TP. In light
of this a coach may consider introducing TP and P postures quite early into the training
programme for 12-or-under ‘athletes’ since many are expected to find it easier than
postures TT or T. As the athlete grows, if their training regime can build the athlete’s
ability to generate angular momentum sufficient to maintain posture P, they will find
that the transition to multiple tuck somersaults from posture P is easier, since posture T
will become progressively easier.

For postures CT vs. TT and postures FT vs. BT the difference in proportions is less
practically significant and it is reasonable for a coach to focus on the gender differences
(Section 5.2.5) and the general posture advice (Section 5.2.4).

### 5.2.7 Do common biomechanic indices identify ‘athletes’ with a natural advantage?

Rather than needing to estimate the full set of inertial properties for an athlete, it would
be desirable to find a single index or ratio that would identify those ‘athletes’ with a
natural advantage. The following common biomechanical indices and ratios were
considered: BMI, Rohrer, Ponderal and Androgyny indices and V-ratio, sitting height-,
arm- and leg-length to stature ratios (Park et al., 2007; Stewart & Sutton, 2012; Bradshaw & Rossignol, 2004).

To determine if any of the indices or ratios had promise, the coefficient of determination, $R^2$, (Phipps & Quine, 2001) and Spearman’s statistic (Lehmann, 1975) between the range of $\tau$ and each index or ratio was determined. None of the common biomechanical indices or ratios showed any value in predicting which ‘athletes’ would have a high or low range of $\tau$; neither in the full range nor the ‘easier’ or ‘harder’ than layout situations. This was true over all 240 inertial property data sets and also over those data sets based on Hanavan-BP, which had only one rejection (Section 3.5.7).

### 5.2.8 Observations from a selection of pure somersaults performed by one example athlete

One athlete (female, 58.8kg, 163.3cm) volunteered to perform one good tuck, pike, layout, high tuck kick-out (a warm-up for a double tuck), and double tuck front somersault off a Gymnova Springalene to land on a crashmat. The athlete was allowed to elect their own run distance, and elected to use a longer run for the high tuck kick-out and double tuck than for the other three skills, presumably to aid the generation of angular momentum and to increase flight time. Since pure somersaults move in a single plane, only one camera placed so that the optical axis was parallel to the somersault axis was required to film the performance. The digitisation process and angle calculation is presented in Appendix C. The somersault angle was measured as the amount of rotation of the line joining the hip joint and the iliac crest from the point of take-off. The joint angle data was smoothed using a fourth order zero lag Butterworth filter, with the cut-off frequency set using Winter’s residual analysis resulting in a cut-off frequency that was between 4 and 7 Hz. Some out-of-plane motion of the arms was observed; this resulted in wild oscillations of the elbow joint angle in the raw data, which was set to zero for this period since, when watching the performances while filming was occurring, it could be seen that the arms were not bending. ICG17 (Section 3.8) was used to determine $\tau = \frac{I_{yy}}{I_{yy,L}}$ each posture, using the estimation method Hanavan3, since it produced the least errors in simulations of pure somersaults as found in Section 3.10 (this is the 14th inertial property data set listed in Appendix A.5). The $I_{yy,L}$ value used in the calculation of $\tau$ is $I_{yy}$ for the reference posture L and should not be confused with the somersault skill actually performed that would be classified as a layout when assigning a difficulty score.

Treating the frames between the last visible contact with the Springalene and the first contact with the crashmat as the airborne phase, differences were observed between the
somersaults in terms of the somersault angle completed and the flight time. The values are given in Table 5-8.

<table>
<thead>
<tr>
<th>Skill</th>
<th>Somersault angle completed (degrees)</th>
<th>Flight time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tuck</td>
<td>330</td>
<td>0.85</td>
</tr>
<tr>
<td>Pike</td>
<td>350</td>
<td>0.84</td>
</tr>
<tr>
<td>Layout</td>
<td>340</td>
<td>0.83</td>
</tr>
<tr>
<td>Tuck kick-out</td>
<td>339</td>
<td>0.94</td>
</tr>
<tr>
<td>Double tuck</td>
<td>691</td>
<td>0.96</td>
</tr>
</tbody>
</table>

The somersault angles are not exact multiples of 360° since some somersault rotation occurred while the athlete was in contact with the Springalene, or as they were in the process of landing. It is clear that when performing, or practicing for, a double somersault off Springalene this athlete deliberately used a longer flight time. The increase in flight time between the tuck and double tuck is 0.11 sec or approximately 13%. This additional flight time appears to be used by this athlete to reduce the angular momentum required to complete the double somersault (Table 5-9).

Figure 5-19 through to Figure 5-23 show plots of $\tau$ for the postures used for each skill over time. Video frames are added at key points to illustrate the posture that gave that value of $\tau$.

Figure 5-19: Somersault relative difficulty, $\tau = I_{yy}/I_{yy-L}$, over time when a tuck somersault was performed

The tuck was entered as a continuation of the take-off throw, the knees were tapped and then the athlete extended out of the tuck in one smooth motion. Less than half the somersault was executed in the tucked posture. The tuck posture that was used reached a minimum $\tau$ value of 0.48, had less hip flexion and greater knee flexion than postures T and FT. Since posture FT was taken based on a previous performance of a forward
somersault by this athlete on the floor, it is apparent that the athlete either adjusts their posture to the new context (Springalene vs. floor) or is not consistent in their performances.

The pike was entered as a continuation of the take-off throw, the ankles tapped and then the pike was exited in one smooth motion. Around half of the somersault was in a posture that could be called ‘pike’. The pike posture with minimum $\tau$ of 0.52 had slightly less flexion at the hips ($\sim 5^\circ$) than posture P and straight arms were reaching towards the ankles rather than the forearms lying along the lower legs.
The layout posture is held almost the whole time, with a slight pike used as part of the take-off throw. The arms were lowered at the start then raised on exit from the somersault. Just before the $\frac{1}{4}$ somersault position $\tau$ reaches its minimum value of 0.84. This is less than one since there is slight flexion of the hips and through the torso. As the arms are raised, $\tau$ increases again. The fact that the value of $\tau$ is less than 1.0 is not a ‘failure’ in the performance of a layout, since by definition a layout allows some flexion through the torso.

![Somersault relative difficulty](image)

**Figure 5-22:** Somersault relative difficulty, $\tau = I_y/I_{yy-t}$, over time when a tuck kick-out lead up was performed

In the tuck kick-out, the knees are touched at a similar entry point to the double tuck. The tuck is held for approximately 50° of somersault rotation and then a clear kick-out phase is seen; the kick-out is completed just before the half-somersault mark. The remaining somersault is completed predominantly in a layout position with the arms raised—the greatest value of $\tau$ an athlete can obtain—presumably to prevent over-rotation since the angular momentum used would be closer to that needed for a double tuck than rather than a tuck.
The quasi-rigid phase

Figure 5-23: Somersault relative difficulty, $\tau = I_y/I_{y-z-L}$ over time when a double tuck somersault was performed

In the double tuck somersault, the tucked posture is entered and then held for approximately 1½ somersaults. There is some oscillation in $\tau$ while the tuck is held, due to some oscillation in the hip and torso joint angles. The oscillation in the joint angles could reflect the athlete slightly losing the posture and then having to pull back in due to high centrifugal forces. The posture with the minimum $\tau$ value of 0.43 is between postures FT and TT: there is less flexion at the hips, and greater flexion at the knees than in posture TT and slightly more flexion at the hips and less at the knees than posture FT.

Figure 5-24 shows the variation of $\tau$ with time for all five performances.

Figure 5-24: Values of Somersault relative difficulty, $\tau = I_y/I_{y-z-L}$ over time for all of the observed skills
The double tuck displays the lowest value of $\tau$ for the whole time. There is a clear entry phase, quasi-rigid phase, and exit phase. At time zero the athlete becomes airborne and since the double-tuck has the lowest value of $\tau$ at the start, it appears that the athlete starts to enter the tuck during the recoil of the Springalene, possibly in order to reach the tightest posture sooner than in the other somersaults.

The tuck kick-out has similar timing and posture to the double tuck on entry, showing it to be a reasonable skill to use as a warm-up for a double tuck. The extension from the tuck is sooner and performed more quickly than in the single tuck somersault, and then the tuck kick-out displays the greatest value of $\tau$; this is to be expected as the kick-out should have similar angular momentum to the double tuck, but the athlete only wants to complete a single somersault. Holding a tucked posture would result in over-rotation. The kick-out had the greatest value of $\tau$ on take-off, which is quite different to the double tuck; this may also be partly that the athlete is seeking to avoid over-rotation and so is not starting the somersault on the recoil of the Springalene as in the double tuck.

The tuck enters and exits more slowly than the kick-out. The slower entry is understandable since it is not a practice for a double tuck, which requires a fast entry. The slower exit probably reflects the need to hold a tighter posture slightly longer, since a shorter flight time is used and, based on the fact that the athlete used a shorter run-up, most likely less angular momentum is used.

There is no obvious pattern between the tuck, pike, and layout. Each reaches their minimum $\tau$ at different times. These three skills were well within the capacity of the athlete and the postures and timing may be dominated by comfort and personal preference rather than mechanics.

A quasi-rigid phase is most clearly seen in the double tuck, both in Figure 5-23 and when the original videos were watched. The other skills, stop in the tightest posture only momentarily so explorations of the quasi-rigid phase are not directly applicable. Nevertheless by understanding the consequences of holding a quasi-rigid phase it is possible to consider why this athlete chooses to show continual movement. To be recognised as the skill required to be performed they needed to “show”, not hold, the tuck, or pike posture; judging allows a long exit phase without deduction. If the tuck or the pike had been held then, due to the low relative difficulties of these postures, the athlete would have over-rotated and failed to land safely. In the layout the same principal applies, with the arms being used to adjust the value of $\tau$ to control rotation.

Let us now consider the angular momentum required to complete the whole skill, both when allowing for differences in flight time and when compressing the skill into the
same flight time. This was determined by the same process as in Section 3.10. Some ratios of angular momentum are given in Table 5-9 below.

**Table 5-9: Ratios of angular momentum required to achieve a skill.**

$H_L$ is the angular momentum required in posture L to complete 360° of somersault rotation. Thus $H/H_L$ gives the proportionate increase in angular momentum required over a layout if both are performed in the same time. The flight time adjustments have assumed that the postural changes could be performed proportionally faster. In reality, the entry and exit phases may be fixed in time and any reduction in flight time will reduce the time in the most compact posture and hence the relative difficulty will be even greater.

<table>
<thead>
<tr>
<th>H ratio</th>
<th>Tuck</th>
<th>Plke</th>
<th>Layout</th>
<th>Double tuck</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H/H_L$</td>
<td>0.82</td>
<td>0.85</td>
<td>1.11</td>
<td>1.03</td>
</tr>
<tr>
<td>$H/H_L$ when the flight time is the same as for the layout.</td>
<td>0.85</td>
<td>0.87</td>
<td>1.11</td>
<td>1.20</td>
</tr>
<tr>
<td>$H/ H_L$ when the flight time is the same as for the double tuck.</td>
<td>0.73</td>
<td>0.75</td>
<td>0.96</td>
<td>1.03</td>
</tr>
</tbody>
</table>

Comparing the second and third rows of the table shows there is a clear reduction in the angular momentum required and this has been achieved by increasing the flight time. Since the double somersault is the most difficult of the skills performed it is no wonder that to achieve the skill the athlete increased flight time when performing the double somersault.

The ratios for tuck are not half of the ratios for the double tuck. This is to be expected since the athlete did not hold the tuck position for a similar proportion of the somersault in the two cases. They found the tuck sufficiently easy and so elected to use it only for a small portion of the somersault. They could have generated less angular momentum but presumably personal preference, or general sporting expectations determined the choice.

### 5.2.9 Observations of the pure 4½ tuck somersault dive

The NSW Institute of Sport provided footage of two elite male springboard divers (height: 168 & 172 cm; mass 71.6 & 67.1kg) attempting a 4½ tuck somersault dive during an intensive training camp in Sydney. These two athletes were part of the group whose inertial properties were estimated in Chapter 4. The postures used by these divers while performing the 4½ tuck somersault dive were extracted, using the same method as in Section 5.2.8. To determine the values of $\tau = I_{yy}/I_{yy,L}$ the inertial property estimation method used was Hanavan3. This was the method selected for the example athlete in Section 3.10, whose performances were analysed in Section 5.2.8, and so for consistency was used for these two athletes as well. Reasonable fidelity when simulating the performance was observed: estimating the somersault angle error produced a maximum magnitude error of 27.35° for Athlete 1 and 45.31° for Athlete 2.
This is larger than in Section 3.10, and is probably due to the lower quality of the footage. Nevertheless patterns and differences in posture can be seen.

Athlete 1 completes less somersault rotation than Athlete 2 (~4.08 vs. ~4.14 somersaults); neither achieves a vertical entry. Athlete 1 also starts with a greater lean forward on take-off than athlete 2 (50.2° rather than 34.8°). The flight time of the two divers was very similar; the difference in achievement is thus the result of the angular momentum generated and the postures used by the athletes. Figure 5-25 plots $\tau$ over time for the two athletes performing their attempts at a 4½ tuck somersault dive.

From Figure 5-25, it is clear that Athlete 1 has a quasi-rigid phase, while Athlete 2 appears to be unable to hold steady in their desired posture. This does not mean that Athlete 1 has no postural change, only that the variations in the joint angles are not sufficiently large, or cancel each other out, so that $\tau$ remains a fairly constant value. For the majority of the time Athlete 2 uses postures with a higher $\tau$ value. It is not clear from this figure if the difference in $\tau$ is due to the athlete’s inertial properties or the postures they used. Figure 5-26 plots the value of $\tau$ for the four combinations of postures used and the inertial properties of the two athletes; i.e. Athlete 1’s inertial properties were applied to their own postures as well as Athlete 2’s postures and vice versa for Athlete 2. This allows the effect of posture choice and inertial properties to be separated.
The quasi-rigid phase

Figure 5-26: Combinations of posture and inertial properties
Applying the inertial property estimate for Athlete 1 to the joint angles extracted from the footage of Athlete 1 (Red dashed line) or Athlete 2 (Red solid line); and applying the inertial property estimate for Athlete 2 to the joint angles extracted from the footage of Athlete 1 (Blue dashed line) or Athlete 2 (Blue solid line). The key times are marked the same as Figure 5-25.

From Figure 5-26 it can then be seen that Athlete 1 has a natural advantage due to their inertial properties alone, since they have a lower $\tau$ value for both posture sequences. For Athlete 1’s posture sequence Athlete 1’s inertial properties result in a $\tau$ value 5.5-8.7% smaller than for Athlete 2 using the same posture sequence during the phase where the tuck is held.

At the times of approximately 0.25 and 0.55 seconds where Athlete 2’s posture sequence allows $\tau$ to be less than its value in Athlete 1’s posture sequence, the hip angles of the two postures are similar and the torso flexion is towards the maximum observed. Apart from these times Athlete 1 used slightly greater hip and knee flexion than Athlete 2; further the torso flexion, although similar when the angles of the two torso joints are summed was more evenly spread across the two joints for Athlete 1 rather than Athlete 2. Thus, a more ‘rounded back’ showing an even spread of flexion, and flexing at the hips and knees appears to be a ‘better’ tuck posture and should be encouraged.

The inertial properties of Athlete 1 and his ability to enter and hold a tighter posture than Athlete 2 suggests that Athlete 1 has greater potential to achieve the 4½ tuck somersault dive. However, he achieves less somersault than Athlete 2. Using the same simulation as in Section 5.2.8, the angular momentum possessed by each athlete was estimated; dividing this by $I_{y-L}$ gives a relative angular momentum of, 360º/sec for
Athlete 1, and 393°/sec for Athlete 2. Athlete 2 generates approximately 9% more relative angular momentum; Athlete 2 has overcome his natural disadvantage due to their inertial properties and his inability to hold as tight a tuck posture as Athlete 1, by generating greater angular momentum. If Athlete 2 could hold the same posture as Athlete 1 he could gain an extra approximately 26° of somersault rotation.

From this comparison the obvious area of focus for improvement for Athlete 1 is angular momentum generation, and for Athlete 2 is holding a steady and tighter tuck posture.

If the inertial property data for the athlete in Section 5.2.8 is applied to Athlete 1’s posture sequence they show a very similar pattern of $\tau$ to Athlete 1. Neither athlete would have a natural advantage over the other. The value of $\tau$ during the quasi-rigid phase is ~0.35; this is clearly lower than ~0.45 used by the athlete in Section 5.2.8 when performing a double somersault. Comparing joint angles the reduction in $\tau$ was due to slightly greater knee flexion, slightly greater flexion at the abdomen-chest joint, and clearly greater flexion at the pelvis-abdomen joint; in general terms a tighter posture.

### 5.2.10 Stability

A disturbance may alter the magnitude and direction of the angular momentum vector with respect to the world and/or the orientation of the athlete with respect to the angular momentum vector. An athlete may also slightly alter posture and hence alter the moments of inertia of the posture held. A change in the magnitude of $\mathbf{H}$, from Equation (4-3), will change the speed of the somersault only. Changes in the relative magnitudes of the moments of inertia, and/or a change in the orientation angles, $\phi$ and/or $\psi$, will alter the rotational behaviour in terms of $\psi$ and $\phi$ within the somersault, as described by Equation (4-4).

It was identified in Section 4.2.3 that rotating about the axis corresponding to the intermediate-valued moment of inertia may be considered unstable in a theoretical sense; that is, that large changes in twist will be observed for small changes in $\phi$ or $\psi$. For this instability to be of practical significance appreciable twist must be observed within a reasonably achievable number of somersaults. That is, if the athlete lands before any appreciable twist has occurred as a result of the instability, then the instability is practically insignificant.

Let us consider the value of $N_0$ for examples of a small disturbance resulting in $\phi_o = 89^\circ$ when $\psi_o = 0$ producing case 3, and $\psi_o = 1^\circ$ when $\phi_o = 90^\circ$ producing Case 4. Figure 5-27 plots $N_0$ for these two situations. The x and y axes are the ratios $I_{yy}/I_{zz}$ and $I_{xx}/I_{yy}$ respectively, and the domain chosen is that where $I_{yy}$ is the intermediate-valued moment.
of inertia and covers only values of interest considering the ratios for different postures in Section 5.1.2: $I_{yy}/I_{zz}$ from 1 to 30 and $I_{xx}/I_{yy}$ from 1 to 1.6. The scale is greater in Figure 5-27 b) than Figure 5-27 a), since with oscillating twist one cycle covers 358° twist: 179° in one direction and then return.

Figure 5-27: $N_\theta$ when a) $\phi_o = 89°$ and $\psi_o = 0$ b) $\psi_o = 1°$ and $\phi_o = 90°$

From the colour pattern in Figure 5-27 it is clear that twist will become apparent earlier ($N_\theta$ is smaller) for smaller values of $I_{yy}/I_{zz}$ and larger values of $I_{xx}/I_{yy}$. From the scales—taking into consideration the fact that b) is for oscillating twist—it is clear that a disturbance in $\phi$ will produce a half twist in fewer somersaults than will a disturbance in $\psi$. This means that the instability is more sensitive to disturbances in $\phi$ than $\psi$.

The laid out postures would be the least stable, and double somersaults in laid out positions are listed in the code of points in gymnastics and in diving so it would be reasonable to conclude that the instability may be of practical concern for some athletes. The disturbances used to plot Figure 5-27, are examples only, and the magnitude of the disturbances will affect the number of somersaults required, so this would need to be considered before deciding if the instability has a practical significance in a particular circumstance. Figure 5-28 plots $N_\theta$ as a function of $\phi_o$ when $\psi_o = 0$ and Figure 5-29, and Figure 5-30 plots $N_\theta$ as a function of $\psi_o$ when $\phi_o = 90°$, for the example athlete from Section 3.10 in postures L, A, and LAP.
Figure 5-28: \( N_\theta \) as a function of \( \phi_0 \) when \( \psi_0 = 0 \)
The curves are for the example athlete in Section 3.10 in postures L, A, and LAP. Posture L has \( I_{zy}/I_{zx} = 15.24 \), \( I_{xx}/I_{yy} = 1.03 \); posture A has \( I_{zy}/I_{zx} = 10.76 \), \( I_{xx}/I_{yy} = 1.01 \); and posture LAP has \( I_{zy}/I_{zx} = 6.64 \), \( I_{xx}/I_{yy} = 1.12 \).

Figure 5-29: \( N_\theta \) as a function of \( \psi_0 \) when \( \phi_0 = 90^\circ \)
The curves are for the example athlete in Section 3.10 in postures L and LAP.
The quasi-rigid phase

Figure 5-30: $N_\theta$ as a function of $\psi$, when $\phi_o = 90^\circ$

The curves are for the example athlete in Section 3.10 in posture A.

For this athlete in posture L disturbances in $\phi$ greater than one degree cause a half-twist in the first somersault. Even a very small disturbance of $0.2^\circ$ produces a half twist in a double layout somersault. For this athlete the instability would be practically significant. Using posture A or LAP does increase the number of somersaults required before a half twist is observed, and so using either posture A or LAP is a reasonable strategy for reducing the consequences of the instability. Using posture A would be the better choice since for small disturbances it increases $N_\theta$ more than posture LAP; for a disturbance less than $1.0^\circ$, $N_\theta$ is greater than two, and for a disturbance less than $0.2^\circ$ $N_\theta$ is greater than three-and-a-half. Based on Figure 5-30, a disturbance in $\psi$ is not as problematic as a disturbance in $\phi$, since a disturbance in $\psi$ of equal magnitude to the one in $\phi$ will require more somersaults before a half twist is observed. Posture A, of the three postures considered, is again the better option for reducing the consequences of the instability.

Referring to the twist produced following a disturbance as “instability” implies that the response is undesired. It should be remembered that the interpretation of what is undesirable depends on the required purpose of the motion, and is not an innate property of the postures. The best postures for maximising twist, and hence the more desirable when performing a twisting somersault, are the postures that are more ‘unstable’ and as a result should be avoided when seeking to perform a large number of pure somersaults. Of course if slight postural changes are allowed which would allow the consequences of instability to be controlled then these postures could be used. It is thus not necessary to compare postures here; the same rankings as for maximising twist may be used to compare instability.
5.3 The twisting somersault

To be awarded a twisting somersault skill an athlete must complete a multiple of a half twist, with a minimum of a half twist, whilst somersaulting. As a result, to perform a twisting somersault it is necessary that the posture used displays continuous twist. Postures will be compared based on the amount of somersault required to complete a half twist under the same initial conditions. The most efficient twist posture will be the one that requires the minimum somersault rotation per half twist; it may be considered efficient since the athlete can perform the skill of greatest difficulty for the least change in $\phi$. This section will analyse the sport-specific postures defined in Section 3.9, seeking to recommend twist postures to use individually or in combination as was proposed in Section 4.2.6. Oscillations in $\phi$ are an inevitable part of achieving a twisting somersault and simply have to be accepted. As a result the oscillations in $\phi$ will only be considered as part of transitioning between postures.

When exiting the twist phase of a somersault an athlete may perform actions to move $\phi$ back to $\pi/2$, or they may change posture to one that displays only small amounts of twist. This will be considered in Section 5.3.3.

Assuming that the medial and transverse moments of inertia are equal greatly simplifies the equations of motion, and so aids the estimation of the value of $\phi$ required to achieve various skills. In Section 4.2.5 the error due to this assumption was explored in terms of the ratios of the moments of inertia. In Section 5.3.6 the error for the sport specific postures of Section 3.9 will be considered.

5.3.1 Continuous twist or oscillating twist

As was seen in Section 4.2.3, the inertial properties of the posture along with $\psi_o$ determines the value of $\phi_{crit}$ which separates continuous ($\phi_o < \phi_{crit}$) and oscillating ($\phi_o > \phi_{crit}$) twist. When $I_{zz}$ is the intermediate-valued moment of inertia, no matter what the initial conditions, oscillating twist will occur: this may be viewed as $\phi_{crit} = 0$. When $I_{yy}$ or $I_{xx}$ is the intermediate-valued moment of inertia then $\phi_{crit} = \phi_{crit1}$ or $\phi_{crit2}$, respectively, from Equation (4-7 a and b). Figure 5-31 plots the value of $\phi_{crit}$ when $\psi_o = 0$ as a function of the ratios, $I_{xx}/I_{yy}$ and $I_{yy}/I_{zz}$. Then in Figure 5-32 box plots are given showing the spread across athletes of the values of $\phi_{Crit}$ for the 20 postures defined in Section 3.9. Even though $\psi_o$ will not always be zero, the relationship between $\phi$ and $\psi$, given by Equation (4-4), may be used to determine the value of $\phi$ when $\psi$ does equal zero and compare it to $\phi_{crit}$. If $\psi$ cannot equal zero then oscillating twist will obviously occur.
From Figure 5-31 the regions where only continuous \( (\phi_{crit} = 90^\circ) \) and only oscillating \( (\phi_{crit} = 0^\circ) \) twist will occur are very clear. When \( I_{yy} = I_{zz} \) there is a discontinuity causing \( \phi_{crit} \) to go from \( 90^\circ \) to \( 0^\circ \); this follows from the fact that when \( I_{yy} \) is the intermediate-valued moment of inertia continuous twist results and when \( I_{zz} \) is the intermediate-valued moment of inertia oscillating twist results. This discontinuity may be viewed as Case 6 in Section 4.2.3 growing in twist oscillations as \( I_{zz} \) approaches \( I_{yy} \), until finally \( \psi \) can reach \( \pi/2 \) and thereby allow a transition to continuous twist. Also from Figure 5-31 a steady transition between a \( \phi_{crit} \) value of \( 90^\circ \) and \( 0^\circ \) may be seen through the region where \( I_{xx} \) is the intermediate-valued moment of inertia. As \( I_{xx} \) increases with respect to \( I_{yy} \), the oscillations in \( \phi \), seen in Case 7 in Section 4.2.3, grow and this means that \( \phi_{crit} \) must decrease by the same number of degrees. It is of particular interest to observe that the transition between when \( \phi_{crit} \) equals \( 90^\circ \) and when it equals \( 0^\circ \), through the region where \( I_{xx} \) is the intermediate-valued moment of inertia, is quicker when \( I_{zz} \) is closer to \( I_{yy} \). This shows that when the three moments of inertia are close to each other, small changes in any of them will result in greater changes in the rotational behaviour than when at least one is quite different.

Figure 5-32 confirms that the postures LAU through to LHF could reasonably be classed as continuous twist postures. Posture Pu is also a continuous twist posture for the majority of ‘athletes’, although some require more tilt to overcome the initial tendency to display oscillating twist. This is as expected since these postures showed their inertial property ratios scattered around \( I_{xx}= I_{yy} \) and with larger \( I_{yy}/I_{zz} \) ratios. Postures with increasing hip flexion, but without knee flexion, show decreasing values
of $\phi_{\text{Crit}}$, until posture TP, which shows all positive values of $\phi_{\text{Crit}}$. Posture TP has a large range of $\phi_{\text{Crit}}$, because all of its moments of inertia are similar, with a different moment of inertia being the intermediate-valued one for different ‘athletes’.

![Figure 5-32: Box plots of $\phi_{\text{Crit}}$ when $\psi_o = 0$ for the key somersault postures.](image)

The key somersault postures were defined in Section 3.9. The maximum value $\phi_{\text{Crit}}$ can be is 90° and the minimum is 0°, hence this is the limit of the values given on the y-axis. Postures LAU, LAP, 1U1DBLF, L, and LHF all have $\phi_{\text{Crit}} = 90°$ and so no boxes are visible.

Posture P would be a good general choice when seeking to prevent continuous twist since for approximately 83% of ‘athletes’ $\phi_{\text{Crit}} = 0°$ and for approximately 86% of ‘athletes’ $\phi_{\text{Crit}} < 70°$. However, when $\phi_o > 80°$ posture OP would be sufficient to prevent further continuous twist. The postures with flexion at the knees and hips show a large range of $\phi_{\text{Crit}}$ values. This is due to the broad scatter of inertial property ratios for different ‘athletes’, and the relative closeness of the inertial properties to each other. Postures with knee and hip flexion are thus not easy to categorise as displaying oscillating or continuous twist; however, the moderate difference of $\phi_{\text{Crit}}$ from 90° for the majority of ‘athletes’ in postures T, FT and BT means these would not be good postures to choose when intending to twist but could be used to prevent further twist if $\phi_o > 80°$ for the majority of ‘athletes’.

### 5.3.2 The amount of somersault required per half twist

The most efficient posture for twisting somersaults will be the one that displays continuous twist and has the minimum value of $N_0$. In Section 4.2.4, it was shown that increasing the ratio $I_{yy}/I_{zz}$ would decrease $N_0$ and, as $\phi_o$ decreases, it would become more important than the ratio $I_{xx}/I_{yy}$.

Figure 5-33 and Figure 5-34 are scatter plots showing the relationship between $N_0$ from Section 4.2.4, and the ratio $I_{yy}/I_{zz}$ for the common twist postures.
The quasi-rigid phase

Figure 5-33: Relationship between \( N_\theta \) and \( I_{yy}/I_{zz} \) when \( \phi_o = 85^o \) and \( \psi_o = 0 \).
The markers on the x-axis are those that display oscillating twist. Each symbol represents an ‘athlete’/posture combination.

Figure 5-34: Relationship between \( N_\theta \) and \( I_{yy}/I_{zz} \) when \( \phi_o = 70^o \) and \( \psi_o = 0 \)

There is a clear hyperbolic relationship between \( N_\theta \) and \( I_{yy}/I_{zz} \). The reduction in \( \phi_o \) between Figure 5-33 and Figure 5-34 has resulted in a much reduced scatter, as \( I_{yy}/I_{zz} \) becomes the key inertial parameter. It is thus reasonable to seek postures with smaller values of \( I_{yy}/I_{zz} \). Despite the strong relationship between \( N_\theta \) and \( I_{yy}/I_{zz} \), from the scatter plots, how the postures would be ordered based on their values of \( N_\theta \) is not clear, since there is a considerable spread of the values of \( I_{yy}/I_{zz} \) across the ‘athletes’. Before recommending a posture it is necessary to determine the rank of the postures by the value of \( N_\theta \) for each ‘athlete’. Then before ranking postures it is necessary to consider the most likely values of \( \phi_o \) (when \( \psi_o = 0 \)) since, different postures will see greater or
lesser change in $N_\theta$ with the change in initial conditions. Considering published observed values of tilt and the published predicted values of tilt for simulated actions it is reasonable to focus on the range $\phi_o$ from $90^\circ$ down to $70^\circ$ when $\psi_o = 0$.

The postures shown are those that predominantly display continuous twist for the example athlete in Section 3.10. As $\phi_o$ decreases, so will $N_\theta$: a plot of $N_\theta$ against $\phi_o$ will have a vertical asymptote at $\phi_o = 90^\circ$, and will approach zero as $\phi_o$ approaches zero. The curve is monotonically increasing, although the gradients of the tangents to the curve at each point will vary according to the posture; curves for different postures will either never cross or cross only once. Figure 5-35 illustrates the curves for the common twist postures for one example athlete, some curves cross and others do not cross. Based on Figure 5-35 the most efficient twist posture would always be posture L. Postures JL and EP are the least efficient twist postures and do not even display continuous twist for all values of $\phi_o$. Of those postures that do display continuous twist for all values of $\phi_o$ the least efficient would be posture A when $\phi_o$ was greater than $\sim 87^\circ$ and posture LAP when $\phi_o$ was less than $\sim 87^\circ$.

Rackham (1970) suggests $\sim 10^\circ$ of tilt may be achieved by a kick-out action and so a value of $\phi_o$ of $80^\circ$ could potentially be achieved. Frohlich (1979) predicts tilts of $10.86^\circ$ and $19.96^\circ$ for his simulated arm action, and so a value of $\phi_o$ of $79.14^\circ$ and $70.04^\circ$ could potentially be achieved. Sanders (1995) observed $5$ to $14^\circ$ and $12$ to $23^\circ$ of tilt during the airborne phase of a $\frac{1}{2}$-twist and a $\frac{1}{2}$-twist forward somersault respectively; these tilt values were determined using anatomical axes not principal axes and so only suggest the general range of the complement of $\phi$. Yeadon has made numerous observations (Yeadon, 1989; Yeadon, et al., 1990; Yeadon, 1993d; Yeadon, 1994; Yeadon, 1997; Yeadon & Kerwin, 1999) where he reports tilt values, with almost all values giving $\phi_o$ greater than or equal to $70^\circ$. 

---

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---

**Figure 5-35: Example of $N_\theta$ as a function of $\phi_o$, when $\psi_o = 0$**

The postures shown are those that predominantly display continuous twist for the example athlete in Section 3.10. The curves of postures JL and EP stop because they will display oscillating twist at higher values of $\phi_o$. 
Ranking the postures for each ‘athlete’ by $N_\theta$ when $\psi_\theta = 0$ and $\phi_\theta = 85^\circ$, $80^\circ$, $75^\circ$, and $70^\circ$ shows that the order of the postures does change as $\phi_\theta$ decreases, as does the spread of the ranks for any particular posture across ‘athletes’. Nevertheless posture L was a clear choice for the most suitable twist posture, since it ranked the highest when $\psi_\theta = 0$ and $\phi_\theta = 85^\circ$ and $70^\circ$ for 88% and 79% of ‘athletes’ respectively. For any particular ‘athlete’ when posture L was not ranked the highest posture LAU, 1U1DB, 1U1DBLF or Pu was ranked the highest. Posture EP was the least suitable posture since it ranked the lowest of the common twist postures when $\psi_\theta = 0$ and $\phi_\theta = 85^\circ$ and $70^\circ$, for 100% and 63% of ‘athletes’ respectively. When posture EP was not ranked the lowest posture JL or LAP was ranked the lowest.

Table 5-10: Proportion of ‘athletes’ that can achieve each skill

The twists column gives the number of twists in a somersault and hence defines the skill that may be performed. To make the table easier to read, the cell is blank when the proportion is zero. The largest value in each row has the cell coloured in light green. Further the column order is by the median value of $N_\theta$ when $\phi_\theta = 85^\circ$ rather than the proportion of ‘athletes’ that can achieve a skill.

<table>
<thead>
<tr>
<th>$\phi_\theta$</th>
<th>Twists</th>
<th>L</th>
<th>LAU</th>
<th>1U1DB</th>
<th>1U1DB</th>
<th>LHF</th>
<th>1UID</th>
<th>Pu</th>
<th>HVL</th>
<th>V</th>
<th>A</th>
<th>LAP</th>
<th>JL</th>
<th>EP</th>
</tr>
</thead>
<tbody>
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<td>0.99</td>
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<td>0.05</td>
<td>0.03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3/2</td>
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<td>0.05</td>
<td>0.02</td>
<td>0.00</td>
<td>0.02</td>
<td>0.01</td>
<td>0.11</td>
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<td>0.67</td>
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<td>0.30</td>
<td>0.05</td>
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</table>
Since being able to achieve more difficult skills is the goal of good posture selection, then the proportion of ‘athletes’ that may achieve various skills in each posture can be used to compare postures. From Table 5-10 it is clear that posture L is the most suitable posture for performing twisting somersaults since it allows the greatest proportion of ‘athletes’ to achieve each of the six skills listed under four different initial conditions, in all but two of the skills listed. For these two skills posture L is still within 1% of the greatest proportion of ‘athletes’. The advantage that posture L has over the other postures decreases as $\phi_o$ decreases.

It is interesting to observe from Table 5-10 that the position of the arms, even when held tight to the body, makes quite a difference to the proportion of ‘athletes’ able to complete a skill. It is of particular interest to note that posture LHF is considerably less effective than posture L but postures 1U1DB LF and 1U1DB are quite similar to each other.

There is a large decrease in the proportion of ‘athletes’ predicted to achieve each skill in postures LAP, JL, or EP rather than in posture L. These postures would be the least suitable twist postures, but may be used in situations where the athlete may wish to slow the twist. In the same way that posture LAP was suggested for use in flying somersaults (Section 5.2.4), it could be used to produce a phase of very little twist, which could be followed by fast twist when the arms are adducted. Alternatively, fast twist could follow a phase of slow twist when extending from postures EP or JL; in this way any contact twist would not be easily seen. Further, if the twist cannot be easily removed by aerial actions, posture LAP would slow the twist to provide the appearance of at least some twist removal. Similarly postures JL and EP could be used. Using any of these three postures would slow twist, while still allowing the posture to be described, for judging purposes, as ‘laid-out’.

The abducted arms which increase $I_{zz}$ in posture LAP are obvious to the naked eye and so it is not surprising that posture LAP is not an effective twist posture. The flexion of postures EP and JL are less dramatic to the eye, and so it could easily be overlooked that these postures are not efficient twist postures, and are in fact even less efficient than posture LAP. Presumably it is the larger mass of the torso and legs, compared to the arms that means that the flexion in postures EP and JL would slow the twist more than posture LAP.

Sanders (1995) observed trampolists of varying ages and abilities performing $\frac{1}{2}$ and $1\frac{1}{2}$ twisting somersaults. The tilt values reported were measured using anatomical features. Although he did not estimate the principal directions, the tilt values that he reports may be used as a guide to what values of $\phi_o$ could be expected. Based on his
results $\phi_o$ of around 76 to 85 degrees for a $\frac{1}{2}$ twisting forward somersault, and $\phi_o$ of around 75 to 87 degrees for a $1\frac{1}{2}$ twisting forward somersault could reasonably be expected. This observation is in agreement with what would be expected from Table 5-10 to achieve these skills.

Twist initiation actions may restrict the postures that may be used during twisting. In addition, to achieve the somersault required, an athlete may be forced to choose a posture with a lower relative difficulty and accept any reduction in twist efficiency. Section 5.3.5 will focus on some specific posture comparisons of sporting interest, with reference back to Table 5-10. The results of these posture comparisons will guide the choice of end postures which will be considered in Chapter 6 where twist initiation actions are compared. Further in Chapter 6 the skills that may be achieved for various twist initiation and posture combinations will be determined; the effect of gender and squad will be considered then.

5.3.3 ‘Ending’ the twist by changing posture

Prior to landing a twisting somersault the athlete should, and the better athlete will, have a phase where the twist “appears” to stop. The twist does not need to stop entirely; any further twist simply needs to be small in magnitude. Rather than returning $\phi$ to $\pi/2$, the twist may appear to ‘stop’ following a change in posture to one that displays a much slower twist rate, or one that displays oscillating twist where the maximum twist achieved is small. When performing an odd multiple of a half-twist, aerial actions used for twist initiation cannot be reversed to return $\phi$ to $\pi/2$, and so changing posture to one that displays oscillating twist or considerably slows the twist is a reasonable method by which to ‘stop’ twist in preparation for landing (Section 4.2.6.2).

In Section 5.3.2, postures LAP, JL and EP were identified as postures that could be used to slow twist. In this section the postures which show oscillating twist for values of $\phi_o$ from $90^\circ$ to $70^\circ$ when $\psi_o = 0$ for at least some ‘athletes’, will be considered. It is necessary to consider both the maximum twist that may be achieved in an oscillation and the amount of somersault required for an oscillation to occur. More suitable postures will be those which have small amplitudes of twist oscillation or require large amounts of somersault for an oscillation, so that no appreciable twist occurs within the remainder of the skill. The magnitude of twist oscillations is not addressed in the gymnastics or diving judging criteria, and so what amount is acceptable is not known. Based on discussions with judges and coaches, postures sought for use in ‘ending’ twist were those which display less than $20^\circ$ of twist, or less than an average of $10^\circ$ of twist per somersault since following the desired amount of twist no more than two somersaults are expected. Figure 5-36 and Figure 5-37 are scatter plots showing the
maximum twist and the number of somersaults required to reach this maximum twist value across ‘athlete’/posture combinations of interest. As $\phi_o$ decreases the maximum twist reached increases for all the postures. For those postures with $I_{xx}$ (or in a few cases $I_{yy}$) as the intermediate-valued moment of inertia the number of somersaults required to reach the maximum decreases. As a result the number of ‘athlete’/posture combinations that are no longer in the acceptable region increases between the cases where $\phi_o = 85^\circ$ and $\phi_o = 70^\circ$.

![Figure 5-36: Maximum twist vs. $N\theta/4$, $\phi_o = 85^\circ$, $\psi_o = 0$](image)

The amount of somersault required for $\psi$ to move from zero to its maximum value is $N\theta/4$. Postures plotted are those displaying oscillating twist for at least some ‘athletes’ when $\phi_o$ is in the range 90° down to 70° and $\psi_o = 0$. The black lines “Max20” and “Avg 10 Deg/Salto” are plotted to aid visualisation of the regions of acceptable and unacceptable amounts of twist.
The quasi-rigid phase

Figure 5-37: Maximum twist vs. $N\theta/4$, $\phi_o = 70^\circ$, $\psi_o = 0$

The amount of somersault required for $\nu$ to move from zero to its maximum value is $N\theta/4$. Postures plotted are those displaying oscillating twist for at least some ‘athletes’ when $\phi_o$ is in the range 90° down to 70° and $\psi_o = 0$. The black lines “Max20” and “Avg 10 Deg/Salto” are plotted to aid visualisation of the regions of acceptable and unacceptable amounts of twist.

From Figure 5-36 and Figure 5-37 it is clear that there is considerable scatter due to the variation between ‘athletes’ for each posture, rather than a clear clustering of the postures, although, postures P and FT seem to show lower values of maximum twist than the other postures. Yeadon predicts a maximum twist angle of 13° when $\phi_o = 80^\circ$ and $\psi_o = 0$ (Yeadon, 1993a) for a piked posture similar to OP and 9° when $\phi_o = 85^\circ$ and $\psi_o = 0$ (Yeadon, 1997b) for a piked posture: these values fall in the upper quartile of those observed in the current investigation for posture P. Thus, posture P is very useful for preventing continuous twist. Table 5-11 gives the proportions of ‘athletes’ that display acceptable amounts of twist for each of the postures of interest for values of $\phi_o = 85^\circ$, 80°, 75°, and 70° when $\psi_o = 0$.

<table>
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<tr>
<th>$\phi_o$</th>
<th>OP</th>
<th>P</th>
<th>T</th>
<th>BT</th>
<th>TP</th>
<th>FT</th>
<th>TT</th>
<th>CT</th>
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<tr>
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<td>0.15</td>
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<tr>
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<td>0.43</td>
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<td>0.78</td>
<td>0.15</td>
<td>0.08</td>
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The low proportion of ‘athletes’ in Table 5-11 that display acceptable amounts of twist in the tight postures TP, TT, and CT makes them poor choices for ‘ending’ twist; this is
not surprising since, in these postures a greater portion of ‘athletes’ have $I_{yy}$ as the intermediate-valued moment of inertia.

Posture P is the most suitable posture for ‘ending’ continuous twist when $\phi_o = 70^\circ$ and $\psi_o = 0^\circ$, and reasonable for the lower values of $\phi_o$. This is not surprising as P showed the largest portion of ‘athletes’ with $I_{zz}$ as the intermediate-valued moment of inertia (Table 5-1). The 84% of ‘athletes’ for whom posture P was predicted as being able to stop twist when $\phi_o = 70^\circ$ were predicted as being able to perform between 3/2 and 6/1 twists in a somersault in posture L. Postures OP and T were unsuitable for all ‘athletes’ for whom posture P was unsuitable when $\phi_o = 70^\circ$; these ‘athletes’ were predicted as being able to perform between 1/1 and a 7/2 twists in a somersault.

‘Athletes’ for whom posture P was predicted to show continuous, rather than oscillating twist, for some values of $\phi_o$, showed oscillating twist and lower maximum twist values for more of the considered initial values of $\phi_o$ in posture OP than posture P: such ‘athletes’ should simply use posture OP to stop continuous twist. A smaller portion of ‘athletes’ displayed an acceptable amount of twist in posture TP rather than posture P; for the majority of ‘athletes’ posture TP actually displayed continuous twist.

In the absence of knowledge of an athletes actual inertial properties a coach could thus suggest using posture P to end twist and then, if too much twist occurs in posture P, suggest using a more open pike posture. The suitability of posture P for “ending” twist supports the instructions given to divers to pike, after completing the desired number of twists, with the stated reason that it slows twist (Batterman, 1968). Further it may explain why Sanders (1995; 1999) observed that, at the same time as hip flexion occurred, continuous twist ceased and only a slow rate of twist remained. In particular Sanders (1995) observed that trampolinists used greater hip flexion when ending a 1½ twisting forward somersault than a ½ twisting forward somersault. Further, more-skilled divers show greater post-twist hip flexion (Sanders, 1999) with less twist continuing in the pike. These observations are in line with Table 5-11 which shows that posture P is more effective than posture OP for ‘ending’ twist for a greater proportion of ‘athletes’ as $\phi_o$ decreases.

Posture FT is the most suitable of the tucked postures T, BT, and FT. Even though, to the naked eye, these postures do not seem very different for the lower values of $\phi_o$, the difference in proportions of ‘athletes’ showing acceptable twist is considerable. If a tuck-based posture may be used to ‘end’ twist a coach could allow the athlete to start with whichever tuck posture is comfortable and then if that posture shows too much extra twist, encourage the athlete to increase the torso flexion, whilst reducing hip flexion to move towards a FT posture rather than a BT or T posture.
The posture FT has a higher proportion of ‘athletes’ predicted to display acceptable twist than does the posture P, for $\phi_o = 85^\circ$, $80^\circ$, and $70^\circ$; however, in posture P, 79% of ‘athletes’ were predicted to show smaller values of extra twist than when in FT. It is thus reasonable for a coach to have their ‘athletes’ start by using posture P and then, if the extra twist is not acceptable, try postures OP or FT instead. For different ‘athletes’, sometimes posture OP would be acceptable, while at other times posture FT would be acceptable. Thus, unless the athlete’s inertial properties are known, the athlete would need to try both postures FT and OP; which posture is tried first is inevitably driven by the angular momentum requirements, since posture FT has a much lower relative difficulty than posture OP.

5.3.3.1 Opening prior to landing

Prior to landing/entry it is expected that a gymnast will open out to a laid-out posture to prepare for landing, while a diver will open out to posture LAU to prepare for entry. Ideally the athlete will land/enter vertically and with minimal deviation from the specified number of twists. Since $I_{yy}$ is the intermediate-valued moment of inertia for the vast majority of the laid-out postures (Table 5-1), then opening out again will result in the return of continuous twist (Case 3 in Section 4.2.3), or if $\phi$ did reach $90^\circ$, the presence of twist will result in oscillations about $\psi = 90^\circ$ (Case 4 in Section 4.2.3).

If an ‘athlete’ was using any of the postures OP, P, T, BT, or FT, which have $I_{xx}$ or $I_{zz}$ as the intermediate-valued moment of inertia, then had the ‘athlete’ entered these postures when $\psi = 0$, the values of $\phi$ and $\psi$ will increase from this point; unless $\phi$ has reached its supplementary value or returned to $\phi_o$, then upon opening out $\phi_o$ will be closer to $90^\circ$ and $\psi_o$ will be greater than zero. If the posture upon opening has $I_{yy}$ as the intermediate-valued moment of inertia then values of $\phi_o$ closer to $90^\circ$ and $\psi_o$ greater than zero, will result in a reduced twist rate within the somersault than prior to entering the posture used to ‘end’ twist (Section 4.2.3).

Further, to minimise any twist when opening out, an ‘athlete’ should open out with as little remaining somersault as possible, so any twist that occurs is minimal. The key is minimising the amount of remaining somersault, not necessarily the remaining time. Longer phases in a laid-out posture prior to entry are associated with better diving performances (Sanders, 1999), but very little somersault need actually occur. Changing from posture P to posture LAU would slow the somersault. The lower the relative difficulty value of posture P the greater the difference in speed between postures P and LAU, and so the less somersault that would occur in posture LAU. ‘Athletes’ with a lower relative difficulty value in P would thus naturally have long pre-entry phases with very little somersault.
When performing multiple somersaults with twist only in the first somersault the piked or tucked posture is typically used for at least one somersault. Generally pike showed the lowest number of somersaults required to reach the maximum twist value (and $\phi = 90^\circ$); this number is greater than 0.29 of a somersault for all ‘athletes’, and greater than 0.5 of a somersault for ~69–72% of the ‘athletes’. In posture FT the number of somersaults required was greater than 0.29 of a somersault for all ‘athletes’, and greater than 0.5 somersaults for 84-86% of ‘athletes’. Thus for the majority of ‘athletes’ $\phi$ would not have reached its supplementary value in one somersault. Hence a slower twist rate is expected on opening out than prior to entering posture P.

Based on the above discussion, twist build-up upon opening out again is not an expected practical concern, and it would appear as though the entry to the piked or tucked based postures has ‘removed’ the twist.

### 5.3.4 Relative difficulty

The relative difficulty of a continuously-twisting somersault compared to a pure somersault in the same posture was introduced in Section 4.2.7. Equations were given for the relative difficulty during the twisting phase. Relative difficulty describes what additional angular momentum is required as a result of twist being introduced to a somersault, if the same posture was used.

Considering the same postures as in Table 5-10, across all ‘athletes’, the relative difficulty increased as $\phi_o$ decreased ($\psi_o = 0$). The relative difficulty when $\phi_o = 70^\circ$ may be used as an upper bound to the relative difficulty expected, and it is reasonable to expect that an ‘athlete’ would achieve a value of $\phi_o$ which is much less than $70^\circ$. Table 5-12 gives the proportion of ‘athletes’, for a series of upper bounds of relative difficulty, when twist is introduced into the somersault.

<table>
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<th>LUIDBLF</th>
<th>LUIDBLHF</th>
<th>LUIDPu</th>
<th>HVLV</th>
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<th>LAP</th>
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<td>0.98</td>
<td>0.03</td>
</tr>
<tr>
<td>1.04</td>
<td>0.98</td>
<td>0.99</td>
<td>0.98</td>
<td>1.00</td>
<td>0.80</td>
<td>1.00</td>
<td>1.00</td>
<td>0.87</td>
<td>1.00</td>
<td>0.09</td>
</tr>
<tr>
<td>1.05</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.98</td>
<td>1.00</td>
<td>1.00</td>
<td>0.97</td>
<td>1.00</td>
<td>0.28</td>
</tr>
</tbody>
</table>

From Table 5-12 it is clear that, for the vast majority of ‘athletes’, introducing twist increases the angular momentum required to complete the skill. This is as expected since $I_{xx} > I_{yy}$ for the majority of ‘athletes’ in these twist postures. As a result, the...
common belief that adding twist increases somersault speed is incorrect. Of course if the athlete alters the postures they use or the angular momentum they generate, then the somersault may increase in speed.

The relative difficulty obtained from the equations in Section 4.2.7 may be multiplied by \( \tau \), from Section 5.2, for the posture of interest, to give the relative difficulty of the posture when twisting in that posture as compared to a pure somersault in posture L. Table 5-13 gives the proportion of ‘athletes’ who have a lower relative difficulty than the values given in the first column.

<table>
<thead>
<tr>
<th>Relative difficulty less than</th>
<th>L</th>
<th>LAU</th>
<th>1U1DB</th>
<th>LF</th>
<th>1U1DB</th>
<th>LHF</th>
<th>1U1D</th>
<th>Pu</th>
<th>HVLVA</th>
<th>LAP</th>
<th>JL</th>
<th>EP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.94</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.33</td>
<td>0.00</td>
<td>0.66</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.97</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.59</td>
<td>0.00</td>
<td>0.92</td>
<td>0.00</td>
<td>0.87</td>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1</td>
<td>0.00</td>
<td>0.00</td>
<td>0.23</td>
<td>0.18</td>
<td>0.95</td>
<td>0.03</td>
<td>0.99</td>
<td>0.02</td>
<td>1.00</td>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1.03</td>
<td>0.93</td>
<td>0.00</td>
<td>0.47</td>
<td>0.31</td>
<td>1.00</td>
<td>0.21</td>
<td>1.00</td>
<td>0.16</td>
<td>1.00</td>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1.06</td>
<td>1.00</td>
<td>0.02</td>
<td>0.83</td>
<td>0.55</td>
<td>1.00</td>
<td>0.33</td>
<td>1.00</td>
<td>0.30</td>
<td>1.00</td>
<td>0.01</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 5-12 and Table 5-13 will be referred to when comparing specific postures in Section 5.3.5; for now some general observations may be made.

When twisting in posture L there is only a small increase in the relative difficulty; the additional angular momentum required is not expected to be of concern, and so it would still be the recommended twist posture. Using posture LAU dramatically increases the relative difficulty and so even though it is the second most effective twist posture, the increase in relative difficulty is likely to be a major limitation. There are no general sport factors that would make posture LAU desirable over posture L, and so posture LAU is not recommended as the posture to use during the twisting phase of a twisting somersault.

Posture 1U1D has a much increased relative difficulty, but flexing at the elbows to end in 1U1DB lowers the relative difficulty and from Table 5-10 also increase the twist efficiency. Thus posture 1U1DB is recommended over posture 1U1D.

Postures LAP and HVLV, result in a considerable increase in the relative difficulty over posture L. Since these postures are also not particularly efficient twist postures they are not recommended as postures to use during the phase where the majority of the twist needs to be completed. Using them would reduce the amount of twist that could be completed and require the athlete to generate greater angular momentum. These postures may still be used at the end of a skill to slow a twist. The considerable increase
in relative difficulty may also be advantageous at this time, slowing the somersault and hence allowing a clear phase where the twist appears to have ended and little rotation occurs as the athlete stretches to the floor for landing or pool for entry; that is, they “float” down.

The advantage of using the posture Pu to reduce the angular momentum requirements of a twisting somersault is quite clear. Only postures JL and EP show a larger reduction in angular momentum required; however, these are far less efficient twist postures. Pu is thus a very useful posture, for the situations where angular momentum is more of a limiting factor than is reducing \( \phi_o \).

The relative difficulty presented here is only for the twisting phase of the twisting somersault. It is also possible to reduce the angular momentum requirements of the whole skill, by using faster twist—achieved by a smaller value of \( \phi_o \)—so that a smaller portion of the somersault is used. The athlete may then enter a much more ‘compact’ posture to speed up the somersault. In practice a coach must balance a number of factors and make a judgement based on phases within a skill where their athlete struggles: reducing \( \phi_o \), generating angular momentum, or changing postures.

5.3.5 Specific postures

The twist initiation actions, the expectations within a particular sport, and the angular momentum requirements may restrict the postures under consideration for use when twisting. This section considers a few postures individually and compares some specific postures of interest to address some of these restrictions. When comparing postures the hypothesis in Section 4.2.6, where \( \phi_o \) could be reduced further by changing between postures at each \( \frac{1}{4} \) twist position is also discussed.

5.3.5.1 The 1U1D, 1U1DB, and 1U1DB LF postures

When considering the end position of twist initiation actions involving asymmetrical arm actions (Section 6.1), postures 1U1D, 1U1DB and 1U1DB LF are all more likely end postures than posture L. In Chapter 6 the change in \( \phi \) produced by the slightly different asymmetric arm actions required to end in each posture will be considered. Here the postures themselves are compared under the same initial conditions.

From Table 5-10 posture 1U1D shows lower proportions of ‘athletes’ able to perform each skill under the same initial conditions as compared to postures 1U1DBLF and 1U1DB; postures 1U1DB LF and 1U1DB allow similar proportions, with posture 1U1DB LF allowing a greater proportion of ‘athletes’, to perform the skills with less twist and posture 1U1DB showing a greater proportion of ‘athletes’ able to perform the skills with more twist.
When considering the difference in $N_\theta$, posture 1U1D requires a greater value of $N_\theta$ than does posture 1U1DB for all but ~5–7% of ‘athletes’ when $\phi_o$ is between 70° and 90°. The advantage of posture 1U1DB decreases as $\phi_o$ decreases. The advantage is small when considering a half twist—a median difference in $N_\theta$ of 0.01–0.03 somersaults for $\phi_o$ between 70° and 90°—although as the number of twists increases this advantage multiplies. It is thus reasonable for a coach to recommend using posture 1U1DB over posture 1U1D, provided that it does not inhibit the twist initiation method. It is thus worthwhile to explore twist initiation actions ending in postures 1U1D and 1U1DB; they are considered in Sections 6.1 and 6.2, along with the effect of the posture. Posture 1U1D shows a greater amplitude of oscillation in $\phi$ than does posture 1U1DB, and $I_{yy}$ is the intermediate-valued moment of inertia, for ~80% of ‘athletes’. If an athlete holds posture 1U1D until the quarter-twist position, and then flexes at the elbows, this will reduce the value of $\phi$, when $\psi = 0$, by the difference in the oscillations and will hence produce a faster twist as was discussed in Section 4.2.6. The gain would be less than 0.5° with a median of 0.19°, when $\phi_o$ started at 85°; the gain would be reduced to less than 0.13° with a median of 0.05°, when $\phi_o$ started at 70°. For these two postures this potential advantage may be considered inconsequential, and so this strategy can be ignored.

Considering the difference in $N_\theta$ between postures 1U1DB and 1U1DB LF reveals that the latter requires less somersault for ~60–68% of ‘athletes’ for $\phi_o$ between 70° and 85°. The difference in $N_\theta$ between the two postures decreases as $\phi_o$ decreases. The spread of this difference across the ‘athletes’ is larger than the difference between postures 1U1DB and 1U1D. Posture 1U1DB LF also has a lower relative difficulty when twisting, as compared to a pure somersault in posture L. This can be seen in Table 5-13. Posture 1U1DB LF is thus expected to reduce the angular momentum requirements of the twisting somersault. Posture 1U1DB LF shows greater oscillations than does posture 1U1DB and so if an athlete is in the posture 1U1DB LF at the zero twist position they should not change to posture 1U1DB part way through the twist, since it will shift the $\phi$-$\psi$ curve (Section 4.2.3) upwards and hence produce a slower twist.

### 5.3.5.2 Posture LAP

In Section 5.3.2 it was identified that posture LAP could be used to slow twist, either in preparation for landing, or to create the appearance that contact twist had not been used. In addition, posture LAP was suggested by Yeadon (1993b) as the posture to be held until a quarter-twist, followed by a change to posture L, so as to shift the $\phi$-$\psi$ curve downwards and thereby increase the rate of twist. From Table 5-1 it can be seen that $I_{yy}$ is the intermediate-valued moment of inertia for postures L and LAP across all the ‘athletes’. Thus, from Section 4.2.6.1, the change in $\phi$ that would be achieved is the
difference between the amplitudes of oscillation of postures LAP and L. Table 5-14 gives descriptive statistics for this difference. The difference decreases with $\phi_o$, but even at 70° it is still practically significant for the majority of ‘athletes’.

Table 5-14: The difference in the amplitude of oscillation in $\phi$ between postures LAP & L (degrees)

<table>
<thead>
<tr>
<th>$\phi_o$</th>
<th>85°</th>
<th>80°</th>
<th>75°</th>
<th>70°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>14.4</td>
<td>11.5</td>
<td>9.3</td>
<td>7.6</td>
</tr>
<tr>
<td>Upper Quartile</td>
<td>5.0</td>
<td>3.5</td>
<td>2.6</td>
<td>2.0</td>
</tr>
<tr>
<td>Median</td>
<td>4.1</td>
<td>2.7</td>
<td>2.0</td>
<td>1.5</td>
</tr>
<tr>
<td>Lower Quartile</td>
<td>3.1</td>
<td>2.1</td>
<td>1.5</td>
<td>1.1</td>
</tr>
<tr>
<td>Minimum</td>
<td>1.0</td>
<td>0.6</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>Mean</td>
<td>4.3</td>
<td>3.0</td>
<td>2.2</td>
<td>1.7</td>
</tr>
<tr>
<td>S.D</td>
<td>1.8</td>
<td>1.5</td>
<td>1.2</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Waiting for the quarter-twist position in posture LAP means that more somersault is completed during the first quarter-twist than would be the situation in posture L. For this strategy to be worthwhile, once the posture has changed to L, the twist needs to use less somersault by an amount equal to the extra used in posture LAP. The difference in $N_\theta$ between postures LAP and L increases as $\phi_o$ decreases and at the same time the difference in the oscillations of postures LAP and L decreases as $\phi_o$ decreases. This means that the strategy would become progressively less worthwhile as the number of twists to be completed increases.

As an example of how this strategy may be used let us consider one for whom this strategy is beneficial. This ‘athlete’ is close to the upper quartile for the difference in the oscillations of $\phi$ between postures L and LAP. For this ‘athlete’ $N_\theta = 0.79$ in LAP while $N_\theta = 0.42$ in posture L when $\phi_o = 85^\circ$. Changing from posture LAP to posture L at the quarter-twist position reduces $\phi$ by ~5°. In posture L, now that $\phi_o = 80^\circ$, $N_\theta = 0.22$.

Table 5-15 compares the situations of posture L being used the whole time and the combination of postures LAP and then L.

Table 5-15: Example of using the strategy of changing between posture LAP and posture L at the ¼ twist position compared to using only posture L.

This scenario is for the ‘athlete’ close to the upper quartile of oscillations in $\phi$ when holding posture LAP, $\phi_o = 85^\circ$.

<table>
<thead>
<tr>
<th>Twist</th>
<th>Somersault used for strategy posture LAP then posture L</th>
<th>Somersault used for strategy of just using posture L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>0.39 in LAP</td>
<td>0.21</td>
</tr>
<tr>
<td>1/2</td>
<td>0.39 in LAP then 0.11 in L, gives a total of 0.5</td>
<td>0.42</td>
</tr>
</tbody>
</table>
| 1/1   | An extra 0.22 in L  
Gives a total of 0.72   | 0.84                                               |
| 3/2   | 0.94   | Cannot complete in a single somersault  |
The oscillations in $\phi$ and the value of $N_\theta$ depend on both $\phi_o$ and the posture used by the ‘athlete’. As $\phi_o$ decreases the difference in both these quantities for postures LAP and L approach zero. As a consequence the strategy of changing from posture LAP to L at a quarter twist rather than holding posture L throughout become less effective. For the example ‘athlete’ in Table 5-15, the strategy will only be effective when performing more than a triple twist when $\phi_o = 75^\circ$.

This example was for an ‘athlete’ in the upper quartile for the difference in the oscillations of $\phi$ between postures L and LAP; it is thus not expected to be worthwhile for most ‘athletes’. It would not be advisable to suggest this strategy as a general coaching approach. However, with knowledge of the athlete’s inertial properties, a specific investigation could be conducted and for some ‘athletes’ this strategy may be worthwhile.

5.3.5.3 Posture Pu: Puck

Puck is an allowable posture when performing multiple somersaults with multiple twists. The intention of using posture Pu is to reduce the angular momentum required while still allowing appreciable numbers of twists to be completed. Due to the arm position in posture Pu (Section 3.9.10) it must be compared with posture 1U1DB to determine what advantage results from flexing at the hip and knees.

From Table 5-13 the relative difficulty $\tau$ of posture Pu is clearly less than that of posture 1U1DB, and from Section 5.3.2 and Table 5-10 posture Pu has a larger value of $N_\theta$; thus less twist could be achieved in a somersault. In addition, for 37% of ‘athletes’, the value of $\phi_o$ separating continuous and oscillating twist is less than $90^\circ$, while for posture 1U1DB this is true for only 10% of ‘athletes’. From Table 5-13 posture Pu would be easier than a pure layout somersault for the vast majority of ‘athletes’. When considering the skills that the vast majority of ‘athletes’ can complete from Table 5-10 posture Pu allows the same skills to be completed as does posture 1U1DB. However, there is a more dramatic drop in the proportions of ‘athletes’ able to complete more difficult skills for posture Pu than posture 1U1DB. For example, when $\phi_o = 70^\circ$, a 3/2 twist in a somersault may be performed by 98% of ‘athletes’ in posture Pu and 97% in posture 1U1DB; while a 3/1 twist in a somersault may be performed by only 25% of ‘athletes’ in posture Pu but 70% of ‘athletes’ in posture 1U1DB.

Comparing posture Pu and postures A or JL reveals that flexing at the knees to reduce the angular momentum required is a better option than flexing through the torso. Posture Pu has a lower relative difficulty as compared to a pure somersault in posture L (Table 5-13) and allows a greater proportion of ‘athletes’ to achieve each multiple of a half twist than either postures A or JL.
Whether or not posture Pu is useful depends on the number of somersaults and the number of twists that are performed, and whether the angular momentum or the value of $\phi_o$ that an athlete can achieve is preventing the desired skill from being performed. Even if angular momentum generation is the limiting factor, using posture Pu is not necessarily the solution; an athlete could use posture 1U1DB, complete the twist sooner, and then enter posture P or T, both of which have much lower relative difficulties, for the remaining somersault. Whether or not using posture Pu, or posture 1U1DB followed by posture P or T, requires less angular momentum depends on the difference in the somersault required in postures 1U1DB and Pu, and hence on the specific athlete. For example, the athlete in Section 3.10 has $N_0$ equal to 0.14 in posture Pu, and 0.11 in posture 1U1DB; this means that in the somersault an athlete could complete a half twist in posture Pu or a half twist in posture 1U1DB plus 0.03 of a somersault in posture P or T. The number of layouts in each of these situations is equivalent to approximately 0.125 in either case; thus for this athlete posture Pu provides no advantage.

A decision would need to be made on a case by case basis with a good knowledge about the specific athlete’s inertial properties, or an athlete will need to try both situations to see which is easier.

### 5.3.5.4 Posture P: Pike

Posture P has $I_{zz}$ as the intermediate-valued moment of inertia for approximately 83% of ‘athletes’ (Table 5-1): for these ‘athletes’ posture P always displays oscillating twist. For the remaining ‘athletes’ posture P may show continuous or oscillating twist because $\phi_{crit} > 0$. For ~14% of ‘athletes’ $\phi_{crit}$ is between 70° and 90°. In Section 5.3.3, posture P was identified as a useful posture for “ending” twist, since the twist that would occur within such a somersault was acceptable for the vast majority of ‘athletes’ (Table 5-11).

In Section 4.2.6.3, a strategy was discussed whereby a diver changes posture from their chosen continuously-twisting posture to one displaying oscillating twist on completion of the desired number of half twists, and then waits until $\phi$ reaches $\pi/2$ to open out for entry. For this strategy to be useful there are two considerations. Firstly, the amount of somersault required for $\phi$ to reach $\pi/2$ must be less than the remaining amount of somersault. In diving, the last half somersault in a twisting somersault dive is typically a piked somersault. For posture P, the number of somersaults required for $\phi$ to reach $\pi/2$ is less than half a somersault for 28–31% of ‘athletes’ when $\phi_o$ is between 70° and 90°. Thus, for the majority of ‘athletes’, $\phi$ will not reach $\pi/2$ prior to entry; it thus will not achieve the goal of $\phi$ being $\pi/2$ on entry although it will be still closer to $\pi/2$ than when entering posture P.
For some ‘athletes’ $\phi$ will reach $\pi/2$, when it does, the second consideration is the amount of twist that will occur after opening out. The example athlete in Section 3.10 is one ‘athlete’ for whom $\phi$ would reach $\pi/2$. It takes 0.38–0.41 of a somersault for $\phi$ to reach $\pi/2$ when $\phi_o$ is between $70^\circ$ and $90^\circ$. This leaves 0.12–0.09 of a somersault that would need to be completed while moving to the entry posture. The twist present when $\phi$ reached $\pi/2$ is $3.7^\circ$–$14.8^\circ$. This means, that if this athlete opened to posture L, twist would start about $\psi = \pi/2$: that is Case 4. For this athlete Case 4 in posture L has $N_\theta$ as always greater than four somersaults. Thus, in approximately a tenth of a somersault, less than $10^\circ$ of twist is expected when preparing for entry. The total twist would then be $\sim$10–25$^\circ$, which is not excessive, although this is probably observable to the naked eye. If the athlete opened via posture OP or EP then the additional twist that occurs while preparing for entry would be reduced.

Using posture P after completing the desired amount of twist in a twisting somersault is a useful strategy to ‘end’ twist; it prevents continuous twist for the majority of ‘athletes’ and only produces small amounts of undesired twist. It will, however, never truly remove twist, and so the entry/landing will not be ‘perfect’, with $\phi = \pi/2$ and $\psi = 0$.

5.3.6 Error associated with assuming $I_{xx} = I_{yy}$ in a twisting somersault

In Section 4.2.5 it was shown that the error due to assuming that $I_{xx} = I_{yy}$ depends on the relative magnitudes of all three moments of inertia and the initial conditions. Initial conditions of interest, as identified in Section 5.3.2, are those that produce a motion such that, when $\psi = 0$, $\phi_o$ is between $70^\circ$ and $90^\circ$ inclusive. The most basic requirement for the assumption of equality to be reasonable is that the posture displays continuous twist with initial conditions in this range. From Figure 5-32 the vast majority of ‘athletes’ would show continuous twist in this range when holding the postures LAU, LAP, 1U1D, HVLV, 1U1DB, 1U1DB LF, L, A, and LHF. As a result these postures will be the focus of this section. Based on Table 5-1 these postures predominantly have $I_{yy}$ as the intermediate-valued moment of inertia.

The error due to assuming that $I_{xx} = I_{yy}$ when predicting $N_\theta$, decreases as $\phi$ decreases, when $\psi = 0$. This is the result of the ratio $I_{yy}/I_{xz}$ becoming more important than $I_{xx}/I_{yy}$ as $\phi$ decreases; this effect was seen in Section 4.2.4 and in the reduction in scatter between Figure 5-33 and Figure 5-34. As can be seen in Figure 5-38, which plots the error $N_{\theta(error)}$ when assuming $I_{xx} = I_{yy}$, $N_{\theta(error)} \to 0$ as $\phi \to 0$, and $N_{\theta(error)} \to \infty$ as $\phi \to 90^\circ$. This general behaviour will occur for all postures, but, how sharply the curves approach these asymptotes, and whether or not the difference is negative or positive will depend on the ‘athlete’ and the posture.
The quasi-rigid phase

Figure 5-38: $N_\theta (\text{error})$ due to assuming equality of $I_{xx} = I_{yy}$

$N_\theta (\text{error}) = N_\theta (\text{Assumption}) - N_\theta (\text{Actual})$ as in Section 3.3.5. Positive values mean that the assumption of equality overestimates the number of somersaults required. When applying the assumption of equality $I_{xx}$ and $I_{yy}$ are both set to their average values. The error is plotted against the value of $\phi_o$ when $\psi_o = 0$. The inertial properties used are for the example athlete from Section 3.10.

In Figure 5-38, ranking the postures by $N_\theta (\text{error})$ would give the least error for posture 1U1DB, followed by postures A, 1U1D, L, 1U1DB LF, LAU, LHF, HVLV, and the greatest value of $N_\theta (\text{error})$ is for posture LAP. The errors for all but postures HVLV and LAP are small when $\phi_o \leq 85^\circ$, and so it would be reasonable to conclude that, for this athlete, the assumption of equality could be used to predict $N_\theta$ when $\phi_o \leq 85^\circ$. Across all ‘athletes’ the ranking varies, although posture 1U1DB shows the least error when assuming equality for ~67% of ‘athletes’ when $\phi_o = 85^\circ$. The values of $\phi_o$ required for $N_\theta (\text{error})$ to be small in magnitude also vary; the difference for the athlete in Figure 5-38, falls in the lower quartile of $N_\theta (\text{error})$ for all the postures when $\phi_o = 85^\circ$.

The magnitude of $N_\theta (\text{error})$ that is acceptable depends on the value of $N_\theta$. As was stated in Section 4.2.5, assuming $I_{xx} = I_{yy}$ could also be used to predict the value of $\phi_o$ required to achieve certain skills. This is possibly the more useful way of assessing when assuming $I_{xx} = I_{yy}$ is reasonable for specific postures since it relates directly back to the skills that athletes wish to perform.

The error in the number of somersaults required was less than or equal to 0.03 ($10.8^\circ$) for all ‘athletes’ performing a double twist in a single somersault when using posture 1U1DB, and for all ‘athletes’ when performing 2½ twists in a single somersault when using postures LAU, 1U1D, HVLV, 1U1DBLF, L, A, and LHF. These are large
numbers of twists, and quite difficult skills; thus the assumption of equality is not particularly useful.

Posture 1U1DB had the lowest value of $N_{\theta(error)}$ when assuming $I_{xx} = I_{yy}$. Figure 5-39 shows box plots illustrating the spread of the number of somersaults required to achieve the set numbers of twists when using the value of $\phi_{o}$ when $\psi_{o} = 0$, predicted by assuming $I_{xx} = I_{yy}$. It can be seen in Figure 5-39 that for 1.5 twists ~85% have a magnitude of error less than 0.03. Thus, only when an ‘athlete’ is performing at least 1.5 twists could the assumption that $I_{xx} = I_{yy}$ for 1U1DB be considered reasonable. Even though the assumption that $I_{xx} = I_{yy}$ would not be reasonable for half and full twists, since ~90% of ‘athletes’ required less somersaults in reality, assuming $I_{xx} = I_{yy}$ could be used to provide a conservative estimate of the initial conditions required.

![Figure 5-39: The number of somersaults required to achieve increasing multiples of a half twist using posture 1U1DB.](image)

The assumption that $I_{xx} = I_{yy}$ was used to predict the value of $\phi_{o}$ required to complete each skill when $\psi_{o} = 0$. The number of somersaults given is the number required to achieve the set number of twists using the predicted value of $\phi_{o}$. The red lines mark ±0.03 of a somersault either side of 1 (the value achieved if $I_{xx}$ and $I_{yy}$ were in fact equal); this is an acceptable error. For 0.5 twists the top whisker goes off the scale because the number of somersaults required to complete the skill is infinite since for some ‘athletes’ posture 1U1DB displays oscillating not continuous twist at the predicted value of $\phi_{o}$.

In Section 5.3.2, posture L was identified as the posture that would enable the most twists to be completed in a somersault. It is thus of particular interest whether or not the assumption of equality would be reasonable for posture L. Figure 5-40 gives box-and-whisker plots, similar to those in Figure 5-39 but for the posture L. It is not until the ‘athlete’ is performing a double twist that ~83% of ‘athletes’ have a magnitude of error less than 0.03, and so the assumption that $I_{xx} = I_{yy}$ would only then be reasonable. For 0.5, 1.0, and 1.5 twists, less somersault is required than when assuming $I_{xx} = I_{yy}$. The amount of somersault is much less than for posture 1U1DB. Unlike posture 1U1DB the
error is too big to be useful even as a conservative estimate when predicting the $\phi_o$ value required.

![Figure 5-40: The number of somersaults required to achieve multiples of a half twist when using posture L.](image)

The assumption that $I_{xx} = I_{yy}$ was used to predict the value of $\phi_o$ required when $\psi_o=0$. The number of somersaults given is the number required to achieve the set number of twists using the predicted value of $\phi_o$. The red lines mark $\pm 0.03$ of a somersault either side of 1 (the value achieved if $I_{xx}$ and $I_{yy}$ were in fact equal); this is an acceptable error. For 0.5 twists the top whisker goes off the scale because the number of somersaults required to complete the skill is infinite since for some ‘athletes’ posture L displays oscillating not continuous twist at the predicted value of $\phi_o$.

The exploration in this section shows that assuming $I_{xx} = I_{yy}$ would be reasonable for common twist postures when performing high numbers of twists, it is not a reasonable assumption for the easier, more widely performed twisting somersault skills.
Chapter 6

Idealised aerial twist initiation

Twist may be introduced into a pure somersault by performing actions that reduce $\phi$ so it no longer equals $\pi/2$. Section 4.3 derived the equations for the re-orientation that allows the calculation of the change in $\phi$ due to an action.

The main actions known or proposed to reduce $\phi$ may be classed as asymmetric rotation of the arms in a frontal plane (Batterman, 1968; Rackham, 1970; Frohlich, 1979; Yeadon, 1993c); flexion through the torso including, lateral flexion and a full (McDonald, 1961; Van Gheluwe & Duquet, 1977; Van Gheluwe, 1981) or partial “hula” (Yeadon, 1993c; Yeadon, 1999; Yeadon, 2001); and “kick-out” which involves torsion between the trunk and legs when extending from a pike (Rackham, 1970). This chapter will explore each of these classes of actions assuming an ‘athlete’ can perform an idealized version of each action. It will then explore slight variations of these idealized actions as well as combinations of these actions that naturally connect.

The inertial property data sets collated in Chapter 3 (referred to as ‘athletes’) were applied to the equations describing each action, in order to estimate the expected magnitude and direction of the re-orientation. The exploration was entirely theoretical: a safe, accurate, and practical method was not available to measure the tilt produced by isolated actions. Allowing an athlete to fall backwards off a ledge onto very soft mats (a slow quarter somersault) while performing some of the idealised arm actions, did produce observable tilt and some resulting twist. On landing the soft matting altered the landing position and, although tilt could be seen, this observation could not be considered an accurate measure of the tilt produced by the arm actions. This observation may only be used to confirm that tilt does produce twist in a somersault.
6.1 Idealized arm actions

Arm actions in the frontal plane will result in an angular displacement about the $\text{Pri}_x$-axis; let this displacement be known as tilt. A positive tilt is an anticlockwise rotation about $\text{Pri}_x$ and will be seen as the shoulders tilting to the right. If no appreciable twist occurs while the action is performed then the tilt will only alter $\phi$. Thus the calculations presented here are expected to produce the greatest change in $\phi$.

Four idealized asymmetrical arm actions are investigated here. They will be known as DiverS, FullS, Drop, and Raise, and each is described below. The body was held in a layout posture and the arms moved in the frontal ($\text{Pri}_x$-$\text{Pri}_y$) plane of this posture. To investigate these four actions, independent of any somersault, it is assumed that the arm movements are made instantaneously. They may then be represented by a planar three segment model: the two arms will be segments 1 and 3 from Section 4.3.2, the rest of the body will be the middle reference segment, with the shoulder joints as the joints between the segments. Thus Equations (4-32) and (4-34) may be used to evaluate the angular displacement due to these actions.

**DiverS and FullS**

DiverS and FullS are actions where the arms are moved with odd symmetry; that is, one raises while the other lowers, both lateral to the body in the frontal plane. DiverS starts with the arms laterally outstretched, while FullS starts with one arm raised above the head and the other by the side. DiverS is common in diving (Frohlich, 1979)\textsuperscript{14}. FullS is not as common but has been used in diving (Rackham, 1970)\textsuperscript{15}; it is however, commonly used by aerial skiers performing a double twist in the first somersault\textsuperscript{16}. Figure 6-1 illustrates the actions and gives the values of $\alpha$ required in Equation (4-32).

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\textsuperscript{14} Frohlich, (1979), describes DiverS and estimates the amount of tilt that could be produced. However, for simplicity Frohlich assumes the arms rotate about a point between the shoulders and so his estimates are not useful for comparison with the results for DiverS determined in this thesis. O’Brien, (1992), describes an action of moving the arms in a circular pattern, while imagining one is turning a steering wheel. Based on the pictures of the final twist positions it is likely O’Brien is describing the same action as DiverS.

\textsuperscript{15} Rackham (1970) reports that Robert Clotworthy used this technique in the Melbourne 1956 Olympics to perform his forward 1½ somersault with double twist. Since the last ½ somersault is generally used to prepare for entry the double twist would almost certainly been performed in the initial somersault.

\textsuperscript{16} From private E-mail correspondence with Denita Preston former Australian aerial ski team member.
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Figure 6-1: The a) FullS and b) DiverS asymmetric arm actions
The odd symmetric arm action is defined by the path $\alpha_3 = \alpha_1 - 180^\circ$. The value of $\alpha_1$ decreases during the action. The solid outline of the arms shows the start position and the dotted outline of the arms shows the end position. a) Full S moves the right arm (segment 1) from the fully lowered ($\alpha_1 = 0^\circ$) position to the fully raised ($\alpha_1 = -180^\circ$) position, and the left arm (segment 3) from the fully raised position ($\alpha_3 = 180^\circ$) to the fully lowered position ($\alpha_3 = 0^\circ$). b) DiverS moves the arms from posture LAP ($\alpha_1 = -90^\circ$, $\alpha_3 = 90^\circ$) to posture 1U1D ($\alpha_1 = -180^\circ$, $\alpha_3 = 0^\circ$).

Drop

Drop is the action where an athlete starts with both arms raised above the head, and then lowers one arm laterally in the frontal plane. This is often seen in gymnasts, divers, and aerial skiers performing backward somersaults with twist since the arms are already up at the end of the throw for a backward somersault. O’Brien (2003) suggests using Drop in back somersaults, and also indicates that both arms may be used by first twisting the chest and then lowering both arms in front of the body; this action is considered in Section 6.1.3, as part of adjusting the arm actions. Figure 6-2 illustrates the Drop action and gives the values of $\alpha$ required in Equation (4-34).

Figure 6-2: The drop asymmetric arm action
The left arm (segment 3) moves from the fully raised position ($\alpha_3 = 180^\circ$) to the fully lowered position ($\alpha_3 = 0^\circ$). The path describing the action is $\alpha_1 = -180^\circ$. 
**Raise**

Raise is the action where both arms start by the sides and then one arm is raised laterally. This technique is seen in somersaults with a “half-out” (a half-twist completed past the halfway point of the somersault rotation) or in a forward twisting somersault where the twist is initiated after the first somersault. Figure 6-3 illustrates the action and gives the values of $\alpha$ required in Equation (4-34).

![Figure 6-3: The Raise asymmetrical arm action](image)

The athlete moves from the posture L ($\alpha_1 = \alpha_3 = 0^\circ$) to the posture 1U1DB ($\alpha_1 = -180^\circ$, $\alpha_3 = 0^\circ$) with the path describing the action being $\alpha_3 = 0^\circ$.

**Drop vs. Raise**

The reverse action to Drop is very close to Raise; the difference is whether the right or left arm is moved and the position of the non-moving arm. Opposite arms are moved so that a positive tilt, that is to the athlete’s right, is produced by both actions. The tilt produced will not be the same since the non-moving arm is in a different position. The position of the non-moving arm will alter the location of the centre of gravity and the moment of inertia of the non-moving arm and body combination; this alters how the non-moving arm and body combination move in response to the action of the moving arm.

**6.1.1 Contribution to tilt over the course of the action**

Since each of the idealized asymmetrical arm actions is in the frontal plane, then the tilt produced is the angular displacement for a planar action as given in Equation (4-32) and Equation (4-34) by substituting in the appropriate path of $\alpha_1$ and $\alpha_3$ then integrating with respect to the remaining variable. Assuming that the arms are identical and symmetrical about their longitudinal axis, $V_{1X} = V_{3X} = 0$, $V_{1Y} = V_{3Y}$, $m_1 = m_3$, and $I_1 = I_3$ and so $G = 0$, and $A = B$. Due to the structure of the body, the centre of gravity of the body segment will be below the shoulders. Since layout is a posture showing symmetry about the mid-sagittal plane then the centre of gravity will be on the longitudinal axis of the middle segment. This means that $V_{21Y} = V_{23y}$ and both are positive, $V_{23x}$ is positive and $V_{21x} = - V_{23x}$, and $V_{1Y}$ and...
\[V_{3Y}\] are both positive. It follows then that \(E\) is positive, \(C = -E\), \(D = F\) and both \(D\) and \(F\) are positive, and \(L\) is negative. Equations (6-1) to (6-3) give the integrals and solutions for each action. A substitution of \(t = \tan \alpha / 2\) (t-results) was used when integrating and the bounds of the integrals are restricted to angles between \(-180^\circ\) and \(180^\circ\). The negative signs in Equations (6-1) to (6-3) simply indicate that the angular displacement, \(\gamma\), is in the opposite direction to the arm action.

The equation for the tilt angular displacement as a result of performing an odd symmetric arm action is

\[
\gamma_{H=0, ref = 2} = -\frac{1}{2} \int_{0}^{\alpha \_\text{start}} \left( \frac{4I_f + 4A - 2L - N}{(N - 2L + 4C \sin \alpha)} + 1 \right) d\alpha
\]

\[
= -\frac{1}{2} \left[ \alpha \_\text{start}^{-180^\circ} \right] + \frac{-(4I_f + 4A - 2L - N)}{\sqrt{(N - 2L)^2 - 16C^2}} \left[ \tan^{-1}\left( \frac{(N - 2L) \tan \left( \frac{\alpha}{2}\right) + 4C}{\sqrt{(N - 2L)^2 - 16C^2}} \right) \right]^{180^\circ}_{\text{start}}
\]

(6-1)

The tilt due to odd symmetric arms actions will have FullS starting at \(0^\circ\) and DiverS starting at \(-90^\circ\). FullS produces double the tilt of DiverS. This is because the integrand in Equation (6-1) has symmetry about \(\alpha_1 = -90^\circ\) due to the symmetry of the sine function.

The equation for the tilt angular displacement as a result of performing the action Drop is

\[
\gamma_{H=0, ref = 2} = -\frac{1}{2} \int_{0}^{\alpha \_\text{start}} \left( \frac{2I_f + 2A - 2D - N}{(N + 2D - 2C \sin \alpha - 2(D + L) \cos \alpha)} + 1 \right) d\alpha
\]

\[
= -\frac{1}{2} \left[ \alpha \_\text{start}^{-0^\circ} \right] + \frac{-(2I_f + 2A - 2D - N)}{\sqrt{N^2 + 4ND - 8DL - 4L^2 - 4C^2}} \left[ \tan^{-1}\left( \frac{(N + 4D + 2L) \tan \left( \frac{\alpha}{2}\right) - 2C}{\sqrt{N^2 + 4ND - 8DL - 4L^2 - 4C^2}} \right) \right]^{0^\circ}_{-180^\circ}
\]

(6-2)

The equation for the tilt angular displacement as a result of performing the action Raise is

\[
\gamma_{H=0, ref = 2} = -\frac{1}{2} \int_{0}^{\alpha \_\text{start}} \left( \frac{2I_f + 2A + 2D - N}{(N - 2D + 2(L - D) \cos \alpha + 2C \sin \alpha)} + 1 \right) d\alpha
\]

\[
= -\frac{1}{2} \left[ \alpha \_\text{start}^{180^\circ} \right] + \frac{-(2I_f + 2A + 2D - N)}{\sqrt{N^2 - 4ND - 4L^2 + 8LD - 4C^2}} \left[ \tan^{-1}\left( \frac{(N - 2L) \tan \left( \frac{\alpha}{2}\right) + 2C}{\sqrt{(N - 2D)^2 - 4(L - D)^2 - 4C^2}} \right) \right]^{180^\circ}_{0^\circ}
\]

(6-3)
When actually performing these actions, an athlete may choose to perform only part of the action, or be only able to complete part of the action due to time constraints. It is thus of interest to know the nature of these functions including the locations of the maxima. As a result, if there is any option to slightly adjust the start position of the arms, then suggestions may be made regarding where the arms should start in order to maximise the tilt produced. Figure 6-4 plots the negative integrands, let them be called NI, from Equations (6-1) to (6-3) for the example athlete in Section 3.10. Since the NI curves were plotted the area under the curve for an action should be read from the end to the start position and it represents the tilt produced by the action. The plot covers the full periods of the negative integrands, although, to keep the arms lateral to the body, \( \alpha \) should be negative for FullS, DiverS and Raise, and positive for Drop.

![Figure 6-4: NI for the idealized arm actions as a function of \( \alpha \)](image)

The maximum value of NI for the odd symmetric actions will always occur at \(-90^\circ\). Thus, if the athlete can only move the arms through a limited range, they should centre their action about the laterally outstretched position.

Unlike the odd symmetric actions the location of the maximum for Drop and Raise will depend on the inertial properties of the particular athlete.

NI for Drop may be rewritten as

\[
\frac{1}{2} \left( \frac{2I_i + 2A - 2D - N}{N + 2D + 2\sqrt{C^2 + (D + L)^2} \sin(\alpha_i - \mu_d)} + 1 \right)
\]
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where, \( \cos \mu_D = \frac{-C}{\sqrt{C^2 + (L + D)^2}} \) and \( \sin \mu_D = \frac{D + L}{\sqrt{(L + D)^2 + C^2}} \).

NI for Raise may be rewritten as

\[
\frac{1}{2} \left( 1 + \frac{2I_1 + 2A + 2D - N}{N - 2D + 2\sqrt{C^2 + (L - D)^2} \sin(\alpha_r - \mu_R)} \right)
\]

Where, \( \sin \mu_R = \frac{-(L - D)}{\sqrt{(L - D)^2 + C^2}} \) and \( \cos \mu_R = \frac{C}{\sqrt{(L - D)^2 + C^2}} \).

In this form it is clear that both NIs will have turning points when \( \alpha = 90^\circ + \mu \) and \(-90^\circ + \mu\).

For all the inertial property data sets collated in Chapter 3 the numerator in the fraction of NI is a negative number, while the denominator is a positive number. Thus the maxima of NI for Drop and Raise occur when \( \alpha = 90^\circ + \mu \).

The five-figure summary (Max, UQ, Median, LQ, Min) of the value of \( \alpha \) giving the maximum value of NI occurs at \((177^\circ, 157^\circ, 154^\circ, 151^\circ, 142^\circ)\) for Drop and \((-144^\circ, -152^\circ, -155^\circ, -159^\circ, -177^\circ)\) for Raise. Figure 6-5 illustrates the maximum, minimum and median values of \( \alpha \) with respect to an athlete to aid visualisation of the actual arm positions.

The difference in \( \alpha \) between the position giving a maximum NI for Drop and Raise for any particular ‘athlete’, apart from being the other arm, is slight; the five figure summary for the difference in position is \((-0.2^\circ, -0.9^\circ, -1.1^\circ, -1.4^\circ, -3.0^\circ)\). This means that the position of the non-moving arm has only a very slight effect on the location of the maximum value of NI.
For 97% of the ‘athletes’, the Odd symmetric arm actions produced the greatest maximum value of NI and for 95% DiverS produced a greater total tilt than Raise. For all the inertial property data sets Raise had a greater maximum value of NI and a greater total tilt for the full action than Drop. Since Raise always produces more tilt than Drop it is clear that placing the non-moving arm by the athlete’s side is better than having it raised.

For all inertial property data sets the minimum value of NI was smaller in magnitude than the maximum value of NI and so the arms should be moved opposite to the desired direction of tilt and be centred about the maximum value of NI in order to produce maximise tilt.

NI for the odd symmetric actions will always be positive while $\alpha$ is negative. This is because when $\alpha = 0$ or $-180^\circ$ then NI becomes $(2I_1 + 2A - 2L)/(N - 2L)$, which is positive. NI for Drop and Raise may become negative and so cross the $\alpha$-axis, depending on the inertial properties of the athlete. When NI is negative the arm action in this domain of $\alpha$ will produce a tilt in the same direction as the arm action. The $x$ intercepts will be

\[
\alpha_i = \mu + \sin^{-1}\left(\frac{-I_1 - A}{\sqrt{C^2 + (D + L)^2}}\right) \text{ for Drop and }
\]

\[
\alpha_i = \mu + \sin^{-1}\left(\frac{-I_1 - A}{\sqrt{C^2 + (L - D)^2}}\right) \text{ for Raise.}
\]

An $\alpha$-axis intercept at a value greater than zero for Drop or less than zero for Raise will mean that the action should be stopped early, lest the tilt that had been produced is undone. The earliest that Drop should stop is $42^\circ$ before the fully lowered position, or the latest Raise should start from the fully lowered position is $45^\circ$. The medians are $17.13^\circ$ and $20.59^\circ$ for Drop and Raise respectively.

Considering the maximum value of NI and its corresponding value of $\alpha$, considering when NI becomes negative, and remembering that Drop is typical in back somersaults, while Raise and the Odd symmetric actions are more typical in front somersaults, a coach should emphasise

- Stretching the arms above the head when entering the aerial phase for a backward somersault so as to start the Drop technique with the arms raised high.

- Stopping the arms in the Drop action when they move below a low V position. To bring the arms close to the body, bending them may be considered, or the final
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drop could be performed after some twist has occurred so the rotation contra to the desired rotation alters the somersault angle \( \theta \), rather than the angle \( \phi \).

- Moving the arms directly out from holding a tuck or pike posture in half-out techniques and raising the arms rather than endeavouring to start with the arms by the sides.

- Using an Odd symmetric technique rather than Raise, for the same length arc of arm movement, wherever possible; for example, ending the arm throw for a forward layout in posture LAP which is the start position for DiverS.

- Centering an odd symmetric arm action around the laterally outstretched position, that is starting in posture LAP, while still moving the arms through the greatest arc.

### 6.1.2 Full action tilt and the twisting somersault skills achievable

To aid discussion regarding which twisting somersaults could be achieved by each of the idealized arm actions, let us evaluate the tilt produced and the consequent value of \( N_\theta \) when the full actions are performed.

Figure 6-6 shows the tilt produced by the actions DiverS, Raise, and Drop across all ‘athletes’, ordered by the tilt achieved with DiverS. From the plot it is clear that Drop produces the least tilt for all ‘athletes’; the difference between the tilt produced by Drop and DiverS or Raise reduces as the tilt produced reduces. Raise can be seen to produce more tilt than DiverS, for 95% of the ‘athletes’, although the advantage is slight. The ‘athlete’ producing the most tilt would appear to be an outlier, yet there is still a large difference in the tilt produced across all ‘athletes’: from 2.28° up to 11.05°, ignoring the outlier. This large range suggests that some ‘athletes’ have a very clear natural advantage. It is, though, important to determine how this tilt translates into skills achievable, and if the difference is still large.
The end posture for all four idealised asymmetric arm actions is 1U1D with the right arm raised. The value of \( N_\theta \) for each idealized arm action and their natural end postures may be determined from Section 4.2.4. The moments of inertia are those of the posture 1U1D and the value of \( \phi \) is determined by subtracting the tilt produced by the action and any tilt of the principal axes of posture 1U1D compared to the entry posture L or LAU, from 90°. All ‘athletes’ displayed continuous twist in posture 1U1D following any of the idealized asymmetric arm actions. As seen in Figure 6-7 there was a general trend between the tilt produced by the arm action and \( N_\theta \) but there is still moderate scatter; this scatter is a result of some ‘athletes’ having more advantageous inertial properties in posture 1U1D and others having an advantage due to the tilt they can produce for the idealised asymmetric arm actions.

Figure 6-6: Tilt produced by each ‘athlete’ ranked in descending order of the tilt produced by DiverS

The jitters in the Drop and Raise curves reflect the fact that the order of the ‘athletes’ is not identical to DiverS.

Figure 6-7: Tilt due to the actions plotted against \( N_\theta \) in 1U1D
The order of the actions in terms of maximising tilt and twist in a somersault is the same: FullS, Raise, DiverS, and Drop. DiverS and Raise achieve similar values of $N_\theta$, as well as tilt. Table 6-1 gives the proportions of ‘athletes’ can achieve a selection of skills using the idealized asymmetric arm actions.

<table>
<thead>
<tr>
<th></th>
<th>FullS</th>
<th>DiverS</th>
<th>Raise</th>
<th>Drop</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>1.00</td>
<td>0.82</td>
<td>0.81</td>
<td>0.83</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.76</td>
<td>0.67</td>
<td>0.63</td>
<td>0.67</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.70</td>
<td>0.26</td>
<td>0.22</td>
<td>0.26</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.55</td>
<td>0.05</td>
<td>0.03</td>
<td>0.07</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.30</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.20</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Based on Table 6-1 it is reasonable to teach any of the above techniques for a full twisting somersault: beyond this, the majority of ‘athletes’ would need to use FullS. Further, in actual performances it is likely that the complete action would not be able to be performed, in which case even less twist would be achieved.

To achieve a double twist in a somersault FullS has been used in aerial skiing\textsuperscript{17} and diving\textsuperscript{18}. However, 3/2 and double twist skills have been performed in numerous gymnastic and diving competitions where the athlete cannot perform FullS. Further the asymmetric arm actions do not explain the observation by Frohlich (1979) of athletes achieving twist rates of 3.8 to 6 twists per somersault at the 1972 Olympic trials. Consequently when performing twisting somersaults these athletes must be using other aerial techniques either entirely or in addition to these asymmetrical arm actions, or they are using some contact twist.

The large variation in $N_\theta$ across the ‘athletes’ (Figure 6-7) means that some ‘athletes’ do have a clear natural advantage when performing twisting somersaults. Considering the genders of the ‘athletes’, there was no significant difference between genders for any of the idealized asymmetric arm actions. Considering the squads, it was found that the squads were not from the same homogenous population under the H-test ($p << 0.01$); the literature data had a significantly lower value of $N_\theta$, and the 12-or-under data required a significantly greater value of $N_\theta$. Figure 6-8 illustrates the proportion of ‘athletes’ who require at least a specific value of $N_\theta$ for the action Raise, which was the action with the least difference between the squads under the H-test. From Figure 6-8 the distinctly different behaviour of

\textsuperscript{17} From private E-mail correspondence with Denita Preston, former Australian aerial ski team member.

\textsuperscript{18} Rackham (1970) reports that Robert Clotworthy used this technique in the Melbourne 1956 Olympics to perform his forward 1½ somersault with double twist.
the literature data and 12-or-under squads from the Teens, Seniors and Masters can readily be seen.

![Figure 6-8: Proportions of ‘athletes’ in each squad with at least each value of $N_\theta$ for the action Raise](image)

### 6.1.3 Adjusting the plane of the arm action as twist begins

As the body begins to twist in response to $\phi$ no longer being $\pi/2$, the frontal plane will no longer contain the angular momentum vector. Continuing the arm actions in the frontal plane will now change both $\phi$ and $\theta$. Adjusting the arm action, so that it no longer moves in the frontal plane, but remains in a plane that contains the angular momentum vector should produce a greater change in $\phi$ than continuing the arm action in the frontal plane; this was discussed in Section 4.3.3. The medial and transverse moments of inertia of the body are not equal, and the shoulders are not on the longitudinal axis of the body, and so even when adjusting the arm action as described above, $\phi$ is not the only orientation angle that will change value.

At the quarter-twist position the arms will need to move in the sagittal plane of the body to continue to reduce $\phi$. For Drop and Raise the moving arm will be in front of the body, while in the odd symmetric actions one arm will need to move in front of the body and the other behind. When moving in the sagittal plane the arm action may be reduced to a planar action. The inertial properties of the body are not the same as when the arms move in the frontal plane and so the tilt achieved will differ between the actions. The moment of inertia required is now the transverse moment of inertia of the body and the shoulders may be treated as though they were on the longitudinal axis of the body; thus, $V_{21x} = - V_{23x} = 0$. 

Equations (6-1), (6-2), and (6-3), may still be used to determine the angular displacement, although C (and E) will now equal zero, and the value of N will differ slightly from the value previously used when the actions were performed in the frontal plane. Since C = 0, the NI function for the odd symmetric actions becomes a constant value: \( \frac{2I_1 + 2A - 2L}{N - 2L} \). This means that the same amount of tilt will be produced by moving the arm in an arc of the same length and so it is not necessary to centre the action about a particular value of \( \alpha \) to maximise tilt when the full arm action cannot be completed.

With C = 0, NI from Drop becomes

\[
\frac{1}{2} \left( \frac{2I_1 + 2A - 2D - N}{N + 2D - 2(D + L)\cos \alpha} + 1 \right)
\]

and Raise becomes

\[
\frac{1}{2} \left( \frac{2I_1 + 2A + 2D - N}{N - 2D + 2(L - D)\cos \alpha} + 1 \right)
\]

which shifts the turning points of NI to \( \alpha \) equals 0 and 180°. For all the inertial property data sets collated in Chapter 3 (‘athletes’) the maximum value of NI for Drop and Raise occurs when \( \alpha = 180^\circ \). This is similar to the frontal plane in that the maximum values of NI are obtained when the arms are raised high, but, unlike in the frontal plane the value of \( \alpha \) giving the maximum value of NI is the same for all ‘athletes’.

When performing the full arm action in the sagittal plane, Raise still produces greater tilt than Drop for all ‘athletes’, and DiverS produces greater tilt than Raise for 90% of the ‘athletes’. However, Raise only has a maximum value of NI greater than Drop for 63% of ‘athletes’, and DiverS only has a maximum value of NI greater than Raise for 15% of ‘athletes’. Thus, it is not possible to say which action will produce more tilt unless the arc length of the action is known.

The NI curves for the actions in the sagittal plane cross the x-axis at different values of \( \alpha \) than for the corresponding actions in the frontal plane: the Drop action must be stopped earlier for all ‘athletes’, and the Raise action started later for all but 2% of the ‘athletes’ performing Raise. The earliest that Drop should stop is now 54° before the fully lowered position, and the latest Raise should start from the fully lowered position is 52°. The medians are 38.4° and 35.4° respectively.

The frontal plane actions produce a greater tilt than the same actions in the sagittal plane for all ‘athletes’ performing an odd symmetric action, and almost all (99.6%) ‘athletes’ performing Drop or Raise. The frontal plane actions also have a greater maximum value of NI than the same actions in the sagittal plane for all ‘athletes’ performing an odd
symmetric action: 92% of ‘athletes’ performing Drop and 87% of ‘athletes’ performing Raise. As a result, to maximise the tilt, ‘athletes’ should aim to perform as much of the action as possible in the frontal plane. Figure 6-9 plots NI in the sagittal plane on the same graph as NI in the frontal plane (which was plotted in Figure 6-4) for the example athlete from Section 3.10.

![Figure 6-9: NI as a function of α for the idealized arm actions in the frontal and sagittal plane](image)

The α-axis gives α which should be interpreted as \(\alpha_1\) or \(\alpha_3\) depending on which arm is moved when performing the action of interest. To keep the arms lateral to the body α should be negative for Raise and the odd symmetric actions, FullS and DiverS; while α will need to be positive for Drop. Plots are for the example athlete in Section 3.10.

The shifting of the location of the maxima of NI when an action is performed in the sagittal rather than frontal plane and the change in amplitude may be clearly seen in Figure 6-9. The value of NI for odd symmetric actions in the sagittal plane is a constant and for this athlete is very close to the average, over the full range of \(\alpha\), of the amplitudes of NI for odd symmetric actions in the frontal plane; their equations are not the same, but the inertial properties of this athlete mean that the values are very close. From Figure 6-9 it is clear that to maximise the tilt this athlete should, as far as possible, perform the arm actions in the frontal plane, since the frontal plane curves are above those for the sagittal plane in the domains of interest.

In the frontal plane Drop and Raise were actions performed with only one arm; the second arm could not be dropped or raised because it would collide with the body. Both arms, however, may be raised or lowered together in the sagittal plane; when moving the arms together raising and dropping of the arms would be true opposites. It is sufficient to model
both arms dropping, and then both arms raising would have the same magnitude of $\gamma$ but the opposite direction. Let this action be called “DroppingT”.

The effect of DroppingT will not be precisely double Drop or Raise in the sagittal plane, since there is no longer a non-moving arm, although the tilt produced is expected to be of similar magnitude. The angular displacement produced in the sagittal plane may be determined from Equation (4-32) by letting $\alpha_1 = \alpha_3$, and as previously mentioned applying $I_1 = I_3$, $G = 0$, $A = B$, $C = E = 0$, $D = F$. For fully dropping the arms the bounds are $180^\circ$ at the start and $0^\circ$ at the finish. This gives

$$\gamma_{H=0, \text{ref}=2} = -\int_{\alpha_{\text{Start}}}^{\alpha_{\text{End}}} \frac{(2I_1 + 2A - 2D \cos \alpha_i)}{(N - 4D \cos \alpha_i)} d\alpha_i$$

$$= -\frac{1}{2} \left[ \alpha_{\text{End}}^{180^\circ} - \left( \frac{4I_1 + 4A - N}{\sqrt{N^2 - 16D^2}} \right) \tan^{-1}\left( \frac{N + 4D \tan \left( \frac{\alpha_1}{2} \right)}{\sqrt{N^2 - 16D^2}} \right) \right]^{180^\circ}_{0^\circ}$$

$$= 90^\circ \left( \frac{4I_1 + 4A - N}{\sqrt{N^2 - 16D^2}} + 1 \right)$$

(6-4)

NI for this action is $\frac{1}{2} \left( \frac{4I_1 + 4A - N}{\sqrt{N^2 - 16D \cos \alpha_i}} + 1 \right)$ and has turning points when $\alpha = 0^\circ$ and $180^\circ$, as for the other actions in the sagittal plane: the maximum occurs when $\alpha = 180^\circ$. The maximum value of NI for DroppingT is greater than any of the other actions performed in the sagittal plane; DroppingT produces a greater angular displacement than DiverS, Drop and Raise all in the sagittal plane, but a smaller tilt than FullS in the sagittal plane. The effect of FullS is greater than DroppingT for the full actions because, although it has a smaller maximum value of NI, NI has a much smaller amplitude of oscillation, maintaining a higher value for a greater arc of the arm movement; anatomically FullS in the sagittal plane is not possible since the arm moving behind the body will need to deviate from the sagittal plane. It is thus more reasonable for the athlete to focus on performing DroppingT as an adjusted action. Figure 6-10 shows NI for the actions in the sagittal plane, for the example athlete in Section 3.10. The symmetry about $\alpha = 0$ means that the same angular displacement is obtained if the arms are lowered in front of the body or raised behind the body; although in reality it is not anatomically possible to move the arms in the sagittal plane behind the body for a large portion of the $180^\circ$ arc.
The end posture of DroppingT is L. The proportion of ‘athletes’ that could achieve the same skills as given in Table 6-1 is given in Table 6-2.

**Table 6-2: Proportion of ‘athletes’ predicted to achieve a selection of skills when using DroppingT.**

<table>
<thead>
<tr>
<th>Skill</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>0.97</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.74</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.46</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.33</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.21</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.08</td>
</tr>
</tbody>
</table>

The proportion of ‘athletes’ able to achieve each skill is greater when using DroppingT than with Drop, Raise, or DiverS in the frontal plane. Using DroppingT rather than Drop now means that almost half the ‘athletes’, rather than around a quarter, can achieve a somersault with 3/2 twists. DroppingT is still, however, inadequate for most ‘athletes’ wishing to complete a double twist.

Since the action of DroppingT allows more twists to be completed than any of the frontal plane actions, when performing a twisting somersault, an athlete could use some torsion of the chest so that the arms can be moved together in the plane including the angular momentum vector sooner than if waiting until the whole body reached the quarter-twist position. The torsion could be progressively reduced to allow the arms to continue in the same plane as the body twists.
If sufficient flight is available a greater angular displacement will be obtained if Drop is performed in the frontal plane, followed by dropping the second arm in the sagittal plane, rather than if DroppingT is performed. Thus, to maximise the change in $\phi$ when starting with both arms raised, an athlete should start Drop in the frontal plane, moving as fast as possible to complete as much of the action in the frontal plane; then as they start twisting progressively adjust the Drop action and begin dropping the second arm. The second arm will contribute the greatest angular displacement when it starts after the first is lowered and also if it could be lowered after a half twist (which means it is again in the frontal plane).

When performing a skill with the constraint of limited time, a single initiation action and achieving the greatest rate of twist as soon as possible, rather than delaying to the half-twist position, is desirable. The second arm should therefore start as soon as possible and be moved as fast as possible, so that both arms end lowered by the side.

Based on the above exploration, let us add to our coaching suggestions. Again realising that arms are typically raised at the start of a backward somersault Drop and DroppingT are the most suitable actions; while for forward somersaults the arms are partly or fully lowered and so Raise and DiverS actions are more typical:

- As the athlete twists the asymmetrical arm actions should each be adjusted to continue to tilt towards the angular momentum vector, thereby reducing $\phi$. The arms should be moved as fast as possible, so that more of the action is performed in the frontal plane.

- When using Drop or Raise, as the twist begins the second arm should begin to move as soon as anatomically possible, so that both are moving together on the same side of the body. This increases the angular displacement in comparison to using only one arm.

## 6.2 Adjustments to the idealized actions

### 6.2.1 General actions with straight arms

FullS, DiverS, Drop and Raise, described in Section 6.1, are set actions performed with straight arms, starting in posture 1U1D, LAP, LAU, or L and each ending in the posture 1U1D. While still preserving the impression of moving between these postures and performing FullS, DiverS, Drop and Raise, an athlete may slightly adjust the action. Plots of Green’s function for the 3-segment planar model used to model the asymmetric arm actions thus far (Section 4.2.4), may be used to suggest slight adjustments to the actions which would increase the tilt produced.
To start the exploration Figure 6-11, illustrates the paths for the actions FullS, DiverS, Drop and Raise, and presents schematics of postures at some crucial coordinates, on a square representing the full range of $\alpha_1$ and $\alpha_3$. The quadrant where the arms would be moving lateral to the body is marked in yellow. All the dashed or dotted lines are overlayed plots of Green's function to assist visualising how the positive and negative areas are located with respect to these arm actions.

![Figure 6-11: Postures and actions plotted as functions of $\alpha_1$ and $\alpha_3$.](image)

The quadrant in yellow is that in which the arms move lateral to the body. Movement of the arms in this quadrant will be in the frontal plane. In other quadrants, anatomical limitations mean that the arm actions will not be strictly in the frontal plane; nevertheless Green’s function in this region may be used to suggest whether or not some extension of the action beyond the lateral actions may be worthwhile. The + and – signs are written in front of the action names to indicate if the tilt produced is to the right (+) or left (-). This will be the result of moving the opposite arm e.g. Drop the right rather than the left arm. As a result of $360^\circ$ being a full circle the corners of the square will all be the posture L, and the postures and action along the parallel sides will be the same. The dotted lines aid visualisation of diagonal paths of interest, which will help when seeking to generally describe plots of Green’s function.

When analysing plots of Green’s function for the arm actions there are a few features that are beneficial to realise. Firstly, due to 0 and $360^\circ$ being the same posture in terms of joint angles, the points on opposite sides of the square represent the same posture. As a result an action whose path traces around the square will produce zero tilt; this means that integrating over the whole square will give the value zero, and so the absolute maximum integral will be about a contour that encloses all the positive area on the left or all the
negative area on the right. Secondly, the line $\alpha_3 = -\alpha_1 + 360^\circ$ represents the action where both arms drop symmetrically and, since the body posture is symmetrical, no tilt is produced. It may thus be used to complete the contour enclosing an area, in the knowledge that there will be no contribution to the tilt produced. In this case Green’s function will be symmetric about $\alpha_3 = -\alpha_1 + 360^\circ$, as mirrored paths from $\alpha_3 = -\alpha_1 + 360^\circ$ will represent actions with the left and right arm swapped; the arms are assumed to be identical, and the layout posture is symmetrical, thus the direction of the tilt would be equal in magnitude but opposite in direction.

In this section, plots of Green’s function will be given for the example athlete of Section 3.10. The plots of Green’s function are specific to each ‘athlete’, although the general shape is similar: the maximum value of the function lies between $\alpha_3 = \alpha_1$ and $\alpha_3 = \alpha_1 + 180^\circ$ and the minimum value lies between $\alpha_3 = \alpha_1$ and $\alpha_3 = \alpha_1 - 180^\circ$. One part of the zero contour lies close to $\alpha_3 = \alpha_1$, while the other is a concave curve from the point $(180^\circ, 0^\circ)$ to the point $(0^\circ, 180^\circ)$. The extent of the elongation of the oval-shaped contours and the extent of the concavity of the zero contours varies with the ‘athlete’. The similarity in the general shape means there will be similarities in the actions that maximise tilt across the ‘athletes’. It is thus sufficient in this section to only provide example plots of Green’s function for one athlete. Figure 6-12 shows Green’s plot for the example athlete from Section 3.10.

![Figure 6-12: Plots of Green’s function for two straight arms and the body in layout](image)

Green’s function has the units degrees$^{-1}$. The inertial property data used was for the example athlete identified in Section 3.10. The colour scale used designates the largest number as deep red and the smallest number as deep blue.

The idealized arm actions are direct routes between fixed postures which will be fixed values of $\alpha_1$ and $\alpha_3$. The value of Green’s function near the direct route between these points may be used to suggest slight adjustments to the idealized arm actions. The tilt
produced by an alternate action will be greater than the idealized arm actions (direct route) if a positive area is enclosed on the left, or a negative area on the right, of the alternative route and the negative of the direct route (Section 4.3.2 and Figure 4-33).

From Figure 6-12 it is clear why DiverS produces more tilt than Drop: there is a large negative area on the right of DiverS and enclosed by the paths representing DiverS, Drop and the path $\alpha_3 = -\alpha_1 + 360^\circ$. For this athlete Raise was only slightly better than DiverS, since only slightly more negative than positive area is on the right of Raise and enclosed by the paths representing Raise, DiverS and the path $\alpha_3 = -\alpha_1 + 360^\circ$. It is also interesting to observe that Raising the lowered arm, then Dropping the originally raised arm will produce less tilt than FullS, but Dropping the raised arm, then Raising the originally lowered arm will produce slightly more tilt than FullS. The order in which the arms are moved is thus important.

The tilt produced by Drop may be increased by starting with both arms not quite fully raised (on the line $\alpha_3 = -\alpha_1 + 360^\circ$), then Dropping one arm as before and Raising the other one towards the end of the movement. Doing this means negative area is enclosed by this alternative action, the path $\alpha_3 = -\alpha_1 + 360^\circ$, and the original Drop action, and it is on the right of the alternative action. This path is illustrated in Figure 6-13 a). Obviously the more abducted the arms are at the start, the closer the action is to DiverS. The greater the abduction the greater the tilt produced. The abduction, however, will be limited by how the new start position will affect the take-off actions. A similar approach may be considered for Raise, although the dropping arm should be moved closer to the start of the action so as to enclose only positive area on the left of the alternative action. This path is illustrated in Figure 6-13 b). The greater the abduction the closer the action is to DiverS, but unlike Drop, the abduction should be limited so as to enclose only positive area on the left. The limit depends on the ‘athlete’ and will lie between $\sim 30^\circ$ and $\sim 50^\circ$. 
Figure 6-13: Adjusting the idealised asymmetric arm actions, Drop and Raise
A possible alternative path is shown in red, starting at the red cross. a) Adjusting Drop: start with both arms raised but with slight symmetrical abduction, then drop one while raising the other. The slower the raising arm raises the more negative region on the right may be enclosed and the greater the tilt that may be produced. b) Adjusting Raise: Start with both arms lowered but with slight symmetrical abduction, then raise one while dropping the other. The dropping arm should drop early in the action, particularly when using greater values of abduction to start.

To adjust DiverS to increase tilt the arms should start lower than in posture LAP, so that the Raising arm is moved through a greater arc. In this way DiverS approaches the adjustment to Raise, from Figure 6-13 b), from the other side. To adjust FullS, the dropping arm should initially lead, and the raising arm catch-up. In this way the path encloses some of the negative region of Green’s function on the right. The limit to the path adjustment is the contour of the Green’s function plot that equals zero; the location of this contour depends on the specific athlete’s inertial properties. Figure 6-14 illustrates this concept.

Figure 6-14: Adjusting FullS
The dropping arm leads and the raising arm catches up to simultaneously reach their end positions in posture 1U1D. The contour of Green’s function that has the value of zero is the limit of the adjustment, and depends on the athlete.
The pattern of the Green’s function plot may also be used to explain how some tilt may be produced by only small asymmetry when both arms are lowered. Sanders (1995) observed that trampolinists used the technique just described when performing ½ twisting forward somersaults. If the left ($\alpha_3$) arm leads when dropping, a right (positive) tilt is produced; this occurs as the path balloons out and encloses some negative region on the right. The exact path to follow to maximise the tilt will depend on the athlete; a general description would be, start dropping both arms, after about 10° of drop the left arm should start to progressively lead until it passes the laterally out-stretched position, and then the right arm should catch-up before ~60° from the fully lowered position, and then finally both arms lower together. Figure 6-15 illustrates. The limit of the left arm’s lead will be determined by what will not detract from the performance where close to symmetrical lowering of the arms is expected; based on the sketches provided by Sanders (1995) the maximum lead of one arm appears to be approximately 45°.

![Figure 6-15: Slight asymmetry when lowering both arms](image)

The slight asymmetry starts where the left arm begins to lead and ends with the right arm catching up to return to a symmetrical situation. The red curve is one of a few that could be drawn. The contour of Green’s function which has the value zero is the limit of the adjustment, and depends on the athlete.

### 6.2.2 Allowing bending of the arms

The straight arm actions described in Section 6.1 and Section 6.2.1 all end in the posture 1U1D. In Section 5.3.2, posture 1U1DB was found to be a more efficient twist posture than posture 1U1D. Further, it was observed in Section 6.1.1 that portions of the actions Drop and Raise produced tilt in the opposite direction to that required, and it was suggested that this effect could potentially be reduced if the arms are flexed at the elbow. The action of flexion at the elbow to bend the arm from posture 1U1D to posture 1U1DB will also affect the tilt, although the direction of the change in tilt is unclear. As a result it is of interest to investigate the effect of flexion at the elbows during the idealized asymmetrical arm actions.
The three segment planar model, with the reference segment as segment 1, as in Equation (4-33) may be used to explore flexing at the elbow of the raising arm during Raise. The three segments would be 1) the body with the left arm by the side 2) the right upper arm, and 3) the right lower arm. Assuming that the centres of gravity of the upper and lower arms lie on their longitudinal axes then $V_{3x} = V_{21x} = V_{23x} = 0$, and so $E = 0$. Since it is desirable to end in posture 1U1DB the question of interest is: when should the flexion at the elbows occur? As was used in Section 6.2.1 plots of Green’s function for this model, Equation (4-36) may be used to compare paths between the set start and end positions of posture L and posture 1U1DB for the action Raise.

Flexion at the elbows is ‘possible’ in both directions $\pm 90^\circ$, because the arm may be rotated about its longitudinal axis; both the positive and negative directions are, however, not anatomically possible for all upper arm positions. Flexing at the elbow up to $+90^\circ$ will be considered throughout the full drop or raise of the arm and flexing at the elbow to $-90^\circ$ will only be considered as possible when the upper arm is between $0^\circ$ and $-90^\circ$. Figure 6-16 plots an example of a Green’s function with paths of interest marked.

**Figure 6-16: Actions and postures when one arm is allowed to bend**
The Green’s function plot is drawn for the example athlete from Section 3.10. The units of Green’s function are degrees$^{-1}$.
A similar pattern of Green’s function is seen across all the ‘athletes’. To move from the posture L (0°, 0°) to the posture 1U1DB (-180°,-90°) the path giving the greatest tilt will be the one joining the points (0°, 0°), (0°,-90°), (-90°,-90°), (-90°,0°), (-180°, 0°), (-180°,-90°), in that order. This flexion and extension at the elbows of the arms is not an easy action to perform quickly. Flexing at the elbows when the arm is laterally outstretched, that is from (0°, 0°) to (0°,-90°), is particularly challenging. It is more reasonable to expect the athlete to flex at the elbow after the laterally outstretched position; if the arms are kept straight until the laterally outstretched position then the athlete should keep them straight as long as possible before flexing at the elbow. Flexing at the elbow so that the arm is bent across the body at the start of the arm action should be avoided; although one may think that flexion of the elbow in either direction would reduce the negative region of NI (Sections 6.1.1 and 6.1.3), from Figure 6-16, it is clear that this is not true.

Drop is opposite to Raise, except for the position of the non-moving arm; the magnitude of the resulting tilt produced will differ slightly, as seen by the slight shift and reduction in amplitude of NI in Figure 6-4, but the general advice is the same.

The recommendation of keeping the arm straight as long as possible is based on the assumption that the athlete has sufficient time and coordination to perform the two-step action of moving the straight arm and then flexing at the elbow. In practice, a single smooth action to end in posture 1U1DB may be easier for an athlete to perform. Single smooth actions have been previously observed (Pike, 1980). A smooth idealized action that ends in posture 1U1DB would be to start flexing at the elbows as the upper arm passes the laterally outstretched position and smoothly flex at the elbows so that the upper and lower arms simultaneously reach their end positions in posture 1U1DB. As can be seen in Figure 6-17, for the example athlete from Section 3.10, the flexing at the elbows increases the tilt produced from the base action with straight arms: the action of flexing at the elbows has added more tilt than is lost by the fact that the arms being moved are progressively more bent. Considering all ‘athlete’ inertial property data sets, the clear majority (> 95%) produce greater tilt due to the action of DiverS, Drop, and Raise ending in posture 1U1DB than when ending in posture 1U1D.
Idealised aerial twist initiation

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Figure 6-17: Cumulative tilt produced by moving straight arms vs. allowing bending
Flexion at the elbows to bend the arms is allowed after the arms reach the laterally outstretched position, and is at a speed enabling the ending to finish at the same time as the upper arm reaches its final position; hence the dotted lines start only at the laterally outstretched position. The $\alpha$-axis shows the angle moved by the upper arm. For Diver$S$ both arms only move 90° while for Drop and Raise the upper arm moves 180°. Lower arms only ever move 90° as they bend. The curves are plotted for the example athlete from Section 3.10. Note: Full $S$ would simply be Diver$S$ plus Diver$S$ with bending.

Tilt produced by the action is not the only thing to consider. Postures 1U1D and 1U1DB also differ in tilt due to the asymmetry of the postures themselves. This must be added to the tilt produced by the action. For all ‘athletes’ posture 1U1D has greater tilt due to the posture being in the desired direction than posture 1U1DB. This difference in the tilt due to the posture is greater, for at least 98% of ‘athletes’, than the difference due to the action; thus the total tilt due to the action of moving smoothly to posture 1U1DB is less than the total tilt due to moving to posture 1U1D, for all three actions: Diver $S$, Raise and Drop. Nevertheless, posture 1U1DB was found to be a more efficient twist posture (Section 5.3.2). Since the value of $N_0$ determines the skills that may be achieved this will be the means by which the actions will be compared. Figure 6-17 is a scatter plot showing how the difference in total tilt translates into a difference in $N_0$. 
Idealised aerial twist initiation

Figure 6-18: Difference in tilt when using an idealized arm action and ending in posture 1U1DB rather than posture 1U1D.

Each symbol represents an ‘athlete’. The difference in the total tilt is that due to the action and the posture. The difference is a subtraction and so a positive value means that ending in posture 1U1DB had the larger tilt, or larger value of $N_\theta$

The smooth action to posture 1U1DB and then holding this posture required a smaller value of $N_\theta$ than moving to posture 1U1D for ~13% of ‘athletes’, when using DiverS, for ~23% when using Drop, and for ~43% when using Raise. For these ‘athletes’, as can be seen in Figure 6-17, the decrease in $N_\theta$ was small. Thus it appears that even though posture 1U1DB is a more efficient twist posture, the reduction in net tilt will mean that posture 1U1D has a lower value of $N_\theta$. The implication of these values of $N_\theta$ in terms of the proportion of ‘athletes’ that may perform various skills, is given in Table 6-3.

<table>
<thead>
<tr>
<th>End 1U1D</th>
<th>End 1U1DB</th>
</tr>
</thead>
<tbody>
<tr>
<td>DiverS</td>
<td>Raise</td>
</tr>
<tr>
<td>1/2 twisting somersault</td>
<td>0.82</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.67</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.26</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.05</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.01</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.00</td>
</tr>
</tbody>
</table>

From Table 6-3 it can be seen that the reduction in the number of ‘athletes’ that can perform each skill is only a few percent. Remembering that posture 1U1DB has lower angular momentum requirements than posture 1U1D (Section 5.2.1 and 5.3.4), moving smoothly to posture 1U1DB is a useful technique. If the ‘athlete’ is able to keep their arms straighter for longer using DiverS, Drop, or Raise, and then bend them to end in posture 1U1DB, they would increase the tilt they achieve and so ending in posture 1U1DB would
progressively reduce $N_0$ to below the situation which ends in posture 1U1D and hence be a better option.

### 6.3 Kick-out twist initiation

An athlete may initiate continuous twist as part of extending or kicking-out from a piked posture. This could be the result of the piked posture preventing continuous twist (Section 5.4.3), which then starts when the athlete extends, or due to actions occurring during the extension from a piked posture. In this section an idealised kick-out action, intended to tilt the longitudinal axis towards the angular momentum vector is considered.

Rackham (1970) describes an action were the diver starts in a piked somersault, then turns “his shoulders to the right and his legs will react by swinging to the left...He opens out into the straight position and his body will have tilted sideways” (Rackham, 1970, p. 265)

It is this general action that will be considered in this section: torsion through the torso and then extension from the piked position. Considering the anatomy of the body, torsion through the torso, was taken to occur at the abdomen-chest joint, about the longitudinal axis of the chest. The flexibility of athletes does vary, but to allow comparison across athletes, 20° of torsion was used. The extension was taken to occur at the pelvis-abdomen, abdomen-chest, and hip joints as appropriate to move from the piked position to a straight position. Piked postures EP (Section 3.9.12) and P (Section 3.9.14) were considered to assist understanding the effect of the depth of the pike on the tilt that may be produced by the action. The arms were assumed to straighten from their starting piked position, as if the athlete were performing piked somersaults prior to the twisting one, to the sides to finish in the posture L, but with torsion remaining between the abdomen and chest; it is reasonable that an athlete will maintain this torsion, as they look in the direction of the twist.

Two kick-out actions were considered: the first where the torsion occurs followed by the extension and the second where the torsion and extension occur simultaneously.

Figure 6-19 shows the somersault, tilt and twist progressively during the first kick-out action for the example athlete from Section 3.10. Figure 6-20 just shows the tilt produced.

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19 An example of the active range of motions of the spine expected during an assessment is given at [https://lumbarspineassessment.wordpress.com/examination/active-range-of-motion/](https://lumbarspineassessment.wordpress.com/examination/active-range-of-motion/). This gives the value of rotation as between 3 and 18 degrees; thus 20 degrees would be an upper bound of what is reasonably anatomically possible for active rotation. Active rotation is what could be expected of an athlete performing the action while airborne. Passive or assisted range of motion in rotation may be higher.
Figure 6-19: Angular displacement due to performing a kick-out twist initiation from posture EP and posture P.

The plot is for a smooth action. The x-axis is scaled so that the torsion component of the action is the first half of the scale and the extension of the body is the second half of the scale, and so that the actions finish together regardless of whether extending from posture P or posture EP. When the somersault curve diverges from close to zero, marked by the vertical black line, this is where the extension from the posture EP or posture P occurs. The inertial property data used was for the example athlete from Section 3.10.

Figure 6-20: Tilt produced when performing a kick-out twist initiation from posture EP and posture P.

This is a repeat of Figure 6-19, but only shows the tilt so the scale on the y-axis may be reduced.
The posture P has some additional flexion through the torso and a large amount of additional flexion at the hips compared to posture EP. As can be seen in Figure 6-19 the difference in the twist produced when kicking out from posture EP or P is quite small, while the difference in the tilt and somersault produced is considerable. The somersault angular displacement when extending from posture P is negative while from posture EP it is positive. This is only a concern if the athlete entered the aerial phase in posture P; in most cases the athlete entered the posture P after take-off from a posture similar to EP and at that point in time positive somersault of similar magnitude would have been gained; thus the net somersault angular displacement would be small. For this ‘athlete’ the kick-out technique from posture P produces $\sim 2.34^\circ$ of tilt while from posture EP it produces $0.19^\circ$. The tilt produced when using posture EP was found to produce only a slow rate of twist. It is thus essential, when using this technique, that the ‘athlete’ uses posture P and not posture EP.

Considering all the ‘athletes’, posture EP produced less tilt than posture P for all except two ‘athletes’; for one of these two ‘athletes’ kick-outs from posture P produced negative tilt. The kick-out action ends in a layout position with torsion through the torso. This is a symmetrical posture about the medial axis, and so there is no tilt due to the posture; the net tilt will be the tilt from the kick-out twist initiation action only. Table 6-4 gives the proportion of ‘athletes’ that can perform various skills using the kick-out from posture P action. The one ‘athlete’ that produced negative tilt is counted as not being able to achieve a half twisting somersault, since the twist would be in the incorrect direction.

<table>
<thead>
<tr>
<th>Skill</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>0.62</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.25</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.05</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.04</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.00</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.00</td>
</tr>
</tbody>
</table>

As can be seen from Table 6-4, the kick-out action will allow the majority of ‘athletes’ to perform only a half twist within a single somersault. These results are contrary to Rackham’s observation, following his description of the kick-out, that

“George Nissen [known as the inventor of the trampoline] was able to demonstrate on the trampoline, a front somersault with 1½ twists *with his hands in his pockets* using his shoulders only to initiate the twist” (Rackham, 1970, p. 265).

Rackham’s observation was by eye and so it is impossible to know if the twist produced was due to the kick-out action Rackham previously described, some other action that was
occurring, or George Nissen used contact twist that was not observed until he extended out of the pike. The only thing that is clear is that asymmetrical arm actions, such as described in Section 6.1, were not used since Nissen’s hands where in his pockets.

The second kick-out twist initiation action where the torsion and extension occur simultaneously is even less effective than the first. Figure 6-21 is a scatter plot showing the value of $N_\theta$ of the two kick-out actions for each ‘athlete’. It is clear from this plot that using the simultaneous action increases $N_\theta$ so that the majority for ‘athletes’ would not even be able to perform a half-twist in a somersault.

![Figure 6-21: $N_\theta$ produced by the two different kick-out twist initiation actions](image)

Each dot is an ‘athlete’. The x-coordinate is the value of $N_\theta$ for when the torsion and extension of the kick-out action are performed sequentially and the y-coordinate is the value of $N_\theta$ for when the torsion and extension of the kick-out action are performed simultaneously.

The kick-out twist initiation technique as described by Rackham and modelled here is not a useful twist initiation technique. This is not entirely surprising, since the legs in posture P would be expected to have a large moment of inertia compared to the torso, about the axis of the torsion; thus a large movement of the legs is not expected. Rackham (1970) indicates that the legs should swing ~10°, but it is not clear how he arrived at this figure, which seems unrealistic. When ‘athletes’ are observed to perform a twisting somersault after a piked somersault, some other action(s) must be occurring.
6.3.1 Additional torsion at the chest-abdomen joint

If additional torsion could be anatomically achieved during the kick-out actions, it is possible to achieve greater tilt values. It has been suggested[^20] that 60° of torsion is anatomically possible. To see the effect of greater torsion the tilt produced when using the kick-out action, where the torsion and extension are performed sequentially, was calculated for 60° of torsion and the example athlete from Section 3.10. The results of the tilt produced over the action is presented in Figure 6-23.

![Figure 6-22: Tilt produced when performing a kick-out twist initiation using 60° of torsion from posture EP and posture P. This figure is similar to Figure 6-20, but now 60° of torsion is applied in the first stage.](image)

By comparing Figure 6-23 and Figure 6-20 it is clear that the additional torsion does dramatically increase the tilt produced by this action. For this example athlete the tilt produced is much greater than the tilt produced by DiverS, Raise and Drop, but not quite as large as that produced by FullS. If such a kick-out could be anatomically achieved then this action would be an effective twist initiation action.

6.4 Torso flexion with left-right asymmetry

If torso flexion has left-right asymmetry then it may produce a change in $\phi$ or $\psi$. To produce continuous twist in a somersault, a change in the angle of $\phi$ is required. A change in $\psi$, will add a single finite angular displacement to the twist; continuous movement would be required to continue twisting.

[^20]: In a private communication with Yeadon, he stated that 80° of torsion was anatomically possible and so his model in his 1993 paper (Yeadon, 1993c) which used 60° of torsion he believes is anatomically reasonable. Although the current author does not agree that such large ranges of torsion are anatomically possible, this section does show that if torsion could be achieved then the tilt produced would increase.
This section will consider both lateral flexion, which is a single movement, and the “hula”, which is a continuous movement, involving flexion of the torso and hips, with the intent of producing twist in a somersault.

Using the 17 segment model (Section 3.1.1) and the posture LHF (Section 3.9.9) the flexion through the torso occurs at the pelvis-abdomen and the abdomen-chest joints. Since the flexion needs to occur at both torso joints then only inertial property data sets with three torso segments will be applied to the actions.

### 6.4.1 Lateral flexion

Lateral flexion is a single movement from the posture L (Section 3.9.7) to the posture LHF (Section 3.9.9). Although most authors (Van Gheluwe, 1981; Yeadon, 1984; Sanders, 1995) who have observed this movement believe it is simply a part of a “hula” movement, George (1980) describes an “unequal radius technique” which appears to be lateral flexion, with the athlete described as shortening their body on one side to produce twist in a somersault. George (1980) indicated that the twist produced by such a method will be slow; however, it is prudent to consider lateral flexion separately from the “hula” to ascertain if it is a useful technique in and of itself, and whether or not the “hula” adds anything beyond lateral flexion.

Any change in \( \phi \) due to lateral flexion while airborne will be the sum of the change due to the difference in the location of the principal axes with respect to the pelvis by nature of the LHF posture, and the actual movement. The programme ICG17 was used to determine the direction of the principal axes with respect to the pelvis in posture LHF. The magnitude of the tilt is then the inverse cosine of the dot product of the principal y-axis of postures LHF and L; the direction of the tilt has the same sign as the z-component of the principal y-axis of posture LHF with respect to the principal frame of posture L: a tilt to the left is negative.

Since the flexion of interest occurs at the pelvis-abdomen and the abdomen-chest joints, and lateral flexion is a single movement within the frontal plane, then modelling lateral flexion was reduced to a three segment planar model. The three segments in the planar model for the actual movement were 1) the pelvis and legs, 2) the abdomen, and 3) the chest, neck, head, and arms combined. The three segments are symmetrical about their longitudinal axes, and so \( V_{1x} = V_{21x} = V_{23x} = V_{3x} = 0 \), meaning that the constants \( C, E, \) and \( G \) are all zero. The reference segment will be segment one in this case, since it contains the pelvis. Thus, to determine the angular displacement due to the movement from posture L to posture LHF, Equation (4-33) was used. Posture LHF has 20° of flexion at each of the pelvis-abdomen and the abdomen-chest joints. Assuming a smooth action where both joints flex simultaneously and move in sync from posture L to posture LHF
(Section 3.9.9) then the path is along $\alpha_{2/1} = \alpha_{3/2}$ from $(0^\circ, 0^\circ)$ to $(-20^\circ, -20^\circ)$. Thus the equation for the angular displacement is

$$
\gamma_{H=\alpha_{\text{ref}, -1}} = -\left[\alpha_{2/1}\right]_{-20^\circ}^{20^\circ} + \int_{0}^{-20^\circ} \frac{(I_1 + A - D \cos \alpha_{2/1} + L \cos(2\alpha_{2/1}))}{(N + 2L \cos(2\alpha_{2/1}) - 2D \cos \alpha_{2/1} - 2F \cos \alpha_{2/1})} \, d\alpha_{2/1}
$$

$$
- \int_{0}^{-20^\circ} \frac{(I_1 + B - F \cos \alpha_{2/1} + L \cos(2\alpha_{2/1}))}{(N + 2L \cos(2\alpha_{2/1}) - 2D \cos \alpha_{2/1} - 2F \cos \alpha_{2/1})} \, d\alpha_{2/1}
$$

$$
= 20^\circ \int_{0}^{-20^\circ} \frac{I_1 - I_1 + A - B + (F - D) \cos \alpha_{2/1}}{(N + 2L \cos(2\alpha_{2/1}) - 2(D + F) \cos \alpha_{2/1})} \, d\alpha_{2/1}
$$

(6-5)

For all of the inertial properties collated in Chapter 3 (‘athletes’), the integrand from $-20^\circ$ to $0^\circ$ is negative, monotonically increasing, and concave up; thus over this domain, as the lateral flexion increases so does the angular displacement, by ever increasing amounts.

Since the starting posture is fixed, as posture L, the maximum tilt produced occurs when the athlete is able to flex all the way to posture LHF while the flexion is still in the frontal plane.

The tilt angles for postures L and LHF, due to the different position of the principal axes with respect to the pelvis, and the tilt due to the action of moving from posture L to posture LHF are in opposite directions. There is a strong correlation between these tilt values (Figure 6-23) and so the net tilt is small. The five figure summary of (Minimum, LQ, Median, UQ, Maximum) is (-4.63, -1.10, -0.54, 0.04, 2.73). Since the net tilt is positive for some ‘athletes’ and negative for others, some ‘athletes’ will twist to the left and others will twist to the right; unless the athlete’s inertial properties are specifically calculated the athlete will not know if they will twist left or right without actually attempting the skill: this makes for a disconcerting situation mid-air! Further, not only is the tilt small, but also the posture LHF is less effective as a twist posture than L or 1U1D (Section 5.4.2), and so the twist produced would be a slow twist, confirming the statement by George (1980); as a result, lateral flexion performed while airborne is not recommended.
Idealised aerial twist initiation

Figure 6-23: Tilt due to posture compared to tilt due to the action.
The tilt due to the posture is a result of the differing position of the principal axes with respect to the transverse axis of the pelvis for posture L and posture LHF. The tilt due to the action is the result of moving from posture L to posture LHF. There is a strong correlation between these two tilts, as shown by the black line and the stated value of $R^2$ as 0.9711. For this plot, a positive tilt is to the right, and a negative to the left. Thus the action of moving the shoulders to the left and hips to the right tilts the longitudinal axis of the pelvis to the right; while the location of the principal axes of the posture means that the principal axes are tilted to the left of the longitudinal axis of the pelvis.

Individually the tilt due to posture and the tilt due to the action each produce moderate values of tilt. If the two tilts could be separated then the twist would be quite fast. The two tilts may be separated by laterally flexing on the ground so that the athlete enters the airborne phase already laterally flexed, but with the longitudinal axis of the pelvis vertical. If the athlete stays in the posture LHF then the tilt will be simply due to the position of the principal axes with respect to the pelvis; if the athlete moves from posture LHF to posture L then the tilt that remains will be due to the action. In either situation the tilt that remains will be negative. This means that when the flexion is such that the shoulders move to the left, as pictured in Figure 3-24 which has the angles $\alpha_{2/1}$ and $\alpha_{3/2}$ negative, then the tilt that remains is to the left; in a forward somersault this will produce a twist to the left, and in a backward somersault the twist produced is to the right. As a result of the direction of flexion and the final direction of twist, this technique is more useful in a forward somersault, where the athlete can laterally flex on the ground and easily look in the direction of their final twist.

Maintaining posture LHF or removing it in the air will mean that the athlete is twisting in a different posture, and so it is necessary to compare these two techniques by the skills
achievable, rather than assuming that the technique that gives the larger tilt will maximise the twist in a somersault. Obviously only the inertial property data sets with three torso segments can be compared. Table 6-5 gives the proportions of ‘athletes’ using each technique that can perform various twisting somersault skills.

<table>
<thead>
<tr>
<th></th>
<th>Remove lateral flexion</th>
<th>Maintain lateral flexion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.96</td>
<td>0.91</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.80</td>
<td>0.53</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.51</td>
<td>0.16</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.24</td>
<td>0.05</td>
</tr>
</tbody>
</table>

It can be seen from Table 6-5 that removing the lateral flexion rather than maintaining it allows more difficult twisting somersault skills to be performed. Removing the lateral flexion produced less tilt than maintaining it for 71% of the ‘athletes’ considered, but the difference was small compared to the advantage of using posture L over posture LHF. The ‘athletes’ considered were a subset (160 rather than 240) of those considered for the arm actions in Section 6.1 since only those sets with a three-segment torso could be used. Nevertheless comparing the proportions aids in determining whether or not laterally flexing on the ground and then removing the lateral flexion in the air would be a technique to recommend; it does allow more ‘athletes’ to perform each twisting somersault skill than any of the arm actions, except that FullS allows a slightly greater proportion of ‘athletes’ to achieve a double twist. Laterally flexing on the ground and removing the lateral flexion in the air is thus quite a useful technique when attempting higher numbers of twist. This would be an alternative hypothesis to the ‘hula’ suggested by Sanders’ (1995) to explain the observation that trampolinists showed lateral flexion near take-off and then again at landing.

The genders were not found to be statistically different from each other. The squads were found to be not part of the same homogeneous population (p < 0.005). The 12-or-under squad tended to have smaller values of $N_\theta$ and the Senior squad tended to have greater values of $N_\theta$.

When generating angular momentum for the somersault, an athlete will generally have some forward or backward flexion at take-off, with the direction of flexion depending on the direction of rotation of the somersault. It is thus likely that the athlete will leave the ground flexed about an axis that is directed somewhere between the $\text{Ref}_x$ and $\text{Ref}_y$ axes; that is, an axis between the one about which lateral flexion occurs and the one about which forward flexion occurs.
Figure 6-24 plots the somersault, tilt, and twist produced when extending from any flexed position. The plot was generated from results created using AngleDisp17.m (Section A.6) and for the example athlete from Section 3.10.

As already identified, extending from a laterally-flexed position is more useful in forward somersaults, since the lateral flexion would be such that the athlete could look in the direction of the twist. In Figure 6-24 the domain of most interest is a forward and left flexed posture—beta is between 270 and 360°. Somersault, tilt and twist are all positive in this domain. Since in a forward somersault the angular momentum vector is to the athlete’s left then the tilt produced would cause twist to the left; the twist produced by this action is a finite amount in the desired direction. The somersault increases in the forward direction. These both aid performance of a left-twisting forward somersault. Obviously more tilt, and hence a faster twist in the somersault, will be produced when the flexion is more lateral, but any lateral flexion on take-off will aid tilt production.

For a backward somersault the angular momentum vector is to the right, thus a negative tilt is desired for a twist to the left. This would not be achieved by a reasonable take-off
position for a backward somersault; starting in an arched and left flexed position would have beta between 180 and 270°, and produce a positive tilt.

The position of the arms while laterally-flexed would affect the tilt that can be achieved. One could investigate different arm positions. However, since it is desirable to have the arms close to the body in the twist posture, in order to maximise the twist rate, it would be of greater interest to consider what tilt could be achieved when an arm action is combined in a smooth motion with extending from lateral flexion. This will be done in Section 6.5.2.

### 6.4.2 “Hula”

The “hula” is a movement named such by Van Gheluwe (Van Gheluwe & Duquet, 1977; Van Gheluwe, 1981) due to perceived similarities to Hawaiian hula dance movements. The hula is “a continuous bending of the body at the hips without torsion” (Van Gheluwe & Duquet, 1977). Van Gheluwe and Duquet (1977) separated the hula out as a separate technique from the traditional “cat-twists” or “two-axis” theory twists since it involved no torsion. They believed that it was used in addition to the “two-axis” method by Belgium’s best gymnasts of their time. The hula was initially believed to produce a finite amount of twist per “hula”; this means that it could not be an action used to initiate continuous twist. Yeadon (1993c) later suggested that the hula could be used to produce tilt, and hence could be used to initiate continuous twist. Yeadon focused on a quarter hula action, and reported that it produced tilt. This section will investigate the twist and tilt, if any, that would be produced by a quarter hula movement using the inertial properties collated in Chapter 3.

In a chronological sense, the first description of the hula action was of two cylinders with conical ends rolling over each other. These cylinders could be equal (Van Gheluwe, 1981; Frohlich, 1979) or unequal (Yeadon, 1984; Yeadon, 1993c). The two cylinders roll towards each other, and hence the hula has no torsion. Although Frohlich (1979) accepts that such a mechanical system would produce twist rotation, he rejects it as a method ever used by ‘athletes’. Further, such an action is incompatible with the anatomy of an athlete. Thus, this description will not be used. Instead the worded description provided by Yeadon (2000) will be used.

“The body moves from a forward flexed position, through side arch [a laterally-flexed position] over the right hip in to a back arch [backward flexion], through a side arch over the left hip [a laterally-flexed position on the other side] and ends in a forward flexed position” (Yeadon, 2000)

The quarter hula will then be the part of the circle where the body moves from a forward flexed position to a laterally flexed position, and then extends (Yeadon, 1997b), to end in the posture L.
As discussed in Section 3.9.9 there is no universal agreement regarding the equations to use to describe either the laterally-flexed position, or the description of the joint angles during the hula. Considering the postures A (Section 3.9.8) and LHF (Section 3.9.9), flexion will be applied to the pelvis-abdomen and abdomen-chest joints. To create a smooth motion between the forward-, lateral-, and back-flexed postures, and to also produce a slightly conservative estimate of the angular displacement produced, the motion will be described as moving between postures with equal flexion angles about an axis which circles the pelvis: the axis about which flexion occurs, may be described in frame Ref by $[\sin \beta(t); \cos \beta(t); 0]$, where the angle $\beta$ is the angle between the flexion axis and the $y$-axis of frame Ref, in a clockwise direction when viewed from above: the angle $\beta$ increases smoothly from zero (forward flexion) to 90° (flexed to the right) for a quarter hula. Since posture A shows the least flexion, it will be considered to be the limit of flexion for the hula circle, and thus the flexion observed there will be the flexion applied at any point around the hula circle: 10° of flexion will be applied to the pelvis-abdomen joint and 20° of flexion will be applied to the abdomen-chest joint.

At any point during the hula action the rotation, $\alpha$, of the abdomen and chest will be

\[
\begin{align*}
\alpha_{\text{Pelvis}} &= 10[\sin \beta(t); \cos \beta(t); 0]° \\
\alpha_{\text{Abdomen}} &= 20[\sin \beta(t); \cos \beta(t); 0]°
\end{align*}
\]

The time derivatives of the rotation, where $\alpha$, $\beta$, and their derivatives are in degrees, is thus,

\[
\begin{align*}
\frac{\partial \alpha_{\text{Pelvis}}}{\partial t} &= \frac{\pi \beta}{18}[\cos \beta \quad -\sin \beta \quad 0]° \\
\frac{\partial \alpha_{\text{Abdomen}}}{\partial t} &= \frac{\pi \beta}{9}[\cos \beta \quad -\sin \beta \quad 0]°
\end{align*}
\]

These derivatives may be used with Angle Disp17.m (Section A.6). The extension from a laterally flexed position would need to occur after the circle action. Although one may be tempted to simply add the tilt produced by extending from lateral flexion, this would give the incorrect results, since some twist has occurred during the circle component of the quarter hula and so the angular displacement due to the extension will no longer be only tilt.

Figure 6-25 plots the somersault, tilt, and twist that would be produced by performing a quarter hula and extending back to the posture L, for one example athlete. Extending from the laterally-flexed posture changes not just the tilt, but also the somersault, and the twist; this is because it produces an angular displacement about the $x$-axis, which is no longer parallel to the $x$-axis of frame Glo.
Twist produced by the hula action is approximately $14^\circ$. This amount would be noticeable to the naked eye, but is not sufficient to mean that the quarter hula is useful in producing twist.

From Figure 6-25 it can be seen that all the tilt produced by the circle part of the quarter hula is effectively eliminated as the athlete extends from the laterally flexed posture. Even though some twist is present it is insufficient to cause the extension to alter the somersault more than the tilt. For all ‘athletes’ the tilt produced by the quarter hula was less than $0.7^\circ$ in magnitude.

![Figure 6-25: Plot of somersault, tilt and twist produced when performing a quarter hula and extending back to L.](image)

The quarter hula starts in a forward or backward flexed position, moves in a circle to a laterally flexed position, and then extends to the posture L. The flexion present at the start and during the circle is $10^\circ$ at the pelvis-abdomen joint and $20^\circ$ at the abdomen-chest joint. The x-axis of this plot gives the value of Beta around the circle from $0^\circ$ to $90^\circ$ and then the amount of extension that has occurred at the pelvis-abdomen joint: $10^\circ$ of extension is from $90^\circ$ to $100^\circ$ on the x-axis. The inertial property data used was of the example athlete from Section 3.10.

Remain in the laterally-flexed posture at the end of the quarter hula would mean there is a tilt due to the action; however, the laterally flexed posture has its principal longitudinal axis tilted in the opposite direction to the action. The net tilt is thus small even if the ‘athlete’ stays in posture LHF: it is less than a degree in magnitude for the majority of ‘athletes’ with the tilt being positive for some and negative for others. The quarter hula technique, as modelled here, is thus not a useful twist initiation technique.

### 6.5 Different body postures or torso movement combined with an arm action

Using a different posture of the non-moving body segments in any action or performing multiple actions simultaneously would alter the tilt produced. Simultaneous actions will
not produce tilt equal to the sum of the individual actions, since the non-moving segments of one action are changing position during the action. It is thus necessary to model simultaneous actions and then compare the results to those of isolated actions.

There are an endless number of different body postures and combinations of actions that could be performed. This section will consider just a few pertinent ones based on the investigation so far.

6.5.1 Different body postures when moving the arms

In Sections 6.1 and 6.2 the tilt produced by the idealized and modified arm actions while the body was in a layout posture was determined. Changing the body posture will alter the inertial properties of the ‘body segment’; that is \( I_2, V_{21}, \) and \( V_{23} \), from Section 4.3.2. If the new body posture held maintains left-right symmetry then \( V_{21x} \) and \( V_{23x} \) will be unaltered. In Equations (6-1), (6-2), and (6-3) any change to \( I_2 \) will only change the value of N, while a change in \( V_{21y} \) and \( V_{23y} \) will change the constants N and D. The changes in N and D will alter the amplitude of the NI curves for all the idealized arm actions, and the phase of the NI curves for Drop and Raise.

To increase the tilt produced by the DiverS action, Frohlich (1979) suggested using a “loose pike position” rather than a layout, since the “loose pike position” has a lower moment of inertia in the frontal plane. Due to the nature of the take-off throws the posture EP (Section 3.9.12) is common on entry to forward twisting somersaults, while the posture A (Section 3.9.8) is common on entry for backward somersaults. Also, by the nature of the arm positions on take-off, Raise and DiverS are more common in forward somersaults, while Drop is more common in backward somersaults. Figure 6-26 adds the plots of NI for the actions DiverS and Raise using the posture EP, and the action Drop using A, to the plots in Figure 6-4. The use of postures EP and A has shifted the NI curves upwards, and increased the amplitude of each curve. When the arms are lateral to the body, that is \( \alpha \) is negative for Raise and DiverS and \( \alpha \) is positive for Drop, then both of these factors increase the value of NI. Using posture A instead of posture L shifts the phase of the NI curve approximately 1° for Drop and using posture EP instead of posture L shifts the phase of the NI curve approximately 4° for Raise. The curves for posture EP and posture A do not cross the curve for posture L and so these postures increase the tilt that would be produced for any part of the curve. The upward shift and the phase shift of the curve also reduce the negative area under the NI curve, thereby further increasing the total tilt, and allowing the ‘athlete’ to be less concerned about starting Raise with the arm partially abducted or ending Drop early.
When using the body posture EP with DiverS and Raise, and the posture A with Drop, NI is shown on the same graph as all three actions in the layout posture. The layout posture curves were previously plotted in Figure 6-4. The inertial property data used was for the example athlete from Section 3.10. To keep the arms lateral to the body $\alpha$ should be negative for Raise and the Odd symmetric actions, FullS and DiverS, while $\alpha$ will need to be positive for Drop.

Figure 6-27 illustrates the relationship between the difference in $N_\theta$, when using posture L rather than either posture EP or A, and the actual value of $N_\theta$ when using posture L. In both cases the end posture used for twisting following the initiating action was posture 1U1D, since the athlete would be expected to extend at the hips from posture EP or A; this extension, when performed before any twist has occurred, will predominantly add to the somersault and any alteration in tilt due to the asymmetric arm position in posture 1U1D is very small. Unlike the example athlete in Figure 6-26, using posture A reduced $N_\theta$ for only approximately 26% of the ‘athletes’. Thus, using posture EP instead of posture L with DiverS and Raise is generally recommended, but using posture A instead of posture L for Drop is not generally recommended.
Figure 6-27: Benefit of using posture P or posture A
The scatter plot shows the relationship between the difference in $N_\theta$ when posture L is used instead of posture EP or A (y-axis), and $N_\theta$ using posture L (x-axis). A positive difference means that using the posture EP or A reduced the number of somersaults required, and so is a beneficial technique. Each dot is an ‘athlete’.

Using posture EP was beneficial for all ‘athletes’, seen by the fact that all dots in Figure 6-27 have a value of the difference in $N_\theta$ greater than zero. A general instruction to use posture EP rather than posture L when performing DiverS or Raise would thus be appropriate. A benefit would be seen by all ‘athletes’, although some would experience a greater benefit than others. In contrast, using posture A rather than L is not beneficial for all ‘athletes’; only approximately 26% would benefit. The difference in $N_\theta$ spreads between approximately ±0.03 (±10.8°). The difference is small, and so a coach may allow the athlete to try both postures and then settle on whichever is the most ‘comfortable’.

The general trend apparent in Figure 6-27 is that ‘athletes’ requiring a larger value of $N_\theta$ experience a greater benefit by using posture EP or A rather than posture L. Nevertheless these ‘athletes’ still have a clear natural disadvantage; Figure 6-28 illustrates a very near linear relationship between $N_\theta$ when using posture L and when using posture EP ($R^2 > 0.99$). The same ‘athletes’ are still at a disadvantage, although the gap between them and the ‘athletes’ with the greatest advantage has reduced.
In Section 6.1.2 it was found that there was no significant difference between genders for any of the idealized asymmetric arm actions. However, the squads were found to be different under the H-test indicating that they are not from the same homogenous population: the literature data required significantly smaller values of \( N_\theta \), and the 12-or-under data required significantly greater values of \( N_\theta \). Due to the linear relationship seen in Figure 6-28 these results will still be valid when using postures EP or A.

Figure 6-27 and Figure 6-28 focused on the values of \( N_\theta \); a clear benefit was obtained when using posture EP rather than posture L. Table 6-6 translates this into the difference in the proportion of ‘athletes’ that can achieve various skills.

<table>
<thead>
<tr>
<th></th>
<th>Diver's (L vs. EP)</th>
<th>Raise (L vs. EP)</th>
<th>Drop (L vs. A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>0.09</td>
<td>0.11</td>
<td>0.00</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.03</td>
<td>0.08</td>
<td>-0.01</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.18</td>
<td>0.23</td>
<td>-0.01</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.17</td>
<td>0.20</td>
<td>-0.03</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.04</td>
<td>0.05</td>
<td>0.00</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
</tr>
</tbody>
</table>

It is clear from Table 6-6 that using posture EP rather than posture L, allows more ‘athletes’ to perform each skill; even though posture A was beneficial for some ‘athletes’ it did not increase the proportion of ‘athletes’ that could perform any of the skills considered. This affirms the value of recommending the use of posture EP rather than posture L, and not recommending using posture A rather than posture L. The greatest increase in the
proportion of ‘athletes’ able to complete each skill when using posture EP was for the 3/2 and 2/1 twisting somersaults, where the increase in proportion was similar to the proportions that were predicted to achieve these skills (Table 6-1). The lower values of the difference for 1/2 and 1/1 twisting somersaults is not surprising, since a large proportion of ‘athletes’ were already predicted to achieve these using posture L. Lower values again for 5/2 and 3/1 twisting somersaults reflect the sheer difficulty of these skills. Table 6-7 gives the proportions of ‘athletes’ that are predicted to now achieve each skill, using posture EP rather than posture L with DiverS or Raise when performing forward somersaults, but maintaining the use of posture L with Drop when performing backward somersaults.

Table 6-7: Proportion of ‘athletes’ able to achieve a selection of skills when using the best body posture and asymmetrical arm action combination.

<table>
<thead>
<tr>
<th>Skill</th>
<th>DiverS EP</th>
<th>Raise EP</th>
<th>Drop L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>0.91</td>
<td>0.92</td>
<td>0.83</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.70</td>
<td>0.71</td>
<td>0.67</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.44</td>
<td>0.45</td>
<td>0.26</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.22</td>
<td>0.23</td>
<td>0.07</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.05</td>
<td>0.06</td>
<td>0.01</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
</tr>
</tbody>
</table>

In Section 6.2.1, plots of Green’s function were used to explore alternative arm actions to the idealised ones in Section 6.1. The change in the body posture from L to EP or A does not change the general shape of the plots of Green’s function, but does shift the scale. This means that the general principals suggested in Section 6.2.1 still apply but the magnitude of the adjustments to maximise tilt, and the benefit gained will differ. Plots of Green’s function for posture EP show a higher peak and lower trough, and the zero contour noticeably shifts towards the lower left-hand corner which corresponds to a reduction in the negative region of NI for the action Raise. Thus, when using posture EP rather than posture L to maximise the tilt, the arms may start in a less abducted position.

In this section, the tilt produced when using postures EP or A was determined using the three-segment planar model from Section 6.1, and applying the principal moments of inertia for postures EP or A. For these results to be valid the arms must be moved in a plane defined by the \( \text{Pri}_y \) and \( \text{Pri}_z \) axes. Since postures EP and A only involve forward or backward flexion, the location of the required plane may be described by an angle between the \( \text{Pri}_x \)-axis and \( \text{Ref}_x \)-axis; let this angle be \( \lambda \). The x-principal direction of the body posture will have its x-component with respect to frame Ref equal to the cosine of \( \lambda \). The sign of \( \lambda \) will be positive if the x-principal direction of the body posture is tipped forwards and negative if it is tipped backwards. The sign of \( \lambda \) may be determined from the z-component with respect to frame Ref of the x-principal direction. Figure 6-29 illustrates the concept with a schematic.
The five figure summary (Minimum, LQ, Median, UQ, Maximum) of the value of $\lambda$ for posture EP across all ‘athletes’ is (-4.21, 2.22, 6.44, 17.80, 32.18). The median value and range of $\lambda$ are much larger than reported by Mikl (2015), which formed conclusions based on only the literature data sets. It is thus clear that the literature data sets alone do not adequately reflect the gymnast and diver population.

The frontal plane of the pelvis was the closest frontal plane of any segment to the plane defined by the $^\text{Pri}_y$ and $^\text{Pri}_z$ axes for 70% of the ‘athletes’; for the remainder, the frontal plane of the abdomen was closer. Thus, the frontal plane of the pelvis was the most helpful plane from which to define $\lambda$ in order to visualise and describe the location of the plane defined by the $^\text{Pri}_y$ and $^\text{Pri}_z$ axes.

It would be logical to assume that if the arm actions are moved in any plane other that the principal frontal plane the amount of tilt produced would be reduced since some twist and somersault re-orientation will also occur; this has not, however, been proved. To explore if moving the arms in a plane other than the principal frontal plane does produce the maximum tilt, let us consider the action DiverS using posture EP for one example athlete. Figure 6-30 and Figure 6-31 plot the tilt and twist produced by performing DiverS using posture EP when the arms are moved in planes at an angle to the frontal plane of the pelvis between $-45^\circ$ and $+45^\circ$. Also marked on the figures, is a dashed vertical line, which indicates the value of $\lambda$ for this example athlete.
It is interesting to observe that the maximum tilt, 6.32°, does not occur when the arms move in the frontal plane. For this athlete the best plane of action is tipped from the principal plane by approximately -12.3°. The twist produced at the same time as the
maximum tilt is negative with a value of -7.19°. This is contrary to the desired twist, and will need to be compensated for by the faster twist resulting from the greater tilt. The additional tilt reduces $N_0$ by ~0.01, which, when performing more than a half twist, will easily make up for the twist in the incorrect direction and allow additional twist to be completed.

Even though moving the arms in the principal frontal plane will produce only tilt, the presence of some twist has allowed the tilt produced at the end of the action to be greater. Posture EP has $I_{yy}$ as the maximum-valued moment of inertia and $I_{xx}$ as the intermediate-valued moment of inertia; this will be one aspect that contributes to the increase by presenting slightly less resistance to the motion of the arm. As the plane deviates further from the principal frontal plane, however, less of the arm action is directed to altering tilt.

This short investigation regarding the plane of action clearly shows that finding the arm action that would produce maximum tilt for any particular body posture is a complex matter and specific to the athlete. A single answer or simple instruction describing the best plane cannot be given.

Even though the maximum tilt will not be produced when moving the arms in the frontal plane of the pelvis it was found to increase the proportion of ‘athletes’ able to perform a selection of skills as may be seen in Table 6-8.

### Table 6-8: Proportion of ‘athletes’ able to achieve each skill when arms are moved in the principal or pelvis plane.

A negative difference means that the action performed in the pelvis plane allows a greater proportion of ‘athletes’ to achieve the skill.

<table>
<thead>
<tr>
<th></th>
<th>In principal plane</th>
<th>In pelvis plane</th>
<th>The difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>0.91</td>
<td>0.92</td>
<td>0.99</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.70</td>
<td>0.71</td>
<td>0.73</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.44</td>
<td>0.45</td>
<td>0.53</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.22</td>
<td>0.23</td>
<td>0.25</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.05</td>
<td>0.06</td>
<td>0.08</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

### 6.5.2 Extension from posture EP or A at a quarter-twist

Extending from an EP posture would produce a somersault angular displacement in the forward direction. If this extension was performed at the quarter-twist position in a forward somersault, then the angular displacement would produce tilt towards the angular momentum vector, reducing $\phi$. Similarly, extending from posture A would cause an angular displacement in the backwards direction, and if this extension was performed at the quarter-twist position in a backward somersault, then the angular displacement would produce tilt towards the angular momentum vector, reducing $\phi$. As a result it is of interest
to consider what tilt, and so what reduction in $\phi$, could be produced by delaying the extension into posture L from posture EP or A. Even though from Section 6.5.1 it was shown that using posture A while performing the idealized arm actions was not beneficial, the benefit of extending at a quarter-twist may make it desirable to use the posture A during the twist initiation for a backward somersault.

To decide if extending at the quarter-twist position is a useful technique, an estimate of the tilt produced may be determined by summing the tilt produced by an arm action and the extension from postures EP or A if each action was performed independently. A summation is a reasonable estimate since these actions are to be performed consecutively, not simultaneously. It is only an estimate since oscillations in $\phi$ will be different between the first quarter-twist and the remaining twist after the extension. Additional tilt could be achieved if flexion of the elbows was allowed to achieve the posture 1U1DB was performed as seen in Section 6.2.2.

Extending from posture EP produces a greater magnitude of somersault angular displacement than extending from posture A; this is as expected since posture EP is a more flexed posture. DiverS and Raise are also more effective actions than Drop (Section 6.5.1), and so performing Drop with the posture A and extending at the quarter-twist is a less effective technique than performing DiverS or Raise using posture EP and then extending at the quarter-twist.

Nevertheless adding the extension from posture A at the quarter-twist overcomes the reduction in tilt when using posture A with Drop. To use this technique it is necessary to be able to reach the quarter-twist position. Unfortunately EP does not display continuous twist for the majority of ‘athletes’ following the arm actions DiverS or Raise, and so prior to the extension the athlete will not be able to reach a quarter-twist, thereby excluding the use of this technique. Posture A displays continuous twist for 99.5% of the ‘athletes’ considered and so it may still be used. Table 6-9 gives the proportion of ‘athletes’ that can perform various skills using Drop and then extending from posture A at the quarter-twist position. The values calculated in Table 6-9 include performing the first quarter-twist in posture A, but ignores any oscillations in $\phi$.

<table>
<thead>
<tr>
<th></th>
<th>Drop in A and extend at $\frac{1}{4}$ twist</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>0.87</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>0.70</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.53</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.27</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.10</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.02</td>
</tr>
</tbody>
</table>
In Table 6-9 the advantage of extending from posture A at the quarter-twist position can be clearly seen by the increase in the proportions of ‘athletes’ able to achieve 3/2 or greater twists in a somersault than using Drop (Table 6-7). Based on these proportions, extending at a quarter-twist is a very useful technique. It is thus advisable for a coach to recommend maintaining posture A which has occurred as part of the backward somersault initiation, for the twist initiation as well. In contrast it is advisable to recommend extending from posture EP as soon as possible after the arm action when performing forward somersaults. Even though extending from posture A at the quarter-twist would increase the number of twists that can be performed, still only around a quarter of ‘athletes’ can achieve a double twist.

When using the posture A with the action Drop and then extending at the quarter-twist position the number of somersaults required to complete any particular number of twists is a combination of the somersault required before and after the extension from posture A. The genders were not significantly different from each other, in somersault required either before or after the extension. The squads were not from the same homogeneous population (p << 0.01) both before or after the extension from posture A. The literature data predicted lower values of \( N_\theta \) before and after the extension from posture A and so predicts a lower number of somersaults required for any number of twists (CLES of 13-19\% when compared to the other squads combined). The 12-or-under squad showed slightly higher values of \( N_\theta \) both before and after the extension from posture A (CLES of 56-61\% when compared to the other squads combined) and so predicts a slightly higher number of somersaults required for any number of twists. The Senior squad showed slightly higher values of \( N_\theta \) (CLES of 59\% when compared to the other squads combined) after the extension from posture A; thus the Seniors would have a slight disadvantage when performing higher numbers of twists.

### 6.5.3 Combining an arm action and lateral flexion

Extending from posture LHF, as considered in Section 6.4.1, could easily and smoothly be combined with the DiverS arm action for a forward somersault, since the rising arm would be on the same side as the shoulder which moves away from the hip when straightening. By combining actions it is expected that tilt production may be boosted. This section assesses such an action: the start position is one were the body is in posture LHF but with the arms extended laterally perpendicular to the body; “DiverS with bending of the arms”, as considered in Section 6.2.2, is performed simultaneously with extending from the laterally-flexed position; the movement of the torso and arm finish simultaneously and so the athlete ends in posture 1U1DB.
Figure 6-32, plots the cumulative tilt produced for DiverS, DiverS ending in posture 1U1DB, extension from posture LHF, and the extension from posture LHF while performing DiverS to end in posture 1U1DB.

It is clear from Figure 6-32 that by combining the extension from posture LHF with DiverS, the result is not simply a sum of the two individual actions, since the position of the segments not involved in the individual action already analysed differ when the actions are performed together. In this situation the interaction works to further boost tilt: this athlete gains approximately 5° of tilt. This additional tilt has, for this athlete, reduced the value of $N_\theta$ to 0.12 from 0.23 when extending from posture LHF only, or from 0.4 when using DiverS only.

Considering the value of $N_\theta$ for all ‘athletes’, the genders where not found to be significantly different, although the squads were found to be not from the same homogeneous population (under the H-test $p = 0.01$): the Senior squad tended to have higher values of $N_\theta$ and the data based on the literature tended to have much lower values of $N_\theta$.

Table 6-10 shows the proportion of ‘athletes’ that can perform a selection of skills when extending from posture LHF with DiverS, as well as when extending from posture LHF only (which was previously given in Table 6-5) for comparison.
Table 6-10: Proportion of ‘athletes’ able to achieve a selection of skills when extending from posture LHF with DiverS compared to only extending from posture LHF.

<table>
<thead>
<tr>
<th>Extend from posture LHF with DiverS</th>
<th>Extending from posture LHF only</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 twisting somersault</td>
<td>1.00</td>
</tr>
<tr>
<td>1/1 twisting somersault</td>
<td>1.00</td>
</tr>
<tr>
<td>3/2 twisting somersault</td>
<td>0.96</td>
</tr>
<tr>
<td>2/1 twisting somersault</td>
<td>0.93</td>
</tr>
<tr>
<td>5/2 twisting somersault</td>
<td>0.74</td>
</tr>
<tr>
<td>3/1 twisting somersault</td>
<td>0.60</td>
</tr>
</tbody>
</table>

As Table 6-10 shows, performing DiverS simultaneously with extending from posture LHF, is a very useful technique. Sixty percent of ‘athletes’ can now perform a triple twist!

Lateral flexion may also occur following an arm action; that is either the shoulder moves towards the hip as the lowering arm lowers or the shoulder moves away from the hip as the raising arm rises. It was found in Section 6.4.1 that laterally flexing in the air was not desirable, and hence such a continuation of the action should be avoided; as a result lateral flexion in this situation will not be further considered.

### 6.6 Technique choice and restrictions

This chapter has presented a selection of different aerial techniques that an athlete may use to perform twisting somersault skills. As has been mentioned some techniques will be more suitable for backward or forward somersault skills. This is due to the expected body posture when entering the aerial phase (Section 6.5.1), or if the direction an athlete turns their chest during the aerial technique corresponds with the direction of twist, allowing the athlete to easily look in the direction of the resulting twist (Section 6.3). This section discusses which techniques may be considered the ‘best’ and what factors may influence whether or not the most effective technique is in fact the most suitable technique.

Even though some techniques allow a greater proportion of ‘athletes’ to achieve a skill, an individual athlete may only be able to achieve the skill using a specific technique, or they may have a choice between a few different techniques; sporting expectations and aesthetics, or personal preference will then have an impact on the technique actually used. Differences in technique choices are thus to be expected even at the elite level. For example Sanders (1999) reported different techniques between the top two athletes in his study of divers performing a full twist in a 1½ somersault: one used an asymmetrical arm action and minimal flexion of the hips, while the other used greater hip flexion and a “stronger kick-out”.

The same technique may not be used for all the twisting somersaults an athlete can perform, even though theoretically they could use the same technique for a greater or lesser
portion of the somersault. For example Sanders (1995) observed that when performing Baranis (1/2 twisting somersaults) trampolinists used “a slow and almost symmetrical action” while, when performing Rudis (3/2 twisting somersaults) they used “a distinctly asymmetrical and vigorous action”. This type of technique difference will be strongly related to the sporting expectations and individual preferences, and so will not be further discussed in this thesis, but left to a coach to make such choices based on their extensive sporting knowledge.

The suggestions in this chapter are based on theoretical explorations. These are expected to be over-estimates since they assume absolute accuracy and no limits on the strength or speed of any postural changes. Rather they can be considered as limits of twisting somersault skills that may be achieved using aerial techniques.

### 6.6.1 ‘Best’ techniques

Techniques may be recommended on the basis of the proportions of ‘athletes’ that may achieve particular skills. With a new or otherwise unknown group of athletes a coach can choose techniques that are predicted to allow the greatest proportion of ‘athletes’ to achieve various skills. Table 6-11 collates the proportions of ‘athletes’ predicted to achieve each skill using the idealised aerial techniques of most interest from Sections 6.1 to 6.5.

The majority of ‘athletes’ would be able to perform a 1/2 twist using any of the techniques in Table 6-11. The asymmetrical arm actions allow the majority of ‘athletes’ to achieve a 1/1 twist in a somersault. The techniques involving torso movement then allow greater numbers of twists to be completed; this is not surprising, since the torso segments have greater mass and moment of inertia than the arms.

From Table 6-11 “Extending from LHF with DiverS” was predicted to allow the greatest proportion of ‘athletes’ to perform each skill; it could thus be considered the ‘best’ technique of those presented in this chapter. It would allow the majority of ‘athletes’ to perform a 3/1 twist. This technique is, however, only suitable for forward twisting somersaults where the twist is performed in the first somersault, since it requires the athlete to leave the ground in a laterally flexed posture; this would not be suitable for any pure somersaults performed prior the one with twist. This is because only in a forward somersault does the lateral flexion allow the athlete to look in the direction of twist, and since the technique occurs at take-off, the twist begins in the first somersault (Section 6.4.1).

For backward somersaults the best technique, based on the proportions of ‘athletes’ predicted to be able to perform each skill, is “Full S”. However, as Ng (1980) identifies, starting in posture 1U1D ready to perform FullS will affect the throw for the somersault
and risks “spoiling” it altogether; thus even though this technique has been reported to have been used in a Diving competition (footnote 2 on page 218) it is not common: this technique is therefore not recommended. When throwing for a backward somersault the arms are expected to be raised, and the body may be in posture A. Thus, when performing a backward somersault, the best technique is “Drop in A & extend at ¼ twist”. Using this technique the majority of ‘athletes’ would be limited to a 3/2 or less twist in a somersault.

Since the best technique for a backward somersault allows less twist than the best technique for a forward somersault, one may conclude that it is easier to add a twist to a forward somersault than to a backward somersault. This does not necessarily mean that a forward twisting somersault is easier than a backward twisting somersault, since the ability to generate angular momentum also contributes to overall difficulty.

When performing twist in any somersault other than the first somersault, the three techniques involving lateral flexion on the ground cannot be used. Any of the actions involving asymmetrical arm action or the “Idealized kick-out from P” could be used. The “Idealized kick-out from P” is the least effective technique, only allowing a 1/2 twist for the majority of ‘athletes’; this may still be useful for “half-out” somersaults where, for aesthetic reasons, arm movement is to be kept to a minimum. If FullS could be used then the majority of ‘athletes’ could achieve a double twist; otherwise, by using the asymmetrical arm actions combined with posture EP or A and extending immediately or at a quarter-twist respectively, would allow the majority of ‘athletes’ to achieve a 3/2 twist.

Table 6-11 presents the proportions of ‘athletes’ who can achieve each skill when the complete idealized arm action is performed. As was discussed in Section 6.2.1 the idealized arm actions can, and should, be refined to increase the tilt production; these refinements are specific to the individual athlete. Nevertheless the refinements would only make small differences and Table 6-11 is a reasonable guide to the limits of the idealized arm actions. These proportions are limits since an athlete would not be able to perform the actions instantaneously or even completely; although more skilled ‘athletes’ would be able to move faster and more accurately. In addition, a slower somersault, if the flight time allows, would also allow the actions to be closer to ideal since, as the twist starts, it would be slower and so more complete actions may be employed with adjustments (Section 6.1.3).

Further, Table 6-11 presents skills where a full somersault is available in which to complete the twist. In reality this does not always happen: take-off and landing constraints may alter the somersault available.
Table 6-11: Proportion of ‘athletes’ that can achieve each skill for each action, presented in Sections 6.1 to 6.5, that was deemed useful.
The maximum proportion for each skill is in bold. This technique would be the ‘best’ provided that other considerations did not preclude its use. Proportions greater than 50% are shaded to aid in the visualisation of the skills that may be achieved by the majority of ‘athletes’ for each technique.

<table>
<thead>
<tr>
<th>Skill</th>
<th>1/2</th>
<th>1/1</th>
<th>3/2</th>
<th>2/1</th>
<th>5/2</th>
<th>3/1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># twists in a single somersault</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2/1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Action class</th>
<th>FullS</th>
<th>DiverS</th>
<th>Raise</th>
<th>Drop</th>
<th>DiverS</th>
<th>Raise</th>
<th>Drop</th>
<th>Dropping together</th>
<th>LHF on ground &amp; remove in the air</th>
<th>LHF on ground and maintain it in the air</th>
<th>DiverS when in EP (principal plane)</th>
<th>Raise when in EP (principal plane)</th>
<th>DiverS when in EP (pelvis plane)</th>
<th>Drop in A &amp; extend at 1/4 twist in LHF with DiverS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1.00</td>
<td>0.82</td>
<td>0.81</td>
<td>0.83</td>
<td>0.80</td>
<td>0.80</td>
<td>0.81</td>
<td>0.97</td>
<td>0.62</td>
<td>1.00</td>
<td>1.00</td>
<td>0.91</td>
<td>0.92</td>
<td>0.99</td>
</tr>
<tr>
<td>1/1</td>
<td>0.76</td>
<td>0.67</td>
<td>0.63</td>
<td>0.67</td>
<td>0.64</td>
<td>0.62</td>
<td>0.66</td>
<td>0.74</td>
<td>0.25</td>
<td>1.00</td>
<td>1.00</td>
<td>0.70</td>
<td>0.71</td>
<td>0.73</td>
</tr>
<tr>
<td>3/2</td>
<td>0.70</td>
<td>0.26</td>
<td>0.22</td>
<td>0.26</td>
<td>0.23</td>
<td>0.21</td>
<td>0.28</td>
<td>0.46</td>
<td>0.05</td>
<td>0.96</td>
<td>0.91</td>
<td>0.44</td>
<td>0.45</td>
<td>0.53</td>
</tr>
<tr>
<td>2/1</td>
<td>0.55</td>
<td>0.05</td>
<td>0.03</td>
<td>0.07</td>
<td>0.07</td>
<td>0.04</td>
<td>0.12</td>
<td>0.33</td>
<td>0.04</td>
<td>0.80</td>
<td>0.53</td>
<td>0.22</td>
<td>0.23</td>
<td>0.25</td>
</tr>
<tr>
<td>5/2</td>
<td>0.30</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.21</td>
<td>0.00</td>
<td>0.51</td>
<td>0.16</td>
<td>0.05</td>
<td>0.06</td>
<td>0.08</td>
</tr>
<tr>
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For example, in Sections 5.2.8 and 5.2.9 the observed ‘athletes’ performing pure somersaults did not complete multiples of 360°. Some of the rotation occurred during the process of take-off and landing. Since the amount of somersault completed during take-off and landing varies with the athlete and the conditions, using a full somersault, as in Table 6-11, allows only general conclusions to be drawn.

To still achieve the desired skills, those ‘athletes’ that fall outside the proportion that can achieve each skill—which is an increasing number as the number of twists to be completed increases—must either use an aerial technique that has not been previously described in the literature, and hence discussed here, or must use contact twist (Section 6.6.3).

Even though a particular action may be better, based on the proportions of ‘athletes’ that may achieve a particular skill, the same action may not be the best for an individual ‘athlete’. Since the techniques involving lateral flexion could only be described using those inertial property data sets which had three torso segments, it is not reasonable to determine the percentage of all data sets for which this was the ‘best’ technique. Nevertheless by determining the technique that allowed an individual ‘athlete’ to achieve the greatest numbers of twists in a somersault can be used to understand which other techniques may be the ‘best’. For the data sets with three torso segments “Extending from LHF with DiverS” was the best for approximately 84% of ‘athletes’, and “LHF on ground & remove in the air” was the ‘best’ for the remainder of those ‘athletes’. For the other data sets “FullS” or “Drop in A & extend at ¼ twist” were the ‘best’ techniques. Figure 6-33 gives the proportions of ‘athletes’ who may achieve various numbers of twists in a somersault using whichever technique was predicted as the most effective for them.

![Figure 6-33: The proportion of ‘athletes’ that may, at best, achieve each skill](image)

The maximum number of twists achievable in a single somersault for each ‘athlete’ was determined based on each ‘athlete’ using the technique that was best for them.
The maximum number of twists in a somersault that any ‘athlete’ may complete using one of the idealized aerial twist initiation actions can be seen in Figure 6-33 to range from 1.5 to 9 twists. It is thus clear that some ‘athletes’ would have a considerable natural advantage over others due to their inertial properties alone. This is considered further in Section 6.6.2.

6.6.2 Natural advantage based on inertial properties

An ‘athlete’ has a natural advantage if they are able to achieve more difficult skills than another ‘athlete’ when performing identical techniques as a result of differing inertial properties; they could have a natural advantage for all or only a few techniques.

Figure 6-33 presented the proportions of ‘athletes’ who may achieve each twisting somersault skill using the techniques most efficient for each individual ‘athlete’: a large range is apparent, meaning that some ‘athletes’ have a clear natural advantage. The genders were not found to be significantly different under the U-test. The squads were, however, found to be not from the same homogeneous population under the H-test; the literature data clearly (p << 0.01, CLES ~80%) allowed greater numbers of twist to be completed. The senior squad (p ~ 0.01, CLES ~61%) tended to be able to achieve less twists in a somersault; this is unfortunate since these ‘athletes’ are the ones competing internationally. Figure 6-34 illustrates the proportions of ‘athletes’, categorised by squad, that may achieve each skill at best.

![Figure 6-34: The proportion of ‘athletes’ by squad that may at best achieve each skill](chart.png)

As for Figure 6-33 but broken into squads: the maximum number of twists achievable in a single somersault for each ‘athlete’ was determined based on each ‘athlete’ using the technique that was best for them.

An ‘athlete’, who has a natural advantage for the most number of twists, may or may not have an advantage across all techniques. In particular, needing to use different techniques, as identified in Section 6.6.1, for forward and backward somersaults means some ‘athletes’ may have a natural advantage for forward and not backward somersaults or vice versa.
The technique “Extending from LHF with DiverS”, which was the ‘best’ technique for forward somersaults, allows ‘athletes’ to achieve 1.5 to 6.5 somersaults; the 9 twists in Figure 6-33 was when the ‘athletes’ with the greatest natural advantage used “LHF on ground & remove in the air” even though for the majority of ‘athletes’ “Extending from LHF with DiverS” allowed the greatest numbers of twists to be achieved. In contrast, “Drop in A & extend at ¼ twist” allows ‘athletes’ to perform 0.5 to 3.5 twists per somersault. This is a much smaller range, which means that the natural advantage of some ‘athletes’, although still present, is smaller for backward twisting somersaults than for forward twisting somersaults. For both “Extending from LHF with DiverS” and “Drop in A & extend at ¼ twist” the genders are not significantly different under the U-test, but the squads are different under the H-test. The literature data in both cases achieves greater numbers of twist (p << 0.01, CLES >80%). The Senior squad tends to achieve fewer numbers of twists (p=0.006 and 0.03; CLES = 62% and 59%; for “Extending from LHF with DiverS” and “Drop in A & extend at ¼ twist” respectively). The 12-or-under squad tends to achieve slightly fewer twists when using “Drop in A & extend at ¼ twist” (p=0.08; CLES = 56%).

By comparing the value of $N_\theta$ achievable using “Extending from LHF with DiverS” and “Drop in A & extend at ¼ twist”, as is done in Figure 6-35, there is a clear trend that those with a natural advantage in one of the two techniques has a natural advantage in the other. There is still moderate scatter, and so the ranking of on ‘athlete’ compared to other ‘athletes’ may jump considerably between the two techniques.

![Figure 6-35: The relationship between $N_\theta$ for the best forward and backward somersault twist initiation techniques](image)

Only the data sets with a three torso segment are shown, since these are the only ones for which the technique “Extending from LHF with DiverS” could be evaluated. The trend line and coefficient of determination $R^2$ are given on the plot.
For twists other than in the first somersault, when the asymmetrical arm actions are the ‘best’ option as specified in Section 6.6.1, the genders are still not statistically different but the literature data and 12-or-under squad do stand out as different from the Teens, Seniors and Masters, as was identified in Section 6.1.2. There is a very strong correlation between the values of \( N_\theta \) for the asymmetric arm actions. Figure 6-36 shows the strong correlation between “DiverS when in EP (pelvis plane)” and “Drop A & extend at ¼ twist” which can both be used in somersaults other than the first for forward and backward somersaults respectively.

Based on the above discussion there is a clear trend that an ‘athlete’ with a natural advantage for the ‘best’ technique for one twisting somersault type would also have a natural advantage for the ‘best’ technique for the other twisting somersault types. The magnitude of this advantage though varies considerably with the technique used. Forward twisting somersaults with the twist in the first somersault show the greatest range in achievement across ‘athletes’.

It is also clear from this discussion that the literature data is quite different from the data sets estimated from current ‘athletes’ in Chapter 3, with ‘athletes’ represented by the literature data achieving much greater numbers of twists in a somersault. The seniors and 12-or-unders have a tendency to achieve fewer twists, although, the magnitude and the statistical significance of this depends on the technique used.

At this point it is of interest to consider whether or not the same ‘athletes’ have a natural advantage when twisting as when performing pure somersaults. From Figure 6-37 it can be
seen that this is not necessarily true, since there is not a clear correlation between the range of $\tau = I_{yy}/I_{yy,L}$, which was used when discussing natural advantage for pure somersaults (Section 5.2.1) and $N_\theta$ for the best twist initiation technique specific to the ‘athlete’ (Section 6.6.1).

![Figure 6-37: The relationship between $N_\theta$ for the best twist initiation for each ‘athlete’ and the range of $\tau$ for that ‘athlete’](image)

All the data sets were used. If the same ‘athletes’ had a natural advantage when twisting and when performing pure somersaults there would be a negative correlation between the two variables. That is an ‘athlete’ with a natural advantage for both pure and twisting somersaults would have a high value of the range of $\tau = I_{yy}/I_{yy,L}$ and a small value of $N_\theta$.

### 6.6.3 Do common biomechanical indices identify ‘athletes’ with a natural advantage?

As in Section 5.2.7, to determine if any of the common indices or ratios had promise, a scatter plot of $N_\theta$ for the technique that gave the lowest value of $N_\theta$ for each ‘athlete’ and the value of each index or ratio was drawn. There was no pattern visible and the coefficient of determination, $R^2$, (Phipps & Quine, 2001) was less than 0.1 for all the indices or ratios. Thus, as with $\tau$, no common biomechanical index or ratio showed promise for identifying ‘athletes’ with a natural advantage.

### 6.6.4 Contact twist

To increase the number of twists beyond that which can be achieved by aerial actions, it is necessary to use some contact twist. For ‘athletes’ and postures requiring smaller values of $\phi$ to achieve a skill, it is thus expected that greater contact twist would be used. Sanders (1995) found a statistically significant difference ($p < 0.01$) between the amount of twist
rotation at last contact, which is a sign of contact twist, when trampolinists performed \( \frac{1}{2} \) and \( 1\frac{1}{2} \) twisting forward somersaults.

Contact twist could be used to enable an athlete to reach a quarter-twist, and thereby allow the aerial actions “Dropping together”, from Section 6.1.3, or extending from posture EP or A at a quarter-twist position, as suggested in Section 6.5.2, to be used to increase the tilt and hence the rate of twist.

Contact twist may also be used as an alternative to aerial twist. A photographic study by Moore and cited in Yeadon (1993b) “found that as more body parts were immobilized, the trampolinists twisted further during the contact phase”. Even though contact twist would allow skills to be achieved, “pancaking” would be observed (Section 2.2) and this is generally deemed to be aesthetically unpleasing. In addition to pancaking, contact twist is associated with lateral displacement (Bangerter & Leigh, 1968). This appears to be an undesired side effect of the actions used to generate twist from the ground.

Even though contact twist could be used instead of aerial techniques, it is possible that the expectation of being “skilled” is the ability to use aerial twist techniques more effectively, thereby relying less on contact twist. In support of this idea Sanders (1995) observed that the amount of chest rotation in the direction of twist, which is related to contact twist (Yeadon, 1993b), was greater for those trampolinists that were deemed less skilled. In addition the two most elite subjects Sanders observed only showed small twist rates at last contact, while the rest of the group showed much higher twist rates at last contact.

The focus of this thesis is on aerial techniques. The theoretical limits discussed in Section 6.6.1 and 6.6.2, are for aerial techniques. Introducing contact twist will allow these limits to increase provided that, when producing contact twist, the actions and end body postures do not preclude the desired aerial actions. The amount of contact twist that can be generated depends on the moments of inertia and the relative speed of various body segments that may be achieved while in contact with the ground.

### 6.6.5 Previous literature

It is always desirable to compare new results with previously published data so that the work from all studies may be used to increase understanding. When considering the tilt that may be produced, Frohlich (1979), Pike (1980), and Yeadon (1984) present their models as well as the values obtained when performing actions they deemed of interest. Each author will be treated separately below. A comparison between the equations used by these authors and those derived in this thesis is presented in Appendix G.
6.6.5.1 Frohlich

Frohlich (1979) derives equations for the angular displacement of the reference segment, when the body may be considered as two rigid segments connected by one joint. To model the “throw” for a twisting dive, which is equivalent to DiverS described in Section 6.1, the two segments Frohlich uses represent the body without arms and the arms. The model and action is illustrated in Figure 6-38.

Frohlich positioned the joint on the longitudinal axis of each segment and the centre of gravity of the segment representing both arms. Applying this scenario to Equation (4-34) gives the same equation as provided by Frohlich. The working may be seen in Appendix G.2.

Frohlich calculated a tilt of 10.85° could be achieved when performing the action DiverS in a layout posture; this value is greater than the tilt values determined for all, except one, ‘athlete’ in the current study. Using the descriptions that Frohlich gave of the body segments in his model to extract inertial property data and then applying these results to the model for DiverS with a posture of L, a tilt of 7.2° is achieved. It is thus clear that modelling the arms pivoting in the centre of the chest as opposed to at the shoulders has overestimated the tilt that may be produced. Further, the ‘athlete’ with the closest whole body moment of inertia in the pre-twist position, equivalent to posture LAP — 19.26 kg/m² compared to the 19.17kg/m² Frohlich used — had a smaller equivalent moment of inertia of an arm segment than Frohlich — 1.85 kg/m² compared to the 2.31kg/m² Frohlich used — and as a result achieved only 5.39° of tilt.

Frohlich obtained a tilt value of 19.96° when using a loose pike position, showing that tilt could be increased by altering the body posture. An increase in the tilt was observed between postures L and EP (Section 6.5.1). The proportionate increase in tilt cannot be compared, since the loose pike position is a much deeper pike than posture EP with the
hips flexed at 90°. Such a deep pike was not investigated in this thesis, since general observations of athletes suggested athletes tended to use an EP posture.

6.6.5.2 Pike

Pike (1980) uses a computer simulation to determine the tilt produced during the “wrapping phase” of a layout twisting somersault. The model, approach, and equations presented are in line with the equations given in Section 4.3; thus results may be reasonably compared. Pike calculates a tilt of approximately 0.18 rad (~10.3°). The arm action Pike used was one in which the diver moved their arms in an odd symmetrical fashion starting with both arms raised in a high V and ending with a position similar to posture 1U1DB.

Pike applied the 50th percentile body segment data from Hanavan (1964). Hanavan’s inertial property data was based on US Air Force Flying personnel in the 1950s. It was not used as part of this thesis, since it only presented two torso segments and so the ‘updated’ inertial property data of air force flying personnel was used instead (Anthropology Research Project, 1988). Applying the inertial property data set representing the 50th percentile male aviator to Diver S, Raise, and Drop in layout position gave a tilt of 9.2°, 9.6°, and 8.5° respectively; then when allowing the bending of the arms (as in Section 6.2.2) the tilt was 7.9°, 8.5°, and 7.5° respectively. This data set was one of the few where ending in posture 1U1DB produced less tilt than ending in posture 1U1D: for most others the tilt was greater. Starting the arm action with the arm slightly abducted, as appeared to be the case in the action Pike modelled, was found in Section 6.2.1 to increase the tilt produced. Thus, it is reasonable to conclude that the results presented in this thesis are in agreement with Pike’s results.

6.6.5.3 Yeadon

Yeadon’s thesis (1984) presents his modelling approach and equations. The principal of the conservation of angular momentum was used to derive the equations underlying the simulation models. In principal Equation (4-27) and Yeadon’s (1984) equation PA on page 168 are in agreement, although they differ in notation, the frames of reference used, and how explicitly the frame of reference for each variable is stated. Yeadon names the two parts of the equation as “whole body”, which occurs even when there is no postural change, and “relative”, which can occur even with zero angular momentum.

Yeadon then wrote the relative angular momentum as a summation of the contributions of 11 segments representing the body, using the angular velocity of the segment with respect to the adjoining segment that is more proximal or closer to the pelvis. During the derivation it appears that terms relating to how the overall centre of gravity moves as a
segment moves are erroneously ignored. A comparison of equations and an investigation of the significance of the difference between the equations for the action Drop is provided in Appendix G.3. Since Yeadon’s equations differ from those derived in this thesis, and the consequences when considering one example (Appendix G.3) are significant, it is not reasonable to compare Yeadon’s results with those presented in this thesis: it would be impossible to tell if differences in results were due to using different inertial properties or because there is an error in the equations presented by Yeadon.

When analysing the techniques of athletes, Yeadon (1993d) presented a method of determining the contribution of each action to the total tilt observed when watching athlete performances. The method involves a series of simulations modelling different parts of the action; Yeadon typically uses the following classes of action: contact, symmetrical, asymmetrical arms, asymmetrical hips, asymmetrical or torsion of the chest. The method relies on the assumption that “if the various aerial techniques are used at the same point in the movement, it may be expected that the sum of the individual contributions will be equal to the tilt angle in the actual performance” (Yeadon, 1993d, p. 220). Yeadon did acknowledge that the actions are not truly independent (Yeadon, 1984) but uses this method in the majority of his observational papers. In Section 6.5.3 it was clearly seen that summing the tilt produced by DiverS and extending from posture LHF did not equal the tilt produced by performing both actions simultaneously: the actions are clearly not independent. As a consequence this approach was not followed in this thesis.
Chapter 7
Conclusions

This thesis has presented a theoretical exploration of aerial techniques used to perform pure and twisting somersaults with the aim of establishing limits to the number of twists and somersaults that may be achieved by aerial techniques alone. These techniques included the choice of posture during the quasi-rigid phase of the somersault and the aerial twist initiation actions performed. Recommendations of which techniques are ‘better’ were made, based on the relative difficulty of skills, and the skills that could be achieved; the range of achievement of different athletes was considered as part of making these recommendations. The recommendations as to which technique is ‘better’ were based on biomechanical conclusions and general sporting expectations; a coach must also consider their athlete and any other factors affecting performance, before concluding which technique they will teach a particular athlete. This chapter summarises the results and resulting recommendations, covering the practical implications of this work as identified in Section Chapter 1.

7.1 Rotation classifications and recommendations

The rotational behaviours of sporting interest that may be observed in the quasi-rigid phase of a somersault may be broadly described as pure somersault, continuously twisting somersault, somersault with a twist oscillation about the quarter-twist position, and somersault with a twist oscillation about the zero twist position. Which rotational behaviour is observed and the speed of the rotation both depend on the relative values of the principal moments of inertia ($I_{xx}, I_{yy}, I_{zz}$) and the initial orientation angles ($\phi_0, \psi_0$).

The pure somersault is the desired rotational behaviour when performing the skill of a “somersault” as specified in the diving and gymnastics rules (FINA, 2009; CoP MAG, 2013; CoP WAG, 2013; CoP TRAMP, 2013). To perform this skill an athlete must generate angular momentum so that they rotate about their transverse axis ($\phi = \pi/2$):
Conclusions

postures with left-right symmetry must be used. The speed of the somersault will increase as the principal transverse moment of inertia, I_{yy}, decreases. This may be achieved by moving to a more ‘compact’ posture, with greater flexion at the hips, knees and through the torso. The relative difficulty of a pure somersault, \( \tau = I_{yy}/I_{yy,L} \), provides a qualitative measure of how hard a somersault skill in a particular posture is for a specific athlete, and allows comparisons across athletes. An athlete with a greater range of \( \tau \) will experience a greater change in rotational speed as they change shape; this may be considered to be a natural advantage.

The continuously twisting somersault rotational behaviour allows an athlete to perform the skill of a “twisting somersault” as specified in the diving and gymnastics rules (FINA, 2009; CoP MAG, 2013; CoP WAG, 2013; CoP TRAMP, 2013). Only postures where I_{yy} is the intermediate-valued moment of inertia and \( \phi_o \) is less than \( \phi_{Crit1} \), or where I_{xx} is the intermediate-valued moment of inertia and \( \phi_o \) is greater than \( \phi_{Crit2} \) (Section 4.2.3) will allow a continuously twisting somersault to be performed. The number of twists that may be achieved in a somersault increases as \( \phi \) decreases and as I_{zz} decreases with respect to I_{yy} and/or I_{xx}. As \( \phi \) decreases the ratio I_{yy}/I_{zz} becomes the dominant parameter when comparing key postures (Section 5.3.2). Good twisting postures are ones that have minimal flexion at the hips, knees, and through the torso.

The rotational behaviour of a somersault with twist oscillation about the quarter-twist position is an undesirable situation. No skill can be awarded since the twist is not continuous as required in a “twisting somersault” skill, and the presence of clearly observable twist means that the “somersault” skill is also not awarded. This situation occurs when I_{yy} is the intermediate-valued moment of inertia, as in postures with extended hips and knees and \( \phi > \phi_{Crit1} \) (Section 4.2.3). It may be viewed as a response to instability in the presence of a disturbance (Section 5.2.10). When performing a layout somersault such “instability” could be a practical problem especially if there is a disturbance in \( \phi \). To reduce the practical consequences a posture with the arms laterally extended, such as LAP, or with flexion through the body, such as posture A, could be used. The flexion through the body is the better option since it reduces the consequences of the instability more than posture LAP and does so for disturbances in both \( \phi \) and \( \psi \).

The rotational behaviour of a somersault with a twist oscillation about the zero-twist position is useful when wishing to prevent continuous twist. Continuous twist will need to be prevented when the athlete has completed the desired amount of twist in a twisting somersault and then needs to complete additional somersaults, or prepare for landing, but they are unable to return \( \phi \) to \( \pi/2 \). Postures with I_{zz} as the intermediate-valued moment of inertia, or postures with I_{xx} as the intermediate-valued moment of inertia and \( \phi > \phi_{Crit2} \),
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(Section 4.2.3) could be used. The more useful postures will be those which display smaller magnitudes of twist oscillation. This will generally be the postures with $I_{zz}$ as the intermediate-valued moment of inertia, since these never display continuous twist; however, the relative magnitudes of the three moments of inertia will determine the magnitude of the twist oscillation. From Section 5.3.3, postures P and FT were generally the best postures for maintaining the twist oscillation in an acceptably small range; thus entering either of these is recommended when performing further somersaults after the desired amount of twist has been completed.

If the medial and transverse moments of inertia are not equal ($I_{xx} \neq I_{yy}$) the somersault rate changes as an athlete twists. To complete the full somersault required the angular momentum must increase if $I_{xx} > I_{yy}$ or decrease if $I_{xx} < I_{yy}$. As $\phi$ decreases, the relative difficulty of a twisting somersault during the twisting phase, compared to a pure layout somersault increases to a plateau of $(I_{xx} + I_{yy})/(2I_{yy}L)$. Additional angular momentum will be required to complete the skill if the postures used when twisting and when performing a pure somersault are different, even if they are both classified as the same posture for purposes of awarding difficulty scores. For example, using posture 1U1DB would require greater angular momentum than would posture L. The proportion of ‘athletes’ predicted to have a relative difficulty $\tau$ less than particular values was given in Table 5-12 (for added twist only) and Table 5-13 (for added twist and using a different posture) respectively. For the majority of ‘athletes’ the angular momentum required increased in postures with extended knees and hips.

7.1.1 Assuming equality of the medial and transverse moments of inertia

Assuming that the medial and transverse moments of inertia are equal ($I_{xx} = I_{yy}$) is an assumption commonly made in the previous literature; it simplifies the equation required to determine the twist-to-somersault ratio that could be achieved for a specific value of $\phi$, and allows the easy determination of the value of $\phi$ required to achieve a specific twist-to-somersault ratio. The error associated with assuming $I_{xx} = I_{yy}$ decreases, as 1) $I_{xx} \rightarrow I_{yy}$, 2) $I_{zz}/I_{yy}$ or $I_{zz}/I_{xx}$ decrease, or 3) when $\phi_0$ decreases—as occurs when attempting greater numbers of twists. It is insufficient to know only the ratio of $I_{xx}/I_{yy}$.

For the postures of interest in Section 3.9, it was reasonable to assume $I_{xx} = I_{yy}$ for the posture 1U1DB for the majority of ‘athletes’ only when performing at least a 1½ twist in a single somersault (Section 5.3.6). For other postures even more twists would need to be performed before the assumption that $I_{xx} = I_{yy}$ would be reasonable. Thus, in general, assuming $I_{xx} = I_{yy}$ is not recommended.
7.2 Recommended techniques for pure somersaults

When performing pure somersaults the principal transverse moment of inertia $I_{yy}$ determines the speed of the somersault for a given angular momentum. In Section 5.2, the parameter used to compare postures and ‘athletes’ was $\tau = I_{yy}/I_{yy,1}$: this specified the number of layout somersaults that could be performed by the same athlete in the same time and with the same angular momentum as a single somersault performed in the posture of interest. Postures with the same value of $\tau$ were considered to be equally difficult.

There was no single order of postures determined by the value of $\tau$ that applied to every ‘athlete’. Posture LAU (Section 3.9.1) was the most difficult posture for all ‘athletes’; thus when an athlete needs to slow rotation as much as possible this is the posture that should be adopted. The value of $\tau$ reduced as the arms were placed close to the torso, with flexion through the torso and at the hips and knees, as occurs when moving to the more ‘compact’ postures. Even though no single order of postures could be obtained, ordering them by the median value of $\tau$ (Table 5-4) could be used to guide the assignment of difficulty scores to various postures.

The postural component of the difficulty of a skill will be either layout, pike or a tuck; each may be achieved by a few different postures defined in Section 3.9. Increasing the hip flexion in a pike, using posture TP rather than posture P, or increasing the hip and knee flexion, using posture TT rather than posture T, would reduce the value of $\tau$, thus making the somersault easier to complete. Using deeper flexion can thus be recommended to achieve greater somersault rotation. When performing a layout the arms should in general be lowered, as in posture L, to reduce $\tau$. Some torso and hip flexion is allowed before deductions apply; using this allowance, although aesthetically less pleasing, would make the skill easier.

A tuck difficulty score would be awarded for both postures FT, and BT even though the flexion through the torso and at the hips differs. Posture FT has greater curvature through the torso, while posture BT has a straighter back with greater hip flexion. Approximately half the ‘athletes’ found one easier than the other, and so no general recommendation could be made; although there was a slight shift to this proportion where more males found posture FT easier and more females found posture BT easier. Allowing some leg separation to use posture CT rather than posture TT is becoming increasingly common, due to the belief that it allows faster rotation; however this is not true for all ‘athletes’. Approximately 50% of ‘athletes’ using posture TT with sufficient angular momentum to complete a somersault in posture CT, would under-rotate by no more than 5°. As a result it
Conclusions

is recommended that ‘athletes’ avoid using posture CT in the early stages of learning, and only consider allowing the leg separation and consequent deduction, later if they are still unable to achieve the desired rotation.

The award of a pike corresponds to a higher difficulty score than a tuck; however, it was not found to be more difficult for all ‘athletes’. Posture TP was easier than posture TT for ~20% of ‘athletes’ and this grew to ~40% when comparing postures P and T. There were a greater number of males, than females, and a greater number of ‘athletes’ from the 12-or-under squad, as opposed to any other squad, who found the piked postures easier than the tucked postures. A coach should thus experiment with both tuck and pike, in case their athletes do in fact find pike easier and thus can readily gain a greater difficulty score. On this basis it is also recommended that a coach introduce a pike posture when their athletes are young since a greater proportion would find it easier. Continuing to train pike postures gives opportunity for the athlete’s power to grow steadily as they seek to maintain achievement in their pike postures. In contrast, if the coach waited till the ‘athletes’ were older the difference in difficulty between tucked and piked postures is greater making a more difficult transition.

Based on the exploration of pure somersaults in Section 5.2, several recommendations may be made to the biomechanics community. Posture LAU (Section 3.9.1) was the most difficult posture for all ‘athletes’; it is thus recommended that LAU, rather than L (Section 3.9.7), should be the common reference posture for normalization and comparison across ‘athletes’. When comparing ‘athletes’ a single parameter, such as \( \tau = \frac{I_{yy}}{I_{yy, L}} \), is insufficient to determine a natural advantage. As was seen in Section 5.2.2, comparing postures based only on \( \tau \) can lead to conclusions of insignificant difference, when in reality there can be a large practical difference in terms of the rotation achieved when moving between these postures. It is recommended instead to compare postures using the notion of “equivalence” (Section 5.2.2) with awareness of the multiplying effect of multiple somersaults.

The variation of \( \tau \) across postures and ‘athletes’, and the equivalence of different postures and ‘athletes’, shows that some ‘athletes’ do have a natural advantage over others: they would achieve more somersault despite using the same postures. Natural advantage, when performing pure somersaults, is summarised in Section 7.4.

7.3 Recommended techniques for twisting somersaults

When performing a twisting somersault an athlete needs to choose a twist initiation action, then a posture to hold during the quasi-rigid phase, and then finally a method to ‘end’ the twist for landing/entry. Section 5.3 discussed which postures would maximise twist within
a somersault. Chapter 6 discussed the twist initiation actions and the logical end postures to use in the quasi-rigid phase. Section 5.3.3 discussed ‘ending’ twist via postural change when the twist initiation actions in Chapter 6 could not be reversed.

Once in the quasi-rigid phase posture L was found to be the most effective twist posture. It allows the greatest proportion of ‘athletes’ predicted to achieve each multiple of a half twist in a somersault (Table 5-10) while requiring no more than a 3% increase in the angular momentum required—with the same flight time—for 93% of ‘athletes’ (Table 5-13).

Flexion at the knees to use posture Pu, is recommended over flexing through the torso, to use postures A or JL, since posture Pu reduces the angular momentum required more than postures A or JL (Table 5-13) and allows clearly higher proportions of ‘athletes’ to achieve each multiple of a half twist in a somersault (Table 5-10). However, as stated in Section 5.3.5.3, it may be better to keep the knees straight to complete the twist sooner and then enter either a P or T posture to complete the somersault, thereby reducing the angular momentum required for the full skill. Which option is better depends on the athlete, the skill, and sporting expectations; one example was given comparing the two situations in Section 5.3.5.3.

The ‘best’ techniques to use for twist initiation depend on the skill being performed. For a forward twisting somersault, with the twist occurring in the first somersault, the action “Extending from LHF with DiverS” (Section 6.5.3) is recommended. If the twist is to be completed in a later somersault either action “DiverS” or “Raise” both using an EP posture and extending to posture L before appreciable twist occurs is recommended (Section 6.5.1). For backward somersaults “Drop in A and extend at ¼ twist” (Section 6.5.2) is recommended; as the twist begins the second arm should also be dropped (Section 6.1.3) to further decrease $\phi$ and hence increase the number of twists that may be completed.

If, following the twist initiation action, the arms are in a position where one is raised and the other lowered, then flexing at the elbows, to hold posture 1U1DB, is recommended (Section 6.2.2). A smooth action for flexing at the elbows from the laterally-outstretched position, when using any of the asymmetrical arm actions, allowed similar percentages of ‘athletes’ to achieve each multiple of a half twist in a somersault. If the flexion could be performed later with respect to the upper arm movement but before appreciable twist has occurred then the change in $\phi$ would increase and so would the proportion of ‘athletes’ able to achieve each multiple of a half twist in a somersault. Using posture 1U1DB also reduced the angular momentum required to achieve the skill.
The calculations of the skills that could be achieved in Chapter 6 were for idealized actions, where the athlete was able to perform them completely and instantaneously. If this is not the case then less twist will be able to be achieved. To still maximise the twist that they can complete the athlete should focus on performing actions to the fullest extent possible in the regions of maximum change in $\phi$ for the smallest postural change. Considering this, the following are some general coaching instructions:

- Use the slowest possible somersault, achieved by increasing flight time. This will allow a twist initiation action to be completed to a greater extent (Section 4.3.3).

- As twist begins, adjust any actions to continue to tilt towards the angular momentum vector, thereby reducing $\phi$. For example, with Drop this might be done by thinking about dropping the arm and allowing oneself to twist into the dropping arm (Section 6.1.3). Actions should occur as fast as possible, so that as much as possible of the action is performed in the frontal plane.

- When using Drop or Raise, as the twist begins the second arm should begin to move as soon as anatomically possible, so that both are moving together on the same side of the body (Section 6.1.3).

- When using the asymmetrical arm actions, the arms should be kept straight as long as possible then bent to hold posture 1U1DB. This will increase the number of twists achievable and reduce the relative difficulty of the skill.

- When using the asymmetrical arm actions, if the full action cannot be completed, the Drop and Raise actions should be centred about the High V position and the odd symmetric actions should be centred about the position with the arms laterally outstretched (Section 6.1.2).

### 7.3.1 Limits to the number of twists achievable within a somersault using aerial techniques

The value of $N_0$ determined for each action in Chapter 6 is the minimum somersault rotation required to complete a half twist; it is the minimum since, in practice, an athlete may not be able to perform the actions fully or exactly. Nevertheless the values of $N_0$ obtained in Chapter 6 provide an estimate of the limits of known aerial techniques.

“Extending from LHF with DiverS” (Section 6.5.3) was the best technique, allowing all ‘athletes’ to achieve a 3/2 twist in a somersault and the majority to achieve a 3/1 twist in a somersault. This is less than the maximum number of twists given as examples in the diving and trampoline gymnastics rules. In diving, 4 twists in a 1½ forward somersault—
allowing a half somersault for entry and exit—is given a difficulty score in Appendix 2 of the diving rules (FINA, 2009). In trampoline gymnastics 3½ twists in a forward somersault is given as the maximum example number of twists (CoP TRAMP, 2013). As a result, if contemplating performing more than three twists in a somersault, an athlete should expect to require some contact twist unless another aerial twist initiation technique, with a higher limit on the number of twists performable, can be found.

The techniques “DiverS when in EP (pelvis plane)”, “Raise when in EP (pelvis plane)” and “Drop in A & extend at ¼ twist” allow the majority of ‘athletes’ to achieve a 3/2 twist in a somersault; this is insufficient to allow the majority of ‘athletes’ to achieve those twisting somersault skills currently being performed in elite competition. For example, at the 2012 London Olympics, some medallists performed 3 twists in the second somersault of a forward 2½ somersault, and 2½ twists in the first somersault of a backward 2½ somersault (International Olympic Commission, 2015). As a result these elite athletes must be using contact twist, an aerial technique outside of those investigated in Chapter 6, or they have very favourable inertial properties.

Even though the aerial techniques explored in Chapter 6 are clearly limited to less than the maximum numbers of twists currently being performed in competition, and given difficulty scores in the sporting rules, they still have value for the lower numbers of twists, and so should not be ignored. The asymmetric arm actions in particular allow the take-off to be the same as for a pure somersault, which is likely to be aesthetically quite pleasing.

### 7.3.2 Strategies for removing continuous twist

To be awarded a twisting somersault the desired amount of twist must be achieved. The techniques that are effective in maximising twist were summarised in Section 7.3. To avoid deductions and land safely it is also necessary to ‘remove’ the twist once the desired amount has been achieved. The twist may be removed by reversing the aerial twist initiation action after an even number of half twists, by using a posture that prevents continuous twist, or through actions on landing that cease both the twist and somersault rotation simultaneously.

When an athlete has completed an even number of half twists the angular momentum vector is on the same side of the body—left or right—as it was when the twist was initiated. The aerial twist initiation action reduced $\phi$ to initiate twist; reversing the twist initiation action will increase $\phi$ back to $\pi/2$, and hence remove the twist.

When an athlete has completed an odd number of half twists the angular momentum vector is on the opposite side of the body—left or right—as it was when the twist was initiated. To return $\phi$ back to $\pi/2$, the aerial twist initiation action would need to be repeated, but the
athlete is not in a position to do so; for example if “Raise” was used to initiate the twist then the arm is already raised and cannot be raised again. As a result different aerial actions or another method for removing twist is required.

For the same value of $\phi$ some postures will display continuous twist and others oscillating twist. Transitioning between two such postures means that continuous twist may be removed and in this way the twist phase of a twisting somersault is ended. This strategy is only useful if the magnitude of the oscillations in the twist observed after the posture change, and prior to landing, is small. Section 5.3.3 considered less than a maximum of 20° or less than 10° per somersault to be small, since it is expected that the amount of somersault remaining would be less than two somersaults. Postures P and FT are the most suitable postures while $\phi$ is between 70° and 90°. A tighter tuck or pike, postures TT or TP, is not effective since continuous twist is still possible for many ‘athletes’. A more open posture, such as posture OP, is only effective for the higher values of $\phi$. Since posture FT is clearly more effective than posture BT a curved torso should be emphasized when using this technique. The magnitude of twist oscillation increases as $\phi$ decreases and a smaller proportion of ‘athletes’ will still display twist oscillation that is small. Thus, as the number of twists to be completed in a twisting somersault increases, and hence a smaller $\phi$ is used, then this strategy becomes less effective. Nevertheless it was predicted that approximately 84% of ‘athletes’ using posture P may still use this strategy while $\phi$ is between 70° and 90°.

To land it is necessary to open from any tucked or piked posture. By opening out continuous twist is expected to start again. Fortunately, the twist rate will be slower than when performing the desired twist (Section 5.3.3.1), since $\phi$ is expected to be closer to 90° than it is prior to entering the tuck or piked posture, and the remaining somersault will be small. Thus, any restart of the twist is not of practical concern.

If there is only a very small amount of somersault remaining prior to landing, then it may be sufficient to slow the twist, and then only remove it on landing. Slowing the twist will create a contrast to the fast twist thereby appearing as if it has been removed, and allow an athlete to prepare for landing. Twist may be slowed by extending the arms laterally, to use posture LAP, or by introducing some flexion at the hips, to use postures JL, EP or OP. In gymnastics, when performing single somersault skills with twist, piking at the end of the twist is considered an error and incurs a deduction (CoP WAG, 2013; CoP MAG, 2013; CoP TRAMP, 2013); as a result posture LAP is the recommended posture to slow twist in gymnastics.
7.4 Spreads of achievement and natural advantage

For each technique there was a spread in predicted achievement due to applying different inertial properties to the equations of motion. This means that some ‘athletes’ would have a natural advantage over others, since they have performed the same technique, but have achieved more. The spread in achievement for pure and twisting somersaults is practically significant. To achieve the desired pure somersault skills, any athlete with a natural disadvantage would need to increase the angular momentum they can generate to overcome their disadvantage. For twisting somersaults ‘athletes’ with a natural disadvantage would need to more accurately and fully perform the twist initiation techniques, or use contact twist if the aerial techniques are insufficient. The practical significance of differences in skill achievement between ‘athletes’ provides evidence in support of the notion of an ‘ideal body type’ for specific skills.

When somersaulting in a posture that would be awarded the tuck or pike difficulty (postures P, T, BT, FT, TP, TT or CT), the ‘athlete’ with the greatest natural disadvantage would achieve less rotation than the median ‘athlete’ by at least 0.31 somersaults, and achieve less rotation than the ‘athlete’ with the greatest natural advantage by at least 0.55 somersaults. This difference multiplies as the number of somersaults increases: for example the ‘athlete’ with the greatest natural advantage is able to perform 4 somersaults in CT, the median ‘athlete’ is able to perform 2.7, and the ‘athlete’ with the greatest natural disadvantage is only able to perform 1.25 somersaults. It is thus abundantly clear that a natural advantage is practically very significant and could be the reason that one athlete succeeds and another does not.

When performing twisting somersaults the spread in achievement is also large. For the most efficient twist posture, L, and for the same value of $\phi_o$, some ‘athletes’ would only be able to achieve a half twist, while others would achieve a triple twist (Table 5-10). The spread in achievement increases as $\phi_o$ decreases. The combination of actions and postures also results in a large spread in achievement across ‘athletes’ (Table 6-11). For the most effective twist initiation action for any specific ‘athlete’ this spread is considerable with the top 10% of ‘athletes’ predicted as being able to perform at least 5 twists, while the lowest 10% of ‘athletes’ predicted as only being able to perform 1½ twists (Figure 6-33).

There was a strong correlation between the predicted values of $N_\theta$ for ‘athletes’ who had a natural advantage for each of the most useful twist initiation techniques (Section 6.6.2). Thus, it is reasonable to expect some ‘athletes’ to be good at twisting in general rather than

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21 The notion of an ideal body type is somewhat vague, in that it is not clear what level of achievement or advantage would be required to designate that body type as ‘ideal’. As a result it is only possible to provide evidence in support rather than prove that there is an ‘ideal body type’.
just for a specific twisting somersault skill. There was no correlation between ‘athletes’ who showed a natural advantage when twisting and when performing pure somersaults (Figure 6-37): it is thus not reasonable to expect that an ‘athlete’ who shows a natural advantage when twisting would have a natural advantage when performing pure somersaults. The two types of somersaults are quite different.

No simple biomechanic index (Section 5.2.7) showed any potential in identifying ‘athletes’ with a natural advantage for pure somersaults or twisting somersaults. It is thus apparent that full inertial property estimates are required to identify ‘athletes’ with a natural advantage.

The characteristics of gender and squad (related to age and years of training) did explain some of the differences between ‘athletes’. The different characteristics showed different proportions of ‘athletes’ for whom certain postures and actions were more or less suitable although there was still considerable overlap between groups defined by the different characteristics. A coach may thus use these sub-population tendencies as a guide, but should be aware that some ‘athletes’ would not follow these tendencies. Some trial and error with different techniques is required since accurate inertial property values are unlikely to be available.

Females tended to have a greater range of $\tau = I_{yy}/I_{yy,L}$ than males, indicating that they would experience a greater change in somersault speed than males when changing from a laid out posture to a more compact posture. This suggests that males would need to generate greater angular momentum than females to perform the same multiple-somersault skills. More females than males found tucked postures easier than piked postures and the leg separation in CT benefited more females than males. This suggests females would find it harder than males to transition to pike from tuck. When considering the type of tuck, more females found that using a straighter torso and greater hip flexion—as in posture BT—is easier than a more rounded torso—as in posture FT; males tended to find the opposite. When performing twisting somersaults neither gender showed a statistically significant advantage over the other.

The squads were found to be from different populations. The literature data set was clearly different from the other squads; this means its suitability for predicting skills achievable is questionable. The literature data is discussed in Section 7.4.1.

For pure somersaults the 12-or-under squad stood out as different. It tended to have lower values of $\tau = I_{yy}/I_{yy,L}$ than the other squads, and so it is expected that these ‘athletes’ would take longer to achieve their first somersault, but then transition more easily to harder postures. Of particular interest is the fact that the majority of the 12-or-under ‘athletes’
would find posture P easier than posture T and a lower proportion than the other squads would find posture TT easier than posture TP. It is thus recommended that a coach introduces piked postures into the training programme quite early for their young ‘athletes’.

For twisting somersaults, the Senior squad and the 12-or-under squad stood out as different for some of the techniques. The Senior squad tended to be able to achieve less twists within a somersault when using the most effective technique for them, when using lateral flexion, with or without DiverS, and when extending from posture A after Drop. The 12-or-under ‘athletes’ tended to achieve more twists within a somersault when using lateral flexion, but less twists when using the asymmetrical arm actions.

7.4.1 The suitability of the current literature data

The inertial property data sets collated in Chapter 3 were intended to provide a range of data which would allow explorations of techniques and hence allow reasonable predictions of skills that may be achieved, and allow reasonable suggestions of which techniques would be better than others. The collated inertial property data sets included estimates based on measurements of current athletes and data extracted from the literature.

Based on the results of Chapter 5 and Chapter 6 it is clear that applying data collated from the literature to the equations of motion resulted in quite different predictions than when inertial property estimates based on measurements of current athletes were applied to the equations of motion. Further, the literature data showed only small variations in achievement across the group. For pure somersaults the literature data had a higher range of $\tau = \frac{I_{yy}}{I_{yy.1}}$ and very little variation in the range of $\tau$; the range of $\tau$ is between 0.85 and 0.96. When considering twisting somersaults, the asymmetrical arm actions, “Drop in A & extend at ¼ twist” and “LHF with DiverS”, displayed small ranges in the predicted values of $N_\theta$ and low values of $N_\theta$. As a result the literature data greatly overestimated the change in speed of a pure somersault when changing posture and the number of twists that may be performed in a somersault, compared to the estimates based on current athletes.

Small ranges suggest that inertial property data sets from the literature represent similarly proportioned athletes. This could be a result of how the data was collected, in that the data sets are averages not actual people, or similarities in the original people sampled; the reason is not known. Nevertheless using only the data sets available in the literature would mean conclusions drawn do not reflect the athlete population in general, and would overestimate what skills may be achieved using the techniques of interest. It is thus recommended that the biomechanics community seek to collate a broader range of inertial properties for use when exploring equations of motion.
7.4.2 Inertial properties and simulations

The accuracy of inertial properties is often assessed by the accuracy of simulations of observed performances using these inertial properties. As was observed in Section 3.10, the accuracy of the simulation, when using different inertial property estimation methods, depended on the skill performed. As a result it is not appropriate to apply inertial properties that were sufficiently accurate for simulating one observed performance to simulations of different performances and assume that the same accuracy will be obtained. Instead, before using the inertial properties to explore features of a specific performance, an accuracy assessment should be conducted and the error deemed acceptable determined. The same inertial property data set was used for the majority of the example plots for the purposes of continuity and to allow discussion of different postures and techniques; it showed good fidelity when observing four different skills, but is not necessarily accurate for all postures and techniques.

From Section 3.10 it was also clear that even inertial property data sets that were deemed “unreasonable” in Section 3.5.7 could still result in better accuracy of simulations for some skills than some inertial property estimates that were not rejected. This indicates that the scaling of the angular momentum can overcome some of the reason that an inertial property estimate was deemed inaccurate; and that fidelity of a simulation and accuracy of inertial properties are in fact different outcomes.

7.5 Further Research

This thesis has focused on the aerial phase of pure and twisting somersaults. The conclusions of this thesis suggest a few areas for further research:

1. Known aerial twist initiation actions did not allow the majority of ‘athletes’ to perform all the twisting skills that are currently observed in competition. As a result
   a. athletes may use some contact twist to allow achievement of skills. An exploration of contact twist would be useful to ascertain what its potential contribution could be.
   b. athletes may use aerial techniques which have not been described in the literature or combinations of aerial techniques which were not considered in Chapter 6. Future research could extract techniques observed to be used by current athletes and compare these to determine if other aerial techniques are being used and what they are.

2. This thesis estimated inertial properties of a broad range of current athletes. Nevertheless, the sample of athletes was limited to a geographic area, and so cannot
be said to represent divers or gymnasts world-wide. Future research could increase the number of athletes whose inertial properties have been estimated, and hence support or challenge the estimates of the limits of achievement for each technique presented here. Further comparing estimates on current divers and gymnasts to athletes from other sports will suggest if diving and gymnastics tend to select certain ‘body types’. Applying athlete data from other sports to the equations describing the somersault would suggest if the ‘body types’ selected for participation in gymnastics and diving are in fact the most suited to these sports.

3. Postures and techniques explored in this thesis were chosen based on previous literature descriptions and discussions with coaches. It was thus assumed that these actions were achievable by an athlete, and the focus was on mathematically analysing the techniques. Future work could determine joint torques to further assess how realistically achievable the various postures and techniques are or to suggest muscle groups that should be the focus of power and strength training for athletes performing somersaults.
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Appendix A.

Body model and inertial property calculations

This appendix expounds on the methods used to collate inertial property data in Chapter 3. In particular it provides a discussion on the use of cadaver data and the uniform density assumption that underlies all estimation methods using geometric models, provides formulas required to apply the inertial property estimation methods from Chapter 3, and explains the function and algorithms of the programme ICG17.

A.1 Uniform density assumption

All geometric models assume uniform density of each individual segment, even if they apply different densities to different segments. The density applied to a segment may a whole body average density, or published density values which are typically based on cadaver data. The uniform density assumption ignores the variation in tissue densities throughout a segment.

The centre of gravity will thus be the centre of volume and the moments of inertia will also differ from actual. Clauser et al. (1969) found that for the limb segments the centre of gravity was actually more distal than the centre of volume. This result is not surprising since the proportion of bone increases distally, the proportion of muscle decreases, and the proportions of skin, fat, and fascia fluctuate (Dempster & Gaughran, 1967). Nevertheless Clauser, et al. (1969) estimated the error in the centre of gravity position to be only 2-3 cm. The error will be minimised by using the smallest possible segments for which different densities are known. The centre of volume and centre of gravity of the body as a whole will not coincide due to differing densities of segments, most notably the chest.

Whether or not the assumption of uniform density is of practical concern depends on the final application of the inertial properties. Further, despite the fact that Ackland, et al. (1988) found “marked variation in cross-section density values” along the lengths of a cadaver and living leg, they concluded that only minor errors were produced when using uniform density and the estimation of the segment volume was more critical than using a
density profile. Hall & Depauw (1982), who compared results from a model assuming uniform density and results from direct measurement, concluded that the uniform density assumption made no significant difference to the calculation of whole body centre of gravity location. Wei & Jensen (1995) found that the density profiles were relatively smooth and could be represented by polynomials. They compared the value of inertial properties obtained by using average densities with using polynomial density functions when using elliptical cylinder slices to represent the body. They did the comparison for a range of different groups of people: children through to the elderly. They found that using their derived density profile generally gave larger inertial property values than using an average density. The average differences were all less than 4%, yet differences for some individuals ranged up to 22.5% (Wei & Jensen, 1995).

The assumption of uniform density also means that the principal axes of the body are taken to be the axes of symmetry; however, the non-uniformity means that this is not actually the case (Chandler, et al., 1975).

### A.1.1 Cadaver data

Measurements taken on cadaver data impacts all inertial property estimation methods. Geometric models and even scans rely on tissues density values (Dempster, 1955), average density values, or regression equations for mass distribution (Barter, 1957), determined based on measurements from cadavers. It is thus important to appreciate how the cadaver data was collected.

Cadaver data is based on a limited sample size, with the subjects tending to be older individuals rather than trained athletes in their prime, as well as being predominantly white males; further there will have been body changes after death, which cannot be fully known.

Cadaver data comes from individuals who have died. The nature of death means that the body undergoes changes from its living state. The most significant reported change is due to progressive fluid loss. The loss of fluid affects mass, standing height, and circumference measurements (McConville, et al., 1980). It is possible to pump a cadaver with preservation fluid to simulate a ‘normal appearance’, although changes in segment inertial properties due to the preservation fluid are unknown (McConville, et al., 1980). In addition, when cadavers are dismembered fluid loss is accelerated, and so the mass of the segments may not total the mass of the whole cadaver (Barter, 1957). Since the proportion of fluid loss between segments is unknown then it is not clear how to re-distribute the difference in masses between the segments. Other tissue changes due to the nature of death and the preservation method have an unknown effect on the inertial properties. It is also unclear if the changes are universal or affect some segments more than others.
(McConville, et al., 1980). However, Clauser, et al. (1969) believed that the differences due to the preservation processes they were familiar with would be negligible when considering mass ratios.

In addition to the changes occurring on death, the sample of cadavers is typically limited in number due to the expense and difficulty of performing direct measurements (McConville, et al., 1980) and predominantly the middle-age to old-age white males (Barter, 1957; Dempster, 1955; Clauser, et al., 1969). Such a sub-set is expected to be quite different to living, generally young, trained athletes. In particular cadaver data has been described as

- Lighter than the living subjects most likely due to weight-reducing disease (Barter, 1957)
- Smaller or shorter than the white male military population at the time of writing (Dempster, 1955; Clauser, et al., 1969)
- Having more mass in the trunk than living subjects. This was most noticeable in the thigh and buttock segments. The difference is most likely due to the accumulation of fat and the changing ratios of tissue types with age (Dempster, 1955).

### A.2 Averages and regression equations

Table A-1 lists a selection of sources of inertial property regression equations with a description of the data provided and the original sample. These are the commonly referenced sources, and aimed at determining inertial properties for all segments of the body. The methods which provided all inertial property data required were used in Section 3.5.6 are bolded. The other regression equations do not provide all inertial property data and so could not be used in Section 3.5.6.

<table>
<thead>
<tr>
<th>Source</th>
<th>Regression equations provided and related comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dempster (1955)</td>
<td>Average percentages. The moment of inertia of body segments is only given about one axis.</td>
</tr>
<tr>
<td></td>
<td>Inputs: Total body mass and segment lengths</td>
</tr>
<tr>
<td></td>
<td>Original sample: Eight male cadavers of elderly individuals</td>
</tr>
<tr>
<td>Clauser et al. (1969)</td>
<td>Linear regression equations</td>
</tr>
<tr>
<td></td>
<td>Inputs: Anthropometric measurements specific to the segment</td>
</tr>
<tr>
<td></td>
<td>Original sample: 13 preserved male cadavers of varying ages who appeared relatively “normal”.</td>
</tr>
<tr>
<td>Chandler et al. (1975)</td>
<td>Linear regression equations</td>
</tr>
<tr>
<td></td>
<td>Inputs: Body mass or segment volume</td>
</tr>
<tr>
<td></td>
<td>Original sample: Male cadaver data</td>
</tr>
<tr>
<td></td>
<td>Chandler et al. (1975) warned against applying the regression equations broadly due to the small size of the sample.</td>
</tr>
<tr>
<td>Source</td>
<td>Regression equations provided and related comments</td>
</tr>
<tr>
<td>----------------------------------------------------------------------</td>
<td>--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
</tbody>
</table>
| “Modified Hanavan man model by the Martin Marietta Corporation” described by Chandler, et al. (1975) | A combination of regression equations were apparently being used in 1975 by NASA simulations: Barter’s regression equations for segment mass, Wooley’s (1972) regression equations for segment moment of inertia based on anthropometric measurements and Hanavan’s (1964) model, and Kurzhals, 1972’s regression equations for the location of the centre of gravity. Unfortunately the regression equations of Kurzhals could not be found. The specific reference cited by Chandler, et al. (1975) only gave equations for finding the centre of gravity of a segment with respect to a reference frame and depended on posture.   
**Input:** Subject’s mass  
**Original sample:** Cadaver data |
| McConville et al. (1976)                                              | Multiple variable linear regression equations. Regression equations for the location of the centre of gravity were not provided.  
**Inputs:** Depend on the specific segment, but are anthropometric measurements.  
**Original sample:** represents a “wide range of body sizes with good representation from the extreme ends of the distribution which are the areas most critical to the solution of design problems”. The average height and mass also still reflected the averages of the larger USAF population. |
| Young, Chandler & Snow (1983)                                       | Multiple variable linear regression equations. No regression equations for the location of the centre of gravity were provided.  
**Inputs:** Regression equations using height and mass as inputs as well as regression equations using various anthropometric measurements depending on the segment are given.  
**Original sample:** 46 women chosen to be “representative of a general United States population as defined by the 1971-74 Public Health Service Health and Nutrition Examination Survey”. |
| Plagenhoef (1983)                                                    | Average percentages.  
Unfortunately, Plagenhoef does not determine the moments of inertia about the longitudinal axis of any segments.  
**Inputs:** Total body mass and segment lengths  
**Original sample:** Collage aged athletes mostly women. |
| Finch (1985)                                                        | Average percentages for three somatotypes  
**Inputs:** Total body mass, segment lengths, somatotype.  
**Original sample:** Fifteen Canadian college-aged females selected to represent the three somatotypes. |
| Jensen (1989)                                                       | Polynomial regression equations to get the proportions to total body mass or segment length, and the moment of inertia about the transverse axis.  
**Inputs:** Age when between 4 and 20 years to get the proportions then the total body mass and segment lengths to obtain the property values.  
**Original sample:** Boys living in Sudbury, Ontario |
| Yeadon & Morlock (1989)                                             | Linear and non-linear regression equations. It is stated, however, that the linear regression equations gave less accurate estimations of inertial properties and as a result the equations were not given in the paper  
**Input:** A combination of width, depth and perimeter measurements  
**Original Sample:** data from Chandler, Clauser, McConville & Young, 1975 |
| Cheng, et al. (2000)                                                | Average percentages and dimensionless moment of inertia. The trunk was given as one segment.  
**Inputs:** Total body mass, Length of each segment  
**Original Sample:** Eight Chinese males with an average age of 26yrs |
| Zatsiorsky (2002)                                                   | Average percentages |
A.3 Geometric shapes and inertial property equations for the geometric model methods used in Section 3.5.6

This section provides clarification of the segment divisions, segment shapes and inertial property equations, for the geometric methods used in 3.5.6 including the modifications.

A.3.1 Methods based on Hanavan (1964)

To help clarify the segment divisions, segment shapes and inertial property equations for the Hanavan-based methods, the shape and associated parameters for each segment are given in Table A-2. The origins of each segment, following Hanavan’s model but adjusted for use and consistency in this thesis, are presented in Table A-3 below. The mass of the segments using HanavanBP method or the Woolley method are given in Table A-4 below. Finally, Table A-5 gives the equations for the inertial properties by shape.
Table A-2: Determining Parameter’s for Hanavan’s model

*Note 1:* When substituting values \( r_1 \) is the more proximal end, so that the Centre of Gravity equations given for the truncated cone are measured from the proximal end.

*Note 2:* Hanavan (1964) chose \( r_2 \) for the foot so that the centre of gravity would be 0.429h. The equation given above is obtained by solving the quadratic equation for equating the centre of gravity formulae with this and taking the positive solution.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Shape</th>
<th>( h )</th>
<th>( R, r_1 ), or Width</th>
<th>( r_2 ) or Depth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>Ellipsoid of revolution</td>
<td>Shoulder to vertex distance</td>
<td>( R = \frac{(Head circumference)}{2\pi} )</td>
<td>n/a</td>
</tr>
<tr>
<td>Upper torso</td>
<td>Elliptical cylinder</td>
<td>Xiphoid process to shoulder.</td>
<td>Width = chest breadth</td>
<td>Depth = Average of chest and waist depth</td>
</tr>
<tr>
<td>Lower torso</td>
<td>(xiphoid process height) – (stature) + (sitting height).</td>
<td>Equivalent to the trochanter to xiphoid length.</td>
<td>Width = hip breadth</td>
<td>Depth = Average of waist and buttock depth</td>
</tr>
<tr>
<td>Upper arm</td>
<td>Truncated Cone</td>
<td>Shoulder to elbow length</td>
<td>( r_1 = \frac{(Arm at arm pit circumference)}{2\pi} )</td>
<td>( r_2 = \frac{(Elbow circumference)}{2\pi} )</td>
</tr>
<tr>
<td>Lower arm</td>
<td>Elbow to wrist length</td>
<td>( r_1 = \frac{(Elbow circumference)}{2\pi} )</td>
<td>( r_2 = \frac{(Wrist circumference)}{2\pi} )</td>
<td></td>
</tr>
<tr>
<td>Hand</td>
<td>Sphere</td>
<td>n/a</td>
<td>( R = \frac{(Fist circumference)}{2\pi} )</td>
<td>n/a</td>
</tr>
<tr>
<td>Upper leg</td>
<td>Truncated cone</td>
<td>Knee – stature + sitting height. But in this thesis will use Hip joint centre to knee ( r_1 = \frac{(Thigh/groin circumference)}{2\pi} )</td>
<td>( r_2 = \frac{(Knee circumference)}{2\pi} )</td>
<td></td>
</tr>
<tr>
<td>Lower leg</td>
<td>Knee to ankle length</td>
<td>( r_1 = \frac{(Knee circumference)}{2\pi} )</td>
<td>( r_2 = \frac{(Ankle circumference)}{2\pi} )</td>
<td></td>
</tr>
<tr>
<td>Foot</td>
<td>Truncated cone</td>
<td>Foot length</td>
<td>( r_1 = \frac{(Ankle circumference)}{2\pi} )</td>
<td>[ r_2 = \frac{-0.275r_1 + \sqrt{(0.275r_1)^2 + 4(1.275)(0.725r_1^2)}}{2.55} ]</td>
</tr>
</tbody>
</table>

Table A-3: Origins of segments for Hanavan’s model

*Note:* All origins are assumed to be in the frontal plane. Also, remember that the z axis of the local frames in vertical when in the reference position, hence the negative values in the limb equations.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Origin distance along previous segment’s longitudinal axis</th>
<th>Distance of the origin along previous segment’s transverse axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>( h_{Upper Torso} )</td>
<td>n/a</td>
</tr>
<tr>
<td>Upper torso</td>
<td>( h_{Lower Torso} ) -{centre of gravity of lower torso (measured from the base of the elliptical}</td>
<td>n/a</td>
</tr>
</tbody>
</table>
Segment | Origin distance along previous segment’s longitudinal axis | Distance of the origin along previous segment’s transverse axis |
--- | --- | --- |
Lower torso | It will be the reference segment and so there is no previous segment. | n/a |
Upper arm | \(h_{\text{Upper Torso}}\) | \((\text{width}_{\text{Upper Torso}})/2+r_{\text{Upper Leg}}\) |
Lower arm | \(-h_{\text{Upper Arm}}\) | n/a |
Hand | \(-h_{\text{Lower Arm}}\) | n/a |
Upper leg | \(-\{\text{Centre of gravity of lower torso (measured from the base of the elliptical cylinder)}\}\) | \(r_{\text{Upper Leg}}\) |
Lower leg | \(-h_{\text{Upper Leg}}\) | n/a |
Foot | \(-h_{\text{Lower Leg}}\) | n/a |

Table A-4: Determining the mass of segments for Hanavan’s model

Note 1: The regression equations are taken from Hanavan (1964) and Woolley (1957), respectively. The multiplication factors for the head and upper torso are 0.079 and 0.079, respectively.

Note 2: For the head, the multiplication factor was taken from Hanavan (1964) as 0.079; however, Miller and Morrison (1975) suggest that Dempster’s subjects, 0.076 would be more appropriate.
<table>
<thead>
<tr>
<th>Segment</th>
<th>Regression equation (Barter, 1957)</th>
<th>HanavnBP</th>
<th>Woolley</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower arm</td>
<td>$W_{LA} = 0.4W - 0.23$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$W_{LA,P} = \frac{W_{LA}}{W_{Seg_total}}$</td>
<td></td>
<td>$W_{LA} = 0.4\left(\frac{W - 5.48}{0.91}\right) - 0.23$</td>
</tr>
<tr>
<td>Hand</td>
<td>$W_{Ha} = 0.01W - 0.32$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$W_{Ha,P} = \frac{W_{Ha}}{W_{Seg_total}}$</td>
<td></td>
<td>$W_{Ha} = 0.01\left(\frac{W - 5.48}{0.91}\right) - 0.32$</td>
</tr>
<tr>
<td>Upper leg</td>
<td>$W_{UL} = 0.18W + 1.45$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$W_{UL,P} = \frac{W_{UL}}{W_{Seg_total}}$</td>
<td></td>
<td>$W_{UL} = 0.18\left(\frac{W - 5.48}{0.91}\right) + 1.45$</td>
</tr>
<tr>
<td>Lower leg</td>
<td>$W_{LL} = 0.11W - 0.86$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$W_{LL,P} = \frac{W_{LL}}{W_{Seg_total}}$</td>
<td></td>
<td>$W_{LL} = 0.11\left(\frac{W - 5.48}{0.91}\right) - 0.86$</td>
</tr>
<tr>
<td>Foot</td>
<td>$W_{F} = 0.02W + 0.68$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$W_{F,P} = \frac{W_{F}}{W_{Seg_total}}$</td>
<td></td>
<td>$W_{F} = 0.02\left(\frac{W - 5.48}{0.91}\right) + 0.68$</td>
</tr>
<tr>
<td>Whole body</td>
<td>$W_{Seg_total} = 0.91W + 5.48$</td>
<td>$W_{Seg_total}$</td>
<td>$W_{Seg_total} = 0.91W + 5.48$</td>
</tr>
</tbody>
</table>

Table A-5: Inertial properties by segment shape

<table>
<thead>
<tr>
<th>Shape</th>
<th>Moments of inertia</th>
<th>Centre of gravity position</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipsoid of revolution</td>
<td>$I_{xx} = \frac{3}{10}(Mass_{Segment})(\frac{r_x^2-r_z^2}{r_x^2-r_y^2})$</td>
<td>At the geometric centre of the ellipsoid</td>
<td>$\frac{4}{3}\pi R^2h$</td>
</tr>
<tr>
<td></td>
<td>$I_{yy} = \frac{3}{10}(Mass_{Segment})(\frac{r_y^2-r_z^2}{r_x^2-r_y^2})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$I_{zz} = \frac{3}{10}(Mass_{Segment})(\frac{r_z^2-r_x^2}{r_x^2-r_y^2})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elliptical cylinder</td>
<td>$I_{xx} = \frac{1}{12}(Mass_{Segment})(h^2 + \frac{3}{2}d^2)$</td>
<td>Centre of gravity is $h/2$ above the base of the cylinder.</td>
<td>$\pi wd h$</td>
</tr>
<tr>
<td></td>
<td>$I_{yy} = \frac{1}{12}(Mass_{Segment})(h^2 + \frac{3}{2}w^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$I_{zz} = \frac{1}{4}(Mass_{Segment})(\frac{d}{2})^2 + (\frac{w}{2})^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Body model and inertial property calculations

<table>
<thead>
<tr>
<th>Shape</th>
<th>Moments of inertia</th>
<th>Centre of gravity position</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truncated cone</td>
<td>$I_x = \frac{3}{20} (\text{Mass}_{\text{segment}}) \left( \frac{r_4^2 - r_5^2}{r_1^2 - r_2^2} \right)$</td>
<td>Centre of gravity is $\frac{\pi h}{3} \left( r_1^2 + r_4 r_5 + r_5^2 \right)$ up from the base circle.</td>
<td>$\frac{\pi h}{3} \left( r_1^2 + r_4 r_5 + r_5^2 \right)$</td>
</tr>
<tr>
<td></td>
<td>$I_y = I_{xx} = I_{zz} = \frac{3}{10} (\text{Mass}_{\text{segment}}) \left( \frac{r_4^2 - r_5^2}{r_1^2 - r_2^2} \right)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sphere</td>
<td>$I_{yy} = I_{xx} = I_{zz} = 0.4 (\text{Mass}_{\text{segment}}) R^2$</td>
<td>At the geometric centre of the sphere.</td>
<td>$\frac{4}{3} \pi R^3$</td>
</tr>
</tbody>
</table>

**A.3.2 GOBD (Baughman, 1983)**

In Baughman (1983) it was not entirely clear how the dimensions of the geometric shapes of segments are obtained from “standard” anthropometric measurements and so slight adjustments were made based on the general descriptions to preserve the perceived intent and to assist comparison between geometric methods. The head, neck, and torso segments overlap by (Neck circumference/2\(\pi\)), even though no explanation other than the need for the model to have “proper appearance” was given. The overlap has been maintained as published (Baughman, 1983). Table A-6 gives the segment shapes and associated parameters.

Table A-7 then gives the equations for the inertial properties of the various shapes not yet presented.

**Table A-6: The segments and dimensions of the GOBD model**

<table>
<thead>
<tr>
<th>Segment</th>
<th>Shape</th>
<th>Semi-axis in the x direction</th>
<th>Semi-axis in the y direction</th>
<th>Semi-axis in the z direction or height</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>Ellipsoid</td>
<td>Head depth/2 (take at below nose level)</td>
<td>Head breadth/2 (take at below nose level)</td>
<td>Average of the vertex-to-chin length and the neck circumference/2(\pi)</td>
</tr>
<tr>
<td>Neck</td>
<td>Ellipsoid</td>
<td>Neck circumference/2(\pi)</td>
<td></td>
<td>Average of the shoulder-to-chin length and the Neck circumference/2(\pi)</td>
</tr>
<tr>
<td>Upper chest sub-</td>
<td>Semi-ellipsoid</td>
<td>(Chest depth at arm pit)/2</td>
<td>(Chest breadth at arm pit)/2</td>
<td>Shoulder height to Armpit</td>
</tr>
<tr>
<td>segment</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Table A-7: Inertial properties by segment shape

**Note:** “a” is the x semi-axis, b the y semi axis, c the z semi-axis, h the height of the Truncated Elliptical prism.

**Note 2:** The inertial properties for the ellipsoid and semi-ellipsoid were derived from the definitions of the shapes and inertial properties, since Baughman (1983) did not give equations.

**Note 3:** Where, the transfer term (same for $I_{xx}$ and $I_{yy}$ as moving along longitudinal axis):

<table>
<thead>
<tr>
<th>Segment</th>
<th>Shape</th>
<th>Semi-axis in the x direction</th>
<th>Semi-axis in the y direction</th>
<th>Semi-axis in the z direction or height</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower chest sub-segment</td>
<td>Truncated Elliptical prism</td>
<td>(Chest depth at arm pit)/2 and tenth rib depth/2 for the two ends</td>
<td>(Chest breadth at arm pit)/2 and tenth rib breadth/2 for the two ends</td>
<td>Armpit to tenth rib</td>
</tr>
<tr>
<td>Abdomen</td>
<td>Truncated Elliptical prism</td>
<td>Iliocristale depth/2 and tenth rib depth/2 for the two ends</td>
<td>Iliocristale breadth/2 and tenth rib breadth/2 for the two ends</td>
<td>Tenth rib to iliocristale</td>
</tr>
<tr>
<td>Pelvis</td>
<td>Semi-ellipsoid</td>
<td>Buttock depth/2</td>
<td>Hip breadth/2</td>
<td>Hip to iliocristale</td>
</tr>
<tr>
<td>Upper leg</td>
<td>Ellipsoid</td>
<td>(Thigh Circumference + Upper leg Circumference)/4π</td>
<td>(Hip joint centre to knee joint centre length)/2</td>
<td></td>
</tr>
<tr>
<td>Lower leg</td>
<td>Ellipsoid</td>
<td>(Calf Circumference)/2π</td>
<td>Knee joint centre to ankle joint centre length/2</td>
<td></td>
</tr>
<tr>
<td>Foot – ankle sub-segment</td>
<td>Elliptical cylinder</td>
<td>(Heel to Ankle joint height)/2</td>
<td>(Ankle Circumference)/4π</td>
<td></td>
</tr>
<tr>
<td>Foot – rest of foot sub-segment</td>
<td>Truncated Elliptical prism</td>
<td>(Heel to Ankle joint height)/2 and zero as each end</td>
<td>(Ankle Circumference nce + Foot Bredt h.)/2 for each end. (Foot breadth being the widest point)</td>
<td>Foot length (Ankle Circumference nce)/2π (Take as heel to longest toe)</td>
</tr>
<tr>
<td>Upper arm</td>
<td>Ellipsoid</td>
<td>Biceps Circumference/2π</td>
<td>Shoulder to elbow/2</td>
<td></td>
</tr>
<tr>
<td>Lower arm and hand</td>
<td>Ellipsoid</td>
<td>Forearm Circumference/2π</td>
<td>Forearm to hand length/2</td>
<td></td>
</tr>
</tbody>
</table>
\[
T = \rho \pi \left( h \left( \frac{F_1}{5} + \frac{F_2}{4} + \frac{a_b h}{3} \right) - h^2 \left( \frac{2F_1}{3} + \frac{F_2}{2} + a_b \right) + h^3 \left( \frac{F_1}{3} + \frac{F_2}{2} + a_b \right) \right)
\]

and,
\[
F_1 = (a_x - a_y)(b_z - b_y)
\]
\[
F_2 = (a_x - a_x)b_y + (b_z - b_x)a_y
\]
\[
F_3 = (a_x - a_y)^2 + (b_z - b_y)^2
\]
\[
F_4 = (a_x - a_x)a_y + (b_z - b_x)b_y
\]

<table>
<thead>
<tr>
<th>Shape</th>
<th>Moments of inertia</th>
<th>Centre of gravity position</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipsoid</td>
<td>( I_x = \frac{\text{Mass}}{5} (b^2 + c^2) )</td>
<td>At the geometric centre</td>
<td>( \left( \frac{4\pi abc}{3} \right) )</td>
</tr>
<tr>
<td></td>
<td>( I_y = \frac{\text{Mass}}{5} (a^2 + c^2) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( I_z = \frac{\text{Mass}}{5} (a^2 + b^2) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Semi-ellipsoid</td>
<td>( I_x = \frac{\text{Mass}}{5} \left( b^2 + \frac{19}{64} c^2 \right) )</td>
<td>(( \frac{3c}{8} )) up from the base and at the centre of the elliptical cross-sections.</td>
<td>( \left( \frac{2\pi abc}{3} \right) )</td>
</tr>
<tr>
<td></td>
<td>( I_y = \frac{\text{Mass}}{5} \left( a^2 + \frac{19}{64} c^2 \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( I_z = \frac{\text{Mass}}{5} (a^2 + b^2) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Truncated Elliptical prism</td>
<td>( I_x = \frac{\rho \pi h}{4} \left( \frac{(a_x - a_y)(b_z - b_y)}{5} + \frac{(a_x - a_x)b_y + (b_z - b_x)a_y}{2} \right) )</td>
<td>up from the base and at the centre of the elliptical cross-sections.</td>
<td>( \frac{\pi h^3}{4} \left( \frac{F_1}{3} + \frac{F_2}{2} + a_b \right) )</td>
</tr>
<tr>
<td></td>
<td>( I_y = \frac{\rho \pi h}{4} \left( \frac{(a_x - a_x)(b_z - b_y)}{5} + \frac{(a_x - a_x)b_y + (b_z - b_x)a_y}{2} \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( I_z = \frac{\rho \pi h}{4} \left( \frac{F_3}{3} + \frac{F_4}{2} + \frac{(a_x^2 + b_y^2)}{3} \right) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**A.3.3 Yeadon (1984)**

Yeadon (1984) uses a semi-ellipsoid of revolution, truncated right circular cones, “stadium solids”, and a right circular cone. Equations for the inertial properties of truncated right circular cones were given in Appendix A.3.1; equations for the inertial properties of the semi-ellipsoid and stadium solid are presented in Yeadon (1984). Adaptations of these equations, to maintain notational consistency with the rest of this thesis are repeated below.
The semi-ellipsoid of revolution used by Yeadon (1984) is a solid formed by revolving half an ellipse about its semi-major axis. The base is a circle and cross-sections cut perpendicular to this base are semi-ellipses. The inertial properties are given in Table A-8.

<table>
<thead>
<tr>
<th>Centre of Gravity location</th>
<th>Mass</th>
<th>(I_{zz})</th>
<th>(I_{xx} = I_{yy})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{3h}{8}) from the circular base along the z axis.</td>
<td>(\frac{2\rho \pi^2 h}{3})</td>
<td>(\frac{4\rho \pi^4 h}{15})</td>
<td>(\frac{2\rho \pi^2 h}{15}(r^2 + h^2) - \frac{3\rho \pi^2 h^3}{32})</td>
</tr>
</tbody>
</table>

Yeadon presents a ‘new’ shape, the right stadium solid for modelling torso segments. It is a smooth sided solid that has the cross-section resembling an athletics stadium, and longitudinal axis perpendicular to each stadia cross-section. Figure A-1 illustrates the stadium solid and Table A-9 gives the inertial property equations.

**Figure A-1: The “stadium” solid**
Adapted from figures 43 and 44 in Yeadon (1984)

**Table A-9: Inertial properties of the stadium solid**

*Note: \(a=(r_2-r_1)/r_1\) and \(b=(t_2-t_1)/t_1\)*

<table>
<thead>
<tr>
<th>Centre of Gravity location along longitudinal axis from proximal end (Z)</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\rho h^2}{3(Mass)} \left( t_1r_1(6+4a+4b+3ab) + \frac{\pi}{4} r_1^2(6+8a+3a^2) \right))</td>
<td>(\frac{\rho h^2}{3} \left( t_1(12+6a+6b+4ab) + \pi r_1(3+3a+a^2) \right))</td>
</tr>
</tbody>
</table>
When segments are combined it is necessary to determine the combined inertial properties. The new mass will simply be the sum of the masses of the segments forming the combined segment. Since each of Yeadon’s segments that combine to from a segment of the 17 segment model have their longitudinal axes aligned, the combined segment’s centre of gravity from the proximal end of the combined segment along its longitudinal axis is

$$Z_{\text{Combined}} = \frac{\sum (\text{Mass})_{\text{Contributing segment}} (Z_{\text{Contributing segment}} + \sum h_{\text{Preceeding segments}})}{\sum (\text{Mass})_{\text{Contributing segment}}}$$

And the combined moments of inertia are

$$I_{xz} = \sum \{I_{xz} \}$$

$$I_{xy} = \sum \{I_{xy} + m(Z_{\text{Combined}} - Z - \sum h_{\text{Preceeding segments}})^2 \}$$

$$I_{yy} = \sum \{I_{yy} + m(Z_{\text{Combined}} - Z - \sum h_{\text{Preceeding segments}})^2 \}$$

Yeadon (1984) doesn’t explicitly state the method of calculating the origins, however, the following is a logical choice: the origins of each combined segment may be found by summing the segment heights from the last segment. For the upper arm and upper leg the
transverse component of the origin vector will be taken as the radius of s5 and the radius of
the upper leg respectively.

**A.3.4 Nikolova**

The method “Nikolova” is described in two papers: Nikolova & Toshev (2007) and
Nikolova (2010a). Equations for the inertial properties are provided explicitly in those
papers, although it is important to note a few points:

1. The volume of the upper torso is not quite the volume of a truncated elliptical cone.
   That is the given parameter R2 is not used in the volume but rather the given
   parameter R*.

2. Moments of inertia for the right truncated elliptical cone are about the R1 end of the
   segment and so the parallel axis theorem must be applied to transform them to
   being about the centre of gravity of the segment.

**A.4 The programme ICG17**

The programme ICG17 was introduced in Section 3.8, and was used to aid the calculation
of whole body or grouped segment inertial properties. The programme requires the inertial
properties of the 17-segments defined in Section 3.1 and the posture specified as a set of
rotation matrices, such as given for any of the key postures in Section 3.9. The programme
will output the total body mass, overall moment of inertia with respect to frame Ref
(Section 4.1.2) and with respect to frame Pri (Section 4.1.3), the principal directions with
respect to frame Ref, and the locations of each segments centre of gravity and origin with
respect to frame Ref. These inertial properties may then be used in further modelling as
described in Section 3.8.

To call ICG17 in Matlab enter the command

```
[Mass, Overall_I_aligned, Principal_directions, Principal_I, Local_CGs, Joints] = ICG17(Masses, Local_I, R, O, CG)
```

Each of the inputs and outputs is described in Sections A.4.1 to A.4.12.

Local_CGs and Joints give the coordinates of each segment’s centre of gravity, and the
joints between segments respectively; the coordinates are given with respect to the overall
centre of gravity. The data contained in Local_CGs and Joints is then used to plot a stick
figure of the body. The body links between joints are plotted as red lines and the centre of
gravity of each segment is plotted as a blue dot. Since there is no joint at the end of the
hands, feet, and head, their centres of gravity will not be on a red line. The stick figure
representation may be used to confirm that the correct posture has been entered, and may also be used to identify any errors in the length inputs of segments.

A stick figure connecting the joints is plotted in red, and each local Centre of Gravity is plotted as filled blue circles. (For the head, hands, and feet, their Centre of Gravities will be past the last joint plotted, since the ends of the final segments are unknown.)

If the Principal_I and Principal_directions are not required then ICG17short, which is otherwise identical to ICG17, may be run instead. ICG17short may be called using the command

\[
\text{[Mass, Overall\_I\_aligned, Local\_CGs, Joints]} = \\
\text{ICG17short(Masses, Local\_I, O, CG, R)}.
\]

### A.4.1 Input: Masses

The Masses input is a 1 X 17 matrix where the column numbers match the segments numbers 1 to 17 as specified in Section 3.1.1. The masses should be entered in kg.

### A.4.2 Input: Local\_I

The Local\_I input is a 1 X 17 object, with each entry corresponding to the segment numbers 1 to 17 as specified in Section 3.1.1. Each entry is the inertia tensor (a 3 x 3 matrix) aligned with the segment’s local frame (Section 4.1.1) but about the segments centre of gravity. Local\_I for each segment should be in kg.cm\(^2\).

### A.4.3 Input: O

The input O is a 3 X 17 matrix with the columns corresponding to the segment numbers 1 to 17 as specified in Section 3.1.1. The column gives the origin (Section 3.1.2) of a segment with respect to the previous segment’s local frame as a vector (Section 4.1.1). The first row is the x-, second the y-, and third the z- component of the origin vector. Since the Pelvis has no joint closer to itself, then its Centre of Gravity will be the origin of its local frame. Table A-10 describes each entry of the input matrix O in relation to the 17-segment model and anatomical features. The entries to O should be in cm.

<table>
<thead>
<tr>
<th>Origins Matrix entry number</th>
<th>Segment name</th>
<th>Vector (x, y, z) between local frames</th>
<th>Description of Origin vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>From</td>
<td>To</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table A-10: Origin vectors for each segment
### Origins Matrix entry number

<table>
<thead>
<tr>
<th>Segment name</th>
<th>Vector ((x, y, z)) between local frames</th>
<th>Description of Origin vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pelvis (0,0,0)</td>
<td>The Pelvis local frame is on the Pelvis’s Centre of Gravity.</td>
<td></td>
</tr>
<tr>
<td>Abdomen (1^{O_2})</td>
<td>Pelvis’s Centre of Gravity</td>
<td>Lower surface of the Abdomen.</td>
</tr>
<tr>
<td>Chest (2^{O_3})</td>
<td>Lower surface of the abdomen</td>
<td>Lower Section of the chest</td>
</tr>
<tr>
<td>Neck (3^{O_4})</td>
<td>Lower surface of the chest</td>
<td>Lower surface of the neck</td>
</tr>
<tr>
<td>Head (4^{O_5})</td>
<td>Lower surface of the neck</td>
<td>Lower surface of the head</td>
</tr>
<tr>
<td>Left Upper Arm (5^{O_6})</td>
<td>Lower surface of the chest</td>
<td>Left shoulder</td>
</tr>
<tr>
<td>Left Forearm (6^{O_7})</td>
<td>Left shoulder</td>
<td>Left elbow</td>
</tr>
<tr>
<td>Left Hand (7^{O_8})</td>
<td>Left elbow</td>
<td>Left wrist</td>
</tr>
<tr>
<td>Left upper leg (8^{O_9})</td>
<td>Vector from the Pelvis’s Centre of Gravity</td>
<td>Left hip</td>
</tr>
<tr>
<td>Left lower leg (9^{O_{10}})</td>
<td>Left hip</td>
<td>Left knee</td>
</tr>
<tr>
<td>Left foot (10^{O_{11}})</td>
<td>Left knee</td>
<td>Left ankle</td>
</tr>
<tr>
<td>Right upper arm (11^{O_{12}})</td>
<td>Lower surface of the chest</td>
<td>Right shoulder</td>
</tr>
<tr>
<td>Right forearm (12^{O_{13}})</td>
<td>Right shoulder</td>
<td>Right elbow</td>
</tr>
<tr>
<td>Right hand (13^{O_{14}})</td>
<td>Right elbow</td>
<td>Right wrist</td>
</tr>
<tr>
<td>Right upper leg (14^{O_{15}})</td>
<td>Vector from the Pelvis’s Centre of Gravity</td>
<td>Right hip</td>
</tr>
<tr>
<td>Right lower leg (15^{O_{16}})</td>
<td>Right hip</td>
<td>Right knee</td>
</tr>
<tr>
<td>Right foot (16^{O_{17}})</td>
<td>Right knee</td>
<td>Right ankle</td>
</tr>
</tbody>
</table>

A.4.4 Input: CG

The input \(CG\) is a 3 X 17 matrix with the columns corresponding to the segment numbers 1 to 17 as specified in Section 3.1.1. Each column gives the location of the local centre of...
gravity of a segment with respect to its local frame. The first row is the x-, second the y- and third the z- component of the centre of gravity vector.

Table A-11 describes each entry of the input matrix CG in relation to the 17-segment model and anatomical features. The CG entries should be in cm.

<table>
<thead>
<tr>
<th>CG column entry</th>
<th>Name</th>
<th>Description of CG vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 = Reference Segment</td>
<td>Pelvis</td>
<td>Will be (0,0,0) since the Centre of Gravity of the pelvis is at origin of the local frame of the Pelvis</td>
</tr>
<tr>
<td>2</td>
<td>Abdomen</td>
<td>Lower surface of the abdomen Abdomen centre of gravity</td>
</tr>
<tr>
<td>3</td>
<td>Chest</td>
<td>Lower surface of the chest Chest centre of gravity</td>
</tr>
<tr>
<td>4</td>
<td>Neck</td>
<td>Lower surface of the neck Neck centre of gravity</td>
</tr>
<tr>
<td>5</td>
<td>Head</td>
<td>Lower surface of the head Head centre of gravity</td>
</tr>
<tr>
<td>6</td>
<td>Left Upper Arm</td>
<td>Left shoulder Left upper arm centre of gravity</td>
</tr>
<tr>
<td>7</td>
<td>Left Forearm</td>
<td>Left elbow Left forearm centre of gravity</td>
</tr>
<tr>
<td>8</td>
<td>Left Hand</td>
<td>Left wrist Left hand centre of gravity</td>
</tr>
<tr>
<td>9</td>
<td>Left Upper Leg</td>
<td>Left hip Left upper leg centre of gravity</td>
</tr>
<tr>
<td>10</td>
<td>Left Lower Leg</td>
<td>Left knee Left lower leg centre of gravity</td>
</tr>
<tr>
<td>11</td>
<td>Left Foot</td>
<td>Left ankle Left foot centre of gravity</td>
</tr>
<tr>
<td>12</td>
<td>Right Upper Arm</td>
<td>Right shoulder Right upper arm centre of gravity</td>
</tr>
<tr>
<td>13</td>
<td>Right Forearm</td>
<td>Right elbow Right forearm centre of gravity</td>
</tr>
<tr>
<td>14</td>
<td>Right Hand</td>
<td>Right wrist Right hand centre of gravity</td>
</tr>
<tr>
<td>15</td>
<td>Right Upper Leg</td>
<td>Right hip Right upper leg centre of gravity</td>
</tr>
<tr>
<td>16</td>
<td>Right Lower Leg</td>
<td>Right knee Right lower leg centre of gravity</td>
</tr>
<tr>
<td>17</td>
<td>Right Foot</td>
<td>Right ankle Right foot centre of gravity</td>
</tr>
</tbody>
</table>

**A.4.5 Input: R**

The input $R$ is a 1 X 16 object, which describes the posture that the body is holding. Each entry gives the rotation matrix (a 3 x 3 matrix) to the segment from the more proximal segment. Table A-12 describes each rotation matrix by the segments it is describing a rotation between. Each rotation matrix will be written with respect to the more proximal segment. If there is no rotation between two segments then that entry to R will be the identity matrix.

<table>
<thead>
<tr>
<th>CG column entry</th>
<th>Name</th>
<th>Description of rotation matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pelvis</td>
<td>Will be the identity matrix since the Centre of Gravity of the pelvis is at origin of the local frame of the Pelvis</td>
</tr>
<tr>
<td>2</td>
<td>Abdomen</td>
<td>Lower surface of the abdomen Abdomen centre of gravity</td>
</tr>
<tr>
<td>3</td>
<td>Chest</td>
<td>Lower surface of the chest Chest centre of gravity</td>
</tr>
<tr>
<td>4</td>
<td>Neck</td>
<td>Lower surface of the neck Neck centre of gravity</td>
</tr>
<tr>
<td>5</td>
<td>Head</td>
<td>Lower surface of the head Head centre of gravity</td>
</tr>
<tr>
<td>6</td>
<td>Left Upper Arm</td>
<td>Left shoulder Left upper arm centre of gravity</td>
</tr>
<tr>
<td>7</td>
<td>Left Forearm</td>
<td>Left elbow Left forearm centre of gravity</td>
</tr>
<tr>
<td>8</td>
<td>Left Hand</td>
<td>Left wrist Left hand centre of gravity</td>
</tr>
<tr>
<td>9</td>
<td>Left Upper Leg</td>
<td>Left hip Left upper leg centre of gravity</td>
</tr>
<tr>
<td>10</td>
<td>Left Lower Leg</td>
<td>Left knee Left lower leg centre of gravity</td>
</tr>
<tr>
<td>11</td>
<td>Left Foot</td>
<td>Left ankle Left foot centre of gravity</td>
</tr>
<tr>
<td>12</td>
<td>Right Upper Arm</td>
<td>Right shoulder Right upper arm centre of gravity</td>
</tr>
<tr>
<td>13</td>
<td>Right Forearm</td>
<td>Right elbow Right forearm centre of gravity</td>
</tr>
<tr>
<td>14</td>
<td>Right Hand</td>
<td>Right wrist Right hand centre of gravity</td>
</tr>
<tr>
<td>15</td>
<td>Right Upper Leg</td>
<td>Right hip Right upper leg centre of gravity</td>
</tr>
<tr>
<td>16</td>
<td>Right Lower Leg</td>
<td>Right knee Right lower leg centre of gravity</td>
</tr>
<tr>
<td>17</td>
<td>Right Foot</td>
<td>Right ankle Right foot centre of gravity</td>
</tr>
</tbody>
</table>
A.4.6 Intermediate step: $\text{Ref}R_{\text{Local}}$

The rotation matrix that may be used between the local frame of a segment and frame $\text{Ref}$ ($\text{Ref}R_{\text{Local}}$) is the product of the rotation matrices between each connecting segment between the segment of interest and the pelvis. Table A-13 gives the equations for determining $\text{Ref}R_{\text{Local}}$ for each segment from the appropriate columns in the input matrix $R$.

Table A-13: The Rotation matrices between the local frame and the reference frame

<table>
<thead>
<tr>
<th>$\text{Ref}R_{\text{Local}}$ number</th>
<th>Name</th>
<th>$\text{Global}R_{\text{Local}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRL1 = Reference Segment</td>
<td>Pelvis</td>
<td>n/a</td>
</tr>
<tr>
<td>GRL2</td>
<td>Abdomen</td>
<td>R1</td>
</tr>
<tr>
<td>GRL3</td>
<td>Chest</td>
<td>R2*R1</td>
</tr>
<tr>
<td>GRL4</td>
<td>Neck</td>
<td>R1<em>R2</em>R3</td>
</tr>
<tr>
<td>GRL5</td>
<td>Head</td>
<td>R1<em>R2</em>R3*R4</td>
</tr>
<tr>
<td>GRL6</td>
<td>Left Upper Arm</td>
<td>R1<em>R2</em>R5</td>
</tr>
<tr>
<td>GRL7</td>
<td>Left Forearm</td>
<td>R1<em>R2</em>R5*R6</td>
</tr>
<tr>
<td>GRL8</td>
<td>Left Hand</td>
<td>R1<em>R2</em>R5<em>R6</em>R7</td>
</tr>
<tr>
<td>GRL9</td>
<td>Left Upper Leg</td>
<td>R8</td>
</tr>
<tr>
<td>GRL10</td>
<td>Left Lower Leg</td>
<td>R8*R9</td>
</tr>
<tr>
<td>GRL11</td>
<td>Left Foot</td>
<td>R8<em>R9</em>R10</td>
</tr>
<tr>
<td>GRL12</td>
<td>Right Upper Arm</td>
<td>R1<em>R2</em>R11</td>
</tr>
<tr>
<td>GRL13</td>
<td>Right Forearm</td>
<td>R1<em>R2</em>R11*R12</td>
</tr>
<tr>
<td>GRL14</td>
<td>Right Hand</td>
<td>R1<em>R2</em>R11<em>R12</em>R13</td>
</tr>
<tr>
<td>GRL15</td>
<td>Right Upper Leg</td>
<td>R14</td>
</tr>
<tr>
<td>GRL16</td>
<td>Right Lower Leg</td>
<td>R14*R15</td>
</tr>
</tbody>
</table>
Body model and inertial property calculations

<table>
<thead>
<tr>
<th>Segment number</th>
<th>Joint</th>
<th>Location of the origin of the Segment’s Local frame w.r.t Pelvis’s local frame (1Oi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pelvis centre of gravity</td>
<td>(0,0,0) = CG1 = O1</td>
</tr>
<tr>
<td>2</td>
<td>Lower surface of abdomen</td>
<td>O2</td>
</tr>
<tr>
<td>3</td>
<td>Lower surface of chest</td>
<td>O2+R1*O3</td>
</tr>
<tr>
<td>4</td>
<td>Lower surface of neck</td>
<td>O2+R1*(O3+R2*O4)</td>
</tr>
<tr>
<td>5</td>
<td>Lower surface of head</td>
<td>O2+R1*(O3+R2*(O4+R3*O5))</td>
</tr>
<tr>
<td>6</td>
<td>Left shoulder</td>
<td>O2+R1*(O3+R2*O6)</td>
</tr>
<tr>
<td>7</td>
<td>Left elbow</td>
<td>O2+R1*(O3+R2*(O6+R5*O7))</td>
</tr>
<tr>
<td>8</td>
<td>Left wrist</td>
<td>O2+R1*(O3+R2*(O6+R5*(O7+R6*O8)))</td>
</tr>
<tr>
<td>9</td>
<td>Left hip</td>
<td>O9</td>
</tr>
<tr>
<td>10</td>
<td>Left knee</td>
<td>O9+R8*O10</td>
</tr>
<tr>
<td>11</td>
<td>Left ankle</td>
<td>O9+R8*((O10+R9*O11))</td>
</tr>
<tr>
<td>12</td>
<td>Right shoulder</td>
<td>O2+R1*(O3+R2*O12)</td>
</tr>
<tr>
<td>13</td>
<td>Right elbow</td>
<td>O2+R1*(O3+R2*(O12+R11*O13))</td>
</tr>
<tr>
<td>14</td>
<td>Right wrist</td>
<td>O2+R1*(O3+R2*(O12+R11*(O13+R12*O14)))</td>
</tr>
<tr>
<td>15</td>
<td>Right hip</td>
<td>O15</td>
</tr>
<tr>
<td>16</td>
<td>Right knee</td>
<td>O15+R14*O16</td>
</tr>
<tr>
<td>17</td>
<td>Right ankle</td>
<td>O15+R14*(O16+R15*O17)</td>
</tr>
</tbody>
</table>

A.4.7 Output: Mass

The output Mass is the total mass of the body and is simply the sum of the segment masses. It is in kg.

A.4.8 Output: Joints

The output Joints is a matrix whose columns are the coordinates of each joint, which is the origin of each segment’s local frame, given with respect to the frame Ref. The column number matches the segment number. The vector components are in cm.

The entries for the matrix Joints is found as follows. First, the location of each joint (origin of each local frame) is given with respect to local frame of the pelvis, may be found by summing the origin vectors and applying the appropriate rotation matrices. Table A-14 gives a description of the Joint and the equation used to determine each entry of the Joints matrix using the inputs O and R.

<table>
<thead>
<tr>
<th>Segment number</th>
<th>Joint</th>
<th>Location of the origin of the Segment’s Local frame w.r.t Pelvis’s local frame (1Oi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GRL17 Right Foot</td>
<td>R14<em>R15</em>R16</td>
</tr>
</tbody>
</table>

Second, once the joint location is known with respect to the local frame of the pelvis (frame 1) then the centre of gravity of each segment with respect to frame 1 may be determined as
$^{1}CG_{i} = {^{Ref}R_{Local}}^{Local}CG + ^{1}O_{i}$

Third, now that the location of each segment’s centre of gravity is known with respect to the local frame of the Pelvis then the location of the centre of gravity of the whole body respect to the local frame of the pelvis may be calculated by (realising that frame Ref and frame 1 are aligned, only their origins are in different locations)

$^{Ref}CG_{Overall} = \frac{\sum_{body} masses(i) * ^{1}CG_{i}}{\sum_{body} masses(i)}$

Finally the location of each joint with respect to frame Ref is

$^{1}O_{i} = ^{Ref}CG_{Overall}$.

**A.4.9 Output: Local_CGs**

The output Local_CGs is a 1X17 matrix, with each columns corresponding to each segment of the 17-segment body model. The columns give the coordinates of each segment’s centre of gravity with respect to frame Ref.

Each entry of the Local_CGs matrix will be $^{1}CG_{i} - ^{Ref}CG_{Overall}$ where these two vectors are defined in Appendix A.4.8.

**A.4.10 Output: Overall_I_aligned**

The output Overall_I_aligned gives the inertia tensor for the whole body with respect to frame Ref defined in Section 4.1.2. It is the summation of all the local inertia tensors rotated to align with the frame Ref and then translated to the overall Centre of Gravity via the parallel axis theorem.

(Local Inertia Tensor aligned with global frame)

$= ^{Ref}R_{Local} (\text{Local Inertia Tensor})^{Ref}R_{Local}^{T}$

The rotation matrix $^{Ref}R_{Local}$ required has already been found above as an Appendix A.4.6.

To move the inertia tensor of each segment to the Overall Centre of Gravity then

$m_{Segment} = \begin{bmatrix} Y^2 + Z^2 & -XY & -XZ \\ -XY & X^2 + Z^2 & -ZY \\ -XZ & -ZY & X^2 + Y^2 \end{bmatrix}$
needs to be added to the rotated local tensor of inertia. (X, Y, Z) are the coordinates the local centre of gravity location w.r.t the frame Ref. These coordinates are the columns of the output Local_CGs (Appendix A.4.9).

**A.4.11 Output: Principal_directions and Principal_I**

The output `Principal_directions` is a matrix whose columns are the principal directions with respect to frame Ref (Section 4.1.2). It is equal to $^{\text{Ref}}R_{p\nu}$. The column order was chosen so that the first, second and third columns are the principal directions with the greatest component in the positive x, y, or z directions of frame Ref. The output `Principal_I` is the diagonal matrix which gives the principal inertia tensor. The diagonal elements are the principal moments of inertia.

`Overall_I_aligned` has already been found in Appendix A.4.10. To find the principal directions (that could form a set of axes) and the associated principal moments of inertia the eigenvectors and values of the `Overall_I_aligned` need to be found.

The built-in Matlab function EIG(X) finds the eigenvectors and eigenvalues of a square matrix X. The commands are `[V, D] = EIG(X)` to generate a diagonal matrix D of eigenvalues and a full matrix V whose columns are the corresponding eigenvectors. This means that $X*V = V*D$. The eigenvectors given by EIG(X) are unit vectors, that are mutually perpendicular.

When two eigenvalues are equal, EIG(X) gives the distinct eigenvector, and then two possible eigenvectors in the plane, that are perpendicular each other. The choice of the two in the plane will be such that any direction in the old axis system that could be a principal direction is maintained.

When all three eigenvalues are equal, then the eigenvectors may be in any direction. This means that the current axis is already aligned with principal directions and the inertia tensor is already a diagonal matrix. EIG(X) in this case will return X as the diagonal matrix of eigenvalues (D) and the Identity matrix for the eigenvector matrix (V), since the current axis system is already aligned with the principal directions.

It is convenient to define a new set of axes aligning with the principal directions representing the least rotation from the global axes of the body (aligned with the reference segment). As a result ICG17 will order, and make positive or negative, the eigenvectors returned by EIG(X), so as to give three eigenvectors that specify the direction the x, y, and z axis of a coordinate system aligned with the principal directions and could be obtained by the minimum rotation from frame Ref. As a result the output matrix `Principal_directions` will be created.
The angle between a principal direction and frame Ref is the inverse cosine of the component of that direction along that axis. To form a new right-hand coordinate system using the principal directions, the principal direction (or its negative) given by EIG(X) will be named as the x, y, or z axis (the column order in the Principal_directions matrix will reflect this) according to the closest reference frame axes. For example, the principal direction 0.607 i + 0.1685 j -0.7762 k is at an angle of \( \cos^{-1}(0.67) \approx 53^\circ \) from the X axis, \( \cos^{-1}(0.1685) \approx 80^\circ \) from the Y axis, and \( \cos^{-1}(-0.7762) \approx 140^\circ \) from the Z axis. Since this example principal direction is closest to the negative Z axis then -0.607 i - 0.1685 j + 0.7762 k will be the z axis principal direction. It is possible that more than one different eigenvectors may be ‘closest’ to the same axis. In this case the eigenvector that has the largest component in that axis direction will be named according to that axis.

**A.4.12 Plots**

As an aid to visualising the posture a stick figure created by connecting the joints: pelvis to head, each leg to pelvis, one arm to shoulder to the other shoulder and down the other arm. The plot is drawn in 3-D with the centre of gravity of the whole body at the origin.

**A.5 Reasonable inertial properties**

The reasonable inertial properties collated using the methodology presented in Chapter 3 are provided in a spreadsheet submitted along with this thesis. The spreadsheet is named: Reasonable Inertial Properties.xls

**A.6 Numerical evaluation of the angular displacement in 3-D for the 17 segment model (Angle_disp17.m)**

To solve Equation (4-30), in any of the following situations: when \( H \neq 0 \), actions are performed in three dimensions, or when the action involves moving more than three of segments, the Matlab function “Angle_disp17.m” was written. It was written such that, the action being modelled, is input as angular velocities of each segment with respect to the more proximal segment.

The inputs required for Angle_disp17.m are (Masses, Local_I, O, CG, R_Start, Proximal_alphadot, H, delta_t, Pre_Proximal_alphadot, Post_Proximal_alphadot, fig):

- The inertial properties, Masses, Local_I, O, and CG are matrices and objects containing the inertial properties (Section 4.1.1) for all of the 17 segments. The format should be the same as for ICG17 (Appendix A.4).
- \textit{R\textsubscript{start}} gives the rotation matrices between each segment from one segment to the next, following the form of the body, in terms of the current segment’s local frame. The format should be the same as for ICG17.

- \textit{Proximal\_alphadot} is a (1 X 16) object where each entry gives the relative rotational velocity of each segment with respect to the more proximal as a 3-by-n matrix: The rows are the x, y, and z components with respect to the local frame of the more proximal segment, and each column is a new time step. Angular velocities are given in degrees per second.

- \textit{H} is the constant angular momentum vector. It a 3 X 1 vector giving the constant angular momentum with x, y, and z components for each row with the axes aligned with the reference segment (pelvis) in the starting position.

- \textit{Delta\_t} is the value of the time step between each angular velocity entry

- \textit{Pre\_Proximal\_alphadot} and \textit{Post\_Proximal\_alphadot} give the angular velocity just prior to and just after the observation time. These values are used in the calculation of the rate of change of the angular displacement at the endpoints

- \textit{Fig}, is the figure number for the stick figure plots of the postures at each time step and the final posture remaining at the end of the evaluation.

The outputs are \{\textit{Angular\_velocity\_deg, Salto, Tilt, Twist, GloRRef\_final}\}

- \textit{Angular\_velocity\_deg} gives the angular velocity vector, in degrees/sec, of the reference segment at each time step.

- \textit{Salto} gives the somersault orientation, in degrees, at each time step.

- \textit{Tilt} gives the value of (90 - \textit{ϕ})\degree at each time step.

- \textit{Twist} gives the twist angle, in degrees, at each time step.

- \textit{InRRef\_final} gives the rotation matrix between the initial and final position of the reference segment.

Figure A-2 presents a flow chart of the mathematical process that Angle\_disp17.m follows. Salto, Tilt and Twist are determined at each time step from \textit{GloRRef}, using the same procedure as was given in Section 4.3.1. Salto, Tilt and Twist should be reviewed to find whether or not there is a jump from close to 360\degree to 0\degree. For this time step and all later steps, 360\degree should be added to achieve the final result. Angle\_disp17.m is structured so that the column number of the inertial property, angular velocity, and \textit{R\_start} match the segments numbers in Section 3.1.1. Actions should thus be modelled, and results interpreted, as if they were performed in a forward rotating somersault. To model a backward rotating somersault the segment numbers for the left and right side should be swapped, so that the values of Salto and Tilt output match the definitions of \textit{θ} and (\pi/2 - \textit{ϕ}) given in Section 4.1.5.
Inertial properties: Masses, Local_I, O, and CG

Proximal_alphadot (\( \alpha_i \)) as well as the pre and post values.

Calculate \( \alpha_i \)
Sum the appropriate proximal angular velocities inputs, after they have been multiplied by the rotation matrices relating the proximal segment back to the reference frame.

Determine rotation matrices (R) defining posture for each time step.
Use input values at \( t=0 \) then, each successive R is the matrix multiplication of previous R and the equivalent axis-angle rotation matrix due to the angular velocity (Section B.1.1). i.e.

\[
R = R^{\text{Tracked} \Delta \theta} \cdot R^{\text{Tracked} \Delta \phi} \cdot R^{\text{Tracked} \Delta \psi}
\]

At each time step \( \text{ref}R_i, D_i \) and \( \text{ref}I_{\text{Overall}} \) are calculated using code equivalent to that in ICG17.

\( \dot{D}_i \) is approximated as the difference between \( D_i \) at the time step after and the time step before the current divided by \( 2 \Delta t \). At the end points the pre and post values of \( D_i \) are used so that \( \dot{D}_i \) is still the difference between the time step after and the time step before.

At \( t=0 \), \( \text{Glo}R_{\text{Ref}} \) is set to the identity matrix and the function to be integrated in Equation (4-30) is evaluated to obtain \( \dot{\gamma} \).

The next \( \text{Glo}R_{\text{Ref}} \) is the matrix multiplication of the equivalent axis-angle rotation matrix and the previous value of \( \text{Glo}R_{\text{Ref}} \) which uses \( \Delta \theta \): \( R^{\Delta \theta}_{\text{ref}} = R^{\Delta \theta}_{\text{ref}} \cdot R^{\text{Tracked} \Delta \phi} \cdot R^{\text{Tracked} \Delta \psi} \).

Each new value of \( \dot{\gamma} \) is calculated by evaluating the function to be integrated in Equation (4-30) using the newly found \( \text{Glo}R_{\text{Ref}} \) matrix.

From each \( \text{Glo}R_{\text{Ref}} \) matrix \( 0, \phi, \psi \) is extracted.

Figure A-2: Flow chart showing the mathematical process used in Angle Disp17.m
Appendix B.

Mathematical formulae and notation

This appendix provides reference for mathematical formulae and notation used in Chapter 4. It also describes the plots and statistical tests that were used in Chapter 5 and Chapter 6 to compare athletes and postures. A basic understanding of vectors, matrices, trigonometry, differentiation and integration has been assumed. Since notation may vary across the literature, this appendix, sets the notation that will be used in this thesis and the reader should hence use this appendix as their reference for interpreting notation.

B.1 Frames of reference

The frames of reference used in this thesis were defined in Section 4.1. The notation used to specify the frame of reference that a vector is measured in will be to prepend the name of the frame as superscript before the vector name. For example, vector $\mathbf{V}$ with respect to the frame “A” would be written $A\mathbf{V}$.

To change the frame of reference, it is necessary to consider both the rotation and the translation of the new frame with respect to the old. If the coordinates of point $P$ are known with respect to frame $A$, then the its coordinates with respect to frame $B$ will be

$$B\mathbf{P} = B^A \mathbf{P} + B^A \mathbf{O}_A,$$

where $B^A$ is the 3-by-3 rotation matrix from $A$ to $B$ and $B^A \mathbf{O}_A$ is the 3-by-1 position vector for the origin of frame $A$ with respect to frame $B$. 
B.1.1 Rotation matrices

A rotation matrix describes the final rotational position of one frame relative to another. It does not say how the frame actually reached that position. The basic form of a rotation matrix is

\[
^{b}R_{a} = \begin{bmatrix}
{i_{b}} \cdot i_{a} & j_{b} \cdot i_{a} & k_{b} \cdot i_{a} \\
{i_{b}} \cdot j_{a} & j_{b} \cdot j_{a} & k_{b} \cdot j_{a} \\
{i_{b}} \cdot k_{a} & j_{b} \cdot k_{a} & k_{b} \cdot k_{a}
\end{bmatrix}
= \begin{bmatrix}
^{b}i_{a} & ^{b}j_{a} & ^{b}k_{a}
\end{bmatrix}
\]

It is useful to note that the transpose is also the inverse, \(R^T = R^{-1}\), and that the determinant of \(R\) is one. If the rotation is about a single Cartesian axis then \(^{a^\prime}R_{b}\) is

\[
x \text{ axis } \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
y \text{ axis } \rightarrow \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

\[
z \text{ axis } \rightarrow \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where, \(\theta\) is the angle of rotation from the original position, with respect to frame \(A\), to the new position, with respect to frame \(B\). \(^{a^\prime}R_{b}\) thus allows reference back to the original frame.

There are other useful ways of constructing a rotation matrix; namely, the equivalent angle-axis method, using Euler rotations, and using fixed axis rotations. The form of the rotation matrix to choose will depend on the application; both the ease of the computations and also the ease of interpretation.

The equivalent angle-axis method constructs the matrix that rotates a vector from frame \(A\) to frame \(B\) through an angle of rotation \(\theta\) about an axis defined by the vector \(k\).

\[
^{a^\prime}R_{b} = \begin{bmatrix}
\dot{k}_x(1-\cos \theta) + \cos \theta & k_x k_y (1-\cos \theta) - k_z \sin \theta & k_x k_z (1-\cos \theta) + k_y \sin \theta \\
k_x k_y (1-\cos \theta) + k_z \sin \theta & \dot{k}_y(1-\cos \theta) + \cos \theta & k_y k_z (1-\cos \theta) - k_x \sin \theta \\
k_x k_z (1-\cos \theta) - k_y \sin \theta & k_y k_z (1-\cos \theta) + k_x \sin \theta & \dot{k}_z(1-\cos \theta) + \cos \theta
\end{bmatrix}
\]
If the rotation matrix is known the equivalent angle and axis may be extracted using the equations:

$$\theta = \cos^{-1}\left(\frac{R_{11} + R_{22} + R_{33} - 1}{2}\right)$$

$$k = \frac{1}{2\sin\theta} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

when \(\sin\theta = 0\) there is a singularity (no unique solution). This is because when \(\sin\theta = 2n\pi\), where \(n = 0, 1, 2\) etc… the rotation could be about any vector and when \(\theta = \pi\) a rotation about the \(k\) or \(-k\) axis is the same.

**Euler angle representations** describe consecutive rotations about an axis of the frame being rotated. If the rotating frame was attached to a person the Euler rotations could be described in terms of rotating about the anatomical axes. The rotation matrix that would allow any rotation to be described would be:

$$^B R_A = ^B R_{B_1}(\alpha)^{B_2}(\beta)^{B_3}(\gamma),$$

where the final position of the frame \(B\) is after rotating from \(A\) to \(B_1\) about an axis of \(A\) by angle \(\alpha\), then \(B_1\) to \(B_2\) about an axis of \(B_1\) by angle \(\beta\), then \(B_2\) to \(B_3\) about an axis of \(B_2\) by angle by the angle of \(\gamma\).

For example the XYZ Euler rotation from frame \(A\) to \(B\) (a rotation of \(\alpha\) about the x-axis, a rotation of \(\beta\) about the y-axis, and then a rotation of \(\gamma\) about the z-axis, where all axes are the axes of the rotating body) is

$$^B R_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Fixed axis rotations** are consecutive rotations about an axis of a fixed frame. This fixed frame should be the reference inertial frame chosen, or the earth-fixed frame. The rotation could be described in terms of rotating about vertical or horizontal axes. The rotation matrix that would allow any rotation to be described would be:

$$^B R_A = R(\gamma)R(\beta)R(\alpha),$$

where the rotation is about one fixed axis by \(\alpha\), then another fixed axis by \(\beta\), then another fixed axis by \(\gamma\).

For example, a rotation about XYZ fixed axes from frame \(A\) to frame \(B\) would be (a rotation of \(\alpha\) about the x-axis, a rotation of \(\beta\) about the y-axis, and then a rotation of \(\gamma\) about the z-axis, where all axes are the axes of the non-rotating frame) is
\[
{^A}_B R = 
\begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix}
\]

**Note:** Fixed axis rotations are applied in the reverse order to Euler rotations.

A rotation matrix has 9 elements, yet because the columns are all unit vectors, and the vectors representing the columns are all perpendicular (represent an orthogonal frame), then there are 6 constraints. This means that there are only 3 independent parameters in a rotation matrix. These 3 parameters may be used to construct the desired rotation matrix. It is not necessary that the position was obtained by those 3 steps. Three rotations about mutually perpendicular axis are commonly used as these three parameters. These angles could be either for the Euler angle representation or Fixed axis rotations, where the same axis of rotation is not used consecutively. There are 12 such rotation orders that may be used: XYZ, XYX, XZX, XZY, YXZ, YXY, YZX, YZY, ZYX, ZYZ, ZXZ, ZXY (Paul, 1981; Khatib & Kolarov, 2006).

This thesis used YXZ Euler or ZXY Fixed axis rotations (Section 4.1.5) since these could be most easily equated with the angles \( \theta, \phi, \psi \) used to describe a twisting somersault.

### B.1.2 Rotation matrices and angular velocity

The position of a point, \( P \), as shown in

Figure B-1 is

\[
{^A}r_{P/O_A} = {^A}r_{B/O_A} + {^A}_B R {^B}r_{P/O_B}.
\]

![Figure B-1: Vectors relating frame A and B and the point P in a rotating frame B](image)

The velocity of point \( P \) may be determined by differentiation:

\[
{^A}\dot{r}_{P/O_A} = {^A}\dot{r}_{O_A} + {^A}_B R {^B}\dot{r}_{P/O_B} + {^A}_B R \left( {^B}\Omega \times {^B}r_{P/O_B} \right)
\]

\[
= {^A}\dot{r}_{O_A} + {^A}_B R {^B}\dot{r}_{P/O_B} + \left( {^A}\Omega \times {^A}_B R {^B}r_{P/O_B} \right).
\]
This equation may be described as the (velocity due to translation of frame B) + (velocity due to movement of P in frame B) + (velocity due to the rotation of frame B)

If frame B was attached to a rigid segment and point P was a point on that segment, then, the location of point P in frame B is constant and so $^B \mathbf{r}_{P/O_A} = 0$. Thus,

$$^A \mathbf{r}_{P/O_A} = ^A \mathbf{r}_{O_B} + \left( ^A \mathbf{\Omega}_B \times ^A R_B \mathbf{r}_{P/O_A} \right).$$

If frame B only rotates with respect to frame A, that is they share an origin, then the vector $^A \mathbf{r}_{P/O_A}$ equals $^A R_B \mathbf{r}_{P/O_A}$. This means that $^A \mathbf{r}_{P/O_A}$ may be written as $^A R_B \mathbf{r}_P + ^A \mathbf{\dot{R}}_B \mathbf{r}_P$. Thus the velocity due to rotation is

$$^A \mathbf{\dot{r}}_B = ^A R_B \left( ^B \mathbf{\Omega}_B \times ^B \mathbf{r}_P \right),$$

$$^A \mathbf{\dot{r}}_B = ^A R_B \left[ ^A R_B \right]^{-1} \mathbf{\dot{r}}_B = ^A \mathbf{\dot{R}}_B \left[ ^A R_B \right]^{-1} \mathbf{\dot{r}}_P,$$

$$^A \mathbf{\dot{r}}_B \left[ ^A R_B \right]^{-1} \mathbf{\dot{r}} = ^A \mathbf{\Omega}_B \times ^A \mathbf{r}_P.$$

Thus,

$$^A \mathbf{\dot{R}}_B \left[ ^A R_B \right]^{-1} \mathbf{\dot{r}} = ^A \mathbf{\Omega}_B \times ^A \mathbf{r}_P.$$

This means that the cross product action, $^A \mathbf{\Omega}_B \times$ (sometimes written $\hat{\mathbf{\Omega}}$), may be viewed as pre-multiplication by $^A \mathbf{\dot{R}}_B \left[ ^A R_B \right]^{-1}.$

**B.2 Tensor of inertia and principal directions**

The tensor of inertia is a matrix that describes the rotational inertia of a body measured in a specific frame of reference. The principal directions are directions parallel to the axes of the frame where the tensor of inertia is a diagonal matrix.

**B.2.1 Principal directions**

By definition of a principal direction, an angular velocity in a principle direction, say $\mathbf{\omega}$, means that the angular momentum that results will be in the same direction.

$$\mathbf{H} = \begin{bmatrix} I_{aa} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & 0 & I_{cc} \end{bmatrix} \mathbf{\omega}_a = I_{aa} \mathbf{\omega}_a.$$
If $\omega_a$ is in a principle direction and equals $\omega_{ax}i + \omega_{ay}j + \omega_{az}k$.

Thus if $a$ is a principle direction, then $Ia=I_{aa}a$, showing that the principle directions are the eigenvectors for the Inertia tensor, and the eigenvalues, $I_{aa}$, are the corresponding moment of inertia. The eigenvalues and eigenvectors may be found as follows.

By definition an eigenvector $a$ for matrix $A$ obeys $Aa=[\text{Identity}]a$, which may also be written as $(A-\lambda[\text{Identity}])a=0$. There is only one solution to this equation other than the trivial solution if the determinant of $(A-\lambda[\text{Identity}])$ is zero. The eigenvalues are found by solving $\text{det}(A-\lambda[\text{Identity}])=0$, known as the characteristic equation. Once the eigenvalues, $\lambda$, are known then $(A-\lambda[\text{Identity}])a=0$ may be solved to obtain the eigenvectors, $a$. The eigenvectors will be parameterised to show the direction, but the magnitude could be any value. For example, if $[2;1]$ is an eigenvector $[4;2]$ will also be an eigenvector. For a $3 \times 3$ matrix $A$, with $3$ eigenvectors and values then.

$$
[A] \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = [a_1 & a_2 & a_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

Therefore

$$
\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^{-1} [A] \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

The matrix $[a_1 \\ a_2 \\ a_3]$ can be thought of as a rotation matrix, which describes the rotation between a frame in which the property described by matrix $A$ is referred to the original frame and on in which the same property now gives a diagonal matrix. If $A$ represented the tensor of inertia of a body then the diagonalised version gives the principal moments of inertia and $[\bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3]$ describes the rotation matrix from the original orientation to the principal directions. That is,

$$
Princ_i \text{al } I = \left[R_{Pr \text{} principal}\right]^{-1} \text{Origional } I \text{Origional } R_{Pr \text{} principal}.
$$

Knowing the principal directions allows the reference frame chosen to be aligned with these axes and a diagonal matrix inertia tensor used. This simplifies calculations by matching components of angular momentum and angular velocity:

$$
H = I_{xx} \omega_x + I_{yy} \omega_y + I_{zz} \omega_z.
$$
If there are three distant eigenvalues there will be three distinct eigenvectors. These eigenvectors will also be mutually perpendicular and so form a basis (Beer & Johnston, 1999). Thus the principal directions will be set.

If two eigenvalues are equal, the only one eigenvector is distinct (it will be perpendicular a plane in which the other two lie) and thus only one principal direction is fixed. This occurs in shapes such as circular cylinders and square prisms. If three eigenvalues are equal then principal direction has no meaning since the inertia tensor is the same for all orientations. This occurs in shapes such as the sphere and the cube. When the eigenvectors are not distinct, but a principal frame needs to be constructed then the choice of axes should be such that they are mutually perpendicular and the unit vector along the x axis, crossed with the unit vector along the y-axis, gives the unit vector along the z-axis.

It is also possible to determine the direction of the principle axes based on symmetry, since a plane of symmetry will be parallel to an eigenvector (Huston, 2009). This is true since if the x axis is an axis of symmetry then every small amount of mass at a point \( (x, y, z) \) will have a corresponding amount of mass at the point \( (-x, y, z) \). The integrals \( \int_{\text{body}} xydm \) and \( \int_{\text{body}} xzdm \) will thus be zero. This makes the x axis a principle axis. Even if only one principle direction can be found, the process of finding the other two principle involves finding the eigenvalues and vectors of a 2–by-2 matrix, which is easier than for a 3-by-3 matrix.

**B.2.2 The inertia tensor in different frames of reference**

The rotation of a reference axis does not change a body’s mass distribution, only how it is described in terms of the axes. If the inertia tensor about a specific set of axes \((B)\) is known, then it can be written in terms of another frame \((A)\) as follows:

1. \( \mathbf{\omega} \) is known in frame \( A \)
2. \( ^B \mathbf{R}_A \mathbf{\omega} \) will give the angular velocity in frame \( B \)
3. \( ^B \mathbf{I} ^B \mathbf{R}_A \mathbf{\omega} \) will then give the angular momentum in frame \( B \)
4. \( ^A \mathbf{R}_B ^B \mathbf{I} ^B \mathbf{R}_A \mathbf{\omega} = ^B \mathbf{R}_A ^T ^B \mathbf{I} ^B \mathbf{R}_A \mathbf{\omega} \) will give the angular momentum in frame \( A \)
5. Hence the inertia tensor w.r.t frame \( A \) is

\[
^A \mathbf{I} = ^B \mathbf{R}_A ^T [ ^B \mathbf{I} ] ^B \mathbf{R}_A = ^A \mathbf{R}_B [ ^B \mathbf{I} ] ^A \mathbf{R}_B ^T
\]
**B.2.3 Parallel axis theorem**

If the body is rotating about a specific point, the if ‘G’ is the body’s Centre of Gravity and ‘o’ is the point that the body is rotating

\[ H = m_{body} r_G \times v_G + I_G \omega_G = I_o \omega_o. \]

To obtain \( I_o \) from \( I_G \); that is moving the axis of rotation from the Centre of Gravity to a point \((a, b, c)\) known with respect to a frame at the Centre of Gravity means that

\[
m_{body} \begin{bmatrix} b^2 + c^2 & ab & ac \\ ab & a^2 + c^2 & bc \\ ac & bc & a^2 + b^2 \end{bmatrix}
\]

must be added to \( I_G \) to get \( I_o \).

If the Centre of Gravity is known w.r.t the axis at the point of rotation/new axis as \((X, Y, Z)\) then

\[
m_{body} \begin{bmatrix} Y^2 + Z^2 & - XY & - XZ \\ - XY & X^2 + Z^2 & - ZY \\ - XZ & - ZY & X^2 + Y^2 \end{bmatrix}
\]

must be added to \( I_o \) to get \( I_G \).

The parallel axis theorem adds to \( I_G \) when moving to \( I_o \). Thus, \( I_G \) is the minimum moment of inertia about any parallel axis.

**B.3 Integration**

The integrals presented here appear at various points in this thesis, but are not widely known integrals in the broader field of biomechanics. By presenting them here, the reader has one reference point and in the body of the thesis it is not necessary to explain each integral each time it is used.

**B.3.1 Cosecant integral used in Appendix E**

The integral of the cosecant can be evaluated as follows,
\[
\int \frac{d\theta}{\sin \theta} = \int -\frac{1}{\sin \theta} \left( \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \right) d\theta = \int -\left( \frac{-1}{\sin^2 \theta} + \frac{-\cos \theta}{\sin^2 \theta} \right) d\theta = -\ln \left( \frac{1 + \cos \theta}{\sin \theta} \right) + C
\]

\[
= -\ln \left( \frac{1 + \cos \theta}{\sin \theta} \right) + C = \ln \left( \frac{\sin \theta}{1 + \cos \theta} \right) + C
\]

### B.3.2 Integrals in Appendix E solvable by t-results

The substitution \( t = \tan(\alpha/2) \), in the standard t-results is useful for evaluating several integrals. Since, \( t = \tan(\alpha/2) \)

Thus \( d\alpha = \frac{2}{1+t^2} dt \), \( \cos \alpha = \frac{1-t^2}{1+t^2} \), \( \sin \alpha = \frac{2t}{1+t^2} \), and \( \tan \alpha = \frac{2t}{1-t^2} \).

Further, the use of t-results means that \( \alpha \) may only be integrated in the bounds \((-\pi, \pi)\). Any integrals that cover a larger interval will need to be split into components which keep \( \alpha \) between \( \pm\pi \).

In the following \( S, Q, R \) are constants that may be positive or negative, but cannot be zero.

**The first integral of interest**

\[
\int_{\alpha_{\text{start}}}^{\alpha_{\text{stop}}} \frac{1}{S+2Q \sin \alpha - 2R \cos \alpha} d\alpha
\]
\[
\tan(\alpha_{\text{trough,}1/2}) = \int_{\tan(\alpha_{\text{start,}1/2})}^{\infty} \frac{2}{1+t^2} \, dt \\
\quad = \int_{\tan(\alpha_{\text{start,}1/2})}^{\infty} \frac{2 (S+2Q - 2R(1-t^2))}{S(1+t^2) + 2Q(2t) - 2R(1-t^2)} \, dt \\
\quad = \int_{\tan(\alpha_{\text{start,}1/2})}^{\infty} \frac{2}{t^2(S+2R) + (2t)2Q + (S-2R)} \, dt \\
\quad = \frac{2}{(S+2R)} \int_{\tan(\alpha_{\text{start,}1/2})}^{\infty} \frac{1}{\left(t + \frac{2Q}{(S+2R)}\right)^2 + \left(\frac{S^2 - 4R^2 - 4Q^2}{(S+2R)}\right)^2} \, dt
\]

The integral is in the form of a \(\tan^{-1}\) primitive, and so may be evaluated using the table of standard integrals

\[
= \frac{2}{(S+2R)} \left[ \tan^{-1} \left( \frac{t + \frac{2Q}{(S+2R)}}{\sqrt{\frac{S^2 - 4R^2 - 4Q^2}{(S+2R)}}} \right) \right]^{\tan(\alpha_{\text{trough,}1/2})}_{\tan(\alpha_{\text{start,}1/2})}
\]

\[
= \frac{2}{\sqrt{S^2 - 4R^2 - 4Q^2}} \tan^{-1} \left( \frac{(S+2R)\tan \left( \frac{\alpha}{2} + 2Q \right)}{\sqrt{S^2 - 4R^2 - 4Q^2}} \right)^{\tan(\alpha_{\text{trough,}1/2})}_{\tan(\alpha_{\text{start,}1/2})}
\]

The second integral of interest

\[
\int_{\alpha_{\text{start}}}^{\alpha_{\text{trough}}} \frac{1}{S + 4Q \sin \alpha} \, d\alpha
\]
\[ \tan(\frac{\alpha_{\text{final}}}{2}) \int \frac{2}{1 + t^2} dt \]
\[ = \tan(\frac{\alpha_{\text{final}}}{2}) \int \frac{2}{\tan(\frac{\alpha_{\text{final}}}{2}) S + 4Q \frac{2t}{1 + t^2}} dt \]
\[ = \tan(\frac{\alpha_{\text{final}}}{2}) \int \frac{2}{\tan(\frac{\alpha_{\text{final}}}{2}) S(1 + t^2) + 4Q(2t)} dt \]
\[ = \tan(\frac{\alpha_{\text{final}}}{2}) \int \frac{2}{t^2(S) + (2t)4Q + (S)} dt \]
\[ = \frac{2 \tan(\alpha_{\text{final}}/2)}{S} \int \frac{1}{\left(t + \frac{4Q}{S}\right)^2 + \left(\frac{\sqrt{S^2 - 16Q^2}}{S}\right)^2} dt \]

The integral is in the form of a \( \tan^{-1} \) primitive, so may be evaluated using the table of standard integrals

\[ = \frac{2}{S} \tan^{-1} \left( \frac{\left(t + \frac{4Q}{S}\right)}{\sqrt{S^2 - 16Q^2}} \right) \tan(\frac{\alpha_{\text{final}}}{2}) \]
\[ = \left[ \frac{2}{\sqrt{S^2 - 16Q^2}} \tan^{-1} \left( \frac{S \tan\left(\frac{\alpha}{2}\right) + 4Q}{\sqrt{S^2 - 16Q^2}} \right) \right]_{\alpha_{\text{final}}}^{\alpha_{\text{initial}}} \]

\[ \text{The third integral of interest} \]

\[ \int_{\alpha_{\text{final}}}^{\alpha_{\text{initial}}} \frac{1}{S - 2R \cos \alpha} d\alpha \]
\[ \tan(\alpha_{\text{peak}}/2) \int \frac{2}{1 + t^2} \frac{1 - t^2}{S - 2R} dt \]

\[ = \tan(\alpha_{\text{peak}}/2) \int \frac{2}{\tan(\alpha_{\text{peak}}/2)} S(1 + t^2) - 2R(1 - t^2) dt \]

\[ = \tan(\alpha_{\text{peak}}/2) \int \frac{2}{\tan(\alpha_{\text{peak}}/2)} \frac{(S + 2R)t^2 + (S - 2R)^2}{dt} \]

\[ = \tan(\alpha_{\text{peak}}/2) \frac{2}{(S + 2R)} \int \frac{1}{t^2 + \left(\frac{S - 2R}{S + 2R}\right)^2} dt \]

The integral is in the form of a tan\(^{-1}\) primitive, so may be evaluated using the table of standard integrals

\[ \tan(\alpha_{\text{peak}}/2) \int \frac{1 - t^2}{S - 2R} \frac{1}{\sqrt{S + 2R}} \frac{1}{\sqrt{S + 2R}} \frac{1}{\tan(\alpha_{\text{peak}}/2)} \]

\[ = \frac{2}{(S + 2R)} \left[ \tan^{-1} \left( \frac{\sqrt{S + 2R} \tan \left( \frac{\alpha}{2} \right)}{\sqrt{S - 2R}} \right) \right]_{\alpha_{\text{peak}}}^{\alpha_{\text{peak}}} \]

\[ = \frac{2\sqrt{S + 2R}}{(S + 2R)\sqrt{S - 2R}} \left[ \tan^{-1} \left( \frac{\sqrt{S + 2R} \tan \left( \frac{\alpha}{2} \right)}{\sqrt{S - 2R}} \right) \right]_{\alpha_{\text{peak}}}^{\alpha_{\text{peak}}} \]

\[ = \frac{2}{\sqrt{S^2 - 4R^2}} \left[ \tan^{-1} \left( \frac{\sqrt{S + 2R} \tan \left( \frac{\alpha}{2} \right)}{\sqrt{S - 2R}} \right) \right]_{\alpha_{\text{peak}}}^{\alpha_{\text{peak}}} \]

**B.3.3 Jacobian elliptical Functions**

The Jacobian elliptical function sn(u,k) is defined by the integral

\[ u = \int_{0}^{\frac{1}{\sqrt{(1 - y^2)(1 - k^2 y^2)}}} dy = sn^{-1}(y, k), \]

where k is the modulus of the function and \( 0 < k < 1 \).
The functions \(cn(u,k)\) and \(dn(u,k)\) are then defined so that

\[
\begin{align*}
\text{sn}^2(x,k) &= 1 - \text{cn}^2(x,k) \\
\text{sn}^2(x,k) &= \frac{1 - \text{dn}^2(x,k)}{k^2},
\end{align*}
\]

with \(cn\) and \(dn\) taken as the positive square root.

As a result,

\[
\begin{align*}
\text{dn}^2(x,k) &= 1 - k^2 + k^2 \text{cn}^2(x,k), \\
\text{cn}^{-1}(y,k) &= \int_{cn(u,k)}^{y} \frac{1}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} dy \\
\text{dn}^{-1}(y,k) &= \int_{dn(u,k)}^{y} \frac{1}{\sqrt{(1 - y^2)(y^2 - (1 - k^2))}} dy,
\end{align*}
\]

provided \(k \neq 0\).

The derivatives of the functions are such

\[
\begin{align*}
\frac{d(\text{sn}(x,k))}{dx} &= cn(x,k)dn(x,k) \\
\frac{d(\text{cn}(x,k))}{dx} &= -\text{sn}(x,k)dn(x,k) \\
\frac{d(\text{dn}(x,k))}{dx} &= -k^2 \text{sn}(x,k)cn(x,k)
\end{align*}
\]

Figure B-2 provides a sketch of these functions.
When \( k = 0 \), then \( sn(x,0) = \sin(x) \); \( cn(x,0) = \cos(x) \); and \( dn(x,0) = 1 \)

When \( k = 1 \), then \( sn(x,1) = \tanh(x) \); \( cn(x,1) = sech(x) \); \( dn(x,1) = \frac{2}{e^x - e^{-x}} \)

\[ \text{sech}(x) = e^{x - x} = 2 \]

\[ \text{B.3.4 Elliptical integrals of the first and third kind} \]

Elliptical integrals may be described as of the first, second, or third kind. The first and third kinds are used in this thesis and so are detailed here. They are “incomplete” if the bounds of the integrals are zero and a variable upper bound, and they are “complete” if the bounds are 0 and \( \pi/2 \). The notation used for elliptical integrals does vary across the literature. The notation used here will allow for the easiest translation into the code required by Mathematica, but avoids confusion with other angles and symbols used in this thesis.

The incomplete elliptical integrals of the first, second, and third kind where the elliptical modulus \( k \) is such \( 0 \leq k \leq 1 \); and \( 0 \leq \varphi \leq \pi/2 \) are as follows,

First kind:

\[ F(\varphi, k^2) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} \]

Third kind:

\[ \Pi(n, \varphi, k^2) = \int_0^{\varphi} \frac{d\alpha}{\left(1 - n \sin^2 \alpha\right)^{1/2} \left(1 - k^2 \sin^2 \alpha\right)^{1/2}} \]

The value \( n \) is known as the elliptical characteristic. If \( n \) is set to 0 then the elliptical integral of the third kind becomes the Elliptic integral of the first kind.

Using the substitution \( y = \sin \alpha \), \( dy/d\alpha = \cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - y^2} \) then these can also be written as

\[ F(\sin^{-1} y, k^2) = \int_0^{\sin^{-1} y} \frac{dy}{\sqrt{1 - y^2} \sqrt{1 - k^2 y^2}} \]

\[ \Pi(n, \sin^{-1} y, k^2) = \int_0^{\sin^{-1} y} \frac{dy}{\left(1 - ny^2\right)^{1/2} \sqrt{1 - y^2} \sqrt{1 - k^2 y^2}} \]

Relating these to the Jacobian elliptical function \( y = \text{sn}(u) \),
The complete Elliptical integrals of the first and third kinds are the

\[
F(\sin^{-1} y, k^2) = \int_{0}^{y} \frac{dy}{\sqrt{1 - y^2} \sqrt{1 - k^2 y^2}} = \text{sn}^{-1}(y, k)
\]

\[
\Pi(n; \sin^{-1}(\text{sn}(u), k)) = \int_{0}^{\text{sn}^{-1}(y)} \frac{du}{1 - n \cdot \text{sn}^2(u, k)}
\]

When \( n = 0 \) then the complete Elliptical integral of the third kind equals the complete elliptical integral of the first kind.

When \( k = 0 \) then the complete Elliptical integral of the first kind equals \( \frac{\pi}{2} \) and complete Elliptical integral of the third kind equals \( \frac{\pi}{2 \sqrt{1 - n}} \) (Weisstein, 2014).

Using the standard notation means that the lower bound of the integral is set to zero, however, when evaluating an integral in the form of an Elliptical integral the lower bound may be shifted upwards as follows

\[
\int_{y_{\text{start}}}^{y_{\text{end}}} \frac{dy}{\sqrt{1 - y^2} \sqrt{1 - k^2 y^2}} = \text{sn}^{-1}(y_{\text{end}}, k) - \text{sn}^{-1}(y_{\text{start}}, k) = F(\sin^{-1}(y_{\text{end}}), k^2) - F(\sin^{-1}(y_{\text{start}}), k^2)
\]
The functions \( \text{EllipticF}[\psi, m] \), \( \text{EllipticK}[m] \), \( \text{EllipticPi}[n, \psi, m] \), and \( \text{EllipticPi}[n, m] \) in Mathematica may be used to evaluate \( F(\psi, k^2) \), \( F(\pi/2, k^2) \), \( \Pi(n, \varphi, k^2) \), and \( \Pi(n, \pi/2, k^2) \) respectively. The inputs \( n \) and \( \psi \) are as presented in the equations above and \( m = k^2 \).

The standard Elliptical integrals, relate directly to \( sn(x, k) \). Elliptical integrals may relate directly to \( cn(x, k) \) and \( dn(x, k) \); these can be written in terms of the standard Elliptical integrals using the relationships between \( sn(x, k) \), \( cn(x, k) \) and \( dn(x, k) \).

**Relating to Cn**

\[
\int_{\psi_{\text{start}}}^{\psi_{\text{end}}} \frac{dy}{\sqrt{1 - y^2} \sqrt{1 - k^2 + k^2 y^2}} = \frac{1}{k} \int_{\psi_{\text{start}}}^{\psi_{\text{end}'}} \frac{dy}{\sqrt{1 - k^2}} \sqrt{\frac{1 - k^2}{k^2} + y^2}
\]

\[
= -\text{cn}^{-1}(y, k)_{\text{End}}^{\text{Start}}
\]

\[
= -\text{sn}^{-1}\left(\sqrt{1 - y^2}, k\right)_{\text{End}}^{\text{Start}}
\]

\[
= -F\left(\sin^{-1}\left(\sqrt{1 - y_{\text{start}}^2}\right), k^2\right)_{\text{End}}^{\text{Start}}
\]

Using the substitution \( y = \text{cn}(u) \), this may be written as the Elliptical integral of the third kind.
\[
\int_{y\text{end}}^{y\text{out}} \frac{dt}{\sqrt{(1-qy^2)^2 \left(1-y^2 + \frac{(1-k^2)}{k^2}\right)}}
\]

\[
= \frac{cn^{-1}(y_{\text{out}})}{cn^{-1}(y_{\text{out}})} \frac{-sn(u,k)dn(u,k)du}{1-q*cn^2(u,k)\sqrt{(1-cn^2(u,k)) \left(cn^2(u,k) + \frac{(1-k^2)}{k^2}\right)}}
\]

\[
= \frac{cn^{-1}(y_{\text{out}})}{cn^{-1}(y_{\text{out}})} \frac{-sn(u,k)dn(u,k)du}{1-q*cn^2(u,k)\sqrt{sn^2(u,k) \frac{dn^2(u,k)}{k^2}}}
\]

\[
= \frac{cn^{-1}(y_{\text{out}})}{cn^{-1}(y_{\text{out}})} \frac{kdu}{(q*cn^2(u,k)-1)}
\]

\[
= \frac{cn^{-1}(y_{\text{out}})}{cn^{-1}(y_{\text{out}})} \frac{kdu}{(q*(1-sn^2(u,k))-1)}
\]

\[
= \frac{cn^{-1}(y_{\text{out}})}{cn^{-1}(y_{\text{out}})} \frac{kdu}{(q-1-q*sn^2(u,k))}
\]

\[
= \frac{k}{q-1} \frac{\sin^{-1}(\sqrt{1-y_{\text{out}}^2})}{\sin^{-1}(\sqrt{1-y_{\text{out}}^2})} \frac{du}{\left(1-q \cdot \frac{1}{q-1} sn^2(u,k)\right)}
\]

\[
= \frac{k}{q-1} \left( \Pi \left( \frac{q}{q-1}, \sin^{-1}\left(\sqrt{1-y_{\text{end}}^2}\right), k^2\right) - \Pi \left( \frac{q}{q-1}, \sin^{-1}\left(\sqrt{1-y_{\text{start}}^2}\right), k^2\right) \right)
\]

**Relating to Dn:**

\[
\int_{y\text{start}}^{y\text{end}} \frac{dy}{\sqrt{(1-y^2)(y^2-(1-k^2))}} = \left[-dn^{-1}(y,k)\right]_{\text{start}}^{\text{end}}
\]

\[
= \left[-\sin^{-1}\left(\frac{\sqrt{1-y^2}}{k}\right), k\right]_{\text{start}}^{\text{end}}
\]

\[
= \left[-F\left(\sin^{-1}\left(\frac{\sqrt{1-y^2}}{k}\right), k^2\right)\right]_{\text{start}}^{\text{end}}
\]
Using the substitution \( y = \text{dn}(u) \), this may be written as the Elliptical integral of the third kind

\[
\int_{\gamma_{\text{start}}}^{\gamma_{\text{end}}} \frac{dy}{(1 - py^2)\sqrt{(1 - y^2)(y^2 - (1-k^2))}}
\]

\[
= \int_{\gamma_{\text{start}}}^{\gamma_{\text{end}}} \frac{-k^2 \text{sn}(u,k)\text{cn}(u,k)du}{(1 - p^* \text{dn}^2(u,k))\sqrt{(1 - \text{dn}^2(u,k))(\text{dn}^2(u,k) - (1-k^2))}}
\]

\[
= \int_{\gamma_{\text{start}}}^{\gamma_{\text{end}}} \frac{du}{(p^* \text{dn}^2(u,k) - 1)}
\]

\[
= \int_{\gamma_{\text{start}}}^{\gamma_{\text{end}}} \frac{du}{(p^* (1 - k^2 \text{sn}^2(u,k)) - 1)}
\]

\[
= \int_{\gamma_{\text{start}}}^{\gamma_{\text{end}}} \frac{du}{(p - 1 - pk^2 \text{sn}^2(u,k))}
\]

\[
= \frac{1}{p-1} \int_{\gamma_{\text{start}}}^{\gamma_{\text{end}}} \frac{du}{\text{sn}^{-1}\left(\frac{1 - y_{\text{end}}^2}{k^2}\right)}
\]

\[
= \frac{1}{p-1} \left( \text{sn}^{-1}\left(\frac{1 - y_{\text{end}}^2}{k^2}\right), k^2 \right) - \Pi\left(\frac{pk^2}{p-1}, \sin^{-1}\left(\sqrt{\frac{1 - y_{\text{end}}^2}{k^2}}\right), k^2 \right)
\]

### B.3.5 Line integrals and work integrals

A line integral is an integral that is evaluated along a particular path. It will be in the form

\[\int_C F ds\]

where \( F \) is the function to be integrated, \( ds \) is the infinitesimal part of the curve and \( C \) is the curve along which the integral is being evaluated. If the function may be written in terms or a single parameter, then the line integral reduces to a definite integral of one
variable. If the curve moves through 3-D space, the length $ds$ may be found from the
distance equation and so may be written as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$ 

Thus, the line integral becomes

$$\int_{c} F ds = \int_{c} F(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$ 

A work integral is one that sums all the infinitesimal amounts of ‘work’ that are done to
move along a path. The work done is $F \cdot dr$ : force times distance in the direction of the
force, although the concept may be expanded to include vector fields where the resulting
function is in the same form. If the path taken is through 3-D space then

$$F \cdot dr = F_1 dx + F_2 dy + F_3 dz.$$ 

Thus a work integral may be written as

$$\int_{c} F \cdot dr = \int_{c} F_1 dx + F_2 dy + F_3 dz.$$ 

Work integrals may also be parameterised,

$$\int_{c} F_1(x, y, z) dx + F_2(x, y, z) dy + F_3(x, y, z) dz = \int_{c} \left( F_1(t) \frac{dx}{dt} + F_2(t) \frac{dy}{dt} + F_3(t) \frac{dz}{dt} \right) dt$$ 

and then the integral evaluated directly with respect to this single parameter.

**B.3.6 Green’s theorem**

Green’s theorem states that

$$\oint_{C} F_1(x) dx + F_2(y) dy = \iint_{R} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy,$$

where $C$ is a closed curve surrounding $R$ with direction such that $R$ is always on the left of
the curve (anti-clockwise). $R$ must be a simply connected (no holes) region. Figure B-1
illustrates. The integrand of the double integral will be referred to Green’s function.
When curve $C$ is not fixed, but may vary then the local maximum work integral achieved by a cyclic path in a specific field ($F_1i + F_2j$) will be the curve $C$ that follows the zero contour of Green’s function. This path gives the maximum because this path will enclose the maximum positive area on the left or the greatest negative area on the right (Dullin, 2011). This is known as the Green-Miele method (Mastroeni, 2001). Figure B-2 illustrates this concept.

Surrounding a clump of positive area on the left gives a local maximum; to move towards a global maximum it is necessary to enclose all the positive areas. Figure B-3 illustrates the concept of enclosing all the positive area.

**Figure B-3: The global maximum encloses all the positive area on the left**
A back and forth path, which has a net contribution of zero may be used to move through a negative region to the location of the positive areas. This may be used in order to surround two positive clumps as in (a) or to move from the necessary start/end position and back as in (b).
Further, to find the global maximum it is necessary to consider if the negative or the positive area is greater and enclose it on the left or right respectively. If there are no constraints on the joint angles, that is, they may take any value from 0 to $2\pi$, then surrounding the negative or positive regions will produce the same result, so either path may be chosen. This is because tracing a box defined by the bounds of the region $0$ to $2\pi$ for the two angles, performs an action returning to the same posture, and then on the opposite side of the box reverses it. When there are constraints on the joint angles then it is necessary to consider if the negative or positive area is larger in the allowed region.

Depending on the final application the positive or negative regions may be circled multiple times to continue to increase the angular displacement. Enclosing all the positive areas or all the negative areas simply maximises the work integral for one cycle.

**B.4 Plots to aid visualising the distribution of a sample**

Plots are useful to visualise the distribution of a measured or calculated item, and to indicate the presence of any trends. This section describes some of the plots used in this thesis.

**B.4.1 Box and whisker plots; the five-figure summary**

Box and whisker plots visually present the five-figure summary: minimum, lower quartile (LQ), median, upper quartile (UQ), and maximum of the sample (Phipps & Quine, 2001). Figure B-3 is an example. This is the form of these plots that will be used in this thesis and for brevity will simply be referred to as “box plots”.

![Box and whisker plot](image)

**Figure B-3: A box and whisker plot.**

**B.4.2 Survival-agreement plots**

A survival-agreement plot (Luiz, et al., 2003), plots the proportion of the sample with at least a specific parameter value. Luiz et al. used difference between two items as their
parameter of interest and hence the, “agreement” in the name. In reality any parameter value may be used, not just the difference.

Figure B-4 illustrates a survival-agreement plot for two differences. The plot is stepped, since each athlete represents a distinct finite proportion of athletes. The parameter value for any percentile or the percentile for any parameter value may be read from the plot. Multiple comparisons may be plotted on the same graphs, to compare them, however, the plotting more and more lines will make the plot harder to read. This means that even though a survival-agreement plot provides more detail than a box plot the latter allows easier comparison across a large number of different samples.

![Figure B-4: Schematic of survival-agreement plot](image)

The survival-agreement plot may be plotted in reverse so as to show the proportion of athletes with less than a specific value of a parameter. These plots will be referred to as “disagreement” plots. Figure B-5 illustrates. The nature of the decision to be made will determine if a survival-agreement or a disagreement plot is more useful.

![Figure B-5: Schematic of a disagreement plot](image)

In some cases the sign as well as the magnitude of a value is important. In such cases the survival-agreement plot may be modified to show the proportions of athletes with a parameter value on the positive and negative side and at that value or closer to zero. Figure B-6 illustrates. This plot is particularly useful when comparing two postures or techniques.
to select which to perform. The sign of the difference is essential so that the correct posture or technique is chosen.

![Figure B-6: Schematic of modified survival-agreement plot](image)

Survival-agreement plots, disagreement or modified survival-agreement plots are used in this thesis to visualise the magnitude of the difference between sub-populations and aid interpretation of the practical significance of any difference.

**B.5 Non-parametric statistical tests**

Non-parametric tests make no assumptions about the nature of the sample distribution; they can thus be used with any distribution. They are used in this thesis since the sample sizes are small, each item is not truly independent from each other, and there is no reason to expect a specific distribution type, such as the normal distribution. Non-parametric tests are based on ranking items and then testing if the ranks are randomly distributed or if the way that the ranks cluster indicates that one sub-population is statistically different from another.

The p-values quoted in this thesis are the probabilities that the observed, or a more extreme observation would be made under the assumed probability distribution; that is the probability that where you believe you have found something significant but it was actually only due to chance. Ideally the p-values chosen should reflect the consequences of incorrectly concluding you have found a significant result. In this thesis p-values are quoted and values of less than 5% flagged as significant and less than 10% flagged as a somewhat significant and indicative of a tendency.

**B.5.1 Kruskal-Wallis H-test**

The Kruskal-Wallis H-tests tests if ‘k’ independent samples come from the same population. Since it is the only H-test that is used in this thesis, for brevity it will simply be referred to as the H-test.
The null hypothesis is that the median of the difference between values from different samples is zero. The alternative is that the 'k' samples are not from the same population. It does not indicate which samples are different. The H-test can thus be used to generally compare samples and suggest if further investigation would be worthwhile. Each sample must have more than five values for the H-test to be used. The test statistic is

\[ H = \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(N+1), \]

where \( N \) is the total number of data points and \( R_i \) is the sum of the ranks for the \( i \)th sample. It may be approximated as a chi-squared distribution with \((k-1)\) degrees of freedom (Siegal, 1956; Freund & Simon, 1997).

When there are only two samples (k=2) then the H-test, is the two-tailed version of the Mann-Whitney U-test (Appendix B.5.2). Thus, the probability it gives is double the probability the U-test will give.

When seeking to determine if sub-populations are statistically different, the H-test may be applied first to determine if there is a statistically significant difference between any of the sub-populations being considered. If a difference is identified then the Mann-Whitney U-test may then be used to test each sub-population against the remainder of the sample to identify which sub-population is distinct from the rest of the sample.

**B.5.2 Mann-Whitney U-test**

The Mann-Whitney U-test tests if one of two independent samples is larger than the other by considering the location of the data points within each sample relative to the combined median. The test statistic is

\[ U = R - \frac{n(n+1)}{2}, \]

where \( n \) is the number of data points and \( R \) is the sum of the ranks for sub-population.

With sample sizes greater than 20, it may be approximated as a normal distribution:

\[ U \sim N \left( \frac{n_1n_2}{2}, \sqrt{\frac{n_1n_2(n_1+n_2+1)}{12}} \right), \]

where \( n_1 \) and \( n_2 \) are the number of data points in the two sub-populations (Siegal, 1956).

The p-value determined is for a one-tailed test with the null hypothesis that the samples are identical and the alternative that one sample larger than the other.
Since no other U-tests are used in this thesis then for brevity the Mann-Whitney U-test will be referred to as the U-test.

**B.5.3 Spearman’s statistic – trend**

A trend means that the measured or calculated values increases or decreases when ordered by some attribute. Spearman’s statistic is a non-parametric test for trend, and considers the number of pairs of ordered observations for which a higher-ordered observation is larger than a lower-ordered one. If there is an upward trend the number of such pairs will be large. The test statistic is

\[ D = \frac{1}{3} \left( N + 1 \right) \left( 2N + 1 \right) - 2 \sum_{i=1}^{N} i T_i \]

where \( N \) is the number of observations and \( T_i \) is the rank based on the parameter value assigned to the \( i \)th observation. The observation order is based on some attribute of the observation (Lehmann, 1975).

\( D \) may be approximated with the normal distribution \( \mathcal{N} \left( \frac{N^3 - N}{6}, \frac{N^2(N + 1)^2(N - 1)}{36} \right) \) when \( N \geq 11 \) (Lehmann, 1975).

Spearman’s statistic for trend is a good first test to determine if an index or other ranking system is a predictor for the ranks of the measured or calculated values. Spearman’s statistic will identify if the ranks of the measured value tend to move with the ranks according to an index or other ranking system.

**B.6 Non-parametric descriptions of the effect**

**B.6.1 Dominance statistic \( d \)**

The dominance statistic \( d \) (Cliff, 1993) is a non-parametric distribution-free measure of effect size. It looks at the amount of overlap of two samples and is defined as the proportion of individuals in one sample having a larger measured or calculated value than individuals in the other sample, minus the reverse situation:

\[ d = \frac{c(x_i > y_i) - c(x_i < y_i)}{mn}, \]

where \( m \) and \( n \) are the sizes of samples \( x \) and \( y \), and \( c \) stands for the count obeying the specific condition.
A value of \( d = -1 \) means that all of the \( x \) values are less than all of the \( y \) values; \( d = 0 \) means half the \( x \) values are greater than the \( y \) values and half are less; and \( d = 1 \) means all \( x \) values are greater than all \( y \) values (Cliff, 1993).

The dominance statistic may be determined from the Mann-Whitney U statistic (Appendix B.5.2) by the formulae:

\[
d = \frac{2U}{n_1 n_2} - 1 \quad \text{(Cliff, 1993).}
\]

### B.6.2 Common language effect size (CLES)

The common language effect size (CLES) gives the proportion of a subjects in one sample group that have a larger value of the parameter of interest than the other for any random pairing of individuals between sample groups. For example, CLES of 0.92 for if males are taller than females means that 92 out of 100 blind dates will mean that the male is taller than the female (Coe, 2002).

The common language effect size relates directly to the dominance statistic (Appendix B.6.1 above) as follows.

\[
CLES = \frac{c(x_i > y_i)}{c(x_i)c(y_i)} = \frac{d + 1}{2}.
\]

Where the sample groups are \( x \) and \( y \), and \( c \) stands for the count obeying the specific condition.

### B.6.3 Spearman’s rank correlation coefficient and coefficient of determination

Spearman’s rank correlation coefficient, \( r_s \), describes the relationship between two rankings. A value of 1 indicates a perfect correlation and a value of 1 a perfect negative correlation.

To determine \( r_s \), the difference \( d_{\text{ranks}} \) in ranks from the two sets of rankings for each of the \( n \), individuals are determined and then the following equation applied (Lehmann, 1975; Freund & Simon, 1997).

\[
r_s = 1 - \frac{6 \sum d_{\text{ranks}}^2}{n(n^2 - 1)}.
\]
Spearman’s rank correlation coefficient combined with a scatter plot with the two sets of rankings on each axis is very useful in identifying if there is a relationship between ranking and what type of relationship this may be.

If a linear relationship looks possible from plotted data then the linear regression line and associated coefficient of variance may be determined. Since the intention of trying to write an equation for the regression line is to determine if the index or ranking system is a good predictor of the parameter value then the regression of the measured or calculated value on the index should be determined.

For each regression line the coefficient of determination variability, $R^2$, should be calculated to indicate the goodness of fit of the regression line. The coefficient of determination gives the proportion reduction in variance in $Y$ as a result of using the regression line to predict $Y$ from $X$ (Feinstein & Kramer, 1980). In other words it indicates how much of the variance in $Y$ is explained by $X$ under a linear model. The residuals of $Y$ may then be plotted to visually confirm that the remaining variance is random rather than indicating a non-linear relationship between $X$ and $Y$. 
Appendix C.

Collecting and processing observational data

Observations of performances of pure somersaults described in Sections 3.10, 5.2.8 and 5.2.9 where captured on video and then analysed. The set-up of the cameras, digitization, and the extraction and calculation of joint angles, are described in this appendix.

Filming of athlete performances was approved by the University of Sydney Human Research Ethics Committee (HREC) protocol number 14974 on 29 June 2012.

C.1 Digitization

“Digitization” extracts image coordinates from an image. This process may be manual, semi-automatic or automatic. By digitizing successive images this allows key points to be tracked through a video sequence. When these image coordinates are combined with information about the camera position, joint angles and/or world coordinates may be determined.

C.1.1 Marking key points on athletes

Uniquely marking some key points of the body means that some automatic tracking programmes may be used to aid digitization. Since the segments of the body are assumed to be rigid it is only necessary to track the joints. Further since it is assumed that any movement at the wrists and ankles, is small or otherwise insignificant, and that any movement of the head is related to sighting rather than exploiting desirable mechanics, it is not necessary to track the ends of all segments. Pure somersaults must have are assumed to have left-right symmetry and so only one side of the body needs to be tracked; this will be the side closest to the camera. Table C-1 lists the key points that will be tracked, the short name that will be used to refer to that marker later (Appendix C.3), the anatomical
landmarks used to set the position, and how these correspond to the seventeen segment model (Appendix 3.1).

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<thead>
<tr>
<th>Key point</th>
<th>Designator</th>
<th>Anatomical Landmarks used</th>
<th>Relationship to the seventeen segment model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ankle-Left</td>
<td>A</td>
<td>Malleoli</td>
<td>Distal end of segment 10 or 16</td>
</tr>
<tr>
<td>Knee</td>
<td>K</td>
<td>Tibiale Laterale</td>
<td>Joint between segment 9 and 10 or 16 and 17</td>
</tr>
<tr>
<td>Hip</td>
<td>H</td>
<td>Trochanterion</td>
<td>Proximal end of segment 9 or 15</td>
</tr>
<tr>
<td>Pelvis</td>
<td>P</td>
<td>Iliocristale</td>
<td>Joint between segment 1 and 2</td>
</tr>
<tr>
<td>Shoulder</td>
<td>S</td>
<td>On longitudinal axis of the arm and down from acromion height by approximately the radius of the arm</td>
<td>Proximal end of segment 6 or 12</td>
</tr>
<tr>
<td>Elbow</td>
<td>E</td>
<td>Radiale</td>
<td>Joint between segment 6 and 7 or 12 and 13</td>
</tr>
<tr>
<td>Wrist</td>
<td>W</td>
<td>Stylion</td>
<td>Distal end of segment 7 or 13</td>
</tr>
</tbody>
</table>

**C.1.2 Digitising and tracking key points**

The programme Tracker (Brown, 2012) was used to extract pixel coordinates from the video footage of each skill. Tracker is “a free video analysis and modelling tool built on the Open Source Physics (OSP) Java framework” (Brown, 2012). Although there is documentation for Tracker, some instruction directly related to its utilization in this thesis is provided in this section.

To aid cutting of footage and rotating images, video was converted to JPG files before loading into Tracker. The images were given the same base-name and were consecutively numbered, using the same number of digits. The Tracker file was saved with the name of the first image, so that the link with the appropriate photographs was maintained.

Points were defined in Tracker as “Point masses” for each of the key markers given in Section 3.5.3. No mass was assigned since Tracker was used only to track the movement of the key marker. The reference frame was positioned so that the origin was in the top left hand corner of the image. The frame rate was 120fps and this was set using “Clip settings”.

Once the points were defined each point was marked in each image and Tracker automatically compiled a table giving the pixel coordinates of each key point against time. The automatic tracking, “auto-tracker” was used as far as possible. The auto-tracker feature, trackers a point based on the colour, using the RGB content of an 8 pixel radius.
circle around the point selected and so the greater the contrast between the marker and the background the more accurately the auto-tracking will be. Where auto-tracker failed to track correctly, the tracking was done manually. Sometimes markers were obscured. In these situations the marker location was chosen using surrounding anatomical features. For example, as the arm passed the torso, the shape of the torso, the marker location just before and just after being obscured, and the fact that marker must be behind the arm or it would not be obscured, were used to estimate the marker location.

After digitizing the video was replayed with the tacked points marked as a means of visually checking that the correct points were being tracked. The graphs that Tracker produces displaying the x or y coordinates over time were used to identify any frames where spikes in the plot occurred, which are checked for the accuracy of digitization. After reviewing the footage the coordinates obtained were deemed to be the most accurate coordinates that can be reasonably attained. Multiple digitisations were unnecessary due to the ability to review and correct errors.

**C.2 Camera set-up**

Since the somersaults observed in Sections 3.10, 5.2.8 and 5.2.9 were pure somersaults, the rotation occurs in one plane and the major postural change was in that plane and so only one camera set with its optical axis perpendicular to the horizontal direction of travel was used. The principal of scaled orthography is then used. A Casio EX-FH100 camera was used to film performances. It was set to film at 120fps which gave a resolution of 640X480. Based on previous experience with the same cameras (Mikl, 2014), and estimating intrinsic parameters using the Matlab camera calibration toolbox (Bouguet, 2010) it was deemed unnecessary to undo camera distortion, and instead the images were scaled using the length of a known object.

**C.3 Calculation of joint angles**

Since image distortion was considered negligible the protractor tool in tracker was used to measure joint angles $\hat{AKH}$, $\hat{KH}$, $\hat{HPS}$, $\hat{PSE}$, $\hat{SWE}$ directly from the video footage. The angles $\hat{HP}$, $\hat{PC}$, $\hat{CS}$, were determined based on the distance from P to S and the lengths of the abdomen and chest segments for each inertial property data set used. The reason for using this process and not putting a marker at the abdomen-chest joint is that different methods define the abdomen-chest joint slightly differently. Considering the postures in Section 3.9 the pelvis-abdomen and abdomen-chest will be taken to flex in the same direction. Figure C-1 illustrates the body-links for the torso with the abdomen and pelvis flexing in the same direction.
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Flexion occurs about an axis into the page in Figure C-1. The length PS was determined from the coordinates of the points P and S extracted from Tracker, and the lengths PC and CS that were taken as the segment lengths found during the estimation of inertial properties for each athlete. Using the cosine rule

\[
\begin{align*}
S\hat{CP} &= \cos^{-1}\left(\frac{PC^2 + CS^2 - PS^2}{2(PC)(CS)}\right), \\
S\hat{PC} &= \cos^{-1}\left(\frac{PC^2 + PS^2 - CS^2}{2(PC)(PS)}\right), \text{ and} \\
P\hat{SC} &= 180 - S\hat{CP} - S\hat{PC}.
\end{align*}
\]

If PS > PC + CS, which may occur due to errors in the reconstruction, then \(S\hat{CP}\) and \(S\hat{PC}\) will be taken to be \(\pi\) and 0 respectively; that is, there is no flexion. These calculated angles were adjusted to give the angles of rotation from the reference position so that the rotation matrices, \(R\), required to enter the posture in ICG17 could be obtained. The angle of flexion of the abdomen, which may be used to construct \(R_1\), is thus \(180 - H\hat{PS} - S\hat{PC}\). The angle of flexion of the chest, which may be used to construct \(R_2\), is \(S\hat{CP}\). The flexion of the torso also alters the shoulder angles. The angle \(S\hat{CP}\) must be added to the angle of the shoulder to give the angle of flexion for \(R_5\); since the protractor tool only measures the angle \(P\hat{SE}\).

Some inertial property estimation methods have only two torso segments, namely the methods based on Finch (1985) and Hanavan (1964). The rotation matrix describing the posture of the abdomen, \(R_1\), will be the identity matrix. With these methods the length PC is not known. Thus, rather than using the cosine rule the sine rule is used to determine
$S\hat{C}P$ as $\sin^{-1}\left(PS \sin\left(C\hat{P}S\right)/CS\right)$. The rotation matrix describing the posture of the chest, $R2$, may then be found as before.

The joint angles were converted into rotation matrices to define the posture in each frame. These rotation matrices were then input into Angle_disp17P.m to simulate the skill. Angle_disp17P.m was functions the same as Angle_disp17.m except that it took a series of postures defined by rotation matrices rather than requiring angular velocities of segments with respect to the proximal segment. The accuracy of Angle_disp17P.m is limited by the fact that postures are only known at set points in time, and how the athlete changes between the postures must be assumed. This is why Angle_disp17P.m is used only here and not when exploring idealized actions in Chapter 6.

From Equation (4-29), in the planar situation the angular displacement is the sum of two components; one which is due to the postural change and is independent of the angular momentum and the other which is a directly proportional to the angular momentum with the constant of proportionality determined by the postures assumed. Thus to determine the angular momentum that would ensure completion of a skill when using each of the inertial properties estimated the following procedure was used:

1. Run Angle Disp17P.m with $H = 0$ to determine that part of the angular displacement due to postural change.

2. Run Angle Disp17P.m again with $H$ set to an initial estimate of the angular momentum as the amount required to complete a single somersault in posture $L$ in the same flight time; this would be $2\pi I_{yy,L}/\text{Flight_time}$. Any estimate could be used although this will help with estimating at the correct order of magnitude.

3. Find the ratio between the difference in angular displacement between the actual somersault achieved and the somersault achieved in step 1, and the difference in angular displacement output from step 1 and step 2. This gave the scaling factor to use.

4. Run Angle Disp17P.m with $H$ equal to the value used in step 2 multiplied by the scaling factor determined in step 4. With this $H$, the simulation will complete the same somersault rotation as observed.

The magnitude of the difference between observed and simulated somersault angle at each frame will be the parameter used to assess the accuracy of the simulation.
C.4 Smoothing data

The measured and calculated joint angles will inevitably contain some noise. Noise can cause large errors in calculations that involve differentiation. This occurs because noise tends to occur at high frequencies and when differentiating the frequency becomes a multiplier, thus increasing the contribution of the noise in the differentiated data (Griffiths, 2006).

The aim of smoothing is to remove noise and so allow the underlying process to be analysed. Smoothing algorithms commonly assume that the noise is random with a zero mean. That is it ‘jitters’ about the actual signal in a fashion independent of the signal magnitude. It will not help with systematic errors, since without specific knowledge systematic errors will not be distinguishable from the signal. Ideally all the noise is removed and the signal remains undistorted.

In an attempt to reduce the noise in the data it was smoothed. A fourth order, zero Lag, recursive Butterworth filter, with the cut-off frequency set using Winter’s (2005) residual analysis was used when smoothing data. A recursive filter takes the original data as input, applies a weighted average of past and present input data, as well as adding past output data to determine an output intended to remove noise from the signal (Wood, 1982). That is, for the input \( y_i \) the output \( \tilde{y}_i \) of a recursive filter is given by

\[
\tilde{y}_i = \sum_{j=0}^{n} a_j (y_j - y_i) + \sum_{k=1}^{m} b_j (\tilde{y}_j - \tilde{y}_i)
\]

where, \( a_j \) is the weighting, and \( n \) and \( m \) are the orders of the filter. The order indicates the number of past input items and the output items (Wood, 1982). Since past data and present data are used it is essential that data points are equally spaced, without any frames missed. Further, the filtering will distort the data at the beginning and end of a data set, since there are no previous data points to use. To overcome this data was collecting on either side of the time interval of interest so that after filtering the end points were not actually used (Robertson, et al., 2004).

A recursive Butterworth filter uses the equation

\[
\tilde{y}_i = a_0 y_i + a_1 (y_i - y_1) + a_2 (y_i - y_2) + b_1 (\tilde{y}_1 - \tilde{y}_1) + b_2 (\tilde{y}_2 - \tilde{y}_2)
\]

The values of the constants, \( a_i \) and \( b_i \), may be obtained from tables such as those presented in Winter (2005, pp. 46-47) using the ratio of the desired cut-off frequency to the sampling frequency. The recursive Butterworth filter uses a second order equation. Applied only once will result in a time lag. To overcome this, the filter is applied twice, feeding the data through the second order filter in the forward and then reverse direction. The added benefit
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of applying the filter twice is that the attenuation increases and the cut-off of the filter being more abrupt, thus effectively doubling the order of the filter. (Wood, 1982; Robertson, et al., 2004; Winter, 2005) This results in a forth order, zero lag, Butterworth filter (Winter, 2005).

A low- or high-pass filter attenuates the high or low frequencies respectively, leaving the low or high to ‘pass’ through. The gain of a filter is the ratio of the amplitude of the output over the input at each frequency (Wood, 1982). The frequency where the gain is $2^{-0.5}$ is known as the cut-off frequency (Griffiths, 2006). For an effective filter the cut-off frequency cannot be higher than half the sampling frequency (Wood, 1982). The order of the filter determines the steepness of the cut-off (Griffiths, 2006).

The cut-off frequency should be chosen such that the noise frequencies will be attenuated, but not the signal. A filter with a sharper the cut-off will allow a more precise separation between frequencies allowed to pass and those attenuated. If the signal and noise have similar frequencies the filter will attenuate both, and thus a compromise needs to be made between eliminating noise and distorting the signal (Wood, 1982; Winter, 2005). Winter’s (2005) residual analysis was used to set the cut-off frequency. The residual is determined by:

$$\sqrt{\frac{1}{N} \sum_{i=1}^{N} (y_i - \tilde{y}_i)^2}$$

for each cut-off frequency. A plot of the residuals will be a straight line at high frequencies and transition to an asymptote to the y-axis. The cut-off frequency is taken as the lowest frequency that the residuals is still a straight line. The cut-off frequency was between 4 and 7 Hz for all joint angles.
Appendix D.

Conservation of energy during the quasi-rigid phase

The principle of the conservation of energy was not used in the derivation of the equations of motion and the relationship between $\phi$ and $\psi$. One can show that energy is conserved by substituting the derived equations into the equation for rotational kinetic energy as follows,

Rotational kinetic energy $= \frac{1}{2} \omega^{Glo} \cdot H^{Glo}, \text{ (Meriam & Kraige, 1998)}$

From Section 4.2.1, $H^{Glo} = H^{Glo_j}$, then from the start of 4.2.2,

$$\omega_{\text{total}} = \dot{H}^{Glo_j} j + \dot{\psi} i + \dot{\psi}^{pn} k.$$ 

Thus

$$\frac{1}{2} \omega^{Glo} \cdot H^{Glo} = \frac{1}{2} \left( \dot{H}^{Glo_j} j + \dot{\psi}^{Glo} R_{S_i} S_i i + \psi^{Glo} R_{P_i} P_i k \right) \cdot H^{Glo_j}.$$ 

Applying the definitions rotation matrices from Section 4.1.5 gives

$$\frac{1}{2} \omega^{Glo} \cdot H^{Glo} = \frac{1}{2} \left( \dot{H}^{Glo_j} j + \dot{\psi}^{Glo} R_{S_i} S_i i + \psi^{Glo} R_{P_i} P_i k \right) \cdot H^{Glo_j} = H^{Glo_j} \left( \dot{\theta} + \dot{\psi} \cos \phi \right).$$ 

Eliminating $\dot{\psi}$ by using the relationship between $\dot{\psi}$ and $\dot{\theta}$ in Equation (4-2)

$$\frac{1}{2} \omega^{Glo} \cdot H^{Glo} = \frac{H}{2} \left( 1 + \cos \phi \left( \frac{\cos \phi}{I_{zz}} \right) \right) \left( \frac{I_{xx}}{I_{xx} - \left( I_{xx} - I_{yy} \right) \sin^2 \psi} - I_{zz} \right)$$
Eliminating $\dot{\theta}$ using Equation (4-3)

\[
\frac{1}{2} \omega \mathbf{G} \mathbf{G} \mathbf{G} \mathbf{G} \mathbf{G} \mathbf{H} = \frac{H^2}{2} \left( \frac{I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi}{I_{xx} I_{yy}} \right) \left( 1 + \cos \phi \frac{(\cos \phi)}{I_{zz}} \left[ \frac{I_{xx} I_{yy}}{I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi} - I_{zz} \right] \right)
\]

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi \right) + \frac{\cos^2 \phi}{I_{zz}} \left[ I_{xx} I_{yy} - I_{zz} \left( I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi \right) \right]
\]

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi \right) \sin^2 \phi + \frac{I_{xx} I_{yy} \cos^2 \phi}{I_{zz}}
\]

Eliminating $\psi$ from the equation using Equation (4-10)

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( I_{xx} - (I_{xx} - I_{yy}) \frac{\sin^2 \phi}{(g - b \sin^2 \phi)} \sin^2 \phi + \frac{I_{xx} I_{yy} \cos^2 \phi}{I_{zz}} \right)
\]

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( I_{xx} \sin^2 \phi - \frac{g - b \sin^2 \phi}{I_{zz}} \right) + \frac{I_{xx} I_{yy} \cos^2 \phi}{I_{zz}}
\]

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( \frac{-g}{I_{zz}} + I_{xx} I_{xx} + b) \sin^2 \phi + I_{xx} I_{yy} \cos^2 \phi \right)
\]

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( \frac{-g}{I_{zz}} + I_{xx} I_{yy} \sin^2 \phi + I_{xx} I_{yy} \cos^2 \phi \right)
\]

\[
= \frac{H^2}{2I_{xx} I_{yy}} \left( \frac{-g}{I_{zz}} + I_{xx} I_{yy} \right)
\]

This is a constant and so the conservation of energy is demonstrated.
Appendix E.

Rigid body rotation with respect to time

In Section 4.2.2 it was elected to solve for $\phi$ and $\psi$ with $\theta$ as the dependant variable, so that skills could be defined. When seeking to determine the angular momentum required it is necessary to first solve for $\phi$ and $\psi$, and then $\theta$ with respect to time. This Appendix derives equations for $\phi$, $\psi$, and $\theta$ with respect to time. It also highlights which equations of $\phi$ with respect to time are equivalent to any similar equations previously published.

Substituting Equation (4-3) into Equation (4-1) gives

$$\dot{\phi} = -H \left( \frac{I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi}{I_{yy}} \right) \sin \phi \left( \frac{\sin \psi \cos \psi (I_{xx} - I_{yy})}{I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi} \right)$$

$$= -H \sin \phi \left( \frac{\sin \psi \cos \psi (I_{xx} - I_{yy})}{I_{xx} I_{yy}} \right)$$

Then using the relationship between $\phi$ and $\psi$ from Equation (4-4),

$$\sin \phi = \sin \phi \sqrt{\frac{I_{xx} (I_{yy} - I_{zz}) + I_{yy} (I_{xx} - I_{yy}) \sin^2 \psi}{I_{xx} (I_{yy} - I_{zz}) + I_{yy} (I_{xx} - I_{yy}) \sin^2 \psi}}$$

$$\sin^2 \psi = \frac{\sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{yy} (I_{xx} - I_{yy}) \sin^2 \psi \right)}{I_{zz} (I_{xx} - I_{yy}) \sin^2 \phi} - \frac{I_{xx} (I_{yy} - I_{zz})}{I_{zz} (I_{xx} - I_{yy})}$$

$$\sin \psi = \pm \sqrt{\frac{\sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{yy} (I_{xx} - I_{yy}) \sin^2 \psi \right)}{I_{zz} (I_{xx} - I_{yy}) \sin^2 \phi} - \frac{I_{xx} (I_{yy} - I_{zz})}{I_{zz} (I_{xx} - I_{yy})}}$$

and so
Rigid body rotation with respect to time

\[
\cos \psi = \pm \sqrt{1 - \left( \frac{\sin^2 \phi (I_{xx}(I_{yy}-I_{zz}) + I_{zz}(I_{xx}-I_{yy}) \sin^2 \psi_o)}{I_{zz}(I_{xx}-I_{yy}) \sin^2 \phi} - \frac{I_{xx}(I_{yy}-I_{zz})}{I_{zz}(I_{xx}-I_{yy})} \right)}
\]

\[
\cos \psi = \pm \sqrt{1 + \frac{I_{xx}(I_{yy}-I_{zz})}{I_{zz}(I_{xx}-I_{yy})} + \frac{-\sin^2 \phi (I_{xx}(I_{yy}-I_{zz}) + I_{zz}(I_{xx}-I_{yy}) \sin^2 \psi_o)}{I_{zz}(I_{xx}-I_{yy}) \sin^2 \phi}}
\]

Both \( \sin \phi \) and \( \cos \phi \) are positive when \( \phi \) is in the first quadrant. The nature of the sine function we can derive equations for the first quadrant and then mirror them in the other quadrants. Let us use the first quadrant substitution.

\[
\dot{\phi} = \left( -\frac{H(I_{xx}-I_{yy})}{I_{xx}I_{yy}} \right) \sin \phi \times 
\]

\[
\sqrt{\left( \sin^2 \phi (I_{xx}(I_{yy}-I_{zz}) + I_{zz}(I_{xx}-I_{yy}) \sin^2 \psi_o) - \frac{I_{xx}(I_{yy}-I_{zz})}{I_{zz}(I_{xx}-I_{yy})} \times \right)}
\]

\[
\dot{\phi} = \left( -\frac{H(I_{xx}-I_{yy})}{I_{xx}I_{yy}I_{zz}(I_{xx}-I_{yy})} \right) \sin \phi \times 
\]

\[
\sqrt{\left( \sin^2 \phi (I_{xx}(I_{yy}-I_{zz}) + I_{zz}(I_{xx}-I_{yy}) \sin^2 \psi_o) - \frac{I_{xx}(I_{yy}-I_{zz})}{I_{zz}(I_{xx}-I_{yy})} \times \right)}
\]

\[
\dot{\phi} = \left( -\frac{H(I_{xx}-I_{yy})}{I_{xx}I_{yy}I_{zz}(I_{xx}-I_{yy})} \right) \sin \phi \times 
\]

\[
\sqrt{\left( \sin^2 \phi (I_{xx}(I_{yy}-I_{zz}) + I_{zz}(I_{xx}-I_{yy}) \sin^2 \psi_o) - \frac{I_{xx}(I_{yy}-I_{zz})}{I_{zz}(I_{xx}-I_{yy})} \times \right)}
\]

Both \( \sin \phi \) and \( \cos \phi \) are positive when \( \phi \) is in the first quadrant. The nature of the sine function we can derive equations for the first quadrant and then mirror them in the other quadrants. Let us use the first quadrant substitution.
Rigid body rotation with respect to time

\[
\frac{dt}{d\phi} = \frac{-I_{xx}I_{yy}I_{zz}}{\sqrt{\left[I_{yy}I_{zz}I_{xx} + I_{xx}I_{zz}I_{yy} - I_{xx}\left(I_{yy}I_{zz} - I_{yy}I_{zz}\right)\sin^2 \phi\right] - I_{xx}\left(I_{yy}I_{zz} - I_{yy}I_{zz}\right)\sin^2 \phi}}
\]

Integrating with respect to \( \phi \) we obtain

\[
t = \left(\frac{I_{xx}I_{yy}I_{zz}}{H}\right) \int_{\phi_0}^{\phi} \sin \phi d\phi \frac{\cos \phi}{\sqrt{\left[I_{yy}I_{zz}I_{xx} + I_{xx}I_{zz}I_{yy} - I_{xx}\left(I_{yy}I_{zz} - I_{yy}I_{zz}\right)\sin^2 \phi\right] - I_{xx}\left(I_{yy}I_{zz} - I_{yy}I_{zz}\right)\sin^2 \phi}}
\]

\[
= \left(\frac{I_{xx}I_{yy}I_{zz}}{H}\right) \int_{\phi_0}^{\phi} \frac{du}{d\phi} \left\{ \frac{\sin^2 \phi \left[ I_{xx}(I_{yy}I_{zz} - I_{yy}I_{zz}) + I_{yy}(I_{xx}I_{yy} - I_{yy}I_{zz})\sin^2 \phi\right]}{\left[ I_{xx}(I_{yy}I_{zz} - I_{yy}I_{zz}) + I_{yy}(I_{xx}I_{yy} - I_{yy}I_{zz})\sin^2 \phi\right] - I_{xx}I_{yy}I_{zz}\sin^2 \phi} + I_{xx}(I_{yy}I_{zz} - I_{yy}I_{zz})\right\} d\phi
\]

\[
= \left(\frac{I_{xx}I_{yy}I_{zz}}{H}\right) \int_{\phi_0}^{\phi} \frac{du}{d\phi} \left\{ \frac{\sin^2 \phi \left[ I_{xx}(I_{yy}I_{zz} - I_{yy}I_{zz}) + I_{yy}(I_{xx}I_{yy} - I_{yy}I_{zz})\sin^2 \phi\right]}{\left[ I_{xx}(I_{yy}I_{zz} - I_{yy}I_{zz}) + I_{yy}(I_{xx}I_{yy} - I_{yy}I_{zz})\sin^2 \phi\right] - I_{xx}I_{yy}I_{zz}\sin^2 \phi} + I_{xx}(I_{yy}I_{zz} - I_{yy}I_{zz})\right\} d\phi
\]

Let \( u = \cos \phi \) then \( du/d\phi = -\sin \phi \) and \( \sin 2\phi = (1-u^2) \)

A cosine substitution, and the resulting elliptical integral, was used by Yeadon (1993a) for the equivalent angular velocity. The constants, however, presented differently from Yeadon as the result of focusing on angular momentum rather than using the conservation of energy.
Rigid body rotation with respect to time

\[
\left( \frac{H}{I_{xx}I_{yy}I_{zz}} \right) \left( \frac{du}{\cos \phi} \right) = \int_{\cos \phi_a}^{\cos \phi} \left( I_{xx} (I_{yy} - I_{zz}) \rho^2 - \left[ I_{xx} (I_{yy} - I_{zz}) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right) \right]
\right)
\]

\[
\left( I_{xx} (I_{yy} - I_{zz}) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right) \right)
\]

\[
\left[ \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \right) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right) \right]
\]

\[
\left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \right) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right)
\]

\[
\left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \right) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a
\]

Let use define the following constants to simplify the form of the equations

\[
a = I_{xx} (I_{yy} - I_{zz}) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right)
\]

\[
b = I_{xx} (I_{yy} - I_{zz}) \cos \phi - I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \sin^2 \phi_a
\]

\[
c = (I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy})) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right)
\]

\[
d = I_{yy} (I_{xx} - I_{zz}) - \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_a \right)
\]

\[
f = I_{aa} (I_{xx} - I_{yy}) + I_{zz} (I_{xx} - I_{yy}) = I_{yy} (I_{xx} - I_{zz})
\]

The previous equation can be written as,

\[
\left( \frac{H}{I_{xx} I_{yy} I_{zz}} \right) = \int_{\cos \phi_a}^{\cos \phi} \frac{du}{\sqrt{bu^2 - a}}
\]

(E-1)

**E.1 Case 3, 4, 6, 7, 8**

Equation (E-1) must be rearranged differently depending on the signs of the constants a, b, c, and f. This will lead to two different substitutions which allow them to be identified as Jacobian elliptical integrals. Let us first consider the cases when a, b, c, and f are non-zero.

\[
\left( \frac{H}{I_{xx} I_{yy} I_{zz}} \right) \left( \frac{du}{\cos \phi} \right) = \int_{\cos \phi_a}^{\cos \phi} \frac{du}{\sqrt{bu^2 - a}}
\]

\[
\left( \frac{H}{I_{xx} I_{yy} I_{zz}} \right) \left( \frac{du}{\cos \phi} \right) = \int_{\cos \phi_a}^{\cos \phi} \frac{du}{\sqrt{bu^2 - a}}
\]

Let

\[y = u, \quad du = dy, \quad \frac{C}{f} \]

Let

\[y = u, \quad du = dy, \quad \frac{a}{b} \]
\[
\begin{align*}
\left( \frac{H}{I_{xx}I_{yy}I_{zz}} \right)^f &= \frac{1}{c} \sqrt{\frac{\int_{\cos \phi} T \cos \phi \, dy}{y^2 - \frac{af}{bc}}(1 - y^2)} \\
&= \frac{1}{\sqrt{bc}} \int_{\cos \phi} \frac{dy}{y^2 - \frac{af}{bc}}(1 - y^2)
\end{align*}
\]

If a, b, c & f are all the same sign and \(af/bc < 1\); let 1-k^2 = af/bc

\[
= \frac{1}{\sqrt{bc - af}} \left( cn^{-1} \left( \frac{T}{c} \cos \phi, k \right) - cn^{-1} \left( \frac{T}{c} \cos \phi, k \right) \right)
\]

An alternative further rearrangement

\[
\begin{align*}
\left( \frac{H}{I_{xx}I_{yy}I_{zz}} \right)^f &= \frac{1}{\sqrt{bc}} \int_{\cos \phi} \frac{dy}{y^2 - \frac{bc}{af}}(1 - y^2) \\
&= \frac{1}{\sqrt{(-a)f}} \int_{\cos \phi} \frac{dy}{y^2 - \frac{bc}{af}}(1 - y^2)
\end{align*}
\]

If c & f are the same sign and then both a & b are of the opposite sign and bc/af < 1 let k^2 = bc/af

\[
= \frac{1}{\sqrt{(-a)f}} \left( sn^{-1} \left( \frac{T}{c} \cos \phi, k \right) - sn^{-1} \left( \frac{T}{c} \cos \phi, k \right) \right)
\]

To match the cases to the correct rearrangement of the integral and then the correct Jacobian elliptical function it is important to know the signs of the constants and if \(af/bc\) is greater than or less than 1. To determine if \(af/bc\) is greater or less than one, it is possible to compare \(af\) and \(bc\) and determined which is larger: if \(af > bc\) then \(af/bc > 1\).
Rigid body rotation with respect to time

<table>
<thead>
<tr>
<th>af</th>
<th>bc</th>
</tr>
</thead>
</table>
| \( \left( \begin{array}{c}
I_{xx} (I_{yy} - I_{zz}) (1 - \sin^2 \phi_o) \\
-I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o \\
(I_{zz} (I_{xx} - I_{yy}) + I_{xx} (I_{yy} - I_{zz}))
\end{array} \right) \times \) | \( \left( \begin{array}{c}
I_{xx} (I_{yy} - I_{zz}) (1 - \sin^2 \phi_o) \\
-I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o \\
(I_{zz} (I_{xx} - I_{yy}) + I_{xx} (I_{yy} - I_{zz}))
\end{array} \right) \times \) |

Adding
\[
I_{xx}^{2} (I_{yy} - I_{zz})^{2} (1 - \sin^2 \phi_o) - I_{xx} (I_{yy} - I_{zz}) (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o
\]
to both af and bc

Subtracting
\[
I_{xx} (I_{yy} - I_{zz}) (I_{xx} - I_{yy})
\]
from both af and bc

Each combination of the order of the moments of inertia matched in Table E-1. It is given after the special cases where \( a, b, c, \) or \( f \) is zero, which are evaluated first.

### E.2 Case 5

Case 5 is the transition between continuous and oscillating twist when \( I_{yy} \) is the intermediate moment of inertia. It occurs when \( a \) equals 0 and \( b, c \& d \) all have the same sign. Equation (E-1) then becomes
Rigid body rotation with respect to time

\[ \left( \frac{H}{I_x I_y I_z} \right) t = \int \frac{\cos \phi}{\sqrt{\cos^2 \phi \left( \frac{f}{c} - \frac{f^2}{c^2} \right)}} \, du \]

\[ = \frac{1}{\sqrt{bf}} \int \frac{du}{\cos \phi \left( \frac{c}{f} - u^2 \right)} \]

Let

\[ u = \frac{c}{f} \sin \alpha, \quad du = \frac{c}{f} \cos \alpha d\alpha \]

Then,

\[ \left( \frac{H}{I_x I_y I_z} \right) t = \frac{1}{\sqrt{bf}} \int \frac{c}{f} \cos \alpha d\alpha \]

\[ = \frac{1}{\sqrt{bc}} \int \frac{\cos \alpha d\alpha}{\sin \alpha \sqrt{\cos^2 \alpha}} \]

\[ = \frac{1}{\sqrt{bc}} \int \frac{d\alpha}{\sin \alpha} \]

This can be integrated using the cosecant integrals in Appendix B.3.1

\[ \left( \frac{H}{I_x I_y I_z} \right) t = \frac{1}{\sqrt{bc}} \ln \left( \frac{\sin \alpha}{1 + \cos \alpha} \right) + K \]

where K is the constant of integration

Undoing both substitution used to assist integration to realising

\[ \sin \alpha = \sqrt{\frac{f}{c}} u = \sqrt{\frac{c}{f}} \cos \phi \]

gives

\[ \left( \frac{H}{I_x I_y I_z} \right) t = \frac{1}{\sqrt{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{1 + \sqrt{1 - \frac{f^2}{c^2} \cos^2 \phi}} \right) + K \]

\[ \exp \left( \sqrt{bc} \left( \frac{H_I}{I_x I_y I_z} - K \right) \right) = \frac{\cos \phi \sqrt{f}}{\sqrt{c + \sqrt{c - f \cos^2 \phi}}} \]

\[ \exp \left( -\sqrt{bc} \left( \frac{H_I}{I_x I_y I_z} - K \right) \right) = \left( \frac{\sqrt{c + \sqrt{c - f \cos^2 \phi}}}{\cos \phi \sqrt{f}} \right) \]
Rigid body rotation with respect to time

\begin{align*}
\sqrt{f} \exp\left(-\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) \cos \phi &= \sqrt{c} + \sqrt{c - f \cos^2 \phi} \\
\sqrt{c - f \cos^2 \phi} &= \sqrt{f} \exp\left(-\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) \cos \phi - \sqrt{c} \\
c - f \cos^2 \phi &= f \exp\left(-2\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) \cos^2 \phi + c - 2\sqrt{f} \exp\left(-\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) \cos \phi \\
0 &= f \exp\left(-2\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) + f \cos^2 \phi - 2\sqrt{f} \exp\left(-\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) \cos \phi
\end{align*}

In Case 5 $\phi \neq \pi/2$ until the end of the motion, thus for the equations describing the motion $\cos \phi \neq 0$. Thus,

\begin{align*}
0 &= f \exp\left(-2\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) + 1 \cos \phi - 2\sqrt{f} \exp\left(-\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) \\
\cos \phi &= \frac{2\sqrt{f} \exp\left(-\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right)}{f \exp\left(-2\sqrt{bc\left(\frac{Ht}{I_{xx}I_{yy}} - K\right)}\right) + 1}
\end{align*}

The form of this equations means that $\phi$ will asymptotically approach $\pi/2$

Now when $t=0$, since $a=0$, from Equation (4-4)

\[ \phi = \phi_0 = \sin^{-1}\left(\frac{I_{xx}(I_{xx} - I_{yy})}{\left(I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy})\right)\sin^2 \psi}\right) \]

So,

\[ K = -\frac{1}{\sqrt{bc}} \ln\left(\frac{\cos \phi \sqrt{f}}{\sqrt{c - d \cos^2 \phi}}\right) \]

**E.3 Case 9**

Case 9 (Section 4.2.3) is the transition between continuous and oscillating twist when $I_{xx}$ is the intermediate moment of inertia. It occurs when $c = 0$ and $a$, $b$, & $d$ all have the same sign.

From Equation (E-1) with $c = 0$

\[ \left(\frac{H}{I_{xx}I_{yy}}\right)^t = \int \frac{du}{\sqrt{(bu^2 - a)(-f)u^2}} \]

\[ = \frac{1}{\sqrt{bf}} \int \frac{du}{u \sqrt{\left(\frac{a}{b} - u^2\right)}} \]

Let
Rigid body rotation with respect to time

\[ u = \frac{a}{b} \sin \alpha, \quad du = \frac{a}{b} \cos \alpha \]

Then

\[
\left( \frac{H}{I_a I_y I_z} \right)_{\phi} = \frac{1}{\sqrt{b}} \int \frac{\alpha}{\sqrt{b} \sin \alpha \sqrt{a - \frac{a}{b} \sin^2 \alpha}} \]

\[
= \frac{1}{\sqrt{a}} \int \frac{\cos \alpha}{\sin \alpha \sqrt{\cos^2 \alpha}} d\alpha
\]

This can also be integrated using the cosecant integrals in Appendix B.3.1

\[
\left( \frac{H}{I_a I_y I_z} \right)_{\phi} = \frac{1}{\sqrt{a}} \ln \left( \frac{\sin \alpha}{1 + \cos \alpha} \right) + K
\]

where \( K \) is the constant of integration

Undoing both substitution used to assist integration to realising

\[
\sin \alpha = \frac{b}{a} - u = \frac{b}{a} \cos \phi
\]

Gives,

\[
\left( \frac{H}{I_a I_y I_z} \right)_{\phi} = \frac{1}{\sqrt{a}} \ln \left( \frac{\cos \phi \sqrt{b}}{\sqrt{a} + \sqrt{a - b \cos^2 \phi}} \right) + K
\]

\[
\exp \left( \frac{Ht}{I_a I_y I_z} - K \right) = \frac{\cos \phi \sqrt{b}}{\sqrt{a} + \sqrt{a - b \cos^2 \phi}}
\]

\[
\exp \left( - \sqrt{a \cos^2 \phi} \right) = \exp \left( - \sqrt{a \cos^2 \phi} \right)
\]

\[
\sqrt{a - b \cos^2 \phi} = \sqrt{b} \exp \left( - \sqrt{a \cos^2 \phi} \right)
\]

\[
a - b \cos^2 \phi = b \exp \left( - 2 \sqrt{a \cos^2 \phi} \right) \cos^2 \phi + a - 2ab \exp \left( - \sqrt{a \cos^2 \phi} \right) \cos \phi
\]

\[
0 = b \exp \left( - 2 \sqrt{a \cos^2 \phi} \right) + b \cos^2 \phi - 2ab \exp \left( - \sqrt{a \cos^2 \phi} \right) \cos \phi
\]

As with Case 5, Case 9 also means \( \phi \neq \pi/2 \) until the end of the motion, thus for the equations describing the motion, \( \cos \phi \neq 0 \). Thus,
Rigid body rotation with respect to time

\[ 0 = b \left( \exp \left( -2\sqrt{ab} \left( \frac{Ht}{I_{xx}I_{yy}I_{zz}} - K \right) \right) + 1 \right) \cos \phi - 2\sqrt{ab} \exp \left( -\sqrt{af} \left( \frac{Ht}{I_{xx}I_{yy}I_{zz}} - K \right) \right) \]

\[ \cos \phi = \frac{2\sqrt{ab} \exp \left( -\sqrt{af} \left( \frac{Ht}{I_{xx}I_{yy}I_{zz}} - K \right) \right)}{b \exp \left( -2\sqrt{af} \left( \frac{Ht}{I_{xx}I_{yy}I_{zz}} - K \right) \right) + 1} \]

The form of this equations means that \( \phi \) will asymptotically approach \( \frac{\pi}{2} \)

Now when \( t = 0 \), since \( c = 0 \), from Equation (4-4)

\[ \phi = \phi_o = \sin^{-1} \left( \frac{I_{yy} \left( I_{xx} - I_{zz} \right)}{I_{xx} \left( I_{yy} - I_{zz} \right) + I_{zz} \left( I_{xx} - I_{yy} \right)} \sin^2 \psi_o \right) \]

So,

\[ K = -\frac{1}{\sqrt{af}} \ln \left( \cos \phi_o \sqrt{\frac{b}{a}} \right) \]

In the situation where \( \psi_o = 0 \), then

\[ \phi_o = \sin^{-1} \left( \frac{I_{yy} \left( I_{xx} - I_{zz} \right) + I_{zz} \left( I_{xx} - I_{yy} \right)}{I_{xx} \left( I_{yy} - I_{zz} \right) + I_{zz} \left( I_{xx} - I_{yy} \right)} \right) = \sin^{-1} \left( \frac{f}{b} \right) \]

\[ K = -\frac{1}{\sqrt{af}} \ln \left( \frac{b - f}{b} \sqrt{\frac{b}{a}} \right) = -\frac{1}{\sqrt{af}} \ln \left( \frac{b - f}{a} \right) \]

Since \( c = 0 \), (defined this case) then \( a = I_{zz} \left( I_{yy} - I_{xx} \right) \); \( b - f = I_{zz} \left( I_{yy} - I_{xx} \right) \); so the constant of integration \( K = 0 \).

**E.4 Case 11**

Case 11 (Section 4.2.3) occurs when \( I_{xx} = I_{zz} \) and so \( f = 0 \). This case will always display oscillating twist about \( \psi = 0 \). If \( I_{yy} \) is the maximum or minimum moment of inertia changes the signs of the other constants and so both situations will be considered.

From Equation (E-1) with \( f = 0 \)

\[ \left( \frac{H}{I_{xx}I_{yy}I_{zz}} \right)^{\phi} = \int_{\cos \phi_o}^{\cos \phi} \frac{du}{\sqrt{(b - f - a)u - a}} \]

If \( I_{yy} \) is the max, \( c < 0 \), \( b > 0 \), \( a > 0 \)

\[ \left( \frac{H}{I_{xx}I_{yy}I_{zz}} \right)^{\phi} = \frac{1}{\sqrt{-c}} \int_{\cos \phi_o}^{\cos \phi} \frac{du}{\sqrt{(a - bu^2)}} \]
This is the form of the standard integral for $\frac{1}{\sqrt{b}} \sin^{-1} \left( \frac{\sqrt{b} u}{\sqrt{a}} \right)$ thus

\[
\left( \frac{H}{I_{xx} I_{yy} I_{zz}} \right) \int = \frac{1}{\sqrt{-bc}} \left[ \sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right) - \sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right) \right]_{\cos \phi}
\]

\[
= \frac{1}{\sqrt{-bc}} \left[ \sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right) - \sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right) \right]
\]

\[
\sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right) = \int \frac{H \sqrt{-bc}}{I_{xx} I_{yy} I_{zz}} \int + \sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right)
\]

\[
\cos \phi = \frac{a}{b} \int \frac{H \sqrt{-bc}}{I_{xx} I_{yy} I_{zz}} \int + \sin^{-1} \left( \frac{\sqrt{b} \cos \phi}{\sqrt{a}} \right)
\]

If $I_{yy}$ is the min, $c > 0$, $b < 0$, $a < 0$

\[
\left( \frac{H}{I_{xx} I_{yy} I_{zz}} \right) \int = \frac{1}{\sqrt{-c}} \int \frac{du}{\sqrt{-(-b)a^2} \sqrt{(-a)}}
\]

This is the form of the standard integral for $\frac{1}{\sqrt{-b}} \sin^{-1} \left( \frac{\sqrt{-b} u}{\sqrt{-a}} \right)$ thus

\[
\left( \frac{H}{I_{xx} I_{yy} I_{zz}} \right) \int = \frac{1}{\sqrt{-bc}} \left[ \sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right) - \sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right) \right]_{\cos \phi}
\]

\[
= \frac{1}{\sqrt{-bc}} \left[ \sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right) - \sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right) \right]
\]

\[
\sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right) = \int \frac{H \sqrt{-bc}}{I_{xx} I_{yy} I_{zz}} \int + \sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right)
\]

\[
\cos \phi = \frac{a}{b} \int \frac{H \sqrt{-bc}}{I_{xx} I_{yy} I_{zz}} \int + \sin^{-1} \left( \frac{\sqrt{-b} \cos \phi}{\sqrt{-a}} \right)
\]

Both when $I_{yy}$ is the maximum- and minimum-valued moment of inertia then the same equation is obtained.

**E.5 Case 12**

Case 12 (Section 4.2.3) occurs when $I_{yy} = I_{zz}$ and so $b = 0$. This case will always display oscillating twist about $\psi = \pi/2$. If $I_{xx}$ is the maximum or minimum moment of inertia changes the signs of the other constants and so both situations will be considered. For any twist to occur it is also necessary that $a \neq 0$ ($\phi, \neq 0$).

From Equation (E-1) with $b = 0$
If $I_{xx} > I_{yy}$ then $a < 0$, $c > 0$, $f > 0$

$$\left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t = \int_{\cos \phi}^{\cos \phi} \frac{du}{\sqrt{(-a)(c - fu^2)}}$$

This is the form of the standard integral for $\frac{1}{\sqrt{u}} \sin^{-1} \left( \frac{\sqrt{a} u}{\sqrt{c}} \right)$ thus

$$\left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t = \frac{1}{\sqrt{-a}} \left[ \frac{1}{\sqrt{f}} \sin^{-1} \left( \frac{\sqrt{f} \cos \phi}{\sqrt{c}} \right) \right]_{\cos \phi}$$

$$= \frac{1}{\sqrt{-a}} \left[ \sin^{-1} \left( \frac{\sqrt{f} \cos \phi}{\sqrt{c}} \right) - \sin^{-1} \left( \frac{\sqrt{f} \cos \phi}{\sqrt{c}} \right) \right]$$

$$\sin^{-1} \left( \frac{\sqrt{f} \cos \phi}{\sqrt{c}} \right) = \left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t + \sin^{-1} \left( \frac{\sqrt{f} \cos \phi}{\sqrt{c}} \right)$$

$$\cos \phi = \frac{c}{f} \sin \left( \left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t + \sin^{-1} \left( \frac{\sqrt{f} \cos \phi}{\sqrt{c}} \right) \right)$$

If $I_{xx} < I_{yy}$ then $a > 0$, $c < 0$, $f < 0$

$$\left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t = \int_{\cos \phi}^{\cos \phi} \frac{du}{\sqrt{(-a)(-f)u^2}}$$

This is the form of the standard integral for $\frac{1}{\sqrt{-u}} \sin^{-1} \left( \frac{\sqrt{-a} u}{\sqrt{-c}} \right)$ thus

$$\left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t = \frac{1}{\sqrt{-a}} \left[ \frac{1}{\sqrt{-f}} \sin^{-1} \left( \frac{-\sqrt{-f} u}{\sqrt{-c}} \right) \right]_{\cos \phi}$$

$$= \frac{1}{\sqrt{-a}} \left[ \sin^{-1} \left( \frac{-\sqrt{-f} \cos \phi}{\sqrt{-c}} \right) - \sin^{-1} \left( \frac{-\sqrt{-f} \cos \phi}{\sqrt{-c}} \right) \right]$$

$$\sin^{-1} \left( \frac{-\sqrt{-f} \cos \phi}{\sqrt{-c}} \right) = \left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t + \sin^{-1} \left( \frac{-\sqrt{-f} \cos \phi}{\sqrt{-c}} \right)$$

$$\cos \phi = \sqrt{\frac{c}{f}} \sin \left( \left( \frac{H}{I_{x}, I_{y}, I_{z}} \right) t + \sin^{-1} \left( \frac{-\sqrt{-f} \cos \phi}{\sqrt{-c}} \right) \right)$$

The same equation describes the motion regardless of if $I_{xx}$ is the minimum- or maximum-valued moment of inertia.
E.6 Summary of equations

Each of the cases from Section 4.2.3 is matched the equation of $\phi$ with respect to time in Table E-1. The cases are matched to the equations by the values of $a$, $b$, $c$, or $f$.

Some of these equations in Table E-1 are known and have been published previously by Yeadon (1993a). Namely

- Case 7: Continuous twist when $I_{yy} \geq I_{xx} \geq I_{zz}$ was given in Equation 18 in Yeadon (1993a).
- Case 8: Oscillating twist about $\psi = 0$, when $I_{yy} \geq I_{xx} \geq I_{zz}$ and $\psi_o = 0$, was given in Equation 29 in Yeadon (1993a).
- Case 9: Twist to $\pi/2$ and stop when $I_{yy} \geq I_{xx} \geq I_{zz}$, when $\psi_o = 0$ the constant of integration in Case 9 is 0 and was given in equation 33 in Yeadon (1993a).
Table E-1: The equation for $\phi$ with respect to time for each of the cases.

<table>
<thead>
<tr>
<th>Case (Section 4.2.3)</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>f</th>
<th>$af/bc$</th>
<th>$k^2$</th>
<th>Equation relating $\phi$ to time</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 and $I_{xx} &gt; I_{yy} &gt; I_{zz}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1</td>
<td>$1 - \frac{af}{bc}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Dn \left( \int Dn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) - \left( \frac{H \sqrt{bc}}{I_{xx} I_{yy} I_{zz}} \right) t, k \right)$</td>
</tr>
<tr>
<td>3 and $I_{zz} &gt; I_{yy} &gt; I_{xx}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>&lt;1</td>
<td>$\frac{bc}{bc - af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Cn \left( \int Cn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) - \left( \frac{H \sqrt{bc - af}}{I_{xx} I_{yy} I_{zz}} \right) t, k \right)$</td>
</tr>
<tr>
<td>4 and $I_{xx} &gt; I_{yy} &gt; I_{zz}$</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = 2 \sqrt{\frac{f}{c}} \exp \left( - \sqrt{\frac{bc}{H_t I_{xx} I_{yy} I_{zz}} - K} \right)$ where $K = -1 \sqrt{\frac{af}{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{\sqrt{c - d \cos^2 \phi}} \right)$</td>
</tr>
<tr>
<td>4 and $I_{zz} &gt; I_{yy} &gt; I_{xx}$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = 2 \sqrt{\frac{f}{c}} \exp \left( - \sqrt{\frac{bc}{H_t I_{xx} I_{yy} I_{zz}} - K} \right)$ where $K = -1 \sqrt{\frac{af}{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{\sqrt{c - d \cos^2 \phi}} \right)$</td>
</tr>
<tr>
<td>5 and $I_{xx} &gt; I_{yy} &gt; I_{zz}$</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = 2 \sqrt{\frac{f}{c}} \exp \left( - \sqrt{\frac{bc}{H_t I_{xx} I_{yy} I_{zz}} - K} \right)$ where $K = -1 \sqrt{\frac{af}{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{\sqrt{c - d \cos^2 \phi}} \right)$</td>
</tr>
<tr>
<td>5 and $I_{zz} &gt; I_{yy} &gt; I_{xx}$</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = 2 \sqrt{\frac{f}{c}} \exp \left( - \sqrt{\frac{bc}{H_t I_{xx} I_{yy} I_{zz}} - K} \right)$ where $K = -1 \sqrt{\frac{af}{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{\sqrt{c - d \cos^2 \phi}} \right)$</td>
</tr>
<tr>
<td>6 and $I_{yy} &gt; I_{zz} &gt; I_{xx}$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>&gt;1</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Sn \left( \int \frac{f}{c} \cos \phi, k \right) t + Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) \right)$</td>
</tr>
<tr>
<td>6 and $I_{xx} &gt; I_{zz} &gt; I_{yy}$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>&gt;1</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Sn \left( \int \frac{f}{c} \cos \phi, k \right) t + Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) \right)$</td>
</tr>
<tr>
<td>7 and $I_{yy} &gt; I_{xx} &gt; I_{zz}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>&gt;1</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Sn \left( \int \frac{f}{c} \cos \phi, k \right) t + Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) \right)$</td>
</tr>
<tr>
<td>7 and $I_{zz} &gt; I_{xx} &gt; I_{yy}$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>&gt;1</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Sn \left( \int \frac{f}{c} \cos \phi, k \right) t + Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) \right)$</td>
</tr>
<tr>
<td>8 and $I_{yy} &gt; I_{xx} &gt; I_{zz}$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>&gt;1</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Sn \left( \int \frac{f}{c} \cos \phi, k \right) t + Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) \right)$</td>
</tr>
<tr>
<td>8 and $I_{zz} &gt; I_{xx} &gt; I_{yy}$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>&gt;1</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{\frac{f}{c}} \cos \phi = Sn \left( \int \frac{f}{c} \cos \phi, k \right) t + Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, k \right) \right)$</td>
</tr>
<tr>
<td>9 and $I_{yy} &gt; I_{xx} &gt; I_{zz}$</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = 2 \sqrt{\frac{f}{c}} \exp \left( - \sqrt{\frac{bc}{H_t I_{xx} I_{yy} I_{zz}} - K} \right)$ where $K = -1 \sqrt{\frac{af}{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{\sqrt{c - d \cos^2 \phi}} \right)$</td>
</tr>
<tr>
<td>9 and $I_{zz} &gt; I_{xx} &gt; I_{yy}$</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = 2 \sqrt{\frac{f}{c}} \exp \left( - \sqrt{\frac{bc}{H_t I_{xx} I_{yy} I_{zz}} - K} \right)$ where $K = -1 \sqrt{\frac{af}{bc}} \ln \left( \frac{\cos \phi \sqrt{\frac{f}{c}}}{\sqrt{c - d \cos^2 \phi}} \right)$</td>
</tr>
<tr>
<td>Case (Section 4.2.3)</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>f</td>
<td>$\frac{af}{bc}$</td>
<td>$k^2$</td>
<td>Equation relating $\phi$ to time</td>
</tr>
<tr>
<td>---------------------</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>-----------------</td>
<td>------</td>
<td>---------------------------------</td>
</tr>
<tr>
<td>10 ($I_{xx} = I_{yy}$) and $I_{zz}$ max</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>n/a</td>
<td>n/a</td>
<td>$\phi$ is constant</td>
</tr>
<tr>
<td>10 ($I_{xx} = I_{yy}$) and $I_{zz}$ min</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>n/a</td>
<td>n/a</td>
<td>$\phi$ is constant</td>
</tr>
<tr>
<td>11 ($I_{xx} = I_{zz}$) and $I_{yy}$ max</td>
<td>+</td>
<td>+</td>
<td>$\geq$ 0</td>
<td>0</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = \sqrt[\frac{a}{b}] \sin \left( \frac{H \sqrt{-bc}}{I_{xx} I_{yy} I_{zz}} t \right) + \sin^{-1} \left( \sqrt[\frac{b}{a}] \cos \phi \right)$</td>
</tr>
<tr>
<td>11 ($I_{xx} = I_{zz}$) and $I_{yy}$ min</td>
<td>-</td>
<td>-</td>
<td>$\leq$ 0</td>
<td>0</td>
<td>n/a</td>
<td>n/a</td>
<td>$\cos \phi = \sqrt[\frac{c}{f}] \sin \left( \frac{H \sqrt{-af}}{I_{xx} I_{yy} I_{zz}} t \right) + \sin^{-1} \left( \sqrt[\frac{f}{c}] \cos \phi \right)$</td>
</tr>
<tr>
<td>12 ($I_{yy} = I_{zz}$) and $I_{xx}$ max</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>n/a</td>
<td>n/a</td>
<td>$\psi$ is constant</td>
</tr>
<tr>
<td>12 ($I_{yy} = I_{zz}$) and $I_{xx}$ min</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>n/a</td>
<td>n/a</td>
<td>$\psi$ is constant</td>
</tr>
<tr>
<td>13 ($I_{xx} = I_{yy} = I_{zz}$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>n/a</td>
<td>n/a</td>
<td>$\psi$ is constant</td>
</tr>
</tbody>
</table>
E.7 Twist $\psi$ and somersault $\theta$ with respect to time

The value of $\psi$ as a function of time may be determined by substituting the appropriate function for $\phi$ into Equation (4-4). The most useful form of Equation (4-4) for the substitution is

$$\sin^2 \psi = \frac{b \cos^2 \phi - a}{(f - b)(1 - \cos^2 \phi)}.$$

Equation (4-10) may also be substituted into Equation (4-1) to determine the somersault angular velocity as:

$$\dot{\theta} = \frac{H}{I_{yy}} \left(\frac{l_{xx} - l_{yy}}{l_{xx}} \left(\frac{b - a}{(f - b)(1 - \cos^2 \phi)} - \frac{b}{(f - b)}\right)\right)$$

$$\dot{\theta} = \frac{H}{I_{yy}} \left(\frac{1}{l_{xx}l_{yy}} \left(b - a \right) \frac{1}{(1 - \cos^2 \phi)} - b\right)$$

$$\dot{\theta} = \frac{H}{I_{xx}I_{yy}} \left(l_{xx} + b + \frac{a - b}{\sin^2 \phi}\right)$$

$$\dot{\theta} = \frac{H}{I_{xx}I_{yy}} \left(l_{xx} + b + \frac{-g}{\sin^2 \phi}\right)$$

The form of the equation for the somersault angular velocity means that somersault will increase by a steady amount with an oscillation determined by the oscillations of $\phi$ superimposed on top. Integrating with respect to time gives

$$\theta = \frac{H}{I_{xx}I_{yy}} \left(l_{xx} + b + \frac{-g}{\sin^2 \phi}\right) dt$$

$$\theta = \frac{H}{I_{xx}I_{yy}} \left(l_{xx} + b + \frac{-g}{\sin^2 \phi}\right) \int_{\text{Start}}^{\text{End}} dt$$

Substituting any of the functions for $\sin \phi$ determined above and integrating with respect to time will give the somersault angle $\theta$. The integration requires the use of the Elliptical integral of the third kind (except for the case when $c = 0$). If desired an average somersault rate may be obtained by evaluating the integral over a period and dividing by the period length. The integral

$$\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi}$$

is evaluated for Cases 3, 7, 6 and 8 below. The solutions could be substituted into the equations of $\theta$ above or, as was done in Section 4.2.7, into the equations for relative difficulty.
Case 3:  
From Table E-1, Case 3  
\[ \cos \phi = \frac{c}{f} Dn\left( Dn^{-1}\left( \int \cos \phi \, \sqrt{1 - \frac{af}{bc}} \right) - \frac{H \sqrt{bc}}{I_x I_y I_z} \right) \]  

To simplify the equations, let  
\[ \alpha = -\frac{H \sqrt{bc}}{I_x I_y I_z} \]  
\[ \beta = Dn^{-1}\left( \int \cos \phi \, \sqrt{1 - \frac{af}{bc}} \right) \]  
\[ k = \sqrt{1 - \frac{af}{bc}}. \]  

Then let  
\[ u = \alpha + \beta, \quad \frac{du}{dt} = \alpha. \]  

Thus,  
\[ \int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{du}{1 - \frac{c}{f} Dn^2(u,k)} \]  

Using the Jacobian elliptical function identities (Appendix B.3.3)  
\[ \int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{c du}{1 - \frac{c}{f} \left( 1 - k^2 \sn^2(u,k) \right)} \]  
\[ = \alpha \int_{\text{Start}}^{\text{End}} \frac{du}{\left[ 1 - \frac{c}{f} \right] + \frac{c}{f} k^2 \sn^2(u,k)} \]  
\[ = \frac{\alpha f}{f - c} \int_{\text{Start}}^{\text{End}} \frac{du}{c - f} \]  
\[ = \frac{ck^2}{c - f}. \]  

This is now in the form of an Elliptical integral of the third kind where the Elliptical characteristic n equals  
\[ \frac{ck^2}{c - f}. \]  

Also since \( g = f - c \) then we have,
Rigid body rotation with respect to time

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \frac{a\phi}{g} \left[ \prod \left( -\frac{e^{2k}}{g}, \sin^{-1}(sn(u)), k \right) \right]_{\text{Start}}^{\text{End}}.
\]

Substituting back the values of \( u \), and \( k \), we obtain

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \frac{-H \sqrt{bc}}{l_{xy}l_{yz}l_{zx}} \left[ \frac{1 - \frac{af}{bc}}{c - f} \cdot \sin^{-1} \left( \frac{-H \sqrt{bc}}{l_{xy}l_{yz}l_{zx}} \right) t + Dn^{-1} \left( \sqrt{H \frac{af}{bc}} \cdot \frac{1 - \frac{af}{bc}}{\sqrt{1 - \frac{bc}{af}}} \right) \right]_{\text{Start}}^{\text{End}}
\]

\[
= \frac{-Hf \sqrt{bc}}{l_{xy}l_{yz}l_{zx}} \left[ \frac{bc - af}{-bg} \cdot \sin^{-1} \left( \frac{-H \sqrt{bc}}{l_{xy}l_{yz}l_{zx}} \right) t + Dn^{-1} \left( \sqrt{H \frac{af}{bc}} \cdot \frac{1 - \frac{af}{bc}}{\sqrt{1 - \frac{bc}{af}}} \right) \right]_{\text{Start}}^{\text{End}}
\]

**Case 7:**

From Table E- 1, Case 7

\[
\cos \phi = \sqrt{\frac{a}{b}} \cdot Dn^{-1} \left( \sqrt{\frac{b}{a}} \cdot \cos \phi \cdot \sqrt{1 - \frac{bc}{af}} \right) - \frac{H \sqrt{af}}{l_{xy}l_{yz}l_{zx}} \cdot \sqrt{1 - \frac{bc}{af}}
\]

To simplify the equations, let

\[
\alpha = \frac{-H \sqrt{af}}{l_{xy}l_{yz}l_{zx}}
\]

\[
\beta = Dn^{-1} \left( \sqrt{\frac{b}{a}} \cdot \cos \phi \cdot \sqrt{1 - \frac{bc}{af}} \right)
\]

\[
k = \sqrt{1 - \frac{bc}{af}}.
\]

Then let

\[
u = \alpha + \beta, \quad \frac{du}{dt} = \alpha.
\]

Thus,

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{dt}{1 - \left( \frac{a}{b} \cdot Dn^{-1} \left( \sqrt{\frac{b}{a}} \cdot \cos \phi \cdot \sqrt{1 - \frac{bc}{af}} \right) - \frac{H \sqrt{af}}{l_{xy}l_{yz}l_{zx}} \cdot \sqrt{1 - \frac{bc}{af}} \right) - \frac{a \alpha u}{b}}
\]

\[
= \int_{\text{Start}}^{\text{End}} \frac{a \alpha u}{b} \cdot Dn^{-1}(a, k)
\]

Very similar working to Case 3 may be followed. Using the Jacobian elliptical function identities.
Rigid body rotation with respect to time

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{\alpha du}{1 - \frac{a}{b} \left(1 - k^2 \sin^2(u, k)\right)}
\]

\[
= \int_{\text{Start}}^{\text{End}} \frac{\alpha du}{1 - \frac{a}{b} \left(1 - k^2 \sin^2(u, k)\right)} + \frac{a}{b} k^2 \sin^2(u, k)
\]

\[
= \frac{\alpha}{1 - \frac{a}{b}} \int_{\text{Start}}^{\text{End}} \frac{du}{1 + \frac{a}{b} \left(1 - k^2 \sin^2(u, k)\right)}
\]

\[
= \frac{\alpha b}{b - a} \int_{\text{Start}}^{\text{End}} \frac{du}{1 - \frac{ak^2}{a-b} \sin^2(u, k)}
\]

This is now in the form of an Elliptical integral of the third kind where the elliptical characteristic, n, equals

\[
\frac{ak^2}{a-b}.
\]

Also since \( g = b - a \) then we have,

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \frac{\alpha b}{g} \left[ \Pi \left( \frac{ak^2}{g} \sin^{-1}(sn(u), k) \right) \right]_{\text{Start}}^{\text{End}}
\]

Substituting back the values of \( u \), and \( k \), we obtain

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \frac{H \sqrt{af}}{l_x l_y l_z} \left[ \Pi \left( \frac{af - bc}{-fg} \sin^{-1} \left( sn \left( \frac{H \sqrt{af}}{l_x l_y l_z} \right) \right) + Dn^{-1} \left( \frac{b}{a \cos \phi}, \sqrt{1 - \frac{bc}{af}} \right) \right) \right]_{\text{Start}}^{\text{End}}
\]

Case 6:

From Table E-1, Case 6

\[
\cos \phi = \sqrt{\frac{c}{f}} \sin \left( \frac{H \sqrt{af}}{l_x l_y l_z} t + Sn^{-1} \left( \frac{f}{c \cos \phi}, \frac{bc}{af} \right) \right)
\]

Thus,

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{H \sqrt{af}}{l_x l_y l_z} \left[ \frac{c}{f} \sin \left( \frac{H \sqrt{af}}{l_x l_y l_z} t + Sn^{-1} \left( \frac{f}{c \cos \phi}, \frac{bc}{af} \right) \right) \right] dt
\]

To simplify the equations, let

\[
\alpha = \frac{H \sqrt{af}}{l_x l_y l_z},
\]
Rigid body rotation with respect to time

\[ \beta = Sn^{-1} \left( \sqrt{\frac{f}{c}} \cos \phi, \sqrt{\frac{bc}{af}} \right) \]

\[ k = \sqrt{\frac{bc}{af}} \cdot \]

Then let

\[ u = \alpha + \beta, \quad \frac{du}{dt} = \alpha \cdot \]

Thus,

\[ \int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{a du}{1 - \frac{c}{f} Sn^{-1}(u, k)} \]

This is in the form of an elliptical integral of the third kind where the elliptical characteristic, \( n = \frac{c}{f} \). Thus

\[ \int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = a \Pi \left( \frac{c}{f}, \sin^{-1}(\sin(u)), k \right) \]

Substituting back the values of \( \alpha, \beta, u, \) and \( k \) we obtain

\[ \int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \left( H \sqrt[3]{\frac{af}{bc}} \frac{1}{1, 1, 1} \right) \left[ \Pi \left( \frac{c}{f}, \sin^{-1}\left( \frac{H \sqrt[3]{af - bc}}{I_{yy} I_{zz}} I_{yy} I_{zz} \right) \right) + \frac{af}{bc} \right] \]

Case 8:

From Table E-1, Case 8

\[ \cos \phi = \sqrt{\frac{a}{b}} Cn^{-1}\left( \frac{b}{a} \cos \phi, \sqrt{\frac{af}{bc}} \right) - H \sqrt[3]{\frac{af - bc}{I_{yy} I_{zz}} I_{yy} I_{zz} \right) \]

Thus,

\[ \int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{dt}{1 - \frac{a}{b} Cn^{-1}\left( \frac{b}{a} \cos \phi, \sqrt{\frac{af}{bc}} \right) - H \sqrt[3]{\frac{af - bc}{I_{yy} I_{zz}} I_{yy} I_{zz} \right) \]

To simplify the equations, let

\[ \alpha = -H \sqrt[3]{\frac{af - bc}{I_{yy} I_{zz}}} \]

\[ \beta = Cn^{-1}\left( \frac{b}{a} \cos \phi, \sqrt{\frac{af}{bc}} \right) \]

\[ k = \sqrt{\frac{af}{bc}} \cdot \]

Then let

\[ u = \alpha + \beta, \quad \frac{du}{dt} = \alpha \cdot \]
Thus,
\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \int_{\text{Start}}^{\text{End}} \frac{\alpha du}{1 - \frac{a}{b} Cn(u, k)}
\]
\[
= \int_{\text{Start}}^{\text{End}} \frac{\alpha du}{1 - \frac{a}{b} (1 - Sn^2(u, k))}
\]
\[
= \int_{\text{Start}}^{\text{End}} \frac{\alpha du}{\left(1 - \frac{a}{b}\right) + \frac{a}{b} Sn^2(u, k)}
\]
\[
= \frac{\alpha}{b-a} \int_{\text{Start}}^{\text{End}} \frac{du}{1 + \frac{a}{b-a} Sn^2(u, k)}
\]
\[
= \frac{\alpha b}{g} \int_{\text{Start}}^{\text{End}} \frac{du}{1 + \frac{a}{b-a} Sn^2(u, k)}
\]

This is in the form of an Elliptical integral of the third kind where the elliptical characteristic, \( n = -a/g \). Thus,
\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = \frac{\alpha b}{g} \Pi \left( \frac{-a}{g}; \sin^{-1}(sn(u)), k \right)
\]

Substituting back the values of \( \alpha, \beta, u, \) and \( k \).

\[
\int_{\text{Start}}^{\text{End}} \frac{dt}{\sin^2 \phi} = -\frac{Hb}{gI_{xx}I_{yy}I_{zz}} \Pi \left( \frac{-a}{g}; \sin^{-1} \left( sn \left( \frac{Cn^{-1} \left( \frac{b}{a} \cos \phi, \sqrt{\frac{af}{af-bc}} \right) - H \sqrt{\frac{af-bc}{I_{xx}I_{yy}I_{zz}} t} \right), \sqrt{\frac{af}{af-bc}} \right) \right)
\]
Appendix F.

Plots of error due to assuming $I_{xx} = I_{yy}$

In Section 4.2.5 the consequences of assuming $I_{xx} = I_{yy}$ was discussed. Only a couple of example plots were given there. This appendix presents additional plots, so that the pattern of the change in the number of somersaults required may be more easily seen when $\phi_o$ decreases, and $\phi_o$ is either set to a fixed value of is chosen based on predicting what would be required to complete a skill. Figure F-1 plots $N_\theta (error) = N_\theta (Assumption) - N_\theta (Actual)$ when $\phi_o$ is set to a fixed value. Figure F-2 plots the number of somersaults required to complete various twisting somersault skills when $\phi_o$ is either set to a fixed value of is chosen based on predicting what would be required to complete a skill. In both figures the error clearly decreases as $\phi_o$ decreases, but the error pattern is quite different as discussed in Section 4.2.5.
Error when assuming $I_{xx}=I_{yy}$

Figure F-1: $N_{\theta(error)}$ when $\phi_0$ equals a) 85°, b) 80°, c) 75°, d) 70°.

$N_{\theta(error)} = N_{\theta(Assumption)} - N_{\theta(Actual)}$. A difference from a value of zero for $N_{\theta(error)}$ is the error due to assuming $I_{xx}=I_{yy}$. 
Error when assuming $I_{xx}=I_{yy}$

Figure F-2: The number of somersaults required to complete a) 1/1, b) 3/2, c) 2/1, d) 5/2, e) 3/1, f) 7/2 twist.

The values of $\phi_0$ were set to the values predicted, by using the assumption that $I_{xx}=I_{yy}$, to achieve the desired twist. A difference from a value of one for the somersault required is the error due to assuming $I_{xx}=I_{yy}$. 
Appendix G.

Postural change and the previous literature

The equations derived in Section 4.3, used the conservation of angular momentum. Previous authors have used this principal to derive equations for specific situations, this section highlights the agreement or disagreement between the equations presented in Section 4.3, and those previously published.

G.1 Two segment planar and Smith & Kane (1967)

The two planar integral may be solved analytically using t-results as follows;

From (4-34)

$$ \gamma_{H=0, ref \ = \ 2} = - \int_{\alpha_{start}}^{\alpha_{end}} \frac{(I_i + A + C \sin \alpha_i - D \cos \alpha_i)}{(N + 2C \sin \alpha_i - 2D \cos \alpha_i)} d\alpha_i $$

This may be divided into two integrals

$$ \gamma_{H=0, ref \ = \ 2} = - \int_{\alpha_{start}}^{\alpha_{end}} \frac{1}{2} \left( N + 2C \sin \alpha_i - 2D \cos \alpha_i \right) - \frac{N}{2} + I_i + A \right) d\alpha_i $$

$$ = - \frac{1}{2} \int_{\alpha_{start}}^{\alpha_{end}} 1 d\alpha_i - \int_{\alpha_{start}}^{\alpha_{end}} \frac{-N}{2} + I_i + A \right) d\alpha_i $$

$$ = - \frac{1}{2} \left[ \alpha_i \right]_{\alpha_{start}}^{\alpha_{end}} - \left( - \frac{N}{2} + I_i + A \right) \int_{\alpha_{start}}^{\alpha_{end}} \frac{1}{(N + 2C \sin \alpha_i - 2D \cos \alpha_i)} d\alpha_i $$

The remaining integral is in the form of the first integral of interest in Appendix B.3.2, where $S = N; Q = C, R = D$. Thus,
In the notation that Smith and Kane (1967) used this is

\[ \gamma_{H=0, \text{ref}=2} = -\frac{1}{2} \left[ \alpha_i \right]_{\text{ref}=\text{start}} \left( -\frac{N}{2} + I_i + A \right) \left[ \frac{2}{\sqrt{N^2 - 4D^2}} \tan^{-1} \left( \frac{(N + 2D) \tan \left( \frac{\alpha}{2} \right) + 2C}{\sqrt{N^2 - 4D^2}} \right) \right]^{\text{finish}}_{\text{start}} \]

\[ = -\frac{1}{2} \left[ \alpha_i \right]_{\text{ref}=\text{start}} + \frac{N - 2I_i - 2A}{\sqrt{N^2 - 4D^2}} \left[ \tan^{-1} \left( \frac{(N + 2D) \tan \left( \frac{\alpha}{2} \right) + 2C}{\sqrt{N^2 - 4D^2}} \right) \right]^{\text{finish}}_{\text{start}} \]

Smith and Kane (1967) only consider the situation when the centre of gravity is on the longitudinal axis of a segment; which corresponds to \( c = 0 \) in the above equation:

\[ \gamma_{H=0, \text{ref}=2} = -\frac{1}{2} \left[ \alpha_i \right]_{\text{ref}=\text{start}} + \frac{N - 2I_i - 2A}{\sqrt{N^2 - 4D^2}} \left[ \tan^{-1} \left( \frac{(N + 2D) \tan \left( \frac{\alpha}{2} \right) + 2C}{\sqrt{N^2 - 4D^2}} \right) \right]^{\text{finish}}_{\text{start}} \]

\[ = -\frac{\alpha_i}{2} + \frac{N - 2I_i - 2A}{\sqrt{N^2 - 4D^2}} \left[ \tan^{-1} \left( \frac{(N + 2D) \tan \left( \frac{\alpha}{2} \right) + 2C}{\sqrt{N^2 - 4D^2}} \right) \right]^{\text{finish}}_{\text{start}} \]

\[ = -\frac{\alpha_i}{2} + \frac{\left( \frac{N}{D} \right) - 2 \left( \frac{I_i + A}{D} \right)}{\sqrt{\left( \frac{N}{D} \right)^2 - 4}} \left[ \tan^{-1} \left( \frac{\left( \frac{N}{D} + 2 \right) \tan \left( \frac{\alpha}{2} \right)}{\sqrt{\left( \frac{N}{D} \right)^2 - 4}} \right) \right]^{\text{finish}}_{\text{start}} \]

In the notation that Smith and Kane (1967) used this is

\[ \gamma_{H=0, \text{ref}=2} = \left[ -\frac{\varphi}{2} + \frac{b - 2a}{\sqrt{b^2 - 4}} \tan^{-1} \left( \frac{\left( \frac{b - 2}{\sqrt{b^2 - 4}} \right) \tan \left( \frac{\varphi}{2} \right)}{b + 2} \right) \right]^{\varphi(1)}_{\varphi(0)} \]

\[ = \left[ -\frac{\varphi}{2} + \frac{b - 2a}{\sqrt{b^2 - 4}} \tan^{-1} \left( \frac{\left( \frac{b - 2}{\sqrt{b^2 - 4}} \right) \tan \left( \frac{\varphi}{2} \right)}{b + 2} \right) \right]^{\varphi(1)}_{\varphi(0)} \]

This agrees with Equation 3.2.11 given by Smith and Kane (1967).
G.2 Frohlich (1979)

Frohlich (1979) uses two segments to model an action similar to DiverS from Section 6.1. To simplify the calculations and hence use a two segment model, Frohlich assumed that the arms were a single rigid segment hinged at shoulder level on the midline of the torso. The arm action was then represented by a rotation of the arm segment through $\pi/2$. Figure G-1 illustrates Frohlich’s model.

![Figure G-1: Frohlich’s model of simultaneously raising one arm while lowering the other from a position with both laterally outstretched.](image)

Let us determine the equation describing Frohlich’s model using the equations derived in Chapter 4. We begin with Equation (4-34) since it is a two segment planar model.

$$\frac{\gamma_{H=0, \text{ref}=2}}{I_1} = - \int_{\alpha_1}^{\alpha_1} \frac{1}{N} \left( I_1 + A + C \sin \alpha_1 - D \cos \alpha_1 \right) \left( N + 2C \sin \alpha_1 - 2D \cos \alpha_1 \right) d\alpha_1$$

The torso is segment 2 and the arms are segment 1. Since the axis of rotation of the arms is at their combined centre of gravity, and on the longitudinal axis of the torso then $V_{1x} = V_{1y} = 0$ and $V_{21x}=0$. This means that $A = C = D = 0$ and $N=(I_1+I_2+(m_1m_2/(m_1+m_2))V_{21y})^2$.

If the arms laterally outstretched is the $\alpha_1 = \pi/2$ position then,

$$\frac{\gamma_{H=0, \text{ref}=2}}{I_1} = - \int_{\alpha_1}^{\pi/2} \frac{1}{N} d\alpha_1$$

$$= - \frac{\alpha_1}{N} \left[ \alpha_1 \right]_{\pi/2}^{\pi/2}$$

$$= - \frac{I_1}{N} \left[ \frac{\alpha_1}{\pi/2} \right]_{\pi/2}^{\pi/2}$$

$$= - \frac{I_1}{I_1 + I_2 + \left( \frac{m_1m_2}{m_1+m_2} \right) V_{21y}^2} \left[ \frac{\pi}{2} \right]$$

This is now equivalent to equation A6 in Frohlich (1979). Thus, the current model is in agreement with Frohlich for the simplified case Frohlich presents.
G.3 Yeadon (1984)

Equation 1 on p166 in Yeadon’s (1984) is equivalent to Equation (4-26) and Yeadon’s equation PA at the bottom of page 168 is in agreement with Equation (4-27). However, when expanding this equation Yeadon discards the terms relating to the movement of the joints with respect to the overall centre of gravity of the body, stating that “the origins of the frame of reference [for each segment] need not to be specified since the vector quantities in the angular momentum equations are unaffected by the movement of the origin”. Even though the angular velocity of the segments is independent of the origin of the segment’s frame of reference, the determination of the velocity of the centre of gravity requires the origin of the segment’s frame to be defined. As a result of this oversight it appears that Yeadon determines the angular momentum about each joint and sums these rather than determining the angular momentum of each segment about the overall centre of gravity and summing these.

To clarify the meaning of this difference, consider a two-segment planar model, with only one straight arm moving the arm is segment 1 and the rest of the body is segment 2. Following the equations on page 172 of Yeadon’s (1984) all of the joint angular velocities other than the left upper arm (a1) with respect to the chest are zero. Thus Yeadon’s equation for the total angular momentum reduces to

\[ h = I_{ff} \omega_{ff} + I_{pf} \omega_{pf} + I_{aa} \omega_{a1c} + m_s a_f \times (\omega_{a1c} \times a_f), \]

which may also be written as

\[ h = I_{ff} \omega_{ff} + I_{aa} \omega_{a1c} + m_s a_f \times (\omega_{a1c} \times a_f) \]

The angular momentum is in the direction of the z-axis of the whole body. For comparison with the equations derived in this thesis, let us write these equations with the notation used in Section 4.3.2 for a planar two segment model, with segment 1 as the arm, and segment 2 as the body, chosen so that the body will be the reference segment, and \( \alpha_2 = \alpha \), since \( \alpha_1 = 0 \):

\[ h = I_{\text{Overall}} \dot{\gamma} + I_1 \dot{\alpha} + m_s D_1 \times (\dot{\alpha} \times V_1), \]

substituting the value of \( I_{\text{Overall}} \) and \( D_2 \), found in Appendix H for Section 4.3.2 using the constants found there as well.
Postural change and previous literature

When angular momentum is zero then the angular displacement is

\[ \gamma_{R,ref} = \int_{a_{low}}^{a_{high}} \frac{(I_1 + D - A \cos \alpha)}{(N + 2C \sin \alpha - 2D \cos \alpha)} \, d\alpha \]

Let us compare this with Equation (4-34):

\[ \gamma_{R,ref} = \int_{a_{low}}^{a_{high}} \frac{(I_1 + A + C \sin \alpha - D \cos \alpha)}{(N + 2C \sin \alpha - 2D \cos \alpha)} \, d\alpha \]

The functions to be integrated are distinctly different. Figure G-2 plots the two curves of the integral to be evaluated for the inertial property data set of the example athlete, from Section 3.10, across the range $-180^o$ to $0^o$, which would be starting with the right arm raised and lowering it laterally to the side while keeping it straight.

![Figure G-2: Plots of the functions to be integrated following Yeadon’s and equations from this thesis for laterally lowering the left arm](image)

The two curves are quite different; Yeadon’s equation will result in an underestimation for the first $90^o$ of movement of the arm and then an overestimation for the remaining $90^o$ of rotation. The overestimation will be less than the underestimation and so over a full $180^o$
Yeadon’s equation will overestimate the angular displacement: numerically integrating\textsuperscript{22}, using the average Riemann sums gives a magnitude of \( \sim 5.6^\circ \) for the full action following Yeadon’s equations and \( \sim 4.8^\circ \) for the full action when using the equations in this thesis. Figure D-2 clearly shows that the differences in the equations presented by Yeadon are derived in this thesis can have a significant effect on the predicted angular displacements due to an action. As a result it is not reasonable to compare the values of tilt predicted by Yeadon, with those predicted in this thesis.

\textsuperscript{22} Analytic evaluation of these equations is possible, but this was deemed unnecessary since the difference in results obtained from these equations is clear with a numerical evaluation.
Appendix H.

Full derivations of equations

This appendix provides the full derivations of equations presented in Chapter 4. This section is divided by Equation numbers given in chronological order.

Equation (4-1), (4-2) and (4-3)

From the text in Section 4.2.2 the angular momentum is

\[
\begin{align*}
0 & \begin{bmatrix} H \ 0 \end{bmatrix} = R_{\psi \theta}^\text{Glo} H R_{\psi \theta}^\text{Glo} \begin{bmatrix} 0 \
H \
0 \end{bmatrix} \\
& = \begin{bmatrix} 
\cos \theta \cos \psi - \sin \theta \cos \phi \sin \psi & -\cos \theta \sin \psi - \sin \theta \cos \phi \cos \psi & \sin \theta \sin \phi \\
\sin \phi \sin \psi & \sin \phi \cos \psi & \cos \phi \\
-\sin \theta \cos \psi - \cos \theta \cos \phi \sin \psi & \sin \theta \sin \psi - \cos \theta \cos \phi \cos \psi & \cos \theta \sin \phi 
\end{bmatrix} \begin{bmatrix}
I_x & I_y & I_z \\
I_y & I_z & I_x \\
I_z & I_x & I_y 
\end{bmatrix} \begin{bmatrix}
\theta \sin \phi \sin \psi + \phi \cos \psi \\
\theta \cos \phi \cos \cos \psi - \sin \theta \cos \phi \sin \psi \\
\sin \phi \sin \psi \sin \phi \sin \psi - \cos \theta \cos \phi \cos \psi + \cos \theta \sin \phi \cos \psi + \cos \theta \sin \phi 
\end{bmatrix}
\end{align*}
\]

Thus, the components of angular momentum in each direction of frame Glo are,

Component in the x-direction

\[
\begin{align*}
0 & = I_x (\theta \sin \phi \sin \psi + \phi \cos \psi (\cos \theta \cos \psi \sin \phi \sin \psi - \sin \theta \cos \phi \sin \psi) \\
& \quad + I_y \left(\sin \phi \cos \psi \cos \psi \sin \phi - \cos \theta \cos \phi \cos \psi \sin \phi \sin \psi\right) \right) \\
& \quad + I_z \left(\cos \phi \sin \theta \sin \phi \sin \phi \sin \psi \right) \\
& = I_x \left(\sin \phi \cos \psi \cos \theta \cos \psi \sin \phi \sin \psi - \sin \theta \cos \phi \sin \psi \right) \\
& \quad - I_y \left(\sin \phi \cos \psi \cos \theta \cos \psi \sin \phi \sin \psi \right) + I_z \left(\sin \phi \cos \psi \cos \theta \cos \psi \sin \phi \sin \psi \right) \\
& \quad + I_z \left(\cos \phi \sin \theta \sin \phi \sin \phi \sin \psi \right) \\
& = \theta \left((\sin \phi \sin \psi \cos \theta \cos \psi) (I_x - I_y) - (\sin \theta \cos \phi \sin \phi)(I_y \sin \phi \sin \psi + I_z \cos \phi \sin \psi - I_z) \right) \\
& \quad + \phi \left(- \sin \theta \cos \phi \sin \psi \cos \psi (I_x - I_y) + \cos \theta \left(I_x \cos \psi \sin \psi \sin \phi \right) \right) \\
& \quad + \psi (I_z \sin \theta \sin \phi)
\end{align*}
\]
Component in the y-direction

\[ H = I_x (\theta \sin \phi \sin \psi + \phi \cos \phi) (\sin \phi \sin \psi) + I_y (\theta \sin \phi \cos \psi - \phi \sin \psi) (\sin \phi \cos \psi) + I_z (\psi + \theta \cos \phi) (\cos \phi) \]

\[ = I_x \theta (\sin^2 \phi \sin^2 \psi) + I_z \phi (\cos \psi \sin \phi \sin \phi) + I_y \theta (\sin^2 \phi \cos^2 \psi) + I_z \phi (-\sin \psi \sin \phi \cos \phi) + I_z \theta (\cos^2 \phi) + I_y \psi (\cos \phi) \]

\[ = \theta (\sin^2 \phi (I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi) + I_{zz} \cos^2 \phi) + \phi \cos \psi \sin \phi \sin \psi (I_{xx} - I_{yy}) + \psi (I_{zz} \cos \phi) \]

Component in the z-direction

\[ 0 = I_x (\theta \sin \phi \sin \psi + \phi \cos \phi) (-\sin \theta \cos \psi - \cos \theta \cos \phi \sin \psi) + I_y (\theta \sin \phi \cos \psi - \phi \sin \psi) (-\sin \theta \sin \psi - \cos \theta \cos \phi \cos \psi) + I_z (\psi + \theta \cos \phi) (\cos \theta \sin \phi) \]

\[ = I_x \theta (-\sin \phi \sin \psi \sin \theta \cos \psi - \sin \phi \sin^2 \psi \cos \theta \cos \phi \sin \phi) + I_x \phi (-\sin \theta \cos^2 \psi - \cos \psi \cos \theta \cos \phi \sin \psi) + I_y \theta (\sin \phi \cos \psi \sin \theta \sin \psi - \sin \phi \cos^2 \psi \cos \theta \cos \phi \cos \psi) + I_y \phi (-\sin \theta \sin \phi \sin \psi + \sin \phi \cos \psi \cos \phi \cos \psi) + I_z \theta (\cos \phi \cos \theta \sin \phi) + I_z \phi (\cos \phi \cos \psi \cos \phi \cos \psi) \]

\[ = -\theta (\sin \phi \sin \psi \sin \theta \cos \psi (I_{xx} - I_{yy}) + (\sin \phi \cos \theta \cos \phi)(I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi - I_{zz})) - \phi (\cos \theta \sin \psi \sin \phi \sin \psi (I_{xx} - I_{yy}) - (\sin \phi \cos \theta \cos \phi)(I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi - I_{zz})) + \psi (I_{zz} \cos \phi \sin \phi) \]

The equations for each component must hold simultaneously.

The x-component of angular momentum may be re-written as

\[ 0 = \cos \theta [\phi (\sin \phi \sin \psi \cos \psi)(I_{xx} - I_{yy}) + \phi (I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi)] + \sin \theta [\phi (\cos \phi \sin \psi \cos \psi)(I_{xx} - I_{yy}) + \phi (\cos \phi \sin \psi \cos \psi)(I_{xx} - I_{yy}) + \psi (I_{zz} \sin \phi)] \]

which has the form

\[ 0 = \cos \theta [f(\theta, \psi, \phi)] + \sin \theta [g(\theta, \psi, \phi)] \]

The z-component may be re-written as

\[ 0 = -\sin \theta [\phi (\sin \phi \sin \phi \cos \phi)(I_{xx} - I_{yy}) + \phi (I_{xx} \cos^2 \phi + I_{yy} \sin^2 \phi)] + \cos \theta [-\phi (\cos \phi \sin \phi \cos \phi)(I_{xx} \sin^2 \phi + I_{yy} \cos^2 \phi - I_{zz}) - \phi (\cos \phi \cos \phi \cos \phi)(I_{xx} - I_{yy}) + \phi (I_{zz} \sin \phi)] \]

which has the form

\[ 0 = -\sin \theta [f(\theta, \psi, \phi)] + \cos \theta [g(\theta, \psi, \phi)] \]

For both to hold simultaneously \([f(\theta, \psi, \phi)] \text{ and } [g(\theta, \psi, \phi)]\) must equal zero.
Thus

\[ 0 = \left[ \dot{\theta} (\sin \phi \sin \psi \cos \psi) (I_{xx} - I_{yy}) + \dot{\phi} (I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi) \right] \]

(H- 1)

And

\[ 0 = \left[ -\dot{\theta} (\sin \phi \cos \phi) (I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi - I_{zz}) - \dot{\phi} (\cos \psi \cos \phi \sin \psi) (I_{xx} - I_{yy}) + \psi (I_{zz} \sin \phi) \right] \]

(H- 2)

must be simultaneously true.

Rearranging (H- 1)

\[ \dot{\phi} = -\dot{\theta} \frac{\sin \phi \sin \psi \cos \psi (I_{xx} - I_{yy})}{I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi} \]

Which also may be written

\[ \frac{\dot{\phi}}{\dot{\theta}} = -\sin \phi \frac{(\sin \psi \cos \phi (I_{xx} - I_{yy}))}{(I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi)} \]

This is Equation (4-1).

Substituting Equation (4-1) into Equation (H- 2) to eliminate \( \dot{\phi} \)

\[ 0 = \left( -\dot{\theta} (\sin \phi \cos \phi) (I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi - I_{zz}) \right) \]

\[ = \left[ -\dot{\theta} \left( \sin \phi \sin \psi \cos \psi \frac{(I_{xx} - I_{yy})}{(I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi)} \right) \right] \left( \cos \psi \cos \phi \sin \psi (I_{xx} - I_{yy}) + \psi (I_{zz} \sin \phi) \right) \]

\[ \psi (I_{zz} \sin \phi) = \dot{\theta} \left( \sin \phi \cos \phi \left( I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi - I_{zz} \right) \right) + \left( \sin \phi \cos \phi \frac{(I_{xx} - I_{yy})}{(I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi)} \right) \]

Since we are investigating the case when \( \phi \neq 0 \), then \( \sin \phi \neq 0 \) and so

\[ \frac{\psi}{\dot{\theta}} = \frac{(\cos \phi)}{I_{zz}} \left[ (I_{xx} \sin^2 \psi + I_{yy} \cos^2 \psi - I_{zz}) + \left( \frac{\psi (I_{zz} \sin \phi)}{(I_{xx} \cos^2 \psi + I_{yy} \sin^2 \psi)} \right) \right] \]

Expanding and simplifying
\[
\psi = \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}\sin^2\psi + I_{yy}\cos^2\psi \right) \left( I_{xx}\cos^2\psi + I_{yy}\sin^2\psi \right) \right] - I_{zz}
\]

\[
= \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}\sin^2\psi\cos^2\psi + I_{yy}\sin^4\psi + I_{xx}I_{yy}\cos^4\psi + I_{xx}^2\sin^2\psi\cos^2\psi \right) \right] - I_{zz}
\]

\[
= \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}\sin^4\psi + 2I_{xx}I_{yy}\cos^2\psi + I_{yy}\cos^4\psi \right) \right] - I_{zz}
\]

\[
= \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}\cos^2\psi + I_{yy}\sin^2\psi \right) \right] - I_{zz}
\]

\[
= \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}I_{yy} \left( \frac{\sin^2\psi + \cos^2\psi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \right) \right] - I_{zz}
\]

\[
= \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}I_{yy} \left( \frac{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \right) \right] - I_{zz}
\]

\[
= \frac{(\cos\phi)}{I_{zz}} \cdot \left[ \left( I_{xx}I_{yy} \left( \frac{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \right) \right] - I_{zz}
\]

This is Equation (4-2)

Substituting Equation (4-1) and Equation (4-2) into the y-component of angular momentum,

\[
H = \dot{\theta}(\sin^2\phi(I_{xx}\sin^2\psi + I_{yy}\cos^2\psi) + I_{zz}\cos^2\phi)
\]

\[
+ \dot{\psi} \cdot \left( \frac{-\sin\phi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \sin^2\phi \cos^2\phi \sin\phi \left( I_{xx} - I_{yy} \right)
\]

\[
+ \dot{\phi} \cdot \left( \frac{(\cos\phi)}{I_{zz}} \cdot \left( I_{xx}I_{yy} \left( \frac{I_{xx}\sin^2\psi + I_{yy}\cos^2\psi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \right) \right)
\]

\[
= \dot{\theta}(\sin^2\phi(I_{xx}\sin^2\psi + I_{yy}\cos^2\psi) + I_{zz}\cos^2\phi)
\]

\[
+ \dot{\psi} \cdot \left( \frac{-\sin\phi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \sin^2\phi \cos^2\phi \sin\phi \left( I_{xx} - I_{yy} \right)
\]

\[
+ \dot{\phi} \cdot \left( \frac{(\cos\phi)}{I_{zz}} \cdot \left( I_{xx}I_{yy} \left( \frac{I_{xx}\sin^2\psi + I_{yy}\cos^2\psi}{I_{xx}\cos^2\psi + I_{yy}\sin^2\psi} \right) \right) \right)
\]
Making \( \theta \) the subject of the equation, since \( H \) is constant gives

\[
\dot{\theta} = H \left( \frac{I_{xx} - (I_{xx} - I_{yy}) \sin^2 \psi}{I_{xx} I_{yy}} \right).
\]

This is Equation (4-3).
**Equation (4-4)**

From the text in Section 4.2.3

\[
\frac{\cos \phi \psi}{\sin \phi} = \frac{-I_{zz}(I_{xx} - I_{yy}) \sin \psi \cos \psi}{[I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi]}.
\]

Considering the two cases for each sign of the right hand side denominator,

<table>
<thead>
<tr>
<th>[I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi]</th>
<th>[I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[&gt; 0]</td>
<td>[&lt; 0]</td>
</tr>
</tbody>
</table>

Integrating with respect to time gives (where \(C\) is the constant of integration)

\[
\sin \phi = \frac{\frac{e^C}{\sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}}{\sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}
\]

\[
\sin \phi = \frac{\frac{e^C}{\sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}}{\sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}
\]

\[
\sin \phi \sqrt{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi} = e^C
\]

\[
\sin \phi \sqrt{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi} = e^C
\]

Thus,

\[
\sin \phi = \sin \phi \sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}{\sqrt{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}
\]

\[
\sin \phi = \sin \phi \sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}{\sqrt{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}
\]

Putting the fraction together under the square root sign means that the negatives cancel, leaving a positive below the square root sign, and thus situation a) and b) may be combined:

\[
\sin \phi = \sin \phi \sqrt{\frac{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}{\sqrt{I_{xx}(I_{yy} - I_{zz}) + I_{zz}(I_{xx} - I_{yy}) \sin^2 \psi}}}
\]

**This is Equation (4-4).**

**Equation (4-4) through to the values in Table 4-1**

Re-writing Equation (4-4) using the constants defined in Equation (4-9) gives

\[
\sin^2 \phi + \frac{g}{b + (f - b) \sin^2 \psi}
\]
\[
\sin^2 \psi = \frac{g - b \sin^2 \phi}{(f - b) \sin \theta}. \\
\]

Thus,
\[
\sin \psi = \sqrt{\frac{g - b \sin^2 \phi}{(f - b) \sin \phi}},
\]

Choosing the positive root, since it is assumed that \( \psi \) starts at zero or in the first quadrant gives
\[
\cos \psi = \sqrt{\frac{1 - g - b \sin^2 \phi}{(f - b) \sin^2 \phi}} = \sqrt{\frac{(f - b) \sin^2 \phi}{(f - b) \sin^2 \phi} + \frac{g \sin^2 \phi}{(f - b) \sin^2 \phi}} = \sqrt{\frac{f \sin^2 \phi - g}{(f - b) \sin^2 \phi}}.
\]

Substituting this into Equation (4-1) yields
\[
\frac{\dot{\phi}}{\theta} = -\sin \phi \left( \frac{\sin \psi \cos \psi (I_{xx} - I_{yy})}{(I_{xx} - I_{yy}) \sin^2 \psi} \right)
\]
\[
= -\sin \phi \left( \frac{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - g \right)} (I_{xx} - I_{yy})}{(f - b) \sin^2 \phi - (I_{xx} - I_{yy}) g - b \sin^2 \phi} \right)
\]
\[
= -\sin \phi \left( \frac{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - g \right)} (I_{xx} - I_{yy})}{(I_{xx} - I_{yy}) (f - b) + b (I_{xx} - I_{yy}) \sin^2 \phi - (I_{xx} - I_{yy}) \sin^2 \phi} \right)
\]
\[
= -\sin \phi \left( \frac{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - g \right)} (I_{xx} - I_{yy})}{(I_{xx} - I_{yy}) (f - b) + b (I_{xx} - I_{yy}) \sin^2 \phi - (I_{xx} - I_{yy}) \sin^2 \phi} \right)
\]
\[
= -\sin \phi \left( \frac{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - e \right)} (I_{xx} - I_{yy})}{(I_{xx} - I_{yy}) (f - b) + b (I_{xx} - I_{yy}) \sin^2 \phi - (I_{xx} - I_{yy}) \sin^2 \phi} \right)
\]
\[
= -\sin \phi \left( \frac{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - g \right)} (I_{xx} - I_{yy})}{(I_{xx} - I_{yy}) (f - b) + b (I_{xx} - I_{yy}) \sin^2 \phi - (I_{xx} - I_{yy}) \sin^2 \phi} \right)
\]
\[
= -\sin \phi \left( \frac{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - g \right)} (I_{xx} - I_{yy})}{(I_{xx} - I_{yy}) (f - b) + b (I_{xx} - I_{yy}) \sin^2 \phi - (I_{xx} - I_{yy}) \sin^2 \phi} \right)
\]

Inverting and integrating gives
\[
\theta = \int_{\phi_0}^{\phi} \frac{g - I_{xx} I_{yy} \sin^2 \phi}{\sqrt{g - b \sin^2 \phi \left( f \sin^2 \phi - e \right)}} d\phi
\]

Now \( \phi \neq 0 \) as specified in Section 4.2.1 and so \( \sin \phi \neq 0 \) thus,
\[ \theta = e^{\int \frac{1}{\sin \phi \sqrt{(g - b \sin^2 \phi)(f \sin^2 \phi - g)}} \, d\phi - I_{xx} \int \frac{\sin \phi}{\sqrt{(g - b \sin^2 \phi)(f \sin^2 \phi - g)}} \, d\phi} \]

This is Equation (4-11).

Let \( u = \cos \phi \); then \( du/d\phi = -\sin \phi \) and \( d\phi = -du/\sqrt{1 - u^2} \).

This is the substitution used by Yeadon (1993a) when seeking an equation for \( \phi \) with respect to time. It allows the equation to be written in a form that can be more easily identified with Elliptic integrals of the first and third kind. Following this substitution, Equation (4-11) becomes

\[ \theta = g \int_{\cos \phi}^{\cos \theta} \frac{1}{\sqrt{1 - u^2 \sqrt{(g - b(1 - u^2))(f(1 - u^2) - g)}}} \, du - I_{xx} I_{yy} \int_{\cos \phi}^{\cos \theta} \frac{du}{\sqrt{(g - b(1 - u^2))(f(1 - u^2) - g)}} \, d\theta \]

\[ = -g \int_{\cos \phi}^{\cos \theta} \frac{du}{(1 - u^2) \sqrt{(bu^2 + g - b)(f - g - fa^2)}} + I_{xx} I_{yy} \int_{\cos \phi}^{\cos \theta} \frac{du}{\sqrt{(bu^2 + a)(c - fa^2)}} \]

This is Equation (4-12).

Let us evaluate the two parts of Equation (4-12) separately.

The first integral – The elliptical integral of the third kind

\[ I_{n1} = -g \int_{\cos \phi}^{\cos \theta} \frac{du}{(1 - u^2) \sqrt{(bu^2 - a)(c - fa^2)}} \]

This can be re-arranged into the form of the elliptical integral of the third kind as presented in Appendix B.3.4.

\[ \Pi(n; \phi, k) \int_{0}^{\sin \phi} \frac{dy}{(1 - n \sin^2 \phi)(1 - k \sin^2 \phi)} \]

which can be found in tables of integrals or using Mathematica: \( 0 \leq k \leq 1 \).

The way to rearrange \( I_{n1} \) depends on the signs of the constants and the value of \( k \) in the re-arrangement:

<table>
<thead>
<tr>
<th>If ( f ) and ( c ) have the same sign.</th>
<th>If ( a ) and ( b ) have the same sign.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{n1} = -\frac{g}{c} \int_{\cos \phi}^{\cos \theta} \frac{du}{(1 - u^2) \sqrt{\left(\frac{fa^2}{c} - \frac{af}{bc}\right) \left(\frac{1 - fa^2}{c}\right)}} )</td>
<td>( I_{n1} = -\frac{g}{a} \int_{\cos \phi}^{\cos \theta} \frac{du}{\left(\frac{bu^2}{a} - 1\right) \sqrt{\left(\frac{bc - bu^2}{a}\right)}} )</td>
</tr>
</tbody>
</table>

Let

\[ y = u \sqrt{\frac{f}{c}}, \quad du = dy \sqrt{\frac{c}{f}} \]

\[ y = u \sqrt{\frac{b}{a}}, \quad du = dy \sqrt{\frac{a}{b}} \]
Full derivations

\[ I_n = -\frac{g}{\sqrt{bc}} \left[ \left( \frac{bc - af}{b(c - f)} \right) \sin^{-1} \left( \frac{bc(1 - y^2)}{bc - af} \right) \right] \frac{y^2 - af}{bc} \]

If both

a & b also have the same sign as c & f

(af/bc) < 1 and so 1 - k^2 = (af/bc); p = c/f are true the integral becomes

\[ I_n = -\frac{g}{\sqrt{bc}} \left[ \left( \frac{bc - af}{b(c - f)} \right) \sin^{-1} \left( \frac{bc(1 - y^2)}{bc - af} \right) \right] \frac{y^2 - af}{bc} \]

or if

b has the same sign as c & f but a has the opposite sign

Let (k^2 - 1)/k^2 = (af/bc); k^2 = be/(bc-af) and q = c/f

\[ I_n = -\frac{g}{\sqrt{bc}} \left[ \left( \frac{bc - af}{b(c - f)} \right) \sin^{-1} \left( \frac{bc(1 - y^2)}{bc - af} \right) \right] \frac{y^2 - af}{bc} \]
### If f and c have the same sign.

The integral becomes

$$n_i = -g \sqrt{1 - \frac{bc}{af}} \times \int \frac{\frac{c}{f} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af}}{\frac{bc - af}{bc}} \, dy$$

$$\left[ \frac{c}{f - 1} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af} \right]_{\frac{bc}{af}}^{1}$$

$$In_i = \left(\frac{-gf}{bc - af}\right) \times$$

$$\int_{\frac{bc}{af}}^{1} \frac{\frac{c}{f} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af}}{\frac{bc - af}{bc}} \, dy$$

Now if b has the opposite sign to c, then the integral needs to be written as

$$In_i = -g \sqrt{1 - \frac{bc}{af}} \times$$

$$\int_{\frac{bc}{af}}^{1} \frac{\frac{c}{f} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af}}{\frac{bc - af}{bc}} \, dy$$

### If a and b have the same sign.

The integral becomes

$$In_i = \left(\frac{-gb}{(a-b)\sqrt{af - bc}}\right) \times$$

$$\int \frac{a}{a-b \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{af}{af - bc}} \, dy$$

$$\left[ \frac{\frac{a}{a-b} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{af}{af - bc}}{\frac{af - bc}{af}} \right]_{\frac{af}{bc}}^{1}$$

$$In_i = \left(\frac{b}{\sqrt{af - bc}}\right) \times$$

$$\int_{\frac{af}{bc}}^{1} \frac{a}{a-b \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{af}{af - bc}} \, dy$$

Now if a has the opposite sign to c, then the integral needs to be written as

$$In_i = -g \sqrt{1 - \frac{af}{bc}} \times$$

$$\int_{\frac{af}{bc}}^{1} \frac{a}{a-b \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{af}{af - bc}} \, dy$$

In this form the integral may be solved for the case when c & f have the same sign and then a & b have the same sign which is opposite to c & f.

If (af/bc) <1, Let k² = (af/bc); n = a/b

The integral becomes

$$In_i = -g \sqrt{1 - \frac{af}{bc}} \times$$

$$\int_{\frac{af}{bc}}^{1} \frac{a}{a-b \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{af}{af - bc}} \, dy$$

In this form the integral may be solved for the case when c & f have the same sign and then a & b have the same sign which is opposite to c & f.

If (bc/af) <1, Let k² = (bc/af); n = c/f

The integral becomes

$$In_i = -g \sqrt{1 - \frac{bc}{af}} \times$$

$$\int_{\frac{bc}{af}}^{1} \frac{c}{f} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af} \, dy$$

$$\left[ \frac{c}{f - 1} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af} \right]_{\frac{bc}{af}}^{1}$$

$$In_i = \left(\frac{-gf}{bc - af}\right) \times$$

$$\int_{\frac{bc}{af}}^{1} \frac{c}{f} \cdot \sin^{-1}\left(\sqrt{1 - y^2}\right) \cdot \frac{bc}{bc - af} \, dy$$
The second integral – the Jacobian elliptical integral

\[ I_{n_2} = I_{xy} \int_{\cos \varphi}^{\cos \varphi} \frac{du}{b \sqrt{\frac{u^2}{a} - \frac{af}{bc}} \left(1 - \frac{u^2}{c}\right)} \]

The way to rearrange \( I_{n_2} \) depends on the sign of the constants and value of \( k \) in the rearrangement:

<table>
<thead>
<tr>
<th>If ( d ) and ( c ) have the same sign.</th>
<th>If ( a ) and ( b ) have the same sign.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{n_2} = \frac{I_{xy}}{\sqrt{bc}} \int_{\varphi}^{\varphi} \frac{dy}{bc \sqrt{1 - \frac{af}{bc}y^2}} )</td>
<td>( I_{n_2} = \frac{I_{xy}}{\sqrt{bc}} \int_{\varphi}^{\varphi} \frac{dy}{bc \sqrt{1 - \frac{af}{bc}y^2}} )</td>
</tr>
<tr>
<td>Let ( y = u \sqrt{\frac{f}{c}} ), ( du = dy \sqrt{\frac{c}{f}} )</td>
<td>Let ( y = u \sqrt{\frac{b}{a}} ), ( du = dy \sqrt{\frac{a}{b}} )</td>
</tr>
<tr>
<td>( I_{n_2} = \frac{I_{xy}}{\sqrt{af}} \int_{\varphi}^{\varphi} \frac{dy}{af \sqrt{1 - \frac{bc}{af}y^2}} )</td>
<td>( I_{n_2} = \frac{I_{xy}}{\sqrt{af}} \int_{\varphi}^{\varphi} \frac{dy}{af \sqrt{1 - \frac{bc}{af}y^2}} )</td>
</tr>
<tr>
<td>( \text{If } a &amp; b \text{ also have the same sign as } c &amp; f (af/bc) &lt; 1 \text{ and so } 1-k^2 = (af/bc) )</td>
<td>( \text{If } c &amp; f \text{ also have the same sign as } a &amp; b (bc/ad) &lt; 1 \text{ and so } 1-k^2 = (bc/ad) )</td>
</tr>
<tr>
<td>The integral becomes</td>
<td>The integral becomes</td>
</tr>
<tr>
<td>( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{af}{bc}}}, 1 - \frac{af}{bc}\right) )</td>
<td>( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{bc}{af}}}, 1 - \frac{bc}{af}\right) )</td>
</tr>
<tr>
<td>( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{bc}{af}}}, 1 - \frac{bc}{af}\right) )</td>
<td>( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{af}{bc}}}, 1 - \frac{af}{bc}\right) )</td>
</tr>
</tbody>
</table>

or if \( b \) has the same sign as \( c \& d \) but \( a \) is the opposite sign

| or if \( d \) has the same sign as \( a \& b \) but \( c \) has the opposite sign |
|------------------------------------------|------------------------------------------|
| \( I_{n_2} = \frac{I_{xy}}{\sqrt{bc - af}} \int_{\varphi}^{\varphi} \frac{dy}{bc - af \sqrt{1 - \frac{bc}{af}y^2}} \) | \( I_{n_2} = \frac{I_{xy}}{\sqrt{bc - af}} \int_{\varphi}^{\varphi} \frac{dy}{bc - af \sqrt{1 - \frac{bc}{af}y^2}} \) |
| Let \( k^2 = (bc - af) \) \( k^2 = (bc - af) \) | Let \( k^2 = (bc - af) \) \( k^2 = (bc - af) \) |
| The integral becomes | The integral becomes |
| \( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{bc}{af}}}, 1 - \frac{bc}{af}\right) \) | \( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{af}{bc}}}, 1 - \frac{af}{bc}\right) \) |
| \( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{bc}{af}}}, 1 - \frac{bc}{af}\right) \) | \( I_{n_2} = -F\left(\frac{1}{\sqrt{1 - \frac{af}{bc}}}, 1 - \frac{af}{bc}\right) \) |

Now if \( c \& f \) have opposite signs to \( a \& b \) then it is
If d and c have the same sign.

Now if a & b have the same sign, then it is necessary to rearrange the integral as follows

\[
I_1 = \frac{I_{xy}}{\sqrt{bc}} \int \frac{\cos \theta}{\sqrt{1 - \frac{bc}{af} \sin^2 \theta}} \, dy
\]

\[
I_2 = \frac{I_{xy}}{\sqrt{bc}} \int \frac{\cos \theta}{\sqrt{1 - \frac{bc}{af} \sin^2 \theta}} \, dy
\]

Then if

(bc/af) < 1

Let \(k_2 = (bc/af)\)

The integral becomes

\[
\frac{I_{xy}}{\sqrt{bc}} \left[ F \left( \sin^{-1} \frac{bc}{af} \right) \right]_{\theta = \cos \phi}^{\cos \phi}
\]

To correctly map the cases to the integrals of the correct form it is essential to know the signs of a, b, c, and d, and then the combination of constants required so \(k^2\) will be between 0 and 1. Table H-1 below summarises the options and the working below the table, aids comparison between af and bc.

<table>
<thead>
<tr>
<th>Case</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>f</th>
<th>(k^2) options from considering the form of the integrals</th>
<th>(k^2) (which is between 0 &amp; 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 I_{xx} the max</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1 - \frac{af}{bc} OR 1 - \frac{bc}{af}</td>
<td>1 - \frac{af}{bc}</td>
</tr>
<tr>
<td>3 I_{xz} the max</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>\frac{bc}{bc - af}</td>
<td>bc is positive and af is negative, thus bc-ad &gt; bc and so (0 &lt; \frac{bc}{bc - af} &lt; 1)</td>
</tr>
<tr>
<td>4 I_{xx} the max</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>\frac{bc}{af} OR \frac{af}{bc}</td>
<td>\frac{af}{bc}</td>
</tr>
<tr>
<td>4 I_{xz} the max</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>\frac{bc}{af} OR \frac{af}{bc}</td>
<td>\frac{af}{bc}</td>
</tr>
<tr>
<td>6 I_{yy} the max</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>1 - \frac{af}{bc} OR 1 - \frac{bc}{af}</td>
<td>\frac{bc}{af}</td>
</tr>
<tr>
<td>6 I_{yz} the max</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>1 - \frac{af}{bc} OR 1 - \frac{bc}{af}</td>
<td>\frac{bc}{af}</td>
</tr>
<tr>
<td>7 I_{yy} the max</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>\frac{af}{af - bc}</td>
<td>ad is positive and bc is negative, thus af-bc &gt; af and so (0 &lt; \frac{af}{af - bc} &lt; 1)</td>
</tr>
<tr>
<td>7 I_{yz} the max</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>\frac{af}{af - bc}</td>
<td>ad is positive and bc is negative, thus af-bc &gt; af and so (0 &lt; \frac{af}{af - bc} &lt; 1)</td>
</tr>
</tbody>
</table>
In the last column of Table H-1 the value of $k^2$ was specified; it must be between 0 and 1.

The value of $k^2$ depends on the value whether or not $bc/af < 1$. If $af/bc < 1$ then

$$ \frac{af}{bc} = \frac{(I_{xx} - I_{zz})(I_{yy} - I_{zz}) \cos^2 \phi_o - I_{yy}(I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o}{I_{xx}(I_{yy} - I_{zz})(I_{xx} - I_{yy}) \cos^2 \phi_o + I_{zz}(I_{xx} - I_{yy})(1 - \sin^2 \phi_o \sin^2 \psi_o)} < 1 $$

**In case 3 and 7**, $a$, $b$, $c$ & $f$ all have the same sign, and so $af$ & $bc$ will be positive, as will be $af/bc$ (and $bc/af$). Then continuing to test the inequality,

$$ \left( I_{xx} (I_{yy} - I_{zz}) \cos^2 \phi_o - I_{yy} (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o \right) (I_{yy} - I_{zz}) < $$

$$ I_{xx} (I_{yy} - I_{zz}) \left( I_{xx} (I_{yy} - I_{zz}) \cos^2 \phi_o + I_{zz} (I_{xx} - I_{yy}) (1 - \sin^2 \phi_o \sin^2 \psi_o) \right) $$

$$ \left( I_{zz} (I_{xx} - I_{yy}) \right) \left( I_{yy} - I_{zz} \right) \left( I_{xx} (I_{yy} - I_{zz}) \cos^2 \phi_o + I_{zz} (I_{xx} - I_{yy}) (1 - \sin^2 \phi_o \sin^2 \psi_o) \right) $$

Subtracting $I_{xx}^2 (I_{yy} - I_{zz}) (1 - \sin^2 \phi_o) - I_{xx} (I_{yy} - I_{zz}) I_{yy} (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o$ from both sides gives

$$ I_{zz} (I_{xx} - I_{yy}) (I_{xx} - I_{yy}) (1 - \sin^2 \phi_o) - I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o < I_{zz} (I_{xx} - I_{yy}) (I_{xx} - I_{yy}) $$

Subtracting $I_{xx} (I_{yy} - I_{zz}) I_{zz} (I_{xx} - I_{yy})$ from both sides gives

$$ -I_{zz} (I_{xx} - I_{yy}) (I_{xx} - I_{yy}) \sin^2 \phi_o - I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_o \sin^2 \phi_o < 0 $$

$$ -I_{zz} (I_{xx} - I_{yy}) \sin^2 \phi_o (I_{xx} (I_{yy} - I_{zz}) + I_{zz} (I_{xx} - I_{yy}) \sin^2 \psi_o) < 0 $$

Since $\psi_o$ is small, the inequality will be true in case 3, and false in case 7,

thus for case 3,

$$ \frac{af}{bc} < 1 $$

and for case 7

$$ \frac{bc}{af} < 1 $$

**In case 6**, $a$ & $b$ have the same sign as so $c$ & $f$, but the two pairs have opposite sign. Thus $af$ & $bc$ are negative, and so $af/bc$ (or $bc/af$) are still positive as for Cases 3 and 7. Using
similar steps to the inequality for Cases 3 and 7, but swapping the inequality sign since $bc$
is now negative then if

$$\frac{af}{bc} < 1,$$

The inequality becomes

$$- I_{xx} (I_{xx} - I_{yy}) \sin^2 \phi \left( I_{xx} (I_{yy} - I_{zz}) + I_{xx} (I_{yy} - I_{yy}) \sin^2 \psi_o \right) > 0.$$ 

This is inequality is true for case 6, since $\psi_o$ is small. Thus

$$\frac{af}{bc} < 1.$$
Table H-2: The Cases mapped to the equations of $\phi$ with respect to the somersault angle $\theta$.

The value of $\theta$ for each of the cases is the sum of the two integrals $\mathrm{In}1$ and $\mathrm{In}2$. These sums are Equation (4-13) through to Equation (4-18).

<table>
<thead>
<tr>
<th>Case</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>$k^2$</th>
<th>$y$</th>
<th>$\mathrm{In}_1 = -g \int \frac{du}{(1-u^2)\sqrt{bu^2 - (c - fu^2)}}$</th>
<th>$\mathrm{In}<em>2 = I</em>{x, y} \int \frac{du}{\sqrt{bu^2 - (c - du^2)}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1 - $\frac{af}{bc}$</td>
<td>$y = x \sqrt{\frac{f}{c}}$</td>
<td>$\frac{f}{bc} \left[ \Pi \left( bc - af, b(c - f) \cos \left( \sqrt{1 - y^2} \right) \right) \right] \frac{\sqrt{c}}{\sqrt{bc}}$</td>
<td>$-\frac{I_{x, y}}{bc} \sqrt{bc} \left[ F \left( \sin^{-1} \left( \sqrt{1 - y^2} \right) \right) \right] \frac{\sqrt{c}}{\sqrt{bc}}$</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>$\frac{bc}{bc - af}$</td>
<td>$\left[ \frac{f}{\sqrt{bc - af}} \left[ \Pi \left( \frac{c}{c - f}, \sin^{-1} \left( \sqrt{1 - y^2} \right) \right) \frac{bc}{bc - af} \right] \right] \frac{\sqrt{c}}{\sqrt{bc}}$</td>
<td>$-\frac{I_{x, y}}{bc} \sqrt{bc} \left[ F \left( \sin^{-1} \left( \sqrt{1 - y^2} \right) \right) \right] \frac{\sqrt{c}}{\sqrt{bc}}$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>$\frac{bc}{af}$</td>
<td>$\sqrt{-bc} \left[ \Pi \left( a, b \sin^{-1} y, \frac{af}{bc} \right) \right] \frac{\sqrt{a}}{\sqrt{bc}}$</td>
<td>$\frac{I_{x, y}}{\sqrt{-bc}} \left[ F \left( \sin^{-1} \left( \frac{af}{bc} \right) \right) \right] \frac{\sqrt{a}}{\sqrt{bc}}$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>1 - $\frac{af}{bc}$</td>
<td>$y = \frac{\sqrt{b}}{\sqrt{a}} \cos \phi$</td>
<td>$\left( \frac{b}{\sqrt{af}} \left[ \Pi \left( \frac{af}{a - bc}, \sin^{-1} \left( \sqrt{1 - y^2} \right) \frac{af}{af - bc} \right) \right] \right) \frac{\sqrt{a}}{\sqrt{af}}$</td>
<td>$-\frac{I_{x, y}}{\sqrt{af}} \sqrt{af} \left[ F \left( \sin^{-1} \left( \sqrt{1 - y^2} \right) \frac{af}{af - bc} \right) \right] \frac{\sqrt{a}}{\sqrt{af}}$</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>$\frac{af}{af - bc}$</td>
<td>$\left( \frac{b}{\sqrt{af - bc}} \left[ \Pi \left( \frac{a}{a - b}, \sin^{-1} \left( \sqrt{1 - y^2} \right) \frac{af}{af - bc} \right) \right] \right) \frac{\sqrt{a}}{\sqrt{af - bc}}$</td>
<td>$-\frac{I_{x, y}}{\sqrt{af}} \left[ F \left( \sin^{-1} \left( \frac{af}{af - bc} \right) \right) \right] \frac{\sqrt{a}}{\sqrt{af - bc}}$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>$\frac{af}{af - bc}$</td>
<td>$\left( \frac{b}{\sqrt{af - bc}} \left[ \Pi \left( \frac{a}{a - b}, \sin^{-1} \left( \sqrt{1 - y^2} \right) \frac{af}{af - bc} \right) \right] \right) \frac{\sqrt{a}}{\sqrt{af - bc}}$</td>
<td>$-\frac{I_{x, y}}{\sqrt{af}} \left[ F \left( \sin^{-1} \left( \frac{af}{af - bc} \right) \right) \right] \frac{\sqrt{a}}{\sqrt{af - bc}}$</td>
<td></td>
</tr>
</tbody>
</table>
Evaluation of the periods of $\phi$ with respect to $\theta$

From Table H-2, $\phi$ with respect to the somersault angle $\theta$, is known. To determine the somersault angle required for a full oscillation of $\phi$ then the integrals may be evaluated with appropriate bounds. Now, since $\phi$ oscillates it is sufficient to evaluate the integrals in Table H-2 for between the extremes of $\phi$, known from Equation (4-4) and Section 4.2.3, and then use symmetry to somersault required for the full oscillation of $\phi$.

The bounds are chosen to be the minimum to maximum value of $y$ (as given in Table H-2), so that positive somersault is given by the integrals; somersault only ever increases in value. The somersault required then needs to be multiplied by 2 for a case displaying continuous twist and multiplied by 4 for a case displaying oscillating twist.

**Case 3**

The twist is continuous and so a half twist will be completed in twice the somersault required, when $\psi_o = 0$, for $\phi$ to move from $\phi_o$ (which is its maximum value) to its minimum value the bounds are

$$\phi = \sin^{-1}\left(\frac{e}{\sqrt{b}}\right) = \cos^{-1}\left(\frac{a}{\sqrt{b}}\right)$$

$$\phi = \sin^{-1}\left(\frac{e}{\sqrt{f}}\right) = \cos^{-1}\left(\frac{c}{\sqrt{f}}\right)$$

These correspond to bounds of $y$ of

$$\frac{af}{\sqrt{bc}}$$

and 1 respectively.

Substituting these bounds into Equation (4-13) gives,
Applying the bounds of $\pi$
The somersault required for one cycle of $\phi$ is thus

$$\theta = \left( \frac{f}{\sqrt{bc}} \right) \left[ \prod \left( \frac{bc - af}{b(c - f)} \right)^{0.1} \frac{1}{\sqrt{bc}} \right] + I_{\psi} \left( \frac{1}{\sqrt{bc}} \right)$$

The somersault required for one cycle of $\phi$ is thus

$$\frac{-2f}{\sqrt{bc}} \prod \left( \frac{bc - af}{b(c - f)} \right)^{0.1} \frac{1}{\sqrt{bc}} + \frac{2}{\sqrt{bc}} F \left( \frac{\pi}{2}, 1 \right)$$

**Case 4**

The twist is oscillating and so a full oscillation will occur in four times the somersault required for $\phi$ to move from $\pi/2$ (when $\psi$ is the smallest value) to its minimum value.

Applying the bounds of $\pi/2$

$$\phi = \sin^{-1} \left( \frac{g}{\sqrt{f}} \right) = \cos^{-1} \left( \frac{c}{\sqrt{f}} \right)$$

These correspond to bounds of $y$ of 0 to 1.

Applying these bounds to Equation (4-14) gives
Thus the period is
\[
\frac{-4d}{\sqrt{bc - af}} \Pi \left( \frac{c}{c - f}, \frac{\pi}{2}, \frac{bc}{bc - af} \right) + \frac{4 I_{\cos} I_{\cos} \sqrt{bc - af}}{bc} F \left( \frac{\pi}{2}, \frac{bc}{bc - af} \right)
\]

Case 6

The twist is oscillating and so a full oscillation will occur in four times the somersault required when \( \psi_{o} = 0 \), for \( \phi \) to move from \( \pi/2 \) to \( \phi_{o} \) (which is its minimum value).

Applying the bounds of

\( \pi/2 \)

\[
\phi = \sin^{-1} \left( \sqrt{\frac{g}{b}} \right) = \cos^{-1} \left( \frac{a}{\sqrt{b}} \right)
\]

These correspond to bounds of \( y \) of 0 to 1

Applying these bounds to Equation (4-15) gives

\[
\frac{-g}{\sqrt{bc - af}} \left[ \Pi \left( \frac{a}{b}, \sin^{-1} \frac{af}{bc} \right), + \frac{I_{\cos} I_{\cos} \sqrt{bc - af}}{bc} F \left( \sin^{-1} \frac{af}{bc} \right) \right]
\]

\[
\frac{-g}{\sqrt{bc - af}} \left[ \Pi \left( \frac{a}{b}, \sin^{-1} 1, \frac{af}{bc} \right) + \frac{I_{\cos} I_{\cos} \sqrt{bc - af}}{bc} F \left( \sin^{-1} 1, \frac{af}{bc} \right) \right]
\]

\[
\frac{-g}{\sqrt{bc - af}} \left[ -\Pi \left( \frac{a}{b}, \sin^{-1} 0, \frac{af}{bc} \right) + \frac{I_{\cos} I_{\cos} \sqrt{bc - af}}{bc} F \left( \sin^{-1} 0, \frac{af}{bc} \right) \right]
\]
\[
\theta = \frac{-g}{\sqrt{-bc}} \left[ \pi \left( \frac{a}{b^2} \frac{\pi}{2} \frac{af}{bc} \right) + \frac{J_{\mu} J_{\nu}}{\sqrt{-bc}} \left( \frac{\pi}{2} \frac{af}{bc} \right) \right] \\
\theta = \frac{-g}{\sqrt{-bc}} \left[ \pi \left( \frac{a}{b^2} \frac{\pi}{2} \frac{af}{bc} \right) + \frac{J_{\mu} J_{\nu}}{\sqrt{-bc}} \left( \frac{\pi}{2} \frac{af}{bc} \right) \right]
\]

Thus the period is

\[
\frac{-4g}{\sqrt{-bc}} \left[ \pi \left( \frac{a}{b^2} \frac{\pi}{2} \frac{af}{bc} \right) + \frac{4J_{\mu} J_{\nu}}{\sqrt{-bc}} \left( \frac{\pi}{2} \frac{af}{bc} \right) \right]
\]

**Case 7**

The twist is continuous and so a half twist will be completed in twice the somersault required when \(\psi_o = 0\), for \(\phi\) to move from its maximum to minimum (\(\phi_o\)) value. Applying the bounds of \(\phi_o\)

\[
\phi = \sin^{-1} \left( \frac{g}{\sqrt{b}} \right) = \cos^{-1} \left( \frac{a}{\sqrt{b}} \right)
\]

These correspond to bounds of \(y\) of

\[
\frac{bc}{\sqrt{af}}
\]

and 1 respectively.

Applying these bounds to Equation (4-16) gives
Applying these bounds to Equation (4-17) gives

\[
\theta = \left( \frac{b}{\sqrt{af}} \right) \left[ \pi \left( \frac{af - bc}{f(a-b)} \sin^{-1} \left( \frac{\sqrt{af(1-y^2)}}{af - bc} \right) \right) \right]^1 + \frac{I_{\alpha} I_{\gamma}}{\sqrt{af}} \left[ F \left( \sin^{-1} \left( \frac{\sqrt{af(1-y^2)}}{af - bc} \right) \right) \right]^1
\]

Thus the period is

\[
-\frac{2b}{\sqrt{af}} \left( \frac{af - bc}{f(a-b)} \frac{\pi}{2} \frac{1}{af} - \frac{bc}{af} \right) + \frac{2I_{\alpha} I_{\gamma}}{\sqrt{af}} F \left( \frac{\pi}{2} \frac{1}{af} - \frac{bc}{af} \right)
\]

**Case 8**

The twist is oscillating and so a full oscillation will occur in four times the somersault required for \( \phi \) to move from \( \pi/2 \) to \( \phi_n \). Applying bounds of \( \pi/2 \)

\[
\phi = \sin^{-1} \left( \frac{\sqrt{a}}{b} \right) = \cos^{-1} \left( \frac{\sqrt{a}}{b} \right)
\]

These correspond to bounds of \( y \) of 0 to 1.

Applying these bounds to Equation (4-17) gives
\[ \theta = \left( \frac{b}{\sqrt{af - bc}} \right) \left( \prod_{a-b} \sin^{-1}(1-0) \frac{af}{af - bc} \right) + \frac{-I_{xx}I_{yy}}{af} \sin^{-1}(1-0) \frac{af}{af - bc} \]

Thus the period is

\[ \left( -\frac{4b}{\sqrt{af - bc}} \right) \prod_{a-b} \left( \frac{a}{a-b} \frac{\pi}{2} \frac{af}{af - bc} \right) + \frac{4I_{xx}I_{yy}}{af} \frac{af}{2} \frac{af}{af - bc} \]

**Evaluation of relative difficulty**

Relative difficulty of a twisting somersault in the twisting phase compared to a pure somersault in the same posture was determined in Section 4.2.7. This section gives the full derivation of the relative difficulties from Equation (4-22) for Case 3 and Equation (4-23) for Case 7. Only Case 3 and 7 need to be considered since they are the only cases that display continuous twist.

**Case 3**

From Equation (4-22)

\[ \frac{[\theta_{\text{Pure}}_{\text{Start}}]_{\text{End}}}{[\theta_{\text{Twisting}}_{\text{Start}}]_{\text{End}}} = \frac{I_{xx}I_{yy}}{I_{xx}I_{yy} + b} \left[ \prod_{a-b} \left( \frac{af}{bc} \right) \sin^{-1} \left( \frac{f}{c} \cos \phi \right) \right]_{\text{Start}}^{\text{End}} \]

Where

\[ k = \sqrt{1 - \frac{af}{bc}}. \]

The start is when \( t = 0 \) and \( \phi = \phi_o \); the end is where

\[ \phi = \cos^{-1} \left( \frac{c}{f} \right). \]
which means from the equation for $\phi$ with respect to time for Case 3 from Appendix E.7 is

$$1 = D_n \left( \frac{H \sqrt{bc}}{I_{in} I_{zz}} t - D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right) \right)$$

$$H \sqrt{bc} = D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right) = 0$$

$$D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right)$$

$$t = \frac{H \sqrt{bc}}{I_{in} I_{zz}}$$

Substituting these start and end values into Equation (4-22) gives

$$I_{in} I_{zz} \left[ \frac{D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right) - 0}{H \sqrt{bc}} \right]$$

$$= \left( I_{in} I_{zz} + b \right) D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right) - f \left( \frac{af - bc}{bg} \right) \left( \sin^{-1} \left( \frac{D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right)}{k} \right) \right)$$

To assist with the evaluation using Mathematica, $D_n^{-1}$ will be rewritten in terms of the elliptical integral of the first kind: $\text{F}(\sin^{-1} y, k^2)$, see Appendix B.3.4. That is,

$$D_n^{-1} \left( \frac{\mathcal{F}}{c} \cos \phi, k \right)$$

$$= \text{Sn}^{-1} \left( \frac{1 - \frac{f \cos^2 \phi}{c}}{1 + \frac{af}{bc}}, k \right)$$

$$= \text{Sn}^{-1} \left( \frac{bc - f \cos^2 \phi}{bc - af}, k \right)$$

$$= \text{F} \left( \sin^{-1} \frac{bc - f \cos^2 \phi}{bc - af}, k^2 \right)$$
When then have the relative difficulty over a period of $\phi$ as

$$I_{\text{as}}I_{\text{zz}}F \left( \sin^{-1} \sqrt{\frac{b(c-f \cos^2 \phi)}{bc-af}}, k^2 \right)$$

$$= (I_{\text{as}}I_{\text{zz}} + b)F \left( \sin^{-1} \sqrt{\frac{b(c-f \cos^2 \phi)}{bc-af}}, k^2 \right) - f \Pi \left( a \frac{f \cos^2 \phi - c}{bc-af}, \sin^{-1} \left( \sqrt{\frac{b(c-f \cos^2 \phi)}{bc-af}} \right), k \right), k$$

Where $\psi_o = 0$

$$= I_{\text{as}}I_{\text{zz}}F \left( \frac{\pi}{2}, 1 - \frac{f \cos^2 \phi}{c} \right)$$

This is Equation (4-24)

**Case 7**

From Equation (4-23)

$$\left[ \theta_{\text{start}} \right]_{\text{end}} - \left[ \theta_{\text{end}} \right]_{\text{start}} = I_{\text{as}}I_{\text{zz}}F \left( \sin^{-1} \left( \sqrt{\frac{b(c-f \cos^2 \phi)}{bc-af}} \right), k \right)$$

$$= (I_{\text{as}}I_{\text{zz}} + b)F \left[ \sin^{-1} \left( \sqrt{\frac{b(c-f \cos^2 \phi)}{bc-af}} \right), k \right]_{\text{start}} + Hb \sqrt{\frac{af}{g}} \Pi \left( a \frac{f \cos^2 \phi - c}{bc-af}, \sin^{-1} \left( \sqrt{\frac{b(c-f \cos^2 \phi)}{bc-af}} \right), k \right), k$$

where

$$k = \sqrt{1 - \frac{bc}{af}}$$

The start is when $t=0$ and $\phi = \phi_0$; the end is where

$$\phi = \cos^{-1} \left( \sqrt{\frac{c}{f}} \right),$$

which means from the equation for $\phi$ with respect to time Case 7 from Appendix E.7 is
Thus the relative difficulty over the twisting phase is

\[
\frac{[\theta_{\text{pure}}]_{\text{start}}}{[\theta_{\text{terminal}}]_{\text{start}}} = \left[ \frac{Dn^{-1}\left( \frac{b}{a} \cos \phi_x, k \right) - Dn^{-1}\left( \frac{bc}{af}, k \right)}{H \sqrt{af} \over I_{I_0} I_{I_0} I_{I_0}} - 0 \right]
\]

\[
= \left( I_{I_0} I_{I_0} + b \right) \left[ Dn^{-1}\left( \frac{b}{a} \cos \phi_x, k \right) - Dn^{-1}\left( \frac{bc}{af}, k \right) \right] \left[ \Pi \left( \frac{bc - af}{fg} \right) \sin^2 \left( \frac{Dn^{-1}\left( \frac{bc}{af}, k \right)}{k} \right) \right] 
\]

When \( \psi_x = 0 \) this reduces to

\[
\left( I_{I_0} I_{I_0} + b \right) \left[ Dn^{-1}\left( \frac{c}{f \cos \phi_x}, k \right) - b \right] \left[ \Pi \left( \frac{c - f \cos \phi_x}{f \sin \phi_x} \right) \sin^2 \left( \frac{Dn^{-1}\left( \frac{c}{f \cos \phi_x}, k \right)}{k} \right) \right]
\]

\[
= \left( I_{I_0} I_{I_0} + b \right) Dn^{-1}\left( \frac{c}{f \cos \phi_x}, k \right) - b \left[ \Pi \left( \frac{c - f \cos \phi_x}{f \sin \phi_x} \right) \sin^2 \left( \frac{Dn^{-1}\left( \frac{c}{f \cos \phi_x}, k \right)}{k} \right) \right]
\]
To assist with the evaluation using Mathematica, $D_n - 1$ will be re-written in terms of the elliptical integral of the first kind: $F(\sin^{-1}y, k^2)$, see Appendix B.3.4. Remembering that $\psi_o$ has been set to zero, then

$$D_n^{-1}\left(\frac{c}{f \cos^2 \phi_o}, k\right) = Sn^{-1}\left(\frac{1 - \frac{c}{f \cos^2 \phi_o}}{1 - \frac{bc}{af}}, k\right)$$

$$= Sn^{-1}\left(\frac{af - \frac{ac}{\cos^2 \phi_o}}{af - bc}, k\right)$$

$$= Sn^{-1}(1, k)$$

$$= F(\sin^{-1}1, k^2)$$

$$= F\left(\frac{\pi}{2}, k^2\right)$$

This means the relative difficulty over a period of $\phi$ is

$$\frac{I_{ss} I_{zz} F\left(\frac{\pi}{2}, k^2\right)}{(I_{ss} I_{zz} + b) F\left(\frac{\pi}{2}, k^2\right)} = \frac{I_{ss} I_{zz} F\left(\frac{\pi}{2}, k^2\right)}{(I_{ss} I_{zz} + b) F\left(\frac{\pi}{2}, k^2\right)} - b \prod \left(\frac{f - b, \pi}{f, \frac{\pi}{2}, k}\right)$$

$$= \frac{I_{ss} I_{zz} F\left(\frac{\pi}{2}, k^2\right)}{(I_{ss} I_{zz} + b) F\left(\frac{\pi}{2}, k^2\right)} - b \prod \left(\frac{f - b, \pi}{f, \frac{\pi}{2}, k}\right)$$

This is Equation (4-25)

**Equation (4-27) from (4-26)**

From the text just after Equation (4-26) the angular momentum possessed by the 17-segment model may be written as

$$\Gamma_{\text{H}} = \sum_{i=1}^{17} m \left(\Gamma_{\text{Ref}} \times (\vec{R}_{\text{Ref}} D_i + \vec{R}_{\text{Ref}} D_i)\right) + \left(\Gamma_{\text{Ref}} \times (\vec{R}_{\text{Ref}} \times F_{\text{Ref}} R_{\text{local}, i})\right) + \left(\Gamma_{\text{Ref}} \times \vec{R}_{\text{Ref}} \times a_i\right)$$
The following facts will allow the above equation to be simplified:

1. The derivative of a rotation matrix (Appendix B.1.2) is

\[ \frac{\partial R_{\text{Ref}}}{\partial \gamma} = R_{\text{Ref}} \left[ \frac{\partial \theta_{\text{Ref}}}{\partial \gamma} \right] \]

2. The cross product of two vectors rotated by the same rotation matrix will be equal to the rotates cross product, that is

\[ R_{\text{Ref}} \times R_{\text{Ref}} = R_{\text{Ref}} \times \frac{\partial \theta_{\text{Ref}}}{\partial \gamma} \]

This means

\[ \frac{\partial R_{\text{Ref}}}{\partial \gamma} = R_{\text{Ref}} \left[ \frac{\partial \theta_{\text{Ref}}}{\partial \gamma} \right] \]

3. \( D_i \) is a vector in three dimensions, and may be written as

\[ D_i = D_{i_x} \mathbf{i} + D_{i_y} \mathbf{j} + D_{i_z} \mathbf{k} \]

thus

\[ D_i \times \left[ \frac{\partial R_{\text{Ref}}}{\partial \gamma} \right] \gamma \times D_i = \begin{bmatrix} D_{i_x}^2 + D_{i_z}^2 & -D_{i_x}D_{i_z} & -D_{i_x}D_{i_c} \\ -D_{i_x}D_{i_z} & D_{i_x}^2 + D_{i_z}^2 & -D_{i_x}D_{i_c} \\ -D_{i_x}D_{i_c} & -D_{i_x}D_{i_c} & D_{i_x}^2 + D_{i_c}^2 \end{bmatrix} \]

Using these facts the angular momentum equation may then be written as

\[ H = \sum_{i=1}^{17} \left( m_i \left[ \frac{\partial R_{\text{Ref}}}{\partial \gamma} \right] \gamma \times D_i \right) \]

Using the parallel axis theorem (Appendix B.2.3), and letting \( I_{\text{Overall}} \) be the moment of inertia for the body as a whole in a specified posture gives

\[ H = \frac{\partial R_{\text{Ref}}}{\partial \gamma} I_{\text{Overall}} \left[ \frac{\partial R_{\text{Ref}}}{\partial \gamma} \right] \gamma + \frac{\partial R_{\text{Ref}}}{\partial \gamma} \sum_{i=1}^{17} (m_i \mathbf{D}_i) \times \frac{\partial \theta_{\text{Ref}}}{\partial \gamma} \]

This is Equation (4-27)
**Equation (4-32) from Equation (4-31)**

The equations of the planer 3-segment model are derived from Equation (4-31) using the model illustrated in Figure 4-31.

From Equation (4-31)

\[
\gamma_{H=0} = - \int_{\text{Start}}^{\text{End}} \sum_{i=1}^{3} \left( m_i \mathbf{D}_i \times \dot{\mathbf{D}}_i + I_i \ddot{\alpha}_i \right) dt
\]

To evaluate this integral it is necessary to write the integrand as a function of the variable \( \alpha_i \) and the inertial constants.

**Step 1:** The value of \( \sum m_i (\mathbf{D}_i \times \dot{\mathbf{D}}_i) \) as a function of the \( \alpha_i \)'s and the inertial constants.

In this step the frame of reference will be the local frame of the reference segment, that is segment 2, since this is the way in which the \( \alpha_i \) values were defined.

The location of each segment’s centre of gravity in terms is

\[
CG_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

since the origin of segment 2’s local frame is at its Centre of Gravity.

\[
CG_1 = V_{21}^{-1} R_1 V_1 = \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} \begin{bmatrix} \cos \alpha_1 & -\sin \alpha_1 \\ \sin \alpha_1 & \cos \alpha_1 \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix}
\]

\[
CG_3 = V_{23}^{-1} R_3 V_3 = \begin{bmatrix} V_{3x} \\ V_{3y} \end{bmatrix} \begin{bmatrix} \cos \alpha_3 & -\sin \alpha_3 \\ \sin \alpha_3 & \cos \alpha_3 \end{bmatrix} \begin{bmatrix} V_{3x} \\ V_{3y} \end{bmatrix}
\]

The location of the overall centre of gravity with respect to segment 2’s local frame is

\[
CG_{\text{Overall}} = \frac{m_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} V_{21x} \\ V_{21y} \end{bmatrix} - \begin{bmatrix} \cos \alpha_1 & -\sin \alpha_1 \\ \sin \alpha_1 & \cos \alpha_1 \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} + m_3 \begin{bmatrix} V_{23x} \\ V_{23y} \end{bmatrix} - \begin{bmatrix} \cos \alpha_3 & -\sin \alpha_3 \\ \sin \alpha_3 & \cos \alpha_3 \end{bmatrix} \begin{bmatrix} V_{3x} \\ V_{3y} \end{bmatrix}}{m_1 + m_2 + m_3}
\]

\[
= \frac{m_1 \begin{bmatrix} V_{21x} \\ V_{21y} \end{bmatrix} - \begin{bmatrix} \cos \alpha_1 & -\sin \alpha_1 \\ \sin \alpha_1 & \cos \alpha_1 \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} + m_3 \begin{bmatrix} V_{23x} \\ V_{23y} \end{bmatrix} - \begin{bmatrix} \cos \alpha_3 & -\sin \alpha_3 \\ \sin \alpha_3 & \cos \alpha_3 \end{bmatrix} \begin{bmatrix} V_{3x} \\ V_{3y} \end{bmatrix}}{m_1 + m_2 + m_3}
\]

The vectors \( \mathbf{D}_i \) are from the centre of gravity of each segment to the overall centre of gravity of the 3 segment body known with respect to the local frame of segment 2. Thus,
\[ D_1 = CG_1 - CG_{\text{bend}} \]
\[ = \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} m_1 V_{1x} + m_2 V_{2x} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \]

\[ D_2 = CG_2 - CG_{\text{bend}} \]
\[ = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} m_1 \begin{bmatrix} V_{2y} V_{1x} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \]

\[ D_3 = CG_3 - CG_{\text{bend}} \]
\[ = \begin{bmatrix} V_{1y} \\ V_{2y} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} m_1 V_{1y} + m_2 V_{2y} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} V_{1y} \\ V_{2y} \end{bmatrix} \]

Each value of \( D_1, D_2, \) and \( D_3 \) have a similar form of
\[ \frac{d}{dt} \begin{bmatrix} V_{ax} \\ V_{ay} \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} V_{ax} \\ V_{ay} \end{bmatrix} \]

This similarity may be used to assist in determining the derivatives of \( D_1):\]
\[ \frac{d}{dt} \begin{bmatrix} m_1 + m_2 \\ m_1 + m_2 + m_3 \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} m_1 V_{1x} + m_2 V_{2x} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \]

\[ \frac{d}{dt} \begin{bmatrix} m_1 + m_3 \\ m_1 + m_2 + m_3 \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} m_1 V_{1x} + m_2 V_{2x} \end{bmatrix} \begin{bmatrix} \cos \alpha \sin \alpha \\ -\sin \alpha \cos \alpha \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{2y} \end{bmatrix} \]
\[ \dot{D}_3 = \frac{d}{dt} \left( \begin{array}{c} -m_i \\ m_i + m_j + m_k \end{array} \right) \left( \begin{array}{c} V_{121} \\ V_{121} \end{array} \right) \left[ \begin{array}{cc} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{array} \right] \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) + \frac{d}{dt} \left( \begin{array}{c} m_i + m_j \\ m_i + m_j + m_k \end{array} \right) \left( \begin{array}{c} V_{231} \\ V_{231} \end{array} \right) \left[ \begin{array}{cc} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{array} \right] \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \]

All three \( D_i \)'s and their derivatives have a similar form differing only in the mass fractions.

Using a general form with the constants \( M_A \) and \( M_B \) standing in for the mass fractions which multiply by

\[ \left( \begin{array}{c} V_{121} \\ V_{121} \end{array} \right) \left[ \begin{array}{cc} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{array} \right] \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \] and

\[ \left( \begin{array}{c} V_{231} \\ V_{231} \end{array} \right) \left[ \begin{array}{cc} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{array} \right] \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \]

respectively.

Then the general expression of \( D_i \times \dot{D}_i \) can be written as

\[ M_A \left( \begin{array}{c} V_{121} \\ V_{121} \end{array} \right) \left[ \begin{array}{cc} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{array} \right] \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) + M_B \left( \begin{array}{c} V_{231} \\ V_{231} \end{array} \right) \left[ \begin{array}{cc} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{array} \right] \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \]

\[ \times M_A \dot{\alpha}_i \left( \begin{array}{c} \cos \alpha_i \\ \sin \alpha_i \end{array} \right) \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) + M_B \dot{\alpha}_i \left( \begin{array}{c} \cos \alpha_i \\ \sin \alpha_i \end{array} \right) \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \]

\[ = M_A \dot{\alpha}_i \left( \begin{array}{c} \cos \alpha_i \\ \sin \alpha_i \end{array} \right) \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \times \left( \begin{array}{c} \cos \alpha_i \\ \sin \alpha_i \end{array} \right) \left( \begin{array}{c} V_{11x} \\ V_{11y} \end{array} \right) \]

Substituting in the appropriate mass fractions to the general form of \( D_i \), then the

The sum \( \sum m_i (D_i \times \dot{D}_i) \) becomes
\[
\sum m_i (D_i \times D_i) = \left( \frac{m_i (m_i + m_j) \alpha_i + m_i (m_i + m_j) \alpha_j}{m_i + m_j} \right) \left( v_{ix}^2 + v_{iy}^2 + \sin \alpha_i (V_{ix} V_{ix} - V_{iy} V_{ix}) + \cos \alpha_i (V_{ix} V_{ix} + V_{iy} V_{iy}) \right)
\]

\[
+ \left( \frac{m_j (m_i + m_j) \alpha_i + m_j (m_i + m_j) \alpha_j}{m_i + m_j} \right) \left( v_{ix}^2 + v_{iy}^2 + \sin \alpha_j (V_{ix} V_{ix} - V_{iy} V_{ix}) + \cos \alpha_j (V_{ix} V_{ix} + V_{iy} V_{iy}) \right)
\]

\[
+ \left( \frac{m_i (m_i + m_j) \alpha_i + m_i (m_i + m_j) \alpha_j}{m_i + m_j} \right) \left( v_{ix}^2 + v_{iy}^2 + \sin \alpha_i (V_{ix} V_{ix} - V_{iy} V_{ix}) + \cos \alpha_i (V_{ix} V_{ix} + V_{iy} V_{iy}) \right)
\]

\[
= \left( \frac{m_i (m_i + m_j)}{m_i + m_j} \right) v_{ix}^2 + v_{iy}^2 + \sin \alpha_i (V_{ix} V_{ix} - V_{iy} V_{ix}) + \cos \alpha_i (V_{ix} V_{ix} + V_{iy} V_{iy})
\]

\[
+ \left( \frac{m_j (m_i + m_j)}{m_i + m_j} \right) v_{ix}^2 + v_{iy}^2 + \sin \alpha_j (V_{ix} V_{ix} - V_{iy} V_{ix}) + \cos \alpha_j (V_{ix} V_{ix} + V_{iy} V_{iy})
\]

\[
+ \left( \frac{m_i (m_i + m_j)}{m_i + m_j} \right) v_{ix}^2 + v_{iy}^2 + \sin \alpha_i (V_{ix} V_{ix} - V_{iy} V_{ix}) + \cos \alpha_i (V_{ix} V_{ix} + V_{iy} V_{iy})
\]

\[
= \alpha_i (A + C \sin \alpha_i + D \cos \alpha_i + G \sin (\alpha_i - \alpha_j) + L \cos (\alpha_i - \alpha_j))
\]

\[
+ \alpha_j (B + E \sin \alpha_j - F \cos \alpha_j + G \sin (\alpha_i - \alpha_j) + L \cos (\alpha_i - \alpha_j))
\]

where,

\[
A = \left( \frac{m_i (m_i + m_j)}{m_i + m_j} \right) v_{ix}^2 + v_{iy}^2
\]

\[
B = \left( \frac{m_j (m_i + m_j)}{m_i + m_j} \right) v_{ix}^2 + v_{iy}^2
\]

\[
C = \left( \frac{m_i (m_i + m_j)}{m_i + m_j} \right) V_{ix} V_{ix} - V_{ix} V_{ix} + \left( \frac{-m_i}{m_i + m_j} \right) V_{ix} V_{ix} - V_{ix} V_{ix}
\]

\[
D = \left( \frac{m_i (m_i + m_j)}{m_i + m_j} \right) V_{ix} V_{ix} + V_{iy} V_{iy} + \left( \frac{-m_i}{m_i + m_j} \right) V_{ix} V_{ix} + V_{iy} V_{iy}
\]

\[
E = \left( \frac{m_j (m_i + m_j)}{m_i + m_j} \right) V_{ix} V_{ix} - V_{ix} V_{ix} + \left( \frac{-m_j}{m_i + m_j} \right) V_{ix} V_{ix} - V_{ix} V_{ix}
\]

\[
F = \left( \frac{m_j (m_i + m_j)}{m_i + m_j} \right) V_{ix} V_{ix} + V_{iy} V_{iy} + \left( \frac{-m_j}{m_i + m_j} \right) V_{ix} V_{ix} + V_{iy} V_{iy}
\]

\[
G = \left( \frac{-m_i}{m_i + m_j} \right) V_{ix} V_{ix} - V_{ix} V_{ix}
\]

\[
L = \left( \frac{-m_j}{m_i + m_j} \right) V_{ix} V_{ix} + V_{iy} V_{iy}
\]

The second step is to determining the value of \( I_{\text{Overall}} \) as a function of the \( \alpha_i \)'s and the inertial constants.
The value of the overall moment of inertia using the parallel axis theorem is

$$I_{\text{overall}} = I_1 + I_2 + I_3 + m_i|D_1|^2 + m_2|D_2|^2 + m_3|D_3|^2$$

Using the general form of $D_i$

$$|D_i|^2 =$$

$$M_i \left[ \begin{array}{cc} V_{ix}^2 - \sin \alpha_i \cos \alpha_i & V_{iy} \sin \alpha_i \\ \sin \alpha_i \cos \alpha_i & V_{iz}^2 \end{array} \right] + M_{ii} \left[ \begin{array}{cc} V_{ix}^2 - \sin \alpha_i \cos \alpha_i & V_{iy} \sin \alpha_i \\ \sin \alpha_i \cos \alpha_i & V_{iz}^2 \end{array} \right]$$

$$= M_i \left( V_{ix}^2 - V_{ix} \cos \alpha_i + V_{iy} \sin \alpha_i \right)^2 + M_{ii} \left( V_{ix}^2 - V_{ix} \cos \alpha_i + V_{iy} \sin \alpha_i \right)^2$$

$$+ 2M_i M_{ii} \left( V_{ix} - V_{iy} \sin \alpha_i \cos \alpha_i \right)$$

$$= M_i \left( V_{ix}^2 + V_{iy}^2 + V_{iz}^2 \right)^2 + M_{ii} \left( V_{ix}^2 + V_{iy}^2 + V_{iz}^2 \right)^2$$

$$+ 2M_i M_{ii} \left( V_{ix} V_{iy} + V_{ix} V_{iy} \right)$$

Therefore,

$$I_{\text{overall}} = I_1 + I_2 + I_3 + m_i|D_1|^2 + m_2|D_2|^2 + m_3|D_3|^2$$
\[ I_{\text{overall}} = I_1 + I_2 + I_3 \\
+ \left( \frac{m_1(m_2 + m_3)^2 + m_2m_4^2 + m_3m_4^2}{m_1 + m_2 + m_3} \right) \left( V_{1x}^2 + V_{2x}^2 + V_{3x}^2 + V_{1f}^2 \right) \\
+ \left( \frac{m_2(m_1 + m_3)^2 + m_2m_4^2 + m_3m_4^2}{m_1 + m_2 + m_3} \right) \left( V_{2x}^2 + V_{2y}^2 + V_{3y}^2 + V_{2f}^2 \right) \\
+ 2 \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{1x}V_{2x} + V_{2x}V_{2y} \right) \\
+ \sin(\alpha_1 - \alpha_2) \left( \frac{-m_1m_2(m_1 + m_3) + m_1m_2m_4 - m_2m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{1x}V_{3y} - V_{1x}V_{3f} \right) \\
+ \cos(\alpha_1 - \alpha_2) \left( \frac{-m_1m_2(m_1 + m_3) + m_1m_2m_4 - m_2m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{1x}V_{3x} + V_{2x}V_{3y} \right) \\
+ \sin(\alpha_1 - \alpha_2) \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{21x}V_{1y} - V_{21y}V_{1x} \right) \\
+ 2 \left( \frac{m_1(m_2 + m_3)^2 + m_2m_4^2 + m_3m_4^2}{m_1 + m_2 + m_3} \right) \left( V_{21x}V_{1y} + V_{21y}V_{1x} \right) \\
+ 2 \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{21y}V_{2y} + V_{23x}V_{1x} \right) \\
+ \sin(\alpha_1 - \alpha_2) \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{21y}V_{3y} - V_{21x}V_{3f} \right) \\
+ 2 \left( \frac{m_1(m_1 + m_3)^2 + m_2m_4^2 + m_3m_4^2}{m_1 + m_2 + m_3} \right) \left( V_{23x}V_{3y} - V_{23y}V_{3x} \right) \\
+ 2 \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{3y}V_{3f} - V_{3x}V_{3f} \right) \\
+ \cos(\alpha_1 - \alpha_2) \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{23y}V_{3y} + V_{23x}V_{3x} \right) \\
+ 2 \left( \frac{m_1(m_1 + m_3)^2 + m_2m_4^2 + m_3m_4^2}{m_1 + m_2 + m_3} \right) \left( V_{3y}V_{2y} + V_{3y}V_{2x} \right) \\
+ 2 \left( \frac{-m_1m_2(m_2 + m_3) + m_2m_3m_4 - m_1m_3(m_1 + m_2)}{(m_1 + m_2 + m_3)^2} \right) \left( V_{3x}V_{2y} + V_{3x}V_{2x} \right) \]
\[ I_{\text{overall}} = I_1 + I_2 + I_3 + \left( m_1 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) V_{21x}^2 + V_{21y}^2 + V_{1x}^2 + V_{1y}^2 \right) \\
+ \left( m_2 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) V_{23x}^2 + V_{23y}^2 + V_{3x}^2 + V_{3y}^2 \right) \\
+ 2 \left( \frac{-m_3}{m_1 + m_2 + m_3} \right) \left( V_{21x} V_{23x} + V_{21y} V_{23y} \right) \\
+ \sin(\alpha_1 - \alpha_2) \left( 2 \left( \frac{-m_3}{m_1 + m_2 + m_3} \right) \left( V_{1x} V_{3y} - V_{3x} V_{1y} \right) \right) \\
+ \cos(\alpha_1 - \alpha_2) \left( 2 \left( \frac{-m_3}{m_1 + m_2 + m_3} \right) \left( V_{1x} V_{3y} + V_{3x} V_{1y} \right) \right) \\
+ \sin \alpha_2 \left( 2 \left( \frac{-m_3}{m_1 + m_2 + m_3} \right) \left( V_{21x} V_{23y} - V_{23x} V_{21y} \right) \right) \\
+ \cos \alpha_2 \left( 2 \left( \frac{-m_3}{m_1 + m_2 + m_3} \right) \left( V_{21x} V_{23y} + V_{23x} V_{21y} \right) \right) \]

where the constants have all been previously defined except \( N \) which is

\[ N = I_1 + I_2 + I_3 + \left( m_1 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) V_{21x}^2 + V_{21y}^2 + V_{1x}^2 + V_{1y}^2 \right) \\
+ \left( m_2 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) V_{23x}^2 + V_{23y}^2 + V_{3x}^2 + V_{3y}^2 \right) \\
+ 2 \left( \frac{-m_3}{m_1 + m_2 + m_3} \right) \left( V_{21x} V_{23x} + V_{21y} V_{23y} \right) \]

Substituting the functions for \( \dot{\alpha}_2 = 0 \) and \( I_{\text{overall}} \) into Equation (4-31) and remembering that \( \alpha_2 = 0 \) gives

\[ \gamma_{t=0} = \int_{t_{\text{start}}}^{t_{\text{end}}} \left( \frac{m_1 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) D_1 \times D_1 + m_2 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) D_2 \times D_2 + m_3 \left( \frac{m_1 + m_3}{m_1 + m_2 + m_3} \right) D_3 \times D_3 + I_1 \dot{\alpha}_1 + I_2 \dot{\alpha}_2 + I_3 \dot{\alpha}_3 \right) \frac{dt}{I_{\text{overall}}} \\
= \int_{t_{\text{start}}}^{t_{\text{end}}} \left( \frac{\dot{\alpha}_1 (I_1 + A + C \sin \alpha - D \cos \alpha + G \sin(\alpha_1 - \alpha_2) + L \cos(\alpha_1 - \alpha_2))}{N + 2G \sin(\alpha_1 - \alpha_2) + 2L \cos(\alpha_1 - \alpha_2)} \right) \frac{dt}{(2 \sin \alpha_1 - 2D \cos \alpha_1 + 2E \sin \alpha_1 - 2F \cos \alpha_1)} \]
Thus from Equation (4-32)

\[ \gamma_{H=0, \text{ref}-1} = \int_{\alpha_1}^{\alpha_{\text{end}}^{\text{ref}}} (I + A + C \sin \alpha_2 - D \cos \alpha_2 + G \sin(\alpha_1 - \alpha_2) + L \cos(\alpha_1 - \alpha_2)) d\alpha_2 \]

\[ - \int_{\alpha_1}^{\alpha_{\text{end}}^{\text{ref}}} (I + B + E \sin \alpha_3 - F \cos \alpha_3 + G \sin(\alpha_1 - \alpha_3) + L \cos(\alpha_1 - \alpha_3)) d\alpha_3 \]

This is Equation (4-32)

\[ \text{Equation (4-33) from Equation (4-31)} \]

Once the equation for the angular displacement is known with one segment as the reference segment, it is possible to change reference segments. This section shows the derivation for changing from segment 2 as the reference segment, Equation (4-32), to segment 1 as the reference segment, Equation (4-33).

From the text, \( \gamma_{H=0, \text{ref}-1} = \gamma_{H=0, \text{ref}-2} + \left[ r_1 \alpha_2 \right]_{\text{start}}^{\text{end}} \) and \( \alpha_2 = -r_1 \alpha_2 \)

Thus from Equation (4-32)

\[ \gamma_{H=0, \text{ref}-1} = \int_{\alpha_1}^{\alpha_{\text{end}}^{\text{ref}}} (I + A + C \sin \alpha_2 - D \cos \alpha_2 + G \sin(\alpha_1 - \alpha_2) + L \cos(\alpha_1 - \alpha_2)) d\alpha_2 \]

\[ - \int_{\alpha_1}^{\alpha_{\text{end}}^{\text{ref}}} (I + B + E \sin \alpha_3 - F \cos \alpha_3 + G \sin(\alpha_1 - \alpha_3) + L \cos(\alpha_1 - \alpha_3)) d\alpha_3 \]

\[ + \left[ r_1 \alpha_2 \right]_{\text{start}}^{\text{end}} \]

This is Equation (4-33)
Equation (4-35) from Equation (4-32)

When segment 2 is the reference segment then from Equation (4-32)

\[
F_1(\alpha_i, \alpha_i) = \frac{-\left(I_1 + A + C \sin \alpha_i - D \cos \alpha_i + G \sin(\alpha_i - \alpha_i) + L \cos(\alpha_i - \alpha_i)\right)}{(N + 2G \sin(\alpha_i - \alpha_i) + 2L \cos(\alpha_i - \alpha_i) + 2C \sin \alpha_i - 2D \cos \alpha_i + 2E \sin \alpha_i - 2F \cos \alpha_i)}
\]

\[
F_2(\alpha_i, \alpha_i) = \frac{-\left(I_1 + B + E \sin \alpha_i - F \cos \alpha_i + G \sin(\alpha_i - \alpha_i) + L \cos(\alpha_i - \alpha_i)\right)}{(N + 2G \sin(\alpha_i - \alpha_i) + 2L \cos(\alpha_i - \alpha_i) + 2C \sin \alpha_i - 2D \cos \alpha_i + 2E \sin \alpha_i - 2F \cos \alpha_i)}
\]

Applying Green’s theorem

\[
\gamma_{n=0,ref=2} = \oint_{\Gamma} F_1(\alpha_i, \alpha_i) d\alpha_i + F_2(\alpha_i, \alpha_i) d\alpha_i = \iint_N \frac{\partial F_1(\alpha_i, \alpha_i)}{\partial \alpha_i} - \frac{\partial F_2(\alpha_i, \alpha_i)}{\partial \alpha_i} d\alpha_i d\alpha_i
\]

Using the quotient rule and realising that the denominators are the same gives

\[
\frac{G \cos(\alpha_i - \alpha_i) - L \sin(\alpha_i - \alpha_i)}{(N + 2G \sin(\alpha_i - \alpha_i) + 2L \cos(\alpha_i - \alpha_i) + 2C \sin \alpha_i - 2D \cos \alpha_i + 2E \sin \alpha_i - 2F \cos \alpha_i)}
\]

Considering just the Numerator within the integral and expanding gives

Numerator =
\[
\begin{align*}
&= \cos(\alpha_i - \alpha_j) \left[ GN - 2I_i G - 2BG - GN - 2GL_j - 2GA \right] \\
&+ \cos(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ 2G^2 - 2L^2 - 2G^2 + 2L^2 + 2G^2 - 2L^2 - 2G^2 + 2L^2 \right] \\
&+ \cos^2(\alpha_i - \alpha_j) \left[ 2GL - 2GL - 2GL \right] \\
&+ \cos(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ 2GC - 2DL + 2CG - 2GC \right] \\
&+ \cos(\alpha_i - \alpha_j) \cos(\alpha_j - \alpha_i) \left[ -2GD - 2CL - 2GD + 2GD \right] \\
&+ \cos(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ 2GE - 2GE + 2GE + 2FL \right] \\
&+ \cos(\alpha_i - \alpha_j) \cos(\alpha_j - \alpha_i) \left[ -2GF + 2GF - 2FG + 2EL \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ -LN + 2LI_j + 2LB - LN + 2LI_i + 2LA \right] \\
&+ \sin^2(\alpha_i - \alpha_j) \left[ -2GL + 2GL - 2GL + 2GL \right] \\
&+ \sin(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ -2LC - 2GD - 2CL + 2LC \right] \\
&+ \sin(\alpha_i - \alpha_j) \cos(\alpha_j - \alpha_i) \left[ 2LD - 2CG + 2DL - 2LD \right] \\
&+ \sin(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ -2EL + 2EL - 2EL + 2FG \right] \\
&+ \sin(\alpha_i - \alpha_j) \cos(\alpha_j - \alpha_i) \left[ 2FL + 2FL - 2FL + 2EG \right] \\
&+ \cos(\alpha_i - \alpha_j) \left[ -2CI_j - 2CB \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ -2DL_j - 2DB \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ -2CE - 2FD \right] \\
&+ \sin(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ -2DE + 2FC \right] \\
&+ \cos(\alpha_i - \alpha_j) \cos(\alpha_j - \alpha_i) \left[ 2CF - 2ED \right] \\
&+ \cos(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \left[ 2DF + 2EC \right] \\
&+ \cos(\alpha_i - \alpha_j) \left[ 2FL_i + 2FA \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ 2GI_j + 2AI_i + 2GI_i + 2AI_j \right] \\
&= \cos(\alpha_i - \alpha_j) \left[ 2G^2 - 2G(I_j + B + I_i + A) \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ -2LN + 2LI_j + B + I_i + A \right] \\
&+ \left[ 2LD - 2CG \right] \sin(\alpha_j - \alpha_i) \cos(\alpha_i - \alpha_j) + \cos(\alpha_i - \alpha_j) \sin(\alpha_j - \alpha_i) \sin(\alpha_i - \alpha_j) \\
&+ \left[ 2GD - 2CL \right] \cos(\alpha_j - \alpha_i) \cos(\alpha_i - \alpha_j) + \sin(\alpha_j - \alpha_i) \sin(\alpha_i - \alpha_j) \cos(\alpha_j - \alpha_i) \\
&+ \left[ 2GE + 2FL \right] \cos(\alpha_j - \alpha_i) \sin(\alpha_j - \alpha_i) + \sin(\alpha_j - \alpha_i) \cos(\alpha_j - \alpha_i) \sin(\alpha_j - \alpha_i) \\
&+ \left[ -2FG + 2EL \right] \cos(\alpha_j - \alpha_i) \sin(\alpha_j - \alpha_i) - \cos(\alpha_j - \alpha_i) \sin(\alpha_j - \alpha_i) \cos(\alpha_j - \alpha_i) \\
&+ \left[ 2DF + 2EC \right] \cos(\alpha_j - \alpha_i) \sin(\alpha_j - \alpha_i) + \cos(\alpha_j - \alpha_i) \cos(\alpha_j - \alpha_i) \sin(\alpha_j - \alpha_i) \\
&+ \left[ 2CF - 2ED \right] \sin(\alpha_j - \alpha_i) + \cos(\alpha_j - \alpha_i) \cos(\alpha_j - \alpha_i) \\
&- \cos(\alpha_i - \alpha_j) \left[ 2GI_j + B \right] \\
&- \sin(\alpha_i - \alpha_j) \left[ 2DI_j + B \right] \\
&+ \cos(\alpha_i - \alpha_j) \left[ 2EI_i + A \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ 2FI_i + A \right] \\
&= \cos(\alpha_i - \alpha_j) \left[ 2G^2 - 2G(I_j + B + I_i + A) \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ -2LN + 2LI_j + B + I_i + A \right] \\
&+ \left[ 2LD - 2CG \right] \sin(-\alpha_j) \\
&+ \left[ 2GD - 2CL \right] \cos(-\alpha_j) \\
&+ \left[ 2GE + 2FL \right] \sin(\alpha_i - \alpha_j) \\
&+ \left[ -2FG + 2EL \right] \cos(\alpha_i - \alpha_j) \\
&+ \left[ 2DF + 2EC \right] \sin(\alpha_i - \alpha_j) \\
&+ \left[ 2CF - 2ED \right] \cos(\alpha_i - \alpha_j) \\
&- \cos(\alpha_i - \alpha_j) \left[ 2GI_j + B \right] \\
&- \sin(\alpha_i - \alpha_j) \left[ 2DI_j + B \right] \\
&+ \cos(\alpha_i - \alpha_j) \left[ 2EI_i + A \right] \\
&+ \sin(\alpha_i - \alpha_j) \left[ 2FI_i + A \right] \\
\end{align*}
\]
\[
= \cos(\alpha_r - \alpha_i) \left[ 2GN - 2G(I_1 + B + I_1 + A) + 2CF - 2ED \right] \\
+ \sin(\alpha_r - \alpha_i) \left[ -2LN + 2L(I_1 + B + I_1 + A) + 2DF + 2EC \right] \\
+ \cos \alpha_r \left[ -2C(I_1 + B) - 2FG + 2EL \right] \\
+ \sin \alpha_r \left[ -2D(I_1 + B) + 2GE + 2FL \right] \\
+ \cos \alpha_i \left[ 2E(I_1 + A) - 2[GD + CL] \right] \\
+ \sin \alpha_i \left[ 2F(I_1 + A) - (2LD - 2CG) \right]
\]

Substituting this simplified version of the numerator back to the integral means that

\[
\frac{1}{c} F_1(\alpha_r, \alpha_i) d\alpha_r + F_2(\alpha_r, \alpha_i) d\alpha_i =
\left( \cos(\alpha_r - \alpha_i) \left[ 2G(N-I_1-B-I_1-A)\right] + 2CF - 2ED \right) \\
+ \sin(\alpha_r - \alpha_i) \left[ 2L(I_1 + B + I_1 + A - N)\right] + 2DF + 2EC \right) \\
- \cos \alpha_r \left[ -2C(I_1 + B) - 2FG + 2EL \right] \\
+ \sin \alpha_r \left[ -2D(I_1 + B) + 2GE + 2FL \right] \\
+ \cos \alpha_i \left[ 2E(I_1 + A) - 2[GD + CL] \right] \\
+ \sin \alpha_i \left[ 2F(I_1 + A) - (2LD - 2CG) \right]
\]

\[
\int_\gamma \int \left[ N + 2G \sin(\alpha_r - \alpha_i) + 2L \cos(\alpha_r - \alpha_i) + 2C \sin \alpha_r - 2D \cos \alpha_r + 2E \sin \alpha_r - 2F \cos \alpha_r \right] d\alpha_r d\alpha_i \]

This is Equation (4-35)

**Equation (4-36) from Equation (4-33)**

When segment 1 is the reference segment then from Equation (4-33) we have

\[
F_1(\alpha_{s1}, \alpha_{s2}) = -1 + \frac{\left( I_1 + A - C \sin^2 \alpha_i - D \cos^2 \alpha_i - G \sin(\alpha_i + \alpha_s^2) + L \cos(\alpha_i + \alpha_s^2) \right)}{\left( N - 2G \sin(\alpha_i + \alpha_s^2) + 2L \cos(\alpha_i + \alpha_s^2) - 2C \sin^2 \alpha_i - 2D \cos^2 \alpha_i + 2E \sin^2 \alpha_i - 2F \cos^2 \alpha_i \right)}
\]

\[
F_2(\alpha_{s1}, \alpha_{s2}) = -\frac{\left( I_1 + B + E \sin^2 \alpha_i - F \cos^2 \alpha_i - G \sin(\alpha_i + \alpha_s^2) + L \cos(\alpha_i + \alpha_s^2) \right)}{\left( N - 2G \sin(\alpha_i + \alpha_s^2) + 2L \cos(\alpha_i + \alpha_s^2) - 2C \sin^2 \alpha_i - 2D \cos^2 \alpha_i + 2E \sin^2 \alpha_i - 2F \cos^2 \alpha_i \right)}
\]

Applying Green’s theorem gives

\[
\gamma_{\alpha_{s1}, \alpha_{s2}} = \frac{1}{c} F_1(\alpha_r, \alpha_i) d\alpha_r + F_2(\alpha_r, \alpha_i) d\alpha_i = \int_\gamma \left( \frac{\partial F_2(\alpha_r, \alpha_i)}{\partial \alpha_r} - \frac{\partial F_1(\alpha_r, \alpha_i)}{\partial \alpha_i} \right) d\alpha_r d\alpha_i
\]

\[
\partial \int \left( \begin{array}{c}
\partial^2 \alpha_i^2 \\
-1 + \frac{\left( I_1 + A - C \sin^2 \alpha_i - D \cos^2 \alpha_i - G \sin(\alpha_i + \alpha_s^2) + L \cos(\alpha_i + \alpha_s^2) \right)}{\left( N - 2G \sin(\alpha_i + \alpha_s^2) + 2L \cos(\alpha_i + \alpha_s^2) - 2C \sin^2 \alpha_i - 2D \cos^2 \alpha_i + 2E \sin^2 \alpha_i - 2F \cos^2 \alpha_i \right)}
\end{array} \right) d\alpha_r d\alpha_i
\]

The function to be integrated upon evaluation of the partial derivatives becomes
\[
\left( 2\left( G \cos^{(\alpha_i, \omega)} \alpha_i + L \sin^{(\alpha_i, \omega)} \alpha_i \right) \right) \times \\
\left( N - 2G \sin^{(\alpha_i, \omega)} \alpha_i + 2L \cos^{(\alpha_i, \omega)} \alpha_i - 2C \sin^{\alpha_i} \alpha_i - 2D \cos^{\alpha_i} \alpha_i + 2E \sin^{\alpha_i} \alpha_i - 2F \cos^{\alpha_i} \alpha_i \right) \\
+ \left( I_1 + B + E \sin^{\alpha_i} \alpha_i - F \cos^{\alpha_i} \alpha_i - G \sin^{\alpha_i} \alpha_i \right) + L \cos^{(\alpha_i, \omega)} \alpha_i \\
- 2G \cos^{(\alpha_i, \omega)} \alpha_i - 2L \sin^{(\alpha_i, \omega)} \alpha_i - 2C \cos^{\alpha_i} \alpha_i + 2D \sin^{\alpha_i} \alpha_i \\
+ \left( I_1 + A - C \sin^{\alpha_i} \alpha_i - D \cos^{\alpha_i} \alpha_i - G \sin^{(\alpha_i, \omega)} \alpha_i \right) + L \cos^{(\alpha_i, \omega)} \alpha_i \\
- 2G \cos^{(\alpha_i, \omega)} \alpha_i - 2L \sin^{(\alpha_i, \omega)} \alpha_i + 2E \cos^{\alpha_i} \alpha_i + 2F \sin^{\alpha_i} \alpha_i \right) \\
\left( N - 2G \sin^{(\alpha_i, \omega)} \alpha_i + 2L \cos^{(\alpha_i, \omega)} \alpha_i - 2C \sin^{\alpha_i} \alpha_i - 2D \cos^{\alpha_i} \alpha_i + 2E \sin^{\alpha_i} \alpha_i - 2F \cos^{\alpha_i} \alpha_i \right)
\]

Considering the numerator and expanding gives

\[
= \cos^{(\alpha_i, \omega)} \alpha_i [2GN - 2G(I_1 + B + I_1 + A)] \\
+ \cos^{(\alpha_i, \omega)} \alpha_i \sin^{(\alpha_i, \omega)} \alpha_i [4GL - 2GL - 2GL] \\
+ \cos^{(\alpha_i, \omega)} \alpha_i \sin^{\alpha_i} \alpha_i [-4GC + 2DL + 2GC] \\
+ \cos^{(\alpha_i, \omega)} \alpha_i \cos^{\alpha_i} \alpha_i [-4GD + 2CL + 2GD] \\
+ \cos^{(\alpha_i, \omega)} \alpha_i \sin^{\alpha_i} \alpha_i [4GE - 2GE + 2FL] \\
+ \cos^{(\alpha_i, \omega)} \alpha_i \cos^{\alpha_i} \alpha_i [-4GF + 2GF + 2EL] \\
+ \sin^{(\alpha_i, \omega)} \alpha_i [2LN - 2L(I_1 + B + I_1 + A)] \\
+ \sin^{(\alpha_i, \omega)} \alpha_i [-4GL + 2GL + 2GL] \\
+ \sin^{(\alpha_i, \omega)} \alpha_i \sin^{\alpha_i} \alpha_i [-4CL + 2GD + 2LC] \\
+ \sin^{(\alpha_i, \omega)} \alpha_i \cos^{\alpha_i} \alpha_i [-4DL + 2GC + 2LD] \\
+ \sin^{(\alpha_i, \omega)} \alpha_i \sin^{\alpha_i} \alpha_i [4EL - 2LE + 2FG] \\
+ \sin^{(\alpha_i, \omega)} \alpha_i \cos^{\alpha_i} \alpha_i [-4FL + 2LF - 2GE] \\
+ \cos^{\alpha_i} \alpha_i [2D(I_1 + B)] \\
+ \sin^{\alpha_i} \alpha_i [2D(I_1 + B)] \\
+ \sin^{\alpha_i} \alpha_i \cos^{\alpha_i} \alpha_i [-2CE + 2FD] \\
+ \sin^{\alpha_i} \alpha_i \sin^{\alpha_i} \alpha_i [2DE - 2FC] \\
+ \cos^{\alpha_i} \alpha_i \cos^{\alpha_i} \alpha_i [2FC - 2ED] \\
+ \cos^{\alpha_i} \alpha_i \sin^{\alpha_i} \alpha_i [-2FD - 2EC] \\
+ \cos^{\alpha_i} \alpha_i [2E(I_1 + A)] \\
+ \sin^{\alpha_i} \alpha_i [2F(I_1 + A)]
\]
\[= \cos(\alpha_1 + \alpha_2) \left[ -2G(I_3 + B + I_1 + A - N) \right] + \cos(\alpha_1 + \alpha_2) \sin \alpha_2 \left[ -2GC + 2DL \right] + \cos(\alpha_1 + \alpha_2) \cos \alpha_2 \left[ -2GD - 2CL \right] + \cos(\alpha_1 + \alpha_2) \sin^2 \alpha_2 \left[ 2GE + 2FL \right] + \cos(\alpha_1 + \alpha_2) \cos^2 \alpha_2 \left[ -2GF + 2EL \right] + \sin(\alpha_1 + \alpha_2) \left[ -2L(I_3 + B + I_1 + A - N) \right] + \sin(\alpha_1 + \alpha_2) \sin \alpha_2 \left[ -2CL - 2GD \right] + \sin(\alpha_1 + \alpha_2) \cos \alpha_2 \left[ -2DL + 2GC \right] + \sin(\alpha_1 + \alpha_2) \sin^2 \alpha_2 \left[ 2EL - 2FG \right] + \sin(\alpha_1 + \alpha_2) \cos^2 \alpha_2 \left[ -2FL - 2GE \right] + \cos \alpha_2 \left[ -2C(I_3 + B) \right] + \sin \alpha_2 \left[ 2D(I_3 + B) \right] + \sin^2 \alpha_2 \cos \alpha_2 \left[ -2CE - 2FD \right] + \sin^2 \alpha_2 \sin \alpha_2 \left[ 2DE - 2FC \right] + \cos^2 \alpha_2 \cos \alpha_2 \left[ 2FC - 2ED \right] + \cos^2 \alpha_2 \sin \alpha_2 \left[ -2FD - 2EC \right] + \cos^2 \alpha_2 \left[ 2E(I_1 + A) \right] + \sin^2 \alpha_2 \left[ 2F(I_1 + A) \right] + \cos^2 \alpha_2 \left[ -2L(I_3 + B + I_1 + A - N) \right] + \sin \alpha_2 \left[ 2D(I_3 + B) \right] + \sin^2 \alpha_2 \cos \alpha_2 \left[ -2CE - 2FD \right] + \sin^2 \alpha_2 \cos \alpha_2 \left[ -2DE - 2FC \right] + \cos^2 \alpha_2 \sin \alpha_2 \left[ -2FD - 2EC \right] + \cos^2 \alpha_2 \left[ 2E(I_1 + A) \right] + \sin^2 \alpha_2 \left[ 2F(I_1 + A) \right] \]
Thus back to the integral

\[
\int \left( \cos^i \alpha_2 + \cos^i \alpha_3 \right) \left[ -2G(I_3 + B + I_1 + A - N) + 2FC - 2FD \right] + \sin^i \alpha_2 + \sin^i \alpha_3 \left[ -2L(I_3 + B + I_1 + A - N) - 2CE - 2FD \right] + \left[ 2DL + 2GC + 2F(I_1 + A) \right] \sin^i \alpha_3 + \left[ 2GD - 2CL + 2E(I_1 + A) \right] \cos^i \alpha_3 + \left[ 2FL - 2GE + 2D(I_3 + B) \right] \sin^i \alpha_2 + \left[ 2GF + 2EL - 2C(I_3 + B) \right] \cos^i \alpha_2
\]

\[Y_{H, \text{ref}-2} = \int \int \left( N - 2G \sin^{i^2} \alpha_3 + 2L \cos^{i^2} \alpha_3 - 2C \sin^{i^2} \alpha_3 - 2D \cos^{i^2} \alpha_3 + 2E \sin^{i^2} \alpha_3 - 2F \cos^{i^2} \alpha_3 \right)\]

**From Equation (4-32) to (6-1), (6-2), (6-3)**

Equations (6-1), (6-2), (6-3) are for the idealized asymmetric arm actions. The path is known and so this path may be simply substituted into Equation (4-32) and then evaluated since the integral will now only have one variable.

From Equation (4-32)
\[\gamma_{\mu_0} = \int_{\alpha_{\mu_0}}^{\alpha} \left( I_1 + A \cos \alpha - D \sin \alpha + G \sin(\alpha_1 - \alpha) + L \cos(\alpha_1 - \alpha) \right) d\alpha,\]

Due to the nature of the model as described in Section 4.3.2, substituting \( I_3 = I_1, G = 0, A = B, E = -C, \) and \( D = F. \)

\[\gamma_{\mu_0} = \int_{\alpha_{\mu_0}}^{\alpha} \left( I_1 + A \cos \alpha - D \sin \alpha + G \sin(\alpha_1 - \alpha) + L \cos(\alpha_1 - \alpha) \right) d\alpha,\]

For DiverS and FullS: \( \alpha_3 = \alpha_1 - 180^\circ \)

\[\gamma_{\mu_0,\alpha_3} = \int_{\alpha_{\mu_0}}^{\alpha} \left( I_1 + A \cos \alpha - D \sin \alpha + G \sin(\alpha_1 - \alpha) + L \cos(\alpha_1 - \alpha) \right) d\alpha,\]

then using the second integral in Appendix B.3.2 with \( S = N - 2L; \) and \( Q = C. \)
For Drop, substituting $\alpha_1 = 180^\circ$ (and $d\alpha_1 = 0$)

$$
\gamma_{H=0, ref2} = \int_{\alpha_{ref}^-}^{\alpha_{ref}^+} \frac{-\left( I_1 + A - C \sin \alpha_i - D \cos \alpha_i + L \cos(180^\circ - \alpha_i) \right)}{(N + 2L \cos(180^\circ - \alpha_i) + 2C \sin(180^\circ - \alpha_i) - 2D \cos(180^\circ + \cos \alpha_i))} d\alpha_i
$$

then using the first integral in Appendix B.3.2 with $S = N+2D$, $Q=-C$, $R=D+L$

$$
= \int_{\alpha_{ref}^-}^{\alpha_{ref}^+} \frac{(N + 2L \cos \alpha_i + 2C \sin \alpha_i - 2D \cos(180^\circ + \cos \alpha_i)) d\alpha_i
$$

For Raise substituting $\alpha_3 = 0^\circ$ (and $d\alpha_3 = 0$)

$$
\gamma_{H=0, ref2} = \int_{\alpha_{ref}^-}^{\alpha_{ref}^+} \frac{-\left( I_1 + A + C \sin \alpha_i + L \cos(0^\circ) \right)}{(N + 2L \cos(\alpha_i - 0) + 2C \sin(\alpha_i - 0) - 2D \cos(\alpha_i + 0))} d\alpha_i
$$

then using the first integral in Appendix B.3.2 with $S = N-2D$, $Q=C$, $R=L-D$ a
From Equation (4-32) to (6-4)

To get the Equation for DroppingT applying $\alpha_1=\alpha_3$, $I_1=I_3$, $G = 0$, $A = B$, $C = E = 0$, $D = F$ to Equation (4-32)

$$\gamma_{H-0,ref-2} = - \int_{a_{close}}^{a_{ext}} \frac{(2I_1 + 2A - 2D\cos \alpha_i)}{(N - 4D\cos \alpha_i)} d\alpha_i$$

$$= - \int_{a_{close}}^{a_{ext}} \frac{2I_1 + 2A - \frac{N}{2} + \frac{1}{2} (N - 4D\cos \alpha_i)}{(N - 4D\cos \alpha_i)} d\alpha_i$$

$$= - \frac{1}{2} \int_{a_{close}}^{a_{ext}} \frac{4I_1 + 4A - N}{N - 4D\cos \alpha_i} d\alpha_i$$

This is in the form of the third integral in Appendix B.3.2 with S = N, R=2D and so evaluated is

$$= - \frac{1}{2} (\alpha_{H-0,ref-2}^0 - (4I_1 + 4A - N) \left[ \tan^{-1} \left( \frac{N + 4D}{\sqrt{N - 4D} \tan \left( \frac{\alpha}{2} \right)} \right) \right]_{a_{close}}^{a_{ext}}$$

The bounds will be 0 and 180° when fully raising or dropping both arms.

This is the integral evaluation in Equation (6-4).