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Convection Driven Dynamos in Rotating Spheres

Andrew Jon Eugene Stephan

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

School of Mathematics and Statistics
University of Sydney

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Abstract

Of the objects in the solar system the Earth, Mercury, Jupiter, Saturn, Uranus, Neptune, Ganymede, and the Sun exhibit a magnetic field. These magnetic fields are believed to be generated by the magnetohydrodynamic dynamo process, in which current, generated as electrically conducting fluid crosses magnetic field lines, regenerates the magnetic field. Although most of the bodies listed above are believed to consist of a fluid outer core with a solid inner core, i.e. a spherical shell geometry, the full sphere dynamo problem is of physical interest as the dynamo of the early Earth, the ancient dynamo of Mars, and possibly Venus, the Moon and (currently) Mercury, are believed to have had no solid inner core.

In this thesis we consider numerically the problem of magnetic field generation in a full sphere of rotating uniformly conducting fluid driven by a volumetric heat source. In order to numerically integrate the governing system of equations we combine the poloidal-toroidal field representation of Elsasser (1946) and Bullard & Gellman (1954) with an implicit/explicit multi-step Gear time-stepping method and finite differences in radius. For the implicit radial differencing we develop a generalised compact finite-difference method which results in high order/low bandwidth time-stepping systems, and we demonstrate that this method is competitive with other finite-difference methods: standard finite differences, Padé finite-differences, and the combined compact finite-difference schemes of Chu & Fan (1998).

The numerical integrator is applied to three physical problems of interest. The first is kinematic dynamo action in a sphere. We investigate the possibility of dynamo action for flows with a missing component in spherical polar coordinates and find the growth rates are highly sensitive to changes in the truncation level. Nevertheless, we do find a working kinematic dynamo with axisymmetric velocity with no azimuthal component which demonstrates convincing convergence. The second problem we consider is that of thermal convection in the absence of a magnetic field in a rotating sphere. We fix the Ekman and Prandtl number \((E, Pr) = (5 \times 10^{-4}, 0.7)\) and obtain an estimate of the critical Rayleigh number \(Ra_c\) for the onset of convection, and describe the main characteristic of the flow for the convection solutions for \(Ra \approx 1.4 Ra_c\) and \(Ra \approx 5 Ra_c\). These solutions are primarily for comparison for solutions computed in the third problem: dynamical dynamo action in a rotating sphere. The primary aim is to survey dynamo solutions for the fixed Ekman and Prandtl numbers \((E, Pr) = (5 \times 10^{-4}, 0.7)\), for magnetic Prandtl number varied from 1 to 40 and the modified Rayleigh number varied up to a few times the critical value for the onset of convection. We consider the solutions through the lens of dynamo scaling laws, but find no universally satisfactory theoretical or numerical scaling law. We also consider a weak/strong field classification of the solutions, finding highly localised force balances. We finish by considering three solutions in detail which represent three distinct classes of dynamo solution: an oscillating dipolar solution, an oscillating quadrupolar solution and a chaotic solution which oscillates between two different hemispherical states.

Finally, we develop a first approach to the problem of dynamo action in a fluid sphere as it cools (with no internal heat source), and we present some first convective solutions which function exactly as we expect: the convection dieing down as the fluid cools.
Acknowledgements

First I would like to thank my supervisor, Associate Professor David Ivers, for his guidance, especially when overcoming the more subtle technical obstacles, for his patient, especially in the preparation of this document, and for the direction necessary to produce a coherent work in such a wide field.

I would also like to thank my family for their support, and my friends for both the technical discussions, as well as the (all too) welcome distractions.

Last, but far from least, I would like to thank Aviea, who now knows more about the geodynamo, numerical methods, and parallel programming than could ever possibly be useful to her, and the value of whose patience, encouragement and support could not be overstated.
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CHAPTEOR 1

Introduction

1.1. The dynamo hypothesis

In 1838 Gauss showed, from magnetic field measurements at the Earth’s surface, that the Earth’s main magnetic field is of internal origin. It is known, however, that the temperature of the Earth’s interior is well above the Curie temperature, at which ferromagnetic materials lose their permanent magnetisation (Moffatt 1978). Thus, without a regenerative process, the magnetic field of the Earth would vanish with an inferred decay time of the order of $10^4$ years. The study of magnetised rock, for example red dacite and pillow basalt (McElhinny & Senanayake 1980) and silicate crystals (Tarduno et al. 2007), indicates that the Earth has possessed a magnetic field for at least $3.5 \times 10^9$ years. The current theory of magnetic field generation in the Earth was first proposed by Larmor (1919) to explain the magnetic field of the Sun. The basic premise of this theory is that current, generated as an electrically conducting fluid crosses magnetic fields lines, regenerates the magnetic field. This model is now generally accepted as the explanation for the magnetic fields observed in many astrophysical bodies: the Sun (e.g. Charbonneau 2014); its planets and possibly their satellites, presently Mercury, Earth, Jupiter, Saturn, Uranus, Neptune and perhaps Ganymede, and in the past Mars and possibly the Moon (e.g. Russell 1993, Connerney 1993, Russell & Dougherty 2010, Stevenson 2003, Stevenson 2010); other stars; and also (contentiously) galaxies (e.g. Kulrud 1999, Shukurov 2004).

1.2. The study of the dynamo

The study of the magnetohydrodynamic (MHD) dynamo began with discouraging results: proofs of several theorems describing conditions on the flow or the magnetic field under which self-exciting dynamo action (i.e. without an external source of energy) is impossible: the so-called anti-dynamo theorems. The first and most well known of these is Cowling’s theorem, which precludes the generation of an axisymmetric magnetic field, which vanishes at infinity, by dynamo action (Cowling 1933, Ivers & James 1984). This was followed by the toroidal flow theorem which precludes dynamo action for flows of the form $\mathbf{u} = \nabla \times (\mathbf{r} \times \mathbf{r})$ (Elsasser 1946, Bullard & Gellman 1954, Backus 1958), where $\mathbf{u}$ is the flow velocity and $\mathbf{r}$ is the position vector, and the planar velocity anti-dynamo theorem which precludes dynamo action by a velocity with $\mathbf{u} = (u_x, u_y, 0)$ in Cartesian co-ordinates $(x, y, z)$, where $u_x = 1_x \cdot \mathbf{u}$ etc, in an infinite fluid (Zel’dovich 1957, Zel’dovich & Ruzmaikin 1980). These antidynamo theorems were accompanied by bounding theorems, which place necessary (but not sufficient) conditions for a flow to act as a dynamo (e.g. Backus 1958, Childress 1969, Busse 1975).

The first major advance in modern dynamo theory was the work of Bullard & Gellman (1954) where, following the (flawed) formulation of Elsasser (1947), a kinematic dynamo was presented. This dynamo was later found to be non-converged with the apparent growth of the magnetic field
caused by insufficient numerical resolution. It has still (April 2016) not been shown if their flow is a dynamo or antidynamo. It was only in 1958 with the stasis dynamo of Backus (1958) and the spherical rotor dynamo of Herzenberg (1958) that rigorous proofs of dynamo action were given. Subsequently, many dynamo solutions have been found either analytically or numerically, such as the nearly axisymmetric dynamo of Braginsky (1964a, 1964b, 1964c) and the helical dynamos of Lortz (1968, 1972) and Ponomarenko (1980). There are many review papers and books such as Moffatt (1978) and Gubbins & Herrero-Bervera (2007), in which more detail can be found.

Parker (1955) proposed a different approach to the dynamo problem with turbulent flows, now known as mean-field dynamo theory. In his paper he argued that the inductive effects in non-axisymmetric fluid upwellings, when averaged, contributed to the mean magnetic field. This idea gained traction in the 1960’s with the publishing of the papers of Braginsky (1964a, 1964b) and Steenbeck, Krause & Rädler (1966), who took different approaches to similar problems. In a series of papers Braginsky (1964a, 1964b) demonstrated, by series expansion in $\eta^{1/2}$ and averages over the azimuth, that when the magnetic diffusivity $\eta$ is small (or the magnetic Reynolds number is large), weak departures from axisymmetry in a spherical dynamo can generate an electromotive force which counteracts the ohmic decay of the magnetic field (see Soward 1972 and Moffatt 1978 for further development). In contrast to this, Steenbeck, Krause & Rädler were interested in turbulent flows, decomposing the velocity into mean and fluctuating components based on an ensemble average. These authors were then able to derive, with some simplifying assumptions such as assuming the two components have widely separated length scales, an expression for the mean interaction of the fluctuating magnetic field and flow components, drawing a similar conclusion to Braginsky: the interaction of the fluctuating magnetic field and fluctuating velocity field is able to generate a mean electromotive force parallel to the local mean magnetic field, what is now termed the alpha-effect. This is now seen as a key element in understanding the dynamics of dynamos (e.g. Olson, Christensen & Glatzmaier 1999), although its validity for turbulent dynamos has recently been questioned (Cattaneo & Hughes 2006, Hughes & Proctor 2009). This idea of decomposing the flow and magnetic field into mean and fluctuating components by some averaging operation has found many subsequent applications such as the periodic dynamos of Roberts (1970, 1972), the random wave dynamos of Soward (1975) and the periodic dynamo of Soward & Childress (1990), all of which used spatial averages.

Apart from mean field dynamo theory many authors have continued to consider different aspects of the dynamo problem from a theoretical viewpoint. A large body of work, for example, has re-considered and extended previous results: e.g. Bachtiar, Ivers & James (2006) who reconsidered the planar velocity anti-dynamo theorem, showing that it fails for a uniformly conducting fluid confined to a sphere with insulating exterior; Ivers & James (1988) who extended the toroidal flow theorem to flows with $L_r \cdot \mathbf{u} = 0$ when $\nabla \cdot \mathbf{u} \neq 0$; Ivers & James (1984) who extended Cowling’s theorem to include compressible flows and non-uniformly conducting fluids; and Proctor (2004) who derived a lower bound on the necessary poloidal velocity (for a fixed toroidal velocity) required for dynamo action. Other than the antidynamo theorems and the bounding theorems other authors have derived more novel results; e.g. the reversible flow theorem of Favier & Proctor (2013) which showed that under certain conditions (flow reversibility) the magnetic growth rate in a sphere is the same for pseudo-vacuum and perfectly conducting electromagnetic boundary conditions. These papers demonstrate that this area of research is far from exhausted.
As computers became more powerful it became possible to solve the dynamo problem numerically, beginning with the kinematic dynamo problem. This necessarily began with the development of numerical methods, with most authors building from the methods of Elsasser (1946) and Bullard & Gellman (1954), the Bullard-Gellman formalism, where the velocity and magnetic induction are expressed in terms of poloidal/toroidal fields with a spectral Galerkin method in angle using spherical harmonic basis functions, or the compact spectral forms of the magnetic induction equation of James (1973), where the velocity is expressed in terms of vector spherical harmonics (see also James 1974). This then proceeded to the numerical solution of the kinematic dynamo problem, concerning which there is a large number of studies (Gubbins 1973, Pekeris, Accad & Shkoller 1973, Dudley & James 1989 and Latter 2004 for a sample). Both of these processes are ongoing with authors developing new numerical methods; e.g. the hybrid time-stepping/Arnoldi method of Willis & Gubbins (2004) for the kinematic dynamo problem with periodic flows, and the use of one-sided Jacobi polynomials to solve the kinematic dynamo equations developed by Li, Livermore & Jackson (2010); and solving the kinematic dynamo problem; e.g. the study of kinematic dynamo action in helical flows by Plunian, Marty & Alemany (1999) and Zabielski & Mestel (2005) for application to liquid sodium coolant in nuclear reactors.

The advances in computation have been paralleled by experimental dynamo studies. Perhaps the most famous are the Lowes & Wilkinson (1963) solid two cylinder dynamo (which mimics the Herzenberg 1958 spherical rotor dynamo), the first (simply-connected) experimental dynamo, and the liquid sodium duct experiment of Steenbeck et al. (1967) (see also Roberts & Stix 1971), where the \( \alpha \)-effect described by the mean field theory discussed above was experimentally demonstrated, albeit in a restricted form. Most modern dynamo experiments use liquid sodium in various geometries and with different driving mechanisms (see Colgate 2006 for a recent summary). One of the great triumphs of the modern experimental dynamo is the marriage of experimentation, theoretical considerations and numerical computation. Two examples of this are the Von Karman sodium (VKS) experiment (Monchaux et al. 2007, Ravelet et al. 2008, Monchaux et al. 2009) and DRES-DYN liquid sodium experiment currently under construction (Stefani et al. 2014). For the VKS experiment kinematic dynamo models have been used to describe the dynamics, and in particular why dynamo action failed when the driving impellers were stainless steel, but succeeded when the impellers were changed to soft iron (Giesecke et al. 2013). For the DRES-DYN liquid sodium cylindrical precession driven dynamo experiment both kinematic (Giesecke et al. 2014) and non-linear (Nore et al. 2011) computations have been performed as preliminary studies of the expected dynamics.

Further advances in computation in the late 20th and this century have allowed authors to shift from numerically solving reduced problems, such as the kinematic dynamo problem, to considering numerically the full 3D dynamical dynamo problem. This began with with mean field / nearly 3D models; e.g. the unsuccessful "\( E_\omega \)-dynamo" of Fearn & Proctor (1987) and the mean field dynamos of Jones, Longbottom & Hollerbach (1995) and Sarson, Jones & Longbottom (1998), in which the evolution of a single non-axisymmetric mode and the resulting axisymmetric component, were considered. This then proceeded to the computation of the full non-linear equations without averaging, but possibly with hyperdiffusion. The most well known of these are the spherical shell models of Glatzmaier & Roberts (1995), Kuang & Bloxham (1997) and Kageyama et al. (1995), which were able to generate dipole dominant magnetic fields qualitatively similar to that
of the Earth, the first two models also having magnetic field reversals. The vast majority of computational work has considered spherical shell geometry, modelling different thermodynamic condition, energy sources, viscous boundary boundary conditions and thermal boundary conditions. To indicate the variety of spherical shell models published we list some of the better known models below.

- Glatzmaier & Roberts (1995) considered dynamo action in a Boussinesq fluid with isothermal conditions at the core-mantle boundary (CMB) and homogeneous heat flux conditions at the inner-core boundary (ICB), an insulating exterior, and no-slip viscous boundary conditions.
- Kageyama et al. (1995) considered dynamo action in a compressible fluid with fixed temperature thermal and no-slip viscous boundary conditions at the CMB and ICB.
- Kuang & Bloxham (1997, 1999) considered a Boussinesq fluid with either no-slip or stress-free viscous boundary conditions and either fixed temperature conditions at both the ICB and CMB, or spherically symmetric heat flux conditions at both the ICB and the CMB.
- Dormy, Cardin & Jault (1998) who consider MHD flow in an incompressible fluid confined to a spherical shell with either insulating or conducting inner core which rotates faster than the mantle. The outer boundary was taken to be insulating and rigid.
- Tilgner (2005) considered precession driven dynamo action in an incompressible fluid in a thick spherical shell (small inner core) with no-slip viscous boundary conditions at the CMB.
- Heimpel et al. (2005) considered the effect of changing the shell thickness on dynamo solutions for a Boussinesq fluid with no-slip viscous boundary conditions. The dynamo was driven by maintaining a temperature difference across the fluid shell.
- Willis, Sreenivasan & Gubbins (2007) and Dietrich & Wicht (2013) considered the effect of inhomogeneous heat flux to study thermal locking and hemispherical dynamo action in a Boussinesq fluid with no-slip viscous boundary conditions and a volumetric heat source.
- Schrinner et al. (2014) considered an anelastic fluid with stress-free viscous boundary conditions driven by maintaining an entropy difference between the ICB and the CMB.

Studies of this type have culminated in the systematic review of spherical shell dynamos, authors methodically varying the dynamical parameters in order to determine the effect on the solutions obtained (e.g. Christensen, Olson & Glatzmaier 1999, Grote, Busse & Tilgner 2000, Simitev & Busse 2005, Grote, Busse & Tilgner 2000).

Another important product of the numerical simulation of fully dynamical dynamos are the benchmark dynamo models. These models are extremely useful as they allow for the cross-validation of existing numerical integrators and present an easy way to test newly developed codes. Numerous dynamo benchmarks have been proposed, with different geometries, boundary conditions and thermodynamic simplifications considered. The standard current benchmark models are:

- the Boussinesq spherical shell model of Christensen et al. (2001) which imposes no-slip viscous and insulating exterior magnetic boundary conditions, and drives the dynamo with a fixed temperature difference between the ICB an CMB, for both a co-rotating insulating inner core and for a differentially rotating conducting inner core;
- the anelastic spherical shell model of Jones et al. (2011) which imposes stress-free impenetrable viscous, insulating exterior and insulating inner core conditions, and drives the dynamo by fixing the entropy difference between the inner core boundary and outer core-mantle boundaries;
1.2. THE STUDY OF THE DYNAMO

- the Boussinesq spherical shell benchmark of Jackson et al. (2013) which imposes no-slip viscous and pseudo-vacuum (purely radial) magnetic conditions at the inner core boundary and outer core boundaries, driving the dynamo with a fixed temperature difference across the fluid shell; and
- the full sphere Boussinesq model of Marti et al. (2014), which imposes stress-free viscous and insulating exterior magnetic boundary conditions and drives the dynamo by a uniform volumetric heat source with isothermal boundary conditions.

These benchmarks have been used to test a number of codes (refer to the references above for lists of contributing codes), originally for validation and also, more recently, for accuracy and performance (e.g. Matsui & Heien 2013).

As with the study of the kinematic dynamo problem, the development of numerical methods has been a vital part of the study of dynamical dynamos. A few methods for solving the dynamical problem have been known for some time; e.g. the method of Taylor (1963) for a Boussinesq rapidly rotating fluid, which has yet to be implemented, and the fully spectral formulation of the dynamo equations for a homogeneous incompressible fluid of Pekeris & Accad (1975) which, on current computers, is infeasible. Most integrators begin with the field representation developed by Elsasser (1946) and Bullard & Gellman (1954), combined with transforms similar to those of Glatzmaier (1984), in which fast Fourier transforms (FFT’s) and Legendre transforms, and their inverses, are used in the azimuthal and latitudinal directions respectively. The primary differences between numerical integrators lie in the treatment of the radial dependence of the scalar potentials, authors such as Glatzmaier (1984), Olson & Glatzmaier (1995), Christensen, Olson & Glatzmaier (1999), Tilgner (1999), Hollerbach (2000) and Ivers (2003) using Chebyshev polynomials in radius, Dormy, Cardin & Jault (1998) using standard finite difference methods, Sakuraba & Kono (1999) using a Chebyshev tau method, Farmer & Ivers (2011) using an hp-finite element method, Takahashi (2012) using a combined compact finite difference method, and Marti (2012) using a Galerkin spectral method with one sided Jacobi polynomial basis functions (extending the representation of Li, Livermore & Jackson 2010 to the non-linear problem). This, of course, is far from universal, with later authors such as Kageyama & Sato (2004) solving the equations directly in real space using finite differences, Harder & Hansen (2005) using a finite volume method, Chan et al. (2007) implementing a Galerkin weighted residual formulation of the governing equations with a tetrahedral domain decomposition, and Zhan, Zhang & Zhu (2011) using a finite element method. More specific aspects of some of these methods and implementations are discussed in Chapter 3.

A fundamental issue with numerical dynamo simulations is how far the dynamical parameters at which computations are performed are necessarily removed from those estimated for physical bodies. Consider, for example, the time-scales on which different processes of the geodynamo vary. The time-scales implied by different physics of the dynamics span everything from a day, implied by the Coriolis force, to a time-scale exceeding that of the age of the Earth, implied by the molecular, viscous and thermal diffusion times. Using the estimates given in Gubbins (2007), reproduced in Tables 1.1 – 1.3 we see a wide range of time-scales. This makes direct numerical simulation of the physical processes impossible as the time-step must be made short enough to resolve the smallest time-scale, but the dynamo needs to be advanced in time long enough for effects which occur at the long time-scales to evolve. This problem of modern dynamo theory has led to extensive consideration of dynamo scaling laws.
<table>
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<tr>
<th>Property</th>
<th>Symbol</th>
<th>Molecular</th>
<th>Turbulent</th>
</tr>
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<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$10^4 \text{kg m}^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>$g$</td>
<td>$0–10 \text{ m s}^{-2}$</td>
<td></td>
</tr>
<tr>
<td>Core radius</td>
<td>$c$</td>
<td>$3484 \text{ km}$</td>
<td></td>
</tr>
<tr>
<td>Inner core radius</td>
<td>$r_i$</td>
<td>$1215 \text{ km}$</td>
<td></td>
</tr>
<tr>
<td>Outer core depth</td>
<td>$d = c - r_i$</td>
<td>$2269 \text{ km}$</td>
<td></td>
</tr>
<tr>
<td>Angular velocity</td>
<td>$\Omega$</td>
<td>$7.272 \times 10^{-5} \text{s}^{-1}$</td>
<td>$1 \text{(room temp)}$</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>$\nu$</td>
<td>$10^{-6} \text{ m}^2 \text{s}^{-1}$</td>
<td>$10^{-6} \text{ m}^2 \text{s}^{-1}$</td>
</tr>
<tr>
<td>Electrical conductivity</td>
<td>$\sigma$</td>
<td>$1.25 \times 10^6 \text{ S m}^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Thermal expansivity</td>
<td>$\alpha$</td>
<td>$5 \times 10^{-6} \text{ K}^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Specific heat capacity</td>
<td>$C_p$</td>
<td>$700 \text{ J kg}^{-1} \text{K}^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>$k$</td>
<td>$100 \text{ W m}^{-1} \text{K}^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>$\kappa = k(\rho C_p)^{-1}$</td>
<td>$1.4 \times 10^{-6} \text{m}^2 \text{s}^{-1}$</td>
<td>$1.6 \text{m}^2 \text{s}^{-1}$</td>
</tr>
<tr>
<td>Magnetic diffusivity</td>
<td>$\eta = (\mu_0\sigma)^{-1}$</td>
<td>$0.64 \text{m}^2 \text{s}^{-1}$</td>
<td>$1.6 \text{m}^2 \text{s}^{-1}$</td>
</tr>
<tr>
<td>Core heat flux</td>
<td>$Q_{\text{core}}$</td>
<td>$5 \text{T W}$</td>
<td>$5 \text{mW}$</td>
</tr>
<tr>
<td>Temperature gradient</td>
<td>$T'$</td>
<td>$0.5 \text{K km}^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Mass diffusion constant</td>
<td>$D$</td>
<td>$10^{-6} \text{m}^2 \text{s}^{-1}$</td>
<td>$1.6 \text{m}^2 \text{s}^{-1}$</td>
</tr>
</tbody>
</table>

Table 1.1. Parameter estimates for the Earth’s core taken from Gubbins (2007). Updated estimates for $\sigma$ and $k$ (and hence $\kappa$ and $\eta$) are taken from Pozzo et al. (2012). The turbulent values of 1.6 are used for calculating the turbulent values in Tables 1.2 and 1.3 and are intended for illustrative purposes.

Perhaps the best known dynamo scaling law is the magnetic analogue of the Titius-Bode law, which without physical reasoning does a (contentiously) reasonable job of relating the magnetic moment of planetary dynamos to their angular momentum (Russell 1978). Modern scaling law studies of numerical dynamo results take one of two approaches. The first approach, which represents the majority, is to derive scaling laws from the relevant system of equations by assuming some asymptotic regime, fundamental force balance or energy considerations, and then test the derived law against some set of numerical dynamos (e.g. Christensen & Aubert 2006, Christensen 2010 and Oruba & Dormy 2014b). The second approach is to derive scaling laws directly from the numerical dynamo solutions (e.g. Stelzer & Jackson 2013). This work has met with limited success (Christensen, Holzwarth & Reiners 2009) and is currently an area of active research interest.
1.3. The full sphere dynamo problem

Most naturally occurring dynamos are thought to function in a spherical shell topology. In the case of the Earth, for example, the dynamo is considered to be driven primarily by convection through the processes of cooling at the outer core-mantle boundary, and thermal and compositional convection from the release of a light component and latent heat by freezing of the heavy component at the inner core boundary. For this reason the majority of contemporary dynamo studies (in particular numerical simulation of fully non-linear dynamical dynamos and scaling laws studies discussed above) have focused on dynamos in spherical shells. In the case of the Earth, however, it is generally agreed that the age of the Earth’s dynamo exceeds that of the inner core (e.g. McElhinny & Senanayake 1980 and Labrosse, Poirier & Le Mouël 1997), and thus the geodynamo...
would have once functioned in a full sphere. The ancient dynamo of Mars is also thought to have functioned without an inner core. It is unknown if Mercury has an inner core (Smith et al. 2012). In a full sphere topology the freezing out of heavy elements onto an inner core is no longer a source of energy, and instead other driving mechanisms such as internal heating by radioactive decay, secular cooling, precession, libration, and tidal forcing are considered (see, e.g., Buffett et al. 1996 and references therein).

In this thesis we build on the work of authors such as Sakuraba & Kono (1999), Zhan, Zhang & Zhu (2011), and Marti (2012), and consider the problem of self-sustaining dynamo action in a full sphere. Within this framework we proceed from the study of the kinematic dynamo to the study of scaling laws as outlined above. Beginning with the development of a numerical integrator built on the methods of Elsasser (1946) and Bullard & Gellman (1954), we consider the stationary kinematic dynamo problem, and non-magnetic convection and dynamo action in a rotating full sphere of a Boussinesq uniformly conducting fluid with homogeneous volumetric heating, isothermal and no-slip viscous boundary conditions and an insulating exterior. The primary goals here are to develop efficient numerical methods for the dynamo equations, and to use these methods to first study classes of stationary kinematic dynamos with a missing flow components in spherical polar co-ordinates, and second to begin a systematic survey of full sphere dynamo solutions as the dynamical parameters are varied by applying the methods described above, previously used to analyse spherical shell dynamo solutions, to the incipient full sphere database.

### 1.4. Thesis outline

We begin by defining the full model of magnetic field generation in a uniformly conducting rotating Boussinesq fluid in an aperiphric volume (without an internal bounding surface) with uniform (constant) volumetric heat source in Chapter 2. The three important problems of interest are then defined and their governing equations non-dimensionalised: the stationary kinematic dynamo problem, thermally driven convection in the absence of a magnetic field, and thermally driven dynamo action, all in a full sphere. Of importance here are the thermodynamic simplification and dimensional scaling adopted.

In Chapter 3 we outline the numerical methods used to integrate the systems of equations given in Chapter 2. The methods given here are mostly standard: the field representation of Elsasser (1946) and Bullard & Gellman (1954) is combined with standard finite differences in radius for explicitly evaluated derivatives and a multi-step time-stepping method. We develop a new generalised form of the Padé (compact) finite difference methods for the implicit radial differencing. These finite difference methods are used instead of standard finite difference methods as for the same truncation error they require solving a smaller linear system at each time-step at the cost of a banded matrix multiplication. They also have the appeal over other compact finite difference methods in that they result in a purely banded time-stepping system, as opposed to a block-banded system of greater rank, at the cost of having to evaluate any derivatives needed explicitly.

Even with the thermodynamic simplifications made in Chapter 2 the task of integrating the dynamo equations is daunting. In Chapter 4 different aspects of the numerical integrator are discussed, with particular focus on the methods of parallelisation. As most of the computers available at the time were shared memory machines, the primary focus is on the development of a scalable
shared memory implementation. We develop a staggered synchronisation method with dynamically allocated linear solves which produces near perfect scaling up to the maximum number of available cores, with the added benefit of negligible memory overhead. In order to increase the number of machines available for computation, a very simple but less efficient distributed memory algorithm is also implemented. We conclude the chapter by outlining a possible hybrid shared/distributed memory algorithm.

After defining the full model problem, the numerical methods and important features of the numerical integrator, we then consider the three smaller problems detailed in Chapter 2. The first is the stationary kinematic dynamo problem in Chapter 5. Three kinematic dynamo models are considered. The first is the kinematic dynamo of Pekeris, Accad & Shkoller (1973). This dynamo was used as an initial test of the numerical integrator. The remaining two are stationary kinematic dynamos for flows with a missing component in spherical polar co-ordinates: the zero theta component (ZTC) and the zero azimuthal component (ZAC) flows. Simple working stationary kinematic dynamos are sought to demonstrate that dynamo action can be driven by non-axisymmetric flows of these forms. Although only the simplest ZTC and ZAC flows are considered, the solutions are found to be highly sensitive to truncation levels and no convergent growing solutions were obtained. Finally, we turn attention to the ZAC dynamo of Moss (2006), who considered an axisymmetric Gailitis (1970) type flow consisting of two ring vortices, embedded in a spherical shell with insulating exterior. For these flows, both in a spherical shell and full sphere, we find working dynamo solutions with weak to reasonable convergence of the growth rate for increased truncation levels.

The second problem, that of thermally driven convection in the absence of a magnetic field, is considered in Chapter 6. After reproducing the benchmark model of Marti et al. (2014), the problem of the onset of convection as well as two convective models, for parameters comparable to those at which many of the dynamo solutions were obtained, are considered to compare with the flows of the dynamo solutions obtained later in Chapter 7.

The third problem, that of thermally driven dynamo action in a rotating sphere, is considered in Chapter 7. After reproducing the benchmark model of Marti et al. (2014), two sets of dynamo solutions are considered. The first, arising from a preliminary search through the four dimensional (Ekman number, Prandtl number, magnetic Prandtl number and Rayleigh number) parameter space, corresponds to dynamos computed with imposed symmetries. These computations were performed as a preliminary search for dynamo action and hence only one model is discussed in detail. The second set of solutions are generated by a more systematic study of a slice of the parameter space, without any imposed symmetry. We begin by considering these results collectively, and attempt to fit both theoretical and numerical based scaling laws to these dynamos. We then go on to classifying the dynamos as strong or weak field following Dormy (2014). From this classification we conclude that the theoretical scaling law analysis of these solutions would necessarily fail. Finally, we consider in detail three of the dynamos found, corresponding to three different types of dynamo solution previously identified in the spherical shell studies discussed above, namely a periodic dipolar solution, a periodic quadrupolar solution and a chaotic hemispherical dynamo.

In Chapter 8 we outline a problem of interest as an avenue of future research: the problem of thermally driven convection in a rotating sphere which is cooling. In this chapter the entire problem
is defined, from the dimensional equations, to the non-dimensional and then spectral equations, and two simple first models are presented.

Finally, in Chapter 9 we summarise the results of Chapters 3 – 8 and suggest interesting directions for future research.
The Model Problem

2.1. The rotating Boussinesq dynamo problem

The model problem is that of a uniformly electrically conducting Boussinesq fluid confined to a volume \( V \) (the core) with impenetrable boundary \( \Sigma \) (the mantle) and insulating exterior \( V_e \). The body \( \Sigma \) rotates with uniform angular velocity \( \Omega \). We consider \( V \) in the uniformly rotating frame of \( \Sigma \). The fluid moves under the action of Coriolis, Lorentz and viscous forces, and buoyancy. The reference state is hydrostatic and well mixed, with pressure \( \pi_0 \), uniform temperature \( \Theta_0 \) and uniform density \( \rho_0 \).

The time evolution of the fluid velocity \( \mathbf{u} \) is governed by the Navier-Stokes momentum equation

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho_0} \nabla \pi + \frac{\mathbf{J} \times \mathbf{B}}{\rho_0} + \frac{\rho - \rho_0}{\rho_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}
\]  

(2.1)

where \( \mathbf{r} \) is the position vector, \( \pi \) is the pressure, \( \rho \) is the density, \( \nu \) is the kinematic viscosity, \( \mathbf{B} \) is the magnetic induction, \( \mathbf{J} \) is the electric current density, \( \Theta \) is the temperature and \( \mathbf{g} \) is the gravitational acceleration, modified to include the centripetal acceleration. Variation in the fluid density is ignored except for thermal expansion,

\[
\rho = \rho_0 [1 - \alpha (\Theta - \Theta_0)]
\]  

(2.2)

where \( \alpha \) is the (volumetric) thermal expansivity, in the buoyancy force

\[
\mathbf{F}_g := \frac{\rho - \rho_0}{\rho_0} \mathbf{g} = -\alpha (\Theta - \Theta_0) \mathbf{g}.
\]  

(2.3)

Otherwise the fluid is considered incompressible and there is no mass flux across \( \Sigma \),

\[
\nabla \cdot \mathbf{u} = 0 \text{ in } V, \quad \mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \Sigma
\]  

(2.4)

where \( \mathbf{n} \) is normal to \( \Sigma \). Two types of viscous boundary conditions are considered: no-slip conditions where the fluid velocity is zero on \( \Sigma \)

\[
\mathbf{u} = 0 \quad \text{on } \Sigma,
\]  

(2.5)

and stress-free conditions where the tangential components of the stress and the normal flow vanish on \( \Sigma \)

\[
\rho \nu \mathbf{n} \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \times \mathbf{n} = 0 \quad \text{on } \Sigma.
\]  

(2.6)

The evolution of the temperature \( \Theta \) is governed by the heat equation

\[
\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta = \kappa \nabla^2 \Theta + \frac{Q}{\rho_0 c_p}
\]  

(2.7)
where $\kappa$ is the thermal diffusivity, $c_p$ is the specific heat capacity at constant pressure and $Q$ is the rate of heat production per unit volume, all of which are assumed uniform. The conduction temperature $\Theta_c$ is the equilibrium solution of (2.7) with $u = 0$,

$$\kappa \nabla^2 \Theta_c = -\frac{Q}{\rho_0 c_p}.$$  \hspace{1cm} (2.8)

Unless otherwise stated the mantle $\Sigma$ is a perfect thermal conductor,

$$\Theta(r_{\Sigma}) = \Theta_0$$ \hspace{1cm} (2.9) constant.

The magnetic induction $\mathbf{B}$ is governed by the magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + \nabla \times (u \times \mathbf{B})$$ \hspace{1cm} (2.10)

where $\eta$ is the magnetic diffusivity. Moreover, $\mathbf{B}$ is solenoidal (Gauss’s Law), continuous across $\Sigma$ and is related to the current density $\mathbf{J}$ by Ampere’s Law

$$\nabla \cdot \mathbf{B} = 0 \text{ in } E^3, \quad [\mathbf{B}]_{\Sigma} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ \hspace{1cm} (2.11)

where $\mu_0$ is the magnetic permeability. Since $V_c$ is insulating, Ampere’s law requires

$$\nabla \times \mathbf{B} = \mathbf{0} \text{ in } V_c.$$ \hspace{1cm} (2.12)

Finally, the field is self-exciting (there are no sources of magnetic field at infinity),

$$\mathbf{B} = O(r^{-3}) \text{ as } r \to \infty,$$ \hspace{1cm} (2.13)

so the magnetic energy is finite,

$$\int_{E^3} \mathbf{B}^2 dV < \infty$$ \hspace{1cm} (2.14)

where $E^3$ denotes all space.

The full dynamo problem is described by equations (2.1), (2.7) and (2.10) subject to (2.4), (2.11), (2.12) and (2.13), with boundary conditions (2.5) or (2.6) for the velocity, and (2.9) and (2.11b) for the temperature and magnetic field respectively. Subsets of these equations are used to study three simpler sub-problems:

I the stationary spherical kinematic dynamo problem;

II thermally driven convection in a rotating sphere; and

III the thermally driven dynamo problem in a rotating sphere;

which isolate important aspects of the full model problem in a spherical geometry. The non-dimensional equations governing these problems are discussed below. Henceforth a subscript $\star$ will distinguish a quantity’s dimensional form from its non-dimensional form;

$$t_\star = \tau t, \quad \mathbf{r}_\star = \mathcal{L} \mathbf{r}, \quad \mathbf{u}_\star = \mathcal{U} \mathbf{u}, \quad \mathbf{B}_\star = \mathcal{B} \mathbf{B},$$ \hspace{1cm} (2.15)

where $\mathbf{r}$ is the position vector, $\tau, \mathcal{L}, \mathcal{U}$ and $\mathcal{B}$ are typical time, length, speed and magnetic field scales respectively, and

$$\Omega = \Omega_{1_\Omega}$$ \hspace{1cm} (2.16)
will be used. This will not apply to dimensional constant such as $\kappa$, $\rho_0$ and $\eta$. The spherical polar co-ordinate system in the mantle frame $(r, \theta, \phi)$ is adopted where $\phi$ is the azimuthal angle. Finally, in anticipation of curling the Navier-Stokes equation, the vector identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u})$$

is used to formulate the problem in terms of the modified pressure

$$p_* = \pi_* + \frac{\rho_0 u_*}{2} \mathbf{u}_* \cdot \mathbf{u}_*.$$  

**Problem I – The kinematic dynamo problem**

The kinetic dynamo problem is the study of the stability of the non-magnetic state of a prescribed velocity to perturbation magnetic fields. It is governed by the magnetic induction equation (2.10) and the constraints and boundary conditions (2.10) – (2.14). In this thesis only the stationary kinematic dynamo problem is considered and the velocity does not vary in time. Taking the length $L$ and time scale $\tau$ to be the radius of the sphere and the magnetic diffusion time $\tau_\eta = L^2 / \eta$ respectively, leads to the non-dimensional magnetic induction equation

$$(\partial_t - \nabla^2) \mathbf{B} = R_m \nabla \times (\mathbf{u} \times \mathbf{B})$$

with conditions (2.11), (2.12) and (2.13), and boundary condition (2.11b), unchanged for the non-dimensional $\mathbf{B}$

$$[\mathbf{B}]_\Sigma = 0, \quad \nabla \cdot \mathbf{B} = 0 \text{ in } E^3, \quad \nabla \times \mathbf{B} = 0 \text{ in } V_c, \quad \mathbf{B} = \mathcal{O}(r^{-3}) \text{ as } r \to \infty, \quad \int_{E^3} \mathbf{B}^2 dV < \infty.$$  

The dimensionless number $R_m = UL/\eta$ is the magnetic Reynolds number. In the context of the stationary kinematic dynamo problem the velocity scale $U$ is typically the maximum speed. Care must be taken in comparing values of $R_m$ if the input velocity is not normalised, as is the case here.

Since $\mathbf{u}$ is independent of $t$, equation (2.19) possesses separable solutions of the form

$$\mathbf{B}(r, t) = \mathbf{B}_0(r)e^{\lambda t}$$

where the complex growth rate $\lambda$ depends on $\mathbf{u}$ and $R_m$. The problem can hence be formulated as a linear eigenproblem for $\lambda$ and $\mathbf{B}$

$$\lambda \mathbf{B} = \nabla^2 \mathbf{B} + R_m \nabla \times (\mathbf{u} \times \mathbf{B}),$$

with dynamo action for a particular $\mathbf{u}$ and $R_m$ characterised by $\text{Re } \lambda \geq 0$. Conversely, a velocity $\mathbf{u}$ is said to act as a dynamo if there exists a $\lambda$ with $\text{Re } \lambda \geq 0$ for some $|R_m| < \infty$.

**Problem II – Rotating thermal convection**

The second problem we consider is that of thermally driven convection in a rotating sphere in the absence of a magnetic field. The $z$-axis is aligned with the rotation axis so $\mathbf{1}_z = \mathbf{1}_\Omega$ and we consider the body in the frame which rotates with the rigid boundary $\Sigma$. The length scale $L$ and time scale $\tau$ are taken to be the radius of the sphere and the viscous diffusion time $\tau_\nu = L^2 / \nu$ respectively. For the temperature scale we use the conduction state temperature gradient on the boundary

$$\Delta \Theta = -r_* \cdot \nabla_* \Theta_{c*} |_{r_* \Sigma}.$$
to give the non-dimensional temperature

$$\Delta \Theta \Theta = \Theta_* - \Theta_{0*}.$$  \hspace{1cm} (2.24)

The (dimensional) thermal conduction state temperature $\Theta_{c*}$ is found by solving (2.8) in the sphere. The solution which is analytic at the origin and matches the fixed temperature thermal boundary condition (2.9) is

$$\Theta_{c*}(r) = \Theta_{0*} + \frac{Q}{6 \kappa \rho_0 c_p} (r_{\Sigma*}^2 - r_*^2).$$ \hspace{1cm} (2.25)

Combining this with (2.24) gives

$$\Delta \Theta = \frac{QL^2 r_{\Sigma*}^2}{3 \rho_0 c_p \kappa}.$$ \hspace{1cm} (2.26)

Finally, the non-dimensional spherical symmetric gravitational acceleration is

$$g = -g_* r_* r, \quad g_* = \frac{4\pi G}{3},$$ \hspace{1cm} (2.27)

where $G$ is the gravitational constant.

Combining these gives the non-dimensional Navier-Stokes equation

$$E \left( \partial_t - \nabla^2 \right) u = E u \times (\nabla \times u) + Ra_\nu \Theta r \mathbf{1}_r - 1_z \times u - \nabla p$$ \hspace{1cm} (2.28)

where the dimensionless numbers

$$E = \nu/2\Omega L^2, \quad Ra_\nu = \alpha \Delta \Theta g_* L/(2\Omega U)$$ \hspace{1cm} (2.29)

are the Ekman and modified Rayleigh numbers respectively. The conditions (2.4) remain unchanged for the non-dimensional velocity,

$$\nabla \cdot u = 0 \text{ in } V, \quad \mathbf{n} \cdot u = 0 \text{ on } \Sigma,$$ \hspace{1cm} (2.30)

and the two viscous boundary conditions considered are the no-slip condition (2.5)

$$u = 0 \text{ on } \Sigma,$$ \hspace{1cm} (2.31)

and the stress-free condition (2.6), which reduces on a sphere to

$$\frac{\partial}{\partial r} \left( \frac{u \times r}{r^2} \right) = 0 \quad \text{on } \Sigma.$$ \hspace{1cm} (2.32)

The non-dimensional temperature equation is

$$(Pr \partial_t - \nabla^2) \Theta = S_\nu - Pr u \cdot \nabla \Theta$$ \hspace{1cm} (2.33)

where

$$Pr = \nu/\kappa$$ \hspace{1cm} (2.34)

is the Prandtl number, and the non-dimensional source term is

$$S_\nu = \frac{QL^2}{\kappa \Delta \Theta \rho_0 c_p}.$$ \hspace{1cm} (2.35)

Using (2.26) with $r_{\Sigma} = 1$ reduces (2.35) to

$$S_\nu = 3.$$ \hspace{1cm} (2.36)
Combining (2.24) with (2.9) gives the thermal boundary condition
\[ \Theta(r_\Sigma) = 0 . \] (2.37)

**Problem III – The rotating thermally driven dynamo problem**

The final problem we consider is thermally driven dynamo action in a rotating sphere. In the uniformly rotating mantle frame we consider magnetic field generation in a Boussinesq fluid with constant volumetric heat source, isothermal boundary condition, and an insulating exterior, with either no-slip or stress-free viscous boundary conditions. The governing equations consist of the Navier-Stokes momentum equation (2.1) for an incompressible fluid with impenetrable boundary \( \Sigma \) (2.4), the heat equation (2.7) with perfectly conducting thermal boundary conditions (2.9) and the magnetic induction equation (2.10), subject to the conditions (2.10) – (2.14). The length and time scales are the radius of the sphere and the magnetic diffusion time scale
\[ \tau_\eta = \frac{\mathcal{L}^2}{\eta} \] (2.38)
respectively. We adopt the Elsasser magnetic scale
\[ \mathcal{B} = \sqrt{2\Omega \rho_0 \mu \mathcal{L}} \] (2.39)
and the same temperature scaling used in the spherical rotating thermal convection problem (2.24). The non-dimensional Navier-Stokes momentum equation is
\[ (\text{Ro } \partial_t - \nabla^2) \mathbf{u} = \text{Ro } \mathbf{u} \times (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{B}) \times \mathbf{B} + \text{Ra}_\eta \Theta \mathbf{r} - \mathbf{1}_z \times \mathbf{u} - \nabla p \] (2.40)
where
\[ \text{Ro} = \frac{\eta}{(2\Omega \mathcal{L}^2)} \] (2.41)
is the Rossby (or magnetic Ekman) number, \( E \) is the Ekman number (2.29a) and \( \text{Ra}_\eta \) is the modified Rayleigh number with the same definition as \( \text{Ra}_\nu \) (2.29b) but with the new velocity scale \( \mathcal{U} = \mathcal{L}/\tau_\eta = \eta/\mathcal{L} \). This is equivalent to assuming \( R_m = 1 \). The incompressible and impenetrable boundary conditions are (2.30), and the viscous boundary conditions are either the no–slip conditions (2.31) or the stress-free conditions (2.32).

The non-dimensional magnetic induction equation is
\[ (\partial_t - \nabla^2) \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) \] (2.42)
subject to (2.20).

The non-dimensional transport equation is
\[ (\partial_t - q \nabla^2) \Theta = S_\eta - \mathbf{u} \cdot \nabla \Theta \] (2.43)
where
\[ q = \frac{\kappa}{\eta} \] (2.44)
is the Roberts number and
\[ S_\eta = \frac{QL^2}{\eta \rho_0 c_p \Delta \Theta} . \] (2.45)
Using (2.25) this reduces to
\[ S_\eta = 3q . \] (2.46)
The perfectly conducting thermal boundary conditions are (2.37).

Another dimensionless parameter which does not appear explicitly in the equations but is of importance is the magnetic Prandtl number

\[ Pm := \nu / \eta \] (2.47)

which can be written in terms of the Rossby and Ekman numbers

\[ Pm = E / Ro . \] (2.48)

The Prandtl number (2.34) can be expressed in terms of \( q \), \( E \) and \( Ro \) by

\[ Pr = \frac{E}{q Ro} . \] (2.49)

In many studies in order to limit the size of the parameter space the Prandtl number is fixed at unity (e.g. Christensen, Olson & Glatzmaier 1999, Marti 2012 and Marti et al. 2014). This fixes the ratio \( Pm = E / Ro = q \). Although this is not strictly adopted in this work it should be kept in mind when comparisons are made to other numerical dynamo results. Furthermore in this work the modified Rayleigh numbers for non-magnetic convection \( Ra_\nu \) and the dynamo problem \( Ra_\eta \) differ due to differing time (and hence velocity) scales. The two are related by

\[ Ra_\eta = Pm Ra_\nu . \] (2.50)

Some authors use \( Ra_\nu \) times \( Pm \) (or \( q \) when \( Pr = 1 \)) instead of \( Ra_\eta \) for the dynamo problem. Care must also be taken when comparing integration times as some authors maintain the use of the viscous diffusion time scale for the magnetic problem; the two are related by

\[ \tau_\eta = Pm \tau_\nu . \] (2.51)

### 2.2. The energy equations and diagnostic parameters

In this section the energy equations for the thermally driven rotating spherical dynamo problem are given in both dimensional and non-dimensional form. These are used, along with the heat flux ratio through \( \Sigma, Q_r \), and magnetic dipole moment \( m \) as diagnostics tools for the characterisation and classification of dynamo solutions.

#### 2.2.1. The energy equations

The kinetic energy equation is found by dotting the momentum equation (2.1) with \( u_* \) and integrating over \( V \). Using (2.4) gives

\[
\frac{d}{dt_*} E_{k_*} = \int_V u_* \cdot J_* \times B_* \, dV_* + \int_V (\rho_* - \rho_0) g_* \cdot u_* \, dV_* - \nu \int_V \rho_* \Delta u_* : \nabla u_* \, dV_* \] (2.52)

where the first term on the right is the work done by the Lorentz force on the fluid, the second is the work done by the buoyancy force, the third is the viscous dissipation and

\[ E_{k_*} = \frac{1}{2} \int_V \rho_* u_* \cdot u_* \, dV_* \] (2.53)

is the kinetic energy. The energy scale is taken from the kinetic energy

\[ E_{k_*} = \rho_0 \mathcal{U}^2 \mathcal{L}^3 \, E_k \] (2.54)

where

\[ E_k = \frac{1}{2} \int u^2 \, dV. \] (2.55)
Assuming $V$ is spherical and the fluid is Boussinesq, which gives the buoyancy (2.3) with spherically symmetric gravity (2.27), yields the non-dimensional kinetic energy equation

$$\frac{d}{dt} E_k = \frac{1}{Ro} \int_V \mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} \, dV + \frac{Ra}{Ro} \int \Theta \mathbf{r} \cdot \mathbf{u} \, dV - Pm \int_V \nabla \mathbf{u} : \nabla \mathbf{u} \, dV. \quad (2.56)$$

The total magnetic energy is

$$E_m(\mathcal{E}^3) = \int_{\mathcal{E}^3} \frac{B^2}{2\mu} \, dV_s. \quad (2.57)$$

An application of the Reynolds transport theorem, combined with Faraday’s law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.58)$$

where $\mathbf{E}$ is the electric field, (2.11c), assuming the exterior to $V$ is insulating, and (2.13) give the magnetic energy equation

$$\frac{d}{dt_s} E_m(\mathcal{E}^3) = -\int_V J_s^2 \, dV_s - \int_V \mathbf{u}_s \cdot \mathbf{J}_s \times \mathbf{B}_s \, dV_s, \quad (2.59)$$

where the first term on the right is the Joule dissipation (or Joule heating)

$$P_{J_s} = \int_J \frac{J^2}{\sigma} \, dV_s, \quad (2.60)$$

and the second term is the work done by the Lorentz force. Note that the second term occurs in the kinetic energy equation (2.52) with opposite sign, signifying a perfect conversion between magnetic and kinetic energies. Non–dimensionalising with respect to the same units used for the kinetic energy gives the non–dimensional magnetic energy equation

$$\frac{d}{dt} E_m(\mathcal{E}^3) = \frac{1}{Ro} \int_V (\nabla \times \mathbf{B})^2 \, dV - \frac{1}{Ro} \int \mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} \, dV \quad (2.61)$$

where the non-dimensional magnetic energy is

$$E_m(\mathcal{E}^3) = \frac{1}{2Ro} \int_{\mathcal{E}^3} B^2 \, dV, \quad (2.62)$$

and the non-dimensional Joule dissipation is

$$P_{J} = \frac{1}{Ro} \int_V (\nabla \times \mathbf{B})^2 \, dV. \quad (2.63)$$

In most cases the internal (as opposed to the total) magnetic energy will be quoted. This will be denoted

$$E_m := E_m(V) = \frac{1}{2Ro} \int_V B^2 \, dV. \quad (2.64)$$
2.2.2. Heat flux through the boundary. We consider \( Q_{r*} \), the ratio of the total heat flux (convected plus conducted) \( q_{\text{tot}*} \) to the conducted heat flux \( q_{\text{cond}*} \) at the boundary \( \Sigma \). This can be written
\[
Q_{r*} := \frac{q_{\text{tot}*}(r_{\Sigma}, t)}{q_{\text{cond}*}(r_{\Sigma})} = \frac{\int_{\Sigma} \mathbf{n} \cdot \nabla* \Theta* dS*}{\int_{\Sigma} \mathbf{n} \cdot \nabla* \Theta_{*c*} dS*.} \tag{2.65}
\]
Using the non-dimensional temperature (2.24), where \( \Theta_0 \) is uniform, gives
\[
Q_{r*} = -\frac{1}{4\pi r_{\Sigma}} \int \mathbf{n} \cdot \nabla \Theta dS. \tag{2.66}
\]
For an internally heated full sphere (or, more generally for any internally heated aperiphactic volume), this quantity should average to one. To see this, integrate the temperature equation (2.43) over \( V \)
\[
\int_V \partial_t \Theta dV - q \int_V \nabla^2 \Theta dV = \int_V -\mathbf{u} \cdot \nabla \Theta dV + 3q \int_V dV. \tag{2.67}
\]
Using
\[
\int_V \mathbf{u} \cdot \nabla \Theta dV = \int_V \nabla \cdot (\mathbf{u} \Theta) dV - \int_V \Theta \nabla \cdot \mathbf{u} dV = \oint_{\Sigma} \mathbf{u} \Theta \cdot d\mathbf{S} = 0 \tag{2.68}
\]
(since \( \mathbf{u} \cdot \mathbf{n} = 0 \) on \( \Sigma \) and \( \nabla \cdot \mathbf{u} = 0 \) in \( V \)), and
\[
\int_V \nabla^2 \Theta dV = \oint_{\Sigma} \nabla \Theta \cdot d\mathbf{S} = -4\pi \text{Nu}, \tag{2.69}
\]
along with an application of the Reynolds transport theorem to (2.67), gives
\[
qQ_{r*} = \frac{3q}{4\pi} \int_V dV - \frac{1}{4\pi} \frac{d}{dt} \int_V \Theta dV \tag{2.70}
\]
and hence
\[
Q_{r*} = 1 - \frac{1}{4q\pi} \frac{d}{dt} \int_V \Theta dV. \tag{2.71}
\]
Taking the integral average over some time interval \( t \in (0, T) \) gives
\[
\langle Q_{r*} \rangle := \frac{1}{T} \int_0^T Q_{r*} dt = 1 - \frac{1}{4\pi} \frac{1}{qT} \left( \int_V \Theta(\mathbf{r}, T) dV - \int_V \Theta(\mathbf{r}, 0) dV \right), \tag{2.72}
\]
For sufficiently large \( T \) the last term on the right should tend to zero provided the total heat in \( V \) is bounded. The end result is
\[
\langle Q_{r*} \rangle \approx 1. \tag{2.73}
\]
2.2.3. Magnetic Dipole Moment. Following Amit & Olson (2008) the magnetic dipole moment \( \mathbf{m} = (m_x, m_y, m_z) \) for a sphere is
\[
\mathbf{m}_* = \int_{\Sigma} \rho_{m*} dS*. \tag{2.74}
\]
where \( \rho_{m*} \) is
\[
\rho_{m*} = \frac{3r_{\Sigma*}}{2\mu_0} B_{r*} 1_r. \tag{2.75}
\]
The dipole tilt angle is
\[ \Theta_m = \tan^{-1} \left( \frac{\sqrt{m_{x*}^2 + m_{y*}^2}}{m_{z*}} \right) \] (2.76)
and the longitude of the dipole
\[ \phi_m = \tan^{-1} \left( \frac{m_{y*}}{m_{x*}} \right) \] (2.77)
where
\[ m_{x,*,*} = 1_x \cdot \mathbf{m}_*, \] (2.78)
similarly for \( m_{y,*} \) and \( m_{z,*} \). Since only these angles are of interest the dimensional pre-factor in (2.75) is ignored and the non-dimensional dipole moment is taken as
\[ \mathbf{m} = \int_{\Sigma} \rho_m \, dS \] (2.79)
where
\[ \rho_m = B_r \cdot 1_r. \] (2.80)
CHAPTER 3

Numerical Methods

In this chapter the methods for numerical solution of Problems I – III described in the previous chapter are detailed. The numerical integrator developed here closely resembles that described in Ivers (2003) and is built on the field representation of Elsasser (1947) and Bullard & Gellman (1954), usually referred to as the Bullard-Gellman formalism, which combines a poloidal-toroidal decomposition for solenoidal vector fields (see Section 3.1.2) and a Galerkin method in angle using spherical harmonic basis functions (see Section 3.1.1), and transforms similar to those described in Glatzmaier (1984) for calculating the non-linear interaction terms. These methods are combined with specialised compact finite difference methods in radius for implicitly evaluated derivatives, standard central differences used for all explicitly evaluated radial derivatives, and the implicit-explicit extension of the Gear (1971) multi-step BDF time-stepping formulae developed by Karniadakis, Orszag & Israeli (1991).

Most spherical (and spherical shell) time-stepping codes use methods very similar to those outlined above. The primary differences between implementations arise from the treatment of the radial co-ordinate, the time-stepping method and implementation of adaptive time-step controls, and methods of parallelisation. Some pioneering dynamo codes which function in this way are listed below.

- Dormy, Cardin & Jault (1998) who studied linearised spherical shell dynamos using finite differences in radius with grid points stretched at the boundaries, combined with the Crank-Nicholson scheme for the diffusion terms and the Adams-Bashforth scheme for all other terms.
- Sakuraba & Kono (1999) who used a Chebyshev tau method combined with the Crank-Nicholson scheme for the diffusion terms and a second order Adams-Bashforth scheme for all other terms to study both full sphere and spherical shell dynamo problems.
- Li, Livermore & Jackson (2010) who developed a Galerkin method in radius using one sided Jacobi polynomials for application to the stationary kinematic eigenvalue problem. This method was extended to the full sphere dynamical dynamo problem by Martí (2012) with the use of the Euler predictor and the Crank-Nicholson corrector scheme, with adaptive time-step based on the update schemes of Christensen, Olson & Glatzmaier (1999) supplemented with accuracy constraints.
- Takahashi (2012) who used established spherical shell benchmarks to test a 4th order combined compact finite difference method (see Appendix A) on a Chebyshev grid using the Crank-Nicholson scheme for the diffusion term and a third-order Adams-type predictor-corrector scheme for the explicit time integration.
3.1. Field representation

The use of poloidal-toroidal fields and spherical harmonic basis function in angle is not universally applied to the spherical dynamo problem. Primarily to boost parallel scalability many different formulations, in particular local approximation methods, have been applied to the problem. The following three examples are codes which use different methods from those discussed above.

- Kageyama & Sato (2004) developed a novel \((r, \theta, \phi)\) grid splitting method, the so-called Yin-Yang grid, to efficiently parallelise mantle-convection and spherical shell dynamo computations on distributed memory architecture. In this code the differential equations are solved in real space using second-order finite-differences on a specially distributed discrete \((r, \theta, \phi)\) grid using the fourth order Runge-Kutta time-stepping method.
- Harder & Hansen (2005) developed a finite-volume method for volumes generated by projection of the unit sphere onto the faces of an enclosing cube and building radial blocks for some radial spacing (either equidistant or stretched) based on this projection, to solve the spherical shell convection and dynamo problems. Time-stepping is done with a three-step, second order, variable step-size time-stepper in combination with an iterative non-linear solver.
- Chan et al. (2007) studied the Boussinesq dynamo problem in a spherical shell using a Galerkin weighted-residual formulation of the governing equations for a tetrahedral domain decomposition. Time-advancement is achieved using the second order implicit Gear (1971) method in combination with a second order extrapolation method for the explicitly evaluated terms, with a constant time-step. This code is also distinctive in that the linear time-stepping systems are solved using a BiCGstab(L) method (see e.g. Vorst 1992).

3.1. Spherical harmonics. The solid spherical harmonics of degree \(n\) were defined by Kelvin & Tait (1879) as any solution of

\[
\nabla^2 U(x, y, z) = 0 \tag{3.1}
\]

which is homogeneous and of degree \(n\) in \(x, y, z\). Surface spherical harmonics are found by dividing these solutions by \(r^n\) (MacRobert 1927). Treating these in spherical polar co-ordinates and separating the \(\phi\) dependence of the surface spherical harmonic of degree \(n\) for \(n \in \mathbb{N}\) gives the spherical harmonics of degree \(n\) and order \(m\),

\[
Y^m_n(\theta, \phi) := (-)^m \sqrt{(2n + 1)(n - m)!} P_{n,m}(\cos \theta) e^{im\phi} \tag{3.2}
\]

where

\[
P_{n,m}(z) := (-)^n \frac{(1 - z^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dz^{n+m}} (1 - z^2)^n \tag{3.3}
\]

is the Neumann associated Legendre function, \(m \in \mathbb{N}\) with \(|m| \leq n\) and the pre-factor corresponds to the normalisation and phase of Condon & Shortley (1935). These spherical harmonics are complete and orthonormal in \(n\) and \(m\) with respect to the inner product

\[
(f, g) := \frac{1}{4\pi} \int f g^* d\Omega \tag{3.4}
\]
where \( d\Omega = \sin \theta d\theta d\phi \), integration is over the unit sphere and an asterisk denotes complex conjugation. The spherical harmonics satisfy

\[
Y_n^m = (-)^m (Y_n^{-m})^* \tag{3.5}
\]

and are the eigenfunctions of the operator \( \Lambda^2 \), where

\[
\Lambda := r \times \nabla, \tag{3.6}
\]

with eigenvalue \(-n(n+1)\), i.e.

\[
\Lambda^2 Y_n^m = -n(n+1)Y_n^m. \tag{3.7}
\]

The operator \( \Lambda^2 \) is the angular part of the scalar Laplacian in spherical polar co-ordinates

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\Lambda^2}{r^2}. \tag{3.8}
\]

When applied to the scalar function \( f(r)Y_n^m(\theta, \phi) \) the operator \( D_n \) is generated

\[
\nabla^2(f(r)Y_n^m(\theta, \phi)) = D_n f(r)Y_n^m(\theta, \phi) \tag{3.9}
\]

where

\[
D_n := \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{n(n+1)}{r^2}. \tag{3.10}
\]

Since the \( Y_n^m \) are complete, any scalar function \( f \) on a sphere \( r = \text{const} \) can be expressed as a spherical harmonic series

\[
f_n^m = (f, Y_n^m), \quad f(r, \theta, \phi, t) = \sum_{n \geq 0} \sum_{|m| \leq n} f_n^m(r, t)Y_n^m(\theta, \phi). \tag{3.11}
\]

The inner product (3.4) of a scalar function \( f \) with \( \{Y_n^m\}_{0 \leq n, |m| \leq n} \) as in (3.11a) will be termed the scalar spherical harmonic transform of \( f \), with inverse (3.11b). In practice the series (3.11b) is truncated. In this work a triangular truncation with truncation level \( N \) is used,

\[
f(r, \theta, \phi, t) \approx \sum_{0 \leq n \leq N} \sum_{|m| \leq n} f_n^m(r, t)Y_n^m(\theta, \phi). \tag{3.12}
\]

Use will be made of the vector spherical harmonics (James 1976)

\[
Y_{n,n_1}^m := (-)^{n-m} \sqrt{2n+1} \sum_{\mu,q} \binom{n}{m} \binom{n_1}{-q} \binom{1}{-\mu} Y_n^m e_{\mu}, \tag{3.13}
\]

where the \( 2 \times 3 \) array is a Wigner 3j-coefficient (Brink & Satchler 1968) which can be evaluated as finite series (Wills 1971), and the complex vectors \( e_{\mu} \) are given in terms of the standard Cartesian basis vectors by

\[
e_0 = 1_z, \quad e_1 = -e_{-1}^* = -2^{-1/2}(1_x + i 1_y). \tag{3.14}
\]

These spherical harmonics are complete and orthonormal in \( n, n_1 \) and \( m \) with respect to the inner product

\[
(F, G) := \int F \cdot G^* d\Omega. \tag{3.15}
\]

Further detail on the vector spherical harmonics and their application to the spherical dynamo problem are given in Appendix C.
3.1.2. Poloidal-Toroidal fields. In a sphere concentric with the origin a solenoidal vector field \( \mathbf{F} \) can be decomposed into the sum of a poloidal field \( \mathbf{S} \) and toroidal field \( \mathbf{T} \)

\[
\mathbf{F} = \mathbf{S}\{S\} + \mathbf{T}\{T\}
\]  

(3.16)

where

\[
\mathbf{T}\{T\} := \nabla \times \mathbf{T} \mathbf{r}, \quad \mathbf{S}\{S\} := \nabla \times \mathbf{T}\{S\}.
\]  

(3.17)

The scalar functions \( S \) and \( T \) are the poloidal and toroidal potentials respectively. The vector fields \( S \) and \( T \), and hence the vector field \( F \), are unaffected by adding an arbitrary function of \( r \) to the potentials \( S \) and \( T \)

\[
\mathbf{S}\{S + f(r)\} = \mathbf{S}\{S\}, \quad \mathbf{T}\{T + f(r)\} = \mathbf{T}\{T\}.
\]  

(3.18)

Uniqueness of the potentials \( S \) and \( T \) is imposed by the conditions

\[
\oint S d\Omega = 0 = \oint T d\Omega.
\]  

(3.19)

The spherical polar components of \( F \) are related to the poloidal-toroidal potentials by

\[
1_r \cdot \mathbf{F} = -\Lambda^2 S / r, \quad 1_\theta \cdot \mathbf{F} = \frac{\partial_\phi \partial_r(rS)}{r \sin \theta} + \frac{\partial_\theta T}{\sin \theta}, \quad 1_\phi \cdot \mathbf{F} = \frac{\partial_\phi \partial_r(rS)}{r \sin \theta} - \partial_\theta T.
\]  

(3.20)

We note that only the poloidal potential contributes to the radial component of \( F \).

Re-writing (3.17b)

\[
\mathbf{S}\{S\} = -\nabla^2 S \mathbf{r} + \nabla (\partial_r r S)
\]  

(3.21)

and taking the curl gives

\[
\nabla \times \mathbf{S}\{S\} = \mathbf{T}\{-\nabla^2 S\}.
\]  

(3.22)

The curl of the toroidal field is contained in the definitions (3.17)

\[
\nabla \times \mathbf{T}\{T\} = \mathbf{S}\{T\}.
\]  

(3.23)

Using the identity \( \nabla^2 \mathbf{S} = \nabla (\nabla \cdot \mathbf{S}) - \nabla \times \nabla \times \mathbf{S} \) with (3.22) and (3.23), and noting from (3.17) that \( \nabla \cdot \mathbf{S} = 0 \) the following useful property of the poloidal field is obtained

\[
\nabla^2 \mathbf{S}\{S\} = \mathbf{S}\{\nabla^2 S\}.
\]  

(3.24)

Using these results the poloidal and toroidal potentials of a solenoidal field \( \mathbf{F} \) can be found by solving

\[
\Lambda^2 S = r \cdot \mathbf{F} \quad \text{or} \quad \Lambda^2 \nabla^2 S = r \cdot \nabla \times \nabla \times \mathbf{F}, \quad \text{and} \quad -\Lambda^2 T = r \cdot \nabla \times \mathbf{F}.
\]  

(3.25)

Taking the horizontal divergence of horizontal poloidal-toroidal fields gives

\[
\nabla_h \cdot \mathbf{S}_h\{S\} = \frac{\Lambda^2 \partial_r(rS)}{r^2}, \quad \nabla_h \cdot \mathbf{T}_h\{T\} = 0.
\]  

(3.26)

Finally, if \( \mathbf{F} \) is continuous across a spherical surface \( \Sigma \) then \( S, \partial_r S \) and \( T \) are also continuous across \( \Sigma \).
When applying this field representation to the problems of Chapter 2, the scalar potentials \( S \) and \( T \) are themselves expressed as spherical harmonic series as in (3.11) with the \( n = 0 \) terms omitted by virtue of (3.19):

\[
S(r, \theta, \phi, t) = \sum_{n \geq 1} \sum_{|m| \leq n} S^m_n(r, t) Y^m_n(\theta, \phi), \tag{3.27}
\]

\[
T(r, \theta, \phi, t) = \sum_{n \geq 1} \sum_{|m| \leq n} T^m_n(r, t) Y^m_n(\theta, \phi). \tag{3.28}
\]

Using (3.7), (3.20a) and (3.23) the coefficients \( S^m_n \) and \( T^m_n \) can be found using the inner products

\[
S^m_n = \frac{(r \cdot F, Y^m_n)}{n(n+1)}, \quad T^m_n = \frac{(r \cdot \nabla \times F, Y^m_n)}{n(n+1)}. \tag{3.29}
\]

When \( F \) is not solenoidal (3.29a) is replaced using (3.25) with

\[
D_n S^m_n = -\frac{(r \cdot \nabla \times \nabla \times F, Y^m_n)}{n(n+1)}. \tag{3.30}
\]

The transforms (3.29b) and (3.30) will be referred to as the poloidal-toroidal (spherical harmonic) transform of \( F \) and the notation

\[
S^m_n \{ F \} := \frac{(r \cdot \nabla \times \nabla \times F, Y^m_n)}{n(n+1)}, \quad T^m_n \{ F \} := \frac{(r \cdot \nabla \times F, Y^m_n)}{n(n+1)}, \tag{3.31}
\]

will be used. The inverse transforms are given by (3.16), or in component form by (3.20), with \( S \) and \( T \) calculated using the spherical harmonic series (3.27) and (3.28). Note that this is not a true forward/backward transform pair due to the \( D_n \) in (3.30).

For the remainder of this work upper case \( S \) and \( T \) are used for the magnetic field and lower case \( s \) and \( t \) are used for the velocity. No confusion should occur between the toroidal velocity potential and the time since time occurs only as an argument of a function or as a derivative.

### 3.2. Spectral forms of the governing equations

The spectral equations governing the spherical dynamo problem (Problem III) as described in Section 2.1 are now given. The spectral equations for Problems I and II follow by omission of terms or equations, or by changing the non-dimensional coefficients. In this section repeated use will be made of the result of MacRobert (1927) and Hobson (1931), that if \( f \) is analytic at the origin, then \( f^m_n \) is of the form

\[
f^m_n(r) = r^n (a_0 + a_2 r^2 + a_4 r^4 + \cdots) \tag{3.32}
\]

for \( a_i \in \mathbb{C} \).

#### 3.2.1. The Navier-Stokes equation

A pressure-free formulation of the Navier-Stokes equation is adopted at the cost of increasing the order of the poloidal momentum equation from 2 to 4. The toroidal and poloidal equations are found by taking respectively \( T^m_n \{ \text{NSE} \} \) and \( S^m_n \{ \text{NSE} \} \), where NSE is the Navier-Stokes equation (2.40):

\[
(\text{Ro} \partial_t - E D_n) t^m_n(r, t) = T^m_n \{ -1_z \times u + F \}, \tag{3.33}
\]

\[
(\text{Ro} \partial_t - E D_n) D_n s^m_n(r, t) = S^m_n \{ -1_z \times u + F \} - \text{Ra} \Theta^m_n \tag{3.34}
\]

where \( F = \text{Ro} u \times (\nabla \times u) + J \times B \). The boundary conditions (2.31) and (2.32) become
3.2. Spectral forms of the governing equations

a) no-slip

\[ s^m_n = \partial_r s^m_n = 0, \quad t^m_n = 0, \quad \text{at } r = 1. \]  

(3.35)

b) stress-free

\[ s^m_n = \partial_{rr} s^m_n = 0, \quad \partial_r t^m_n = t^m_n / r, \quad \text{at } r = 1. \]  

(3.36)

The behaviour at the origin follows from (3.32):

\[ s^m_n = O(r^n), \quad t^m_n = O(r^n), \quad \text{as } r \to 0 \]  

(3.37)

which is implemented as

\[ s^m_n(-r, t) = (-)^n s^m_n(r, t), \quad t^m_n(-r, t) = (-)^n t^m_n(r, t); \]  

(3.38)

in particular

\[ s^m_n(0, t) = t^m_n(0, t) = 0. \]  

(3.39)

In the case of stress-free boundary conditions the total angular momentum of the fluid is perfectly conserved at all times. This is usually accounted for by either adjusting the total angular momentum after each time-step with a solid body rotation or, in spherical shell computations, by replacing the viscous boundary conditions on either the ICB or CMB each time-step so the total angular momentum is exactly conserved (see Jones et al. 2011). In this work stress-free boundary conditions were used only for the reproduction of established benchmark results and no special treatment was required for the conservation of angular momentum in these cases.

The spectral Coriolis terms in (3.33) and (3.34) are linear,

\[ T_n^m \{ 1_z \times \mathbf{u} \} = -\frac{n-1}{n} c_n^m \left( \partial_r - \frac{n-1}{r} \right) s^m_{n-1} - \frac{n+2}{n+1} c_{n+1}^m \left( \partial_r + \frac{n+2}{r} \right) s^m_{n+1} - \frac{im}{n(n+1)} t^m_n, \]  

(3.40)

\[ S_n^m \{ 1_z \times \mathbf{u} \} = \frac{n-1}{n} c_n^m \left( \partial_r - \frac{n-1}{r} \right) t^m_{n-1} + \frac{n+2}{n+1} c_{n+1}^m \left( \partial_r + \frac{n+2}{r} \right) t^m_{n+1} - \frac{im}{n(n+1)} D_n s^m_n. \]  

(3.41)

These terms could be treated implicitly in the time-stepping but are treated explicitly to remove coupling (in \( m \)) between the equations (see Hollerbach 2000 for a discussion on treating the Coriolis term implicitly). Since \( \mathbf{u} \) is already calculated in \((r, \theta, \phi)\) space (see Section 3.4) it is simplest to treat the Coriolis force in \((r, \theta, \phi)\) space

\[ 1_z \times \mathbf{u} = -u_{\phi} \sin \theta \mathbf{1}_r - u_{\phi} \cos \theta \mathbf{1}_\theta + (u_\theta \cos \theta + u_r \sin \theta) \mathbf{1}_\phi. \]  

(3.42)

This also becomes advantageous when parallelisation considerations are made since, in a sense, it decouples the storage requirements on the fields (see Chapter 4).

3.2.2. The temperature equation. The spectral heat equation is found by taking the spherical harmonic transform of (2.43)

\[ (\partial_t - q D_n) \Theta^m_n(r, t) = S^m_n + f^m_n \]  

(3.43)

where \( \Theta^m_n, S^m_n \) and \( f^m_n \) are the spherical harmonic components of \( \Theta \), \( S \) and \( f = -\mathbf{u} \cdot \nabla \Theta \) respectively. The perfectly conducting thermal boundary condition (2.37) becomes

\[ \Theta^m_n = 0 \quad \text{at } r = 1, \]  

(3.44)
with the condition at the origin
\[
\Theta^m_n = \mathcal{O}(r^n) \quad \text{as } r \to 0.
\] (3.45)

From (3.32) this is implemented as
\[
\Theta^m_n(-r,t) = (-)^n \Theta^m_n(r,t), \quad \partial_r \Theta^0_0(0,t) = 0; \quad (3.46)
\]
in particular,
\[
\Theta^m_n(0,t) = \partial_r \Theta^0_0(0,t) = 0.
\] (3.47)

### 3.2.3. The magnetic induction equation.

The poloidal and toroidal spectral magnetic induction equations are found by taking the spherical harmonic transform of \( r \cdot \) and \( r \cdot \nabla \times \) the magnetic induction equation (2.42). Using (3.22) and (3.23) gives
\[
(\partial_t - D_n) S^m_n(r,t) = T^m_n \{ \mathbf{F} \} \quad (3.48)
\]
\[
(\partial_t - D_n) T^m_n(r,t) = - S^m_n \{ \mathbf{F} \} \quad (3.49)
\]
where \( \mathbf{F} = \mathbf{u} \times \mathbf{B} \).

The boundary conditions for \( S^m_n \) and \( T^m_n \) are found by solving Ampere’s law in the exterior (2.12) with the condition (2.13), and combining this with the continuity of the magnetic field across \( \Sigma \) (2.11b) (which implies the continuity of \( S \), \( \partial_r S \) and \( T \) across \( \Sigma \)). Substituting the poloidal-toroidal representation of \( \mathbf{B} \)
\[
\mathbf{B} = S \{ S \} + T \{ T \} \quad (3.50)
\]
into (2.12), using (3.22) and (3.23) with (3.29) gives the exterior equations
\[
T^m_n = 0, \quad D_n S^m_n = 0 \quad \text{in } V_e. \quad (3.51)
\]
The continuity of \( T \) across \( \Sigma \) implies the boundary condition
\[
T^m_n(1,t) = 0 \quad (3.52)
\]
whereas the solution of (3.51b) incorporating the continuity of \( S \) across \( \Sigma \) is
\[
S^m_n(r > 1,t) = S^m_n(1,t) \ r^{-(n+1)}. \quad (3.53)
\]
Matching the radial derivative of \( S^m_n \) across \( \Sigma \) gives the local poloidal magnetic boundary condition
\[
\frac{\partial S^m_n}{\partial r} + \frac{n + 1}{r} S^m_n = 0 \quad \text{at } r = 1. \quad (3.54)
\]
The conditions at the origin follow from (3.32)
\[
S^m_n = \mathcal{O}(r^n), \quad T^m_n = \mathcal{O}(r^n) \quad \text{as } r \to 0 \quad (3.55)
\]
which are implemented as
\[
S^m_n(-r,t) = (-)^n S^m_n(r,t), \quad T^m_n(-r,t) = (-)^n T^m_n(r,t); \quad (3.56)
\]
in particular,
\[
S^m_n(0,t) = T^m_n(0,t) = 0. \quad (3.57)
\]

Of use later are the magnetic free decay modes. These are the solutions of (3.48) – (3.49) with \( \mathbf{u} = 0 \) (and hence \( \mathbf{F} \equiv 0 \)) which satisfy (3.52) and (3.54), and can be found in, e.g., Moffatt (1978). Considering first the toroidal problem. Separable solutions are assumed;
\[
T^m_n(r,t) = \hat{T}^m_n(r) \exp(\lambda t), \quad (3.58)
\]
with eigenfunctions regular at the origin

\[ \hat{T}_n^m \propto r^{-1/2} J_{n+1/2} (\sqrt{-\lambda} r), \quad (3.59) \]

where the \( J_n \) are Bessel function of the first kind and the decay rates are determined by the boundary condition (3.52) and are related to the zeros of \( J_{n+1/2} \) by

\[ J_{n+1/2} (\sqrt{-\lambda}) = 0. \quad (3.60) \]

A similar process applied to the poloidal equation gives the eigenfunctions

\[ \hat{S}_n^m \propto r^{-1/2} J_{n+1/2} (\sqrt{-\lambda} r), \quad (3.61) \]

with decay rates \( \lambda \) determined by the boundary condition (3.54)

\[ J_{n-1/2} (\sqrt{-\lambda}) = 0. \quad (3.62) \]

### 3.3. Spectral forms of the energy equations and diagnostic parameters

In this section the diagnostic parameters of Section 2.2 are expressed in terms of the spectral field representations given in Section 3.1. This reduces the cost of evaluating the integrals as the use of spherical harmonics simplifies the evaluation of angular integrals, and all the derivatives of the potentials are already calculated as part of the field transforms discussed in Section 3.4.

#### 3.3.1. The energy equations.

The energy equations of Section 2.2.1 can be expressed in terms of the poloidal-toroidal spherical harmonic components of \( \mathbf{u} \) and \( \mathbf{B} \). Using the decomposition of \( \mathbf{u} \) (3.16) – (3.17) with (3.27) – (3.28) and the result

\[ \oint s \cdot t \, d\Omega = 0 \quad (3.63) \]

for any \( s \) and \( t \) gives the spectral form of the kinetic energy (2.55),

\[ E_k = \frac{1}{2} \int \mathbf{t}^2 \, dV + \frac{1}{2} \int s^2 \, dV, \quad (3.64) \]

where

\[ \int s^2 \, dV = 4\pi \sum_{n,m} n(n+1) \int_0^1 n(n+1) |s_n^m|^2 + |\partial_r (r s_n^m)|^2 \, dr \quad (3.65) \]

and

\[ \int t^2 \, dV = 4\pi \sum_{n,m} n(n+1) \int_0^1 |t_n^m|^2 r^2 \, dr, \quad (3.66) \]

with a similar expression for the magnetic energy (2.64). Using (3.22) and (3.23) a similar expression for the Joule dissipation (2.63)

\[ P_J = \frac{1}{Ro} \int_V (\nabla \times \mathbf{B})^2 \, dV \]

is found by replacing \( s_n^m \) and \( t_n^m \) in (3.64) – (3.66) with \( T_n^m \) and \( D_n S_n^m \) respectively. Finally, the work done by the buoyancy force

\[ P_b = \frac{Ra}{Ro} \int V \Theta \mathbf{r} \cdot \mathbf{u} \, dV, \quad (3.67) \]
is combined with (3.20a) and (3.27) to give

\[ P_b = \frac{Ra}{Ro} \sum_{n,m} 4\pi n(n+1) \int_0^1 \left( s_n^m \right)^* \Theta_n^m r^2 dr. \]  

(3.68)

The work done by the Lorentz force

\[ \frac{1}{Ro} \int_V \mathbf{u} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} dV \]  

(3.69)

is calculated by balancing the magnetic energy equation (2.61). The final term in the energy equations is the viscous dissipation

\[ P_m \int_V \nabla \mathbf{u} : \nabla \mathbf{u} dV, \]  

(3.70)

which is calculated by balancing the kinetic energy equation (2.56) using the work done by the Lorentz force, calculated via the magnetic energy equation. When balancing these equations the total magnetic energy (as opposed to the internal magnetic energy) is required. Following the same process above using (3.51a) and (3.53) the external magnetic energy is

\[ E_m(E^3/V) = \int_{E^3/V} S^2 dV = 4\pi \sum_{n,m} n(n+1) \int_1^\infty n(n+1)|S_n^m|^2 + |\partial_r(rS_n^m)|^2 dr \]

\[ = \sum_{n,m} n |S_n^m(1)|^2 \]  

(3.71)

### 3.3.2. Heat flux through the boundary.

The heat flux through the boundary (2.66)

\[ Q_i(t) = -\frac{1}{4\pi} \int \mathbf{n} \cdot \nabla \Theta dS, \]  

(3.72)

where integration is over the unit sphere, is calculated by noting \( \mathbf{n} = \mathbf{1}_r \) and \( Y_0^0 = 1 \). Substituting the spherical harmonic series for \( \Theta \)

\[ \Theta(r,t) = \sum_{n \geq 0} \sum_{|m| \leq n} \Theta_n^m(r,t)Y_n^m(\theta,\phi) \]  

(3.73)

into the integral above and recalling the spherical harmonics are orthonormal in \( n \) and \( m \) with respect to the inner product (3.4) gives

\[ Q_i(t) = -\partial_r \Theta_0^0(1). \]  

(3.74)

### 3.3.3. The magnetic dipole moment.

The magnetic dipole moment \( \mathbf{m} \) defined in Section 2.2.3 can be expressed in terms of the poloidal magnetic field by first noting

\[ \mathbf{1}_x \cdot \mathbf{1}_r = -\frac{1}{\sqrt{6}} (Y_1^1 - Y_1^{-1}), \quad \mathbf{1}_y \cdot \mathbf{1}_r = -\frac{1}{i\sqrt{6}} (Y_1^1 + Y_1^{-1}), \quad \mathbf{1}_z \cdot \mathbf{1}_r = \frac{1}{\sqrt{3}} Y_1^0. \]  

(3.75)

Using (3.20a) with (3.27) and recalling that the spherical harmonics are orthonormal with respect to the inner product (3.4) with the negative \( m \) harmonics given by (3.5) gives

\[ m_x = \frac{16\pi}{\sqrt{6}} \text{Re} \{ S_1^1 \}, \quad m_y = \frac{16\pi}{\sqrt{6}} \text{Im} \{ S_1^1 \}, \quad m_z = \frac{8\pi}{\sqrt{3}} S_1^0, \]  

(3.76)

where \( \text{Im} \{ z \} \) is the imaginary part of \( z \).
3.4. Non-linear terms and transforms

The non-linear (RHS) terms in the spectral equations above are calculated in \((r, \theta, \phi)\) space using discrete forms of the spherical harmonic transforms of Section 3.1.1 for the temperature equation (3.43) and the discrete forms of the poloidal-toroidal spherical harmonic transforms of Section 3.1.2 for all other equations: the momentum equations (3.33) – (3.34) and (3.42) and the magnetic induction equations (3.48) – (3.49). For convenience the spherical harmonics (3.2) are rewritten as

\[
Y^m_n(\theta, \phi) = P^m_n(\cos \theta)e^{im\phi},
\]

where \(P^m_n\) is the associate Legendre function

\[
P^m_n(z) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{(n+m)!}} P_{n,m}(z)
\]

which are orthonormal with respect to the inner product

\[
\langle f, g \rangle = \frac{1}{2} \int_{-1}^{1} fg \, d\mu,
\]

where the notation \(\mu = \cos \theta\) will persist.

3.4.1. Discrete spherical harmonic transform. The discrete spherical harmonic transform is used to convert the components of \(\nabla \Theta\) from \((r, n, m)\) space to \((r, \theta, \phi)\) space and \(u \cdot \nabla \Theta\) from \((r, \theta, \phi)\) space to \((r, n, m)\) space. The forward transform involves two steps; projection onto the Fourier basis and projection onto the associated Legendre functions

\[
(f, Y^m_n) = \frac{1}{2} \int_{-1}^{1} P^m_n(\mu) \left( \frac{1}{2\pi} \int_{0}^{2\pi} f e^{-im\phi}d\phi \right) d\mu.
\]

The discrete transform is the evaluation of these integrals using \(f\) on the discrete grid \(\{(\theta_k, \phi_l) \mid k = 1 : K, \ l = 0 : L\}\). In this work an \(L\)-panel trapezoidal rule in \(\phi\) and the \(K\)-point Gauss Legendre quadrature in \(\mu \in [0, 1]\) with weights \(w_k\) (see Davis & Rabinowitz 1975 and Section 3.4.3) are used

\[
f^m(\theta_k) = \frac{1}{L} \sum_{l=0}^{L-1} f(\theta_k, \phi_l)e^{-im\phi_l}, \quad f^m_n = \sum_{k=1}^{K} f^m_m(\theta_k)P^m_n(\theta_k)\frac{w_k}{2}.
\]

The summations are computed in the order given to allow the use of a discrete Fourier transform to evaluate the \(l\)-summation.

For a real \(f\) given as a truncated spherical harmonic series the \(l\)-summation in (3.81) is exact provided \(2N + 1 \leq L\) since the DFT is then \(m\)-orthogonal without aliasing. The \(K\)-point Gauss Legendre quadrature is exact for polynomials up to degree \(2K - 1\). Noting that \(P^m_{n_1}P^m_n\) is a polynomial of degree \(n_1 + n \leq 2N\) provided \(m_1 + m\) is even, and that \(m_1 = m\) is selected by the DFT, leads to the condition \(K \geq N + 1\). In application to the non-linear terms in the dynamo problem these conditions become more stringent since \(f\) is quadratic \(f = gh\) where \(g\) and \(h\) are (real) truncated spherical harmonic series. This requires integrating polynomials in \(\mu\) up to degree \(3N\) and taking the discrete Fourier transform of a real function with \(|m| \leq 2N\). This gives \(K \geq 3N/2 + 1\) and \(L \geq 4N + 1\). The values \(L = 4N + 1\) and \(K = \lceil 3N/2 + 1 \rceil\) are taken. This choice of \(L\) contrasts the typical use of the ‘three-halves’ rule of Orszag (1972). In the case of quadratic terms involving derivatives of scalar fields (e.g. \(u \cdot \nabla \Theta\)) it is noted that radial derivatives and \(\phi\) derivatives
do not affect the above arguments and that $\theta$ derivatives do not increase the order of $P_n^m$ in $\mu$.

The inverse spherical harmonic transforms are simply the summations

$$f^m(\theta_k) = \sum_{n=m}^{N} f_n^m P_n^m(\theta_k), \quad f(\theta_k, \phi_l) = \sum_{m=-N}^{N} f_m(\theta_k)e^{im\phi_l}, \quad (3.82)$$

performed in the order given to allow use of an inverse discrete Fourier transform to evaluate the $m$-summation.

### 3.4.2. Discrete poloidal-toroidal (scalar) spherical harmonic transforms. The discrete poloidal-toroidal scalar spherical harmonic transforms (and their inverses) are used to transform $u$, $\omega$, $B$ and $J$ from spectral space to real space, and transform the non-linear terms of the Navier-Stokes equations (3.33)–(3.34) and magnetic induction equations (3.48)–(3.49) from real space to spectral space.

For $\mathbf{F} = (F_r, F_\theta, F_\phi) = \mathbf{T}\{T\} + \mathbf{S}\{S\}$ the backward transforms are found by substituting the truncated series (3.27) and (3.28) into (3.20). This gives

$$F_\zeta(\theta_k, \phi_l) = \sum_{m=-N}^{N} F_m^\zeta(\theta_k)e^{im\phi_l}, \quad \zeta = r, \theta, \phi \quad (3.83)$$

where

$$F_r^m(\theta_k) = \sum_{n=m}^{N} \frac{n(n+1)S_n^m P_n^m(\theta_k)}{r} \quad (3.84)$$
$$F_\theta^m(\theta_k) = \sum_{n=m}^{N} \left( \frac{\partial_r(rS_n^m)}{r} \partial_\theta P_n^m(\theta_k) + T_n^m \frac{im P_n^m(\theta_k)}{\sin \theta_k} \right) \quad (3.85)$$
$$F_\phi(\theta_k) = \sum_{n=m}^{N} \left( \frac{\partial_r(rS_n^m)}{r} \frac{im P_n^m(\theta_k)}{\sin \theta_k} - T_n^m \partial_\theta P_n^m(\theta_k) \right) \quad (3.86)$$

again the summations are evaluated in this order to allow the use of an inverse DFT to evaluate the $m$ summations.

The discrete forward transforms are the evaluation of (3.30)–(3.29) in the same way as in the discrete spherical harmonic transform

$$T_n^m\{\mathbf{F}\} = -\sum_{k=1}^{K} \left( F_\theta^m(\theta_k) \frac{im P_n^m(\theta_k)}{n(n+1) \sin \theta_k} + F_\phi^m(\theta_k) \frac{\partial_\theta P_n^m(\theta_k)}{n(n+1)} \right) \frac{w_k}{2} \quad (3.87)$$
$$S_n^m\{\mathbf{F}\} = \sum_{k=1}^{K} \left( -\frac{F_r^m(\theta_k)}{r} P_n^m(\theta_k) + \frac{\partial_r(rF_r^m(\theta_k))}{r} \frac{\partial_\theta P_n^m(\theta_k)}{n(n+1)} - \frac{\partial_r(rF_\phi^m(\theta_k))}{r} \frac{im P_n^m(\theta_k)}{n(n+1) \sin \theta_k} \right) \frac{w_k}{2} \quad (3.88)$$
3.4.3. Associated Legendre Functions. The nodes of the $K$-point Gauss Legendre quadrature are found using the 3rd order Schröder (1870) method (see also Berezin & Zhidkov 1965). Seeking $x$ for $f(x) = 0$, this method can be found by applying a Newton’s method to $f / f'$

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)} \quad (3.89)$$

To use this method to find nodal points $P_K(z)$ and its first two derivatives are required. These are generated using the recurrence relations

$$(n + 1)P_{n+1}(z) = (2n + 1)zP_n(z) - nP_{n-1}(z), \quad (3.90)$$

$$(1 - z^2)P'_n(z) = -nzP_n(z) + nP_{n-1}(z), \quad (3.91)$$

and the definition of $y = P_n$ as the solution to the DE

$$(1 - z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n + 1)y = 0, \quad (3.92)$$

(Abramowitz & Stegun 1964), with the starting values

$$P_{-1}(z) = 0, \quad P_0(z) = 1. \quad (3.93)$$

Once the nodal points $\{z_k\}_{k=1}^K$ are known the associated Legendre functions and their derivatives are found for $m \geq 0$ using the recurrence relations

$$P_0^m = 1, \quad P_m = -\sqrt{\frac{2m + 1}{2m}} \sin \theta P_{m-1}^{m-1}, \quad m \geq 1 \quad (3.94)$$

$$\partial_\theta P_m^m = m \cot \theta P_m^m, \quad m \geq 0, \quad (3.95)$$

$$c_{n+1}^m P_{n+1}^m = \cos \theta P_n^m c_n^m P_{n-1}^m, \quad n \geq m, \quad (3.96)$$

$$c_{n+1}^m \partial_\theta P_{n+1}^m = (\cos \theta \partial_\theta P_n^m - \sin \theta P_n^m) - c_n^m \partial_\theta P_{n-1}^m, \quad n \geq m \quad (3.97)$$

where

$$c_n^m := \sqrt{\frac{n^2 - m^2}{(2n - 1)(2n + 1)}}. \quad (3.98)$$

The quadrature weights $w_k$ are given by

$$w_k = \frac{2}{(1 - z_k^2)[P_n'(z_k)]^2}. \quad (3.99)$$

The combinations required for the spectral transforms (3.81), (3.82), (3.83)–(3.86) and (3.87)–(3.88) are

$$P_n^m(\theta_k), \quad \frac{mP_n^m(\theta_k)}{\sin \theta_k}, \quad \frac{mP_n^m(\theta_k)}{n(n + 1) \sin \theta_k}, \quad \partial_\theta P_n^m(\theta_k), \quad \frac{\partial_\theta P_n^m(\theta_k)}{n(n + 1)},$$

and the same combinations multiplied by the quadrature weights $w_k$. 
3. Numerical Methods

3.5. Implicit radial differencing

The implicit radial differencing is handled using specialised (generalised) compact finite difference methods. These methods generalise the Padé compact finite difference approximation (given in e.g. Lele 1992) by considering entire linear differential operators instead of pure derivative terms in the finite difference approximation. This results in high-order low bandwidth alternative to standard finite difference methods at the cost of an explicit matrix multiplication. In this section these methods are developed for PDE’s of the form

\[
(\partial_t \chi_1 - \chi_2) f(r, t) = N \tag{3.100}
\]

where \( \chi_1 \) and \( \chi_2 \) are linear differential operators involving only radial derivatives and the inhomogeneous term \( N \) is treated explicitly.

3.5.1. The generalised compact approximation. Starting with a \( p \)-th order linear differential operator

\[
\chi := \psi_0(r) + \psi_1(r) \frac{\partial}{\partial r} + \ldots + \psi_{p-1}(r) \frac{\partial^{p-1}}{\partial r^{p-1}} + \psi_p(r) \frac{\partial^p}{\partial r^p} \tag{3.101}
\]

defined on the possibly non-uniform grid \( \{r_j\}_{j=1}^J \), the approximation for \( \chi \) acting on a scalar function \( f \) at \( j \) has the assumed form

\[
\sum_{l=-l_\alpha}^{u_\alpha} \alpha_{lj} (\chi f)(r_j) = \sum_{l=-l_\beta}^{u_\beta} \beta_{lj} f(r_{j+l}) + \mathcal{E}_j, \tag{3.102}
\]

where \( l_\alpha \) and \( u_\alpha \) are the lower and upper bandwidths of the \( \alpha \) approximation and \( \mathcal{E} \) is the truncation error. The coefficients \( \alpha \) and \( \beta \) are determined by some constraint on \( \mathcal{E} \), usually \( \mathcal{E} = 0 \) for all elements \( f \) in a low order Taylor basis about \( r_j \). The coefficient of the \( k \)th derivative in the Taylor expansion of (3.102) centred \( r_j \) is

\[
\sum_{l=-l_\alpha}^{u_\alpha} \alpha_{lj} \sum_{\tilde{p}=0}^{p} \mathbb{1} \{ \tilde{p} \leq k \} \psi_{\tilde{p}} (r_{j+l}) \frac{h_{\tilde{p}}^k}{(k-p)!} - \sum_{l=-l_\beta}^{u_\beta} \beta_{lj} h_{\tilde{p}}^k = 0 \tag{3.103}
\]

where

\[
\mathbb{1} \{ x \} := \begin{cases} 
1 & \text{if } x \text{ true} \\
0 & \text{if } x \text{ false}
\end{cases} \tag{3.104}
\]

is the indicator function and

\[
h_l := r_{j+l} - r_j. \tag{3.105}
\]

Varying \( k = 0, 1, \ldots, l_\alpha + u_\alpha + l_\beta + u_\beta \) gives a linear system which determines \( \alpha \) and \( \beta \) up to a multiplicative constant. The solution is made unique by the constraint \( \alpha_{0,j} = 1 \). Inspection of the \( k \)th term, where \( k = l_\alpha + u_\alpha + l_\beta + u_\beta + 1 \), gives an expected error

\[
\mathcal{E} = O \left( h_{l_\alpha+u_\alpha+l_\beta+u_\beta+1-p} \right), \quad \hat{h} := \max\{h_l\}. \tag{3.106}
\]

Boundary schemes for non-periodic domains are simple to implement. As an example, a generic scheme to include Robin type conditions at \( J \) for the approximation at \( j \) can be written

\[
\sum_{l=-l_\alpha}^{J-j} \alpha_{lj} (\chi f)(r_{j+l}) = \sum_{l=-l_\beta}^{J-j} \beta_{lj} f(r_{j+l}) + \gamma_{J-j,j} \frac{\partial f}{\partial r}(r_j) + \mathcal{E}_j \tag{3.107}
\]
and the coefficients found as above with $\gamma_{j-j,j}$ another free parameter.

Even for simple operators simple symbolic solutions to the system (3.103) are found only for small bandwidths, uniform grids and/or symmetric approximations [e.g. (3.114)]. Although exact solutions can be found for specific parameters (e.g. fixing the radial grid and bandwidths) using symbolic algebra, this is impractical in a stand alone program and a numerical coefficient finder is developed. Solving the linear system directly was found to generate inaccurate coefficients. To improve the accuracy of the coefficients it is necessary to

- rescale the $\beta$ coefficients by $h^{-p}$ for $h$ some measure of the local grid spacing (e.g. $h = \min_{l \neq 0} \{ h_l \}$);
- rescale the $k \geq p$ equations by $k!/h^{k-p}$;
- remove the $\beta_{0,j}$ dependence and solve the $k = 0$ equation

$$
\beta_{0,j} = \sum_l \alpha_{l,j} \psi_0 (r_l + j) - \sum_{l \neq 0} \beta_{l,j}
$$

after determining the other $\alpha$ and $\beta$ coefficients; and

- use an iterative refinement method.

In practice the linear systems were filled in quadruple precision, solved in double precision using an LU decomposition and refined with quadruple precision residuals. These choices were motivated primarily by the availability of a driver routines which handles the solving and refinement steps automatically (see Chapter 4). The linear systems solved at each radial point are of size $(l_{\alpha} + u_{\alpha} + l_{\beta} + u_{\beta} - 2) \times (l_{\alpha} + u_{\alpha} + l_{\beta} + u_{\beta} - 2)$. The largest compact approximation used in this work is hepta-diagonal ($l_{\alpha} + u_{\alpha} + l_{\beta} + u_{\beta} = 12$), corresponding to solving a $10 \times 10$ system at each radial point. These systems are small and independent, and can be solved in parallel.

The generalised compact scheme (3.102) can be written in matrix form

$$
L (\chi f) (r) = R f + E
$$

where the $(j, l + j)$ entry of $L (R)$ is $\alpha_{l,j} (\beta_{l,j})$, and the vectors $f = \{ f(r_j) \}_{j=1;J}$ and $E$ are the values of $f$ and $E$ on the discrete grid. The following errors, the residual and numerical, are used in assessing these schemes

$$
E_{\text{res}} := |L (\chi f) (r) - R f|, \quad E_{\text{num}} := |(\chi f) (r) - L^{-1} R f|,
$$

the former expected to give $E$ directly from (3.108), the latter being more indicative of the actual error incurred by the approximation. To apply this method to the DE

$$
(\partial_t - \chi) f = \mathcal{N}
$$

first discretise in $r$ and left-multiply by $L$

$$
(L \partial_t - R) f = L \mathcal{N}.
$$

Combining this with a multi-step time-stepping method (see Section 3.6) gives the fully discretised time-stepping equation

$$
(\gamma L - \Delta t R) f^{[k+1]} = b^{[k]}
$$

where $\Delta t$ is the time-step, $\gamma$ is some constant coming from the time-stepping method, superscripts of the form $\{ k \}$ denote time level and the vector $b^{[k]}$ includes the multiplication by $L$ and is fully
known at time \( k \).

As an example, substituting \( \chi = D_0 \) into \( (3.103) \) gives the \( k \)th Taylor condition

\[
\sum_{l=-l_\alpha}^{u_\alpha} \alpha_{l,j} \left( 1 \{ k \geq 2 \} \frac{h_l^{p-2}}{(p-2)!} + 1 \{ k \geq 1 \} \frac{2 h_l^{p-1}}{r_{t+j} (p-1)!} \right) - \sum_{l=-l_\beta}^{u_\beta} \beta_{l,j} \frac{h_l^p}{p!} = 0. \tag{3.113}
\]

Varying \( k = 0, 1, \ldots, l_\alpha + u_\alpha + l_\beta + u_\beta \) gives a linear system in \( \alpha \) and \( \beta \), which is solved as outlined above. On a uniform grid with spacing \( r_{j+1} - r_j = h \), the scheme for \( l_\alpha = u_\alpha = l_\beta = u_\beta = 1 \) can be solved exactly

\[
\alpha_{-1,j} = \frac{r - h}{10r}, \quad \alpha_1 = \frac{r + h}{10r},
\]

\[
\beta_{-1,j} = \frac{6(r - h)}{5h^2 r}, \quad \beta_{0,j} = -\frac{12}{5h^2}, \quad \beta_{1,j} = \frac{6(r + h)}{5h^2 r}, \tag{3.114}
\]

\[
\mathcal{E} = h^4 \left( \frac{3}{100r} \frac{\partial^5 g}{\partial r^5} + \frac{1}{200} \frac{\partial^6 g}{\partial r^6} \right) + \mathcal{O}(h^6).
\]

The set of 4th order schemes, including one sided schemes for the boundary with both Dirichlet and Robin type conditions, are simple to generate symbolically on a uniform grid — e.g. with Mathematica (Wolfram Research 2012), but the coefficients become restrictively complicated on the boundary. This is also true for the 4th order interior schemes on a non-uniform grid and worsens for the 4th order boundary schemes on a non-uniform grid and 6th order interior schemes on a uniform grid. This lack of ability to write the compact coefficients as simple functions of position, grid spacing and bandwidth for all but the simplest cases for even the simple operator \( D_0 \) accents the need for a numerical coefficient solver.

Since the majority of these schemes are generated numerically the behaviour of the associated truncation error is checked numerically. For this purpose we use the operator \( \chi = D_0 \) and the simple test function and boundary conditions

\[
f(r) = r^6 \cos(40\pi r), \quad f(0) = f(1) = 0 \tag{3.115}
\]

for \( r \in (0, 1) \). For the purpose of testing the convergence of these schemes we use interior central difference schemes with \( l_\alpha = u_\alpha = l_\beta = u_\beta \), with one-sided schemes at the boundary which preserve the bandwidth of the interior schemes, and which do not evaluate \( D_0 f(0) \) or \( D_0 f(1) \). Global schemes are labelled according to the bandwidth of the interior schemes — e.g. the pentadiagonal scheme has

\[
l_\alpha = \min\{2, j-1\}, \quad u_\alpha = \min\{2, J - 1 - j\}, \quad l_\beta = \min\{2, j\}, \quad u_\beta = \min\{2, J - j\}
\]

for \( j = 1 : J - 1 \). Convergence curves and error profiles are generated for both \((J\text{-point})\) uniform and Chebyshev \([0, 1]\) radial grids, with convergence rates given in terms of \( h := 1/J \) in all cases. These are shown in Figure 3.1 for the uniform grid and Figure 3.2 for the \([0, 1]\) Chebyshev grid. Important features of these plots are:

1. The use of lower order schemes at the boundaries result in the spikes in the errors seen in Figure 3.1(c) and (d), and to a lesser extent (due to a compacting of the grid at the boundaries) in Figure 3.2(c) and (d);
2. The propagation of this error further into the interior in the numerical error is unsurprising since the inversion in (3.109) couples these points together;

3. \( D_0 f(r) \to 0 \) as \( r \to 0 \), causing the spikes in the relative error near \( r = 0 \) seen in Figures 3.1 (d) and 3.2 (d); and

4. The convergence rates are calculated on the straight line segments of the convergence curves before loss of accuracy of the compact coefficients and round-off errors in the matrix-vector multiplications and linear inversion in (3.109) saturate as \( J \) is increased. These are particularly notable for the interior hepta-diagonal scheme on a uniform grid [Figure 3.1 (a)] and the penta-diagonal boundary scheme on the Chebyshev grid [Figure 3.2 (b)].

When these schemes are used to solve the DE’s of Section 3.2 the loss of accuracy at the origin may be overcome by virtue of the result (3.32). At the outer boundary, however, we can either maintain the order of the interior approximations by increasing the bandwidth of the approximation, or we must accept a loss of accuracy while maintaining the bandwidth. While a simple partitioning method can be used to solve linear banded linear systems with additional bands at the boundary, this is undesirable for the distribution memory implementation discussed in Section 4.2 as the number of parallel processes that can be used for a single linear solve is dependent on the bandwidth of the matrix. We will see in Section 3.5.2, however, that minor modification to the generalised compact scheme can overcome this problem and generate boundary schemes that maintain both the order of the finite difference approximation and the bandwidth of the linear system solved at each time-step.

When \( \chi \) contains only a single derivative term the Padé type compact finite difference schemes are recovered. If \( \chi \) contains derivatives of different orders, then the use of Padé schemes requires combining the schemes for each individual derivative present. This results in a block-band time-stepping system which solves for \( f \) as well as all the derivatives of \( f \) in \( \chi \) except the highest on the discrete grid (e.g. Mohebalhojeh & Dritschel 2007, Liu, Kuang & Tangborn 2009). As an example, consider the fully discretised DE (3.110) with \( \chi = D_0 \)

\[
(\gamma - \Delta t D_0) f^{(k+1)}(r_j) = b^{(k)}(r_j). \tag{3.116}
\]

where all explicit terms have been absorbed into \( b^{(k)} \). Using the Padé schemes for the first and second derivatives

\[
\sum_{l=-l_\alpha}^{u_\alpha} \alpha^{(1)}_{l,j} \frac{\partial}{\partial r} f(r_{j+l}) = \sum_{l=-l_\beta}^{u_\beta} \beta^{(1)}_{l,j} f(r_{j+l}), \quad \sum_{l=-l_\alpha}^{u_\alpha} \alpha^{(2)}_{l,j} \frac{\partial^2}{\partial r^2} f(r_{j+l}) = \sum_{l=-l_\beta}^{u_\beta} \beta^{(2)}_{l,j} f(r_{j+l}) \tag{3.117}
\]

where the upper and lower bandwidths for the two approximations are possibly different, the time-stepping system is found by first taking the linear combination \( \sum_{l=-l_\alpha}^{u_\alpha} \left( \alpha^{(2)}_{l,j} \cdot \text{DDE} \right) \) where DDE is the discretised DE (3.116), and the lower/upper bandwidths will be taken as the largest of all in (3.117). This results in

\[
\sum_{l=-l_\alpha}^{u_\alpha} \alpha^{(2)}_{l,j} \left( \gamma - \Delta t \frac{2}{r_{j+l}} \frac{\partial}{\partial r} \right) f^{(k+1)}(r_{j+l}) - \Delta t \sum_{l=-l_\beta}^{u_\beta} \beta^{(2)}_{l,j} f^{(k+1)}(r_{j+l}) = \sum_{l=-l_\alpha}^{u_\alpha} \alpha^{(2)}_{l,j} b^{(k)}(r_{j+l}). \tag{3.118}
\]
Supplementing this with the Padé scheme for the first derivative, gives the time-stepping equation

$$\sum_t \begin{bmatrix} \gamma \alpha^{(2)}_{t,j} - \Delta t \beta^{(2)}_{t,j} \\ \beta^{(1)}_{t,j} \\ -\alpha^{(1)}_{t,j} \end{bmatrix} \begin{bmatrix} f(r_{t+j}) \\ (\partial_r f)(r_{t+j}) \end{bmatrix}^{k+1} = \begin{bmatrix} \sum_t \alpha^{(2)}_{t,j} \beta_{t+j}^{(1)} \\ 0 \end{bmatrix}^{k}. \tag{3.119}$$

The left of this equation corresponds to a \((2 \times 2)\) block-band system with bandwidth given by the maximum bandwidth in (3.117) and the right of this equation corresponds to a banded matrix vector multiplication which is interlaced with zeros. From (3.119) it is clear that there is significant overhead in using Padé methods directly if the operator \(\chi\) contains many derivative terms or if the derivatives evaluated as part of the time-stepping are not needed, as would be the case when applying these methods to the poloidal momentum equation (3.34) (as the third derivative of \(s^n_m\) is...
Another extension of the Padé compact finite difference methods are the combined compact schemes of Chu & Fan (1998). The general combined compact scheme assumes finite difference approximations for the $p$th derivative of $f$ at $j$ of the form (see e.g. Ghader & Vahid 2011)

$$
\frac{\partial^p}{\partial r^p} f(r_j) = \sum_{i=-l_0}^{u_0} \alpha_{0,i} f(r_{j+i}) + \sum_{i=-l_1}^{u_1} \beta_{1,i} \frac{\partial}{\partial r} f(r_{j+i}) + \sum_{i=-l_2}^{u_2} \beta_{2,i} \frac{\partial^2}{\partial r^2} f(r_{j+i}) + \cdots + \mathcal{E} \quad (3.120)
$$

where all $\beta_{k,0} = 0$ and $\mathcal{E}$ is the truncation error, with the coefficients found by matching terms in the Taylor expansion around $r_j$. These methods were applied to the spherical (shell) dynamo problem by Takahashi (2012) with the inclusion of the first two derivative terms on the right of (3.120) at $r_{j \pm 1}$ in the interior for all required derivatives [up to the 4th for the poloidal momentum equation (3.34)]. This results in a time-stepping system which solves for $f$, $\partial_r f$ and $\partial_{rr} f$ at the grid points which, in the interior, is $3 \times 3$ block tri-diagonal — see Appendix A for more detail.
While these methods ensure that only derivatives needed for the explicit calculation are generated implicitly at each time-step and do not require the matrix multiplication on the right of \((3.112)\), they generate significantly larger time-stepping systems, both in rank and bandwidth, than those constructed from the generalised compact methods developed above.

A spectral analysis of the error associate with the approximation \((3.102)\) is possible using minor modifications to the method outlined in Vichnevetsky & Bowles (1982) and Cain & Bush (1994). Such an analysis is used to quantify the resolution characteristics of a finite difference scheme, to mean the accuracy with with a scheme differences at the shorter length scales resolvable on a computational grid. Historically this type of analysis has often been used to compare Padé type compact schemes to standard finite differences as has been extended to optimise the resolving characteristics of such methods on both uniform (Lele 1992) and non-uniform (Gamet et al. 1999) grids. The improved resolution characteristics of the Padé schemes relative to standard finite differences as well as the subsequent optimisations, both of which are discussed in Lele (1992), have justified the use of Padé compact schemes for applications ranging from the simulation of turbulent flows (e.g. Gamet et al. 1999 and Tejada-Martínez et al. 2009) to modelling shocks (e.g. Lele 1992, Zhong 1998 and Dexun & Yanwen 2001). The spectral analysis, as well as the optimisation for small length scale differencing, is easily extended to the generalised compact schemes given above. For completion this analysis is performed for the general scheme \((3.102)\) and an optimised scheme is found and compared to the ‘near spectral’ Padé schemes of Lele (1992) for the operator \(\chi = D_0\) in Appendix B.

### 3.5.2. More general differential equations.

The generalised compact approximation \((3.102)\) is easily applied to the spectral dynamo equations of Section 3.2 with the exception of the poloidal momentum equation \((3.34)\) which has the form

\[
\left( \partial_t \chi_1 - \chi_2 \right) f = \mathcal{N}
\]

\((3.121)\)

where \(\chi_1\) and \(\chi_2\) are different (non-trivial) linear differential operators. In the framework of the generalised compact finite difference approximations there are two options available for the implicit differencing of this equation. The first is to follow the same methodology used to difference operators with more than one derivative term using Padé finite differences as in \((3.117) - (3.119)\), where the approximations \((3.117)\) are replaced with the generalised compact approximations for \(\chi_1\) and \(\chi_2\)

\[
\sum_{l=-l_{\alpha}}^{u_{\alpha}} \alpha^{(1)}_{l,j} (\chi_1 f) (r_{j+l}) = \sum_{l=-l_{\beta}}^{u_{\beta}} \beta^{(1)}_{l,j} f(r_{j+l}) ,
\]

\[
\sum_{l=-l_{\alpha}}^{u_{\alpha}} \alpha^{(2)}_{l,j} (\chi_2 f) (r_{j+l}) = \sum_{l=-l_{\beta}}^{u_{\beta}} \beta^{(2)}_{l,j} f(r_{j+l}),
\]

\((3.122)\)

and the discretised PDE \((3.116)\) is replaced by the fully discretised form of \((3.121)\)

\[
(\gamma \chi_1 - \Delta t \chi_2) f^{(k+1)}(r_j) = b^{(k)}(r_j),
\]

\((3.123)\)

where again all explicit terms have been absorbed into \(b^{(k)}\). This results in two possible block 2 × 2 band time-stepping systems

\[
\sum_{l} \left[ \begin{array}{cc}
-\Delta t \beta^{(2)}_{l,j} & \gamma \alpha^{(2)}_{l,j} \\
\beta^{(1)}_{l,j} & -\alpha^{(1)}_{l,j}
\end{array} \right] \left[ \begin{array}{c}
f(r_{l+j}) \\ (\chi_1 f) (r_{l+j})
\end{array} \right]^{(k+1)} = \sum_{l} \left[ \begin{array}{c}
\alpha^{(2)}_{l,j} b(r_{l+j}) \\
0
\end{array} \right]^{(k)}
\]

\((3.124)\)
or
\[
\sum_l \begin{bmatrix}
\gamma \beta_{l,j}^{(1)} - \Delta t \alpha_{l,j}^{(1)} \\
\beta_{l,j}^{(2)} - \alpha_{l,j}^{(2)}
\end{bmatrix}
\begin{bmatrix}
f(r_{l+j}) \\
(\chi_2 f)(r_{l+j})
\end{bmatrix}^{(k+1)} = \sum_l \alpha_{l,j}^{(1)} b(r_{l+j}) \right)^{(k)} ,
\]

(3.125)

the choice between the two lying in whether \( \chi_1 f \) or \( \chi_2 f \) is more useful in the explicit portion of the time-stepping and how easily boundary conditions can be found and satisfied. The other option is to discretise in time
\[
(\gamma \chi_1 - \Delta t \chi_2) f^{(k+1)} = b^{(k)}
\]

and then form a generalised compact approximation for
\[
\chi = (\gamma \chi_1 - \Delta t \chi_2)
\]

directly from (3.102). This results in the time-stepping equation
\[
R f^{(k+1)} = b^{(k)}.
\]

(3.128)

where \( R_{j,j+l} = \beta_{l,j} \) and the vector \( b^{(k)} \) includes multiplication by the matrix \( L_{j,j+l} = \alpha_{l,j} \).

Comparison of (3.112) and (3.128) reveals that as long as the compact coefficients can be generated to sufficient precision and the time step \( \Delta t \) is constant throughout a computation it is advantageous to develop the compact scheme in all cases after the time-discretisation since it allows for a finer control of the bandwidth of the time-stepping system and, in particular, allows the formal order of the scheme to be maintained by the one sided boundary schemes without effecting the bandwidth of the linear system to be solved at each time-step (i.e. the bandwidth of \( R \)). The control of the bandwidth is of particular importance for the distributed memory implementation of these methods since it is the bandwidth of the linear systems which determines the minimal radial blocking size (See Chapter 4). In the case of variable time-step applications the cost of re-factorisation of the array \( R \) must be considered in tandem with the cost of generating the compact coefficients.

The most important application of this is the poloidal momentum equation (3.34), which after time discretisation has the form
\[
\chi = \gamma Ro D_n - \Delta t E D_n^2 .
\]

(3.129)

Evaluation of the compact coefficients proceeds as in Section 3.5; \( \chi \) is substituted into (3.102) and the coefficients of the terms in the Taylor expansion found as in (3.103). The corresponding linear system is solved as outlined above except now it is assumed the scaling is determined by the term \( \Delta t E D_n^2 \) and hence a scaling factor of \( \Delta t / h^4 \) is assumed for the \( \beta \) coefficients.

No analytic solutions to (3.102) with (3.129) were found, and hence no symbolic scheme is given. Convergence results are hence tested numerically. The test function and boundary conditions
\[
f(r) = 50r^n(1 - r^2)^3 \sin(80\pi r), \quad f(0) = f(1) = \partial_{rr} f(1) = 0,
\]

(3.130)
are chosen as a simple example of a function which matched the stress-free velocity conditions (3.36) and the condition (3.32). The parameters
\[ \gamma = \frac{11}{6}, \quad \text{Ro} = \frac{5}{7} \cdot 10^{-4}, \quad E = 5 \cdot 10^{-4}, \quad \Delta t = 10^{-6}, \quad n = 4 \] (3.131)
are chosen so E and Ro match the dynamo benchmark considered in Chapter 7 and γ matches the 3rd order time-stepper described in Section 3.5. For compact schemes developed after time discretisation it is pertinent to introduce a modified numerical error
\[ E_{\text{mod, num}} := |f - \mathbf{R}^{-1} \mathbf{L} (\chi f)| \] (3.132)
since it is f that is extracted at each time-step. Convergence rates and error profiles in the errors (3.109) as well as (3.132) are given for the J-point uniform and J-point Chebyshev [0, 1] grids, for the bandwidth preserving compact schemes in Figures 3.3 and 3.4 respectively. As with the compact schemes of Section 3.5, large relative errors at the r = 0 boundary are caused by \( \chi f \to 0 \) as \( r \to 0 \) and the spikes in the error near the \( r = 1 \) boundary are caused by a decrease in order of the approximation, as the bandwidth is being preserved.

Inspection of (3.128) suggests it would be optimal to form a compact scheme with \( l_\beta = u_\beta = 0 \) since it would result in \( \mathbf{R} = \mathbf{I} \). Indeed such schemes for (3.129) and
\[ \chi = 11/6 - \Delta t \ D_n \] (3.133)
can be easily generated and demonstrate good convergence properties in the error (3.109a). However, in cases where \( \beta_0 \neq 0 \) the truncation error was found to scale roughly as \( \beta_0 \), resulting in a horizontal convergence curve in the modified numerical error (3.132). This behaviour is demonstrated for the operator (3.133) with \( \Delta t = 10^{-5}, \ n = 4 \) and \( f \) and its boundary conditions given by (3.115) on a uniform grid at \( r = 0.8 \) for \( l_\alpha = u_\alpha = 2 \) in Figure 3.5. This demonstrates why the residual error, although yielding \( E \) directly, is an insufficient measure of the accuracy of a scheme and why it is crucial to consider the error (3.132) before applying these methods.

As a final note, unlike the schemes analysed above, the band structure of the interior central difference schemes actually used for the implicit difference was maintained at the \( r = 0 \) boundary by virtue of (3.32). This allowed the approximation (3.102) to be extended over the origin into \( r < 0 \) and mapped back into the \( r > 0 \) [e.g. by (3.38)]. In practice only the right of the approximation (3.102) was extended over the origin. This was because the largest bandwidth approximation used in an actual computation was penta-diagonal, and if \( l_\alpha = u_\alpha = l_\beta = u_\beta \leq 2 \) both the bandwidth and the order of the of the global approximation (3.108) could be maintained at this boundary. To show this, let \( l_\alpha = u_\alpha = l_\beta = u_\beta = k \) in the interior. For \( j \leq k \) the bandwidths of the compact approximation is
\[ l_\alpha = j - 1, \quad u_\alpha = k, \quad u_\beta = k, \quad l_\beta = 2k + 1 - j. \] (3.134)
Denoting \( r_{-j} = -r_j \), it can be seen that \( r_{j-l_\beta} = r_{2j-2k-1} \Rightarrow r_{2k+1-2j} \). The largest overlap clearly occurs when \( j = 1 \), in which case \( r_{2k+1-2j} = r_{2k-1} \). This is less than or equal to \( r_{j+k} = r_{k+1} \) provided \( k \leq 2 \). Since when actually time-stepping the dynamo equations the largest scheme used was the interior penta-diagonal, the origin schemes never effect the bandwidth of the global approximation. It is also pertinent at this point to reiterate that these generalised compact schemes were only used for implicit radial differencing. Explicit differencing was handled using standard 4th order centre differences in the interior and 4th order one sided differences at the boundary, with the same mapping across the origin discussed above.
Figure 3.3. Error profiles for the penta-diagonal generalised compact scheme for (3.129) on a uniform grid. (a) Interior convergence rate 6.40. (b) Outer boundary convergence rate 7.55. (c) $\log_{10}$ absolute error for $J = 300$. (d) $\log_{10}$ relative error for the $J = 300$. 

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**Figure 3.4.** Error profiles for the penta-diagonal generalised compact scheme for (3.129) on the Chebyshev $[-1, 1]$ grid. (a) Interior convergence rate 6.25. (b) Outer boundary convergence rate 4.55. (c) $\log_{10}$ absolute error for $J = 300$. (d) $\log_{10}$ relative error for the $J = 300$.

**Figure 3.5.** Convergence of the $(l_\alpha, u_\alpha, l_\beta, u_\beta) = (2, 2, 0, 0)$ compact scheme for (3.133) on a uniform grid at $r = 0.8$. 
3.6. Time-stepping

The dynamo equations of Chapter 2 describe a stiff system, with phenomena occurring over a wide range of time scales (see Hollerbach 2013 and Chapter 1). The spectral forms of the governing equations are hence time advanced using the third-order (3-step) semi-implicit extension of the stiffly stable multi-step Gear (1971) backwards differentiation method of Karniadakis, Orszag & Israeli (1991) (see also Hulsen 1996), with the one and two step methods used for the first two time steps. For the DE

\[ \frac{\partial y}{\partial t} = f(y, t) \] (3.135)

these methods are derived by first splitting \( f \) into linear (implicit) and non-linear (explicit) components \( f = f_L + f_N \) and evaluating

\[ y = y_L + y_N \] (3.136)

where

\[ \frac{\partial y_L}{\partial t} = f_L(y, t), \quad \frac{\partial y_N}{\partial t} = f_N(y, t). \] (3.137)

The implicit time integration is handled using the \( k \)-step Gear (1971) BDF

\[ \gamma y^{(k+1)}_L - \sum_{i=0}^{k} \alpha_i y^{(k-i)}_L = \Delta t f^{(k+1)}_L \] (3.138)

with \( \alpha \) and \( \gamma \) given in Table 3.1.

<table>
<thead>
<tr>
<th>Order</th>
<th>( \gamma )</th>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3/2</td>
<td>2</td>
<td>-1/2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>11/6</td>
<td>3</td>
<td>-3/2</td>
<td>1/3</td>
</tr>
</tbody>
</table>

**Table 3.1. Coefficients for the multi-step Gear BDF.**

The scheme for the explicit integration beings with a BDF for (3.137b)

\[ \frac{\partial}{\partial t} y^{(k+1)}_N = \frac{1}{\Delta t} \left( \gamma y^{(k+1)}_N - \sum_{i \geq 0} \alpha_i y^{(k-i)}_N \right). \] (3.139)

From this and (3.137b) the follow can be obtained

\[ \frac{1}{\Delta t} \left( \gamma y^{(k+1)}_N - \sum_{i \geq 0} \alpha_k y^{(k-i)}_N \right) = \frac{1}{\Delta t} \sum_{i \geq 0} \alpha_i \int_{t_{k-i}}^{t_{k+1}} \partial_t y_N \, dt \]

\[ = \frac{1}{\Delta t} \sum_{i \geq 0} \alpha_i \int_{t_{k-i}}^{t_{k+1}} f_N(y, t) \, dt \] (3.140)
where use has been made of \( \gamma = \sum_{i \geq 0} \alpha_i \), which is obtained by substituting \( y_N \equiv 1 \) into (3.137b). Approximating the integral on the right with a linear combination of \( f_N \) gives

\[
\frac{1}{\Delta t} \left( \gamma y_N^{(k+1)} - \sum_{i \geq 0} \alpha_i y_N^{(k-i)} \right) = \sum_{i \geq 0} \beta_k f_N^{(k-i)} .
\]  

(3.141)

The \( \beta \) coefficients for the explicit integration of (3.141) are found by matching coefficients in the Taylor expansion of (3.141) around \( t_k \) [or, equivalently, substituting \( u = t^p \) into (3.141) for \( p \geq 0 \)]. In order to combine this with (3.138) to solve (3.136) the \( \alpha \) and \( \gamma \) must match those used in (3.138). For application to the dynamo problem the schemes where the implicit and explicit integrations are of the same order are used. This leads to the \( \beta \) coefficients given in Table 3.2. It is this method of finding the explicit \( \beta \) coefficients which differentiates this method from a standard explicit time-stepping method — e.g. an implicit Gear method combined with an explicit Adams-Bashforth method.

<table>
<thead>
<tr>
<th>Order</th>
<th>( \beta_0 )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
</tr>
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<tbody>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>−1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>−3</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE 3.2. Coefficients for the explicit multi-step methods.

Writing the DE (3.135) as

\[
\frac{\partial}{\partial t} y(r, t) = \chi_1 y(r, t) + \mathcal{N}(y, t)
\]  

(3.142)

where the linear operator \( \chi_1 \) is independent of time and is identified with \( f_L \) and \( \mathcal{N} \) is identified with \( f_N \), the \( k \)-step time-stepping equation becomes

\[
(\gamma - \Delta t \chi) y^{(k+1)} = \sum_{i=0}^{k} \alpha_i y^{(k-i)} + \Delta t \sum_{i=0}^{k} \beta_i \mathcal{N}^{(k-i)} .
\]  

(3.143)

Discretising in \( r \) and combining (3.143) with the generalised compact scheme for \( \chi = \gamma - \Delta t \chi_1 \) as in (3.128) gives the time-stepping projection equation

\[
\begin{bmatrix}
R \\
I \\
\vdots \\
I
\end{bmatrix}
\begin{bmatrix}
y^{(k+1)} \\
y^{(k)} \\
\vdots \\
y^{(k-K+1)}
\end{bmatrix}
= \begin{bmatrix}
\alpha_0 L & \alpha_1 L & \cdots & \alpha_{K-1} L \\
I & \cdots & I & 0 \\
\vdots & I & \cdots & I \\
1 & \cdots & 1 & 1
\end{bmatrix}
\begin{bmatrix}
y^{(k)} \\
y^{(k-1)} \\
\vdots \\
y^{(k-K)}
\end{bmatrix}
+ \begin{bmatrix}
\beta_0 L & \beta_1 L & \cdots & \beta_{K-1} L \\
\Delta t & I & \cdots & I \\
\vdots & I & \cdots & I \\
I & \cdots & I & 1
\end{bmatrix}
\begin{bmatrix}
\mathcal{N}^{(k)} \\
\mathcal{N}^{(k-1)} \\
\vdots \\
\mathcal{N}^{(k-K)}
\end{bmatrix}
\]  

(3.144)
where $I$ is the identity matrix, missing entries and $0$ are arrays of zeros and $y$ and $N$ are the vectors of values of $y$ and $N$ on the discrete $r$ grid respectively. The first row of this system is the time-stepping equation solved each time-iteration.

The stability properties of these methods are considered for the prototype equation

$$
\left( \frac{\partial}{\partial t} - \omega \right) y = \lambda y \tag{3.145}
$$

where $\omega$ and $\lambda$ are interpreted as the eigenvalues of the linearised $\chi$ and $N$ in (3.142) respectively with the former treated implicitly and the latter explicitly. Since $\chi$ is associated with the diffusion operator only negative real values of $\omega$ are considered. Making the appropriate substitutions into (3.143) gives

$$
(\gamma - \omega \Delta t) y^{(k+1)} = \sum_{i=0}^{k} \alpha_i y^{(k-i)} + \Delta t \sum_{i=0}^{k} \beta_i y^{(k-i)} . \tag{3.146}
$$

The effect of $\omega$ on the region of stability is considered for the general method (3.143) with the $\alpha$ and $\beta$ coefficients given for both the implicit-explicit Gear method outlined above as well as the implicit Gear, explicit Adams-Bashforth method, since the latter results in a time-stepping equation of the same form but with different coefficients. This analysis is formulated in two ways. The first is the method used by Karniadakis, Orszag & Israeli (1991) in which the non-dimensional implicit eigenvalue $\omega \Delta t$ is given some fixed value and the stability region found in the non-dimensional explicit eigenvalue $\lambda \Delta t$ in the usual way: substitute $y^{(i)} = \zeta^{i-1}$ and $z = \lambda \Delta t$ into (3.146) to get

$$
z = \frac{(\gamma - \omega \Delta t) \zeta^k - \sum_{i=0}^{k} \alpha_i \zeta^{k-i-1}}{\sum_{i=0}^{k} \beta_i \zeta^{k-i-1}}, \tag{3.147}
$$

and solve for $z$ with $\zeta = \exp(i\phi)$ for $\phi \in [0, 2\pi)$. The stability diagrams for this problem are shown in the top row of Figure 3.6 with regions of stability the interior of the closed curves. The second is the method used by Hulsen (1996) in which the implicit eigenvalue $\omega$ is ‘normalised’ against $|\lambda|$. Setting $\mu = \omega / |\lambda|$ gives

$$
(\gamma - \mu |\lambda \Delta t|) y^{(k+1)} = \sum_{i=0}^{k} \alpha_i y^{(k-i)} + \Delta t \sum_{i=0}^{k} \beta_i y^{(k-i)} . \tag{3.148}
$$

Making the same substitutions as above gives the non-linear equation for $z = \lambda \Delta t$

$$
(\gamma - \mu |z|) \zeta^k - \sum_{i=0}^{k} \alpha_i \zeta^{k-i-1} = z \sum_{i=0}^{k} \beta_i \zeta^{k-i-1}, \tag{3.149}
$$

which is solved for $z$ for a fixed $\mu$ and $\zeta = \exp(i\phi)$ for $\phi \in [0, 2\pi]$. Attempts to use the same solution method as Hulsen (1996); a Newton-Raphson iteration scheme on the real and imaginary equations, was found to convergence very poorly for the $\mu$ considered. The method used here was to write $|\lambda| = \exp(-i\theta)\lambda$ where $\theta = \arg(\lambda)$, the principle argument of $\lambda$, and find the roots of

$$
\arg(z) - \theta = 0, \tag{3.150}
$$

where

$$
z = \frac{\gamma \zeta^k - \sum_{i=0}^{k} \alpha_i \zeta^{k-i-1}}{\mu \exp(-i\theta) \zeta^k + \sum_{i=0}^{k} \beta_i \zeta^{k-i-1}}, \tag{3.151}
$$
for a fixed $\mu$ and $\zeta = \exp(i\phi)$ with $\phi \in [0, 2\pi)$. The stability diagrams for this formulation are shown in the bottom row of Figure 3.6 with the regions of stability the interior of the closed curves.

These results are most directly applicable to the kinematic dynamo problem where, for $\mathbf{B}$ a solution, the magnetic induction equation (2.19) can be written in the form (3.145). In interpreting these results it is more useful to begin with the stability problem (3.149); the bottom row of Figure 3.6. The first thing to note is that $z = 0$ is always a solution of (3.149) provided $\gamma = \sum_i \alpha_i$ (substitute $\zeta = 1$). This means for both methods the curves going off along $\mathrm{Re}\{\lambda \Delta t\} > 0$ will, most likely, turn around and close off at the origin. The region of stability in these diagrams behaves very much as expected — as the eigenvalues of the diffusive part of the PDE become more negative the region of stability increases. This indicates that diffusion has a stabilising effect on the scheme with increasing returns as the implicit eigenvalues become increasingly negative (recall $\mu = \omega/|\lambda|$). From these diagrams we also see that only the largest (least negative) implicit eigenvalue need be considered as these generate the smallest regions of stability. A reassuring
features of both these plots is that for most $\lambda$ the time-stepper can be made stable by taking a sufficiently small $\Delta t$. The top row of Figure 3.6 has a less straightforward interpretation since for a fixed truncation (read a fixed $\omega$), decreasing $\Delta t$ corresponds to both moving towards the origin and moving towards larger (less negative) values of $\omega \Delta t$, hence the curve of marginal stability shrinks as well. These plots do, however, show the same basic behaviour as the bottom plots — increasing the damping (magnitude of $\omega$) increases the size of the region of stability and, for the most part, only the largest (least negative) $\omega$ need be considered. Fortunately, as demonstrated by the bottom row, the rate at which the marginal stability curves in the top row shrink as $\Delta t \to 0$ for a fixed $\omega$ is slower than the rate at which $\lambda \Delta t \to 0$.

Comparison of the implicit-explicit Gear and the implicit Gear, explicit Adams-Bashforth methods, the left and right columns of Figure 3.6 respectively, reveals that the splitting method of Karniadakis, Orszag & Israeli (1991) gives a slight improvement of the size of the stability regions for small magnitude $\omega \Delta t$ and $\mu$, in particular along the imaginary axis, at no additional computational cost and without a loss of order. Ideally, the scheme used would be stable for all $\text{Re}(\lambda + \omega) < 0$. Substituting $\zeta = 1$ into (3.147) reveals this is a point where the marginal stability curve for the implicit-explicit Gear method crosses the positive real axis. When using the explicit Adams-Bashforth method this limit becomes $\lambda = -\omega/\gamma$. Whether relative to the implicit Gear, explicit Adams-Bashforth method this is a gain or not would depend on the problem since the increased stretching of the stability region along the position real axis for the implicit-explicit Gear methods comes at the cost of squeezing the curves for $\text{Re} \lambda > 0$ towards the real axis relative to the implicit Gear, explicit Adams-Bashforth method.

The discussion of stability for the spherical dynamo problem in the literature has focused less on the intrinsic stability properties of the time-stepping method and more on Courant-Friedrichs-Lewy (CFL) type conditions for a given spatial discretisation. Different authors have used different CFL type conditions to dynamically modify the time-step throughout a computation in an attempt to guarantee stability, e.g.

- Glatzmaier (1984) took the time-step as the minimum of three limits; the radial diffusion time for the non-linear interaction terms $\mathbf{F} = (F_r, F_\theta, F_\phi)$, and the radial and horizontal CFL advection limits

$$
\Delta t \leq \frac{\Delta r}{r^2 F_r}, \quad \Delta t \leq \frac{\Delta r}{u_r} \quad \text{and} \quad \Delta t \leq \frac{r}{\sqrt{N(N+1)(u_\theta^2 + u_\phi^2)}},
$$

(3.152)

respectively, which were all minimised over the discrete $(r, \theta, \phi)$ grid with $\Delta r$ the local radial grid-spacing;
- Kuang & Bloxham (1999) limited the size of the time step at each time iteration so

$$
\Delta t \lambda \leq 1.1,
$$

(3.153)

where $\lambda$ was estimated from the dynamo equations linearised around the current solution

$$
\lambda \approx -\mathbf{u} \cdot \Delta \mathbf{h}^{-1} \pm \sqrt{(\mathbf{u} \cdot \Delta \mathbf{h}^{-1})^2 + 4 \left[ \frac{1}{\text{Ro}} (\mathbf{B} \cdot \Delta \mathbf{h}^{-1})^2 + \frac{\text{Ra} \partial_r \Theta_c}{\text{Ro}} \right]},
$$

(3.154)

where $\Delta \mathbf{h}^{-1}$ is the local grid spacing; and
Christensen, Olson & Glatzmaier (1999) modified the CFL conditions to consider both the Alfvén velocity \( u_A = (Pm E)^{-1/2} B \), as well as the damping effects of viscous and Ohmic dissipation. This was achieved by replacing \( u \) in the radial and horizontal CFL conditions (3.152b, c) by the modified speed

\[
|u_{r,\text{mod}}| = 2.5|u_r| + u_{A,r}^2 / \sqrt{u_{A,r}^2 + [(1 + Pm^{-1})/(2\Delta r)]^2}, \tag{3.155}
\]

with a similar expression used for the horizontal speed.

In this work the simple but risky route of accounting for such stability constraints by setting a small constant time step was adopted. This has the advantages of being simple to implement, not requiring the re-factorisation of the finite difference matrices when changing the time-step and not requiring the calculation of whatever step-size criterion is adopted [e.g. (3.155)]. This, however, came at the cost of both requiring a small time-step over the full length of a calculation and the loss of computation time to numerical instability. Notionally, implementation of a dynamically modified time-step algorithm similar to those mentioned above would be straightforward in the context of semi-implicit multi-step time-stepping methods, as solving for the \( \alpha \) and \( \beta \) coefficients in e.g. (3.143) is both simple and inexpensive, and nearly all the fields in e.g. (3.152) – (3.155) are already calculated on the \((r, \theta, \phi)\) grid as part of the explicit evaluation of the interaction terms. These methods were, however, not implemented.

### 3.7. Symmetries and problem reduction

All the methods necessary to produce a time-stepping code for the relevant problem have been given above. Some methods of reducing the problem size without effecting the physical system being numerically approximated will now be discussed.

#### 3.7.1. Real fields

The physical vector fields and scalar fields are real. When considering the SH series representation of \( f \in \mathbb{R} \)

\[
f(r, \theta, \phi) = \sum_{n \geq 0} \sum_{|m| \leq n} f_n^m(r) Y_n^m(\theta, \phi)
\]

and the property of the SH’s (3.5)

\[
Y_n^{-m} = (-)^m (Y_n^m)^* \]

it can be seen that

\[
f_n^{-m} = (-)^m (f_n^m)^*. \tag{3.156}
\]

In the context of discrete Fourier transforms this symmetry is called *conjugate even* and allows the scalar \( f \) to be stored in complex format for \( m \geq 0 \) (without the \( m < 0 \) coefficients). This allows a \( 4N + 1 \) real to \( 2N + 1 \) complex DFT to be evaluated as part of the forwards/backwards spectral transforms instead of a \( 4N + 1 \) complex to \( 4N + 1 \) complex DFT, and allows the same memory to be used for the input/output of the DFT’s (see Frigo & Johnson 2005 for implementation).
3.7.2. Reflective symmetry in the equatorial plane. Reflection in the equatorial plane \( \theta = \pi/2 \) is given by the mapping \( \theta \rightarrow \pi - \theta \). A scalar field \( f \) is even/odd symmetric in the equatorial plane if

\[
f(r, \pi - \theta, \phi) = \begin{cases} 
  f(r, \theta, \phi) & \text{if } f \text{ even} \\
  -f(r, \theta, \phi) & \text{if } f \text{ odd}
\end{cases}
\]  
(3.157)

A vector field \( \mathbf{F} = \mathbf{1}_r F_r + \mathbf{1}_\theta F_\theta + \mathbf{1}_\phi F_\phi \) is dipole symmetric if

\[
F_r(r, \pi - \theta, \phi) = -F_r(r, \theta, \phi), \quad F_\theta(r, \pi - \theta, \phi) = F_\theta(r, \theta, \phi), \quad F_\phi(r, \pi - \theta, \phi) = -F_\phi(r, \theta, \phi)
\]  
(3.158)

and quadrupole symmetric if

\[
F_r(r, \pi - \theta, \phi) = F_r(r, \theta, \phi), \quad F_\theta(r, \pi - \theta, \phi) = -F_\theta(r, \theta, \phi), \quad F_\phi(r, \pi - \theta, \phi) = F_\phi(r, \theta, \phi).
\]  
(3.159)

Letting \( e, o, d \) and \( q \) denote even, odd, dipole and quadrupole respectively, the standard vector operations have the following properties with respect to this symmetry

- Scalar-scalar product
  \[
o \cdot o \rightarrow e, \quad o \cdot e \rightarrow o, \quad e \cdot o \rightarrow o, \quad e \cdot e \rightarrow e
\]  
(3.160)

- Scalar-vector product
  \[
o \cdot d \rightarrow q, \quad o \cdot q \rightarrow d, \quad e \cdot d \rightarrow d, \quad e \cdot q \rightarrow q
\]  
(3.161)

- Vector dot product
  \[
d \cdot d \rightarrow e, \quad d \cdot q \rightarrow o, \quad q \cdot q \rightarrow e
\]  
(3.162)

- Vector cross product
  \[
d \times d \rightarrow d, \quad d \times q \rightarrow q, \quad q \times q \rightarrow d
\]  
(3.163)

Operations involving \( \nabla \) are obtained by treating \( \nabla \) as a quadrupole symmetric vector in the above identities.

In application to the scalar \( f \) given as a spherical harmonic series it is noted from the definition \( (3.3) \) that

\[
P_{n,m}(-z) = (-)^{n+m} P_{n,m}(z)
\]  
(3.164)

and hence

\[
Y^m_n(\pi - \theta, \phi) = (-)^{n+m} Y^m_n(\theta, \phi)
\]  
(3.165)

from which it is concluded that \( Y^m_n \), and hence \( f^m_n Y^m_n \) is even (odd) if \( n + m \) is even (odd). Note that this is equivalent to \( n - m \) even (odd).

For a solenoidal \( \mathbf{F}^m_n = \mathbf{S} \{ S^m_n(r) Y^m_n(\theta, \phi) \} + \mathbf{T} \{ T^m_n(r) Y^m_n(\theta, \phi) \} \), the definitions \( (3.17) \) with \( r \) quadrupole symmetric can be combined with \( (3.165) \) and the identities above to conclude

\[
\mathbf{S} \{ S^m_n Y^m_n \} = \begin{cases} 
  \text{dipole if } n - m \text{ odd} \\
  \text{quadrupole if } n - m \text{ even}
\end{cases}, \quad \mathbf{T} \{ T^m_n Y^m_n \} = \begin{cases} 
  \text{dipole if } n - m \text{ even} \\
  \text{quadrupole if } n - m \text{ odd}
\end{cases}
\]  
(3.166)

Implementation of this field decomposition allows for the storage of only half the \( \theta_k \) grid, \( \theta \in (0, \pi/2] \), but requires the separate storage of the even/odd parts of every real stored scalar field.
and the dipole/quadrupole parts of the individual components of each real stored vector field. If the initial conditions contain only $u = u_0, \Theta = \Theta_0$ with an even source $S = S_e$ and only one of $B = B_d$ or $B = B_q$, then using the result that $1_z$ is dipole the equations of Section 3.2 split into two sets which evolve independently: $\{u_q, \omega_d, B_d, J_q, \Theta_e\}$ and $\{u_q, \omega_d, B_q, J_d, \Theta_e\}$.

### 3.7.3. Azimuthal symmetry classes.

If the coefficients for $m \geq 0$ in the SH expansion of a scalar field $f$ are non-zero for $m = \{m_1, m_2, \ldots, m_p\}$ and $m_{hcf}$ is the highest common factor of these then $f$ is invariant under rotations around $1_z$ by $\Phi = 2\pi/m_{hcf}$. If this symmetry is preserved by the (quadratic) interaction terms in the dynamo equations then a saving can be made in the fully spectral code in the storage of $m$ and in the time-stepping code in the size of the forwards/backwards transforms and the storage of the real fields. This symmetry is characterised by the equivalence classes

$$I := \{k \in \mathbb{Z} \mid k = l \mod m_{hcf}\}, \quad (3.167)$$

with the harmonic orders associated with an equivalence class shown in Table 3.3. Since all fields considered are real the spherical harmonics of order $-m$ must be included whenever an order $m$ term is present. The azimuthal symmetry classes are therefore unions of the equivalence classes into what will be called $m$-classes, as given in Table 3.4.

<table>
<thead>
<tr>
<th>Equivalence Class</th>
<th>Associated $m$-Harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>\cdots \cdots 0 0 2m_{hcf} -m_{hcf} -m_{hcf} 2m_{hcf} \cdots</td>
</tr>
<tr>
<td>1</td>
<td>\cdots \cdots (2m_{hcf} - 1) -(m_{hcf} - 1) 1 (m_{hcf} + 1) (2m_{hcf} + 1) \cdots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots \vdots \vdots \vdots</td>
</tr>
<tr>
<td>$m_{hcf} - 1$</td>
<td>\cdots \cdots - (1 - m_{hcf}) (m_{hcf} - 1) \cdots</td>
</tr>
</tbody>
</table>

**Table 3.3.** Azimuthal symmetry (equivalence) classes.

<table>
<thead>
<tr>
<th>$m_{hcf}$ even</th>
<th>$m_{hcf}$ odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$-class</td>
<td>Equivalence Classes</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1 $\cup m_{hcf} - 1$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$m_{hcf}/2$</td>
<td>$m_{hcf}/2$</td>
</tr>
</tbody>
</table>

**Table 3.4.** List of m-classes

In practice this symmetry is only used as stated in the eigenvalue analysis associated with the kinematic dynamo problem. In the dynamical problem only the $m$-class $= 0$ survives. In this case a significant saving can still be made for, if $f$ contains only modes for which $m = k m_{hcf}$, then

$$f(r, \theta, \phi) = \sum_{k \geq 0} f^m(r, \theta) e^{ikm_{hcf} \phi}. \quad (3.168)$$
For the forwards transforms it is noted that $f$ is $2\pi/m_{\text{hcf}}$ periodic then

$$\frac{1}{2\pi} \int_0^{2\pi} f(r, \theta, \phi) e^{-ikm_{\text{hcf}} \phi} d\phi = \frac{m_{\text{hcf}}}{2\pi} \int_0^{2\pi/m_{\text{hcf}}} f(r, \theta, \phi) e^{-ikm_{\text{hcf}} \phi} d\phi \quad (3.169)$$

where $\tilde{f}(r, \theta, \phi) = f(r, \theta, \phi/m_{\text{hcf}})$. Thus the $k \cdot m_{\text{hcf}}$ mode of $f$ is the same as the $k$ mode of $\tilde{f}$ and consideration of $\tilde{f}$ instead of $f$ allows the Fourier series to be truncated at $\lceil \frac{4N+1}{m_{\text{hcf}}} \rceil$, where in this case the ‘odd ceiling’ is taken i.e. the smallest odd number $L$ such that $L \geq \lceil \frac{4N+1}{m_{\text{hcf}}} \rceil$. Equivalently, this can be thought of as considering only the slice of the sphere $\phi \in [0, \pi/m_{\text{hcf}})$. 
CHAPTER 4

Implementation & Parallelisation

All the methods necessary to numerically integrate the dynamo equations of Chapter 2 have been developed in Chapter 3. However, even for modest truncation levels an impractically long computational time would be required to integrate the relevant equations. In this chapter key implementation details of the algorithms described in Chapter 3 and parallelisation paradigms are considered, with two implementations of the same basic algorithm developed; one for shared memory architecture and one for distributed memory architecture.

<table>
<thead>
<tr>
<th>Routines</th>
<th>Intel Routines and Libraries</th>
<th>GPL Library</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete Fourier Transforms</td>
<td>Intel MKL DFT</td>
<td>FFTW</td>
</tr>
<tr>
<td>Basic Linear Algebra</td>
<td>Intel MKL BLAS</td>
<td>NETLIB BLAS</td>
</tr>
<tr>
<td>Factorisation and solution of Linear Systems</td>
<td>Intel MKL LAPACK</td>
<td>NETLIB LAPACK</td>
</tr>
<tr>
<td>Shared Memory Communication</td>
<td>Intel OPENMP</td>
<td>OPENMP Version 3.1 API</td>
</tr>
<tr>
<td>Distributed Memory communication</td>
<td>Intel MPI</td>
<td>MPI Version 2.1</td>
</tr>
<tr>
<td>Communication for Distributed Memory Linear Algebra</td>
<td>Intel MKL BLACS</td>
<td>NETLIB BLACS</td>
</tr>
<tr>
<td>Distributed Memory Linear Algebra</td>
<td>Intel MKL SCALAPACK</td>
<td>NETLIB SCALAPACK</td>
</tr>
</tbody>
</table>

TABLE 4.1. Intel libraries & their GPL base

All user supplied routines were written in FORTRAN95/03 and compiled using the Intel FORTRAN 13.0.1.177 compiler. A brief list of the Intel libraries used and the GPL libraries from which

http://www.fftw.org/ (Frigo & Johnson 2005)
http://www.netlib.org/blas/ (Blackford et al. 2001)
http://www.netlib.org/lapack/ (Anderson et al. 1999)
http://www.netlib.org/blacs/ (Dongarra, Van De Geijn & Whaley 1993)
http://www.netlib.org/lapack/ (Blackford et al. 1997)
4.1. Shared memory architecture

Numerical integration of the dynamo equations was parallelised on shared memory machines using the Intel OPENMP API. To achieve this the dynamical dynamo problem was divided into two consecutive blocks of work as shown in Figure 4.1.

The first block, the \( j \)-block, contains all work associated with transforming the fields from \((r, n, m)\)-space to \((\phi, \theta, r)\)-space, including the evaluation of explicit derivatives (denoted, e.g., Calculate \( B \) and \( J \)), the calculation of the interaction terms in \((\phi, \theta, r)\)-space and the transformation of these fields back to \((r, n, m)\)-space (denoted, e.g., Calculate \( F B^m \)). These calculations are performed in parallel in the radial co-ordinate \( j \), and are distributed amongst the \( N_{CPU} \) computational threads in pre-allocated radial blocks of size \( J_{block} \). These blocks are distributed amongst the computational threads in a cascading manner as outlined in Algorithm 1. Note that the processors are labelled \( 0 : N_{CPU} - 1 \). Allocating the storage of the fields in real space as well as required workspaces as thread private entities allows these transforms to be evaluated in parallel.
Algorithm 1. Blocking of the radial co-ordinate for shared memory architecture. Recall $J$ is the number of radial grid points.

exactly as they would in a serial code once the spherical harmonic coefficients on the radial grid are available from the previous time-step. Splitting the $J$-block work in this manner limits the maximum number of threads to $J$, the number of radial grid points.

The second block of work, the $n$-block, is simply the solving of the linear time-stepping equations. Since all non-linear terms as well as the Coriolis term are treated explicitly, the problem completely decouples in $n$, $m$ and each field, i.e. $s_m^n$, $S_m^n$, $e_m^n$, $T_m^n$ and $\Theta_m^n$ can be treated independently. In this implementation these linear systems are distributed to idle computational threads by field and $n$, i.e. by the coefficient matrix of the time-stepping system. These linear systems are allocated to the computational threads using an openmp parallel for loop with the dynamic scheduler. The use of the no-wait directive ensures execution of the time-stepping loop continues after each of the individual blocks of work in the larger $n$-block is finished, as outlined in Algorithm 2 (see OpenMP Application Program Interface, Version 3.1 2011). Splitting the computational work in the $n$-block in this manner limits the number of computation threads to $5N + 1$. This could easily be increased to $\frac{5}{2}N(N + 3) + 1$ by treating each $m$ (along with $n$ and field) independently. It, however, was found to be unnecessary due to the limitation on the number of threads on the available shared memory machines.

As each thread in the two computational blocks discussed above writes values to memory required by all threads in the subsequent block, some form of synchronisation is required to ensure the threads have a consistent view of the shared memory arrays. The simplest method of achieving such a synchronisation is to place a barrier directive, which not only flushes all shared variables but also suspends computation on faster threads until all threads have reached the directive, between the two blocks. In order to circumvent these costly operations, in particular the suspension of computation, we adopted a staggered synchronisation and flushing mechanism as outlined in Figure 4.1. The basic idea here is similar to that described in Bull & Ball (2003); an integer flag is updated in the routine UPDATE after a particular block of work has been completed, in this case
4.1. Shared memory architecture

!$OMP DO SCHEDULE(DYNAMIC)
    DO n = 1, N
        CALL SOLVER ROUTINE(uS, n)
    END DO
!$OMP END DO NOWAIT

!$OMP DO SCHEDULE(DYNAMIC)
    DO n = 1, N
        CALL SOLVER ROUTINE(uT, n)
    END DO
!$OMP END DO NOWAIT

Algorithm 2. Distribution of linear solves to computational threads. Note that the SOLVER ROUTINE solves all the linear systems for a fixed $n$ and input field (e.g. uS) together.

associated with a particular field, and checked in the routine CHECK before the data is re-accessed. The functioning of these routines is detailed in Algorithms 3 and 4. The notation $L$ and $N$ in Figure 4.1 is used to denote the linear operations, in this case the time-stepping, and the non-linear operations, in particular the evaluation of (most of) the explicit parts of a time-step, respectively. The use of dynamically allocated do loops in the $n$-block is intended to introduce liquidity into the synchronisation model, faster threads having computational work allocated to them as slower threads catch up, as opposed to pre-allocated blocks which at best would only remove the overhead associated with dynamically distributing the work, and at worse would reproduce the same problem a barrier directive would. To see the advantage of this synchronisation model we need only consider the case of $\Theta$ in the transition from the $j$-block to the $n$-block. With a barrier directive all work on all threads in the $j$-block would have to finish before the $\Theta^m_n$ equations were solved. With the model described above, however, all the threads have only to complete up to UPDATE $N\Theta$ before the fastest thread can start solving the $\Theta^m_n$ time-stepping equations, more of the linear solves being allocated to the faster threads while the others catch up.

The parallel memory overhead in this implementation is negligible. There are two primary instances where additional storage is required by the parallel algorithm. The first is in the forward transforms where an additional complex array of size $6K$ is required on each thread to store the left of (3.81), and the second is in the $n$-block where a complex workspace of size $J(N + 3)$ is required on each thread to store the evaluated right side of the first row of (3.144) before solving the time-stepping system. Since most of the explicitly evaluated derivatives are not needed for the time-stepping the latter of these two spaces is aliased into the thread private storage of the $(m, n, j)$-blocked derivatives, and only the larger of $J(N + 3)$ and $J_{block} (6N(N + 3) + 2)$ is allocated. A summary of the major user allocated workspaces and storage spaces are summarised in Table 4.2, where the following assumptions have been made:

1. All explicit finite difference matrices have bandwidth 10.
2. All implicit compact finite difference matrices are penta-diagonal before pivoting.
Algorithm 3. Shared memory synchronisation update. The `FLUSH(FIELD)` call performs a set of flushes conditional on the input `FIELD` and the time-step.

Algorithm 4. Shared memory synchronisation query. The `FLUSH(FIELD)` call performs a set of flushes conditional on the input `FIELD` and the time-step.
3. The pivot indices, which have been counted in the implicit CFD arrays, have $1/4$ the byte size per element of the arrays themselves. This is assuming the coefficients are stored as 16-byte double complex numbers while the indices are stored as 4-byte integers.

These memory requirements are plotted against the number of computational threads for the two truncations $(J, N) = (200, 60)$ and $(J, N) = (1000, 200)$ in Figure 4.2. For illustrative purposes we plot the memory requirements for $N_{\text{CPU}} \leq \min\{J, 5N + 1\}$, the upper bound given by the limitations discussed above. We see that the parallel memory overhead is only a small fraction of the total memory required and, for the two truncations considered, is mostly independent of the number of threads. Finally, we note that the values tabulated and plotted are only the major memory components allocated by the user and do not include internal workspaces allocated by the library routines, e.g. the storage of the Fourier transform coefficients.

<table>
<thead>
<tr>
<th>Object</th>
<th>Shared Memory Count</th>
<th>Thread Private Memory Count</th>
<th>Data Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectral Fields</td>
<td>$17JN(N + 3)$</td>
<td>0</td>
<td>Complex</td>
</tr>
<tr>
<td>Explicit Derivatives*</td>
<td>0</td>
<td>$J_{\text{block}}(6N(N + 3) + 2)$</td>
<td>Complex</td>
</tr>
<tr>
<td>Real fields</td>
<td>0</td>
<td>$34J_{\text{block}} \cdot L \cdot K$</td>
<td>Real</td>
</tr>
<tr>
<td>Implicit CFD arrays</td>
<td>$\frac{49J(5N + 1)}{4}$</td>
<td>0</td>
<td>Complex</td>
</tr>
<tr>
<td>Explicit FD arrays</td>
<td>$159J$</td>
<td>0</td>
<td>Complex</td>
</tr>
<tr>
<td>Legendre Functions</td>
<td>$6K \cdot N(N + 3)$</td>
<td>0</td>
<td>Real</td>
</tr>
<tr>
<td>$j$–block workspace</td>
<td>0</td>
<td>$6K$</td>
<td>Complex</td>
</tr>
<tr>
<td>$n$–block workspace*</td>
<td>0</td>
<td>$J(N + 3)$</td>
<td>Complex</td>
</tr>
</tbody>
</table>

**Table 4.2.** Major components of users allocated memory for shared memory architecture. Thread private memory data is given as count per thread while shared memory data is given as a total over all threads. Only the largest of the workspaces labelled with a * need be allocated.

All effort is wasted if the execution time does not decrease sufficiently fast as the number of threads is increased. To test the scalability of the algorithms discussed above the time per time-step is measured for a number of truncations on $2 \times$ Intel Xeon E5-2690 with hyper threading enabled running Debian Linux 3.2.54-pk6.22-amd64 x86_64, for up to 40 virtual processors. The time per time-step and speed-up relative to the two thread execution, $T_n/T_2$ where $T_i$ is the total execution time on $i$ processors, are shown in Figure 4.3. The algorithm is found to scale extremely well up to 40 processors, the least efficient truncation at 40 threads being $(J, N) = (100, 30)$ with a 92.0% efficiency, and the most efficient being $(J, N) = (200, 60)$ with a 99.8% efficiency.
4. Implementation & Parallelisation

In order to increase the number of machines on which to perform computations it is advantageous to develop an implementation of the methods described in Chapter 3 for distributed memory architecture. For this purpose we take the basic algorithm used for shared memory architecture and re-write the synchronisation routines and linear solver routines to perform all computations in parallel in $j$, using the Intel MPI library to facilitate communications between processes.

The basic work-flow for a single time-step on a distributed memory machine is shown in Figure 4.4 and is identical to that which would be used for a serialised code with the exception of the send/receive blocks. As the practical number of processors available was limited to 40\footnote{Sydney University Silica Cluster \url{http://www.hpcf.chem.usyd.edu.au/index.html}} and the bulk of the computation was intended for the shared memory implementation we took the simplest

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_2.png}
\caption{Memory requirements for shared memory implementation vs $N_{\text{CPU}}$ for two truncations.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_3.png}
\caption{Shared memory scalability tests on $2 \times$ Intel Xeon E5–2690 with hyper threading enabled running Debian Linux 3.2.54–pk06.22–amd64 x86_64.}
\end{figure}

4.2. Distributed memory architecture

In order to increase the number of machines on which to perform computations it is advanta-

approach and arranged the MPI processes into a single vector, distributing the data and workload by contiguous \(j\)-blocks along this vector. This type of data and workload distribution is commonly referred to as a radial-splitting algorithm and is known to have very poor scaling properties for a large number of processors (see e.g. Marti 2012). It is, however, very quick and simple to implement and, because of the restrictions previously indicated, there is no advantage to building a more complex but better scaling algorithm. We do, however, outline how the algorithm given here could be modified in the same vain as the tubular splitting algorithm of Marti (2012) to improve the scalability. This will be discussed at the end of this chapter.

The data distribution is almost identical to the \(j\)-block distribution of the shared memory implementation discussed above except it is now also applied to the \((j, n, m)\)-blocked fields as well. Each MPI process is allocated a radial block of size

\[
J_{\text{block}} = \left\lceil \frac{J}{N_{\text{CPU}}} \right\rceil,
\]

the final processes receiving whatever remains. In the distributed memory code the evaluation of the explicit terms proceeds exactly as in the shared memory implementation. As each process works on non-overlapping radial blocks a communication is required both before the backwards transforms [in order to evaluate explicit derivatives in \((j, n, m)\)-space], and after the forward transforms [in order to evaluate the explicit derivative in (3.88), as well as for the evaluation of the right of the time-stepping equations (3.144)]. The use of compact finite difference methods does not increase the number of communications — both would still be required if standard finite difference methods were used — but does increase the size of the latter communication. This results in the very simple communication pattern shown in Figure 4.5 where the circles represent the MPI processes with lines and arrows giving the direction of a data transfer.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_4}
\caption{Work-flow for single time-step on distributed memory architecture.}
\end{figure}
In order to maintain this pattern and avoid re-distributing the full \((j, n, m)\)-data before and after every time-step we use the banded divide and conquer distributed linear algebra solver provided in the Intel implementation of the Scalable Linear Algebra Package (SCALAPACK) (Blackford et al. 1997, Cleary & Dongarra 1997). For brevity we do not discuss the inner workings of this algorithm and instead refer the interested reader to the sources referenced. Arbenz & Gander (1994) and Arbenz et al. (1999) have considered the scaling properties of these algorithms. For the \((n \times n)\) linear system

\[
A \mathbf{x} = \mathbf{b}
\]

where the matrix \(A\) has upper/lower bandwidth \(k_u\) and \(k_l\), the maximum number of cores that can participate in a single linear solve is

\[
N_{\text{CPU}}^{\text{max}} = \frac{n}{k_l + k_u + 1}.
\]

These authors have found that the algorithm performs a linear solve faster as the number of processes is increased to this limit, but the speed-up very quickly saturates and diminishing efficiency is seen well before the limit (4.2) is reached (e.g. Arbenz et al. 1999, Fig. 2.4). The scalability of the distributed memory time-stepping algorithm therefore hinges on the balance between the decreased wall-time associated with splitting the evaluation of the (explicit) interaction terms, and the parallel overhead and communication time associated with data synchronisation between the processes and the distributed linear algebra.

The use of compact finite difference methods for the implicit derivatives means that the bandwidth of the explicit finite difference matrices dictates the smallest radial block size for which the communication pattern shown in Figure 4.5 can be maintained. The use of hepta-diagonal central difference schemes for all explicitly evaluated radial derivatives puts the condition \(J_{\text{block}} \geq 6\), and hence \(N_{\text{CPU}} \leq \lfloor J/6 \rfloor\). On 40 processors this puts the lower bound \(J = 240\) on the number of radial grid points. Restrictive limits of this form are the primary argument against straight radial splitting algorithms for distributed memory codes. As most computations were performed using the shared memory code and the distributed memory code was used on a maximum of 40 processors, there was no need to increase the complexity of the code in order to relax the radial splitting restrictions.

Unlike the shared memory implementation there is significant memory overhead associated with the distributed memory code. The additional memory requirements stem from three primary sources; the need to store the full set of Legendre functions as well as their derivatives (as outlined in Section 3.4.3) on every process; the additional buffers required to store incoming/outgoing data; and the additional blocks allocated by SCALAPACK used in the matrix re-blocking and pivoting, as well as the buffers for the incoming/outgoing data used as part of each linear solve. The significant user allocated sources of memory consumption for the distributed memory code are tabulated in Table 4.3 and plotted for two truncations in Figure 4.6. As with the shared memory case,
workspaces allocated internally by the library routines (e.g. the Fourier transform coefficients) are not counted. It can be seen from Figure 4.6 that although the memory overheads are large, they are small for each process, and thus of minimal concern.

<table>
<thead>
<tr>
<th>Object</th>
<th>Local Count</th>
<th>Data Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectral Fields</td>
<td>$17(J_{\text{block}} + 6)N(N + 3)$</td>
<td>Complex</td>
</tr>
<tr>
<td>Explicit Derivatives*</td>
<td>$J_{\text{block}} \cdot 6N(N + 3) + 2$</td>
<td>Complex</td>
</tr>
<tr>
<td>Real fields</td>
<td>$34 J_{\text{block}} \cdot L \cdot K$</td>
<td>Real</td>
</tr>
<tr>
<td>Implicit CFD arrays</td>
<td>$\frac{1}{4} \left( 184 + 61 J_{\text{block}} + 1456 N + 325 J_{\text{block}} \cdot N \right)$</td>
<td>Complex</td>
</tr>
<tr>
<td>Explicit FD arrays</td>
<td>$159(J_{\text{block}} + 6)$</td>
<td>Complex</td>
</tr>
<tr>
<td>Legendre Functions</td>
<td>$6K \cdot N(N + 3)$</td>
<td>Real</td>
</tr>
<tr>
<td>$j$–block workspace</td>
<td>$6K$</td>
<td>Complex</td>
</tr>
<tr>
<td>$n$–block workspace*</td>
<td>$J_{\text{block}}(N + 9)$</td>
<td>Complex</td>
</tr>
<tr>
<td>MPI Buffers</td>
<td>$53N(N + 3)$</td>
<td>Complex</td>
</tr>
</tbody>
</table>

**Table 4.3.** Major components of user allocated memory for distributed memory architecture. The $n$-block and $j$-block workspaces are of identical use to the same variables in the shared memory case. As with the shared memory case a * denotes that only the largest of these workspaces need be allocated.

![Graphs](image)

**Figure 4.6.** Memory requirements for distributed memory implementation at two truncation levels. For illustrative purposes theoretical memory requirements are plotted for up to the maximum processes number $N_{\text{CPU}} = \lfloor J/6 \rfloor$. 
This algorithm is found to have very poor scaling properties. Shown in Figure 4.7 are the wall time versus number of processes, and the corresponding speed-up for the distributed memory algorithm. These results are consistent with the scaling test of Marti (2012, Fig. 4.7) for the same problem splitting. The discrete jumps we see in Figure 4.7 (right) are reminiscent of the scaling properties of the SCALAPACK banded matrix solver predicted by Arbenz et al. (1999, Fig. 2.4). To test if the poor scaling properties are driven by the distributed linear algebra we repeat the scaling tests omitting either the distributed linear algebra or the transform routines, in the case of the former while still retaining the synchronisation routines. The results of these tests are shown in Figure 4.8, where we see the linear solver routines scale very poorly relative to the transform routines. The computational time associated with solving the linear systems is dwarfed by the computational time associated with the transform and synchronisation routines, as inferred from the comparison of Figure 4.8 and Figure 4.7(b). From this we conclude that the loss of efficiency by the use of distributed linear algebra is negligible compared to the losses incurred by using a straight radial splitting algorithm. This is further reinforced by the results of Marti (2012), where a radial splitting algorithm without distributed memory linear algebra was tested, yielding similar results to those obtained here.

![Figure 4.7](image_url)

**Figure 4.7.** Scaling tests of the distributed memory implementation on $2 \times$ Intel Xeon E5–2690 with hyperthreading enabled running Debian Linux 3.2.54–pk06.22–amd64 x86_64. Computations labelled *dist mem* was performed on the SILICA cluster.
4.3. Future directions

We now briefly discuss two extensions to the distributed memory algorithm outlined above as possible methods of increasing the number of processes which can perform a single computation in parallel.

4.3.1. Row/Column Splitting Algorithm. The first extension we discuss is a row/column splitting algorithm. The basic idea is almost identical to the tubular splitting algorithm of Marti (2012), with the exception of the retention of distributed linear algebra for solving the time-stepping systems. This would work by arranging the processes into an \((N_{\text{row}} \times N_{\text{col}})\) array, each column of which is designated a contiguous radial block, denoted \(J_{\text{col}}\), and each row designated some subset of the linear solves. The solution of the time-stepping systems would proceed exactly as described above, except now each process row performs independent linear solves in parallel.

\[\text{Figure 4.9. Communication pattern for an extended distributed memory implementation. Circles represent MPI processes and lines correspond to communications with arrowheads indicating the direction of data flow. Note that this is untested.}\]
These steps would result in the communication patten shown in red in Figure 4.9 where each process communicates along the rows only with its nearest neighbours. This communication would be identical to the corresponding communications in the radial splitting implementation, including the update of the boundary blocks, but working on only a subset of the data determined by the distribution of the linear systems down into the rows.

This algorithm differs from the radial splitting discussed in section 4.2 in the treatment of the interaction terms. Like Marti (2012), the idea would be to broadcast the radial data up/down each column so each process within a column had access to the full column-radial block $J_{\text{col}}$, as well as the boundary blocks, for every $(n, m)$ for every field. Each process within a column would then calculate the interaction terms on some pre-allocated subset of $j \in J_{\text{block}}$ and $k = 1 : K$, for the full set $l = 0 : L$, up to the $K$-sums of (3.81b), (3.87) and (3.88), which would be calculated as the $k$-partial sum of whatever block was assigned to each process. The partial sums would then be broadcast up/down each column as necessary and combined to give the non-linear interaction terms in spectral space. The computation would then pick up from the updating of the boundary blocks for the row-based solves and the time advanced one time-step. The evaluating of the non-linear terms would result in the dense communication patten shown in blue in Figure 4.9 where every process with a column communicates to all other processes in the same column. The resulting work-flow diagram for a single time-step is shown in Figure 4.10.
Blocking the data this way greatly increases the maximum number of processes that can be used for a single calculation. As with the radial-splitting, the number of columns is limited to $N_{\text{col}} \leq \lfloor J/6 \rfloor$, but the number of rows is limited by $N_{\text{row}} \leq K \cdot J_{\text{col}}$. Unlike the radial splitting algorithm of Marti (2012), however, it is likely not optimal to set $N_{\text{row}}$ as close to $N_{\text{col}}$ as possible, as the communication patterns between the two are vastly different. Without implementation and numerical testing little more can be said of this splitting algorithm.

4.3.2. Hybrid MPI/OPENMP. We very briefly discuss one more possible extension to the radial splitting distributed memory algorithm discussed above which uses both shared and distributed memory parallelisation. We propose exactly the tubular splitting method of Marti (i.e., without the use of distributed linear algebra), but with each process column parallelised over a shared memory system. Calculation of the interaction terms would proceed exactly as described in the row/column splitting algorithm except now the $K$-partial sums are evaluated on the shared memory architecture, the sum of the partial sums calculated after a data synchronisation. Redistributing the full data set across the individual MPI processes then allows the linear systems to be solved exactly as in the shared memory implementation discussed above and the time-step repeated.

Although this algorithm has yet to be implemented it is clear that without a sufficient number of cores on each individual computer in a distributed memory cluster the added coding and work-flow complexity would not justify the possible gains, especially in light of the good scaling properties of the splitting method of Marti (2012).
In this chapter dynamo action is considered in a full sphere with a prescribed velocity fixed at its initial value. Three classes of flow are considered. The first flow, considered in Section 5.1, is the $s_2^3 t_2^3$ Beltrami flow with the radial dependence of Pekeris, Accad & Shkoller (1973). This model is used as the first test of the numerical integrator described in Chapters 3 and 4. This model was selected as it three-dimensional, has been well studied and shows good convergence properties for modest truncation levels (e.g. Dudley & James 1989).

The second and third flow classes considered form part of a numerical investigation of spherical antidynamo theorems for flows with missing components in spherical polar co-ordinates. We consider flows with zero theta component $\theta u^\star = 0$ (ZTC flows) in Section 5.2, and flows with zero azimuthal component $\phi u^\star = 0$ (ZAC flows) in Section 5.3, with the ZAC dynamos of Moss (2006) considered in Section 5.4. The most general expression for flows of this form is

$$u^\star \cdot \xi = 0$$

for some vector field $\xi$. In this work we consider only cases where $\xi = \nabla \xi$. This is equivalent to an $\xi$ which satisfies

$$\xi \cdot \nabla \times \xi = 0,$$

for, if (5.2) holds then $\xi$ can be written as

$$\xi = \alpha \nabla \beta$$

for scalar functions $\alpha$ and $\beta$ (Brand 1964), and hence (5.1) implies

$$u^\star \cdot \nabla \beta = 0.$$  

If we write

$$\xi = \nabla \xi$$

the ZTC flows has $\xi = \theta$, and the ZAC flows has $\xi = \phi$.

The proof of antidynamo theorems for flows with missing components generally rely on a separation of the magnetic induction equation into two scalar equations for separate field components. In the simplest cases this is achieved by dotting the magnetic induction equation (2.10) with $\xi$, where $\xi \cdot u^\star = 0$, and then showing that this implies the decay of $B^\star \cdot \xi$, which in turn implies the decay of the other part of $B^\star$ (see e.g. Zel’dovich & Ruzmaikin 1980 and Busse & Proctor 2007). These proofs rely on the key simplification

$$\xi \cdot \nabla^2 B^\star = \nabla^2 (B^\star \cdot \xi),$$

which is indeed true when $\xi = 1_r$ and $\xi = const$, as in the toroidal flow and planar flow antidynamo theorems respectively. Equation (5.6) is not true for general $\xi$, and certainly does not hold.
for $\xi = 1_\theta$ and $\xi = 1_\phi$, as in the cases of ZTC and ZAC flows respectively.

Taking $\xi \cdot (2.10)$, and assuming first that components of $\xi$ are differentiable, second that $\xi$ is irrotational, third that the vector $\xi$ is independent of time, and finally that $u_\ast \cdot \xi = 0$, $\nabla_\ast \cdot u_\ast = 0$ and $\nabla_\ast \cdot B_\ast = 0$, gives

$$\partial_t \ast (B_\ast \cdot \xi) = \eta \xi \cdot \nabla_\ast^2 B_\ast - (u_\ast \cdot \nabla_\ast) (B_\ast \cdot \xi)$$

(5.7)
or equivalently,

$$\left( \frac{D}{Dt} \right) \ast (B_\ast \cdot \xi) = \eta \xi \cdot \nabla_\ast^2 B_\ast .$$

(5.8)

In a frame moving with the fluid this reduces to a diffusion equation for $\xi \cdot B_\ast$, with source term

$$\eta \xi \cdot \nabla_\ast^2 B_\ast - \eta \nabla_\ast^2 (B_\ast \cdot \xi) ,$$

(5.9)

which is zero when (5.6) holds. When (5.6) does not hold there is diffusive coupling between $B_\ast \cdot \xi$ and the other part of $B_\ast$, providing a source term in (5.8) for the generation of $B_\ast \cdot \xi$.

As observed by Bachtiar, Ivers & James (2006), when the conducting fluid is confined to a finite volume $V$, even if (5.6) holds, there can be diffusive coupling introduced by the boundary conditions, which can mean an antidynamo theorem proven for an infinite fluid can fail for a finite fluid (in this case for the planar flow theorem for $V$ a sphere with insulating exterior). There is hence no reason to expect that the ZTC and ZAC flows cannot act as dynamos, and we seek to verify this numerically by searching for examples of working dynamos with flows of these forms.

We note that there is no general anti-dynamo theorem for flows of the form (5.1), since $\nabla \cdot u = 0$ implies $u$ can be written as

$$u = \nabla \zeta \times \nabla \xi ,$$

(5.10)

for some scalar functions $\zeta$ and $\xi$ (Brand 1964). The choice $\xi = \nabla \xi$ thus satisfies (5.1), and hence the component $\xi \cdot B_\ast$ can only be generated by diffusive coupling to the other part of $B$, similarly for $\zeta = \nabla \zeta$. This emphasises the importance of both vector diffusion and flow topology in the dynamo process (e.g. Soward 1990); all incompressible dynamos having a magnetic field with some components which are only diffusively coupled.

The reader is reminded that the flows are not normalised before a computation and the quoted magnetic Reynolds numbers need to be scaled by an appropriate measure of the velocity. For all flows considered we give the maximum fluid speed.

### 5.1. PAS flow in a sphere

The first kinematic dynamo we consider is the stationary dynamo of Pekeris, Accad & Shkoller (1973). These authors studied the stationary Beltrami $s_2^2 t_2^2$ flow defined by

$$s_2^2 = (s_2^{-2})^* = \Lambda \sqrt{\frac{6}{5}} j_2 (\Lambda r) , \quad t_2^2 = (t_2^{-2})^* = \Lambda^2 \sqrt{\frac{6}{5}} j_2 (\Lambda r)$$

(5.11)

where $j_2$ is the spherical Bessel function of second order

$$j_2(z) = \left( \frac{3}{z^3} - \frac{1}{z} \right) \sin z - \frac{3}{z^2} \cos z$$

(5.12)
and Λ is any non-zero root of \( j_2 \), taken to be the third root \( Λ = 12.32294097056658 \). Substituting this into (3.20) with (3.2 – 3.3) gives the velocity components

\[
\begin{align*}
    u_r &= \frac{18Λ \sin^2(θ) \cos(2φ) j_2(Λr)}{r} \\
    u_θ &= \frac{3Λ \{ \sin(2θ) \cos(2φ) [Λ j'_2(Λr) + j_2(Λr)] - 2Λr \sin(θ) \sin(2φ) j_2(Λr) \}}{r} \\
    u_φ &= \frac{-6Λ \sin(θ) \{ Λr \sin(2φ) j'_2(Λr) + j_2(Λr) [Λr \cos(θ) \cos(2φ) + \sin(2φ)] \}}{r}
\end{align*}
\]

(5.13)

where \( j'_2 \) is the radial derivative of \( j_2 \). This gives the maximum fluid speed \( U_{\text{max}} = 284.345 \).

This dynamo is considered as a first test of the numerical integrator described in Chapters 3 and 4, but has been of historical interest as it is able to generate a magnetic field with leading harmonics with magnitudes which approximately match those of the geomagnetic field.

Growth rates are calculated by fitting an exponential growth/decay to the internal magnetic energy, initialised with a random seed field. In the first instance the dynamo was integrated using the interior penta-diagonal central difference compact scheme for \( D_n \), with one sided boundary schemes which preserve the bandwidth of both the left and the right compact difference arrays used at the boundary. Initially no azimuthal symmetry was imposed even though the velocity has \( m_{\text{hcf}} = 2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_1.png}
\caption{Growth rates of the dominant magnetic mode as a function of the magnetic Reynolds number for the PAS kinematic dynamo. (Left) Growth rates generated using the time-stepping method superimposed on the growth rates obtained for the \( m_{\text{class}} = 1 \) modes using the eigenvalue code. (Right) relative difference (relative to eigenvalue rates) between the growth rates generated using the two methods.}
\end{figure}

The growth rates obtained for \( R_m \in [0, 0.2] \) using both the time-stepping code as well as an eigenvalue code based on the spectral equations described in Appendix C combined with second order finite differences in radius, are shown in Figure 5.1. These were generated with \( (J, N, Δt) = (300, 30, 10^{-4}) \), with the convergence of the growth rate for \( R_m = 0.2 \) given in Table...
5.1. PAS FLOW IN A SPHERE

Table 5.1. Convergence of the growth rate for the dominant mode at $R_m = 0.2$ for the PAS spherical kinematic dynamo.

<table>
<thead>
<tr>
<th>Interior Scheme</th>
<th>J</th>
<th>N</th>
<th>$\Delta t$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penta-diagonal</td>
<td>200</td>
<td>20</td>
<td>1E-4</td>
<td>1.362658</td>
</tr>
<tr>
<td>Penta-diagonal</td>
<td>300</td>
<td>30</td>
<td>1E-4</td>
<td>1.362269</td>
</tr>
<tr>
<td>Penta-diagonal</td>
<td>300</td>
<td>30</td>
<td>1E-5</td>
<td>1.362265</td>
</tr>
<tr>
<td>Penta-diagonal</td>
<td>400</td>
<td>35</td>
<td>1E-4</td>
<td>1.362132</td>
</tr>
<tr>
<td>Tri-diagonal</td>
<td>300</td>
<td>30</td>
<td>1E-4</td>
<td>1.362265</td>
</tr>
<tr>
<td>Eigen-solver</td>
<td>300</td>
<td>30</td>
<td>–</td>
<td>1.361955</td>
</tr>
</tbody>
</table>

Plots of the field components on azimuthal slices are shown for the dynamo solution at $R_m = 0.2$ in Figure 5.2. At this magnetic Reynolds number the magnetic field is concentrated near the origin. This was the impetus for repeating the calculation with the tri-diagonal compact scheme, as this scheme is fourth order in the interior and at the origin, but is only second order at the boundary. Unsurprisingly, we see this has little effect on the calculated dominant growth rate.

The magnetic growth rates shown in Figure 5.1 differ from those found by Dudley & James (1989) due to differing azimuthal symmetry classes (see Section 3.7.3). Dudley & James considered the $m_{\text{class}} = 0$ modes; the time-stepping code did not split any azimuthal symmetry classes. To reconcile the two sets of results the growth rates for the $m_{\text{class}} = 0$ modes were found using the compact spectral eigenvalue code and compared to the growth rates given in Dudley & James (1989). The growth rates are shown in Figure 5.3 with the growth rate at $R_m = 0.4$ found to be

$$\lambda_{0.4} = 2.02224 \pm 0.021,$$ (5.14)

where the bracketed value is the growth rate calculated by Dudley & James (1989).

The selection of this flow to validate the code is fortuitous in that inspection of Figure 5.2 reveals almost all the field structure is close to the origin, testing the differencing near the coordinate singularity and the implementation of origin conditions (3.55). However, this dynamo has solution branches with close eigenvalues. In a time-stepping code this is manifested as a very slow convergence of the measured growth rate with time. This effect is demonstrated in Figure 5.4 where the time derivative of the measured magnetic growth rate for $R_m = 0.2$ is plotted against time. This dynamo requires around 27 diffusion times to separate the dominant mode from the subdominant mode. Using an eigenvalue approach these are easily separated using a shifted inverse iteration or Arnoldi method, and was found to have (real) growth rate

$$\lambda_{\text{sub-dominant}} = 0.7136434,$$ (5.15)

belonging to the same symmetry class as the dominant mode.

This result highlights the trade-off accepted when using a general time-stepping approach as opposed to an eigenvalue approach. When time-stepping, much higher truncation levels are achievable for the same storage requirements at the expense of computation time. Additionally, with a
Figure 5.2. Magnetic field profiles for the dominant mode of the Pekeris, Accad & Shkoller (1973) kinematic dynamo at $R_m = 0.2$ in azimuthal slices. Rows correspond to $\theta = 0$, $\pi/4$, $\pi/2$ respectively.
5.2. Flows with ZTC in a sphere

We now consider dynamos with missing flow components in spherical polar co-ordinates. The first we consider is the zero theta component (ZTC) flow.

Recently a hybrid method which combines an Arnoldi eigenvalue method with a time-stepper to simultaneously compute multiple eigenvector / eigenvalue pairs has been developed by Willis & Gubbins (2004) to study kinematic dynamo action for periodic flows. Motivated by this combination of standard eigenvalue and time-stepping methods, an attempt was made to formulate a hybrid time-stepping inverse iteration eigenvalue method for the steady kinematic dynamo problem. The development of this method and initial results are given in Appendix D. In summary: in order to achieve accelerated convergence too much prior information was required about the unknown sought eigenvalues.

5.2. Flows with ZTC in a sphere

We now consider dynamos with missing flow components in spherical polar co-ordinates. The first we consider is the zero theta component (ZTC) flow.
5.2.1. Flow definition. The ZTC-flows are flows with no $\theta$ component. The condition $u_\theta = 0$ with (3.20) implies

$$\partial_\theta (rt) = -\sin \theta \partial_\theta \partial_r (rs).$$

(5.16)

and hence

$$\sum_{n,m} im r_n^m s_n^m = -\sum_{n,m} \partial_r (rs^m_n) \{nc_{n+1}^m Y_{n+1}^m - (n+1)c_n^m Y_{n-1}^m \}$$

$$= -\sum_{n,m} \{ (n-1)c_n^m \partial_r (rs_{n-1}^m) - (n+2)c_{n+1}^m \partial_r (rs_{n+1}^m) \} Y_n^m.$$  (5.17)

Taking the inner product with $Y_n^m$ gives

$$im r_n^m = -(n-1)c_n^m \partial_r (rs_{n-1}^m) + (n+2)c_{n+1}^m \partial_r (rs_{n+1}^m).$$  (5.18)

If $m = 0$ the flow is defined by

$$t_0^n f(r),$$  (5.19)

where the only restriction on $f$ is that $t_0^n$ must be of the form (3.32). Flows of this form cannot function as dynamos by the toroidal flow anti-dynamo theorem. If $m \neq 0$ then the flow is defined by

$$t_{n-1}^m = -i \frac{(n+1)c_n^m \partial_r (rf)}{m} \frac{r}{r^2}, \quad s_n^m = f, \quad t_{n+1}^m = i \frac{n c_{n+1}^m \partial_r (rf)}{m} \frac{r}{r^2}$$  (5.20)

where $f$ is some arbitrary function of $r$. To satisfy $1_r \cdot u = 0$ on the boundary $r = 1$, $f(r)$ must satisfy $f(1) = 0$. Additionally, $f$ must be chosen so that both $s_n^m$ and $t_n^m$ satisfy the condition (3.32).

Flows of increasing complexity can be formed by increasing the complexity of $f$ or by taking linear superposition of these flows. The flows considered are generated by a single $s_n^m$ with four different radial structures considered. These are fully described by the ordered quadruple $(n, m, a, b)$ and the radial functions

I – single radial cell flows $[a = 0, b = 0]$

$$f(r) = r^{n+2}(1 - r^2);$$  (5.21)

II – double radial cell flows $[a \neq 0, b = 0]$

$$f(r) = r^{n+2}(1 - r^2)(a^2 - r^2);$$  (5.22)

III – single radial cell flows with a conducting mantle $[a = 0, b \neq 0]$

$$f(r) = r^{n+2}(b^2 - r^2)^2(1 - H(r - b))$$  (5.23)

where $H$ is the Heaviside function; and

IV – double radial cell flows with a conducting mantle $[a \neq 0, b \neq 0]$

$$f(r) = r^{n+2}(a^2 - r^2)(b^2 - r^2)^2(1 - H(r - b)).$$  (5.24)

where $a \in [0, 1]$ and $b \in [0, 1]$ with $b > a$ when $b \neq 0$. Note that in the case of a conducting mantle, this implies $u \neq 0$ when $r < b$, and $u = 0$ when $r > b$. All computations have a perfectly insulating exterior.
5.2. Boundary conditions for flows with a conducting mantle. For flows with a conducting mantle care must be taken to ensure the tangential component of the electric field is continuous across the core-mantle boundary

\[ \mathbf{n} \times \mathbf{E}_{r=b} = \mathbf{0}. \]  
(5.25)

Assuming the conducting mantle has the same electromagnetic properties as the conducting fluid, (5.25) implies

\[ \mathbf{n} \times (\nabla \times \mathbf{B} - \mathbf{u} \times \mathbf{B})_{r=b} = \mathbf{0}. \]  
(5.26)

Having \( \mathbf{u} \) satisfy a no-slip condition at the CMB, enforced by having the flow components (5.20) satisfy (3.35), and hence having \( f \) satisfy

\[ f(r = b) = \partial_rf(r = b) = 0, \]  
(5.27)

reduces (5.26) to

\[ [\mathbf{n} \times (\nabla \times \mathbf{B})]_{r=b} = \mathbf{0}. \]  
(5.28)

The condition (5.27) is satisfied in this work by squaring the factor \((b^2 - r^2)\) in (5.23) and (5.24). Noting that (5.28) implies \[ [\nabla_h \cdot (\nabla \times \mathbf{B})]_{r=b} = 0 \] and using (3.22), (3.23) and (3.26) with \( S \) and \( T \) the spherical harmonic series (3.31) gives

\[ [T_m^m]_{r=b} = [\partial_rT_m^m]_{r=b} = 0. \]  
(5.29)

This condition on the toroidal field is accompanied by an identical condition on the poloidal field

\[ [S_n^m]_{r=b} = [\partial_rS_n^m]_{r=b} = 0. \]  
(5.30)

which is easily obtained from (2.11b). Note that these conditions must be satisfied regardless of any other condition (e.g., fixing \( \mathbf{1}_\theta \cdot \mathbf{u} = 0 \)) placed on the velocity.

There are two obvious ways to enforce these conditions in the implicit time-stepping. The first is to follow the method used by Bachtiar (2009) for the inclusion of an inner core with the same electrical conductivity as the fluid. Bachtiar suggested making the velocity sufficiently smooth at the boundary to allow for differencing of the magnetic field across the boundary to remain unchanged. Noting that the magnetic spectral interaction terms (Appendix C, Equations (C.5) and (C.12)) involve up to the first derivative of \( t_n^m \) and up to the second derivative of \( s_n^m \), the magnetic potentials \( S_n^m \) and \( T_n^m \) can be made \( p \) times continuously differentiable in \( V \) if \( t_n^m \in C^{p+1}(r) \) and \( s_n^m \in C^{p+2}(r) \) for all \((n,m)\). This can be achieved by raising the factor \((b^2 - r^2)\) to the power \( p + 3 \). The finite differencing across the boundary can be used unaltered if \( p \) is greater than or equal to the highest derivative matched in both the implicit and explicit finite differencing. The second method is to difference directly using the conditions (5.29) - (5.30). In the poloidal equation this can be achieved by introducing the extra variables \( \partial_rS_n^m(r = b^+) \), \( S_n^m(r = b^+) \) and \( S_n^m(r = b^-) \), and using one sided differencing at \( r = b^+ \) and \( r = b^- \) supplemented with the matching condition \( S_n^m(r = b^+) = S_n^m(r = b^-) \), with the same process for the toroidal potentials. The latter method is adopted as it places fewer constraints on the function \( f \), in particular allowing for higher shears near the CMB. If the compact differencing is tri-diagonal, implementing this condition does not effect the bandwidth or order of the time-stepping system.

In practice all computations involving a conducting mantle used interior tri-diagonal central difference schemes, with the one-sided boundary scheme which preserved the tri-diagonal structure used at the CMB, for the implicit radial differencing, the equivalent penta-diagonal scheme being
used for all other radial structures. The use of lower order differencing for computations with a conducting mantle was motivated primarily by a loss in accuracy and introduction of modes with large imaginary parts in the free decay spectra. Shown in Figure 5.5 (left) are the numerical free decay spectra for the \( n = 1 \) toroidal free decay problem (see Section 3.2.3) where one sided differencing has been used at the CMB on the \( J = 100 \) uniform grid. These are the eigenvalues of the generalised eigenproblem

\[
R x = \lambda L x
\]  

(5.31)

where \( L \) and \( R \) are the compact difference matrices for \( D_n \) (see Section 3.5), and are found using a sparse Arnoldi eigensolver. Shown in Figure 5.5 (top right) are the relative errors for the first 21 modes. It can be seen that the tri-diagonal scheme generates better results than the penta-diagonal scheme. No obvious reason for this difference could be found, and we simply adopt the tri-diagonal scheme. We note that this effect is not seen when the free-decay spectrum is computed without a conducting mantle (Figure 5.5 bottom).
5.2.3. Streamlines. The streamlines for the ZTC flows are the level contours of the stream function $\psi$. The solenoidal condition on $u$ with $u_\theta = 0$ gives

$$
\begin{align*}
  u_r &= \frac{\partial\phi\psi}{(r \sin \theta)^2}, \\
  u_\phi &= \frac{\partial_r\psi}{r \sin \theta}.
\end{align*}
$$

(5.32)

Using (3.20) with (5.32) the following expression is obtained for the stream function

$$
\psi_n^m = \left( m^2 + (n - 1)(n + 2)c_n^m + n(n + 2)c_{n+1}^m \right) rf.
$$

(5.33)

The streamlines for these flows are confined to the surfaces $\theta = \text{const}$ which are cones embedded in the sphere with apex at the origin and axis along the $z$-axis. The $(n, m)$ single mode flow has $2m$ evenly spaced angular cells with increasing $n$ pushing these cells towards the CMB. Streamlines are plotting for a number of flows below. To help visualise these flows we plot the isosurfaces $\pm \frac{1}{2} \max |\psi_n^m|$ for the $(n, m) = (3, 2)$ single radial cell flow in Figure 5.6 – the flow forming vortices in the longitudinal cross sections of these tubes, the direction given by the sign of $\psi_n^m$.

![Figure 5.6. Isosurfaces of $\pm \frac{1}{2} \max |\psi_n^m|$ for the $(4, 3)$ ZTC single radial cell flow.](image)

5.2.4. Growth rates. A summary of the computations performed with these flows is given in Table 5.3 with streamlines and growth rates as a function of the magnetic Reynolds number $R_m$ for a number of these flows given in Figures 5.7 – 5.9. Unless otherwise stated the $m$-class = 0 magnetic field was considered — i.e. $m_{hcf} = m$, the harmonic order of the stationary velocity, was taken.

The growth rates for these flows were found to be sensitive to the truncation level with all found growing modes failing convergence tests. A typical example is shown Table 5.2 for the $(n, m) = (5, 1)$ single-cell flow with $R_m = 75$. The large truncation levels required for these flows was the primary limiting factor in the number of computations performed and prohibited the use of the spectral eigenvalue code based on the field representation outlined in Appendix C.

It remains an open question as to whether there are any (steady) ZTC flows that can act as a dynamo, and we suggest proceeding by constructing more complicated models; either by taking linear superposition of the models above or by changing the radial dependence of the stream function.
Figure 5.7. Streamlines on $\theta = 40^\circ$ (left) and growth rates (right) for the ZTC $(n, m) = (3, 2)$ and $(a, b) = (0, 0)$ single radial-cell flow with $m_{\text{hcf}} = 1, 2$.

Figure 5.8. Streamlines on $\theta = 40^\circ$ (left) and growth rates (right) for the ZTC $(n, m) = (3, 2)$ and $(a, b) = (0.8, 0)$ double radial-cell flow with $m_{\text{hcf}} = 1$.

Table 5.2. Convergence of the dominant mode for the single cell $(5, 1)$ ZTC flow at $Rm = 75$. 

<table>
<thead>
<tr>
<th>J</th>
<th>N</th>
<th>$\Delta t$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>30</td>
<td>$10^{-6}$</td>
<td>(922.395, 0)</td>
</tr>
<tr>
<td>400</td>
<td>30</td>
<td>$10^{-6}$</td>
<td>(613.8664, 149.894)</td>
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<tr>
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<td>60</td>
<td>$10^{-6}$</td>
<td>(91.4326, 0)</td>
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<tr>
<td>300</td>
<td>60</td>
<td>$10^{-6}$</td>
<td>(12.3587, 0)</td>
</tr>
<tr>
<td>300</td>
<td>100</td>
<td>$10^{-7}$</td>
<td>(−31.9236, 0)</td>
</tr>
</tbody>
</table>
5.2. Flows with ZTC in a sphere

Figure 5.9. Streamlines on $\theta = 40^\circ$ (left) and growth rates (right) for the ZTC $(n, m) = (5, 2)$ and $(a, b) = (0.725, 0.9)$ double radial-cell flow in a sphere with a conducting mantle, with $m_{\text{hcf}} = 2$. 
| Flow Classification  
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td>(n, m)</td>
<td>(a, b)</td>
<td>$U_{\text{max}}$</td>
<td>$\max R_m$</td>
<td>$(J, N, \Delta t)$</td>
<td>$\lambda_{\max R_m}$</td>
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</tr>
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<td>Single cell*</td>
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<td>(0, 0)</td>
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<td>275</td>
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<td>100</td>
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<td>20</td>
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<tr>
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<td>28</td>
<td>200</td>
<td>(300, 60, $10^{-6}$)</td>
</tr>
<tr>
<td>Dual cell*</td>
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<td>(0.8, 0)</td>
<td>10</td>
<td>650</td>
<td>(300, 60, $10^{-6}$)</td>
</tr>
<tr>
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<td>(0.8, 0)</td>
<td>10</td>
<td>1000</td>
<td>(300, 60, $10^{-6}$)</td>
</tr>
<tr>
<td>Dual cell</td>
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<td>(0.7, 0)</td>
<td>35</td>
<td>200</td>
<td>(300, 60, $10^{-6}$)</td>
</tr>
<tr>
<td>Conducting Mantle</td>
<td>(5, 2)</td>
<td>(0.725, 0.9)</td>
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<td>0.30</td>
<td>625</td>
<td>(300, 60, $10^{-6}$)</td>
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</tbody>
</table>

**Table 5.3.** Summary of ZTC computations. Flows marked with * took magnetic $m_{hcf} = 1$. All others took magnetic $m_{hcf} = m$. Truncation levels are those used to generate $\lambda_{\max R_m}$, the growth rate at the largest $R_m$ considered.
5.3. Flows with ZAC in a sphere

5.3.1. Flow definition. The ZAC flows are defined by the condition \( u_\phi = 0 \). Substitution into (3.20) gives
\[
\partial_\phi \partial_r(rs) = \sin \theta \partial_\theta (rt) \tag{5.34}
\]
and hence
\[
\sum_{n,m} im \partial_r(rs^m_n)Y^m_n = \sum_{n,m} rt^m_n \{ nc_{n+1}^m Y^m_{n+1} - (n + 1)c_n^m Y^m_{n-1} \} = \sum_{n,m} r\{(n - 1)c_n^m t^m_{n-1} - (n + 2)c_{n+1}^m t^m_{n+1}\}Y^m_n. \tag{5.35}
\]
From here we follow exactly the same procedure as the ZTC flows. For \( m = 0 \) the flow is defined by
\[
s^0_n = f(r), \tag{5.36}
\]
where \( f(r) \) is required to satisfy \( f(1) = 0 \) so \( 1_r \cdot u = 0 \) on the boundary, and \( s^m_n \) must satisfy (3.32). If \( m \neq 0 \) then
\[
s^m_{n-1} = i \frac{(n + 1)c_n^m}{m} f, \quad t^m_n = \frac{\partial_r(r f)}{r}, \quad s^m_{n+1} = -\frac{nc_{n+1}^m}{m} f \tag{5.37}
\]
where \( f \) is some arbitrary function of \( r \) satisfying the same conditions as in the case where \( m = 0 \).

As with the ZTC flows, four types of ZAC flow are considered, each fully characterised by the same ordered quadruple \((n, m, a, b)\):

I – single cell flows \([a = 0, b = 0]\)
\[
f(r) = r^{n+1}(1 - r^2); \tag{5.38}
\]

II – dual cell flows \([a \neq 0, b = 0]\)
\[
f(r) = r^{n+1}(1 - r^2)(a^2 - r^2); \tag{5.39}
\]

III – single cell flows with a conducting mantle \([a = 0, b \neq 0]\)
\[
f(r) = r^{n+1}(b^2 - r^2)^2(1 - H(r - b)) \tag{5.40}
\]
where \( H \) is the Heaviside function; and

IV – dual cell flows with a conducting mantle \([a \neq 0, b \neq 0]\)
\[
f(r) = r^{n+1}(a^2 - r^2)(b^2 - r^2)^2(1 - H(r - b)). \tag{5.41}
\]
where \( a \in [0, 1) \) and \( b \in [0, 1) \) with \( b > a \) when \( b \neq 0 \). As with the ZTC flows, more complex flows can be constructed by increasing the complexity of \( f \) and by taking a linear supposition of these modes. As with the ZTC flows with a conducting mantle, the factor \((b^2 - r^2)\) in (5.40) and (5.41) is squared to ensure \( u \) satisfied a no-slip condition on the CMB \( r = b \), giving the same magnetic conditions (5.29) and (5.30) at \( r = b \), and hence allows the use of the same compact finite difference matrices for the time-stepping.
5. Kinematic Dynamo Action in a Sphere

### Table 5.4

<table>
<thead>
<tr>
<th>J</th>
<th>N</th>
<th>$\Delta t$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>60</td>
<td>$10^{-6}$</td>
<td>$(6.5488, 0)$</td>
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<tr>
<td>400</td>
<td>60</td>
<td>$10^{-6}$</td>
<td>$(0.68010, 0)$</td>
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<tr>
<td>300</td>
<td>81</td>
<td>$10^{-6}$</td>
<td>$(-54.8314, 0)$</td>
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Convergence of the dominant mode for the single cell $(5, 3)$ ZAC flow with $m_{hc} = 3$ at $R_m = 80$.

### 5.3.2. Streamlines

Following the same procedure outlined in Section 5.2.3, the streamlines for the ZAC flows are the level contours of the stream function

$$im\psi_n^m = \{m^2 + (n - 1)(n + 1)c_n^m + n(n + 2)c_{n+1}^m\}rf,$$

and lie in the azimuthal slices $\phi = \text{const}$. These flows form ring-like vortices scaled by $\cos(m\phi)$, with $n - m + 1$ longitudinal cells, with increasing $n$ pushing these cells closer to the CMB. Streamlines for selected flows are plotted below.

### 5.3.3. Growth rates

A summary of the computations performed for these dynamos is given in Table 5.5 with the streamlines and growth rates for some selected flows given in Figures 5.10–5.13. As with the ZTC flows, unless otherwise stated the magnetic field was assumed to have the same azimuthal symmetry as the velocity ($m_{hc} = m$).

These dynamos suffered from the same convergence problems encountered with the ZTC dynamos, with all growing modes failing convergence tests. A typical example given for the $(5, 3)$ single cell flow in Table 5.4. As with the ZTC flows the high resolution severely limited the number of computations performed and hence the size of the parameter space explored. As with the ZTC flows it remains an open question as to whether there are any (steady) ZAC flows of this form, in particular non-axisymmetric ZAC flows, that can act as a dynamo. We make the same suggestion of constructing more complicated models either by taking linear superposition of the models above or by changing the radial dependence of the stream function, exactly as with the ZTC flows.

![Figure 5.10](image-url) Streamlines (left) and growth rates (right) for the ZAC $(n, m) = (4, 3)$ and $(a, b) = (0, 0)$ single radial-cell flow with $m_{hc} = 1$. 
5.3. Flows with ZAC in a sphere

**Figure 5.11.** Streamlines (left) and growth rates (right) for the ZAC \((n, m) = (4, 4)\) and \((a, b) = (0, 0)\) single radial-cell flow with \(m_{hcf} = 4\).

**Figure 5.12.** Streamlines (left) and growth rates (right) for the ZAC \((n, m) = (3, 2)\) and \((a, b) = (0.7, 0)\) single radial-cell flow with \(m_{hcf} = 2\).

**Figure 5.13.** Streamlines (left) and growth rates (right) for the ZAC \((n, m) = (5, 2)\) and \((a, b) = (0.725, 0.9)\) flow in a sphere with a conducting mantle with \(m_{hcf} = 1\).
<table>
<thead>
<tr>
<th>Flow Classification $(n, m)$</th>
<th>$(a, b)$</th>
<th>$U_{\text{max}}$</th>
<th>$\max R_m$</th>
<th>$(J, N, \Delta t)$</th>
<th>$\lambda_{\text{max}} R_m$</th>
<th>Figure</th>
</tr>
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<td>Single Cell (3, 1)</td>
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<td>55</td>
<td>75</td>
<td>$(300, 60, 10^{-6})$</td>
<td>$(-40.1861, 9.32217)$</td>
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<td>$(300, 60, 10^{-6})$</td>
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<tr>
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<td>200</td>
<td>$(300, 60, 10^{-6})$</td>
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<td>200</td>
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<td>$(300, 60, 10^{-6})$</td>
<td>$(-15.4346, 0)$</td>
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<tr>
<td>Conducting Mantle* (5, 2)</td>
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<td>0.58</td>
<td>1000</td>
<td>$(300, 60, 10^{-6})$</td>
<td>$(-12.4267, 0)$</td>
<td>5.13</td>
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<td>Conducting Mantle* (5, 3)</td>
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<td>250</td>
<td>$(300, 60, 10^{-6})$</td>
<td>$(-10.8057, 0)$</td>
<td>–</td>
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</table>

**Table 5.5.** Summary of ZAC computations. Truncation levels are those used to generate $\lambda_{\text{max}} R_m$. Entries marked with an asterisks were performed with $m_{\text{hcf}} = 1$, otherwise $m_{\text{hcf}} = m$ was chosen. Truncation levels are those used to generate $\lambda_{\text{max}} R_m$, the growth rate at the largest $R_m$ considered.
5.4. The Moss-Gailitis ZAC dynamo

In this section we consider the ZAC dynamo of Moss (2006, 2008). Gailitis (1970, 1993) considered a dynamo driven by a pair of annular vortices with slippery boundaries embedded in an infinite, electrically conducting fluid, showing that these flows are able to sustaining a magnetic field. Moss (2006) considered numerically flows of a similar form, a pair of annular vortices, but embedded in a finite spherical shell with insulating exterior.

This dynamo has velocity
\[ \mathbf{u} = \left( \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}, 0 \right) \]  
with stream function
\[ \psi = (r - b)^2 (1 - r)^2 \sin^\alpha \theta \cos^\beta \theta , \]
where \( \alpha \) and \( \beta \) are positive integers [identified with the \( m \) and \( n \) of Moss (2006) respectively], \( b < 1 \) and \( r \in [b, 1] \). Since
\[ u_r(r, \theta, \phi) = (-)^{\beta+1} u_r(r, \pi - \theta, \phi) , \quad u_\theta(r, \theta, \phi) = (-)^\beta u_\theta(r, \pi - \theta, \phi) , \]
\( \mathbf{u} \) is (equatorially) dipole symmetric if \( \beta \) is even, and quadrupole symmetric if \( \beta \) is odd.

The magnetic boundary conditions used by Moss (2006) at the ICB \( r = b \) are
\[ S_m^m(b, t) = 0 , \quad T_m^m(b, t) = 0 , \]
or equivalently
\[ \mathbf{B}_T = 0 , \quad \mathbf{r} \cdot \mathbf{B}_P = 0 , \quad \text{at } r = b , \]
where \( \mathbf{B}_T \) is the toroidal magnetic field, and \( \mathbf{B}_P \) is the poloidal magnetic field, with the exterior of the sphere \( r = 1 \) taken to be insulating. The conditions at the ICB preserve self-excitation in the energetic sense, as the Poynting flux across \( r = b \) is
\[ \oint_{r=b} \mathbf{E} \times \mathbf{B} \cdot d\mathbf{S} = 0 . \]
To demonstrate this, begin with the poloidal-toroidal decomposition \( \mathbf{B} = \nabla \times \mathbf{T} + \nabla \times \nabla \times \mathbf{S} \mathbf{r} \). Uncurling Faraday’s law
\[ \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \]  
gives
\[ \mathbf{E} = - \dot{T} \mathbf{r} - \nabla \times \dot{S} \mathbf{r} + \nabla \Phi , \]
where \( \dot{S} = \partial_t S \), similarly for \( \dot{T} \), and \( \Phi \) is some scalar function. Noting that the boundary conditions \( (5.46) \) imply
\[ \dot{T}(r = b, t) = 0 \]
and
\[ \nabla \times \dot{S} \mathbf{r} = (\nabla \dot{S}) \times \mathbf{r} = (\nabla_H \dot{S}) \times \mathbf{r} = 0 \quad \text{at } r = b , \]
reduces \((5.48)\) to
\[
\oint_{r=b} \mathbf{E} \times \mathbf{B} \cdot d\mathbf{S} = \oint_{r=b} \nabla \phi \times \mathbf{B} \cdot d\mathbf{S}. \tag{5.53}
\]
Integrating by parts gives
\[
\oint_{r=b} \nabla \phi \times \mathbf{B} \cdot d\mathbf{S} = \oint_{r=b} \nabla \times (\mathbf{B} \phi) \cdot d\mathbf{S} - \oint_{r=b} \mathbf{B} \nabla \phi \times d\mathbf{S}. \tag{5.54}
\]
The first integral on the right is zero by Stokes’ theorem. The second integral on the right is also zero as, from \((3.20)\), \((3.22)\) and \((3.23)\),
\[
\mathbf{r} \cdot \nabla \times \mathbf{B} = \mathbf{r} \cdot \left( \mathbf{S} \{ T \} + \mathbf{T} \{- \nabla^2 S\} \right) = -\Lambda^2 T \tag{5.55}
\]
which is zero on the ICB.

**Figure 5.14.** Streamlines for the Moss-Gailitis \((\alpha, \beta, b) = (6, 3, 0.2)\) spherical shell kinematic dynamo generated with the stream-function \((5.44)\) normalised against its maximum value. (Left) Streamlines in the azimuthal plane, (right) isosufraces \(\pm \frac{1}{2} \text{max} |\psi|\). In this figure the \(z\)-axis is plotted.

We begin by repeating the calculation of Moss for \((\alpha, \beta) = (6, 3)\) and \(b = 0.2\), with the unusual magnetic conditions \((5.46)\) at the ICB. This flow has \(U_{\text{max}} = 2.38 \times 10^{-2}\). Streamlines for this flow are shown in Figure 5.14. Like Moss (2006) we consider only the magnetic modes with azimuthal wave number \(m = 1\). In order to accelerate computation the routines which perform the spectral transforms described in Section \(3.4\) are modified to filter out all \(m \neq 1\) magnetic modes, as well as to allow the Gauss-Legendre points \((K)\) and the number of points in \(\phi (L)\) to vary; the truncation level \(N\) for the spherical harmonic series constructed backwards from the condition on the number of Gauss-Legendre points given in Section \(3.4\)
\[
N = \left\lfloor \frac{2}{3} (K - 1) \right\rfloor. \tag{5.56}
\]
The poloidal-toroidal magnetic field potential are therefore expressed as the series
\[
S = \sum_{n=1}^{N} S_n^0 Y_n^0, \quad T = \sum_{n=1}^{N} T_n^0 Y_n^0. \tag{5.57}
\]
5.4. The Moss-Gailitis ZAC dynamo

We were unable to reproduce the results of Moss (2006) for the \((\alpha, \beta) = (6, 3)\) flow with \(b = 0.2\) and \(R_m = 10^5\). Instead, we obtained a growing dynamo with a different growth rate which demonstrates weak convergence with changing truncation levels. These growth rates are given in Table 5.6. For the same parameters Moss (2006) quoted a growth rate of \(\lambda = 8.2\), obtained for uniform radial and longitudinal grids with \((J, K) = (201, 201)\).

We plot the magnetic field components for this dynamo in Figure 5.15. We see a radial magnetic field profile at the CMB similar to that observed by Moss (2006, Fig. 4a) for the same parameters. We notice two features of interest in these plots. The first, consistent with results of Gailitis (1970) for an infinite conducting fluid, is that the quadrupolar flow has excited a quadrupolar magnetic field. We also see that the magnetic field lines have aligned with the streamlines and magnetic flux has been expelled from the vortices. This is unsurprising as the true magnetic Reynolds number is large \((U_{\text{max}} \cdot R_m = 2380)\).

We change from a spherical shell model to a full sphere model, replacing the stream function (5.44) by

\[
\psi = r^\alpha \beta (1 - r^2)^{\frac{\alpha + \beta}{2}} \sin^\alpha \theta \cos^\beta \theta ,
\]

and consider the \((\alpha, \beta) = (6, 3)\) model, with streamlines shown in Figure 5.16. This flow has \(U_{\text{max}} = 8.82 \times 10^{-4}\).

Growth rates are given as a function of the magnetic Reynolds number in Figure 5.18. We test convergence of the growing mode found for \(R_m = 2 \times 10^6\), finding more convincing convergence with changing truncation level (Table 5.7). We plot the magnetic field components in Figure 5.17.

<table>
<thead>
<tr>
<th>(J)</th>
<th>(N)</th>
<th>(L)</th>
<th>(K)</th>
<th>(\Delta t)</th>
<th>(\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>134</td>
<td>153</td>
<td>202</td>
<td>(2 \times 10^{-7})</td>
<td>38.3082</td>
</tr>
<tr>
<td>150</td>
<td>136</td>
<td>157</td>
<td>206</td>
<td>(2 \times 10^{-7})</td>
<td>38.3095</td>
</tr>
<tr>
<td>300</td>
<td>134</td>
<td>161</td>
<td>202</td>
<td>(2 \times 10^{-7})</td>
<td>39.1582</td>
</tr>
<tr>
<td>200</td>
<td>134</td>
<td>201</td>
<td>202</td>
<td>(2 \times 10^{-7})</td>
<td>38.5354</td>
</tr>
<tr>
<td>200</td>
<td>134</td>
<td>205</td>
<td>202</td>
<td>(10^{-7})</td>
<td>38.5355</td>
</tr>
<tr>
<td>300</td>
<td>134</td>
<td>202</td>
<td>401</td>
<td>(10^{-7})</td>
<td>39.1582</td>
</tr>
<tr>
<td>500</td>
<td>134</td>
<td>201</td>
<td>202</td>
<td>(2 \times 10^{-7})</td>
<td>38.9098</td>
</tr>
<tr>
<td>500</td>
<td>333</td>
<td>241</td>
<td>502</td>
<td>(10^{-7})</td>
<td>38.9098</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(J)</th>
<th>(N)</th>
<th>(L)</th>
<th>(K)</th>
<th>(\Delta t)</th>
<th>(\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>60</td>
<td>182</td>
<td>91</td>
<td>(2 \times 10^{-7})</td>
<td>38.5354</td>
</tr>
<tr>
<td>200</td>
<td>80</td>
<td>241</td>
<td>121</td>
<td>(2 \times 10^{-7})</td>
<td>38.5354</td>
</tr>
<tr>
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<td>60</td>
<td>182</td>
<td>91</td>
<td>(2 \times 10^{-7})</td>
<td>38.7441</td>
</tr>
<tr>
<td>500</td>
<td>80</td>
<td>241</td>
<td>121</td>
<td>(10^{-7})</td>
<td>38.9099</td>
</tr>
</tbody>
</table>

Table 5.6. Growth rates for the Moss-Gailitis \((\alpha, \beta, b) = (6, 3, 0.2)\) ZAC spherical shell kinematic dynamo with boundary conditions (5.46) for \(R_m = 10^5\). Computations in the upper block were performed with \((J, L, K)\) as input parameters, with the value of \(N\) inferred from \(K\). Computations listed in the lower block had \((J, N)\) input parameter, giving the listed values of \(K\) and \(L\).
Figure 5.15. Magnetic field profiles for the Moss-Gailitis \((\alpha, \beta, b) = (6, 3, 0.2)\) spherical shell kinematic dynamo at \(R_m = 10^5\). Top, from left to right: radial magnetic field, latitudinal magnetic field and azimuthal magnetic field on azimuthal slices with flow streamlines superimposed. Middle, from left to right: Magnitude of the magnetic field with streamlines of the flow and meridional magnetic field lines superimposed on the normalised flow speed. Bottom: Hammer projection of the radial magnetic field at the CMB.
5.4. THE MOSS-GAILITIS ZAC DYNAMO

<table>
<thead>
<tr>
<th>$J$</th>
<th>$N$</th>
<th>$\Delta t$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
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<td>$2 \times 10^{-7}$</td>
<td>3.87138</td>
</tr>
<tr>
<td>300</td>
<td>50</td>
<td>$2 \times 10^{-7}$</td>
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</tr>
<tr>
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<td>60</td>
<td>$2 \times 10^{-7}$</td>
<td>3.75098</td>
</tr>
<tr>
<td>300</td>
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</tr>
<tr>
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<td>60</td>
<td>$10^{-7}$</td>
<td>3.74188</td>
</tr>
<tr>
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<td>80</td>
<td>$10^{-7}$</td>
<td>3.74662</td>
</tr>
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<td>1200</td>
<td>60</td>
<td>$10^{-7}$</td>
<td>3.74546</td>
</tr>
<tr>
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<td>40</td>
<td>–</td>
<td>3.86846</td>
</tr>
<tr>
<td>400</td>
<td>80</td>
<td>–</td>
<td>3.86930</td>
</tr>
<tr>
<td>800</td>
<td>80</td>
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</tr>
<tr>
<td>1600</td>
<td>80</td>
<td>–</td>
<td>3.75423</td>
</tr>
</tbody>
</table>

**Table 5.7.** Growth rates for the Moss-Gailitis $(\alpha, \beta) = (6, 3)$ ZAC full sphere kinematic dynamo for $R_m = 2 \times 10^6$. The upper block corresponds to computations performed using the time-stepping method. The lower block corresponds to growth rates computed using the eigenvalue method.

seeing a similar situation to that obtained in the spherical shell case; the quadrupolar velocity has excited a quadrupolar magnetic field, with magnetic field lines aligning with the streamlines of the flow, and magnetic flux begin expelled from the vortices. This is again unsurprising as the Magnetic Reynolds number is large ($U_{\text{max}} \cdot R_m = 1764$).

**Figure 5.16.** Streamlines for the Moss-Gailitis $(\alpha, \beta) = (6, 3)$ full sphere kinematic dynamo generated with the stream-function (5.58) normalised against its maximum value. (Left) Streamlines in the azimuthal plane, (right) isosufraces $\pm \frac{1}{2} \max |\psi|$. In this figure the $z$-axis is plotted.
Figure 5.17. Magnetic field profiles for the Moss-Gailitis $(\alpha, \beta) = (6, 3)$ full sphere kinematic dynamo at $R_m = 2 \times 10^6$. Top, from left to right: radial magnetic field, latitudinal magnetic field and azimuthal magnetic field on azimuthal slices with flow streamlines superimposed. Middle, from left to right: Magnitude of the magnetic field with streamlines of the flow and meridional magnetic field lines superimposed on the normalised flow speed. Bottom: Hammer projection of the radial magnetic field at the boundary.
5.5. Discussion

In this chapter we considered the stationary kinematic dynamo problem for a sphere of uniformly conducting incompressible fluid with insulating exterior. We began by testing of the numerical integrator used to time-step the kinematic equations against the Pekeris, Accad & Shkoller (1973) dynamo in Section 5.1. The magnetic grow rates calculated for this flow were then compared to the magnetic growth rates obtained using an eigenvalue code based on the compact spectral magnetic induction equation derived by James (1973, 1974), given in Appendix C using second order central finite differences in radius. The discrepancy between the growth rates given here and those found for the same flow by Dudley & James (1989) were attributed to the selection of different azimuthally symmetries, the two calculations bridged by recomputing the magnetic growth rates for the second azimuthal symmetry class using the eigenvalue code.

The problem of mode selection in a time-stepping code motivated the development of hybrid time-stepping/eigenvalue approach. Inspired by the hybrid time-stepping/Arnoldi method of Willis & Gubbins (2004), a hybrid time-stepping/inverse iteration method was derived for the kinematic dynamo problem in Appendix D. This was successful as far as reproducing the free decay rates was concerned, but failed to provide accelerated convergence unless the growth rate was known to high precision, making the entire process redundant. It would perhaps be advantageous to combine this method with a standard time-stepping method; changing from one method to the other whenever convergence slowed. In this thesis, this method was not pursued beyond the free decay problem.

The remainder of the chapter concerned the search for working dynamo solutions for flows with missing components in spherical polar co-ordinates, the ZTC and ZAC flows. With the formulation developed for these flows in this chapter we were unable to find convergent growing dynamos. The sensitivity of the growth rates to the truncation level was surprising given both the simplicity of the flows and the relatively small magnetic Reynolds numbers considered, and no obvious reason for this was found.

As part of the consideration of the ZAC flows the axisymmetric Moss-Gailitis dynamos of Moss (2006) were reconsidered. In this context we repeated the calculation of Moss for a spherical

![Figure 5.18. Growth rates for the Moss-Gailitis \((\alpha, \beta) = (6, 3)\) full sphere kinematic dynamo.](image)
shell, with the same unusual magnetic boundary conditions at the ICB. We were unable to reproduce the magnetic growth rates found in Moss (2006), instead finding a growing dynamo with different magnetic growth rate, which demonstrated a weak convergence with changing truncation levels. We then modified the radial dependence of the stream function and repeated the calculation in a full sphere geometry, finding a growing magnetic mode which demonstrates much more convincing convergence. It would be pertinent to formulate classes of ZTC flows in a similar way.
Thermal Convection in a Rotating Sphere

In this chapter convection in a rotating sphere of Boussinesq fluid with homogeneous volumetric heat source with isothermal boundary conditions, and either no-slip or stress-free viscous boundary conditions, is considered in the absence of a magnetic field. Two sets of solutions are considered in this context. The first, which is considered in Section 6.1, is the full sphere stress-free thermal benchmark of Marti et al. (2014), which is used as a validation tool for the methods outlined in Chapters 3 and 4. In particular this benchmark tests the implicit and explicit differencing of the velocity potentials and the temperature, as well as the spherical harmonic transform described in Section 3.4, which are not tested by the PAS kinematic dynamo in Section 5.1. The azimuthal and equatorial symmetries of this solution also allow the implementation of the problem reductions described in Sections 3.7.3 and 3.7.2 to be tested.

The second set of solutions, considered in Section 6.2, is for a very similar problem, but with no-slip viscous boundary conditions. All the models considered in this context are for $E = 5 \cdot 10^{-4}$ and $Pr = 0.7$, and are needed for comparison with the dynamo solutions obtained in the next chapter for the same parameters. The first sub-problem we consider in this context is the calculation of the critical Rayleigh number for the onset of convection $Ra_c$, in Section 6.2.1. The linear onset of convection in the rapid rotation (small $E$) limit was most notably considered in a full sphere by Jones, Soward & Mussa (2000) and in spherical shells by Dormy et al. (2004). These authors built on the local stability criterion of Yano (1992), who assumed a WKBJ type solution and sought a double turning point in the complex $\sigma$–plane (cylindrical radius) for a cylindrical annulus with sloped ends. These extended Yano’s work by considering the spherical geometry directly, instead of the small tilt expansion, at the cost of having to solve a second order DE. It is beyond the scope of the present work to consider the formal asymptotic analysis of this problem, and direct comparison between the results obtained here and those of the aforementioned authors cannot be made due to the use of different Prandtl numbers. In this work a rough estimate of the critical Rayleigh number is sought for a fixed (not asymptotically small) $E$ and $Pr$ using the non-linear code for purposes of normalising the Rayleigh numbers considered in the dynamo problem. Two convective models are then considered in greater detail; that for $Ra \approx 1.4 Ra_c$ in Section 6.2.2 and $Ra \approx 5.7 Ra_c$ in Section 6.2.3.

For convenience the control parameters and diagnostics for the thermal convection problem are given in Table 6.1. The parameter

$$ D := \left\{ J(N + 1)^2 \right\}^{\frac{1}{3}} $$

(6.1)

is an expression for the degrees of freedom; the expression (6.1) corresponding to a discrete radial grid with $J$ points with spherical harmonic series in angle with triangular truncation level $N$. 

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6.1. The stress-free benchmark of Marti et al. (2014)

As a first test of the components of the code untested in the kinematic dynamo problem, we reproduce the convection benchmark of Marti et al. (2014). This model considers a rotating full sphere of Boussinesq fluid with homogeneous thermal and viscous diffusivities, heated by a homogeneous volumetric heat source, with stress-free viscous and perfectly conducting thermal boundary conditions. The dynamical parameters for this model are

\[
Ra = 95, \quad E = 3 \cdot 10^{-4}, \quad Pr = 1. \tag{6.2}
\]

The initial temperature profile is the conduction state temperature with a \( Y_3^3 \) perturbation

\[
\Theta(r, 0) = \Theta_0^0(r, 0)Y_0^0 + \Theta_3^3(r, 0)Y_3^3 + \Theta_3^{-3}(r, 0)Y_3^{-3} \tag{6.3}
\]

where

\[
\Theta_0^0(r, 0) = \frac{1}{2} (1 - r^2), \quad \Theta_3^3(r, 0) = - (\Theta_3^{-3}(r, 0))^* = (-1 + i) \frac{10^{-5}}{4\sqrt{\pi}} r^3 (1 - r^2), \tag{6.4}
\]

and no initial velocity

\[
u(r, 0) = 0. \tag{6.5}
\]

These expressions for the initial temperature differ from those in Marti et al. (2014) due to differing spherical harmonic normalisation. The implicit compact schemes used are the interior tri-diagonal scheme for the second order temperature and toroidal momentum equations, and the interior penta-diagonal scheme for the fourth order poloidal momentum equations. Order preserving one sided schemes which maintain the bandwidth of the time-stepping systems are used at the boundary in
The evolution of the kinetic energy for \((J, N, \Delta t) = (200, 60, 10^{-5})\) on the uniform radial grid with \(m_{hcf} = 1\) is shown in Figure 6.1. The diagnostic parameters for this benchmark are the final steady kinetic energy and the drift rate \(\omega_d\). The flow is expected to be pure quadrupole with a 3-fold azimuthal symmetry \((m_{hcf} = 3)\). The measured steady kinetic energy and drift rates for the uniform and Chebyshev-[0, 1] radial grids for several truncations and assumed symmetries are given in Table 6.2, with the relative differences from the benchmark values tabulated in Marti et al. (2014) given in brackets.

### Table 6.2. Convergence of diagnostics for the Marti et al. (2014) rotating thermal convection benchmark. Bracketed values are relative differences from the benchmark values.

<table>
<thead>
<tr>
<th>Grid Type</th>
<th>(J)</th>
<th>(N)</th>
<th>(m_{hcf})</th>
<th>(D)</th>
<th>(\Delta t)</th>
<th>(\omega_d)</th>
</tr>
</thead>
<tbody>
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<td>60</td>
<td>1</td>
<td>91</td>
<td>(10^{-5})</td>
<td>29.1206</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
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<td>66</td>
<td>(10^{-5})</td>
<td>29.2250</td>
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<td>3</td>
<td>68</td>
<td>(10^{-5})</td>
<td>29.0829</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.30 \times 10^{-3})</td>
</tr>
<tr>
<td>Chebyshev</td>
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<td>3</td>
<td>96</td>
<td>(5 \times 10^{-5})</td>
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<td></td>
<td></td>
<td></td>
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</table>

**Figure 6.1.** Energy series (left) and time-averaged kinetic energy spectrum (right) for the Marti et al. (2014) convection benchmark.
The components of the velocity are plotted on the meridional slice that passes through the maximum value of $|\mathbf{u}|$ in Figure 6.2 and in the equator in Figure 6.3. Note that a quadrupolar velocity implies $\mathbf{1}_\theta \cdot \mathbf{u} = 0$ on the equatorial plane, and this plot is hence omitted. These figures demonstrate good agreement with the same profiles given in the benchmark paper.

**Figure 6.2.** Meridional slices of the flow components for the Marti et al. (2014) convection benchmark containing maximum value of $|\mathbf{u}|$. From left to right; radial component, latitudinal component, azimuthal component.

**Figure 6.3.** Equatorial slices of the flow components for the Marti et al. (2014) convection benchmark. Radial component (left) and azimuthal component (right).

This benchmark tests three aspects of the code not directly tested by the kinetic simulations. The first is the implicit differencing and time-stepping for the poloidal momentum equation. The second is the symmetry preservation for reflection in the equatorial plane. This benchmark having an even initial temperature and no initial velocity means no energy should leak into the dipole velocity modes. The results of Marti et al. (2014) imply this result is stable to dipole velocity and odd temperature perturbations. This symmetry is preserved exactly in this implementation due to the splitting of the two symmetry classes and it is reassuring that the same solution is obtained whether it is assumed or not. Finally this benchmark tests the preservation of the azimuthal symmetry classes inbuilt into the numerics. For this benchmark the initial conditions have an $m_{\text{hcf}} = 3$ symmetry and the solution appears stable to perturbations which break this symmetry. This is tested by the computations with $m_{\text{hcf}} = 1$ for which round-off error has resulted in non-zero energies in the $m \mod 3 \neq 0$ modes [see Figure 6.1(right)].
6.2. Thermal convection for \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\) with no-slip viscous boundary conditions

In this section we consider thermal driven convection in a rotating sphere for an internally heated fluid with perfectly conducting thermal and no-slip viscous boundary conditions with \(E = 5 \cdot 10^{-4}\) and \(Pr = 0.7\). Three problems are considered in this context: the calculation of the critical Rayleigh number \(Ra_c\) for the onset of convection in Section 6.2.1, the convective state for \(Ra \approx 1.4 Ra_c\) in Section 6.2.2, and the convection state for \(Ra \approx 5.7 Ra_c\) in Section 6.2.3. All of these results are required later where they will be compared to dynamo computations performed for similar parameters.

6.2.1. The onset of convection. Fixing \(E = 5 \cdot 10^{-4}\) and \(Pr = 0.7\), we calculate an estimate for the critical Rayleigh number for the onset of convection by testing for the growth/decay of the kinetic energy from the initial state \(u(r,0) = 0\), with initial temperature profile

\[
\Theta_0^0 = \frac{1}{2} (1 - r^2), \quad \Theta_0^3 = -(\Theta_3^{-3})^* = (-1 + i) \frac{10^{-5}}{4\sqrt{\pi}} r^3 (1 - r^2).
\]

To find the critical Rayleigh number a least squared linear model is fitted to the velocity growth rate \(\lambda\) as a function of \(Ra\), with the critical value satisfying \(\lambda(Ra_c) = 0\). The fitted curve is shown in Figure 6.4, with the critical Rayleigh number

\[
Ra_c = 105.979.
\]

6.2.2. \(Ra = 150\). In this section we consider rotating thermal convection for \((Ra, Pr) = (150, 0.7)\) to compare with the dynamo solutions considered in the next chapter. For these parameters \(Ra \approx 1.4 Ra_c\). The flow settles to a pure quadrupole, 3-fold (azimuthally) symmetric [Figure 6.5 (right)] quasi-columnar solution with a prograde tilt near the boundary, in particular at mid latitudes, i.e. the typical Busse (1970) columns configuration. The kinetic energy is steady, \(E_k = 244.639\) [Figure 6.5 (left)], with the columnar structures drifting with angular frequency \(\omega_{\text{drift}} = 8.59022\) in non-dimensional time.

The helicity

\[
H := u \cdot \nabla \times u = u \cdot \omega
\]
where $\omega := \nabla \times u$ is the vorticity, is positive in the southern hemisphere and negative in the northern [Figures 6.6(c), 6.7(a,b)]. As the $z$-vorticity alternates sign between adjacent columns [Figure 6.7(b)], this implies the flow is towards the equator in the vortices with $\omega_z > 0$, what will be termed positive vortices, and away from the equator in vortices $\omega_z < 0$, the negative vortices, in both hemispheres as demonstrated in Figures 6.6(b), 6.7(e) and 6.8.
6.2. Thermal convection for \((E, \text{Pr}) = (5 \cdot 10^{-4}, 0.7)\) with no-slip viscous boundary conditions

Figure 6.7. Flow characteristics for \((Ra, Pm) = (150, 0.7)\) convection model. Black lines on spherical plots are \(z = -0.45 : 0.45 : 0.9\).
6.2.3. **Ra = 600.** In this section we present results from thermally driven convection in the absence of a magnetic field for the parameters \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\) with \(Ra = 600\), corresponding to \(Ra \approx 5.7 Ra_c\).

**Figure 6.8.** Horizontal streamlines on \(z = 0.1\), colours show the \(z\)-velocity.

**Figure 6.9.** Energy series for the \((Ra, Pr) = (600, 0.7)\) convection model.
6.2. THERMAL CONVECTION FOR \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\) WITH NO-SLIP VISCOUS BOUNDARY CONDITIONS

The energy series as well as the energy spectrum are shown in Figure 6.9. After the initial growth of the kinetic energy the flow settles to a regime similar to that obtained just above onset: the flow is organised into quasi-geostrophic convective columns with alternative positive and negative \(z\)-vorticity; the helicity is negative in the northern hemisphere, positive in the southern hemisphere and the whole system is dominated by the 3-fold azimuthally symmetric quadrupolar modes. Compared to the results obtained for \(Ra = 150\), however, we find first that the energy is oscillatory

\[
E_k = E_{k,0} + E_{k,1} \cos(2\pi \omega_k t + \phi_k) \tag{6.9}
\]

with

\[
E_{k,0} = 7.27056 \times 10^3, \quad E_{k,1} = 282.135, \quad \omega_k = 68.8082 \tag{6.10}
\]

with the column structure drifting with angular velocity

\[
\omega_{\text{drift}} = 5.60344, \tag{6.11}
\]

and second that a secondary set of convective columns along the rotation axis and near the boundary at low latitudes have formed. The structure of these columns is most easily seen in the hammer projections of the \(z\)-velocity, \(z\)-vorticity and helicity shown in Figure 6.10 and in equatorial slices of the \(z\)-velocity shown in Figure 6.11.

**Figure 6.10.** Hammer projections of the \(z\)-velocity (top left), \(z\)-vorticity (top right) and helicity (bot) on \(r = 0.9\) for \((Ra, Pm) = (600, 0.7)\).
In this chapter we considered the problem of thermally driven convection in a rotating sphere with homogeneous volumetric heat source and fixed temperature boundary conditions. In the first instance, Section 6.1, we reproduced the stress-free convection benchmark of Marti et al. (2014), testing not only the implementation of the generalised compact scheme for the 4th order poloidal momentum equation, but also testing the implementation of the azimuthal symmetry reduction, the equatorial symmetry decomposition and the spherical harmonic transform described in Chapter 3.

The remainder of this chapter, Section 6.2, was dedicated to convection in a rotating sphere with no-slip viscous boundary conditions for the parameters \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\), for two purposes both related to the dynamo computations considered in the next chapter. The first was to calculate the critical Rayleigh number for the onset of convection, in Section 6.2.1, which is later used to normalise the Rayleigh numbers at which dynamo solutions were found. The second is to give the structure of the flow in non-magnetic convection for comparison with convection in the magnetic case for similar parameters. In particular we considered \(Ra \approx 1.4 Ra_c\) in Section 6.2.2 and \(Ra \approx 5.7 Ra_c\) in Section 6.2.3, finding that both models settled to a quasi-columnar state dominated by the 3-fold (azimuthally) symmetric quadrupolar flow.

**Figure 6.11.** Equatorial slices of the \(z\)-velocity (colours) for \((Ra, Pr) = (600, 0.7)\). (Left) \(z = 0.3\) with horizontal streamlines, (Right) \(z = 0.7\). Note that the colour scales are different.
In this chapter numerical solutions are given to Problem III described in Section 2.1: viz. magnetic field generation in a rotating full sphere of uniformly conducting Boussinesq fluid with homogeneous volumetric heat source and perfectly conducting thermal, and both no-slip and stress-free viscous, boundary conditions and an insulating exterior. We consider the three topics of interest in contemporary dynamo studies which were discussed in Chapter 1: dynamo benchmark solutions; the effect of changing the dynamical parameters on dynamo solutions; and dynamo scaling laws.

In section 7.1 we test the full numerical integrator against the full sphere stress-free benchmark dynamo of Marti et al. (2014). After reproducing the benchmark diagnostics for this dynamo we consider no-slip viscous boundary conditions and, in Section 7.2, we undertake a preliminary search for dynamo solutions through the four parameter \((E, Ra, Pm, Pr)\) space, presuming either an equatorial or an azimuthal symmetry. From among the solutions found we discuss an oscillatory dynamo that requires both the equatorial splitting as well as the azimuthal symmetry reduction to be reproduced.

In Section 7.3 we move from the initial search through \((E, Ra, Pm, Pr)\)-space and focus on \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\), varying \(Pm\) from 1 to 40 and \(Ra\) up to a few times the critical value for the onset of convection. For these parameters we examine the remaining topics of interest: the effect of changing the dynamical parameter on the type of dynamo solution obtained and dynamo scaling laws.

The first of these, the numerical study of how dynamo solutions vary with the dynamical parameters, is a relatively new area of research as enormous computational resources are required to make any significant headway. Despite this there is already a rich body of work dedicated primarily to numerical spherical shell dynamos, albeit in a limited region of parameter space, which unfortunately excludes natural dynamos and laboratory dynamo experiments. Early authors began by focusing on regions of the parameter space corresponding to different classes of dynamo solution: e.g. Grote, Busse & Tilgner (2000) identified separated regions in \((Ra, Pm)\) space corresponding to regular dipolar, chaotic dipolar, hemispherical and quadrupolar dynamos, and Kutzner & Christensen (2002) searched \((Ra, q)\) space and identified regions corresponding to steady dipolar dominant, and reversing dipolar dominant or multi-polar dynamo solutions. This work has been built upon by authors such as Christensen & Aubert (2006) who extended the database built from these studies to lower values of \(E\) and \(Pm\), Morin & Dormy (2009) who identified multiple solution branches characterised by different magnetic energies for the same parameters, and Dormy (2014) who was interested in a strong/weak field classification of the different dynamo branches. We begin by following Dormy (2014) and attempt to classify the dynamo solutions as weak or strong field in Section 7.5 with a subsequent analysis of the local strong/weak field force balance acting in the dynamo solutions. We then consider in detail three particular solutions, which
## Control Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ekman Number</td>
<td>$E$</td>
<td>$\nu/2\Omega L^2$</td>
</tr>
<tr>
<td>Magnetic Prandtl Number</td>
<td>$P_m$</td>
<td>$\nu/\eta$</td>
</tr>
<tr>
<td>Modified Rayleigh Number</td>
<td>$Ra$</td>
<td>$\alpha \Delta \Theta g L/2\Omega U$</td>
</tr>
<tr>
<td>Prandtl Number</td>
<td>$Pr$</td>
<td>$\nu/\kappa$</td>
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## Derived Parameters

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<td>Magnetic Ekman Number (Rossby Number)</td>
<td>$Ro$</td>
<td>$E/Pm$</td>
</tr>
<tr>
<td>Roberts Number</td>
<td>$q$</td>
<td>$P_m/Pr$</td>
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<td>$3q$</td>
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## Diagnostics

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<tr>
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</tr>
<tr>
<td>Internal Magnetic Energy</td>
<td>$E_m$</td>
<td>$1/2 Ro \int V B^2 dV$</td>
</tr>
<tr>
<td>Joule Dissipation</td>
<td>$P_J$</td>
<td>$1/Ro \int (\nabla \times B)^2 dV$</td>
</tr>
<tr>
<td>Buoyancy Power</td>
<td>$P_b$</td>
<td>$\frac{Ra}{Ro} \int \Theta r \cdot u dV$</td>
</tr>
<tr>
<td>Nusselt Number</td>
<td>$Nu$</td>
<td>$-\partial_r \Theta^0</td>
</tr>
<tr>
<td>Magnetic Reynolds Number</td>
<td>$R_m$</td>
<td>$\sqrt{3/2\pi \nu^k}$</td>
</tr>
<tr>
<td>Magnetic Dipole Moment</td>
<td>$(m_x, m_y, m_z)$</td>
<td>$\frac{16\pi}{\sqrt{6}} \left( \text{Re} {S^1_1}, \text{Im} {S^1_1}, S^0_1/\sqrt{2} \right)$</td>
</tr>
</tbody>
</table>

**Table 7.1.** Control parameters and diagnostics for the spherical dynamo problem.
The stress-free benchmark dynamo of Marti et al. (2014) corresponds to three types of dynamo solution found in spherical shell computations; a dipolar dynamo obtained for \((Ra, Pm) = (5000, 35)\) in Section 7.6.1, a quadrupolar dynamo obtained for \((Ra, Pm) = (3000, 20)\) in Section 7.6.2, and an oscillating hemispherical dynamo obtained for \((Ra, Pm) = (3000, 5)\) in Section 7.6.3.

The large database of spherical shell dynamo solutions built from the studies referred to above has facilitated the contemporary study of the final topic considered, that of dynamo scaling laws. There has been extensive study of scaling laws for spherical shell dynamos, with two approaches taken. The first is that of testing theoretical scaling laws. This proceeds exactly as the scaling law analysis of any dynamical system; some asymptotic regime or force balance is assumed and a product/power law is derived from the governing equations. For dynamo studies this usually takes the form of assuming some force balance in the momentum equation which implies a scaling law for the velocity, which is then combined with the magnetic induction equation to derive a scaling law for the magnetic field strength. The derived laws are then tested against the numerical dynamo solutions (e.g., Christensen & Aubert 2006, Christensen 2010). We consider scaling laws of this form for the dynamo solutions obtained here in sections 7.4.1 and 7.4.2. The second approach is to derive scaling laws numerically from the dynamo database. The usual method here is to assume some set of dynamical parameters and dynamo diagnostics (e.g. the strength of the magnetic field, the kinetic energy and the Joule dissipation) are related by a product power law, and to use some numerical curve fitting technique to find the exponents (e.g., Stelzer & Jackson 2013). We consider dynamo scaling laws of this form for the full sphere dynamos solutions obtained here in Section 7.4.4.

For convenience the set of diagnostics used for dynamo calculations are given here in Table 7.1.

### 7.1. The stress-free benchmark dynamo of Marti et al. (2014)

The numerical integrator described in Chapter 4 is tested against the full sphere dynamo benchmark of Marti et al. (2014). This dynamo has the parameters values

\[
Ra = 1400, \quad E = 5 \cdot 10^{-4}, \quad Pm = 7, \quad Pr = 1.
\]  

(7.1)

The initial velocity is a toroidal quadrupole

\[
t_1^0 = \frac{1}{\sqrt{3}} r \left( -\frac{54625}{198} + \frac{350}{2} r^2 - \frac{325}{2} r^6 \right)
\]

\[t_2^1 = -(t_2^{-1})^* = r^2(30 + \frac{1250}{3} r^2 - 130 r^4 - 90 r^6 + i(105 - 245 r^2 + 155 r^4 - \frac{145}{7} r^6)),\]

(7.2)

satisfying stress-free viscous boundary conditions. The initial magnetic field is a toroidal dipole

\[
T_1^1 = -(T_1^{-1})^* = \sqrt{\frac{2}{3}}(1 + i) r \left( \frac{3}{4} - 3 r^2 + \frac{9}{2} r^4 - \frac{9}{4} r^6 \right), \quad T_2^0 = \frac{1}{\sqrt{5}} r^2 \left( \frac{3}{2} - \frac{21}{4} r^2 + \frac{27}{4} r^4 - 3 r^6 \right),
\]

(7.3)

with a perfectly insulating exterior. The temperature is initialised to the conducting state plus a degree 3 and order 3 perturbation,

\[
\Theta_0^0 = \frac{1}{2}(1 - r^2), \quad \Theta_3^3 = -(\Theta_3^{-3})^* = (-1 + i) \frac{10^{-5}}{4\sqrt{\pi}} r^3(1 - r^2),
\]

(7.4)
with perfectly conducting thermal boundary conditions. As with the thermal benchmark discussed in Chapter 6, these expressions differ from those in Marti et al. (2014) due to the different spherical harmonic normalisations.

The time evolution of the kinetic and internal magnetic energies are shown in Figure 7.1. These are obtained for \((J, N, \Delta t) = (216, 60, 5 \cdot 10^{-6})\) on a uniform radial grid. Since the fields on the \((\phi, \theta, r)\) grid were decomposed into the two equatorial symmetry classes (see Section 3.7.2) the dipole fraction of the kinetic energy was exactly zero and the dipole fraction of the magnetic energy was exactly unity for the entire computation. This is consistent with the benchmark solution. The magnetic and kinetic energies of this dynamo are expected to settle to a periodic state characterised by a single oscillatory mode

\[
E_k = E_{k,0} + E_{k,1} \cos (2\pi \omega_k t + \phi_k), \quad E_m = E_{m,0} + E_{m,1} \cos (2\pi \omega_m t + \phi_m). \tag{7.5}
\]

The benchmark diagnostics for this dynamo are the mean (internal) energies, and the amplitude, frequency and phases of the oscillation, i.e. the values of \(E_{k,i}, \omega_k, E_{m,i}, \omega_m\) and \(\phi_k - \phi_m\). The values measured for this computation are tabulated in Table 7.2 and show good agreement with the values quoted in the benchmark paper.

![Figure 7.1. Evolution of the energies for the full sphere stress-free dynamo benchmark of Marti et al. (2014).](image-url)
7.2. A preliminary search for no-slip dynamos in a sphere

In this section dynamo results obtained using no-slip viscous boundary conditions are presented. The results discussed now are obtained from an initial search through \((E, P_m, Pr, Ra)\) parameter space. These computations were performed first to search for working dynamo solutions over a wide range parameters as an initial exploration of the parameter space and second to search for steady or quasi-steady dynamos which could act as possible no-slip full-sphere benchmark models. To hasten the search through the parameter space most computations were performed imposing equatorial or azimuthal symmetries.

The set of computations performed are summarised in Figure 7.2 (for more information see Table E.1, Appendix E). These were initialised with

\[
\Theta_0^0 = \frac{1}{2} (1 - r^2), \quad \Theta_0^3 = -(\Theta_3^3)^* = (-1 + i) \frac{10^{-5}}{4\sqrt{\pi}} r^3 (1 - r^2),
\]

\[
T_1^1 = -(T_{1}^{-1})^* = \sqrt{\frac{2}{3}} (1 + i) r \left( \frac{3}{4} - 3r^2 + \frac{9}{2} r^4 - \frac{9}{4} r^6 \right), \quad T_2^0 = \frac{1}{\sqrt{5}} r^2 \left( \frac{3}{2} - \frac{21}{4} r^2 + \frac{27}{4} r^4 - 3r^6 \right),
\]

\[
t_1^0 = \frac{1}{\sqrt{3}} r \left( \frac{-54625}{198} + 350r^2 + \frac{625}{2} r^4 - 325r^6 \right) (1 - r^2),
\]

\[
t_2^1 = -(t_2^{-1})^* = r^2 \left( 30 + \frac{1250}{3} r^2 - 130r^4 - 90r^6 + i (105 - 245r^2 + 155r^4 - \frac{145}{7} r^6) \right) (1 - r^2).
\]  

(7.6)

These conditions are identical to the stress-free benchmark above except for the squaring of the factor \((1 - r^2)\) in the velocity potentials, required to satisfy the no-slip condition \((3.35)\). Two azimuthal symmetry classes were considered, the \(m_{hcf} = 1\) and the \(m_{hcf} = 3\). For the latter, the \(t_1^1\) and \(T_1^1\) terms were omitted. Note that with or without the azimuthal symmetry reduction these conditions imply a quadrupole velocity, equatorially even temperature and dipole magnetic field.

Only one set of solutions is discussed in detail here, those obtained for the parameters \((Ra, P_m, Pr, E) = (4000, 7, 0.7, 5 \cdot 10^{-4})\). Both \(m_{hcf} = 1\) and \(m_{hcf} = 3\) were considered. The \(m_{hcf} = 3\) case was also

<table>
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<tr>
<th>Diagnostic</th>
<th>Computed Value</th>
<th>Relative Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_{k,0})</td>
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<td>2.85 \times 10^{-4}</td>
</tr>
<tr>
<td>(E_{k,1})</td>
<td>1883.584</td>
<td>2.00 \times 10^{-3}</td>
</tr>
<tr>
<td>(\omega_k)</td>
<td>302.7618</td>
<td>2.01 \times 10^{-4}</td>
</tr>
<tr>
<td>(E_{m,0})</td>
<td>926.3338</td>
<td>1.89 \times 10^{-2}</td>
</tr>
<tr>
<td>(E_{m,1})</td>
<td>38.68822</td>
<td>3.28 \times 10^{-2}</td>
</tr>
<tr>
<td>(\omega_m)</td>
<td>302.7618</td>
<td>2.01 \times 10^{-4}</td>
</tr>
<tr>
<td>(\phi_k - \phi_m)</td>
<td>1.893225</td>
<td>8.78 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Table 7.2. Diagnostics for the stress-free full sphere dynamo benchmark of Marti et al. (2014). Differences are given relative to the tabulated benchmark values.
considered with an odd temperature perturbation, breaking the equatorial symmetry. This dynamo solution is discussed here as it is periodic when the azimuthal and equatorial symmetries are preserved during time-stepping, but was found to be unstable to perturbations which break either of these symmetries. This makes this solution ideal for testing correct implementation of these symmetries. It also serves as a warning against assuming symmetries in dynamo solutions a priori.

7.2.1. Convective dynamo I: a periodic no-slip full sphere dynamo with \((E, Ra, Pm, Pr) = (5 \cdot 10^{-4}, 4000, 7, 0.7)\). The convective dynamo I (CDI) is obtained for the parameters

\[
E = 5 \cdot 10^{-4}, \quad Pm = 7, \quad Pr = 0.7, \quad Ra = 4000 ,
\]  

\( (7.7) \)
with initial conditions (7.6). When the \( m_{hcf} = 3 \) symmetry is enforced this dynamo quickly settles to a periodic state [Figure 7.3, generated with \((J, N, \Delta t) = (240, 60, 10^{-6})\)] with only a small number of modes dominating the spectra of both the magnetic and kinetic energies (Figure 7.4).

**Figure 7.3.** Energy series for the CDI no-slip dynamo.

**Figure 7.4.** Amplitude spectra for the fluctuating part of the kinetic (left) and internal magnetic (right) energies for the CDI no-slip dynamo.

Expressing the kinetic and internal magnetic energies as cosine series
\[
E_k = A_{0,k} + \sum_{i \geq 1} A_{i,k} \cos(2\pi\omega_{i,k}t + \phi_{i,k}), \quad (7.8)
\]
\[
E_m = A_{0,m} + \sum_{i \geq 1} A_{i,m} \cos(2\pi\omega_{i,m}t + \phi_{i,m}), \quad (7.9)
\]

the amplitudes and frequencies are extracted by DFT and the phases by a least squares fit in the time domain. The values for the dominant 4 kinetic and 9 magnetic modes are given in Tables 7.3 and 7.4 respectively. The mode index ranks the magnetic modes by amplitude and matches the kinetic modes of the same approximate frequencies to the magnetic modes. The phases quoted correspond to a time shift which imposes \( \phi_{1,k} = 0 \).
To verify that the dynamo had settled to a periodic steady state the amplitude spectrum is calculated in a moving window of size $t_{\text{window}} = 1$ starting at $t = 2$ and incremented by $\Delta t = 0.01$. The mean relative errors in the amplitudes and frequencies are shown in Figure 7.5, where error bars show the coefficient of variance. The larger error in the amplitudes is attributed to the difficulty in finding peaks in the amplitude spectrum, since the spikes occupy a small frequency window but have poorly resolved maxima.

Determining the phases was nowhere near as robust, most likely due to the least squares fit finding a minimum local to the initial guesses as opposed to the global minimum. Despite this we find that the computed amplitudes, frequencies and phases generate a convincing fit to the original computed energy series. This is shown in Figure 7.6. The superior fit in the magnetic energy is attributed to the use of more modes in the reconstruction of the signal.

To test the convergence of this solution the computation was repeated with $(J, N, \Delta t) = (240, 80, 10^{-6})$. The energy series obtained for the two truncations are superimposed in Figure 7.7, where the series for $N = 80$ has been time-shifted. The agreement between the diagnostics for the two truncations is quantified in Figure 7.8 where the differences, both absolute and relative
7.2. A PRELIMINARY SEARCH FOR NO-SLIP DYNAMOS IN A SPHERE

Figure 7.5. Relative differences in the frequencies (left) and amplitudes (right) of energy series in a moving window for the CDI no-slip dynamo. Error bars show the coefficient of variation.

Figure 7.6. Reconstructed fluctuating component of kinetic (left) and internal magnetic (right) energies for the CDI no-slip dynamo.

to the tabulated values in the measured frequency and amplitude of the modes are plotted. The relatively large error in the amplitude of the index 3 magnetic mode is unexplained but appears not to significantly effect the series shown in Figure 7.7. This agreement between the magnetic and kinetic energy series for the two truncations is reinforced in Figure 7.9, where the averaged energy spectra (in harmonic order \( m \)) for the two truncations are superimposed.

The stability of this dynamo to perturbations which break the azimuthal and equatorial symmetries of the initial conditions was considered. To test the former the computation was repeated with the same initial conditions but without assuming any azimuthal symmetry [i.e. setting \( m_{\text{hcf}} = 1 \) but still omitting the \( t_2^1 \) and \( T_1^1 \) terms in (7.6)]. In infinite precision this symmetry should be maintained exactly, however, round-off error in the DFT’s is expected to artificially leak energy into the \( m \neq 0 \) (mod 3) modes. It is seen in Figure 7.10 that the initial perturbation to the 3-fold azimuthal symmetry grows, and a different solution is obtained. It is impossible to determine if this instability corresponds to the natural state of the dynamo, realised as soon as the 3-fold symmetry is broken, or if it caused by the leaking of energy into \( m \neq 0 \) (mod 3) modes. As a possible cause of this instability we hypothesise that aliasing errors are introduced at each time-step by non-periodic
FIGURE 7.7. Energy series for the no-slip CDI dynamo for \( N = 60 \) and \( N = 80 \).

FIGURE 7.8. Absolute (left) and relative (right) errors in the amplitude spectrum of the kinetic and internal magnetic energies for the CDI no-slip dynamo for \( (J, N, \Delta t) = (240, 60, 10^{-6}) \) and \( (J, N, \Delta t) = (240, 80, 10^{-6}) \). Differences are given relative to values tabulated for \( N = 60 \).

FIGURE 7.9. Time average spectra for the CDI no-slip dynamo for \( (J, N, \Delta t) = (240, 60, 10^{-6}) \) (circles) and \( (J, N, \Delta t) = (240, 80, 10^{-6}) \) (triangles). The harmonics with \( m \neq 0 \) (mod 3) have energy exactly zero and have been omitted.

sampling in the forward DFT. To test this we repeated the computation for the initial half diffusion time with \( N = 58 \) and \( N = 59 \), \( N = (58, 59, 60) \) corresponding to \( L = (2, 0, 1) \) (mod 3).
The energy series obtained for $N = 58$ and $N = 59$ are shown in Figure 7.11, and clearly the instability persists — emphasised by the $m$-energy spectrum where it is seen that a significant amount of energy is contained in the $m \neq 0 \pmod{3}$ modes. These results would suggest that the $\phi$ sample rate in the DFT is not the reason these solutions differ. We comment on the observation that the series for $N = 58, N = 59$ and $N = 60$ are all clearly different for the first half diffusion time, despite having the same initial conditions. We suggest that this is more likely due to the non-symmetric dynamos being chaotic than due to lack of resolution, based primarily on the $m$-energy series dropping by a factor of $10^4$ in both the kinetic and magnetic energies from peak to minimum. This, however, could only be thoroughly tested by running the calculations until a statistically steady state was reached, and then comparing the relevant diagnostics. As the purpose of these calculations was simply to check the stability of the dynamo to perturbation which break the azimuthal symmetry, this was not done. As a final comment we note that filtering the DFT, for example by setting all coefficients below a given threshold to zero, should produce exactly the same result as remapping the modes as outlined in Section 3.7.3 albeit in a less efficient manner.

![Figure 7.10. Energy series for the CDI no-slip dynamo with $m_{hcf} = 1$.](image)

Finally, we tested the stability of this dynamo to perturbations which break the equatorial symmetry of the initial conditions while maintaining the azimuthal symmetry of the initial conditions exactly. To achieve this we repeated the computation retaining $m_{hcf} = 3$, with the initial conditions modified to include the odd temperature perturbation

$$\Theta_3^3 = 10^{-5} r \Theta_3^3.$$  \hspace{1cm} (7.10)

This dynamo was found to settle to a state very similar state to that obtained without the initial $\Theta_3^3$ perturbation. The internal magnetic and kinetic energy series are shown in Figure 7.12 and energy spectra, dipole kinetic energy fraction and quadrupole magnetic energy spectra are shown in Figure 7.13. Although these appear very similar to the energy series shown in Figure 7.3, even a cursory comparison of the frequency spectra shown in Figures 7.4 and 7.14 reveals differences in the both the kinetic energy and internal magnetic energy series.

These results highlight the importance of not assuming any symmetry in a dynamo solution a priori, especially when the most unstable solution for a given set of parameters and initial conditions is the one sought. This raises the question of whether the added coding complexity required in the time-stepping and transform routines to implement the equatorial symmetric/anti-symmetric field decomposition gives any tangible gains. This, of course, depends heavily upon the intended...
Figure 7.11. Energy time series (left) and $m$-spectrum (right) for $N = 58$ (top) and $N = 59$ (bottom) for the candidate no-slip benchmark model with $m_{hcf} = 1$. $m$-spectrum is the average energy for $t \geq 0.25$.

Figure 7.12. Energy series for the CDI no-slip dynamo with odd perturbation to the initial temperature.
The remainder of this chapter is dedicated to the analysis of the dynamo results presented in this section. These dynamos differ from those discussed thus far in that they are all generated for the parameters

\[ E = 5 \cdot 10^{-4}, \quad Pr = 0.7, \quad (7.11) \]
without assuming any symmetry in the solution, for \( Pm \in [1, 40] \) and \( Ra \) up to a few times the critical value for convective onset. Computations were initialised with the conditions

\[
T_2^0 = T_2^1 = -(T_2^{-1})^* = \frac{1}{\sqrt{5}}r^2 \left( \frac{3}{2} - \frac{21}{4}r^2 + \frac{27}{4}r^4 - 3r^6 \right)
\]

\[
t_1^0 = t_1^1 = -(t_1^{-1})^* = \frac{1}{\sqrt{3}}r \left( -\frac{54625}{198} + 350r^2 + \frac{625}{2}r^4 - 325r^6 \right)(1 - r^2)
\]

\[
t_3^0 = t_3^3 = -(t_3^{-3})^* = \frac{1}{\sqrt{5}}r^3 \left( 45 - 575r^2 - 835r^4 - 350r^6 \right)(1 - r^2)
\]

\[
\Theta_0^0 = \frac{1}{2}(1 - r^2), \quad \Theta_3^3 = - (\Theta_3^{-3})^* = (-1 + i) \frac{10^{-5}}{4\sqrt{\pi}} r^3 (1 - r^2), \quad \Theta_4^4 = -(\Theta_4^{-4})^* = r\Theta_3^3
\]

A summary of the results obtained is shown in Figure 7.15, with the working solutions detailed in Table 7.5 and an overview of every computation summarised in Section E2. In this figure open circles represent failed dynamos and asterisks denote working dynamo solutions, their colour giving the average magnetic energy. Crosses are used to distinguish solutions for which dynamo action was questionable. This was either due to computations being terminated too soon to conclude dynamo action or the energy series not settling to a ‘steady’ state (i.e. without large, irregular fluctuations). The lines on Figure 7.15(right) are \( Ra = Ra_c \) and \( Ra = 2Ra_c \).
<table>
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<tr>
<th>Ra</th>
<th>Pm</th>
<th>$E_k$</th>
<th>Dipole Fraction</th>
<th>$P_h$</th>
<th>$P_{\text{viscous}}$</th>
<th>$E_m$</th>
<th>Dipole Fraction</th>
<th>$P_J$</th>
<th>$P_{\text{Lorentz}}$</th>
<th>$\Lambda_2$</th>
</tr>
</thead>
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<tr>
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<td>5</td>
<td>78889.5 ± 14434.9</td>
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<td>2.05393 × 10^{8}</td>
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<td>10473.8 ± 11548.9</td>
<td>1.06 × 10^{-2}</td>
<td>5.98819 × 10^{6}</td>
<td>5.98878 × 10^{6}</td>
<td>0.17</td>
</tr>
<tr>
<td>3000</td>
<td>5</td>
<td>91431.1 ± 14328.4</td>
<td>1.22 × 10^{-2}</td>
<td>1.34872 × 10^{10}</td>
<td>1.34734 × 10^{10}</td>
<td>20941.3 ± 19417.4</td>
<td>3.94 × 10^{-2}</td>
<td>1.38107 × 10^{7}</td>
<td>1.38713 × 10^{7}</td>
<td>0.24</td>
</tr>
<tr>
<td>3000</td>
<td>7</td>
<td>93613.2 ± 18505.7</td>
<td>1.62 × 10^{-2}</td>
<td>3.31685 × 10^{8}</td>
<td>2.95505 × 10^{8}</td>
<td>57814.0 ± 26642.1</td>
<td>3.93 × 10^{-1}</td>
<td>3.62346 × 10^{7}</td>
<td>3.61908 × 10^{7}</td>
<td>0.35</td>
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<tr>
<td>2500</td>
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<td>77327.4 ± 17286.7</td>
<td>3.98 × 10^{-4}</td>
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<td>2.11917 × 10^{8}</td>
<td>6627.18 ± 4825.95</td>
<td>1.34 × 10^{-2}</td>
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<td>5.59364 × 10^{6}</td>
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<tr>
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<td>2.98 × 10^{-5}</td>
<td>5.08546 × 10^{8}</td>
<td>4.92634 × 10^{8}</td>
<td>23081.1 ± 295.455</td>
<td>2.50 × 10^{-4}</td>
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<td>1.59074 × 10^{7}</td>
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<tr>
<td>4000</td>
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<td>174976 ± 21569.9</td>
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<td>9.35612 × 10^{8}</td>
<td>9.35512 × 10^{8}</td>
<td>205.251 ± 258.935</td>
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<td>9.96935 × 10^{4}</td>
<td>9.97056 × 10^{4}</td>
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<tr>
<td>3500</td>
<td>25</td>
<td>109248 ± 55.5840</td>
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<td>5.71085 × 10^{9}</td>
<td>5.43998 × 10^{9}</td>
<td>9807.14 ± 438582</td>
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<tr>
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<td>135088 ± 17042.1</td>
<td>2.71 × 10^{-5}</td>
<td>1.68225 × 10^{9}</td>
<td>1.09448 × 10^{9}</td>
<td>17563.2 ± 6522.01</td>
<td>1.25 × 10^{-4}</td>
<td>5.87773 × 10^{8}</td>
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<td>0.17</td>
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<tr>
<td>6000</td>
<td>30</td>
<td>386120 ± 57411.5</td>
<td>2.25 × 10^{-6}</td>
<td>2.67132 × 10^{9}</td>
<td>2.64182 × 10^{9}</td>
<td>2555.17 ± 4992.53</td>
<td>1.17 × 10^{-4}</td>
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<td>2.96101 × 10^{7}</td>
<td>8.33 × 10^{-3}</td>
</tr>
<tr>
<td>5000</td>
<td>35</td>
<td>16475.8 ± 4571.21</td>
<td>1.03 × 10^{-4}</td>
<td>1.82265 × 10^{11}</td>
<td>1.79406 × 10^{11}</td>
<td>14895.1 ± 7061.15</td>
<td>0.97</td>
<td>2.85913 × 10^{9}</td>
<td>2.85912 × 10^{9}</td>
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**Table 7.5.** Diagnostics for working no-slip dynamo solutions with $E = 5 \cdot 10^{-4}$, $Pr = 0.7$ and $m_{hcf} = 1$. Energies are quoted as mean ± standard deviation. The dipole fraction is the mean fraction of the total energy associated with the dipole component.
Only eleven dynamo solutions were computed in this parameter regime, with the parameters and energy diagnostics for these working dynamos listed in Table 7.5. Because the solutions are so sparsely distributed in the \((\text{Ra}, \text{Pm})\) space it is difficult to make conclusive assertions concerning the gross properties of dynamo solutions as the parameters are varied. In an attempt to do just this we now turn to dynamo scaling laws.

### 7.4. Dynamo scaling laws

We now consider the systematic no-slip dynamo solutions through the lens of scaling laws for spherical dynamos. As so few dynamo solutions were obtained, and these were in a very limited region of the parameter space, the conclusions that can be drawn from a scaling law analysis are very limited. The aims of considering dynamo scaling laws are thus limited first, to classifying the dynamo solutions found, and second to describing the dynamics. We appeal to both theoretically based scaling laws, as well as numerically derived scaling laws.

All scaling laws considered will be multiplicative power laws. We follow the same method as Oruba & Dormy (2014a) when both assessing the fit of a scaling law to the data as well as for derivation of numerical scaling laws. Writing a general power law as

\[
y = \alpha \prod_{k \geq 1} x_{\{k\}}^{\beta_k},
\]

for an endogenous variable \(y\) and exogenous variables \(x_{\{k\}}\), the exponents are found by applying a standard least squares multiple (linear) regression method to the log of \((7.13)\),

\[
\log_{10} y = \beta_0 + \sum_{k \geq 1} \beta_k \log_{10} x_{\{k\}},
\]

where \(\beta_0 = \log_{10} \alpha\). In the study of scaling laws for numerical dynamos it has become customary to characterise the goodness of fit using the mean relative misfit

\[
\chi_{\text{rel}} := \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \frac{y_i - \hat{y}_i}{\hat{y}_i^2} \right)^2},
\]

where \(\hat{y}\) is the fitted response (e.g., Olson & Christensen 2006, Christensen & Aubert 2006, Stelzer & Jackson 2013, Oruba & Dormy 2014a), as opposed to the usual use of the coefficient of determination.

When the exponents \(\beta\) in \((7.13)\) are assumed, the power law can be re-written as

\[
y = \alpha x .
\]

In these cases the coefficient \(\alpha\) can be found by either a least squares minimisation

\[
\alpha_{\text{ls}} = \frac{\sum x_i y_i}{\sum x_i^2},
\]

or by minimising the relative misfit \((7.15)\)

\[
\alpha_{\text{rm}} = \frac{\sum x_i/y_i}{\sum x_i^2/y_i^2}.
\]
The reader is reminded that the scales for the density, magnetic field, velocity and length scales are \( \rho_0, B, U \) and \( L \) respectively, which are all dimensional. An asterisk \( * \) is used to identify the dimensional form of a variable from the non-dimensional form; e.g.

\[
r_* = L r . \tag{7.19}
\]

### 7.4.1. Scaling of the magnetic field.
We first consider the power balance satisfied by every statistically steady dynamo solution, as examined by Oruba & Dormy (2014a). Adding the kinetic energy equation (2.52) to the magnetic energy equation (2.59) we obtain

\[
P_{b*} = P_{J*} + P_{\nu*} , \tag{7.20}
\]

where \( P_{b*} \) is the power generated by the buoyancy force, \( P_{J*} \) is the Joule dissipation and \( P_{\nu*} \) is the viscous dissipation. Writing the fraction of the total power that is lost to ohmic dissipation as

\[
f_{\text{ohm}} := \frac{P_{J*}}{P_{\nu*} + P_{J*}} \tag{7.21}
\]

allows (7.20) to be written as

\[
P_{b*} = \frac{P_{J*}}{f_{\text{ohm}}} \sim \frac{1}{f_{\text{ohm}}} \frac{\eta}{\mu_0} \frac{B^2}{l_{B*}^2} V . \tag{7.22}
\]

where \( l_{B*} \) is the magnetic dissipative (or Ohmic) length scale defined by

\[
l_{B*}^2 := \frac{\int_V \nabla \times B_* dV_*}{\int_V \nabla \times B_*^2 dV_*} . \tag{7.23}
\]

Re-arranging gives the following scaling law for the magnetic field strength;

\[
B^2/\mu_0 \sim f_{\text{ohm}} \frac{l_{B*}^2}{\eta} P_{b*} . \tag{7.24}
\]

Re-arranging the estimate for the magnetic energy

\[
E_{m*} \sim \frac{1}{2\mu_0} B^2 V , \tag{7.25}
\]

and recalling that the time scale is \( \tau_\eta = L^2/\eta \), allows (7.24) to be written in non-dimensional form

\[
E_{m} \sim \frac{f_{\text{ohm}} l_{B*}^2 P_{b}}{2} . \tag{7.26}
\]

This relation can be derived directly from (7.21) and (7.23), with Ampere’s Law (2.11c), the definitions of the magnetic energy (2.57) and the Joule dissipation (2.60), and by taking the magnetic diffusion time for the time-scale. We thus expect (7.26) to hold exactly (with unit pre-factor).

Applying the law (7.26) to the tabulated dynamo solutions results in Figure 7.16 (left). Assuming (7.26) holds, and finding the least squares pre-factor given by (7.17) with abscissa \( x = (f_{\text{ohm}} l_{B*} P_{b}) \) and ordinate \( y = E_{m} \) gives \( \alpha = 0.5002 \) with the relative misfit \( \chi_{\text{rel}} = 0.02 \). It is unsurprising that the dynamo solutions satisfy this law, as it is constructed essentially by re-arranging the relation (7.20), which is approximately satisfied by the dynamo solutions (see Table 7.5).
7. Convection Driven Dynamos in Rotating Spheres

7.4.2. Force balance scaling of the velocity. We now seek to use scaling laws to classify the dynamo solutions. We begin as Christensen (2010) does by considering different force balances in the momentum equation and test the implied scaling laws against the dynamo results obtained. We consider four force balances for the velocity: the mixing length balance, the MAC (Magneto-Archipedean-Coriolis) balance, the CIA (Coriolis-Inertia-Archipedean) balance, and the VAC (mixed Coriolis-Buoyancy and Coriolis-Viscous) balance outlined in King & Buffett (2013). The basic idea is to estimate $l_b^2/\eta$ in (7.24) by $L/\mathcal{U}$, where power laws for $\mathcal{U}$ are derived by assuming one of these four force balances. To remove the portion of the force balance associated with gradient forces (e.g. the pressure) the velocity scaling laws are derived from the equation for the vorticity

$$\omega_* := \nabla_* \times \mathbf{u}_* ,$$

which is derived by taking the curl of the momentum equation (2.1);

$$\left( \partial_t - \nu \nabla^2 \right) \omega_* = -\nabla \times (\mathbf{u}_* \cdot \nabla \mathbf{u}_*) - 2\Omega z \cdot \nabla \mathbf{u}_* + \frac{1}{\rho_0} \nabla \times (\rho - \rho_0) \mathbf{g}_* + \frac{1}{\rho_0} \nabla \times (\mathbf{J}_* \times \mathbf{B}_*) .$$

(7.28)

In this section repeated use will be made of the estimate for the buoyancy power

$$P_b^* = \rho_0 \alpha g \int V \Theta'_* \mathbf{u}_* \cdot \mathbf{1}_r \, dV_* \sim \rho_0 \alpha g \mathcal{U}_* \Theta'_* L V ,$$

(7.29)

to write

$$\alpha \Theta'_* g L \sim \frac{P_b^*}{\rho_0 \mathcal{U} V} .$$

(7.30)

We now consider the four force balances listed above and examine the corresponding scaling laws for the velocity.

I. Mixing length balance. In the mixing length balance the buoyancy is balanced by inertia, which transfers energy from large scales to small scales where it is dissipated by the viscosity. This is assumed to apply in highly turbulent convection in stars (e.g. Kippenhahn & Weigert 1990, Starchenko & Jones 2002), and is thus not expected to apply to the dynamo solutions obtained here.
just above the critical onset for convection. Regardless, we consider this law for completeness. Assuming
\[ \nabla_\star \times (u_\star \cdot \nabla_\star u_\star) \sim \frac{1}{\rho_0} \nabla_\star \times ((\rho_\star - \rho_0)g_\star) \quad (7.31) \]
gives
\[ U^2 \sim \alpha \Theta'_\star g l_\star^2, \quad (7.32) \]
where the length scale \( l_\star \) is the typical scale on which the momentum and temperature are mixed. Assuming this is the same as the depth of the convection region sets \( l_\star \sim \mathcal{L} \). Combining this with (7.30) allows (7.32) to be re-written as
\[ U \sim \left( \frac{\mathcal{L} P_{b_\star}}{\rho V} \right)^{1/3}. \quad (7.33) \]
Re-arranging the estimating for the kinetic energy
\[ E_{k_\star} \sim \frac{\rho_0}{2} U^2 V \quad (7.34) \]
exactly as was done for the magnetic energy above, and simplifying using the estimate (7.29), noting that the units of energy are defined by (2.54), allows (7.33) to be written in dimensionless form
\[ E_k \sim P_b^{2/3}. \quad (7.35) \]
Applying this scaling law to the 11 tabulated dynamo solutions gives Figure 7.17(left), a law which is clearly not universally satisfied by the dynamo solutions obtained.

**Figure 7.17.** Mixing length velocity scaling. Full data set (left) and truncated data set (right).

We see that the dynamo solutions are clustered, with outliers \((Ra, Pm) = (3000, 5), (3500, 25), (5000, 35)\). This clustering of the solutions was found to persist for all the force balance scaling laws considered herein, a feature for which no obvious explanation was found. We note that these solutions do not correspond to the outlier dynamos found in Figure 7.16. For this velocity scaling, as well as those considered below, we retested the proposed scaling with these three points removed, and checked the goodness of fit obtained for the truncated data set — in this case (7.35), using (7.18) to calculate the pre-factor. The resulting fit is shown in Figure 7.17(right), where we
see that truncated data set also gives a poor fit to the numerical solutions.

II. MAC balance. The next force balance we consider is the MAC balance in which the primary force balance in the fluid is between the buoyancy (Archimedean), Coriolis and Lorentz forces. In this balance the buoyancy force works against the Lorentz force to increase the magnetic energy which is lost by Joule dissipation, resulting in a large scale flow (Starchenko & Jones 2002). Since the Coriolis and buoyancy forces are assumed comparable we can find a velocity scaling independent of the magnetic field

\[ \mathcal{U} \sim \frac{\alpha \theta^* g \mathcal{L}}{2\Omega}, \]  

which, using (7.30), can be re-written as

\[ \mathcal{U}^2 \sim \frac{P_b^*}{2\Omega \rho V}. \]  

Using the kinetic energy to estimate \( \mathcal{U} \) gives the non-dimensional law

\[ E_k \sim \text{Ro} \ P_b. \]  

The application of this to the dynamo results is shown in Figure 7.18 (left), where it is made clear that this scaling does not apply either universally or the truncated data set. The large relative misfit in the truncated set seems to be driven primarily by the dynamo at \((\text{Ra}, \text{Pm}) = (4600, 23)\). However there is insufficient justification to remove this datum.

III. CIA balance. The next balance we consider is the CIA scaling for rapidly rotating convection where the Coriolis, inertial and Archimedean forces are assumed to be of the same magnitude. The length scale \( l^* \) is associated with derivatives of the velocity except for the \( z \)-derivative in the Coriolis term which is assumed to have scale \( \mathcal{L} \), as we are considering the rapid rotation limit. For this same reason the scale \( \mathcal{L} \) is assumed for the buoyancy term. In the radial vorticity equation the inertia term balances the Coriolis term which gives

\[ l^2 \sim \frac{\mathcal{U} \mathcal{L}}{2\Omega}, \]  

and the balance between the buoyancy term and the inertia term thus gives

\[ \mathcal{U} \sim \left( \frac{P_b^*}{\rho V} \right)^{2/5} \left( \frac{\mathcal{L}}{\Omega} \right)^{1/5}. \]  

![Figure 7.18. MAC velocity scaling. Full data set (left) and truncated data set (right).](image-url)
These laws can be written in the non-dimensional form

\[ l^2 \sim \text{Ro}, \quad E_k \sim \text{Ro}^{2/5} \text{P}_b^{4/5}. \] (7.41)

The application of (7.41b) to the dynamo solutions is shown in Figure 7.19 (left). Again the same three outliers are seen. A significantly improved, but still inaccurate, fit is found by omitting these. As with the MAC scaling above the large relative misfit seems to be caused mostly by the same dynamo, and again there is insufficient grounds to remove this datum.

Figure 7.19. CIA velocity scaling. Full data set (left) and truncated data set (right).

IV. VAC scaling. The next balance we consider is the VAC balance described in King & Buffett (2013) and Oruba & Dormy (2014a). In this regime the inertia and Lorentz forces are assumed negligible, and the Coriolis force balances the Archimedean force, except in the radial vorticity equation where the viscous term is retained in order to balance the Coriolis force. This would be expected to apply near the onset of dynamo action. The latter balance gives a law for the velocity length scale

\[ l_* \sim \left( \frac{\text{L} \nu}{2\Omega} \right)^{1/3} = E^{1/3} \mathcal{L}. \] (7.42)

Combining this with the Coriolis-Archimedean force balance, where the larger length scale \( \mathcal{L} \) is associated with the Archimedean force, gives

\[ \mathcal{U}^2 \sim \frac{E^{1/3} \text{P}_b}{2\Omega \rho \nu}. \] (7.43)

This can be written in the non-dimensional form

\[ E_k \sim E^{1/3} \text{Ro} \text{P}_b, \] (7.44)

with the application of this law to the dynamo results shown in Figure 7.20. Again we see a poor fit to the numerical data, indicating that this force balance is not satisfied universally by the dynamo solutions.
7.4.3. Discussion of theoretical scaling laws. None of the scaling laws discussed above are satisfied universally by the dynamo solutions obtained. There are many possible explanations for this, for example: the different dynamo solutions have different pre-factors in the relations (7.35), (7.38), (7.41) or (7.44); the different solutions are characterised by different force balances; or the dynamo solutions simply do not satisfy any of the force balances considered. Another possible explanation is that the force balances considered above are all local balances and, by using integrated quantities (e.g., the total buoyancy power), which we have implicitly averaged in time, we have assumed that the force balance is satisfied not only in the bulk of the fluid, but also over the integration time where quantities are calculated. We will see in Section 7.5, where we classify the dynamo solutions as either strong or weak field, that the balance between the Lorentz and Coriolis forces are highly localised in space and, for some dynamos, the balance is not satisfied for all time. The way in which global quantities such as the total kinetic energy and total power generated by the buoyancy force are used introduces a further error as we have made the implicit assumption that integrals of products are the same as the product of the integrals, both in terms of the time-averaging process as well as the integration over the fluid volume. For example, for the VAC scaling going from (7.40) to (7.41) we have assumed both
\[
\int_V \left( \frac{P_b}{V} \right)^{4/5} dV = P_b^{4/5},
\]
and
\[
\langle P_b^{4/5} \rangle = \langle P_b \rangle^{4/5},
\]
where we have introduced the notation \( \langle \cdot \rangle \) to denote a time average over some period. Authors such as Christensen, Holzwarth & Reiners (2009) and Christensen (2010) have suggested integrating the scaling laws over the fluid volume to address the first of these errors. We did not follow this approach as it would require re-calculating the corresponding right sides of (7.35), (7.38), (7.41) and (7.44), a process that could be simplified by simply comparing the forces directly in real space.

The conclusion is that, especially in light of the results of 7.4.1, it is optimistic to suppose these solutions would universally satisfy a single scaling law. This should be unsurprising given the different properties of the solutions obtained that can be gleaned from Table 7.15. This contrasts the studies of scaling laws for spherical shell dynamo solutions mentioned above where dynamos with similar properties, such as dynamical parameters, magnetic field morphology (e.g., dipole selection...
criterion) and in similar dynamical regimes (e.g. Nu based selection of rigorously convecting models) are selected for comparison. This, of course, given the small number of solutions obtained was not possible here.

Despite these limitations we now turn to the use of numerical scaling laws in an attempt to find a single law universally satisfied by the dynamo solutions.

### 7.4.4. Numerical scaling laws

In this section we attempt to characterise the numerical dynamo solutions using numerically based scaling laws. Following primarily Christensen (2010) and Stelzer & Jackson (2013) we test scaling laws for the magnetic field strength, the flow velocity and the magnetic dissipation time

\[
\tau_{\text{diss}} = \frac{E_m^*}{P J^*}, \quad \tau_{\text{diss}} = \frac{E_m}{P J}.
\]  

(7.47)

**I. Flow Velocity**. We first seek scaling laws for the velocity of the form

\[
E_k^{1/2} \sim \text{Ra}^\alpha \text{Pm}^\beta \text{Pb}^\gamma \text{Em}^\delta.
\]  

(7.48)

The first we tested is the equivalent of Stelzer & Jackson (2013, Eqn. 22), with \(\gamma = \delta = 0\). We also tested the more constrained model \(\beta = \gamma = \delta = 0\), and then eased the constraints by testing \(\delta = 0\) and then with the full set of parameters. The parameters found for each model are shown in Table 7.6, with the corresponding plots shown in Figure 7.21.

![Figure 7.21. Numerical scaling laws for the flow strength.](image-url)
As expected, as we increased the number of explanatory factors we achieved a better fit to the numerical data. Of particular interest is that the exponents of the parameters which are included in more than one model change significantly as we add more parameters. These two observations qualitatively indicate that we should retain all the parameters in the numerical model.

Inspection of Figure 7.21, however, reveals that the data spans a small range in both the abscissa and ordinate. We should therefore be very cautious in the interpretation of these results as power based scaling laws. In particular, we note that the numerical fit becomes better as the span of the data in the abscissa is increasingly limited.

II. Magnetic dissipation time. We fit the same law for the magnetic dissipation time as Stelzer & Jackson (2013, Eqn. 34). Retaining the same variables used above gives the law

\[ \tau_{\text{diss}} \sim E_k^\alpha Pm^\beta. \]  

Fitting this to the numerical data gives Figure 7.22 (left), where it is seen that the data clusters into three groups based on \( Pm \in [5, 8], Pm \in [20, 23] \) and \( Pm \in [25, 35] \), a clustering that can be seen when \( \tau_{\text{diss}} \) is plotted against \( Pm \) directly, as in Figure 7.22 (right). Unfortunately this clustering of the data suggests that the scaling law should be fitted to the clusters individually, an unrealistic prospect as each group contains at most four data points and the numerical fitting requires fitting three coefficients.

![Figure 7.22](image-url)

**Figure 7.22.** Numerical Scaling Laws for the magnetic dissipation time. Numerical scaling law (left), and \( \tau_{\text{diss}} \) plotted against \( Pm \) directly (right).
III. Magnetic field strength. The last set of numerical scaling laws we consider are those for the magnetic field strength. For this scaling we scale the magnetic field strength by the ohmic dissipation fraction and seek power laws relating this to Pm and Ra;

\[ \frac{E_m}{f_{ohm}} \sim Ra^\alpha Pm^\beta, \]  

(7.50)

and to Pm and P_b

\[ \frac{E_m}{f_{ohm}} \sim P_b^\alpha Pm^\beta. \]  

(7.51)

Fitting these scaling laws to the numerical data gives the law shown in Figure 7.23, where, unsurprisingly given all the discussion above, it is seen that the data clearly does not satisfy a single law.

\[ \text{Figure 7.23. Numerical scaling laws for the magnetic field strength. Full data set (left), and truncated data set (right).} \]

7.4.5. Discussion of the scaling law analysis. The unfortunate end result is that no scaling law which encompasses all the data was found, the only real contender being (7.26) which, as discussed, should be automatically satisfied by every statistically steady dynamo solution. This highlights the need for further computations, to both expand the region of the parameter space explored (in particular different values for the Ekman number and the Prandtl number, as well as higher Rayleigh and small magnetic Prandtl numbers), as well as to fill out more completely the region considered in this work — an expensive undertaking.
7.5. Weak/Strong field dynamo solutions

We notice in Figure [7.15] high (magnetic) energy dynamos found for relatively small Rayleigh numbers. The cases \( \text{Pm} = 20 \) where increasing \( \text{Ra} \) was found to decrease the magnetic energy, and of \( \text{Pm} = 25 \) where increasing \( \text{Ra} \) led to failing dynamo solutions, are particularly interesting as they are recollective of the sub-critical dynamos studied in spherical shell geometry by Morin & Dormy (2009). Dormy (2014) identified these branches with strong/weak field field classification of the dynamo solutions. We follow this work and apply the same methods to classify the dynamo solutions obtained here.

The relative strength of the Lorentz force to the Coriolis force is given by the (dynamic) Elsasser number

\[
\Lambda := \frac{\{(\mu \rho)^{-1} \nabla \times B \times B\}}{\{2\Omega \times \mathbf{u}\}} = \frac{B^2}{2\mu \rho \Omega U L_B} \quad (7.52)
\]

where \( \{\cdot\} \) denotes the order of magnitude and \( L_B \) is a typical length scale for variation of the magnetic field. A dynamo is classified as weak field if \( \Lambda \ll 1 \) or strong field if \( \Lambda = \mathcal{O}(1) \). The non-dimensionalisation used for the magnetic field in Chapter 2 is based on the assumptions that: first, the dynamos are strong field with \( \Lambda = 1 \); second, the magnetic induction balances the magnetic dissipation

\[
\{\nabla \times (\mathbf{u} \times \mathbf{B})\} \sim \{\eta \nabla \times \nabla \times \mathbf{B}\} \quad (7.53)
\]

which is equivalent to \( \mathcal{R}_{m} = 1 \) (or \( \mathcal{U} = \eta / \mathcal{L} \)); and finally that the magnetic field has a large spatial scale \( L_B = \mathcal{L} \) where \( \mathcal{L} \) the length scale of the problem geometry, in this case the radius of the sphere. Putting these assumptions together gives the magnetic field scale \( B \) (2.39). These assumptions, however, are not necessarily true for a dynamo computation.

Dormy (2014) considered three reasons why the Elsasser number for a computed dynamo (the computed Elsasser number) does not scale to unity, namely that the values for the three scales \( \mathcal{U} \), \( \mathcal{L}_B \) and \( B \) are wrong. Suppose \( \Lambda' \) is the Elsasser number based on the scales \( \mathcal{U}' \), \( \mathcal{L}'_B \) and \( B' \), then \( \Lambda' \) and \( \Lambda \) are related by

\[
\frac{\Lambda'}{\Lambda} = \left( \frac{B'}{B} \right)^2 \frac{\mathcal{U}}{\mathcal{U}'} \frac{\mathcal{L}_B}{\mathcal{L}'_B}. \quad (7.54)
\]

Identifying \( B' \) with the volume average of \( B \)

\[
(B')^2 = \frac{1}{V} \int_V B^2 dV, \quad (7.55)
\]

and recalling that the scale \( B \) has been assumed for the magnetic field, so that the right of (7.55) is

\[
\frac{1}{V} \int_V B^2 dV = B^2 \int_V B^2 dV, \quad (7.56)
\]

gives

\[
\left( \frac{B'}{B} \right)^2 = \int_V B^2 dV = 2 \text{Ro} E_m(V). \quad (7.57)
\]
Following a similar process for $\mathcal{U}$ and $\mathcal{U}'$ gives

$$\frac{\mathcal{U}'}{\mathcal{U}} = \mathcal{R}_m = \sqrt{\int_V \mathbf{u}^2 \, dV} = \sqrt{2 E_k}. \tag{7.58}$$

For the final scale $\mathcal{L}_{B}'$, Dormy (2014) suggested using the Ohmic dissipation length scale, which gives

$$\frac{\mathcal{L}_{B}'}{\mathcal{L}_{B}} = l_{B*} = l_B \tag{7.59}$$

where $l_{B*}$ is given by (7.23).

In this thesis we consider two different realised Elsasser numbers based on the refinements discussed above. The first, $\Lambda'_1$, assumes that only the magnetic scales $B'$ and $B$ are mismatched, giving

$$\Lambda'_1 := \left(\frac{B'}{B}\right)^2 \Lambda = 2 \mathcal{R}_m \mathcal{E}_m(V). \tag{7.60}$$

This is the Elsasser number most commonly quoted for computational dynamos. The second, $\Lambda'_2$ (identified with the $\Lambda'$ of Dormy 2014) assumes that all three scales discussed above as mismatched, giving

$$\Lambda'_2 := \left(\frac{B'}{B}\right)^2 \frac{\mathcal{U}}{\mathcal{U}'} \frac{\mathcal{L}_{B}}{\mathcal{L}_{B}'} \Lambda = \frac{\Lambda'_1}{\mathcal{R}_m l_B}. \tag{7.61}$$

When calculating (7.60) and (7.61) we run into a problem similar to the problem described in Section 7.4.3: if the full series for the internal magnetic energy, kinetic energy or the Joule dissipation are not available then time averages of products of the series must be replaced with products of time averages of the series. The values of $\Lambda'_1$ and $\Lambda'_2$ tabulated in Table 7.7 are determined by calculating $\Lambda'_1$ and $\Lambda'_2$ at each output step and averaging over time. We compute an additional realised Elsasser number, $\Lambda'_3$, which is identical to $\Lambda'_2$, but with estimates

$$\Lambda'_1 = \langle 2 \mathcal{R}_m \mathcal{E}_m(V) \rangle, \quad l_B^2 = 2 \frac{\langle \mathcal{E}_m(V) \rangle}{\langle P \rangle}, \quad \mathcal{R}_m = \sqrt{2 \langle E_k \rangle}, \tag{7.62}$$

used in (7.61), where $\langle \cdot \rangle$ denotes the time average over some period. For most of the solutions there is little difference between $\Lambda'_2$ and $\Lambda'_3$. This is not the case, however, for the dynamos $(\mathcal{R}_a, \mathcal{P}_m) = (2500, 5)$ and $(2500, 8)$.

At the moment we have no sense of how well the estimates for $\Lambda$ given above characterise the balance between the Lorentz and Coriolis forces throughout the bulk of the fluid. We address this by calculating

$$\Lambda(r) = \frac{|\nabla \times \mathbf{B} \times \mathbf{B}|}{|\mathbf{1} \times \mathbf{u}|} \tag{7.63}$$

directly on the $(r, \theta, \phi)$ grid. Unlike the force balances considered in Section 7.4.2, where we were concerned with the roll the velocity plays in the force balance, we do not discount the gradient forces before considering the balance (7.63), as here we are concerned with the relative strengths of the total Lorentz and the total Coriolis forces. Picking the most obvious strong field candidate first, namely $(\mathcal{R}_a, \mathcal{P}_m) = (5000, 35)$, we plot the local Elsasser number given by (7.63) as well as
the norms of the Lorentz force, Coriolis force and Elsasser number in Figure 7.25. We see that there is no strong field balance in the bulk of the fluid. A large Elsasser number / strong field balance is only observed in localised patches near the boundary at high and low latitudes, where the Coriolis force is particularly weak. Even in the weak field solutions this pattern persists: the regions of strong Lorentz force avoid the regions of strong Coriolis force.

We repeat this process for the weak field dynamo \((Ra, Pm) = (3500, 25)\) in Figure 7.26, and for the dynamo \((Ra, Pm) = (3000, 5)\), which was found to oscillate between a weak field state (Figure 7.27) and a strong field state (Figure 7.28), at the times indicated in Figure 7.24.

One direct implication of these results is that the scaling laws based on force balances discussed in the previous section should not work when applied to these dynamos collectively.

Table 7.7. Computed Elsasser numbers for no-slip dynamo solutions.

<table>
<thead>
<tr>
<th>Ra</th>
<th>Pm</th>
<th>(\Lambda_1^\prime)</th>
<th>(\Lambda_2^\prime)</th>
<th>(\Lambda_3^\prime)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>5</td>
<td>2.09</td>
<td>0.17</td>
<td>8.92 \times 10^{-2}</td>
</tr>
<tr>
<td>3000</td>
<td>5</td>
<td>4.19</td>
<td>0.24</td>
<td>0.18</td>
</tr>
<tr>
<td>3000</td>
<td>7</td>
<td>8.26</td>
<td>0.35</td>
<td>0.34</td>
</tr>
<tr>
<td>2500</td>
<td>8</td>
<td>0.83</td>
<td>0.15</td>
<td>3.33 \times 10^{-2}</td>
</tr>
<tr>
<td>3000</td>
<td>20</td>
<td>1.15</td>
<td>5.69 \times 10^{-2}</td>
<td>5.29 \times 10^{-2}</td>
</tr>
<tr>
<td>4000</td>
<td>20</td>
<td>1.03 \times 10^{-2}</td>
<td>2.88 \times 10^{-4}</td>
<td>2.70 \times 10^{-4}</td>
</tr>
<tr>
<td>4600</td>
<td>23</td>
<td>4.39 \times 10^{-2}</td>
<td>9.54 \times 10^{-4}</td>
<td>2.67 \times 10^{-3}</td>
</tr>
<tr>
<td>3500</td>
<td>25</td>
<td>0.39</td>
<td>9.86 \times 10^{-2}</td>
<td>9.86 \times 10^{-2}</td>
</tr>
<tr>
<td>4000</td>
<td>28</td>
<td>0.63</td>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>6000</td>
<td>30</td>
<td>8.52 \times 10^{-2}</td>
<td>8.33 \times 10^{-3}</td>
<td>3.38 \times 10^{-3}</td>
</tr>
<tr>
<td>5000</td>
<td>35</td>
<td>0.43</td>
<td>0.72</td>
<td>0.73</td>
</tr>
</tbody>
</table>

Figure 7.24. Energy series (left) and Elsasser numbers (right) for the dynamo \((Ra, Pm) = (3000, 5)\). Vertical lines are the sample times used for Figures 7.27 and 7.28.
Figure 7.25. Strong/weak field force balance for the dynamo \((Ra, Pm) = (5000, 35)\). Azimuthally averaged components are plotted in (a–c), point wise components are plotted on transverse slice \(z = -0.9 : 0.3 : 0.9\) in (d–f).
FIGURE 7.26. Strong/weak field force balance for the dynamo \((Ra, Pm) = (3500, 25)\). Azimuthally averaged components are plotted in (a–c), point wise components are plotted on transverse slice \(z = -0.9 : 0.3 : 0.9\) in (d–f).
Figure 7.27. Strong/weak field force balance for the dynamo \((Ra, Pm) = (3000, 5)\) in the low energy state at \(t = 3.0\). Azimuthally averaged components are plotted in (a–c), point wise components are plotted on transverse slice \(z = -0.9 : 0.3 : 0.9\) in (d–f).
Figure 7.28. Strong/weak field force balance for the dynamo \((Ra, Pm) = (3000, 5)\) in the high energy state at \(t = 4.2\). Azimuthally averaged components are plotted in (a–c), point wise components are plotted on transverse slice \(z = -0.9 : 0.3 : 0.9\) in (d–f).
7.6. Three no-slip dynamo solutions with \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\)

Having compared different dynamo solutions collectively, we now consider some of the solutions in greater detail. We select three dynamos which display apparent similarity in their field structures, but upon closer inspection belong to three distinct classes of dynamo solution: an oscillating dipolar solution for \((Ra, Pm) = (5000, 35)\); an oscillating quadrupolar solution for \((Ra, Pm) = (3000, 20)\); and a chaotic solution which flips between two hemispherical states for \((Ra, Pm) = (3000, 5)\).

7.6.1. The dipolar dynamo \((Ra, Pm) = (5000, 35)\). The first solution we consider is the dipolar dynamo obtained for \((Ra, Pm) = (5000, 35)\) with \(Ra \sim 1.3 Ra_c\). The energy series for this dynamo (Figure 7.29) shows this dynamo settles to a quasi-periodic state after 2 diffusion times with comparable internal magnetic and kinetic energies.

![Energy series for the dynamo \((Ra, Pm) = (5000, 35)\).](image)

**Figure 7.29.** Energy series for the dynamo \((Ra, Pm) = (5000, 35)\).

![Flow features on azimuthal slices for the dynamo \((Ra, Pm) = (5000, 35)\).](image)

**Figure 7.30.** Flow features on azimuthal slices for the dynamo \((Ra, Pm) = (5000, 35)\). (a) Azimuthally averaged helicity, (b) contours of \(z\)-vorticity in an azimuthal slice through a convective column, (c) contours of \(z\)-velocity in an azimuthal slice through a convective column.
The flow for this dynamo is characterised by nearly 3-fold azimuthally symmetric columnar convection with negative axial-helicity $H$ in the northern hemisphere and positive axial-helicity in the southern, and a prograde tilting at high latitude as shown in Figures 7.30 and 7.31. This flow structure is typical of non-magnetic convection just above onset and is almost indistinguishable...
7.6. Three no-slip dynamo solutions with \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\)

**Figure 7.32.** Energy spectra for the dynamo \((Ra, Pm) = (5000, 35)\).

**Figure 7.33.** Radial magnetic field at the boundary for the dynamo \((Ra, Pm) = (5000, 35)\). Black lines are: right \(z = -0.45 : 0.45 : 0.9\), left \(z = 0.45, 0.9\).

from that obtained in Chapter 6 for non-magnetic convection with similar parameters. A similar pin-wheel pattern has also been observed in spherical shell dynamo computations [Sreenivasan & Jones 2011, Fig. 9 (d)].

The magnetic field for this dynamo is predominately dipolar. The latitude and distribution of the dipole moment are plotted on the surface of the sphere in Figure 7.36. Dipolar dynamos functioning in a columnar convection regime have been studied in spherical shell computations (e.g., Christensen, Olson & Glatzmaier 1998 and Olson, Christensen & Glatzmaier 1999). Unlike these dynamos, however, the one obtained here is not an axial dipole, but instead has emergent field concentrated into three flux patches in each hemisphere as shown in Figure 7.33. The near 3-fold symmetry observed in these plots is somewhat misleading as, even though the majority of the energy is contained in the \(m = 0 \pmod{3}\) modes for both the flow and the magnetic field, the \(m \neq 0 \pmod{3}\) modes contain a significant fraction of the energy. This is shown in Figure 7.32.

The plots of the azimuthally averaged radial magnetic field and the total magnetic field in Figure 7.35 show that the emergent field corresponds to regions of relatively weak magnetic field strength; the radial field only just leaks from the interior through the boundary. As there is no
spatial correlation between the total magnetic field strength and the radial magnetic field it is unsurprising to see in Figure 7.37 that most of the magnetic energy is contained in the toroidal field.

We plot $1_z \cdot B$ in transverse slices in Figure 7.38 and see exactly the effect described in Olson, Christensen & Glatzmaier (1999); the field is concentrated into columns, converging with the fluid into the negative vortices ($1_z \cdot \omega < 0$) at low latitude, and similarly with the positive vortices ($1_z \cdot \omega > 0$) at high latitude, a typical alpha-effect (see, e.g., Moffatt 1978). The role of the helicity in the generation of this large scale dipole field is demonstrated in Figure 7.39 where the generating azimuthal current is seen to be concentrated at high latitudes with the helicity [Figure 7.30(a)].

The magnetic flux patches seen at low latitudes in Figure 7.35 (b,c,d) are generated by an omega-effect. This is demonstrated in Figure 7.40 where the azimuthal average of $s B_p \cdot \nabla (u_\phi / s)$, for $s$ the perpendicular distance from the axis of rotation and $B_p$ the poloidal magnetic field, is plotted; the axisymmetric omega-effect (see, e.g., Moffatt 1978). This is reinforced in Figure 7.35 (d). This feature, a flux patch at low latitudes near the boundary, is found for many of the dynamos computed, and is similar to the solution obtained in spherical shell geometry by Kutzner & Christensen (2000, Fig. 4). The classification of this dynamo as being generated by an $\alpha \omega$-mechanism is consistent with the toroidal magnetic field containing the majority of the magnetic energy (see, e.g., Krause & Rädler 1980).

The final comment about this dynamo concerns the heat flux through the boundary. Plotting $Q_r$ as a function of time in Figure 7.34 we see that even in the transient state this dynamo has $Q_r \approx 1$. This is consistent with the result of Section 2.2.2 where the time-averaging does not require the dynamo to have settled to a statistically steady state.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.34a}
\includegraphics[width=0.5\textwidth]{figure7.34b}
\caption{Boundary heat flux for the dynamo ($Ra, Pm$) = (5000, 35).}
\end{figure}
7.6. Three no-slip dynamo solutions with $(E, Pr) = (5 \cdot 10^{-4}, 0.7)$

**Figure 7.35.** Components of the magnetic field for the dynamo $(\text{Ra}, \text{Pm}) = (5000, 35)$. (Top left) Azimuthally averaged radial magnetic field, (top right) azimuthally averaged magnetic field strength, (bottom left) pointwise magnetic field strength on $z = -0.9 : 0.2 : 0.9$, and (bottom right) pointwise magnetic field strength on $z = 0.2$ (colours) on-top of streamlines of equatorial flow (lines) on $z = 0.1$. 

**Figure 7.36.** Dipole moment for the dynamo $(\text{Ra}, \text{Pm}) = (5000, 35)$: (left) Latitude of the dipole moment, (right) direction (points) and magnitude (colour) of the dipole vector.
Figure 7.37. Fraction of energy contained in the poloidal fields for the dynamo $(Ra, Pm) = (5000, 35)$.

Figure 7.38. The $z$-component of the magnetic field for the dynamo $(Ra, Pm) = (5000, 35)$: (top left) $z = 0.1$, (top right) $z = 0.9$, (bottom) transverse slices $z = -0.9 : 0.2 : 0.9$. 
Figure 7.39. The $\phi$-component of the current for the dynamo $(Ra, Pm) = (5000, 35)$; (top left) $z = 0.1$, (top right) $z = 0.9$, and (bottom) equatorial slices $z = -0.9 : 0.2 : 0.9$.

Figure 7.40. Axisymmetric omega effect for the dynamo $(Ra, Pm) = (3000, 35)$: (left) $|s \, B_p \cdot \nabla (u_\phi / s)|$, (right) azimuthal average of azimuthal magnetic field.
7.6.2. The quadrupolar dynamo \((Ra, Pm) = (3000, 20)\). The second dynamo we consider is the quadrupolar solution obtained for \((Ra, Pm) = (3000, 20)\) with \(Ra \approx 1.4 Ra_c\). The energy series for this dynamo is shown in Figure 7.41 with the energy spectrum and dipole fractions given in Figures 7.42 and 7.43 respectively. We see from these that the dynamo quickly settles to a periodic state with dominant quadrupolar magnetic and kinetic energies after one diffusion time. Although the energy series appear perfectly anti-correlated this is not the case, as shown in Figure 7.44 where the cross-correlation of the kinetic energy and internal magnetic energy series is plotted. Writing the energy series as cosine series

\[
E_m = E_{0,m} + E_{1,m} \cos(2\pi \omega_m t + \phi_{1,m}), \quad E_k = E_{0,k} + E_{1,k} \cos(2\pi \omega_k t + \phi_{1,k}),
\]

we find

\[
E_{0,m} = 23081.2, \quad E_{1,m} = 273.481, \quad \omega_m = 85.7466, \\
E_{0,k} = 82133.3, \quad E_{1,k} = 417.755, \quad \omega_k = 85.7470,
\]

with phase difference

\[
\phi_m - \phi_k = 0.125069.
\]

---

**Figure 7.41.** Energy series for the dynamo for the dynamo \((Ra, Pm) = (3000, 20)\): (left) unscaled series, (right) magnetic energy scaled by ratio of means to have the same mean value as the kinetic energy series.

Dynamos with this field configuration have been found in spherical shell studies (e.g., Grote, Busse & Tilgner 1999, Simitev & Busse 2005 and Kutzner & Christensen 2000). These authors found the Rayleigh number, the Prandtl number and magnetic Prandtl number, as well as the heat source (i.e. a volumetric heat source versus a temperature difference between the inner and outer core boundaries) to be important in the selection of this symmetry.

This dynamo functions in a flow regime very similar to the previously discussed dynamo: convective columns of alternate \(z\)-vorticity with negative helicity in the northern hemisphere, positive helicity in the southern hemisphere concentrated at high latitude with a prograde tilting at the boundary (Figure 7.45). This is unsurprising as the two dynamos have \(Ra_c = 143\) and 150, near the value for which this flow pattern was found in thermal convection computations in Section 6.2.2.
7.6. THREE NO-SLIP DYNAMO SOLUTIONS WITH \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\)  

**Figure 7.42.** Time-averaged energy spectra for the kinetic and internal magnetic energies for the dynamo \((Ra, Pm) = (3000, 20)\) for \(t > 1.6\).

**Figure 7.43.** (Left) Dipole fraction of the energies, (right) kinetic dipole fraction scaled by the ratio of the mean values, to have the same mean value as the magnetic dipole fraction, for the dynamo \((Ra, Pm) = (3000, 20)\).

**Figure 7.44.** Cross-correlation of the kinetic and internal magnetic energy series for the dynamo \((Ra, Pm) = (3000, 20)\).
Since, superficially, there is a high degree of similarity between the flows of this dynamo and that of the dynamo \((Ra, Pm) = (5000, 35)\), it is unsurprising to see (Figure 7.48) that the magnetic energy is distributed in a very similar way: the field is concentrated at the poles and near the equator, with only a very weak radial field leaking through the boundary. While it appears there is a similar generation of magnetic field by an omega-effect near the equator (Figure 7.47), we see, however, no advection of the magnetic field into the convective column (Figure 7.46) as in the previous dynamo.

The components of the axisymmetric magnetic field, as well as the azimuthally averaged magnitude of the magnetic field, in time intervals \(\Delta t = 0.001\) for just over a period, are plotted in Figures 7.49–7.52. We see a process similar to the wave process of, e.g., Grote, Busse & Tilgner
7.6. Three no-slip dynamo solutions with \((E, \text{Pr}) = (5 \cdot 10^{-4}, 0.7)\)

**Figure 7.47.** Magnitude of the magnetic field on \(z = 0\) (left) and \(z = 0.1\) (right) for the dynamo \((R_a, \text{Pm}) = (3000, 20)\). Lines are the horizontal streamlines.

**Figure 7.48.** Magnetic field distribution for the dynamo \((R_a, \text{Pm}) = (3000, 20)\): (top left) azimuthally averaged magnetic field strength, (top right) azimuthally averaged radial magnetic field, (bottom left) pointwise magnetic field on \(z = -0.9 : 0.2 : 0.9\), (bottom right) magnetic field strength (colours) with streamlines of the horizontal flow on \(z = 0.2\).

(2000). Patches of azimuthal flux are generated at the poles and move towards the equator along the rotation axis. These patches weaken as they move towards the equator while patches of opposite polarity are generated at the poles, and the period resets.
Figure 7.49. Axisymmetric radial magnetic field for the dynamo \((Ra, Pm) = (3000, 20)\). Plots have row major ordering and are separated by \(\Delta t = 0.001\).
Figure 7.50. Axisymmetric latitudinal magnetic field for the dynamo \((\text{Ra, Pm}) = (3000, 20)\). Plots have row major ordering and are separated by \(\Delta t = 0.001\).
Figure 7.51. Axisymmetric azimuthal magnetic field for the dynamo \((Ra, Pm) = (3000, 20)\). Plots have row major ordering and are separated by \(\Delta t = 0.001\).
Figure 7.52. Azimuthally averaged magnetic field strength for the dynamo \((Ra, Pm) = (3000, 20)\). Plots have row major ordering and are separated by \(\Delta t = 0.001\).
7.6.3. The chaotic hemispherical dynamo \((Ra, Pm) = (3000, 5)\). The last dynamo we consider is that obtained for \((Ra, Pm) = (3000, 5)\) with \(Ra_c \approx 5.7\). This dynamo was found to flip between a low and high magnetic energy state, as shown in Figure 7.54. It is perhaps optimistic to call this a self-sustaining dynamo given the length of the integration time — authors such as Morin & Dormy (2009) having found chaotic solutions in spherical shell computations that are sustained for around 17 magnetic diffusion times before collapsing. They termed such solutions meta-stable dynamos. It is also impossible to conclude that this dynamo will continue to flip between the low and high energy states, given that only one complete cycle is contained in the run. Nevertheless we comment on the main features of this solution.

The radial magnetic field at the boundary and the pointwise magnetic field strength for this dynamo are shown for the low energy state in Figure 7.58 and for the high energy state in Figure 7.59. We see from these plots that the magnetic field has assumed a configuration reminiscent of the hemispherical dynamos found in both full sphere (Landeau & Aubert 2011), as well as in spherical shell geometries (Dietrich & Wicht 2013), where the dipolar and quadrupolar fields combine in such a way that the majority of the magnetic energy is confined to a single hemisphere. The generation of hemispherical dynamo action has been linked to the excitation of an equatorially anti-symmetric axisymmetric (EEA) flow component, i.e. a dipolar axisymmetric mode. These authors generated this mode either by applying inhomogeneous heat flux boundary conditions at the boundary or by simply driving the flow hard enough. They observed hemispherical
7.6. THREE NO-SLIP DYNAMO SOLUTIONS WITH \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\)

**Figure 7.55.** Kinetic and internal magnetic energy spectra for the dynamo \((Ra, Pm) = (3000, 5)\): (Left) low (magnetic) energy state \(t \in (1.4, 2.7)\), (right) high (magnetic) energy state \(t \in (3.1, 3.75)\).

Dynamo action when the energy contained in the EEA flow component was comparable to the energy contained in the quadrupole flow. For this dynamo the EEA flow component never contains a significant portion of the energy (Figure 7.60, left). In particular, partitioning the kinetic energy spectrum by dipolar/quadrupolar component (Figure 7.57) we see the primary component of the dipolar flow is the two-fold symmetric component in both the high and the low energy states. We also note that the hemispherical field configuration is not seen in the emergent radial magnetic field at the boundary but is seen in the radial magnetic field in the bulk of the fluid; like the dynamo for \((Ra, Pm) = (5000, 35)\) only a small fraction of the radial field leaks through the boundary. This, of course, raises the question of how to classify a dynamo solution as hemispherical. Dietrich & Wicht (2013) used a boundary radial magnetic flux criterion to classify a dynamo as hemispherical, whereas Landeau & Aubert (2011) used a condition based on the distribution of the magnetic energy density based on the **hemispherical factor** \(f_h\) defined by

\[
f_h := \max\{\frac{E_m(V_N)}{E_m}, \frac{E_m(V_S)}{E_m}\}
\]

where \(V_N\) \((V_S)\) is the volume of fluid in the northern (southern) hemisphere. The factor \(f_h\) is bounded by \(1/2 \leq f_h \leq 1\), where \(f_h = 1\) corresponds to a pure hemispherical dynamo. The fraction of energy contained in the northern hemisphere for this dynamo in both the low and high energy states is shown in Figure 7.56. We see that in the low energy state the vast majority of the magnetic energy is confined to the southern hemisphere, and in the high energy state the majority of the magnetic energy is confined to the northern hemisphere.

The flow for this dynamo in both the high and the low energy states is characterised by quasi-geostrophic convective columns containing the majority of the kinetic energy, with negative helicity in the northern hemisphere and positive helicity in the southern hemisphere (Figures 7.61 and 7.62). The primary difference we see between the flow in the low energy state and the flow in the high energy state is the breaking-up of the large scale azimuthal velocity as shown in Figures 7.63 and 7.64. Even the most cursory comparison of these plots and those obtained in the absence of a magnetic field in Chapter 6 Section 6.2.3 shows that, despite the localisation of the magnetic flux, the flow has been significantly modified throughout the whole fluid. In particular we note that the dipole flow has yet to be stimulated and the flow is predominantly 3-fold symmetric in the absence
of a magnetic field.
7.6. Three no-slip dynamo solutions with \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\)

**Figure 7.58.** Pointwise magnetic field strength on \(z = 0.2 : 0.2 \) (top left), azimuthally averaged strength of the radial magnetic field (top right) and Hammer projection of the radial magnetic field at the boundary (bottom) for the dynamo \((Ra, Pm) = (3000, 5)\) in the low energy state \(t = 3.4\).

**Figure 7.59.** Pointwise magnetic field strength on \(z = 0.2 : 0.2 \) (top left), azimuthally averaged strength of the radial magnetic field (top right) and Hammer projection of the radial magnetic field at the boundary (bottom) for the dynamo \((Ra, Pm) = (3000, 5)\) in the high energy state \(t = 4.205\).
Figure 7.60. EEA energy and poloidal energy fractions for the dynamo \((Ra, Pm) = (3000, 5)\): (left) quadrupolar kinetic energy and energy contained in the EEA mode, (right) poloidal energy fraction of the kinetic and internal magnetic energies.

Figure 7.61. Flow characteristics for the dynamo \((Ra, Pm) = (3000, 5)\) in the low energy state \(t = 3.4\). (Top) \(z\)-velocity on \(z = -0.9 : 0.2 : 0.9\) (left) and \(z = 0.7\) (right). (bottom) Helicity on \(r = 0.8\).
Figure 7.62. Flow characteristics for the dynamo \((Ra, Pm) = (3000, 5)\) in the high energy state \(t = 4.205\). (Top) \(z\)-velocity on \(z = -0.9 : 0.2 : 0.9\) (left) and \(z = 0.7\) (right). (bottom) Helicity on \(r = 0.8\).
The azimuthally averaged magnetic field strength and contours of the azimuthally averaged azimuthal magnetic field in the low energy state are plotted for different times in Figures 7.65 and 7.66 respectively. We see here a feature reminiscent of the wave mechanism described by e.g. Grote, Busse & Tilgner (1999), found in quadrupolar dynamo simulations, except now confined to a single hemisphere — two patches of opposite polarity moving towards and eventually passing through each other with a stretching towards the rotation axis. This is not so obvious for the high energy state, where the same plots are shown in Figures 7.67 and 7.68.

**Figure 7.63.** Azimuthal velocity for the dynamo \((Ra, Pm) = (3000, 5)\) in the low energy state \(t = 3.4\). (Left) pointwise azimuthal velocity on \(z = -0.9 : 0.2 : 0.2\), (Right) pointwise radial velocity on \(z = 0.7\) with horizontal streamlines.

**Figure 7.64.** Azimuthal velocity for the dynamo \((Ra, Pm) = (3000, 5)\) in the high energy state \(t = 4.205\). (Left) pointwise azimuthal velocity on \(z = -0.9 : 0.2 : 0.2\), (Right) pointwise radial velocity on \(z = 0.7\) with horizontal streamlines.
7.6. THREE NO-SLIP DYNAMO SOLUTIONS WITH \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\)

**Figure 7.65.** Azimuthally averaged magnetic field strength for the dynamo \((Ra, Pm) = (3000, 5)\) in the low energy state. First plot (top left) is \(t = 3.4002\), each plot (left to right, then to the second row) is separated by \(\Delta t = 2 \cdot 10^{-3}\).

**Figure 7.66.** Axisymmetric azimuthal averaged magnetic field strength for the dynamo \((Ra, Pm) = (3000, 5)\) in the low energy state. First plot (top left) is \(t = 3.4002\), each plot (left to right, then to the second row) is separated by \(\Delta t = 2 \cdot 10^{-3}\).
Figure 7.67. Azimuthally averaged magnetic field strength for the dynamo $(Ra, Pm) = (3000, 5)$ in the high energy state. First plot (top left) is $t = 4.2002$, each plot (left to right, then to the second row) is separated by $\Delta t = 2 \cdot 10^{-3}$.

Figure 7.68. Axisymmetric azimuthal averaged magnetic field strength for the dynamo $(Ra, Pm) = (3000, 5)$ in the high energy state. First plot (top left) is $t = 4.2002$, each plot (left to right, then to the second row) is separated by $\Delta t = 2 \cdot 10^{-3}$.
7.7. Discussion

In this chapter we considered the problem of thermally driven dynamo action in a rotating sphere with homogeneous volumetric heat source and fixed temperature boundary conditions. We began, in Section 7.1, by testing the full non-linear integrator by reproducing the stress-free full sphere dynamo benchmark of Marti et al. (2014).

The remainder of this chapter considered convection driven dynamos with no-slip viscous boundary conditions. In Section 7.2 we outlined a number of computations performed with equatorial symmetry imposed on all the fields (even temperature, dipole magnetic field, quadrupolar velocity), both with and without a 3-fold azimuthal symmetry imposed. We presented in detail a quasi-periodic dynamo model in Section 7.2.1 which is reproducible only if both the equatorial and azimuthal symmetries are implemented, as a possible validation test for the implementation of the no-slip dynamo problem.

The symmetry constraints were then lifted and focus placed on no-slip dynamos with \((E, Pr) = (5 \cdot 10^{-4}, 0.7)\) in Section 7.3. The solutions obtained in this context were then tested against theoretically derived dynamo scaling laws based on energy considerations (Section 7.4.1) and assumed force balances (Section 7.4.2). No universally valid scaling law was found in these contexts, possibly due to the range of parameters considered (and corresponding variability in the solutions obtained). The lack of a universally satisfied scaling law was also found when numerical scaling laws were considered in Section 7.4.4.

One possible reason the theoretical scaling laws of Section 7.4.2 failed for the solutions obtained is that the dynamos found function in regimes corresponding to different force balances. This was shown to be true in Section 7.5 where the criterion of Morin & Dormy (2009) was used to classify the solutions as either strong or weak field dynamos. We found, however, that for all the solutions considered the magnetic field was highly localised. In particular this meant that a dynamo classified as strong field had a weak field balance at play in the bulk of the fluid, confounding classification.

Finally, in Section 7.6, we considered in detail three solutions with very different field morphologies. The first was the dipolar dynamo obtained for \((Ra, Pm) = (5000, 35)\) in Section 7.6.1. This dynamo was found to function in a columnar regime, with the magnetic field generated by an alpha-effect in the columns, and an omega effect at the equator.

The second dynamo considered in detail was the periodic quadrupolar dynamo obtained for \((Ra, Pm) = (3000, 20)\) in Section 7.6.2. This dynamo was found to function in a flow regime almost indistinguishable from the dynamo \((Ra, Pm) = (5000, 35)\), but with a vastly different magnetic field configuration. In particular, while there was still present the omega effect generation of magnetic flux near the equator and flux patches at the poles, no alpha-effect associated with the convective columns could be identified. The field oscillations were instead identified with a mechanism similar to the wave mechanism found in quadrupolar spherical shell computations, with patches of azimuthal flux being generated at high latitudes, and moving along the rotation axis toward equator.
The final dynamo considered was the chaotic hemispherical dynamo obtained for \((Ra, Pm) = (3000, 5)\) in Section 7.6.3. This dynamo was found to oscillate between a weak (magnetic) state with the majority of the magnetic flux concentrated in the southern hemisphere and a strong (magnetic) state with the majority of the magnetic flux concentrated in the northern hemisphere. This dynamo is strange amongst other hemispherical dynamos as this type of field configuration is usually associated with the generation of an equatorial anti-symmetric axisymmetric flow component which was found to be weak in both the high and low energy states. In the solution presented here, however, a wave type mechanism could be identified but confined to regions near the boundary in a single hemisphere when the dynamo is in the low energy state. No such process could be discerned in the high energy state. Clearly further computation exploring regimes of these last two dynamos are required to gain a complete picture of the dynamics.
Chapter 8

Thermal Convection in a Rotating Sphere of Cooling Fluid

In this chapter we consider an offshoot problem to thermally driven dynamo action in a full sphere: the problem of dynamo action as a sphere of fluid cools. In spherical shell dynamo models convection can be driven by volume heat sources in the core or boundary heat sources from the ICB. For the geo-dynamo, the dominant mechanism for driving convection is believed to be the release of latent heat and a light fluid component from the freezing of iron at the ICB (Braginsky 1964d). Some spherical shell models also consider external heat sources, i.e. the mantle, but usually with a net outward heat flux (e.g., the study of convection locked to the CMB by Willis, Sreenivasan & Gubbins 2007 and Gubbins et al. 2011). In the early Earth, prior to the formation of the inner core, there were only volume heat sources, e.g. radiogenic heat, and primordial heat from the planet formation. This is also believed to be true for the ancient dynamo of Mars, and perhaps the Moon and Venus (Stevenson 2003, Stevenson 2010). In this chapter we develop a simple model as a first approach to the problem of dynamo action as a full (Boussinesq) fluid sphere cools in the absence of any volumetric heat source (without solidification) due to a spatially uniform heat flux through the boundary.

We begin by describing the full problem, from the dimensional equations in Section 8.1 to the non-dimensional equations in Section 8.2 and the spectral equations and numerical method in Section 8.3. The intention here is to develop a simple model for this problem and to perform some preliminary computations to check for qualitative agreement with the expected physics. In this context we briefly discuss two computations in Section 8.4, which consider thermally driven convection in the absence of a magnetic field for this problem, and finally give some suggestions to the direction which future consideration of this problem could take in Section 8.5.

8.1. The temperature equation for the cooling dynamo problem

We consider the problem of dynamo action in the uniformly rotating mantle frame for a cooling sphere of Boussinesq fluid. We begin with the problem described in Section 2.1, but with no volumetric heat source, $Q_s = 0$ in (2.7), and a prescribed heat flux boundary condition

$$\mathbf{n} \cdot \nabla_s \Theta_s = \Theta'_s(\theta, \phi, t) \quad \text{on } \Sigma. \quad (8.1)$$

We assume from the onset that $\Theta'_s$ is uniform and non-zero, i.e. $\Theta'_s = \Theta'_s(t)$. First, we consider the conduction state temperature. This is the solution of the DE

$$\nabla^2_s \Theta_{cs} = Q_s \quad \text{in } V. \quad (8.2)$$

Setting $Q_s = 0$ gives the solution for $V$ a sphere which is analytic at the origin

$$\Theta_{cs} = \Theta_{0s} \quad (8.3)$$
constant, which is incompatible with (8.1). In order to satisfy this condition the time-dependence in conduction state temperature must be retained, and hence the conduction state temperature satisfies

\[ (\partial_t + \kappa \nabla^2) \Theta_{c*} = 0. \]  

(8.4)

This has solution

\[ \Theta_{c*} = \Theta_{0*} + \sum_{n,m,\lambda} \hat{\Theta}_{n*,m}^m j_n \left( \sqrt{\frac{q \lambda^2_{(n,m)}}{\kappa}} \cdot r_\star \right) \exp\left(-\lambda^2_{(n,m)} q t_\star \right) Y_n^m, \]  

(8.5)

where \( \Theta_{0*} \) and the \( \hat{\Theta}_{n*,m}^m \) are constants, \( j_n \) is the spherical Bessel function of degree \( n \) of the first kind, and the scaling of the \( \lambda \)'s by \( \sqrt{q} \) is done for later convenience and corresponds to the intended use of the magnetic diffusive time scale. In the absence of a magnetic field \( q \) is replaced by \( Pr^{-1} \).

The only conditions on the \( \lambda_{(n,m)} \) at the moment are that they are real and non-zero. Additionally, the \( \lambda_{(n,m)} \)'s are assumed discrete. This is not a necessary condition and the \( \lambda \) sum can be replaced by an integral (equivalently, by a Laplace transform in \( t \)), but is adopted now in anticipation of fixing the value of \( \lambda_{(n,m)} \) below. The notation \( \lambda_{(n,m)} \) is used to indicate that independent \( \lambda \)'s can be chosen for each \( n \) and \( m \). [in fact multiple independent \( \lambda_{(n,m)} \) can be chosen for a fixed \((n,m)\)].

Matching (8.5) to the boundary condition (8.1) gives

\[ \Theta'_{\star} = \sum_{n,m,\lambda} \hat{\Theta}_{n*,m}^m j_n' \left( \sqrt{\frac{q \lambda^2_{(n,m)}}{\kappa}} \cdot r_\star \right) \exp\left(-\lambda^2_{(n,m)} q t_\star \right) Y_n^m \]  

(8.6)

where \( j_n' \) is the first derivative of \( j_n \). To guarantee that there is a net heat flux out of the sphere (i.e. \( \Theta'_{0*} \neq 0 \)) we require

\[ \hat{\Theta}_{0*,0}^0 \neq 0, \quad \lambda_{(0,0)} \neq 0, \quad \text{and} \quad j_0' \left( \sqrt{\frac{q \lambda^2_{(0,0)}}{\kappa}} \cdot r_\star \right) \neq 0. \]  

(8.7)

The requirement that \( \Theta'_{\star} \) is uniform is satisfied by setting all \( \hat{\Theta}_{n*,m}^m = 0 \) for \((n,m) \neq (0,0)\).

Decomposing the temperature into the conduction state plus the particular solution

\[ \Theta_{\star} = \Theta_{c*} + \Theta_{p*} \]  

(8.8)

where, from (8.1), \( \Theta_{p*} \) has

\[ n \cdot \nabla \Theta_{p*} = 0 \quad \text{on} \quad \Sigma, \]  

(8.9)

and substituting into the temperature equation (2.7) with \( Q_{\star} = 0 \), using (8.4), gives

\[ (\partial_t + \kappa \nabla^2) \Theta_{p*} = -u_\star \cdot \nabla \Theta_{p*} - u_\star \cdot \nabla \Theta_{c*}. \]  

(8.10)

Integrating this over \( V \), assuming that \( u \) is solenoidal and \( \Sigma \) is impenetrable, and using (8.9) and an application of the Reynolds transport theorem, gives

\[ \frac{d}{dt_\star} \int_V \Theta_{p*} dV_\star = 0. \]  

(8.11)

This implies, consistent with (8.9), that the net rate of heat loss is controlled by \( \Theta_{c*} \), and hence by \( \lambda_{(0,0)} \). The temperature equation for the cooling dynamo problem is hence (8.10), with boundary condition (8.9), subject to (8.11), where the conduction temperature is (8.5). This is related to the
heat flux boundary condition (8.1) by (8.6) with $\hat{\Theta}^m = 0$ unless $n = m = 0$, in which case the input parameter $\lambda_{*(0,0)}$ is required to satisfy (8.7).

8.2. Non-dimensional equations for the cooling dynamo problem

This problem is non-dimensionalised using the same time, length and magnetic intensity scales as the internally heated dynamo problem of Section 2.1. The temperature is scaled by

$$\Delta \Theta = -r_{\Sigma*} \cdot \nabla_* \Theta_* (r_{\Sigma*}, t_* = 0) = -r_{\Sigma*} \Theta'_* (t = 0),$$  \hspace{1cm} (8.12)

where $\Theta'_*$ is given by (8.6), so that

$$\Theta = \Theta_c + \Theta_p$$  \hspace{1cm} (8.13)

where

$$(\Delta \Theta) \Theta_c = \Theta_{c*} - \Theta_{0*}, \quad (\Delta \Theta) \Theta_p = \Theta_{p*}.$$  \hspace{1cm} (8.14)

The non-dimensional temperature equation is hence

$$(\partial_t - q \nabla^2) \Theta_p = -u \cdot \nabla \Theta_p - u \cdot \nabla \Theta_c,$$  \hspace{1cm} (8.15)

with boundary condition

$$\partial_r \Theta_p (r = 1) = 0,$$  \hspace{1cm} (8.16)

subject to

$$\frac{d}{dt} \int_V \Theta_p \, dV = 0.$$  \hspace{1cm} (8.17)

Combining (8.12) and (8.14) with $\lambda_{*(0,0)} = \lambda_*$ and all $\hat{\Theta}^m_{n*} = 0$ when $(n, m) \neq (0, 0)$, noting that the dimension of $\lambda_*$ is

$$[\lambda_*] = T^{-1/2},$$  \hspace{1cm} (8.18)

gives the non-dimensional conduction state temperature

$$\Theta_c = \frac{j_0 \left( \sqrt{\lambda^2 r} \right) \exp(-\lambda^2 qt)}{-\sqrt{\lambda^2} j'_0(\sqrt{\lambda^2})}.$$  \hspace{1cm} (8.19)

The magnetic induction equation (2.42) and conditions (2.20) are unchanged from the internally heated dynamo problem described in Chapter 2. The same is true for the Navier-Stokes equation (2.40), the constraints on the fluid velocity (2.30), and the no-slip boundary conditions (2.31). As the conduction state is included in the total temperature the only difference in the non-dimensional momentum equation is in the definition of the Rayleigh number, where $\Delta \Theta$ is now given by (8.12). In order to satisfy (8.1) with a $\Theta'_* \neq 0$, $\lambda$ is required to satisfy

$$\lambda \neq 0, \quad \text{and} \quad j'_0(\sqrt{\lambda^2}) \neq 0,$$  \hspace{1cm} (8.20)

with only $\lambda \in (0, \pi)$ considered. The corresponding conduction state temperatures are shown in Figure 8.1 with the piece $\lambda \in [\pi/2, \pi]$ shown in Figure 8.2.
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Figure 8.1. Conduction state temperature profiles $z = \Theta_c(r, \lambda)$ for the cooling dynamo problem with $\lambda \in (0, \pi)$ truncated at $z \leq 50$. (Left) linear vertical scale, (right) logarithmic vertical scale.

Figure 8.2. Conduction state temperature profiles $z = \Theta_c(r, \lambda)$ for the cooling dynamo problem with $\lambda \in (\pi/2, \pi)$. (Left) surface $z = \Theta_c(r, \lambda)$, (right) contours of the surface $z = \Theta_c(r, \lambda)$.

8.3. Spectral form of the temperature equation for the cooling dynamo problem

Taking the inner product of the non-dimensional temperature equation for the cooling dynamo problem (8.15) with $Y_n^m$ gives the spectral temperature equation

$$\left( \partial_t - qD_n \right) \Theta^m_{pn} = f^m_n, \quad (8.21)$$

where

$$f^m_n = - \left( \mathbf{u} \cdot \nabla (\Theta_p + \Theta_c), Y_n^m \right). \quad (8.22)$$

When $(n, m) \neq (0, 0)$, the boundary condition (8.16) becomes

$$\partial_r \Theta^m_{pn}(r = 1, t) = 0, \quad (8.23)$$

with these modes not contributing to the integral condition (8.17) by virtue of the orthogonality condition on the spherical harmonics discussed in Section 3.1.1. The $(n, m) = (0, 0)$ equation requires more careful treatment as the boundary condition

$$\partial_r \Theta^0_{p0}(r = 1, t) = 0 \quad (8.24)$$
still applies, but the condition (8.17) is no-longer automatically satisfied. The most obvious way of satisfying this condition is to replace the \( j = J \) equation with the integral condition (8.17) satisfied numerically with some quadrature rule with weights \( c_j \)

\[
\int_V \Theta_p(r, t) \, dV = \int_0^1 \Theta_{p0}^0(r, t) r^2 \, dr \approx \sum_{j=1}^J c_j \Theta_{p0}^0(r_j, t), \tag{8.25}
\]

which is equal to its initial value for all time-steps. To apply this directly would destroy the band-structure of the linear system solved at each time-step. To overcome this problem we introduce a second set of variables

\[
\Theta_0^\Sigma(j, t) = \sum_{i=1}^j c_i \Theta_{p0}^0(r_i, t), \tag{8.26}
\]

which are the partial sums of the numerical integral on the right of (8.25). We satisfy the integral condition numerically by introducing the equations

\[
\Theta_0^\Sigma(1, t) = c_1 \Theta_{p0}^0(r_1, t), \tag{8.27}
\]

at \( j = 1 \),

\[
\Theta_0^\Sigma(j, t) = \Theta_0^\Sigma(j - 1, t) + c_j \Theta_{p0}^0(r_j, t), \tag{8.28}
\]

for \( 2 \leq j \leq J - 1 \) in the interior, and with the DE at \( j = J \) replaced by the final integral condition

\[
\Theta_0^\Sigma(J - 1, t) + c_J \Theta_{p0}^0(r_J, t) = \int_0^1 \Theta_{p0}^0(r, t = 0) r^2 \, dr. \tag{8.29}
\]

Ordering the variables into the \( j \)-blocks

\[
\begin{bmatrix}
\Theta_{p0}^0(r_j, t) \\
\Theta_0^\Sigma(j, t)
\end{bmatrix}
\]

has the effect of doubling the rank as well as the upper and lower bandwidths of the linear system that is solved at each time-step when compared to a straight finite difference scheme without the integral condition. We demonstrate this effect for the tri-diagonal compact scheme in Figure 8.3 for \( J = 100 \), where the system becomes \( 199 \times 199 \) with a bandwidth of 5 when the integral condition is introduced in the way outlined above.

We test this method on the DE

\[
(1 - 10^{-6} D_0) f = b \tag{8.31}
\]

which corresponds to the first order Gear time-step method with \( \Delta t = 10^{-6} \), and \( q = 1 \), for the test function

\[
f = r^2(1 - r^2)^2(1/4 - r^2) \cos(\pi r^2), \tag{8.32}
\]

using Simpson’s rule for the numerical integration. Solving this system on the \( J = 100 \) and \( J = 200 \) uniform grids as outlined above gives the errors shown in Figure 8.4. We note that the spike in the error seen at \( r = 1 \) is a side effect of using lower order schemes at the boundary, as discussed in Section 3.5. The error in the numerical integration is seen to be small for each time-step. To avoid build up of this error we purpose scaling the entire \( \Theta_{p0}^0 \) solution vector at every
Figure 8.3. Matrix structure of the time-stepping system for the $\Theta^0_0$ equation with the integral condition (8.25) implemented as (8.27)–(8.29), with the blocking (8.30) and a tri-diagonal compact scheme used for $D_0$. Dots represent non-zero matrix elements with lines drawn over the $\pm2$ bands and the main diagonal.

time step, ensuring the integral condition is satisfied (by the numerical integration method) exactly.

Figure 8.4. Numerical testing of the $\Theta^0_0$ scheme for the cooling dynamo problem. (Top row) $J = 100$-point uniform grid, (bot row) $J = 200$-point uniform grid, (left column) error in the function value, (right column) error in partial sums of the numerical integration.
8.4. Preliminary results

In this section we briefly discuss two results obtained for the problem of thermal convection in a rotating cooling sphere in the absence of a magnetic field. These computations both used the parameters

\[ E = 5 \times 10^{-4}, \quad \Pr = 1, \quad \text{Ra} = 200, \quad (8.33) \]

with the two thermal decay rates

\[ \lambda_1 = 2, \quad \lambda_2 = 3. \quad (8.34) \]

Both computations were initialised with zero fluid velocity

\[ u(r, 0) = 0, \quad (8.35) \]

with the initial temperature

\[ \Theta_0^p = \frac{1}{2}(1 - r^2)^2, \quad \Theta^2_{p,3} = (\Theta^{-2}_{p,3})^* = \Theta^3_{p,3} = -(\Theta^{-3}_{p,3})^* = (1 + i)10^{-5} \frac{r^3}{4\sqrt{\pi}}(1 - r^2)^2. \quad (8.36) \]

No-slip viscous boundary conditions were used for both computations.

**Figure 8.5.** Energy series (left) and buoyancy power (right) for the cooling dynamo problem with \( \lambda = 2 \) (top) and \( \lambda = 3 \) (bottom).
The energy series obtained for these computations are shown in Figure 8.5. These were both obtained with \((J, N, \Delta t) = (200, 60, 10^{-6})\). In these figures we see exactly what we expect to see for this kind of problem; initially the kinetic energy grows exponentially as the fluid begins to convect. Outwardly, this initial growth is indistinguishable from the correspond growth of the kinetic energy in the onset problem considered in Chapter 6, and in the case \(\lambda = 2\), we see the very beginning of the kinetic energy plateau. After this initial growth, however, the thermal driving is no longer sufficient to counteract the viscous dissipation and the power generated by the buoyancy force, and hence the kinetic energy, decays exponentially.

8.5. Future directions

In this chapter we outlined a simple first approach to the problem of dynamo action as a sphere of fluid cools, and considered briefly two convection models, in the absence of a magnetic field.

There are many possible avenues for further study of this problem. Apart from the obvious — including magnetic effects and changing to a spherical shell geometry — it would be pertinent to examine models where the internal heating term is either decreased with time or turned off after a convective or dynamo model had settled to a statistically steady state. It would also be apposite to restore the ohmic heating and viscous heating to the temperature equation. The primary questions of interest are what type of field morphology is seen as convection slows, and for how long convection/dynamo action can be sustained as the fluid cools. This leads into the problem of optimising some functional of the system, e.g. the duration of the magnetic field or the peak magnetic field strength, over the decay rate \(\lambda\).

Linking the results from these types of computations to naturally occurring dynamos carries a difficulty already discussed in Chapter 1, that of a wide range of time-scales. In the Earth, for example, the time-scale associated with the convective motions, of the order of a year, is much shorter than that associated with the cooling and contraction of the core, of the order of 100 My, and the secular cooling is hence usually treated as a steady heat source (Nimmo 2015, Christensen & Wicht 2015). This corresponds to a small \(q\) in (8.19) [See Table 1.3]. This poses stringent limitations on the size of the parameter space explorable in numerical computations, and makes the direct numerical simulation of Earth like dynamos impossible. This leaves avenues such as the analysis of scaling laws and systematic parameter variation to eventually relate computations to physical dynamos.
Conclusion and Future Directions

In this thesis we considered numerically the problem of magnetic field generation in a rotating full sphere of Boussinesq conducting fluid with insulating exterior. Previous researchers have not strongly focused on the full sphere dynamo problem as most naturally occurring dynamos, especially the geodynamo, function in an (approximate) spherical shell geometry. The full sphere dynamo problem, however, is of interest as some naturally occurring dynamo smay have functioned in a full sphere geometry, e.g. the early Earth, Mars and possibly the present dynamo of Mercury. We considered three sub-problems of the full sphere dynamical dynamo problem: the stationary kinematic dynamo problem for an incompressible fluid; thermally driven convection in the absence of a magnetic field; and magnetic field generation powered by thermally driven convection in a rotating Boussinesq fluid with a homogeneous volumetric heat source, an insulating exterior and isothermal boundary conditions. These three problems, as well as both the dimensional and non-dimensional systems of governing equations, were described in Chapter 2.

In Chapter 3 we outlined the numerical methods used to solve the systems of equations given in Chapter 2. Many aspects of the methods presented are commonly used to solve the types of problems considered here: we began by representing all solenoidal vector fields in terms of their poloidal and toroidal components with a spectral Galerkin method in angle with spherical harmonic basis functions — the Bullard-Gellman formalism of Elsasser (1946) and Bullard & Gellman (1954). We combined this field representation with finite differences in radius and the multi-step implicit/explicit time-stepping extension of the Gear (1971) BDF by Karniadakis, Orszag & Israeli (1991) and Hulsen (1996). The primary new method developed here, which distinguishes this numerical integrator from other finite difference/pseudo-spectral integrators, such as Dormy, Cardin & Jault (1998), are the generalised compact finite difference methods used for the implicit radial differentiation. These methods are distinct from other compact finite difference schemes, such as the Padé finite difference methods and the combined compact finite difference methods of Chu & Fan (1998), in that they difference an entire (linear) differential operator, including coefficients which are possibly functions of radius, together. This results in a high-order narrow band approximation for the operator which does not include schemes for individual derivatives. Like the aforementioned compact finite difference methods, this represents a trade-off between the bandwidth and rank of the time-stepping system solved at each time-step, and the cost of (banded) matrix multiplications associated both with the approximation itself as well as the evaluation of any derivative needed for the explicit component of each time-step. The generalised compact scheme developed here represents the minimisation of the rank and bandwidth of the time-stepping system. It was shown that these methods become more advantageous if the compact scheme is developed after applying the time-stepping method to the discretised DE, as both the bandwidth of the time-stepping system and the order of the finite difference approximation can be maintained, even at the boundary. This, however, was found to breakdown when the approximation was made purely diagonal. We also demonstrated, in Appendix B how the spectral analysis of Vichnevetsky &
Bowles (1982) and Cain & Bush (1994) and the spectral matching method outlined in Lele (1992), can be applied to the generalised compact scheme to construct ‘near-spectral’ generalised compact schemes. These were shown to be competitive, in terms of resolution characteristics, with the equivalent ‘near-spectral’ Padé schemes of Lele (1992). This, however, was not applied to the dynamo problem.

In Chapter 4 we outlined the key coding features of the numerical integrator with particular focus on the methods of parallelisation. The primary interest of this chapter was the development of a scalable shared memory integrator. The algorithm developed was based around decomposing the explicit computations into radial blocks which could be computed in parallel, and decomposing the implicit linear solves into blocks based on field and spherical harmonic degree that could be computed in parallel. The primary new development here was the implementation of a staggered flag/flush mechanism using a flagging method similar to that of Bull & Ball (2003). The basic premise of this algorithm was to synchronise threads with dynamically allocated linear solves instead of halting execution, while minimising both the amount of data that needed to be flushed and the parallel overhead in both memory and CPU cycles. The performance of this integrator was found to scale near perfectly for both very small and very large truncation levels up to the limit of 40 cores on the shared memory machines available at the time. In the chapter we also detailed a version of the code for distributed memory architecture. This was developed simply to increase the number of machines on which dynamo calculations could be done. As such, the simplest problem splitting, a straight radial splitting, was combined with distributed linear algebra. This method was found to scale very poorly relative to the shared memory method, which was unsurprising as radial splitting algorithms are known to have poor scaling properties, even for dynamo codes using spectral methods in radius (Marti 2012). By testing the individual components of the code we found that the SCALAPACK banded matrix (distributed) solver had very poor scaling properties when compared to the portion of the code that performed the explicit transforms and data updates. The overall scaling of the full code, however, was found to more closely resemble the scaling of the explicit portion of the code, indicating that it was in these computations that the majority of the wall time was spent. It remains an open question as to whether increasing the complexity of the communication pattern, as well as the amount of data communicated, would outweigh the loss of efficiency incurred by the use of SCALAPACK in this situation.

The remainder of the thesis was dedicate to discussing the results obtained for the three problems outlined in Chapter 2, beginning with the stationary kinematic dynamo problem in Chapter 5. The first result was the benchmarking of the code against the stationary dynamo of Pekeris, Accad & Shkoller (1973), as studied by Dudley & James (1989). To consider different azimuthal symmetry classes we required the use of a compact spectral eigenvalue code, which combined the field representation discussed in Appendix C with second order finite differences in radius, in order to validate the results of the time-stepping. With the time-stepping method the dominant growth rate for this dynamo was found to converge very slowly with time. This was shown to be caused by a second magnetic mode with growth rate close to the dominant mode. Motivated by the hybrid time-stepping/Arnoldi method of Willis & Gubbins (2004) for the periodic kinematic dynamo problem, we attempted to develop a hybrid time-stepping/inverse iteration method to accelerate convergence, in Appendix D. This method was tested for the toroidal free decay problem, yielding the correct eigenvalues. When actually time-stepped, however, the growth rates were found to converge far too slowly, even for initial guesses close to the (well separated) free decay modes.
This was attributed to the method mapping the growth rates of quickly decaying modes to values close to zero, resulting in a clustering of the eigenvalues the method aimed to avoid. It remains an open question as to whether accelerated convergence could be achieved using this method, and we suggest alternating between time-stepping the original problem and time-stepping the shifted inverse problem as a possible avenue of further development.

The remainder of Chapter 5 considered classes of flows with one missing component in spherical polar co-ordinates: the zero theta component (ZTC) and zero azimuthal component (ZAC) flows. We sought examples of simple flows of these forms which could act as a dynamo. The growth rates computed with the flow representation developed in the chapter, for both flow classes, were found to be highly sensitive to the truncation level, with no convergent growing mode found. This was surprising given the simplicity of the flows considered. Due to this lack of progress we considered the ZAC Moss-Gailitis spherical shell dynamo dynamos of Moss (2006). We were unable to reproduce the results of Moss (2006), but instead we found a dynamo with a different growth rate for the same parameters, and demonstrated that the convergence of the growth rate increased with the truncation level. We then modified the radial dependence of the stream function to consider a Moss-Gailitis axisymmetric flow in a full sphere, for which we were able to find a dynamo which demonstrated convincing convergence of the growth rate. This was also verified using the eigenvalue method. From this we were able to conclude that ZAC flows can indeed function as dynamos. It would be pertinent to attempt to reproduce the single roll dynamos of Moss (2008), both with the inner core and modified for the full sphere, as well as constructing an analogue of the Moss-Gailitis flows for the ZTC flows.

In Chapter 6 we considered the problem of convection driven by a homogeneous volumetric heat source in a rotating full sphere of Boussinesq fluid with isothermal boundary conditions in the absence of a magnetic field. In the first section of the chapter we reproduced the stress-free benchmark of Marti et al. (2014). We then changed the viscous boundary conditions to no-slip and performed computations with the same parameters considered for the dynamo solutions discussed in Chapter 7, reproducing the standard Busse (1970) column configuration.

In Chapter 7 we considered the full sphere dynamo problem in a rotating, uniformly conducting, Boussinesq fluid with a volumetric heat source, isothermal boundary conditions and an insulating exterior. After reproducing the stress-free benchmark of Marti et al. (2014) we moved to no-slip viscous boundary conditions and consider two sets of solutions. The first was generated with imposed equatorial or azimuthal symmetry, from which we presented a dynamo solution that is periodic and requires the implementation of both the equatorial and azimuthal symmetries to be reproduced. This dynamo highlights the importance of not assuming symmetry in a dynamo solutions a priori. We then fixed the Ekman number $E = 5 \cdot 10^{-4}$ and the Prandtl number $Pr = 0.7$, and varied the magnetic Prandtl number $1 \leq Pm \leq 40$ and the Rayleigh number up to a few times the critical value for the onset of convection. We investigated the description of these solutions using scaling laws, either based on an assumed force balance or from a purely numerical perspective. We demonstrated, using the weak/strong field classification of Dormy (2014), why the former failed — the solutions have different strong/weak field classification, and therefore different force balances. This applied to the testing of numerical scaling laws. Before scaling laws can be properly tested more computations are required, both in the parameter regime considered, but also for different
Ekman numbers and Prandtl numbers, in order to identify regions of parameter space which correspond to different classes of dynamo solution.

The remainder of Chapter 7 concluded with more thorough descriptions of three dynamo solutions obtained from the systematic parameter search. These dynamos were chosen as they corresponded to three types of dynamo solutions found in other (spherical shell) dynamo studies; a periodic dipolar dynamo solution for \((Ra, Pm) = (5000, 35)\), a periodic quadrupolar solution for \((Ra, Pm) = (3000, 20)\), and a chaotic dynamo which jumped between two hemispherical states obtained for \((Ra, Pm) = (3000, 5)\). Although a clear \(\alpha\omega\) process was identifiable in the first of these, the dynamo mechanisms of the remaining two dynamos were not as clearly discernible. In particular for the hemispherical dynamo no significant EEA mode, which has been associated with hemispherical dynamo action (Landeau & Aubert 2011, Dietrich & Wicht 2013), was present.

As a final proposal for future research we outlined an approach to the problem of dynamo action in a fluid sphere as it cools in Chapter 8. We considered only the simplest case of a rotating fluid sphere with a (spatially) uniform heat flux condition at the boundary, with no volumetric heat source. Even this simple model introduced unforeseen complications in the form of a time-dependent conduction state temperature and an integral condition on the axisymmetric perturbation temperature. We developed a method which solved the integral condition simultaneously (implicitly) with the axisymmetric temperature every time-step, which doubles both the bandwidth and rank of the implicit time-stepping system, and apply this to two thermal convection models (in the absence of a magnetic field) which gave exactly what we expected — initially there was sufficient heat in the sphere to drive convection, but as the sphere cooled the power dissipated by the viscosity became larger than that generated by the buoyancy, and the convection slowed.

The study of computational dynamos is necessarily an open problem as it is impossible to consider the full parameter space and the effect of different initial conditions, including symmetries in the initial conditions, on the solutions. The problems considered in this thesis are no exception: for the kinematic dynamo problem we considered only the simplest flows; for the dynamical dynamo problem we considered only a small region of the parameter space with fixed initial conditions; and for the cooling dynamo problem we considered only two first convection models. Nevertheless, we leave the detailed diagnostics of the full sphere dynamo solutions in Chapter 7 and Appendix E for future researches to incorporate into a full sphere dynamo database.
APPENDIX A

The Combined Compact Finite Difference Approximation

In this appendix the combined compact difference schemes (CCDS) of Chu & Fan (1998) are described to aid the comparison of this method to the generalised compact method described in Chapter 3. These methods have gone through what has become a standard development process for compact finite difference schemes — generalisation to non-uniform grids and higher order derivatives (Chu & Fan considering only the first two derivative terms) (Ghader & Vahid 2011, Takahashi 2012), consideration of stability properties and resolution characteristics (Chu & Fan 1998, Mahesh 1998, Sengupta, Lakshmanan & Vijay 2009) and finally implementation (Mousa, Abadeer & Abbas 2012, Takahashi 2012). In this appendix only the tri-diagonal interior schemes given in Chu & Fan (1998) and Takahashi (2012) are discussed in any detail.

A1. The combined compact difference scheme

Beginning with the scalar function $f(r)$ defined on the discretised grid $\{r_j\}_{j=1,...,N}$ the general combined compact scheme for the $p$th derivative of $f$ at $j$ can be found by first assuming the approximation

$$
\frac{\partial^p}{\partial r^p} f(r_j) = \sum_{i=-l_0}^{u_0} \alpha_{0,i} f(r_{j+i}) + \sum_{i=-l_1}^{u_1} \beta_{1,i} \frac{\partial}{\partial r} f(r_{j+i}) + \sum_{i=-l_2}^{u_2} \beta_{2,i} \frac{\partial^2}{\partial r^2} f(r_{j+i}) + \cdots + \mathcal{E} \quad (A.1)
$$

where all $\beta_{k,0} = 0$ and $\mathcal{E}$ is the truncation error. Limiting the number of derivative terms on the right (usually to just the first two) and matching coefficients in the Taylor expansion around $r_j$ yields the combined compacted difference scheme for the $p$th derivative of $f$ at $r_j$ (see Ghader & Vahid 2011).

When approximating $\chi f$, where $\chi$ is some linear differential operator

$$
\chi = \psi_0(r) + \psi_1(r) \frac{\partial}{\partial r} + \psi_2(r) \frac{\partial^2}{\partial r^2} + \cdots + \psi_k(r) \frac{\partial^k}{\partial r^k} ,
$$

the individual schemes (A.1) for all $p$ with $\psi_p \neq 0$ are combined in exactly the same way Padé schemes are combined for the same operator (see section 3.5). There is greater flexibility in the CCDS methods, however, as each author is free to choose which derivatives are included on the right of (A.1), and hence when applied to a time-stepping problem, which derivatives of $f$ are evaluated implicitly.
As an example, the tri-diagonal schemes involving up to the second derivative of \( f \) on the right of \((A.1)\) on a uniform grid with spacing \( h \) are (Chu & Fan 1998)

\[
\frac{\partial}{\partial r} f(r_j) = \frac{15}{16h} \left( f(r_{j+1}) - f(r_{j-1}) \right) - \frac{7}{16} \left( \frac{\partial}{\partial r} f(r_{j+1}) + \frac{\partial}{\partial r} f(r_{j-1}) \right) + \frac{h}{16} \left( \frac{\partial^2}{\partial r^2} f(r_{j+1}) - \frac{\partial^2}{\partial r^2} f(r_{j-1}) \right) + \frac{h^6}{5040} \frac{\partial^7}{\partial r^7} f(r_j) + \mathcal{O}(h^8),
\]

(A.3)

and

\[
\frac{\partial^2}{\partial r^2} f(r_j) = \frac{3}{h^2} \left( f(r_{j+1}) - 2f(r_j) + f(r_{j-1}) \right) - \frac{9}{8h} \left( \frac{\partial}{\partial r} f(r_{j+1}) - \frac{\partial}{\partial r} f(r_{j-1}) \right) + \frac{1}{8} \left( \frac{\partial^2}{\partial r^2} f(r_{j+1}) + \frac{\partial^2}{\partial r^2} f(r_{j-1}) \right) + \frac{h^6}{20160} \frac{\partial^8}{\partial r^8} f(r_j) + \mathcal{O}(h^8).
\]

(A.4)

Using these schemes to approximate the discretised DE

\[
(\gamma_0 - \Delta t D_n) f(r_j) = b(r_j)
\]

(A.5)

where \( \gamma_0 \) and \( \Delta t \) are constants, and all \( b(r_j) \) are known, gives

\[
\begin{bmatrix}
  f(r_{j-1}) \\
  \partial_r f(r_{j-1}) \\
  \partial_{rr} f(r_{j-1})
\end{bmatrix} + \begin{bmatrix}
  f(r_j) \\
  \partial_r f(r_j) \\
  \partial_{rr} f(r_j)
\end{bmatrix} + \begin{bmatrix}
  f(r_{j+1}) \\
  \partial_r f(r_{j+1}) \\
  \partial_{rr} f(r_{j+1})
\end{bmatrix} = \begin{bmatrix}
  b(r_j) \\
  0 \\
  0
\end{bmatrix}
\]

(A.6)

where

\[
A_{j,j-1} = \begin{bmatrix}
  0 & 0 & 0 \\
  -\frac{15}{16h} & -\frac{7}{16} & -\frac{h}{16} \\
  3 & 9 & 1 \\
  \frac{1}{h^2} & \frac{9}{8h} & \frac{8}{8}
\end{bmatrix}, \quad A_{j,j} = \begin{bmatrix}
  0 & 0 & 0 \\
  -\frac{15}{16h} & -\frac{7}{16} & -\frac{h}{16} \\
  3 & 9 & 1 \\
  \frac{1}{h^2} & \frac{9}{8h} & \frac{8}{8}
\end{bmatrix}, \quad A_{j,j+1} = \begin{bmatrix}
  \gamma_0 + \Delta t \frac{n(n+1)}{r_j^2} & -\Delta t \frac{2}{r_j} & -\Delta t \\
  0 & 1 & 0 \\
  -\frac{6}{h^2} & 0 & 1
\end{bmatrix}
\]

(A.7)

(A.8)

The end result is a 6th order (in space) block \( 3 \times 3 \) tri-diagonal equation (in the interior) which solves for \( f, \partial_r f \) and \( \partial_{rr} f \) at the grid points.
The advantage of these methods over Padé schemes is most obvious when considering the
discretised 4th order poloidal momentum equation
\[(\text{Ro} \gamma_0 - \Delta t E D_n) D_n f(r_j) = b(r_j), \quad (A.9)\]
where Ro, E, \(\Delta t\) and \(\gamma_0\) are all constants, and all the \(b(r_j)\) are known. Using the tri-diagonal
methods for the third and fourth derivative schemes
\[
\begin{align*}
\frac{\partial^3}{\partial r^3} f(r_j) &= \frac{15}{4h^3} \left( f(r_{j-1}) - f(r_{j+1}) \right) + \frac{15}{4h^2} \left( \frac{\partial}{\partial r} f(r_{j+1}) + \frac{\partial}{\partial r} f(r_{j-1}) \right) \\
&\quad + \frac{3}{4h} \left( \frac{\partial^2}{\partial r^2} f(r_{j-1}) - \frac{\partial^2}{\partial r^2} f(r_{j+1}) \right) + \frac{h^4}{280} \frac{\partial^7}{\partial r^7} f(r_j) + \mathcal{O}(h^6),
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial^4}{\partial r^4} f(r_j) &= -\frac{36}{h^4} \left( f(r_{j+1}) + f(r_{j-1}) \right) + \frac{21}{h^3} \left( \frac{\partial}{\partial r} f(r_{j+1}) - \frac{\partial}{\partial r} f(r_{j-1}) \right) \\
&\quad - \frac{3}{h^4} \left( \frac{\partial^2}{\partial r^2} f(r_{j+1}) + \frac{\partial^2}{\partial r^2} f(r_{j-1}) \right) + \frac{h^4}{560} \frac{\partial^8}{\partial r^8} f(r_j) + \mathcal{O}(h^6),
\end{align*}
\]
results in a 4th order block \(3 \times 3\) tri-diagonal scheme which solves for \(f, \frac{\partial}{\partial r} f\) and \(\frac{\partial^2}{\partial r^2} f\) at the radial
points, at each time-step, the corresponding Padé schemes adding the third derivative of \(f\) to this list.
APPENDIX B

Spectral Analysis of the Generalised Compact Approximation

In this appendix a spectral analysis of the differencing errors associated with the generalised compact finite difference schemes of chapter 3 are given with the specific example $\chi = D_0 = \partial_{rr} + 2/r \partial_r$ considered in detail. The basic method is that of Vichnevetsky & Bowles (1982) and Cain & Bush (1994), and follows closely the analysis of Gamet et al. (1999), with the same optimisation method of Lele (1992).

B1. Spectral analysis of the generalised compact finite difference approximation

We consider the $2L$ periodic extension of a real function $f$ and operator $\chi$ for $r \in (0, 2L)$ on a discrete grid which is allowed to deviate locally from a $J$-point uniform grid. A single mode is considered

$$f(r) = \exp \left( \frac{i \omega r}{h} \right)$$  \hspace{1cm} (B.1)

where $h = J^{-1}$ is the reference grid spacing and $\omega \in [0, \pi]$, $\omega = \pi$ being the $2\delta$ wave on the reference $J$-point uniform grid. Substituting this into the compact approximation (3.102) gives the contribution of the $\omega$ mode to the truncation error

$$E_{j}(\omega, \alpha, \beta) := u_{\alpha} \sum_{l=-l_{\alpha}}^{u_{\alpha}} \alpha_{l,j} \exp \left( \frac{i \omega r_{j+l}}{h} \right) \frac{p^{\bar{p}}}{\bar{p}} \psi_{\bar{p}} (r_{l+j}) - \sum_{l=-l_{\beta}}^{u_{\beta}} \beta_{l,j} \exp \left( \frac{i \omega r_{j+l}}{h} \right).$$  \hspace{1cm} (B.2)

Comparing this to the exact value of the left of the compact approximation (3.102) with $f$ replaced by (B.1), which gives the first term on the left of (B.2), gives what will be termed the relative error $E_{j}^{\text{rel}}(\omega, \alpha, \beta)$

$$E_{j}^{\text{rel}}(\omega, \alpha, \beta) := \frac{E_{j}(\omega, \alpha, \beta)}{\sum_{l} \alpha_{l,j} \exp \left( \frac{i \omega r_{j+l}}{h} \right) \sum_{\bar{p}} \frac{p^{\bar{p}}}{\bar{p}} \psi_{\bar{p}} (r_{l+j})} = 1 - \frac{\sum_{l} \beta_{l,j} \exp \left( \frac{i \omega h_{l}}{h} \right)}{\sum_{l} \alpha_{l,j} \exp \left( \frac{i \omega h_{l}}{h} \right) \sum_{\bar{p}} \frac{p^{\bar{p}}}{\bar{p}} \psi_{\bar{p}} (r_{l+j})}.\hspace{1cm} (B.3)$$

Some elementary manipulation reveals this is identical to the relative error in the effective wavenumber $\gamma_{\text{eff}}$ given in e.g. Lele (1992), which is defined in terms of the exact $\gamma : \chi f = \gamma(r, \omega) f$.
by

\[ \gamma_{\text{eff}}(r_j, \omega) := \frac{\sum_l \beta_{l,j} \exp \left( \frac{i\omega h_l}{h} \right) \gamma(r_j, \omega)}{\sum_l \alpha_{l,j} \gamma(r_{j+l}, \omega) \exp \left( \frac{i\omega h_l}{h} \right) \gamma(r_j, \omega)}. \]  

(B.4)

In assessing the resolution characteristics of a finite difference scheme it is customary to introduce a resolving efficiency \( E_{\text{eff}}(e, \alpha, \beta) \) given in terms of a threshold error \( e \) by

\[ E_{\text{eff}}(e, \alpha, \beta) := \frac{\omega_c}{\pi}, \quad \omega_c := \min \{ \omega : |E_{\text{rel}}(\omega, \alpha, \beta)| > e \}. \]  

(B.5)

This gives the portion of grid-resolvable waves for which the relative error \( E_{\text{rel}} \) is less than some prescribed threshold \( e \).

The range of length scales accurately difference by a compact scheme can be improved by reducing the maximum order polynomial exactly differenced by a scheme [i.e. by limiting the number of elements in the Taylor basis matched in (3.103) and replacing these conditions with \( E_j(\omega, \alpha, \beta) = 0 \), for some fixed \( \omega \), in (B.2)]. This is the spectral matching procedure of Lele (1992) and is equivalent to setting \( E_j \) in (3.102) to zero for the polynomials \( \left\{ \frac{(r - r_j)^k}{k!} \right\}_{k=0,1,\ldots,l_\alpha+u_\alpha+l_\beta+u_\beta-2K} \) as well as the elements on the unit circle \( \left\{ \exp \left( \frac{ir\omega_k}{h} \right) \right\}_{k=1,2,\ldots,K} \), where \( K \) is the number of \( \omega_k \) to be matched. In general, two conditions for each such \( \omega \) are generated; one from the real component, and another from the imaginary component of (B.2), which is why \( 2K \) elements are removed from the Taylor basis. These \( \omega_k \) can be prescribed or taken as disposable parameters chosen to optimise some property of the method, usually the resolving efficiency for a given threshold error. For the remained of this chapter we assume the \( \alpha \) and \( \beta \) are real.

Application of these methods to non-periodic domains is not rigorous but is understood to give a good indication of the expected behaviour of a scheme. For the generalised compact scheme care must be taken when interpreting the results from this kind of analysis, as the expression for the spectral error can no longer be obtained by Fourier transform of the compact approximation; the radial dependencies in the \( \Psi_p \)'s no longer maps a Fourier mode into itself. This is why we consider ‘the contribution of the \( \omega \) mode to the truncation error‘ as opposed to ‘the \( \omega \) mode of the truncation error‘. This distinction does not occur in the analysis of the Padé schemes (e.g. Lele 1992 and Gamet et al. 1999) as the two are identical when the \( \Psi_p \) are independent of \( r \).

B2. Application to \( \chi = D_0 \)

B2.1. Spectral analysis & spectral matching. In this section we apply the methods described above to the operator \( \chi = D_0 \). Substituting this into (B.2) gives the error introduced by the \( \omega \) mode by the approximation at \( j \)

\[ E_j(\omega, \alpha, \beta) = \sum_{l=-l_\alpha}^{u_\alpha} \alpha_{l,j} \left( -\frac{\omega^2}{h^2} + \frac{2i\omega}{r_{l+j}} \right) \exp \left( \frac{i\omega r}{h} \right) - \sum_{l=-l_\beta}^{u_\beta} \beta_{l,j} \exp \left( \frac{i\omega r}{h} \right). \]  

(B.6)
To improve the range of length scales accurately differenced by a scheme, $\mathcal{E}_j(\omega, \alpha, \beta)$ is set to zero for some set of $\omega$. The real and imaginary matching conditions are then

\begin{align*}
\sum \alpha_{l,j} \left( -\frac{\omega^2}{h^2} \cos \left( \frac{\omega h_l}{h} \right) - \frac{\omega}{h} \frac{2}{r_{l+j}} \sin \left( \frac{\omega h_l}{h} \right) \right) &= \sum \beta_{l,j} \cos \left( \frac{\omega h_l}{h} \right) \\
\sum \alpha_{l,j} \left( -\frac{\omega^2}{h^2} \sin \left( \frac{\omega h_l}{h} \right) + \frac{\omega}{h} \frac{2}{r_{l+j}} \cos \left( \frac{\omega h_l}{h} \right) \right) &= \sum \beta_{l,j} \sin \left( \frac{\omega h_l}{h} \right)
\end{align*}

(B.7)

respectively, which are solved simultaneously with the Taylor matching conditions (3.113).

**Figure B.1.** Spectral error characteristics of $D_0$ on at $j = 150$ on the $J = 300$-point uniform grid.

Shown in Figure B.1 are the real and imaginary parts of the effective wave numbers plotted against the true wave number as well as the relative errors (B.3) and resolving efficiencies (B.5) as functions of $\omega$, for $D_0$ at $j = 150$ on the $J = 300$-point uniform grid for $r \in (0, 1]$. These are given for the full order tri-diagonal and full order penta-diagonal central difference compact schemes, as well as the penta-diagonal central difference scheme matched at $\omega = 1.8593, 2.3283$ and the $(2 \times 2)$-block hepta-diagonal scheme for this operator built from the near-spectral first and second derivative schemes of Lele (1992). Of particular interest is the comparison of the full order penta-diagonal and spectrally matched penta-diagonal schemes which demonstrates that by
sacrificing accuracy at long length scales — i.e. decreasing the formal order of the finite difference scheme — the range of length scales accurately differenced by a scheme can be extended. Despite that this scheme is out-performed by the near spectral scheme of Lele, the comparison is still favourable when the bandwidths are taken into consideration. The scheme built from the near-spectral schemes of Lele are block $(2 \times 2)$ hepta-diagonal, and require a hepta-diagonal matrix multiplication, for a 4th order truncation error. This contrasts the spectrally matched schemes developed here, which are 3rd order penta-diagonal, and require a penta-diagonal matrix multiplication. It was noted that the behaviour of these schemes was little modified by changing $h$, except for very small values, and by changing $r_j$, except for small $r$ (which is unsurprising as small values of $r$ make the contributions from the first derivative dominate the contribution from the second derivative in $D_0$) and at the boundaries where one sided schemes are used.

**B2.2. Numerical testing.** To demonstrate how this manifests in a computation the errors generated by the full order and spectrally matched penta-diagonal schemes given above are found for the test function

$$f(r) = \sum_{k=1}^{K} \frac{\cos(2\pi k r + \phi_k)}{4\pi^2 k^2}$$

where $\phi_k \sim U[0, 2\pi]$ independent and $K = 16, 32, 48, 64$ on the $J = 128$ point uniform grid. Errors were given as the average over 1000 realisations of $f$ for each $K$ with both schemes using the same (unmatched) one-sided bandwidth preserving boundary schemes. The point-wise average relative errors are shown in Figure [B.2] and correlate well with the behaviour seen in [B.1]; the unmatched scheme is superior when only long length scales are present, the spectrally matched scheme superior as shorted length scales are added until the error saturates with the inclusion of large $\omega$, or equivalently, as we move from left to right on Figure [B.1]:
Figure B.2. $\log_{10}$ pointwise mean relative errors for penta-diagonal scheme for $D_0$ acting on a supposition of cosine modes with uniformly distributed phases on the $J = 128$ point uniform grid, plotted against radial position $r$. 
APPENDIX C

Compact spectral magnetic induction equation

In this appendix the compact spectral form of the magnetic induction equation, as derived by James (1974), are given. The eigenvalue code combines the methods outlined here with second order central differences in radius. Only a very coarse outline of this approach is given here and the interested reader is referred to Brink & Satchler (1968), Wills (1971), James (1973), James (1974) and Ivers & Phillips (2008) for further detail of the derivation, connected theory and implementation.

This formulation begins with the magnetic induction equation (2.19),

\[
(\partial_t - \nabla^2) \mathbf{B} = R_m \nabla \times (\mathbf{u} \times \mathbf{B}).
\] (C.1)

The left of this equation is expressed exactly as was outlined in Chapter 3; the magnetic induction is decomposed into its poloidal/toroidal components, with a spectral Galerkin method in angle with spherical harmonic basis functions. The right of (C.1) is evaluated by expressing both the magnetic induction and the velocity as vector spherical harmonic series

\[
\mathbf{u} = \sum_{\alpha} u_{\alpha} \mathbf{Y}_{\alpha}, \quad \mathbf{B} = \sum_{\beta} B_{\beta} \mathbf{Y}_{\beta}
\] (C.2)

where Greek subscripts denote the 3-index \((n, n_1, m)\) \[i.e. \mathbf{Y}_{\alpha} = \mathbf{Y}_{n_\alpha, n_{1\alpha}}\], and \(n \geq 1, n_1 = n, n \pm 1\) and \(|m| \leq n\), and

\[
u_{\alpha} = (\mathbf{u}, \mathbf{Y}_{\alpha})
\] (C.3)

for the inner product (3.15). This eventually yields the compact spectral magnetic induction equation (James 1974)

\[
(\partial_t - D_\gamma) S_\gamma = R_m \sum_{\alpha, \beta, n_{1\gamma} = n_\gamma} (v_{\alpha} B_{\beta} B_{\gamma}), \quad (\partial_t - D_\gamma) T_\gamma = R_m \sum_{\alpha, \beta, n_{1\gamma} = n_\gamma, n_1 = n_\gamma \pm 1} (v_{\alpha} B_{\beta} B_{\gamma})
\] (C.4)

with the insulating exterior boundary conditions (3.54). The spectral interaction terms are

\[
(v_{\alpha} B_{\beta} B_{\gamma}) := e_\beta(\gamma) f_B(\beta)(\mathbf{Y}_{\alpha} \times \mathbf{Y}_{\beta}, \mathbf{Y}_{\gamma})
\]

\[
\begin{align*}
\partial_t (v_{\alpha} \partial_\beta S_\beta), &\quad n_{1\gamma} = n_\gamma \pm 1, \ n_1 = n_\beta \pm 1; \\
\partial_t (v_{\alpha} T_{\beta}), &\quad n_{1\gamma} = n_\gamma \pm 1, \ n_1 = n_\beta; \\
v_{\alpha} \partial_\beta S_\beta, &\quad n_{1\gamma} = n_\gamma, \ n_1 = n_\beta \pm 1; \\
v_{\alpha} T_{\beta}, &\quad n_{1\gamma} = n_\gamma, \ n_1 = n_\beta;
\end{align*}
\] (C.5)
where $e_B$ is given by
\[
e_B(n, n_1) := \begin{cases} 
1/\sqrt{n/(2n+1)}, & n_1 = n - 1; \\
i/\sqrt{n(n+1)}, & n_1 = n; \\
1/\sqrt{(n+1)(2n+1)}, & n_1 = n + 1; 
\end{cases}
\] (C.6)

the factor $f_B$ is
\[
f_B(n, n_1) := \begin{cases} 
(n + 1)\sqrt{n/(2n+1)}, & n_1 = n - 1; \\
-i\sqrt{n(n+1)}, & n_1 = n; \\
n\sqrt{(n+1)/(2n+1)}, & n_1 = n + 1; 
\end{cases}
\] (C.7)

and
\[
\partial_\gamma = \partial_{n_1, \gamma}, \quad \partial_\gamma = \partial_{n_1, \gamma} \, \partial_\gamma
\] (C.8)

where
\[
\partial_{n_1} := \begin{cases} 
\partial_r + \frac{n + 1}{r}, & n_1 = n - 1 \\
\partial_r - \frac{n}{r}, & n_1 = n + 1 
\end{cases}
\] (C.9)

The coupling integrals $(Y_\alpha \times Y_\beta, Y_\gamma)$ are given in terms of $3j$ and $9j$ symbols (James 1973, 1974)
\[
(Y_\alpha \times Y_\beta, Y_\gamma) = (-)^{n_\alpha + n_\beta + n_1, \gamma + m_\gamma} i \sqrt{6} \Gamma(\alpha, \beta, \gamma) 
\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ n_{1\alpha} & n_{1\beta} & n_{1\gamma} \\ 1 & 1 & 1 
\end{pmatrix} 
\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ 0 & 0 & 0 \\ m_\alpha & m_\beta & -m_\gamma 
\end{pmatrix}
\] (C.10)

where
\[
\Gamma(\alpha, \beta, \gamma) := \sqrt{(2n_\alpha + 1)(2n_{1\alpha} + 1)(2n_\beta + 1)(2n_{1\beta} + 1)(2n_\gamma + 1)(2n_{1\gamma} + 1)}. 
\] (C.11)

It is simple to show that the poloidal/toroidal spherical harmonic representation of $u$ is related to the vector spherical harmonic representation of $u$ by
\[
u_{n,n-1}^m = (n + 1) \sqrt{\frac{n}{2n+1}} \partial_n^{n-1} s_n^m, \quad u_{n,n}^m = -i \sqrt{n(n+1)} t_n^m, \quad u_{n,n+1}^m = n \sqrt{\frac{n+1}{2n+1}} \partial_n^{n+1} s_n^m. 
\] (C.12)
Hybrid time-stepping/inverse iteration scheme

A shifted inverse iteration eigen-method is a useful tool to find and separate the eigenvalues of an eigenproblem

$$Ax = \lambda x.$$  \hfill (D.1)

Instead of the problem (D.1), the shifted inverse eigenproblem

$$(A - \lambda_0 I)^{-1} x = \tilde{\lambda} x,$$  \hfill (D.2)

is considered, where $\lambda$ and $\tilde{\lambda}$ are related by

$$\tilde{\lambda} = \frac{1}{\lambda - \lambda_0}$$  \hfill (D.3)

for $\lambda_0$ some initial guess to $\lambda$. Applying a power method to this problem yields convergence to the $\tilde{\lambda}$ with largest norm, corresponding to the $\lambda$ which minimises $|\lambda - \lambda_0|$. To apply this method to the kinematic dynamo problem the time dependence of the magnetic field is separated

$$B(r, t) = B(r, 0) \exp(\tilde{\lambda} t)$$ \hfill (D.4)

and the magnetic induction equation (2.19) discretised in space, usually after $B$ has been decomposed into poloidal and toroidal component with scalar potentials given as spherical harmonic series (as in Chapter 3). This results in the eigenproblem

$$(\lambda I - A) b = R_m C b.$$ \hfill (D.5)

The hybrid inverse-iteration time-stepping algorithm for the spherical stationary kinematic dynamo problem is built from the observation that applying a shifted inverse iteration eigenvalue method to (D.5) is identical to performing a standard eigenvalue analysis on the semi-discretised (exact time differencing) form of

$$\frac{\partial}{\partial t} \left( \nabla^2 B + R_m \nabla \times (u \times B) - \lambda_0 B \right) = B,$$ \hfill (D.6)

where the separable solutions $B(r, t) = B(r, 0) \exp(\tilde{\lambda} t)$ have been assumed. This DE can be time-stepped using the methods outlined in chapter 3

$$(\partial_t (D_n - \lambda_0) - 1) S_n^m = R_m \frac{\partial}{\partial t} \left( T_n^m \{u \times B\} \right),$$ \hfill (D.7)

$$(\partial_t (D_n - \lambda_0) - 1) T_n^m = -R_m \frac{\partial}{\partial t} \left( S_n^m \{u \times B\} \right)$$ \hfill (D.8)

subject to the standard magnetic boundary conditions

$$T_n^m(r = 1, t) = 0, \ (\partial_r - (n + 1)) S_n^m(r = 1, t) = 0.$$ \hfill (D.9)
Notionally the spectral form of $\frac{\partial}{\partial t} \nabla \times (u \times B)$ can be treated explicitly using the 4th order Padé scheme on the uniform $t$-grid

$$\frac{\partial}{\partial t} f^{(k)} = \frac{3}{\Delta t} (f^{(k)} - f^{(k-2)}) - 4 \frac{\partial}{\partial t} f^{(k-1)} - \frac{\partial}{\partial t} f^{(k-2)}.$$  \hfill (D.10)

As seen below, however, there was no need to develop this method to this point.

As a proof of principle the inverse free decay ($R_m = 0$) problem is considered. Combining the compact scheme for

$$\chi = \gamma (D_n - \lambda_0) - \Delta t$$  \hfill (D.11)

with the third order semi-implicit multi-step method, observing that

$$L(D_n - \lambda_0) f = \frac{1}{\gamma} (R + \Delta tL) f,$$  \hfill (D.12)

gives the time projection equation

$$\begin{bmatrix} R & I \\ I & I \end{bmatrix} x^{(k+1)} = \frac{e^{\tilde{\lambda} \Delta t}}{\gamma} \begin{bmatrix} \alpha_0 (R + \Delta tL) & \alpha_1 (R + \Delta tL) & \alpha_2 (R + \Delta tL) \\ \gamma I & \gamma I \end{bmatrix} x^{(k)},$$  \hfill (D.13)

where $x^{(k)} = [f^{(k)} \ f^{(k-1)} \ f^{(k-2)}]^T$, $I$ is the identity matrix and missing entries are zeros. Unlike standard inverse iteration convergence in this case is to the $\tilde{\lambda}$ which has the largest real part — i.e. the $\tilde{\lambda}$ which maximises the norm of $\exp(\tilde{\lambda} t)$. This corresponds to the $\lambda$ which minimises $\{\text{Re}(\lambda - \lambda_0) \mid \text{Re}(\lambda - \lambda_0) \geq 0\}$ if such a $\lambda$ exists. If such a $\lambda$ does not exist convergence is expected to the fastest decaying mode, that which maximises $\text{Re} |\lambda - \lambda_0|$. This holds true for $R_m \neq 0$.

To test the method the first 3 slowest decaying modes for the $n = 1$ toroidal free decay problem are calculated from the eigenvalues of (D.13) using a sparse Arnoldi eigensolver. The growth rates $\lambda$ obtained are plotting against the initial guess $\lambda_0$ along with the errors relative to the analytic values (see Chapter 3) for $(J, \Delta t) = (300, 10^{-4})$ in Figure D.1, with the analytic growth rates given as vertical lines. The behaviour seen here is exactly as expect; the Eigenvalues of (D.13) are converging to the first (smallest) $\text{Re} (\lambda - \lambda_0) > 0$.

These results, however, do not extended favourably to the time-advancement problem where, despite convergence to the correct rates, long integration times were required. Shown in Figure D.2 is the $\log_{10}$ of absolute error in the measured growth rate of the slowest decaying toroidal mode as a function of time for $(J, \Delta t) = (300, 10^{-4})$ and $\lambda_0 = -22$. The dip at around $t = 1.9$ corresponds to a change in sign of the error. Convergence here was much slower than in the time-stepping of the standard free decay problem (which required less than 1 diffusion time to converge) despite the large separation of the growth rates. The expected cause of this is the mapping of the rapidly decaying modes to modes of near zero rate in $\lambda$, resulting in exactly the same problem this method was intended to overcome, modes with near growth rates.
**Figure D.1.** Inverse iteration $\lambda$ and the relative error in $\lambda$ as a function of the initial guess $\lambda_0$ for the $n = 1$ toroidal free decay problem with $(J, \Delta t) = (300, 10^{-4})$. The vertical lines are $\lambda_0$ equal to the analytic free decay rates.

**Figure D.2.** $\log_{10}$ of the absolute error in the measured growth rate of the first (slowest decaying) mode for the $n = 1$ toroidal free decay problem generated using the time-stepped inverse iteration scheme with $(J, \Delta t) = (300, 10^{-4})$ and $\lambda_0 = -22$.

Despite these drawbacks it should be advantageous to use this inverse iteration type method if the two dominant modes $\text{Re} (\lambda_1) > \text{Re} (\lambda_2)$ of the initial problem have $\text{Re} (\lambda_1 - \lambda_2) < 1$ since a starting guess $\lambda_0 = \lambda_2 + \epsilon$ for $\text{Re} (\epsilon) > 0$ can be made. This should result in accelerated convergence provided $\text{Re} (\lambda_1 - (\lambda_2 + \epsilon)) > 0$. Realistically this requires knowledge of the growth rates of the initial problem that would be generated from time-stepping the initial problem. Fortunately it is exactly the case of slow convergence in the initial problem which would both suggest the use of an inverse iteration scheme and provide an initial guess to $\lambda_0$. This would suggest that in the case of slow convergence it would be prudent to alternate between time-stepping the initial problem and time-stepping the inverse problem, with initial guess given by $\lambda_0 = \lambda_2 + \epsilon$, were $\lambda_2$ is the current growth rate from the standard time-stepping. This, however, was not further developed.
### APPENDIX E

**Summary of no-slip dynamos**

**E1. Preliminary no-slip dynamo results**

<table>
<thead>
<tr>
<th>$E = 10^{-5}$</th>
<th>$E = \frac{5}{7} \cdot 10^{-4}$</th>
<th>$E = 10^{-4}$</th>
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<td>$P_r$</td>
<td>$Ra$</td>
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<td>1</td>
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</tr>
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</tr>
<tr>
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<td>1</td>
<td>180</td>
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<th>$\text{Pm}$</th>
<th>$\text{Pr}$</th>
<th>$\text{Ra}$</th>
<th>$m_{\text{hex}}$</th>
<th>$(J, N, \Delta t)$</th>
<th>Radial Grid</th>
<th>$t_{\text{max}}$</th>
<th>$E_k$</th>
<th>$E_m$</th>
</tr>
</thead>
<tbody>
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<td>0.1</td>
<td>1</td>
<td>5000</td>
<td>1</td>
<td>$(280, 60, 10^{-6})$</td>
<td>Uniform</td>
<td>0.90</td>
<td>2693.81 ± 95.0272</td>
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</tr>
<tr>
<td>1</td>
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<td>3</td>
<td>$(240, 60, 10^{-6})$</td>
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<td>1.00</td>
<td>2173.36 ± 619.511</td>
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</tr>
<tr>
<td>1</td>
<td>750</td>
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<td>$(240, 60, 10^{-6})$</td>
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<td>3620.27 ± 931.997</td>
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<tr>
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<td>1.00</td>
<td>5237.42 ± 1322.27</td>
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</tr>
<tr>
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<td>8265.29 ± 1526.17</td>
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</tr>
<tr>
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<td>3</td>
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<td>1.00</td>
<td>16684.7 ± 3179.94</td>
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</tr>
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<td>3</td>
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<td>1.21</td>
<td>22481.6 ± 4213.31</td>
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</tr>
<tr>
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<td>$(240, 60, 10^{-6})$</td>
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<td>0.95</td>
<td>33859.2 ± 5730.23</td>
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</tr>
<tr>
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<tr>
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<td>3</td>
<td>$(240, 60, 10^{-6})$</td>
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<td>2.87</td>
<td>40325.3 ± 2294.05</td>
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</tr>
<tr>
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<td>0.7</td>
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<td>4.18</td>
<td>158377.8 ± 13695.7</td>
<td>24596.54 ± 1044.51</td>
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<tr>
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<td>29429.84 ± 29429.8</td>
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<td>213605.7 ± 21021.0</td>
<td>29114.95 ± 7319.29</td>
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<td>1400</td>
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<td>Uniform</td>
<td>3.85</td>
<td>25048.05 ± 4155.81</td>
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<td>1750</td>
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<td>Uniform</td>
<td>10.0</td>
<td>39493.12 ± 4641.74</td>
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<tr>
<td>7</td>
<td>1</td>
<td>2100</td>
<td>1</td>
<td>$(320, 30, 10^{-6})$</td>
<td>Uniform</td>
<td>8.13</td>
<td>52995.45 ± 6597.08</td>
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</tr>
<tr>
<td>7</td>
<td>1.4</td>
<td>2500</td>
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<td>Uniform</td>
<td>0.55</td>
<td>54853.74 ± 3665.85</td>
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<tr>
<td>7</td>
<td>2.8</td>
<td>1250</td>
<td>3</td>
<td>$(240, 60, 10^{-6})$</td>
<td>Uniform</td>
<td>0.59</td>
<td>14052.09 ± 1280.32</td>
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<tr>
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<td>7</td>
<td>400</td>
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<td>$(240, 60, 10^{-6})$</td>
<td>Uniform</td>
<td>0.58</td>
<td>1681.402 ± 129.385</td>
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</tbody>
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Continued from previous page

\[ E = 5 \cdot 10^{-3} \]

<table>
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<tr>
<th>Pm</th>
<th>Pr</th>
<th>Ra</th>
<th>( m_{\text{hcf}} )</th>
<th>((J, N, \Delta t))</th>
<th>Radial Grid</th>
<th>( t_{\text{max}} )</th>
<th>( E_k )</th>
<th>( E_m )</th>
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<tr>
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<td>108562 ± O(1)</td>
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<td>12814.6 ± 3269.28</td>
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<td>70</td>
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<td>0.42</td>
<td>438.366 ± 1.65358</td>
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**Table E.1.** Summary of preliminary no-slip dynamos. All dynamos have pure quadrupolar velocity and pure dipolar magnetic field. Energies are gives as mean values ± standard deviation.
E2. Systematic no-slip dynamo results for \((E, \text{Pr}) = (5 \cdot 10^{-4}, 0.7)\)

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<th>Ra</th>
<th>Pm</th>
<th>((J, N, \Delta t))</th>
<th>Radial Grid</th>
<th>(t_{\text{max}})</th>
<th>(E_k)</th>
<th>Dipole Fraction</th>
<th>(E_m)</th>
<th>Dipole Fraction</th>
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<td>766.375 \pm 56.2608</td>
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<td>Chebyshev</td>
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<td>(280, 60, 10^{-6})</td>
<td>Uniform</td>
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<td>3995.36 \pm 388.202</td>
<td>2.16 \times 10^{-19}</td>
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<tr>
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<td>Uniform</td>
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<td>3.01 \times 10^{-14}</td>
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<td>6.01 \times 10^{-2}</td>
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**Table E.2.** Summary of all systematic no-slip dynamos for (E, Pr) = (5 · 10^{-4}, 0.7). Energies are given as mean values ± standard deviation.
Bibliography


