Acknowledgements

I would like to thank my supervisor Anthony Henderson for his assistance with this thesis. I would like to thank him for helping to choose the topic, for his thorough and insightful feedback, and for his guidance with my research. I would like to thank my associate supervisor Anne Thomas for her helpful talks and feedback. I would also like to thank the examiners for their careful reading and suggestions.

I would like to thank my family and friends for their love and support throughout my education. Thank you for always being interested and proud of what I do.
CONTENTS

Acknowledgements ................................................................. iii

Introduction ............................................................................ v

Chapter 1. Preliminary definitions and examples ..................... 1
  1.1. Simplicial complexes ..................................................... 1
  1.2. Polytopes ........................................................................ 5
  1.3. Generalized permutohedra and their vertex posets .......... 7
  1.4. Building sets and nestohedra ........................................ 9
  1.5. Normal fans of nestohedra ............................................ 13
  1.6. Stellar subdivisions of simplicial complexes .................. 16

Chapter 2. The gamma-vector of edge subdivisions of the
  boundary of the cross polytope .............................................. 21
  2.1. Subdivision sequences .................................................. 21
  2.2. The main theorem ....................................................... 28
  2.3. Simple graphic zonotopes ............................................. 34

Chapter 3. The Nevo and Petersen conjecture for nestohedra ..... 36
  3.1. Building sets and nestohedra ........................................ 36
  3.2. Face shavings of flag nestohedra .................................. 39
  3.3. The flag simplicial complex \( \Gamma(O) \) for a flag ordering \( O \) of \( B \) . 40
  3.4. The dual simplicial complex of nestohedra .................... 48
  3.5. The flag simplicial complexes of Nevo and Petersen ....... 51

Chapter 4. Inequalities between gamma-polynomials of
  graph-associahedra ........................................................... 60
  4.1. Tree shifts ................................................................. 61
  4.2. Flossing moves ........................................................... 67

Chapter 5. Further research ..................................................... 74
  5.1. Gal’s conjecture for edge subdivisions ......................... 74
  5.2. Question 14.3 of Postnikov, Reiner and Williams ........... 78

References ............................................................................. 86
Introduction

This thesis relates to the theory of face enumeration of simplicial complexes. To any simplicial complex $\Theta$ that is a homology sphere there is an associated polynomial called the $\gamma$-polynomial, denoted $\gamma(\Theta)$, which is defined in Section 1.1. It is derived from the well known $f$-polynomial (face polynomial) and $h$-polynomial of any simplicial complex. Many studies have been done on upper and lower bounds of $\gamma$-polynomials, and we summarize some of these results in this introduction. This has been the area of study of this thesis. The $\gamma$-polynomial was first defined by Gal in [17], where he conjectured:

Conjecture 0.0.1. [17, Conjecture 2.1.7]. If $\Theta$ is a flag homology sphere, then $\gamma(\Theta) \geq 0$.

The notation $\gamma(\Theta) \geq 0$ means that every coefficient of $\gamma(\Theta)$ is non-negative. We refer to this well known conjecture as Gal’s conjecture. This has been proven to be true for the following classes of flag homology spheres:

- Flag homology spheres of dimension $< 5$, (see [10] and [17]).
- The order complexes of Gorenstein* posets (see [20]).
- Coxeter complexes (see [30]).
- Simplicial complexes obtained by stellar subdivisions in edges of the boundary of the $d$-dimensional cross polytope (see [31]). This set of simplicial complexes is denoted $sd(\Sigma_{d-1})$, and is defined in Section 1.6.

Previous to the result in [31], Gal’s conjecture was shown to hold for the dual simplicial complexes to chordal nestohedra, which are a subset of $sd(\Sigma_{d-1})$ in [26]. The work by Athanasiadis in [4] shows that Gal’s conjecture holds for any homology sphere obtained by edge subdivisions of a homology sphere $\Theta$, such that $\gamma(lk_{\Theta}(F)) \geq 0$ for all $F \in \Theta$, where $lk_{\Theta}(F)$ denotes the link of the face $F$ in $\Theta$. This result is discussed, and examples are given, in Chapter 5.

In [23], Nevo and Petersen conjectured the following strengthening of Gal’s conjecture on the $\gamma$-polynomial of flag homology spheres:
**Conjecture 0.0.2.** [23, Conjecture 6.3]. If $\Theta$ is a flag homology sphere then $\gamma(\Theta)$ satisfies the Frankl-Füredi-Kalai inequalities.

The Frankl-Füredi-Kalai inequalities characterize the $f$-vectors of balanced simplicial complexes (see [14]). Conjecture 0.0.2 is proven for the following list of flag homology spheres; the first four cases were proven by Nevo and Petersen in [23], and the fifth was proven by Nevo, Petersen and Tenner in [24]:

- $\Theta$ is a Coxeter complex (including the simplicial complex dual to $P_B(K_n)$),
- $\Theta$ is the simplicial complex dual to the associahedron ($P_B(\text{Path}_n)$),
- $\Theta$ is the simplicial complex dual to the cyclohedron ($P_B(\text{Cyc}_n)$),
- $\Theta$ has $\gamma_1(\Theta) \leq 3$,
- $\Theta$ is the barycentric subdivision of a homology sphere.

Here $K_n$ denotes the complete graph with $n$ vertices, $\text{Path}_n$ denotes the graph that is a path with $n$ vertices, and $\text{Cyc}_n$ denotes the cycle graph with $n$ vertices. The notation $B(G)$ denotes the graphical building set for the graph $G$, and $P_B(G)$ denotes the corresponding graph-associahedron.

Nevo and Petersen prove the above result in [23] by showing that their $\gamma$-vector is the $f$-vector of a flag simplicial complex. This suffices because Frohmader [15, Theorem 1.1] showed that the $f$-vector of any flag simplicial complex satisfies the Frankl-Füredi-Kalai inequalities. This naturally leads to the following strengthening of 0.0.2, which Nevo and Petersen suggest in [23]:

**Conjecture 0.0.3.** [23, Problem 6.4]. If $\Theta$ is a flag homology sphere then $\gamma(\Theta)$ is the $f$-polynomial of a flag simplicial complex.

We will refer to this conjecture as the Nevo and Petersen conjecture. We have shown in [1] that the Nevo and Petersen conjecture holds for the dual simplicial complexes of all flag nestohedra. This result was then extended to all simplicial complexes that can be obtained by edge subdivisions of the boundary of the cross polytope in [2]. This result was also proven independently by Volodin in [32]. We summarise the results of [2] in Chapter 2, and then the more specific case of the dual simplicial complexes to nestohedra is discussed in Chapter 3. Given any $\Theta \in sd(\Sigma_{d-1})$, we construct a flag simplicial complex $\Gamma(\Theta)$ such that $f(\Gamma(\Theta)) = \gamma(\Theta)$ (see Theorem 2.0.6). This construction is dependent on the sequence of edge subdivisions performed on $\Sigma_{d-1}$ to obtain $\Theta$. When $\Theta$ is the dual simplicial complex to a flag nestohedron, this sequence of subdivisions corresponds to a ‘flag ordering’ of the flag building set, which we define in Chapter 3. In Chapter 3, we give combinatorial descriptions for $\Gamma(\Theta)$ for particular flag orderings,
and we give a recurrence for the $\gamma$-polynomial of the stellohedron, as an example of what one can deduce from the construction in Chapter 3. The contruction of $\Gamma(\Theta)$ when $\Theta$ is the dual simplicial complex of a nestohedron is easier to work with than the more general construction in Chapter 2, and so is worth including in this thesis.

In [26], Postnikov, Reiner and Williams conjectured the following monotonicity property of the $\gamma$-polynomials of the graph-associahedra of trees.

**Conjecture 0.0.4. [26, Conjecture 14.1].** There exists a (nontrivial) partial order $\leq$ on the set of (unlabelled, isomorphism classes of) trees with $n$ vertices, with the following properties:

- Path$_n$ is the unique $\leq$-minimal element,
- $K_{1,n-1}$ is the unique $\leq$-maximal element,
- $T \leq T'$ implies $\gamma(B(T)) \leq \gamma(B(T'))$.

Here $K_{1,n-1}$ denotes the graph with $n$ vertices with exactly one vertex of degree $n-1$ and $n-1$ vertices of degree 1. The author has shown in [3] that Conjecture 0.0.4 holds for a partial order defined using transformations of the graph called tree shifts, and that the $\gamma$-polynomial lowers under transformations called flossing moves which was predicted in [6]. We discuss this work in Chapter 4.

Here is a summary of the contents of this thesis, with references to those parts that have already been published or submitted. Chapter 1 contains the main definitions used in this thesis. It also includes some basic theory relating to these fundamental concepts, along with examples. Chapter 1 includes an original result, Theorem 1.5.4, answering a question of Postnikov-Reiner-Williams, which characterises the normal fans of nestohedra. Chapter 2 contains the content of the paper [2], of which Theorem 2.0.6 is the main result. As mentioned, [2] shows that the Nevo and Petersen conjecture holds for simplicial complexes in $sd(\Sigma_{d-1})$. Chapter 3 includes the content of the paper [1], where we show that the Nevo and Petersen conjecture holds for the dual simplicial complexes to nestohedra in Theorem 3.0.4. Chapter 4 contains the content of the paper [3] in which we prove Conjecture 0.0.4 in Theorem 4.1.2 by showing that tree shifts lower the $\gamma$-polynomial of graph-associahedra. Chapter 4 also includes Theorem 4.2.1, which shows that flossing moves also lower the $\gamma$-polynomial of graph-associahedra. In Chapter 5 we include smaller results that have been made. This chapter includes a result proving Gal’s conjecture for edge subdivisions of the order complexes of Gorenstein* complexes, and shows that this result can be attributed to the work of Athanasiadis in [4].
5 also includes some work we have done towards answering Question 14.3 of [26] for interval building sets.
Preliminary definitions and examples

For further reading on the topics in this chapter the reader is advised to see [26] or [35].

1.1. Simplicial complexes

A simplicial complex Θ with vertex set $V_Θ$ is a set of subsets of $V_Θ$ such that every singleton $\{v\} \in V_Θ$ belongs to Θ, and if $S \in \Theta$ and $I \subseteq S$ then $I \in \Theta$. Elements in Θ are called faces, and the dimension of a face $S$ (denoted $\dim(S)$) is equal to $|S| - 1$. Note that $\emptyset$ is a face of dimension $-1$. An edge of Θ is a face of dimension one, and two distinct vertices are adjacent if they are contained in the same edge. The dimension of a simplicial complex Θ, denoted $\dim(\Theta)$, is the maximal dimension of a face in Θ. The $(d - 1)$-dimensional simplex, denoted $\Delta^{d-1}$ is the set of all subsets of $[d] := \{1, 2, ..., d\}$.

If $\Theta_1$ and $\Theta_2$ are simplicial complexes, then the join of $\Theta_1$ and $\Theta_2$, denoted $\Theta_1 * \Theta_2$, is the simplicial complex on the vertex set $V_{\Theta_1} \cup V_{\Theta_2}$, defined by

$$\Theta_1 * \Theta_2 := \{F_1 \cup F_2 \mid F_1 \in \Theta_1, F_2 \in \Theta_2\}.$$  

Simplicial complexes $\Theta_1$ and $\Theta_2$, are equivalent, denoted $\Theta_1 \cong \Theta_2$, if there is a bijection between their vertices that induces a bijection between their faces.

The underlying graph of a simplicial complex Θ is the 1-dimensional simplicial complex with vertices $V_Θ$ and 1-dimensional faces those of Θ.

A simplicial complex is flag if for every set $S \subseteq V_Θ$ such that any two vertices of $S$ are adjacent, we have $S \in \Theta$. A flag simplicial complex is determined by its underlying graph, since the faces are the cliques in this graph.
Example 1.1.1. The simplicial complex
\[ \Theta = \{ \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \} \} \]
is not flag since its vertex set \( \{ a, b, c \} \) is a set of vertices such that any pair is an edge, however \( \{ a, b, c \} \) is itself not a face of \( \Theta \). The simplicial complex
\[ \Theta' = \{ \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \} \} \]
obtained from \( \Theta \) by adding the set \( \{ a, b, c \} \) is flag.

The link of a face \( F \) in a simplicial complex \( \Theta \), denoted \( \text{lk}_\Theta(F) \), is the following subcomplex of \( \Theta \):
\[ \text{lk}_\Theta(F) := \{ G \in \Theta \mid G \cup F \in \Theta, \ G \cap F = \emptyset \}. \]
If the simplicial complex is flag then the link of a face is the induced subcomplex on the set of vertices that are adjacent to every vertex in \( F \).

We will now recall the definition of the reduced simplicial homology on a simplicial complex \( \Theta \), with coefficients in a field \( k \) (see [19]). We label the vertices of \( \Theta \) by \( \{ 1, 2, \ldots, n \} \). Consider the chain complex
\[
0 \xrightarrow{\delta_n} \bigoplus_{\dim(F)=n-1} k_F \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} \bigoplus_{\dim(F)=1} k_F \xrightarrow{\delta_1} \bigoplus_{\dim(F)=0} k_F \xrightarrow{\delta_0} k_{\delta-1} \to 0.
\]
The maps \( \delta_n \) and \( \delta_{-1} \) are the zero maps. The map \( \delta_0 \) is the augmentation map, that is, the linear extension of the map
\[ kF \mapsto k, \ k \in k. \]
For \( 1 \leq i \leq n-1 \) the map \( \delta_i \) is the linear extension of the map
\[
\bigoplus_{\dim(F)=i} k_F \longrightarrow \bigoplus_{\dim(F)=i-1} k_F,
\]
sending the simplex
\[ F = \{ \epsilon_0, \epsilon_1, \ldots, \epsilon_{i-1} \}, \ \text{where} \ \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{i-1}, \]
to
\[ \sum_{0 \leq j \leq i-1} (-1)^j(\{ \epsilon_0, \epsilon_1, \ldots, \epsilon_{i-1} \} - \{ \epsilon_j \}). \]
Then the $i$th reduced simplicial homology of $\Theta$, denoted $\tilde{H}_i(\Theta, k)$ is equal to
\[ \tilde{H}_i(\Theta, k) := \frac{\text{Ker}(\delta_i)}{\text{Im}(\delta_{i+1})}. \]

Suppose $\Theta$ is a flag simplicial complex, and $F$ is a face in $\Theta$, such that $\dim(F) = l$. The reduced simplicial homology of $lk_\Theta(F)$ (over a field $k$) can be computed from the chain complex
\[
0 \xrightarrow{\delta_n} \bigoplus_{F \subseteq H, \dim(H) = n-1} k_H \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_{i+2}} \bigoplus_{F \subseteq H, \dim(H) = i+1} k_H \xrightarrow{\delta_{i+1}} k_F \xrightarrow{\delta_i} 0.
\]

A simplicial complex $\Theta$ is a homology sphere (over a field $k$) if for every $F \in \Theta$ (including $F = \emptyset$) the reduced simplicial homology of $lk_\Theta(F)$ is:
\[ \tilde{H}_i(lk_\Theta(F), k) = \begin{cases} k, & \text{if } i = \dim(lk_\Theta(F)), \\ 0, & \text{otherwise}. \end{cases} \]

**Example 1.1.2.** Any simplicial complex whose topological realisation is homeomorphic to a sphere is a homology sphere. An example is the simplicial complex $\Theta$ of Example 1.1.1.

**Example 1.1.3.** The Poincaré homology sphere is an example of a manifold with the homology of a sphere that is not homeomorphic to a sphere. It can be constructed as the space
\[ \frac{SO(3)}{I}, \]
where $SO(3)$ is the group of all rotations about the origin in $\mathbb{R}^3$, and $I$ is the icosahedral group (which gives the rotational symmetries of the icosahedron), with presentation $I = \langle s, t \mid s^2, t^3, (st)^5 \rangle$. Any triangulation of the Poincaré homology sphere gives a simplicial complex that is a homology sphere.

### 1.1.1. The $f$-, $h$- and $\gamma$-polynomials.

For a $(d - 1)$-dimensional simplicial complex $\Theta$, the $f$-polynomial is a polynomial in $\mathbb{Z}[t]$ defined as follows:
\[ f(\Theta)(t) := f_0 + f_1 t + \cdots + f_d t^d, \]
where $f_i = f_i(\Theta)$ is the number of $(i - 1)$-dimensional faces of $\Theta$, (so $f_0(\Theta) = 1$). The $h$-polynomial (with the coefficient of $t^i$ denoted $h_i$) is given by
h(Θ)(t) := (1 − t)^d f(Θ) \left( \frac{t}{1 - t} \right).

When Θ is a homology sphere h(Θ) is symmetric (this famous result known as the Dehn-Sommerville relations can be found in [22]) hence it can be written uniquely in the form

\[ h(Θ)(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1 + t)^{d-2i}, \]

for some \( \gamma_i \in \mathbb{Z} \). Then the \( \gamma \)-polynomial is given by

\[ \gamma(Θ)(t) := \gamma_0 + \gamma_1 t + \cdots + \gamma_{\lfloor \frac{d}{2} \rfloor} t^{\lfloor \frac{d}{2} \rfloor}. \]

The vectors of coefficients of the \( f \)-polynomial, \( h \)-polynomial and the \( \gamma \)-polynomial of a simplicial complex are known respectively as the \( f \)-vector, \( h \)-vector and \( \gamma \)-vector. If \( A_1, ..., A_n \) are a set of simplicial complexes, then \( f(A_1 \ast \cdots \ast A_n) = f(A_1) f(A_2) \cdots f(A_n) \), and consequently \( \gamma(A_1 \ast \cdots \ast A_n) = \gamma(A_1) \gamma(A_2) \cdots \gamma(A_n) \). The notation \( \gamma(Θ) \geq 0 \) implies that the coefficients of \( \gamma(Θ) \) are all non-negative, and we say that \( \gamma(Θ) \) is non-negative.

Now that we have learnt the definition of the \( \gamma \)-polynomial we recall Conjecture 0.0.1 (Gal’s conjecture) from the introduction [17, Conjecture 2.1.7]:

**If Θ is a flag homology sphere, then \( \gamma(Θ) \geq 0 \).**

We also recall Conjecture 0.0.3 (the Nevo and Petersen conjecture) [23, Problem 6.4]:

**If Θ is a flag homology sphere then \( \gamma(Θ) \) is the \( f \)-polynomial of a flag simplicial complex.**

**Example 1.1.4.** The boundary of the \((d - 1)\)-dimensional simplex, denoted \( \delta(\Delta^{d-1}) \), consists of all proper subsets of the set \([d]\). It is a homology sphere that is not flag, and it has \( f \)-polynomial

\[ f(\delta(\Delta^{d-1})) = \sum_{i=0}^{d-1} \binom{d}{i} t^i, \]

and \( h \)-polynomial

\[ h(\delta(\Delta^{d-1})) = 1 + t + t^2 + \cdots + t^{d-1}. \]

**Hence, when \( d \geq 3 \), its \( \gamma \)-polynomial is not non-negative.**
1.2. Polytopes

A convex polytope, or polytope for short, is the convex hull of a finite set of points in $\mathbb{R}^n$. The dimension of a polytope $P$, denoted $\dim(P)$, is the dimension of its affine span. A face $F$ of a convex polytope $P$ is a set of points in $\mathbb{R}^n$ of the form

$$F = \{x \in P \mid \lambda(x) = \max\{\lambda(y) \mid y \in P\}\},$$

where $\lambda$ is a linear functional in $(\mathbb{R}^n)^*$. Every face of a polytope is itself a polytope. Zero dimensional faces are called vertices, one dimensional faces are called edges, and $d-1$ dimensional faces of a $d$-dimensional polytope are called facets.

A $d$-dimensional polytope is simple if every vertex is contained in exactly $d$ facets. If $P$ is a simple polytope, the dual simplicial complex $\Theta_P$ is the simplicial complex whose vertices are the facets of $P$, where a set of facets forms a face of $\Theta_P$ if their intersection is non-empty. Note that $\dim(\Theta_P) = \dim(P) - 1$, and that maximal faces of $\Theta_P$ correspond to vertices of $P$. More generally, $i$-dimensional faces of $\Theta_P$ correspond to $(d-i-1)$-dimensional faces of $P$.

**Example 1.2.1.** Suppose that $P$ is the $n$-dimensional cube (the convex hull of the set of all points $(x_1, ..., x_n) \in \mathbb{R}^n$ such that $x_i \in \{-1, 1\}$). Then $\Theta_P$ is the boundary of the $n$-dimensional cross polytope, denoted $\Sigma_{n-1}$. The $n$-dimensional cross polytope is the convex hull of the set of all points in $\mathbb{R}^n$ with exactly one non-zero coordinate equal to 1 or −1. The simplicial complex $\Sigma_{n-1}$ is a flag simplicial complex, with underlying graph described as follows: It has vertices $\{\pm\epsilon_1, ..., \pm\epsilon_n\}$, and $\epsilon_i$ is adjacent to all vertices other than $-\epsilon_i$, whilst $-\epsilon_i$ is adjacent to all vertices except for $\epsilon_i$, for $i = 1, ..., n$. For all $n$, $\gamma(\Sigma_{n-1}) = 1$, (see [26, Section 7.2]).

**Figure 1.** The one skeleton of $\Sigma_2$. 
1. Preliminary definitions and examples

A polytope is flag if any set of facets with pairwise non-empty intersection has non-empty intersection. Note that a simple polytope is flag if and only if its dual simplicial complex is flag.

Two polytopes are combinatorially equivalent if there is an inclusion preserving bijection between their faces. The boundary of an \( n \)-dimensional polytope is its boundary as a subset of \( \mathbb{R}^n \) with the usual topology.

If \( P_1 \) is a polytope in \( \mathbb{R}^n \) and \( P_2 \) is a polytope in \( \mathbb{R}^m \), then the product of \( P_1 \) and \( P_2 \), denoted \( P_1 \times P_2 \), is the polytope in \( \mathbb{R}^{n+m} \) that is the set of points \( \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m} \mid (x_1, \ldots, x_n) \in P_1, \ (y_1, \ldots, y_m) \in P_2 \} \).

Then \( \dim(P_1 \times P_2) = \dim(P_1) + \dim(P_2) \) and the non-empty faces of \( P_1 \times P_2 \) are the products of the non-empty faces of \( P_1 \) and the non-empty faces of \( P_2 \).

A polyhedral cone in \( \mathbb{R}^n \) is a subspace defined by a set of weak inequalities \( \lambda(x) \geq 0 \), for linear forms in \( (\mathbb{R}^n)^* \). A face of a polyhedral cone \( C \), is a subset of \( C \) defined by replacing some of the linear inequalities with the equalities \( \lambda_i(x) = 0 \). Two polyhedral cones intersect properly if their intersection is a face of each. A complete fan of cones \( F \) in \( \mathbb{R}^n \), is a collection of distinct polyhedral cones that pairwise intersect properly, that cover \( \mathbb{R}^n \), and such that if \( F \) is a face of a cone \( C \in F \) then \( F \subseteq F \). If \( F \) and \( F' \) are complete fans of cones such that every cone in \( F \) is the union of cones in \( F' \), then we say that \( F' \) refines \( F \).

**Example 1.2.2.** The braid arrangement fan is a complete fan of cones in \( \mathbb{R}^n \), in which the top dimensional cones are \( \{C_\sigma \mid \sigma \in S_n \} \), where

\[
C_\sigma := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)} \}.
\]

Since each cone is closed with respect to translation by \( \mathbb{R}(1, 1, \ldots, 1) \), we often take the braid arrangement fan to be its image in the quotient \( \mathbb{R}^n / \mathbb{R}(1, 1, \ldots, 1) \).

Let \( P \) be a convex polytope in \( \mathbb{R}^n \) and suppose that \( F \) is a face of \( P \). The normal cone to \( P \) at \( F \) is the set

\[
C_F := \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(x) = \max\{\lambda(y) \mid y \in P \}, \text{ for all } x \in F \}.
\]

For any polytope \( P \), the set of normal cones for all faces of \( P \) form a complete fan of cones, denoted \( \mathcal{N}(P) \). The cones in \( \mathcal{N}(P) \) are all simplicial
if and only if $P$ is simple. When $P$ is simple, the intersection of the unit sphere with $\mathcal{N}(P)$ gives a geometric realisation of the dual simplicial complex $\Theta_P$. Hence any dual simplicial complex $\Theta_P$ is a homology sphere.

1.2.1. The $f$-, $h$- and $\gamma$-polynomials of polytopes.

For a $d$-dimensional polytope $P$ the $f$-polynomial is a polynomial in $\mathbb{Z}[t]$ defined as follows:

$$f(P)(t) := f_0 + f_1 t + \cdots + f_d t^d,$$

where $f_i = f_i(P)$ is the number of $i$-dimensional faces of $P$ (so that $f_d(P) = 1$). The $h$-polynomial (with coefficient of $t^i$ denoted $h_i$) is given by

$$h(P)(t) = f(P)(t - 1).$$

When $P$ is a simple polytope, the Dehn-Sommerville relations guarantee that $h(P)$ is symmetric (the Dehn-Sommerville relations can be found in [22]), so that it can be written

$$h(P)(t) = \sum_{i=0}^{[\frac{d}{2}]} \gamma_i t^i (1 + t)^{d - 2i},$$

for some $\gamma_i \in \mathbb{Z}$. Then the $\gamma$-polynomial is given by

$$\gamma(P)(t) := \gamma_0 + \gamma_1 t + \cdots + \gamma_{[\frac{d}{2}]} t^{[\frac{d}{2}]}.$$  

The vectors of coefficients of the $f$-polynomial, $h$-polynomial and the $\gamma$-polynomial of a polytope are known respectively as the $f$-vector, $h$-vector and $\gamma$-vector. If $P_1, ..., P_n$ are a set of polytopes, then by the definition of the product of polytopes we have $f(P_1 \times \cdots \times P_n) = f(P_1) f(P_2) \cdots f(P_n)$, and consequently $\gamma(P_1 \times \cdots \times P_n) = \gamma(P_1) \gamma(P_2) \cdots \gamma(P_n)$. It is not too hard to see that for any simple $d$-dimensional polytope $P$ we have $t^d f(P)(t^{-1}) = f(\Theta_P)$, $h(P) = h(\Theta_P)$ and $\gamma(P) = \gamma(\Theta_P)$. The $\gamma$-polynomials of some well known polytopes are given in [26].

1.3. Generalized permutohedra and their vertex posets

For further reading on the material in this section we advise the reader to see [26, Section 3]. A poset $P$ on a finite set $S$ is a binary relation $P \subseteq S \times S$ on the set $S$ that is reflexive ($(x, x) \in P$ for all $x \in S$), antisymmetric (if $(x, y) \in P$ and $(y, x) \in P$ then $x = y$) and transitive (if $(x, y) \in P$ and $(y, z) \in P$ then $(x, z) \in P$). We use the notation $x \leq_P y$, or $y \geq_P x$ to denote that $(x, y) \in P$. If $x \leq_P y$ in $P$, and there is no $z \in S$ such that
\( x \leq_P z \leq_P y \) then \( y \) covers \( x \) and we denote this by \( x \prec_P y \), or \( y \succ_P x \).

If \( Q \subseteq S \times S \) is a binary relation on \( S \), then the \textit{opposite relation} denoted \( Q^{\text{op}} \) is the binary relation on \( S \) such that \( (x, y) \in Q^{\text{op}} \) if and only if \( (y, x) \in Q \). If \( Q \subseteq S \times S \) is a binary relation on \( S \), then the \textit{transitive closure} of \( Q \), denoted \( \overline{Q} \), is the smallest transitive binary relation on \( S \) that contains the relations of \( Q \). Suppose that \( Q \subseteq S \times S \) is a reflexive and transitive binary relation. Then \( Q_\sim \) is the poset (on a quotient of \( S \)) that is obtained from \( Q \) by identifying any \( x, y \in S \) such that \( (x, y) \) and \( (y, x) \) are in \( Q \).

If \( P \) is a poset, and \( Q \subseteq P \subseteq S \times S \) is a binary relation, then the poset \((P \cup Q^{\text{op}})_\sim \) is a contraction of \( P \). Note that if a poset \( C \) is a contraction of a poset \( P \), then \( C = (P \cup R^{\text{op}})_\sim \) where \( R \subseteq P \subseteq S \times S \) is a subset of the covering relations in \( P \). In this case, we say that \( C \) is obtained by contracting \( P \) along the relations \((a, b) \in R \).

A \textit{complete fan of posets} \( \mathcal{P} \) on \( S \) is a set of posets on \( S \) such that:

(1) if \( P, Q \in \mathcal{P} \) then \((P \cup Q)_\sim \) is a contraction of \( P \) and of \( Q \),

(2) the linear extensions of every \( P \in \mathcal{P} \) disjointly cover all total orders of \( S \).

A poset \( P \) on \( S \) can be represented by its \textit{Hasse diagram}, which is a directed graph on \( S \), with an edge from vertex \( x \) to \( y \) whenever \( x \prec_P y \). A \textit{tree poset} \( P \) on \( S \) is a poset on \( S \) whose Hasse diagram is a directed tree graph on \( S \). A \textit{rooted tree poset} \( P \) is a tree poset on \( S \) such that there is a unique element \( r \in S \) such that \( r \geq_P x \) for all \( x \in S \). The vertex \( r \) is known as the \textit{root} of \( P \). Note that a contraction of a post corresponds to a contraction of its Hasse diagram.

A \textit{generalized permutohedron} is a polytope whose normal fan is refined by the braid arrangement fan (see Example 1.2.2). Suppose that \( P \) is a generalized permutohedron of dimension \( n - 1 \) with normal fan \( \mathcal{N}(P) \). To every maximal cone \( C_{\{v\}} \in \mathcal{N}(P) \) (which corresponds to the vertex \( \{v\} \)) there is an associated poset \( Q_v \) on \([n]\) where \( i \leq_{Q_v} j \) when \( x_i \leq x_j \) holds for all \((x_1, \ldots, x_n) \in C_{\{v\}} \). The set of posets \( \{Q_v \mid v \text{ is a vertex of } P\} \) form a complete fan of posets on \([n]\). The posets are tree posets exactly when \( P \) is simple.

**Example 1.3.1.** The \((n - 1)\)-dimensional permutohedron is a well known polytope, see [18], [25], [26] and [35]. It is defined by taking a point \((a_1, \ldots, a_n) \in \mathbb{R}^n \) such that \( a_1 < a_2 < \cdots < a_n \), and taking the convex hull of all points in its orbit under the action of the symmetric group \( S_n \).
where \( \sigma(a_1, ..., a_n) = (a_{\sigma(1)}, ..., a_{\sigma(n)}) \) for \( \sigma \in S_n \). It is the polytope whose normal fan is the braid arrangement fan, hence it is a generalized permutohedron.

**Example 1.3.2.** The \( d \)-dimensional cube is a generalized permutohedron, and is described in more detail in Example 1.4.9. A description of its normal fan can be found using the theory in Section 1.4.

Suppose that \( Q \) is a tree poset on \([n]\). Then \( \text{des}(Q) \) denotes the number of descents in \( Q \), which is defined to be the number of pairs \( i, j \in [n] \) such that \( i < j \) in \([n]\), but \( i \not\geq_Q j \).

The following interesting Theorem appears in [26]:

**Theorem 1.3.3.** [26, Theorem 4.2]. Let \( P \) be a simple generalized permutohedron, with vertex posets \( \{Q_v\}_{v \in V_P} \), where \( V_P \) denote the set of vertices of \( P \). Then one has the following expression for its \( h \)-polynomial:

\[
h_P(t) = \sum_{v \in V_P} t^{\text{des}(Q_v)}.
\]

### 1.4. Building sets and nestohedra

In this section we describe nestohedra, an interesting class of simple generalized permutohedra with good combinatorial properties. Nestohedra were first defined in [25], and they are part of a more general theory developed in [12]. Further information on the theory in this section can be found in [25, Section 7] and [26, Sections 3, 4 and 6].

A building set \( B \) on a finite set \( S \) is a set of non-empty subsets of \( S \) such that

- For any \( I, J \in B \) such that \( I \cap J \neq \emptyset, I \cup J \in B \).
- \( B \) contains the singletons \([i]\), for all \( i \in S \).

\( B \) is connected if it contains \( S \). For any building set \( B \), \( B_{\text{max}} \) denotes the set of maximal elements of \( B \) with respect to inclusion. The elements of \( B_{\text{max}} \) form a disjoint union of \( S \), and if \( B \) is connected then \( B_{\text{max}} = \{S\} \).

Building sets \( B_1, B_2 \) on \( S \) are equivalent, denoted \( B_1 \cong B_2 \), if there is a permutation \( \sigma : S \to S \) that induces a one to one correspondence \( B_1 \to B_2 \).

**Example 1.4.1.** Let \( G \) be a graph with no loops or multiple edges, with \( n \) vertices labelled distinctly from \([n]\). Then the graphical building set \( B(G) \) is the set of subsets of \([n]\) such that the induced subgraph of \( G \) is connected.
1. Preliminary Definitions and Examples

(see [8], [11], [26, Sections 7 and 12] and [31]). \( B(G)_{\text{max}} \) is the set of connected components of \( G \).

Let \( B \) be a building set on \( S \) and \( I \subseteq S \). The restriction of \( B \) to \( I \) is the building set

\[
B|_I := \{ J \mid J \subseteq I, \text{ and } J \in B \}
\]

on \( I \).

The contraction of \( B \) by \( I \) is the building set

\[
B/I := \{ J - (J \cap I) \mid J \in B, J \nsubseteq I \}
\]

on \( S - I \).

Example 1.4.2. If \( G \) is a graph on \([n]\), and \( I \in B(G)\), then \( B(G)/I = B(G')\) where \( G'\) is the graph on \([n] - I\) such that any two vertices \( i, j \in [n] - I \) are adjacent if they are adjacent in \( G \), or both \( i \) and \( j \) are adjacent to vertices in \( I \) in the full graph \( G \).

Given a building set \( B \), a subset \( N \subseteq B - B_{\text{max}} \) is a nested set if it satisfies

- For any \( I, J \in N \), either \( I \subseteq J, J \subseteq I \), or \( I \cap J = \emptyset \).
- For any collection of \( k \geq 2 \) disjoint subsets \( J_1, \ldots, J_k \in N \), the union \( J_1 \cup \cdots \cup J_k \notin B \).

The nested set complex \( \Delta_B \) is the simplicial complex on \( B - B_{\text{max}} \) whose faces are the nested sets.

We will now use the term simplicial complex to describe the geometric realisation of a simplicial complex, and we will use the term simplex to describe the geometric realisation of a simplex.

We associate a polytope to a building set as follows. Let \( e_1, \ldots, e_n \) denote the endpoints of the coordinate vectors in \( \mathbb{R}^n \). Given \( I \subseteq [n] \), define the simplex \( \Delta_I := \text{ConvexHull}(e_i \mid i \in I) \). Let \( B \) be a building set on \([n]\). The nestohedron \( P_B \) is a polytope given by the Minkowski sum of the simplices \( \Delta_I \) for all \( I \in B \), i.e.

\[
P_B := \sum_{I \in B} \Delta_I.
\]

The term nestohedra is used for the plural of nestohedron. If \( B(G) \) is a graphical building set for a graph \( G \), then \( P_{B(G)} \) is known as a graph-associahedron. Nestohedra are all simple polytopes (see Theorem 1.4.10) and so each has a dual simplicial complex.

For any building set \( B \), we denote the \( \gamma \)-polynomial of \( P_B \) by \( \gamma(B) \). If \( B \) and \( B' \) are building sets, the notation \( \gamma(B) \leq \gamma(B') \) implies that for all \( i \) the coefficient of \( t^i \) in \( \gamma(B) \) is less than or equal to the coefficient of \( t^i \) in \( \gamma(B') \).
Example 1.4.3. The \((n - 1)\)-dimensional permutohedron of Example 1.3.1, with \(a_i = 2^{i-1}\), is the nestohedron for the building set consisting of every subset of \([n]\). Equivalently, it is the graph-associahedron corresponding to the complete graph on \(n\) vertices \(K_n\), hence it is denoted \(P_{\mathcal{B}(K_n)}\).

Example 1.4.4. \(\text{Path}_n\) denotes the graph that is a path with \(n\) vertices. The graph-associahedron \(P_{\mathcal{B} (\text{Path}_n)}\) is known as the associahedron. This well known polytope has a combinatorial description in terms of bracketings of a string of \(n\) letters (see [35]). The \(h\)-polynomial has been calculated (see [29], [25, Section 8.2] and [26, Sections 10.2 and 11.3]) to be

\[
h_k(\mathcal{B} (\text{Path}_n)) = N(n, k + 1),
\]

for \(k = 0, \ldots, n - 1\), where

\[
N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}
\]

is the Narayana number. The \(\gamma\)-polynomial has been calculated to be

\[
\gamma(\mathcal{B} (\text{Path}_n))(t) = \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} C_r \binom{n-1}{2r} t^r,
\]

where \(C_r\) is the \(r\)th Catalan number.

Example 1.4.5. \(\text{Cyc}_n\) denotes the graph that is a cycle with \(n\) vertices. The graph-associahedron \(P_{\mathcal{B} (\text{Cyc}_n)}\) is the well known cyclohedron (see [29], [25, Section 8.3] and [26, Sections 10.3 and 11.3]). The \(h\)-polynomial and \(\gamma\)-polynomial have been calculated to be

\[
h(\mathcal{B} (\text{Cyc}_n))(t) = \sum_{k=0}^{n} \binom{n}{k} 2^k t^k,
\]

and

\[
\gamma(\mathcal{B} (\text{Cyc}_n))(t) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{r, r, n-2r} t^r.
\]

Example 1.4.6. \(K_{1,n-1}\) is the graph with \(n\) vertices with exactly one vertex of degree \(n - 1\) and \(n - 1\) vertices of degree 1. The graph-associahedron \(P_{\mathcal{B} (K_{1,n-1})}\) is known as the stellohedron, see [26, Section 10.4]. The \(h\)-polynomial has been calculated to be

\[
h(\mathcal{B} (K_{1,n-1}))(t) = 1 + \sum_{r=1}^{m} \binom{m}{r} \sum_{k=1}^{r} A(r, k) t^k,
\]
where $A(r, k)$ denotes the number of permutations of $[r]$ with $k$ descents. If $w = w(1)w(2)\ldots w(n)$ is a permutation of $[r]$, then the number of descents of $w$, is the number of positions $i$ (for $1 \leq i \leq r - 1$) such that $w(i) > w(i + 1)$.

A building set $\mathcal{B}$ is flag if $P_\mathcal{B}$ is flag. Proposition 1.4.7 below, which is proven using Lemma 3.1.3, determines whether a building set is flag. It follows from this that a graphical building set is flag.

**Proposition 1.4.7.** A building set $\mathcal{B}$ is flag if and only if for every non-singleton $I \in \mathcal{B}$, there exist two elements $D_1, D_2 \in \mathcal{B}$ such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = I$.

**Proof.** The proof is immediate from Lemma 3.1.3, and its following discussion. \qed

**Example 1.4.8.** The nestohedron for the minimal connected building set on $[n]$, i.e. $\mathcal{B} = \{\{1\}, \{2\}, \ldots, \{n\}, \{n\}\}$, is $\Delta^{n-1}$.

**Example 1.4.9.** Let $\mathcal{D}$ be the building set $\{\{1\}, \{2\}, \ldots, \{n\}, \{2\}, \{3\}, \ldots, \{n\}\}$. Then $P_\mathcal{D}$ is combinatorially equivalent to an $(n - 1)$-dimensional cube. $\mathcal{D}$ is an example of a connected minimal flag building set on $[n]$. Minimal connected flag building sets are described in more detail in Section 3.1, and all their nestohedra are combinatorially equivalent to cubes. As mentioned in Example 1.2.1, for any minimal connected flag building set $\mathcal{D}$ on $[n]$, $\gamma(\mathcal{D}) = 1$.

For any building set $\mathcal{B}$, the nestohedron $P_\mathcal{B}$ is related to the nested sets of $\mathcal{B}$ as described in the following theorem.

**Theorem 1.4.10.** [25, Theorem 7.4], [12, Theorem 3.14]. Let $\mathcal{B}$ be a building set on $[n]$. The nestohedron $P_\mathcal{B}$ is a simple polytope of dimension $n - |\mathcal{B}_{\text{max}}|$. Also, the dual simplicial complex $\Theta_{P_\mathcal{B}}$ is equivalent to $\Delta_\mathcal{B}$.

The following construction is due to Erokhovets [11]. Let $[i, j]$ denote the interval $\{i, i + 1, \ldots, j\}$. Let $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ be connected building sets on $[n], [k_1], \ldots, [k_n]$ respectively, and let $[k_i]$ denote the interval $[\sum_{j=1}^{i-1} k_j + 1, \sum_{j=1}^{i} k_j]$. Define the connected building set $\mathcal{B}[\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n]$ on $[k_1 + k_2 + \cdots + k_n]$, where $\mathcal{B}|_{[k_i]}$ is equivalent to $\mathcal{B}_i$, and add the elements $[k_1] \cup [k_2] \cup \cdots \cup [k_m]$ for every $\{i_1, i_2, \ldots, i_m\} \in \mathcal{B}$.

**Lemma 1.4.11.** [11, Erokhovets]. Let $\mathcal{B}, \mathcal{B}_1, \ldots, \mathcal{B}_n$ be connected building sets on $[n], [k_1], \ldots, [k_n]$ respectively. Let $\mathcal{B}' = \mathcal{B}[\mathcal{B}_1, \ldots, \mathcal{B}_n]$. Then $P_{\mathcal{B}'}$ is combinatorially equivalent to $P_\mathcal{B} \times P_{\mathcal{B}_1} \times \cdots \times P_{\mathcal{B}_n}$. 

Example 1.4.12. Let $B$ be the building set $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{3\}\}$ and let $B_1 = \{\{1\}\}, B_2 = \{\{1\}, \{2\}\}$ and $B_3 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{4\}\}$. Then

$$B[B_1, B_2, B_3]$$

is the building set on $[7]$ given by

$$\{\{1\}, \ldots, \{7\}, \{2, 3\}, \{1, 2, 3\}, \{5, 7\}, \{6, 7\}, \{5, 6, 7\}, \{4, 5, 6, 7\}, \{7\}\}.$$

1.5. Normal fans of nestohedra

We will now describe the normal fan $\mathcal{N}(P_B)$ of a nestohedron in more detail. First we require some background definitions from [26, Section 8.1]. Suppose that $T$ is a rooted tree poset on $[n]$. For any $i \in [n]$ we let $T_{\leq i}$ denote the set of all elements of $[n]$ that are less than or equal to $i$ in $T$. Let $B$ be a connected building set on $[n]$. Then a rooted tree poset $T$ on $[n]$ is a $B$-tree if:

a) For any $i \in [n]$ we have $T_{\leq i} \in B$.

b) For $k \geq 2$ pairwise incomparable vertices $i_1, \ldots, i_k$ of $T$, we have

$$\bigcup_{j=1}^{k} T_{\leq i_j} \not\in B.$$

If $T$ is a $B$-tree, then $\{I \subseteq [n] \mid I = T_{\leq i}$ for some $i \in [n]\}$ is a maximal nested set in $B$. The set of $B$-trees for any connected building set $B$ form the complete fan of tree posets corresponding to the complete fan of cones $\mathcal{N}(P_B)$ (see Section 1.3). This correspondence defines the equivalence of $\Theta_{P_B}$ and $\Delta_B$ mentioned in Theorem 1.4.10. Hence the normal fan of a nestohedron is a complete fan of rooted tree posets. The following propositions give an interesting relation between the contraction of a building set $B$ and the the set of $B$-trees.

Proposition 1.5.1. Suppose $B$ is a connected building set on $[n]$, and $i \in [n]$. Then the set of $B/\{i\}$-trees is the set of trees obtainable by contracting a $B$-tree along an edge $(i, j)$ such that $j$ is a child vertex of $i$, and labelling the identified vertices as $j$, or obtainable by removing the vertex $i$ of a $B$-tree for which $i$ is a leaf.

Proof. Suppose that $T$ is a $B$-tree. If the vertex $i$ has a child vertex $j$ and we contract along the relation $(i, j)$, or the vertex $i$ is a leaf and we remove $i$, then the resulting tree, which we denote by $\tilde{T}$, is a $B/\{i\}$-tree. This is true since the set $\{\tilde{T}_{\leq k} \mid k \in [n] - \{i\}\}$ is equal to the set $S = \{I - (I \cap \{i\}) \mid I = T_{\leq k}$ for some $k \in [n]\}$. The set $S$ is a maximal nested set in $B/\{i\}$. To see
this, note that it has cardinality $n - 1$, hence is maximal. It is straightforward to show that the first property for nested sets is satisfied. To show that the second property for nested sets holds, suppose for a contradiction that $S_1, ..., S_k$ ($k \geq 2$) are a set of pairwise disjoint elements in $S$, whose union is in $B/\{i\}$, such that no proper subset of size two or more of $S_1, ..., S_k$ has union in $B/\{i\}$. Note that there can be at most one element $S_m$ in this set such that $S_{m} \cup \{i\} \in B$. Suppose that for some $m$, $S_{m} \cup \{i\} = T_{\leq l}$ for some $l$. Then we must have $S_1 \cup \cdots \cup S_k \cup \{i\} \in B$. This contradicts that $S_1, ..., S_k$ is a nested set in $B$. If there is no index $m$ such that $S_{m} \cup \{i\} = T_{\leq l}$ for some $l$, then $S_1, ..., S_k$ are not a nested set in $B$ (since their union is in $B$), a contradiction.

It is also true that any $B/\{i\}$-tree can be obtained this way. For suppose that $N$ is a maximal nested set in $B/\{i\}$. Then the set $P = \{J \in B \mid J \in N, \ or \ J - \{i\} \in N\} \cup \{i\}$ is a maximal nested set in $B$. To show the first condition of a nested set is satisfied, suppose for a contradiction that $P_1$ and $P_2$ are sets in $P$ such that $P_1 \cap P_2 \neq \emptyset, P_1 \not\subseteq P_2$, and $P_2 \not\subseteq P_1$. Then $P_1 \cup P_2 \in B$, which implies that $P_1 \cup P_2 - \{i\} \in B/\{i\}$. This implies that $P_1 - \{i\}$ and $P_2 - \{i\}$ which are both elements of $N$, are disjoint, but their union is in $B/\{i\}$, a contradiction. To show that the second condition for a nested set is satisfied, suppose for a contradiction that $P_1, ..., P_k$ ($k \geq 2$) are disjoint subsets of $P$ such that $P_1 \cup \cdots \cup P_k \in B$. Then $P_1 - \{i\}, ..., P_k - \{i\}$ are a disjoint set of subsets of $N$ whose union is in $B/\{i\}$, a contradiction. The set $P$ must be a maximal nested set since it contains at least $n$ elements. The $B/\{i\}$-tree that corresponds to $N$ can be obtained from the $B$-tree that corresponds to $P$ by the moves described in this proposition.

**Proposition 1.5.2.** Suppose $B$ is a connected building set on $[n]$, and that $I \subseteq [n]$. Then the set of $B/I$-trees is the set of trees obtainable from a $B$-tree $T$ by sequentially doing the following for each $i \in I$:

- contract $T$ along an edge $(i, j)$ such that $j$ is a child vertex of $i$, and labelling the identified vertices by $j$, or
- removing the vertex $i$, if $i$ is a leaf of $T$.

**Proof.** This follows easily from Proposition 1.5.1. For suppose that $I = \{i_1, ..., i_s\}$. Then if we let $B^{i_1}$ denote the building set $B/\{i_1\}$, and for each $j \in \{2, ..., s\}$ let $B^{i_j}$ denote $B^{i_{j-1}}/\{i_j\}$, then $B/I = B^{i_s}$. Since $B$ can be obtained by these sequential contractions, this is clearly how one obtains $B/I$-trees.

Postnikov, Reiner and Williams pose the following interesting question in [26]:
Question 1.5.3. [26, Question 8.3]. Does a simple (indecomposable) generalized permutohedron $P$ come from a (connected) building set if and only if every poset $Q_v$ is a rooted tree, i.e. has a unique maximal element?

We answer this by proving a more general statement:

Theorem 1.5.4. Suppose $\mathcal{P}$ is a complete fan of rooted tree posets. Then for some connected building set $\mathcal{B}$ on $[n]$, $\mathcal{P}$ is the complete fan of posets corresponding to the complete fan of cones $\mathcal{N}(\mathcal{P}_B)$.

To prove this we first we give some new definitions. Let $T$ be a rooted tree poset. A lower ideal of $T$ is the union $\bigcup_{k=1}^k T_{\leq i_k}$, where $i_1, \ldots, i_k$ are a set of pairwise incomparable vertices in $T$. A lower ideal $I$ of $T$ is connected if it is equal to $T_{\leq i}$ for some $i \in [n]$. An upper ideal of $T$ is the complement (as a subset of $[n]$) of a lower ideal of $T$. If $\mathcal{P}$ is a complete fan of tree posets on $[n]$, let $\mathcal{S}_\mathcal{P}$ denote the following set

$$\mathcal{S}_\mathcal{P} := \{I \subseteq [n] \mid I = T_{\leq i} \text{ for some } T \in \mathcal{P} \text{ and some } i \in [n]\}.$$

Suppose that $\mathcal{P}$ is a poset on $S$, $R$ is a binary relation on $S$, and $\alpha \in S$, then we let $\bar{\alpha}$ denote the image of $\alpha$ in the poset $(\mathcal{P} \cup \bar{R})_{\sim}$.

Proposition 1.5.5. Suppose that $\mathcal{P}$ is a complete fan of rooted tree posets on $[n]$, and that $I \in \mathcal{S}_\mathcal{P}$. Then there is no tree $T \in \mathcal{P}$ such that $I$ is a lower ideal of $T$ that is not connected.

Proof. Let $T_I \in \mathcal{P}$ be a tree such that $I = (T_I)_{\leq i}$ for some $i \in [n]$. Suppose for a contradiction that $T \in \mathcal{P}$ is a tree that contains $I$ as a lower ideal that is not connected. Since $(T_I \cup T)_{\sim}$ is a contraction of $T_I$, the set $\{\bar{\alpha} \mid \alpha \in I\}$ is a connected lower ideal of $(T_I \cup T)_{\sim}$. So there exist $\alpha$ and $\beta$ in $I$ such that

- there is no $\gamma \in I$, such that both $\gamma \geq_T \alpha$ and $\gamma \geq_T \beta$,
- $\bar{\alpha} \geq_{(T_I \cup T)_{\sim}} \bar{\beta}$.

The relation $\bar{\alpha} \geq_{(T_I \cup T)_{\sim}} \bar{\beta}$ can only be obtained if there is some $\mu \in [n] - I$ and $\nu \in I$ such that $\mu \succ_T \nu$ and $T$ is contracted along $(\nu, \mu)$ to obtain $(T_I \cup T)_{\sim}$. This is a contradiction since in both $T$ and $T_I$ the elements of $[n] - I$ are either incomparable to, or greater than elements in $I$.

$\square$

Proposition 1.5.6. Suppose that $\mathcal{P}$ is a complete fan of rooted tree posets on $[n]$. Then $\mathcal{S}_\mathcal{P}$ is a connected building set on $[n]$.

Proof. The set $\mathcal{S}_\mathcal{P}$ clearly contains $[n]$ and the singletons $\{i\}$ for all $i \in [n]$. Suppose that $I, J \in \mathcal{S}_\mathcal{P}$ and that $I \cap J \neq \emptyset$. We will show that $I \cup J \in \mathcal{S}_\mathcal{P}$.
We let $T_I$ denote a tree in $\mathcal{P}$ that contains both $I$ and $I \cup J$ as lower ideals. Such a tree must exist in $\mathcal{P}$ since $\mathcal{P}$ is a complete fan of tree posets and so requires an element with a linear extension that begins with the elements of $I$, followed by the elements of $J - I$. By Proposition 1.5.5, $I$ must be a connected lower ideal in $T_I$. Similarly we have a tree $T_J$ in $\mathcal{P}$ that has $I \cup J$ as a lower ideal and has $J$ as a connected lower ideal. Suppose for a contradiction that $I \cup J$ is not a connected lower ideal of $T_J$ (or equivalently of $T_I$). We suppose that the root of the subtree of $T_I$ consisting of the vertices in $I$ is labelled by $\gamma$. Since $(T_I \cup T_J)_{\sim}$ is a contraction of $T_I$, there are vertices $\bar{\alpha}$, $\bar{\beta}$ in $I$ such that $\bar{\alpha} \geq \bar{\gamma}$ in the tree $(T_I \cup T_J)_{\sim}$, $\bar{\beta} \geq \bar{\gamma}$, (i.e. $\alpha$ and $\beta$ are in two disjoint lower ideals of $T_J$ that each consist of elements of $I$).

Then we must have either $\alpha \geq \bar{\gamma}$ or $\beta \geq \bar{\gamma}$ in the tree $(T_I \cup T_J)_{\sim}$. This can only be obtained by contracting $T_J$ along a relation $\nu \preceq T_J \mu$ for some $\nu \in I \cup J$ and some $\mu \in [n] - (I \cup J)$ (since contractions can be obtained by contracting along covering relations). This is a contradiction since in both $T_I$ and $T_J$ the elements of $[n] - (I \cup J)$ are greater than or incomparable to the elements in $I \cup J$. $\square$

**Corollary 1.5.7.** Suppose that $\mathcal{P}$ is a complete fan of rooted tree posets, and $T \in \mathcal{P}$. Then $T$ is an $S_\mathcal{P}$-tree for the building set $S_\mathcal{P}$.

**Proof.** By definition, for any $i \in [n]$ we have $T_{<i} \in S_\mathcal{P}$, so that the first condition for $T$ to be an $S_\mathcal{P}$-tree is satisfied. By Proposition 1.5.5 the second condition for $T$ to be an $S_\mathcal{P}$-tree is satisfied. $\square$

Theorem 1.5.4 clearly follows from Corollary 1.5.7.

### 1.6. Stellar subdivisions of simplicial complexes

The *stellar subdivision*, or *subdivision*, of a simplicial complex $\Theta$ in the face $F$ is the simplicial complex $\Theta'$ given by:

- $\Theta'$ has vertices $V_{\Theta'} = V_\Theta \cup \{s\}$ where $s \not\in V_\Theta$,

- $\Theta'$ contains all sets in $\Theta$ that do not include $F$, and does not contain any set $K \in \Theta$ such that $F \subseteq K$,

- $\Theta'$ contains sets $\tau \cup \{s\}$ for all $\tau \in \Theta$ such that $F \not\subseteq \tau$, and $\tau \cup F \in \Theta$. 

If $F$ is a simplex, we denote by $F^o$ the stellar subdivision of $F$ in the face $F$. If $\Theta$ is a simplicial complex, then we denote by $sd(\Theta)$ the set of simplicial complexes that can be obtained from $\Theta$, by a sequence of edge subdivisions. Note that at each step the edge subdivided need not be an edge of the original simplicial complex $\Theta$.

If $P$ is a $d$-dimensional polytope with face $F$ then the face shaving of $P$ in $F$ is an operation which produces a new polytope $P'$ defined as follows: Let $\lambda \in (\mathbb{R}^d)^*$ be a linear functional such that $\lambda(x) > 0$ for every vertex $x$ of $F$, and $\lambda(y) < 0$ for every vertex $y$ of $P$ not in $F$. Then $P'$ is the set

$$P' := \{x \in \mathbb{R}^d \mid x \in P \text{ and } \lambda(x) \leq 0\}.$$  

To understand more about the definition of polytopes by linear functionals see [35]. If $P$ is a simple polytope, then shaving a face $F$ of codimension $k + 1$ is equivalent to stellar subdividing the corresponding face of dimension $k$ in the dual simplicial complex.

Claim 1.6.1. (See [9, Section 5.3.2] and [17, Proposition 2.4.6]). Suppose $\Theta$ is a flag simplicial complex, and $\Theta'$ is obtained from $\Theta$ by subdivision in an edge $S$. Then $\Theta'$ is a flag simplicial complex.

Proof. Consider a set $L$ of vertices of $\Theta'$ such that any pair of vertices in $L$ is in $\Theta$. We will show that $L \in \Theta'$.

If $s \not\in L$ then every two element subset of $L$ does not include $s$, and so they were all in $\Theta$. This implies that $L$ was in $\Theta$ and since $S \not\subseteq L$ this implies that $L \in \Theta'$.

Suppose that $s \in L$. Let $\tau$ denote $L/\{s\}$. Then all two element sets in $\tau$ are all in $\Theta$ so that $\tau \in \Theta$. Since $\{s\} \cup \{v\} \in \Theta'$ for all $v \in \tau$ this implies that $\{v\} \cup S \in \Theta$ for all $v \in \tau$. This implies that $\tau \cup S \in \Theta$ since $\Theta$ is flag, and hence that $L \in \Theta'$. □

Example 1.6.2. The set of simplicial complexes that can be obtained by stellar subdivisions of the boundary of the $(d-1)$-simplex $\delta(\Delta^{d-1})$, denoted $sd(\delta(\Delta^{d-1}))$, includes the dual simplicial complexes to all $d$-dimensional nestohedra (see [13, Theorem 4]). The set of simplicial complexes that can be obtained by stellar subdivisions in edges of $\Sigma_{d-1}$, denoted $sd(\Sigma_{d-1})$, includes all flag nestohedra (see [31, Lemma 6]).

Corollary 1.6.3. The simplicial complexes in the set $sd(\Sigma_{d-1})$ are flag homology spheres.
**Proof.** The simplicial complexes in $sd(\Sigma_{d-1})$ are flag by Claim 1.6.1. They are homology spheres since stellar subdivisions do not change the topology of the simplicial complex.\[\square\]

**Lemma 1.6.4.** [17, Proposition 2.4.3]. Suppose $\Theta'$ is a flag homology sphere obtained from a flag homology sphere $\Theta$ (of dimension $d-1$) by stellar subdividing an edge $S$. Then
\[\gamma(\Theta') - \gamma(\Theta) = t\gamma(lk_\Theta(S)).\]

**Proof.** If we stellar subdivide a face $F$ in a simplicial complex $\Theta$ to obtain $\Theta'$, the change in the $f$-polynomial is
\[f(\Theta') - f(\Theta) = f(F^\circ \ast lk_\Theta(F)) - f(F \ast lk_\Theta(F)),\]
since the set of faces in $\Theta - \Theta'$ is $F \ast lk_\Theta(F)$ and the set of faces in $\Theta' - \Theta$ is $F^\circ \ast lk_\Theta(F)$.

In general for simplicial complexes $A$ and $B$ we have
\[f(A \ast B) = f(A)f(B).\]
Hence
\[f(\Theta') - f(\Theta) = f(lk_\Theta(S))[f(S^\circ) - f(S)] = f(lk_\Theta(S))[1 + 3t + 2t^2 - (1 + 2t + t^2)] = f(lk_\Theta(S))[t(1 + t)].\]

Then
\[h(\Theta') - h(\Theta) = (1 - t)^d f(lk_\Theta(S)) \left(\frac{t}{1 - t}\right)^2 \left[1 + \frac{t}{1 - t}\right] = (1 - t)^d f(lk_\Theta(S)) \left(\frac{t}{1 - t}\right)^2 \left[\frac{t}{1 - t}\right] = t(1 - t)^d f(lk_\Theta(S)) \left(\frac{t}{1 - t}\right) = th(lk_\Theta(S)).\]

So
\[\gamma(\Theta') \left(\frac{t}{(1 + t)^2}\right) - \gamma(\Theta) \left(\frac{t}{(1 + t)^2}\right) = \frac{t}{(1 + t)^2} \gamma(lk_\Theta(S)) \left(\frac{t}{(1 + t)^2}\right).\]

The result follows. \[\square\]

Suppose that a flag homology sphere $\tilde{\Theta}$ is obtained from a flag homology sphere $\Theta$ by stellar subdivision in the edge $S = \{s_a, s_b\}$ and that the vertex in $V_{\tilde{\Theta}} - V_\Theta$ is labelled $w$. Then the faces of $\tilde{\Theta}$ are in one of the following five sets:
\[ F_1 := \{ F \in \tilde{\Theta} \mid s_a \text{ or } s_b \in F, \text{ and } w \notin F \}, \]

\[ F_2 := \{ F \in \tilde{\Theta} \mid s_a \text{ or } s_b \in F, \text{ and } w \in F \}, \]

\[ F_3 := \{ F \in \tilde{\Theta} \mid s_a, s_b \notin F, \text{ and } w \in F \}, \]

\[ F_4 := \{ F \in \tilde{\Theta} \mid s_a, s_b, w \notin F, \text{ and } \{ w \} \in \text{lk}_{\tilde{\Theta}}(F) \}, \]

\[ F_5 := \{ F \in \tilde{\Theta} \mid s_a, s_b, w \notin F, \text{ and } \{ w \} \notin \text{lk}_{\tilde{\Theta}}(F) \}. \]

Then it is not too hard to show the following:

1. If \( F \in F_1 \) then
   \[ \text{lk}_{\tilde{\Theta}}(F) \cong \text{lk}_{\Theta}(F). \]
   If \( s_a \in F \) then the vertex \( w \) in \( \text{lk}_{\tilde{\Theta}}(F) \) replaces the vertex \( s_b \) in \( \text{lk}_{\Theta}(F) \), and if \( s_b \in F \) then the vertex \( w \) in \( \text{lk}_{\tilde{\Theta}}(F) \) replaces the vertex \( s_a \) in \( \text{lk}_{\Theta}(F) \). Otherwise the links are identical.

2. If \( F \in F_2 \), then
   \[ \text{lk}_{\tilde{\Theta}}(F) = \text{lk}_{\Theta}(F - \{ w \} \cup \{ s_b \}) \]
   if \( s_a \in F \), or
   \[ \text{lk}_{\tilde{\Theta}}(F) = \text{lk}_{\Theta}(F - \{ w \} \cup \{ s_a \}) \]
   if \( s_b \in F \).

3. If \( F \in F_3 \) then
   \[ \text{lk}_{\tilde{\Theta}}(F) = \text{lk}_{\Theta}(F - \{ w \} \cup S) \ast \Sigma_0, \]
   with \( s_a, s_b \) being the vertices of \( \Sigma_0 \) (recall that \( \Sigma_0 \) denotes a 0-sphere, or the boundary of a 1-dimensional cross polytope).

4. If \( F \in F_4 \) then \( \text{lk}_{\tilde{\Theta}}(F) \) is the stellar subdivision of \( \text{lk}_{\Theta}(F) \) in \( S \).

5. If \( F \in F_5 \) then \( \text{lk}_{\tilde{\Theta}}(F) = \text{lk}_{\Theta}(F) \).
We have mentioned these five sets because they are frequently referred to throughout the thesis. The following Corollary follows from these observations:

**Corollary 1.6.5.** Suppose that \( F \in \Theta \) for some \( \Theta \in sd(\Sigma_{d-1}) \). Then \( lk_\Theta(F) \in sd(\Sigma_{d-1-|F|}) \).
Chapter 2

The gamma-vector of edge subdivisions of the boundary of the cross polytope

In this chapter we present the work in [2]. This paper gives a partial solution to the Nevo Petersen conjecture (Conjecture 0.0.3), given here as Theorem 2.0.6. The conjecture is proven for a subclass of flag homology spheres, namely those that can be obtained by subdividing the boundary of the cross polytope in edges (the set $sd(\Sigma_{d-1})$). For any flag simplicial complex $\Theta$ in $sd(\Sigma_{d-1})$, we define a flag simplicial complex $\Gamma(\Theta)$ (dependent on the sequence of subdivisions) whose $f$-vector is the $\gamma$-vector of $\Theta$. In particular, this partially proves the weaker Conjecture 0.0.2, as it shows that the $\gamma$-vector of any $\Theta \in sd(\Sigma_{d-1})$ satisfies the Frankl-Füredi-Kalai inequalities.

Here is a summary of the contents of this Chapter. Section 2.1 contains specific definitions used in the paper [2], as well as some propositions relating to them. Section 2.2 contains the proof of Theorem 2.0.6, which is the main theorem in [2]:

**Theorem 2.0.6.** Suppose that $\Theta \in sd(\Sigma_{d-1})$. Then there is a flag simplicial complex $\Gamma(\Theta)$ such that $f(\Gamma(\Theta)) = \gamma(\Theta)$.

In addition to proving this theorem, we show in Section 2.3 that the Nevo and Petersen conjecture holds for simple graphic zonotopes. However this section is short since this result is fairly trivial.

After this work was completed I learnt that V.D. Volodin has independently proved Theorem 2.0.6. His proof was announced in [33] and appeared in [32].

2.1. Subdivision sequences

For the purposes of this argument, say that a subdivision sequence is a sequence of simplicial complexes

$$(\Theta^0, \Theta^1, ..., \Theta^k)$$
where $\Theta^0$ is equivalent to $\Sigma_{d-1}$ for some $d$ and each $\Theta^i$ ($i = 1, \ldots, k$) is obtained from $\Theta^{i-1}$ by subdividing an edge. (Not up to equivalence, but literally, so the set of vertices of $\Theta^i$ consists of the set of vertices of $\Theta^{i-1}$ together with one new vertex). Note that the edge that gets subdivided is determined by the sequence. Call $\Theta^k$ the result of the subdivision sequence. For $i = 1, 2, \ldots, k$ we label the unique vertex of $\Theta^i$ that is not contained in $\Theta^{i-1}$ by $w_i$, so that $V_{\Theta^k} - V_{\Theta^0} = \{w_1, w_2, \ldots, w_k\}$.

Given a subdivision sequence $(\Theta^0, \ldots, \Theta^k)$, and a face $F$ of $\Theta^k$, there is an induced subdivision sequence

$$(\Phi^0(F), \ldots, \Phi^{l_F}(F))_{(\Theta^0, \ldots, \Theta^k)}$$

where $l_F \leq k$ that we describe next, whose result $\Phi^{l_F}(F)$ is the simplicial complex $lk_{\Theta^k}(F)$. If the subdivision sequence $(\Theta^0, \ldots, \Theta^k)$ is clear we abbreviate this to the notation $(\Phi^0(F), \ldots, \Phi^{l_F}(F))$.

The definition of the induced subdivision sequence is inductive on $k$. If $k = 0$ the subdivision sequence $(\Theta^0)$ consists of a single simplicial complex equivalent to $\Sigma_{d-1}$, so that for all $F \in \Sigma_{d-1}$, $lk_{\Theta^0}(F)$ is equivalent to $\Sigma_{d-1} - |F|$. Hence we define the induced sequence to have no subdivisions and set $\Phi^0(F) = lk_{\Theta^0}(F)$.

If $k \geq 1$, we assume by induction on $k$ that there is an induced subdivision sequence

$$(\Phi^0(F), \ldots, \Phi^{j_F}(F))_{(\Theta^0, \ldots, \Theta^{k-1})}$$

for all faces $F \in \Theta^{k-1}$ whose result is $lk_{\Theta^{k-1}}(F)$. Then for any face $F \in \Theta^k$ we consider which of the five sets $F$ lies in (again we suppose the last edge to be subdivided is $S = \{s_a, s_b\}$). Then the subdivision sequence

$$(\Phi^0(F), \ldots, \Phi^{l_F}(F))_{(\Theta^0, \ldots, \Theta^k)}$$

is defined to be (note that the sets $\mathcal{F}_1, \ldots, \mathcal{F}_5$ that we refer to are defined in Section 1.6):

1. If $F \in \mathcal{F}_1$ then $l_F = j_F$ and the simplicial complexes of the induced subdivision sequence $(\Phi^0(F), \ldots, \Phi^{j_F}(F))_{(\Theta^0, \ldots, \Theta^k)}$ are equivalent to the simplicial complexes of $(\Phi^0(F), \ldots, \Phi^{j_F}(F))_{(\Theta^0, \ldots, \Theta^{k-1})}$. The map on the vertices is the identity, except that $w_k$ replaces $s_a$ or $s_b$ if either is contained in the sequence. In this case, since $F \in \Theta^{k-1}$, we are giving $lk_{\Theta^k}(F)$ (up to equivalence) the subdivision sequence that is given for $lk_{\Theta^{k-1}}(F)$. 
(2) If $F \in \mathcal{F}_2$ and $s_a \in F$ then $l_F = j_{F - \{w_k\} \cup \{s_b\}}$ and the subdivision sequence

$$(\Phi^0(F), ..., \Phi^{l_F}(F))_{(\Theta^0, ..., \Theta^k)}$$

is equal to the subdivision sequence

$$(\Phi^0(F - \{w_k\} \cup \{s_b\}), ..., \Phi^{j_{F - \{w_k\} \cup \{s_b\}}}(F - \{w_k\} \cup \{s_b\}))_{(\Theta^0, ..., \Theta^{k-1})}.$$  

If $s_b \in F$ then the same statements hold with $s_a$ in place of $s_b$. Recalling that $lk_{\Theta^k}(F) = lk_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\})$, we see that we are adopting the subdivision sequence of $lk_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\})$.

(3) If $F \in \mathcal{F}_3$ then $l_F = j_{F - \{w_k\} \cup S}$, and $(\Phi^0(F), ..., \Phi^{l_F}(F))_{(\Theta^0, ..., \Theta^k)}$ is the suspension of the subdivision sequence

$$(\Phi^0(F - \{w_k\} \cup S), ..., \Phi^{j_{F - \{w_k\} \cup S}}(F - \{w_k\} \cup S))_{(\Theta^0, ..., \Theta^{k-1})},$$

meaning that $\Phi^i(F) = \Phi^i(F - \{w_k\} \cup S) * \Sigma_0$. The vertices of $\Sigma_0$ are labelled $s_a$ and $s_b$.

(4) If $F \in \mathcal{F}_4$, then $l_F = j_{F+1}$, and the first $l_F - 1$ simplicial complexes of $(\Phi^0(F), ..., \Phi^{l_F}(F))_{(\Theta^0, ..., \Theta^k)}$ are equal to the simplicial complexes of the induced subdivision sequence $(\Phi^0(F), ..., \Phi^{l_F - 1}(F))_{(\Theta^0, ..., \Theta^{k-1})}$, and $\Phi^{l_F}(F)$ is the subdivision of $\Phi^{l_F - 1}(F)$ in the edge $S$. Recall that in this case $lk_{\Theta^k}(F)$ is the subdivision of $lk_{\Theta^{k-1}}(F)$ in the edge $S$.

(5) If $F \in \mathcal{F}_5$, then $l_F = j_F$ and $(\Phi^0(F), ..., \Phi^{l_F}(F))_{(\Theta^0, ..., \Theta^k)}$ is equal to the subdivision sequence $(\Phi^0(F), ..., \Phi^{j_F}(F))_{(\Theta^0, ..., \Theta^{k-1})}$.

**Example 2.1.1.** Let the subdivision sequence $(\Sigma_3, \Theta^1, \Theta^2, \Theta^3)$ be obtained by:

**Step 1:** subdivide the edge $\{\epsilon_1, \epsilon_2\}$, to obtain the new vertex $w_1$.
**Step 2:** subdivide the edge $\{\epsilon_3, \epsilon_4\}$, to obtain the new vertex $w_2$.
**Step 3:** subdivide the edge $\{\epsilon_1, w_2\}$ to obtain the new vertex $w_3$.

We denote the face $\{\epsilon_1, -\epsilon_2\}$ by $F$. Then $lk_{\Theta^3}(F)$ is a cycle with five vertices as illustrated in Figure 1 (a). The induced subdivision sequence

$$(\Phi^0(F), ..., lk_{\Theta^3}(F))_{(\Sigma_3, \Theta^1, \Theta^2, \Theta^3)}$$

is given by replacing the vertex $w_2$ by $w_3$ in the subdivision sequence

$$(\Phi^0(F), ..., lk_{\Theta^2}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}.$$
$l_{k_{\Theta^3}}(F)$ is the cycle illustrated in Figure 1 (b). The subdivision sequence 
$(\Phi^0(F), ..., l_{k_{\Theta^3}}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}$ is equal to $(\Sigma_1, l_{k_{\Theta^3}}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}$ in which 
$\Sigma_1$ is illustrated in Figure 1 (c). Hence the subdivision sequence 
$(\Phi^0(F), ..., l_{k_{\Theta^3}}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}$ is equal to 
$(\Sigma_1, l_{k_{\Theta^3}}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}$ in which 
$\Sigma_1$ is illustrated in Figure 1 (c). Hence the subdivision sequence 
$(\Phi^0(F), ..., l_{k_{\Theta^3}}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}$ is equal to 
$(\Sigma_1, l_{k_{\Theta^3}}(F))_{(\Sigma_3, \Theta^1, \Theta^2)}$.

**Figure 1.** (a) : $l_{k_{\Theta^3}}(F)$, (b) : $l_{k_{\Theta^3}}(F)$, (c) : $\Sigma_1$.

When $F = \emptyset$ it is obvious by induction on $k$ that the induced subdi-
vision sequence $(\Phi^0(\emptyset), ..., \Phi^k(\emptyset))$ coincides with the subdivision sequence 
$(\Theta^0, ..., \Theta^k)$, since $\emptyset$ is a face in $F_4$.

Given the above induced subdivision sequence 
$(\Phi^0(F), ..., \Phi^l(F))_{(\Theta^0, ..., \Theta^k)}$
define the sets 
$W_{(\Theta^0, ..., \Theta^k)}(F) := V_{\Phi^l(F)} - V_{\Phi^0(F)}$.

When the subdivision sequence is clear from the context we denote this set 
by $W_{\Theta^k}(F)$. We label by $w_{1,F}, w_{2,F}, ..., w_{l,F}$ the vertices of $W_{\Theta^k}(F)$ where 
$w_i,F$ for $i = 1, ..., l$ is the unique vertex in $\Phi^i(F)$ that is not contained in 
$\Phi^{i-1}(F)$. With this notation we have $w_{j,F} = w_j$ for $j = 1, 2, ..., k$. We order 
the sets $W_{\Theta^k}(F), F \in \Theta^k$, by stipulating that if $i < j$ then $w_{i,F} < w_{j,F}$.

**Proposition 2.1.2.** Suppose $(\Theta^0, ..., \Theta^k)$ is a subdivision sequence. For any 
face $F \in \Theta^k$, the set $W_{\Theta^k}(F)$ satisfies one of the following relations:

1. If $F \in F_1$ and $s_a \in F$ then $W_{\Theta^k}(F)$ is equal to $W_{\Theta^{k-1}}(F)$ except 
$s_b$ is replaced by $w_k$ if $s_b \in W_{\Theta^{k-1}}(F)$. The ordering of the set 
$W_{\Theta^k}(F)$ is the same as the ordering of the set $W_{\Theta^{k-1}}(F)$ however the 
vertex $w_k$ takes the position of $s_b$ if $s_b \in W_{\Theta^{k-1}}(F)$. If $s_b \in F$ then 
the same statements hold with $s_a$ in place of $s_b$. 


2.1. Subdivision sequences

(2) If \( F \in \mathcal{F}_2 \) and \( F \) contains \( s_a \), then \( W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\}) \), and the ordering of the sets coincide.

(3) If \( F \in \mathcal{F}_3 \) then \( W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F - \{w_k\} \cup S) \), and the ordering of the sets coincide.

(4) If \( F \in \mathcal{F}_4 \) then \( W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F) \cup \{w_k\} \), and the ordering of \( W_{\Theta^k}(F) - \{w_k\} \) coincides with the ordering of \( W_{\Theta^{k-1}}(F) \), and \( w_k \) is last in the ordering.

(5) If \( F \in \mathcal{F}_5 \) then \( W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F) \), and the ordering of the sets coincide.

Proof. This can be proven easily by induction on \( k \), using the definition of the induced subdivision sequence. \( \square \)

Let \((\Theta^0, ..., \Theta^k)\) be a subdivision sequence where \( \Theta^0 = \Sigma_{d-1} \). For any face \( F \in \Theta^k \) we define a set of vertices

\[ K_{(\Theta^0, ..., \Theta^k)}(F), \]

This is abbreviated to \( K_{\Theta^k}(F) \) when the subdivision sequence is clear from the context. We let

\[ K_{\Theta^k}(F) := \bigcap_{v \in F} K_{\Theta^k}({v}), \]

and for any vertex \( v \in \Theta^k \) we define \( K_{\Theta^k}({v}) \) inductively as follows:

If \( k = 0 \) so that \( \{v\} \in \Sigma_{d-1} \), then \( K_{\Sigma_{d-1}}({v}) = \emptyset \) for all \( \{v\} \in \Sigma_{d-1} \). If \( k \geq 1 \) then \( K_{\Theta^k}({v}) \) is given by:

(1) If \( \{v\} \in \mathcal{F}_1 \) (i.e. \( v = s_a \) or \( s_b \)) or if \( \{v\} \in \mathcal{F}_5 \) (i.e. \( v \not\in \{s_a, s_b, w_k\} \) and \( \{v\} \not\in lk_{\Theta^k}(\{w_k\}) \)) then \( K_{\Theta^k}({v}) = K_{\Theta^{k-1}}({v}) \).

(2) If \( \{v\} \in \mathcal{F}_3 \) (i.e. \( v = w_k \)) then \( K_{\Theta^k}({v}) = K_{\Theta^{k-1}}({s_a}) \cap K_{\Theta^{k-1}}({s_b}) \).

(3) If \( \{v\} \in \mathcal{F}_4 \) (i.e. \( v \not\in \{s_a, s_b, w_k\} \) and \( \{v\} \in lk_{\Theta^k}(\{w_k\}) \)) then \( K_{\Theta^k}({v}) = K_{\Theta^{k-1}}({v}) \cup \{w_k\} \).

We can also give an inductive definition of \( K_{\Theta^k}(F) \).

Proposition 2.1.3. For any face \( F \in \Theta^k \) we have

(1) If \( F \in \mathcal{F}_1 \) then \( K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F) \).
(2) If $F \in \mathcal{F}_2$ and $s_a \in F$ then $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\})$
(by symmetry the same statement hold with $s_a$ and $s_b$ swapped).

(3) If $F \in \mathcal{F}_3$ then $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F - \{w_k\} \cup S)$.

(4) If $F \in \mathcal{F}_4$ then $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F) \cup \{w_k\}$.

(5) If $F \in \mathcal{F}_5$ then $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F)$.

Proof. We show that the claim holds in each of the five cases for $F \in \Theta^k$.

(1) If $F \in \mathcal{F}_1$ and $s_a \in F$ then for any $w \in F$ either $K_{\Theta^k}(\{w\}) = K_{\Theta^{k-1}}(\{w\})$ or $K_{\Theta^k}(\{w\}) = K_{\Theta^{k-1}}(\{w\}) \cup \{w_k\}$. Also, $w_k \notin K_{\Theta^k}(\{s_a\})$. Therefore $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F)$. By symmetry the claim holds in this case when $s_b \in F$.

(2) If $F \in \mathcal{F}_2$ and $s_a \in F$ then

$$K_{\Theta^k}(F) = \left( \bigcap_{w \in F - \{s_a, w_k\}} K_{\Theta^k}(\{w\}) \right) \cap K_{\Theta^k}(\{w_k\}) \cap K_{\Theta^k}(\{s_a\})$$

$$= \left( \bigcap_{w \in F - \{s_a, w_k\}} K_{\Theta^{k-1}}(\{w\}) \right) \cap K_{\Theta^{k-1}}(\{s_a\}) \cap K_{\Theta^{k-1}}(\{s_b\})$$

$$= \bigcap_{w \in F - \{w_k\} \cup \{s_b\}} K_{\Theta^{k-1}}(\{w\}).$$

The second equality uses the fact that for any $w \in F - \{s_a, w_k\}$ we have $K_{\Theta^k}(\{w\}) = K_{\Theta^{k-1}}(\{w\})$ or $K_{\Theta^k}(\{w\}) = K_{\Theta^{k-1}}(\{w\}) \cup \{w_k\}$ yet $w_k \notin K_{\Theta^k}(\{w_k\})$. By symmetry the claim holds when $s_b \in F$. 
2.1. SUBDIVISION SEQUENCES

(3) If $F \in \mathcal{F}_3$ then
\[
K_{\Theta^k}(F) = \left( \bigcap_{w \in F - \{w_k\}} K_{\Theta^k}\{w\} \right) \cap K_{\Theta^k}\{w_k\}
\]
\[
= \left( \bigcap_{w \in F - \{w_k\}} K_{\Theta^{k-1}}\{w\} \right) \cap K_{\Theta^{k-1}}\{s_a\} \cap K_{\Theta^{k-1}}\{s_b\}
\]
\[
= \bigcap_{w \in F - \{w_k\}} K_{\Theta^{k-1}}\{w\}
\]
\[
= K_{\Theta^{k-1}}(F - \{w_k\} \cup S).
\]
The second equality uses the fact that for all $w \in F - \{w_k\}$ we have $K_{\Theta^k}\{w\} = K_{\Theta^{k-1}}\{w\} \cup \{w_k\}$, and that $w_k \not\in K_{\Theta^{k-1}}\{w\}$.

(4) If $F \in \mathcal{F}_4$ then every vertex $w \in F$ is adjacent to $w_k$ and not equal to $s_a$ or $s_b$, so $K_{\Theta^k}\{w\}$ is the union of $K_{\Theta^{k-1}}\{w\}$ and $\{w_k\}$. Taking the intersection over all vertices $w$ of $F$ gives the claim immediately.

(5) If $F \in \mathcal{F}_5$ then there is some vertex $w \in F$ that is not adjacent to both $s_a$ and $s_b$ so that $w_k \not\in K_{\Theta^k}(F)$. Since for every vertex $w \in F$ either $K_{\Theta^k}\{w\} = K_{\Theta^{k-1}}\{w\}$ or $K_{\Theta^k}\{w\} = K_{\Theta^{k-1}}\{w\} \cup \{w_k\}$ the claim clearly holds in this case.

Claim 2.1.4. Given a subdivision sequence $(\Theta^0, \ldots, \Theta^k)$, for any face $F \in \Theta^k$ we have
\[
|K_{\Theta^k}(F)| = |W_{\Theta^k}(F)|.
\]

Proof. The statement is clear by induction, noting that in the recursive rules of Propositions 2.1.2 and 2.1.3 this property is maintained, and that for $k = 0$, both sets are empty, so the two sides of the inequality vanish.

Given a subdivision sequence $(\Theta^0, \ldots, \Theta^k)$, for any $F \in \Theta^k$, the set $K_{\Theta^k}(F)$ is a subset of $W_{\Theta^k}(\emptyset) = V_{\Theta^k} - V_{\Theta^0} = \{w_1, \ldots, w_k\}$. We define an ordering on the set $K_{\Theta^k}(F)$ where for any $w_i, w_j \in K_{\Theta^k}(F)$ we stipulate that if $i < j$ then $w_i < w_j$. Since Claim 2.1.4 holds, for any face $F \in \Theta^k$ we define the following order preserving bijection
\[
\phi_{\Theta^k,F} : K_{\Theta^k}(F) \rightarrow W_{\Theta^k}(F).
\]
In the case where \( F = \emptyset \) this is the identity map \( w_i \mapsto w_{i,0} \).

**Definition 2.1.5.** Given a subdivision sequence \((\Theta^0, \ldots, \Theta^k)\) define a flag simplicial complex 
\[ \Gamma(\Theta^0, \ldots, \Theta^k) \]
on the vertex set \( \{w_1, \ldots, w_k\} \), where the condition for \( w_a \) to be adjacent to \( w_b \) (for \( a < b \)) is that \( w_a \) belongs to \( K_{\Theta^0, \ldots, \Theta^b}(\{w_b\}) \). (When the subdivision sequence is clear we abbreviate this to \( \Gamma(\Theta^k) \)).

**Example 2.1.6.** Let \((\Sigma^3, \Theta^1, \Theta^2, \Theta^3)\) be the subdivision sequence of Example 2.1.1.
Then \( K_{\Theta^1}(\{\pm \epsilon_3\}) = K_{\Theta^1}(\{\pm \epsilon_4\}) = \{w_1\} \), and in the other cases we have
\[ K_{\Theta^1}(\{v\}) = \emptyset, \]
\[ K_{\Theta^2}(\{\pm \epsilon_1\}) = K_{\Theta^2}(\{\pm \epsilon_2\}) = K_{\Theta^2}(\{w_1\}) = \{w_2\}, \text{ and } K_{\Theta^2}(\{\pm \epsilon_3\}) = K_{\Theta^2}(\{\pm \epsilon_4\}) = K_{\Theta^2}(w_2) = \{w_1\}. \]
Finally, \( K_{\Theta^3}(\{-\epsilon_3\}) = K_{\Theta^3}(\{-\epsilon_4\}) = K_{\Theta^3}(\{w_2\}) = \{w_1\}, \)
\[ K_{\Theta^3}(\{\pm \epsilon_1\}) = \{w_2\}, \quad K_{\Theta^3}(\{\epsilon_3\}) = K_{\Theta^3}(\{\epsilon_4\}) = \{w_1, w_3\}, \]
\[ K_{\Theta^3}(\{\pm \epsilon_2\}) = K_{\Theta^3}(\{w_1\}) = \{w_2, w_3\}, \text{ and } K_{\Theta^3}(\{w_3\}) = \emptyset. \]

Hence \( \Gamma(\Sigma^3, \Theta^1, \Theta^2, \Theta^3) \) is the simplicial complex illustrated in Figure 2.

**Figure 2.** The simplicial complex \( \Gamma(\Sigma^3, \Theta^1, \Theta^2, \Theta^3) \).

\[ \bullet \quad \bullet \quad \bullet \]

\( w_1 \quad w_2 \quad w_3 \)

Note that in this example \( K_{\Theta^3}(\{\epsilon_1\}) = \{w_2\} \), whereas \( W_{\Theta^3}(\{\epsilon_1\}) = \{w_3\} \).

**2.2. The main theorem**

The goal of this section is to prove:

**Theorem 2.2.1.** For any subdivision sequence \((\Theta^0, \ldots, \Theta^k)\),
\[ f(\Gamma(\Theta^0, \ldots, \Theta^k)) = \gamma(\Theta^k). \]
In order to prove this theorem we first need to prove Propositions 2.2.2 and 2.2.3.

**Proposition 2.2.2.** Given a subdivision sequence \((\Theta^0, ..., \Theta^k)\) and faces \(F, G \in \Theta^k\) such that \(G \in \text{lk}_{\Theta^k}(F)\), we have that \(K_{\Theta^k}(F \cup G) = K_{\Theta^k}(F) \cap K_{\Theta^k}(G)\) maps to \(K_{\text{lk}_{\Theta^k}(F)}(G)\) under \(\phi_{\Theta^k, F}\) (the set \(K_{\text{lk}_{\Theta^k}(F)}(G)\) is defined using the induced subdivision sequence with result \(\text{lk}_{\Theta^k}(F)\)).

**Proof.** This is a proof by induction on \(k\). If \(k = 0\) then for any face \(F \in \Theta^0 = \Sigma_d\) we have \(W_{\Sigma_d}^d(F) = \emptyset\) and \(K_{\Sigma_d}^d(F) = \emptyset\) so that the proposition holds. If \(k \geq 1\) then will consider all five cases for faces in \(\Theta^k\) and show that the proposition holds in each case. For each case, it is sufficient to assume that \(G \) is a vertex \(\{g\}\). This is sufficient since if this holds then \(\phi_{\Theta^k, F}\) being a bijection implies that the image of \(K_{\Theta^k}(F) \cap K_{\Theta^k}(G)\) is equal to \(\bigcap_{w \in G} K_{\Theta^k}(\{w\})\).

1. Suppose that \(F \in \mathcal{F}_1\), and we may suppose that \(s_a \in F\). Recall that \(K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F)\), and that either \(W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F)\) or \(W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F) \cup \{s_b\}\) where \(w_k\) takes the position of \(s_b\) in the order. Then \(\phi_{\Theta^k, F}\) is the same as \(\phi_{\Theta^{k-1}, F}\) except for the possible replacement of \(s_b\) by \(w_k\) in the codomain.

**Figure 3.** The sets described in the case that \(g \neq w_k\). Note that \(w_k\) and \(s_b\) might not be contained in the sets, and they may be contained in \(K_{\text{lk}_{\Theta^k}(F)}(\{g\})\) and \(K_{\text{lk}_{\Theta^{k-1}}(F)}(\{g\})\).
Assume that $g \neq w_k$. Then $K_{\Theta^k}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{g\})$ (since $F \cup \{g\} \in \mathcal{F}_1$). By the inductive hypothesis we have

$$K_{\Theta^{k-1},F}(K_{\Theta^{k-1}}(F \cup \{g\})) = K_{\Theta^{k-1}}(F \cup \{g\}).$$

By the definition of the induced subdivision sequence we have that $K_{\Theta^{k-1}}(F \cup \{g\})$ is equal to $K_{\Theta^{k-1}}(F \cup \{g\})$ except for the possible replacement of $s_b$ by $w_k$. Hence the proposition holds in this case (see figure 3).

Assume that $g = w_k$. Then $K_{\Theta^k}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{s_b\})$. By the inductive hypothesis $K_{\Theta^{k-1},F}(K_{\Theta^{k-1}}(F \cup \{s_b\})) = K_{\Theta^{k-1}}(F \cup \{s_b\})$. So $K_{\Theta^k,F}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{s_b\})$.

(2) Assume that $F \in \mathcal{F}_2$, and we may assume that $s_a \in F$. In this case $\text{lk}_{\Theta^k}(F)$ does not contain any of $s_a$, $s_b$, or $w_k$ so that $g$ is not equal to any of these vertices. Here $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\})$ and $W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\})$, and $\phi_{\Theta^k,F}$ is the same as $\phi_{\Theta^{k-1},F - \{w_k\} \cup \{s_b\}}$. Now $K_{\Theta^k}(F \cup \{g\}) = K_{\Theta^{k-1}}(F - \{w_k\} \cup \{s_b\} \cup \{g\})$ maps under $\phi_{\Theta^{k-1},F - \{w_k\} \cup \{s_b\}}$ to $K_{\Theta^{k-1}}(F - \{w_k\} \cup \{g\})$, which equals $K_{\Theta^{k-1}}(F \cup \{g\})$ by the definition of the induced subdivision sequence.

(3) Assume that $F \in \mathcal{F}_3$. In this case both $s_a$ and $s_b$ are in $\text{lk}_{\Theta^k}(F)$, $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F - \{w_k\} \cup S), W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F - \{w_k\} \cup S)$, and $\phi_{\Theta^k,F}$ is the same as $\phi_{\Theta^{k-1},F - \{w_k\} \cup S}$.

If $g$ is not equal to either $s_a$ or $s_b$ then $K_{\Theta^k}(F \cup \{g\}) = K_{\Theta^{k-1}}(F - \{w_k\} \cup S \cup \{g\})$, which is mapped to $K_{\Theta^{k-1}}(F - \{w_k\} \cup S \cup \{g\})$ by $\phi_{\Theta^{k-1},F - \{w_k\} \cup S}$, and this is equal to $K_{\Theta^{k-1}}(F \cup \{g\})$ by the definition of the induced subdivision sequence.

If $g = s_a$ then $K_{\Theta^k}(F \cup \{s_a\}) = K_{\Theta^{k-1}}(F - \{w_k\} \cup S)$, and this maps under $\phi_{\Theta^{k-1},F - \{w_k\} \cup S}$ to the set $K_{\Theta^{k-1}}(F - \{w_k\} \cup S)(\emptyset) = W_{\Theta^{k-1}}(F - \{w_k\} \cup S)$ which is the set $K_{\Theta^{k-1}}(F)(\emptyset) = W_{\Theta^k}(F)$. This is the same set as $K_{\Theta^{k-1}}(F \cup \{s_a\})$ since $\text{lk}_{\Theta^k}(F)$ is the suspension of $\text{lk}_{\Theta^{k-1}}(F - \{w_k\} \cup S)$ in the two additional vertices $s_a$ and $s_b$, which are in $\Sigma_{d-1-|F|}$ in the induced subdivision sequence. By symmetry the result also holds when $g = s_b$.

(4) Suppose that $F \in \mathcal{F}_4$. Then $\text{lk}_{\Theta^k}(F)$ is the stellar subdivision of $\text{lk}_{\Theta^{k-1}}(F)$ in $S$ and $W_{\Theta^k}(F) = W_{\Theta^{k-1}}(F \cup \{w_k\})$. We have $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F \cup \{w_k\})$ and $\phi_{\Theta^k,F}$ restricts to $\phi_{\Theta^{k-1},F}$ on
been performed. We suppose that the proposition holds for any subdivision. The proposition is clearly true when no subdivisions have been performed. We show that the proposition holds by induction on the number of subdivisions. We let $\phi^{sd}$ be the mapping of $\Theta^{k-1}$ to $K_{\Theta^{k-1}}(F)$ that maps $w_k$ to $w_k$. We now have to consider the different possibilities for $g$.

Suppose that $\{g\} \in \mathcal{F}_1$. We may suppose that $g = s_a$. Then $K_{\Theta^{k-1}}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{g\})$. Under $\phi^{sd}$, $K_{\Theta^{k-1}}(F \cup \{g\})$ maps to the set $K_{lk_{\Theta^{k-1}}(F)}(\{g\})$, and this is equal to $K_{lk_{\Theta^{k-1}}(F)}(\{g\})$ since $\{g\} \in \mathcal{F}_1$ in $lk_{\Theta^{k}}(F)$.

We cannot have $\{g\} \in \mathcal{F}_2$ since this implies that $|\{g\}| \geq 2$.

Suppose that $\{g\} \in \mathcal{F}_3$, i.e. that $g = w_k$. In this case $K_{\Theta^{k-1}}(F \cup \{w_k\}) = K_{\Theta^{k-1}}(F \cup \{g\})$. Under $\phi^{sd}$ this maps to $K_{lk_{\Theta^{k-1}}(F)}(S)$, and this is equal to $K_{lk_{\Theta^{k-1}}(F)}(\{w_k\})$ since $\{w_k\} \in \mathcal{F}_3$ in $lk_{\Theta^{k}}(F)$.

Suppose that $\{g\} \in \mathcal{F}_4$. Then $K_{\Theta^{k-1}}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{g\}) \cup \{w_k\}$. Now $\phi^{sd}(K_{\Theta^{k-1}}(F \cup \{g\})) = K_{lk_{\Theta^{k-1}}(F)}(\{g\})$, so $\phi^{sd}(K_{\Theta^{k-1}}(F \cup \{g\})) = K_{lk_{\Theta^{k-1}}(F)}(\{g\}) \cup \{w_k\} = K_{lk_{\Theta^{k-1}}(F)}(\{g\})$, since $\{g\}$ is in $\mathcal{F}_4$ in $lk_{\Theta^{k}}(F)$.

Suppose that $\{g\} \in \mathcal{F}_5$. Then $K_{\Theta^{k-1}}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{g\})$ and this maps under $\phi^{sd}$ to $K_{lk_{\Theta^{k-1}}(F)}(\{g\})$ which is equal to $K_{lk_{\Theta^{k-1}}(F)}(\{g\})$.

(5) Suppose that $F \in \mathcal{F}_5$. Then $K_{\Theta^{k-1}}(F) = K_{\Theta^{k-1}}(F)$, $W_{\Theta^{k-1}}(F) = W_{\Theta^{k-1}}(F)$, $K_{\Theta^{k-1}}(F \cup \{g\}) = K_{\Theta^{k-1}}(F \cup \{g\})$, $\phi^{sd} = \phi^{sd}$, and $K_{lk_{\Theta^{k-1}}(F)}(\{g\}) = K_{lk_{\Theta^{k-1}}(F)}(\{g\})$ so that the proposition clearly holds in this case.

\[ \square \]

**Proposition 2.2.3.** Suppose $(\Theta^0, \ldots, \Theta^k)$ is a subdivision sequence. Then for any face $F \in \Theta^k$ the restriction of $\Gamma(\Theta^0, \ldots, \Theta^k)$ to the vertices in $K_{\Theta^{k}}(F)$ is equivalent to $\Gamma((\tilde{\Phi}^0(F), \ldots, \tilde{\Phi}^k(F))_{\Theta^0, \ldots, \Theta^k})$. The map on the vertices is $\phi^{sd}$.

**Proof.** We show that the proposition holds by induction on the number of subdivisions. The proposition is clearly true when no subdivisions have been performed. We suppose that the proposition holds for any $\Theta$ in the set $sd(S_d^{k-1})$, that is obtained by $k-1$ subdivisions. We let $\Theta^k \in sd(S_d^{k-1})$ be obtained by subdividing $\Theta^{k-1}$ in the edge $S = \{s_a, s_b\}$ to give the new vertex $w_k$, and show that the proposition holds for $\Theta^k$. We consider all five cases for a face $F \in \Theta^k$.
2. Boundary of the Cross Polytope

(1) Suppose that $F \in \mathcal{F}_1$, and we may suppose that $s_a \in F$. Then $K_{\Theta^k} (F) = K_{\Theta^{k-1}} (F)$, and by the definition of the induced subdivision sequence

$$(\Phi^0(F), \ldots, \Phi^{l_F} (F))_{(\Theta^0, \ldots, \Theta^k)}$$

we have that

$$\Gamma((\Phi^0(F), \ldots, \Phi^{l_F} (F))_{(\Theta^0, \ldots, \Theta^k)}) \cong \Gamma((\Phi^0(F), \ldots, \Phi^{l_F-1} (F))_{(\Theta^0, \ldots, \Theta^{k-1})})$$

where the map on all vertices is the identity except that $s_b \mapsto w_k$ if $s_b \in W_{\Theta^{k-1}} (F)$. By induction the restriction of $\Gamma(\Theta^{k-1})$ to $K_{\Theta^{k-1}}$ is equivalent to $\Gamma((\Phi^0(F), \ldots, \Phi^{l_F-1} (F))_{(\Theta^0, \ldots, \Theta^{k-1})})$. Hence the proposition holds in this case since $\Phi_{\Theta^{k-1}, F}$ is the same as $\Phi_{\Theta^{k-1}, F}$ except for the possible replacement of $s_b$ by $w_k$ in the codomain.

(2) Suppose that $F \in \mathcal{F}_2$. Then $K_{\Theta^k} (F) = K_{\Theta^{k-1}} (F - \{w_k\} \cup \{s_b\})$ and by induction, the restriction of $\Gamma(\Theta^k)$ to $K_{\Theta^{k-1}} (F - \{w_k\} \cup \{s_b\})$ is equivalent to

$$\Gamma((\Phi^0(F - \{w_k\} \cup \{s_b\}), \ldots, \Phi^{l_F} (F - \{w_k\} \cup \{s_b\}))_{(\Theta^0, \ldots, \Theta^k)}$$

By the definition of the induced subdivision sequences we have that

$$\Gamma((\Phi^0(F), \ldots, \Phi^{l_F} (F))_{(\Theta^0, \ldots, \Theta^k)})$$

is equal to

$$\Gamma((\Phi^0(F - \{w_k\} \cup \{s_b\}), \ldots, \Phi^{l_F} (F - \{w_k\} \cup \{s_b\}))_{(\Theta^0, \ldots, \Theta^k)})$$

Hence the proposition holds in this case.

(3) Suppose that $F \in \mathcal{F}_3$. Then $K_{\Theta^k} (F) = K_{\Theta^{k-1}} (F - \{w_k\} \cup S)$ and by the definition of the induced subdivision sequences we have that

$$\Gamma((\Phi^0(F - \{w_k\} \cup S), \ldots, \Phi^{l_F} (F - \{w_k\} \cup S))_{(\Theta^0, \ldots, \Theta^k)})$$

is equal to

$$\Gamma((\Phi^0(F), \ldots, \Phi^{l_F} (F))_{(\Theta^0, \ldots, \Theta^k)})$$

Hence the desired condition holds in this case.

(4) Suppose that $F \in \mathcal{F}_4$. Then $K_{\Theta^k} (F) = K_{\Theta^{k-1}} (F \cup \{w_k\})$, and by the inductive hypothesis the restriction of $\Gamma(\Theta^0, \ldots, \Theta^k)$ to $K_{\Theta^{k-1}} (F)$ is equivalent to $\Gamma((\Phi^0(F), \ldots, \Phi^{l_F-1} (F))_{(\Theta^0, \ldots, \Theta^{k-1})})$. By the definition of the induced subdivision sequence

$$(\Phi^0(F), \ldots, \Phi^{l_F} (F))_{(\Theta^0, \ldots, \Theta^k)}$$

$\Phi^{l_F} (F)$ is the subdivision of $\Phi^{l_F-1} (F)$ in the edge $S$. Hence the flag simplicial complex $\Gamma((\Phi^0(F), \ldots, \Phi^{l_F} (F))_{(\Theta^0, \ldots, \Theta^k)})$ is obtained
from $\Gamma((\Phi^0(F),...,\Phi^{F-1}(F))_{(\Theta^0,...,\Theta^{k-1})})$ by attaching the vertex $w_k$ to the vertices in

$$K_{(\Phi^0(F),...,\Phi^{F}(F))}(\{w_k\}) = K_{(\Phi^0(F),...,\Phi^{F-1}(F))}(S).$$

The vertex $w_k$ attaches to $K_{\Theta^{k-1}}(S)$ in the 1-skeleton of $\Gamma(\Theta^k)$, and so attaches to the vertices $K_{\Theta^{k-1}}(F) \cap K_{\Theta^{k-1}}(S)$ in the 1-skeleton of

$$\Gamma((\Phi^0(F),...,\Phi^{F-1}(F))_{(\Theta^0,...,\Theta^{k-1})}).$$

By Proposition 2.2.2, $\phi_{\Theta^{k-1},F}$ maps $K_{\Theta^{k-1}}(F) \cap K_{\Theta^{k-1}}(S)$ to the set $K_{lk_{\Theta^{k-1}}}(F)(S)$ in $W_{\Theta^{k-1}}(F)$, so that the proposition holds in this case.

(5) Suppose that $F \in F_5$. Then $K_{\Theta^k}(F) = K_{\Theta^{k-1}}(F)$, hence it is clear by the definition of the induced subdivision sequences and the relevant sets that the proposition holds in this case.

\[ \square \]

**Proof of Theorem 2.2.1.** We assume by induction that the theorem holds for any simplicial complex in $sd(\Sigma_i)$ where $i < d - 1$. For the base case, when $i = 0$, $\Sigma_0$ is a point, and $sd(\Sigma_0)$ contains $\Sigma_0$ only. $\Gamma(\Sigma_0)$ is empty, and so $f(\Gamma(\Sigma_0)) = \gamma(\Sigma_0) = 1$. Suppose a subdivision is made on $\Theta \in sd(\Sigma_{d-1})$ in an edge $S$ to obtain $\Theta'$, and $w$ is the unique vertex in $V_{\Theta'} - V_{\Theta}$. Then by the construction of $\Gamma(\Theta')$, and by Proposition 2.2.3, the faces of $\Gamma(\Theta') - \Gamma(\Theta)$ are $F \cup \{w\}$ for all $F \in lk_{\Theta}(S)$. Therefore,

$$f(\Gamma(\Theta')) - f(\Gamma(\Theta)) = tf(\Gamma(lk_{\Theta}(S))),$$

and by the inductive hypothesis we have

$$f(\Gamma(lk_{\Theta}(S))) = \gamma(lk_{\Theta}(S)).$$

Also, by Proposition 1.6.4

$$\gamma(\Theta') - \gamma(\Theta) = t\gamma(lk_{\Theta}(S)),$$

so that

$$f(\Gamma(\Theta')) - f(\Gamma(\Theta)) = \gamma(\Theta') - \gamma(\Theta).$$

Since $f(\Gamma(\Sigma_{d-1})) = 1$ and $\gamma(\Sigma_{d-1}) = 1$, by induction on the number of subdivisions performed, the theorem holds.

\[ \square \]
2.3. Simple graphic zonotopes

In this section we show that the Nevo and Petersen conjecture holds for graphic zonotopes. Graphic zonotopes are defined in [25, Section 8.6] and [26, Section 5], to which the reader is referred for further details.

A zonotope $Z$ is the Minkowski sum of one-dimensional polytopes. Given a graph $G = (V, E)$ with no loops or multiple edges, with vertex set $V = [n]$ and edge set $E$, the graphic zonotope $Z_G$ is the Minkowski sum of the line segments $\{e_i - e_j \mid \{i, j\} \in E\}$, where $e_1, ..., e_n$ are the endpoints of the coordinate vectors in $\mathbb{R}^n$. A biconnected component of a graph $G$ is a maximal (with respect to number of vertices) induced subgraph $B_G$ of $G$ such that the removal of any vertex of $B_G$ results in a connected graph. The following proposition and corollary are taken from [26, Section 5]:

**Proposition 2.3.1.** [21, Remark 5.2],[26, Proposition 5.2]. The graphic zonotope corresponding to a graph $G = (V, E)$ is a simple polytope if and only if every biconnected component of $G$ is a complete subgraph. If $V_1, ..., V_r$ are the vertex sets for these complete subgraphs, then $Z_G$ is isomorphic to the Cartesian product of usual permutohedra of dimensions $|V_j| - 1$ for $j = 1, ..., r$.

**Corollary 2.3.2.** [26, Corollary 5.4]. Let $Z_G$ be a simple graphic zonotope. Then $Z_G$ is flag, and its $f$, $h$- and $\gamma$-polynomial respectively are equal to the products of the $f$, $h$- and $\gamma$-polynomial of the $(|V_j| - 1)$-dimensional permutohedra, for $j = 1, ..., r$, where the $V_j$ are as mentioned in Proposition 2.3.1.

The following corollary is a simple consequence of this:

**Corollary 2.3.3.** Let $Z_G$ be a simple graphic zonotope. Then $Z_G$ satisfies the Nevo and Petersen conjecture, that is, Conjecture 0.0.3.

**Proof.** Recall from Example 1.6.2 that the dual simplicial complex of the $(d - 1)$-dimensional permutohedron, which is denoted $\Theta_{P_B(K_d)}$, is in the set $sd(\Sigma_{d-2})$. Hence there exists a subdivision sequence

$$(\Sigma_{d-2}, ..., \Theta_{P_B(K_d)}),$$

which we will denote by $S_d$ such that

$$f(\Gamma(\Sigma_{d-2}, ..., \Theta_{P_B(K_d)})) = \gamma(\Theta_{P_B(K_d)}).$$

Therefore if $Z_G$ is a graphic zonotope, and $V_1, ..., V_r$ are the vertex sets of the biconnected components of $G$, then
\[ \gamma(Z_G) = f(\Gamma(S_{|V_1|}) \ast \cdots \ast \Gamma(S_{|V_r|})). \]
Since the join of flag simplicial complexes is flag, the proposition holds.
The Nevo and Petersen conjecture for nestohedra

In this Chapter we discuss the work in [3], in which we prove the Nevo Petersen conjecture for flag nestohedra. Specifically, we prove the following theorem:

**Theorem 3.0.4.** If $P_B$ is a flag nestohedron, there is a flag simplicial complex $\Gamma(B)$ such that $f(\Gamma(B)) = \gamma(P_B)$. In particular $\gamma(P_B)$ satisfies the Frankl-Füredi-Kalai inequalities.

This Theorem is implied by the more general Theorem 2.0.6 in Chapter 2, since nestohedra are included in the class of polytopes $sd(\Sigma_{d-1})$. The work in this chapter is included in this thesis since it was completed before [2], the proofs are simpler than those that lead to Theorem 2.0.6, and they provide insight into the reason Theorem 3.0.4 holds true.

In Section 3.1 we include some background theory on nestohedra that is required in this Chapter. In Section 3.2 we include some theory on face shavings of nestohedra. In Section 3.3 we define the construction $\Gamma(B)$ and include the theory up to and including the proof of Theorem 3.0.4. Our construction of $\Gamma(B)$ depends on the choice of a flag ordering for $B$, which we define in Section 3.3. The flag ordering we define corresponds to a subdivision sequence as defined in Chapter 2. In Section 3.4 we show in Corollary 3.4.2 that the flag simplicial complexes defined in this chapter are equivalent those defined in Chapter 2. In Section 3.5 we compare the flag simplicial complexes $\Gamma(B)$ to those defined by Nevo and Petersen in [23] for particular flag nestohedra. In the special cases considered by [23] our $\Gamma(B)$ does not always coincide with the complex they construct. We also give combinatorial definitions for $\Gamma(B)$ when $B = B(K_n)$ and $B = B(K_{1,n-1})$. This yields a recurrence formula for $\gamma(B(K_{1,n-1}))$ which appears in Section 3.5.4.

### 3.1. Building sets and nestohedra

This section requires knowledge of the material in Chapter 1; the most relevant sections are Section 1.4 and 1.6. Here we discuss definitions and
results on building sets that are used to prove Theorem 3.0.4.

The conditions in Proposition 3.1.1 determine whether a building set is flag, and are the original conditions given in [26].

**Proposition 3.1.1.** [26, Proposition 7.1]. For a building set $B$, the following are equivalent:

1. $P_B$ is flag.
2. If $J_1, \ldots, J_m$, $m \geq 2$, are disjoint and $J_1 \cup \cdots \cup J_m \in B$, then the sets can be reindexed so that for some $k$ such that $1 \leq k \leq m - 1$, $J_1 \cup \cdots \cup J_k \in B$ and $J_{k+1} \cup \cdots \cup J_m \in B$.
3. If $N \subseteq B \setminus B_{max}$ such that
   - for any $I, J \in N$ either $I \subseteq J$, $J \subseteq I$ or $I \cap J = \emptyset$,
   - for any $I, J \in N$ such that $I \cap J = \emptyset$, one has $I \cup J \notin B$,
   then $N$ is a nested set.

A minimal flag building set $D$ on a set $S$ is a connected building set on $S$ that is flag, such that no proper subset of its elements form a connected flag building set on $S$. Minimal flag building sets are described in detail in [26, Section 7.2]. They take the form of a binary tree, where the vertices biject to elements of $D$, and the direct descendants of any non-leaf vertex that represents an element $I \in D$ are the two elements in $D$ whose disjoint union is $I$. For any minimal flag building set $D$, $\gamma(D) = 1$ (see [26, Section 7.2]).

**Example 3.1.2.** Let $D$ be the minimal flag building set on $[4]$ given by
\[
\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}, [4]\}.
\]
The binary tree that corresponds to $D$ is illustrated in Figure 3.1.2 below.

Let $B$ be a building set. A binary decomposition or decomposition of a non-singleton element $I \in B$ is a set $D \subseteq B$ that forms a minimal flag
building set on $I$. Suppose that $I \in \mathcal{B}$ has a binary decomposition $\mathcal{D}$. The two maximal elements $D_1, D_2 \in \mathcal{D} - \{I\}$ with respect to inclusion are the maximal components of $I$ in $\mathcal{D}$. The following lemma gives another definition of when a building set is flag.

**Lemma 3.1.3.** A building set $\mathcal{B}$ is flag if and only if every non-singleton $I \in \mathcal{B}$ has a binary decomposition.

**Proof.** The only if part follows immediately from [26, Proposition 7.3]. For the if part, suppose that $\mathcal{B}$ is a building set and every element has a binary decomposition. We show that $\mathcal{B}$ is flag by showing that part (3) of Proposition 3.1.1 holds. Suppose by contradiction that (3) does not hold so that there exists a set $\mathcal{S} = \{S_1, \ldots, S_k\} \subset \mathcal{B}$, $k \geq 3$, such that $S_i \cap S_j = \emptyset$, $S_i \cup S_j \notin \mathcal{B}$ for all $i \neq j$, and $S_1 \cup \cdots \cup S_k = I \in \mathcal{B}$. Fix a decomposition $\mathcal{D}$ of $I$. Now consider all one element sets of $\mathcal{D}$ (the set of all $\{i\}$ such that $i \in I$). They are each a subset of one element of $\mathcal{S}$. Suppose by induction that all elements in $\mathcal{D}$ that are sets with $\leq i$ elements are a subset of one element of $\mathcal{S}$. Then any $i + 1$ element subset of $\mathcal{D}$ must also be contained in one element of $\mathcal{S}$. This is true since each $i + 1$ element subset of $\mathcal{D}$ is the union of two elements of $\mathcal{D}$ each with less than $i + 1$ elements. These two subsets must be contained in the same element of $\mathcal{S}$ since if they were contained in two distinct elements then their union would intersect two elements $S_i$ and $S_j$ of $\mathcal{S}$ which implies $S_i \cup S_j \in \mathcal{B}$. As the size of the elements of the decomposition increase, they are eventually equal to $I$, which implies that $k = 1$, a contradiction since $k \geq 3$. \hfill $\square$

The previously mentioned criterion for a building set to be flag, Proposition 1.4.7, is a consequence of Lemma 3.1.3. It is clear that if every element in a building set has a decomposition, then every element in the building set is a disjoint union of two elements in the building set. Suppose that $\mathcal{B}$ is a building set in which every element is the disjoint union of two elements in the building set. Then for any element $I \in \mathcal{B}$, we may form a subset $\mathcal{I}$ of $\mathcal{B}$, such that $I \in \mathcal{I}$, and for any $J \in \mathcal{I}$, there are elements $J_1, J_2 \in \mathcal{I}$ such that $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = J$. The set $\mathcal{I}$ contains a decomposition of $I$. Hence every element in $\mathcal{B}$ has a binary decomposition.

**Lemma 3.1.4.** Suppose $\mathcal{B}$ is a flag building set. If $I, J \in \mathcal{B}$ and $J \subseteq I$, then there is a decomposition of $I$ in $\mathcal{B}$ that contains $J$.

**Proof.** Consider the set $\{J, \{i_1\}, \ldots, \{i_k\}\}$ where $\{i_1, \ldots, i_k\} = I - J$. This is a set of disjoint elements whose union is in $\mathcal{B}$. Therefore, by Proposition 3.1.1 part (2) we can reindex these sets until we obtain two disjoint sets each in $\mathcal{B}$ whose union is $I$. We can repeatedly perform this same procedure on
the elements in \( \{ J, \{ i_1 \}, \{ i_2 \}, ..., \{ i_k \} \} \) that are subsets of each of the new sets obtained at each step. All of the new sets obtained with reindexing, together with a decomposition of \( J \) and the element \( I \), are a decomposition of \( I \) that contains \( J \).

\( \square \)

### 3.2. Face shavings of flag nestohedra

The following Theorem is proven by Volodin [31].

**Theorem 3.2.1.** [31, Lemma 6]. Let \( B \) and \( B' \) be connected flag building sets on \([n]\) such that \( B \subseteq B' \). Then \( B' \) can be obtained from \( B \) by successively adding elements so that at each step the set is a flag building set.

Suppose that a connected flag building set \( B' \) on \([n]\) is obtained from a flag building set \( B \) on \([n]\) by adding an element \( I \). Then \( I \) has a binary decomposition in \( B' \) with two maximal components \( D_1, D_2 \). This implies that \( P_{B'} \) can be obtained by shaving the codimension 2 face of \( P_B \) that corresponds to the nested set \( \{ D_1, D_2 \} \).

**Lemma 3.2.2.** Let \( B \) be a building set with nestohedron \( P_B \). Suppose that \( F_0 \) is a facet of \( P_B \) corresponding to a (non-maximal) building set element \( I \). Then the face poset of \( F_0 \) is isomorphic to the poset of faces of \( P_{B|I} \times P_{B/I} \).

**Proof.** The poset of faces of \( F_0 \) is the subposet of the faces of \( P \), consisting of faces that are contained in \( F_0 \). Since the facet \( F_0 \) corresponds to the nested set \( \{ I \} \), the set of faces of \( P \) that are contained in \( F_0 \) correspond to nested sets that contain \( I \). The simplicial complex of nested sets of \( B \) that contain \( I \) is isomorphic to \( \Delta_{B|I} \times \Delta_{B/I} \). The isomorphism is given by

\[
(N_1, N_2) \in \Delta_{B|I} \times \Delta_{B/I} \mapsto N_1 \cup N_2 \cup \{ I \},
\]

where \( N_2 := \{ D \mid D \in N_2 \text{ and } D \cup I \notin B \} \cup \{ D \cup I \mid D \in N_2, \ D \cup I \in B \} \). It is not too hard to see that this is a map to nested sets that contain \( I \), that preserves the inclusion relation, and that is injective and surjective.

\( \square \)

[31, Proposition 5] states that if a polytope \( Q \) can be obtained from a simple \( n \)-dimensional polytope \( P \) by shaving a face \( G \) of dimension \( k \) to obtain a new facet \( F_0 \), then \( F_0 \) is combinatorially equivalent to \( G \times \Delta^{n-k-1} \), where \( \Delta^d \) denotes the \( d \)-dimensional simplex. If \( G \) is of dimension \( n-2 \) then \( F_0 \) is combinatorially equivalent to \( G \times \Delta^1 \), so that \( \gamma(F_0) = \gamma(G) \gamma(\Delta^1) = \gamma(G) \). Hence, in the case that the polytopes are flag nestohedra, using Lemma 3.2.2, we can rewrite [17, Proposition 2.4.3] as:
Lemma 3.2.3. If $B'$ is a flag building set on $[n]$ obtained from a flag building set $B$ on $[n]$ by adding an element $I$ then

$$\gamma(B') = \gamma(B) + t\gamma(B'|I)\gamma(B'/I)$$

$$= \gamma(B) + t\gamma(B|I)\gamma(B/I).$$

Proof. The first identity is a direct consequence of the preceding discussion. From the definition of the contraction of a building set we have that $B'/I = B/I$ so that $\gamma(B'/I) = \gamma(B/I)$. Let $D_1, D_2$ be the maximal components of $I$ in the decomposition of $I$ in $B'$. They are unique since $I \notin B$. Using Lemma 1.4.11 below we have that $B'|I = D[B|D_1, B|D_2]$ where $D$ is the building set $\{\{1\}, \{2\}, \{2\}\}$. Hence

$$\gamma(B'|I) = \gamma(D)\gamma(B|D_1)\gamma(B|D_2) = \gamma(D)\gamma(B|I) = \gamma(B|I).$$

Note that if $B$ is a flag building set on $[n]$ and $I \in B$, then $B/I$ and $B|I$ are flag building sets. This is obvious for $B|I$. For the claim about $B/I$, we let $D \in B/I$. Then if $D \in B$ there exist two elements $D_1, D_2$ in $B/I$ such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = I$. If $D \notin B$ then $D \cup I \in B$, and since $I \subseteq I \cup D$, by Lemma 3.1.4, $I$ is in a decomposition $D$ of $I \cup D$ and this implies there are two elements $D_1, D_2 \in D$ such that $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 = D \cup I$, and $I$ is a proper subset of either $D_1$ or $D_2$. Let $D_i$ denote the image of $D_i$ in the contraction. Then $\overline{D_1 \cap D_2} = \emptyset$ and $\overline{D_1 \cup D_2} = D$.

Using Theorem 3.2.1 and Lemma 3.2.3, [31] shows the following two Theorems. Their proof uses the inductive hypothesis that both $\gamma(B'|I)$ and $\gamma(B'/I)$ of Lemma 3.2.3 are such that $\gamma(B'|I) \geq 0$ and $\gamma(B'/I) \geq 0$.

Theorem 3.2.4. [31, Theorem 2]. For any flag nestohedron $P_B$ we have

$$\gamma(B) \geq 0.$$ 

Theorem 3.2.5. [31, Theorem 3] [8, Theorem 1.1]. If $B$ and $B'$ are connected flag building sets on $[n]$ and $B \subseteq B'$, then $\gamma(B) \leq \gamma(B').$
3.3. THE FLAG SIMPLICIAL COMPLEX $\Gamma(O)$ FOR A FLAG ORDERING $O$ OF $\mathcal{B}$

$$P_{\mathcal{B}} = P_{\mathcal{B}_1} \times P_{\mathcal{B}_2} \times \cdots \times P_{\mathcal{B}_a}$$

which implies that if $\gamma(\mathcal{B}_i) = f(\Gamma(\mathcal{B}_i))$ for some flag simplicial complex $\Gamma(\mathcal{B}_i)$, then

$$\gamma(\mathcal{B}) = \gamma(\mathcal{B}_1) \cdot \gamma(\mathcal{B}_2) \cdots \gamma(\mathcal{B}_a) = f(\Gamma(\mathcal{B}_1) \ast \Gamma(\mathcal{B}_2) \ast \cdots \ast \Gamma(\mathcal{B}_a)).$$

Hence to prove Theorem 3.0.4 we need only consider connected flag building sets.

Suppose that $\mathcal{B}$ is a connected flag building set on $[n]$, $\mathcal{D}$ is a decomposition of $[n]$ in $\mathcal{B}$, and $I_1, I_2, ..., I_k$ is an ordering of $\mathcal{B} - \mathcal{D}$, such that $B_j = \mathcal{D} \cup \{I_1, I_2, ..., I_j\}$ is a flag building set for all $j$. (Such an ordering exists by Theorem 3.2.1). We call the pair consisting of such a decomposition $\mathcal{D}$ and the ordering on $\mathcal{B} - \mathcal{D}$ a flag ordering of $\mathcal{B}$, denoted $O$, or $(\mathcal{D}, I_1, ..., I_k)$. For any $I_j \in \mathcal{B} - \mathcal{D}$, we say an element in $B_j - I_j$ is earlier in the flag ordering than $I_j$, and an element in $B - B_j$ is later in the flag ordering than $I_j$.

**Example 3.3.1.** Consider the building set $\mathcal{B} := \mathcal{B}($Path$_n$). Then a possible flag ordering of $\mathcal{B}$ is

$$(\mathcal{D}, \{1, 2, 3\}, \{2, 3, 4\}, \{2, 3\}),$$

where

$$\mathcal{D} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}, [4]\}.$$  

The ordering of $\mathcal{B} - \mathcal{D}$ given by

$$\{1, 2, 3\}, \{2, 3\}, \{2, 3, 4\}$$

does not give a flag ordering since $\mathcal{D} \cup \{1, 2, 3\} \cup \{2, 3\}$ is not a building set.

For any $j \in [k]$ define

$$U_j := \{i \mid i < j, I_i \not\subseteq I_j\}, \text{ there is no } J \in \mathcal{B}_{i-1} \text{ such that } J \setminus I_j = I_i \setminus I_j\},$$

and

$$V_j := \{i \mid i < j, I_i \subseteq I_j, \exists J \in \mathcal{B}_{i-1} \text{ such that } I_i \subseteq J \subseteq I_j\}.$$  

If $i \in U_j \cup V_j$ then we say that $I_i$ is non-degenerate with respect to $I_j$. If $I_i \in \mathcal{B}_{i-1}$ and $i \not\in U_j \cup V_j$ then $I_i$ is degenerate with respect to $I_j$. Degenerate elements with respect to $I_j$ that are not contained in $I_j$ are elements that we need not consider as contributing to the building set $\mathcal{B}_j/I_j$. The set of
degenerate elements with respect to $I_j$ that are subsets of $I_j$, together with $I_j$, forms a decomposition of $I_j$ in $B_j|I_j$.

Given a flag building set $B$ with flag ordering $O = (D, I_1, ..., I_k)$ define a graph on the vertex set

$$V_O = \{v(I_1), ..., v(I_k)\},$$

where for any $i < j$, $v(I_i)$ is adjacent to $v(I_j)$ if and only if $i \in U_j \cup V_j$. Then define a flag simplicial complex $\Gamma(O)$ whose faces are the cliques in this graph. If the flag ordering is clear then we denote $\Gamma(O)$ by $\Gamma(B)$. For any $I \subseteq [k]$, we let $\Gamma(O)|_I$ denote the induced subcomplex of $\Gamma(O)$ on the vertices $v(I_i)$ for all $i \in I$.

**Example 3.3.2.** Consider the flag building set $B(\text{Path}_5)$ on $[5]$. It has a flag ordering $O$ given by

$$D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{2, 3\}, \{3, 4\}, \{3, 4, 5\}\},$$

and

$$I_1 = \{3, 4\}, I_2 = \{2, 3, 4\}, I_3 = \{2, 3\}, I_4 = \{2, 3, 4, 5\}, I_5 = \{3, 4, 5\}, I_6 = \{4, 5\}.$$ Then $\Gamma(O)$ has only two edges, namely

$$\{v(I_2), v(I_6)\} \text{ and } \{v(I_3), v(I_4)\}.$$ These are edges because $I_2 = \{2, 3, 4\}$ is the earliest element which has image $\{2, 3\}$ in the contraction by $I_6$, and the element $I_3 = \{2, 3\}$ is a subset of $I_2 = \{2, 3, 4\}$ which is in turn a subset of $I_4$.

Now $D/I_k$ is a decomposition of $[n] - I_k$, and we have an induced ordering of $(B/I_k) - (D/I_k)$, where the $i$th element is $I'_u_i := I_{u_i} \setminus I_k$ if $u_i$ is the $i$th element of $U_k$ (listed in increasing order). Then for all $i$, $D/I_k \cup \{I'_u_1, ..., I'_u_k\}$ is a flag building set. Hence we can also define a flag simplicial complex $\Gamma(B/I_k)$. We label the vertices of $\Gamma(B/I_k)$ by $v(I'_{u_1}), v(I'_{u_2}), ..., v(I'_{u_k})$.

**Claim 3.3.3.** Let $B$ be a connected flag building set with flag ordering given by $(D, I_1, ..., I_k)$. For all $J \in B$ let $J' = J \setminus I_k$. If $J' \neq \emptyset$, $j \in U_k$, and $J \in B_{j-1}$ then $J \subseteq I_j$ if and only if $J' \subseteq I'_j$.

**Proof.** $\Rightarrow$: It is clear that $J \subseteq I_j$ implies $J' \subseteq I'_j$.

$\Leftarrow$: Suppose for a contradiction that $J' \subseteq I'_j$ and $J \not\subseteq I_j$. Then $J \cap I_j \neq \emptyset$ and $J \cup I_j \neq I_j$, which implies that (since $B_j$ is a building set) $J \cup I_j \in B_{j-1}$. 

We also have that \((J \cup I_j)' = I_j'\), which implies that \(I_j\) is degenerate with respect to \(I_k\), a contradiction. □

**Proposition 3.3.4.** Let \(\mathcal{B}\) be a connected flag building set with flag ordering given by \((\mathcal{D}, I_1, \ldots, I_k)\). Then \(\Gamma(\mathcal{B}/I_k) \cong \Gamma(\mathcal{B})|_{I_k}\). The map on the vertices is given by \(v(I_k') \mapsto v(I_i)\).

**Proof.** \(\Gamma(\mathcal{B})|_{I_k}\) is a flag simplicial complex with vertex set
\[
v(I_{u_1}), v(I_{u_2}), \ldots, v(I_{U_{I_k}})
\]
and \(\Gamma(\mathcal{B}/I_k)\) is a flag simplicial complex with vertex set
\[
v(I_{u_1}'), v(I_{u_2}'), \ldots, v(I_{U_{I_k}'}).\]

Suppose that \(i < j\) where \(i, j \in U_k\). We need to show that \(\{v(I_i'), v(I_j')\} \in \Gamma(\mathcal{B}/I_k)\) if and only if \(\{v(I_j), v(I_i)\} \in \Gamma(\mathcal{B})|_{I_k}\). We will show the following:

a) If \(I_i \subseteq I_j\) (by Claim 3.3.3, equivalently \(I_i' \subseteq I_j'\)) then \(\{v(I_j'), v(I_i')\} \in \Gamma(\mathcal{B}/I_k)\) if and only if \(\{v(I_j), v(I_i)\} \in \Gamma(\mathcal{B})|_{I_k}\).

b) If \(I_i \not\subseteq I_j\) (by Claim 3.3.3, equivalently \(I_i' \not\subseteq I_j'\)) then \(\{v(I_j'), v(I_i')\} \in \Gamma(\mathcal{B}/I_k)\) if and only if \(\{v(I_j), v(I_i)\} \in \Gamma(\mathcal{B})|_{I_k}\).

(1) \(\Rightarrow\): Suppose that \(\{v(I_j'), v(I_i')\} \in \Gamma(\mathcal{B}/I_k)\), so that there exists \(J \in \mathcal{B}_{i-1}\) such that \(I_i' \subseteq J' \subseteq I_j'\). By Claim 3.3.3, \(J \subseteq I_j\) and since \(I_i \subseteq I_j\) this implies \(J \cup I_i \subseteq I_j\). Since \(J \cap I_i \neq \emptyset\) we have \(J \cup I_i \in \mathcal{B}_{i-1}\). Hence \(I_i \not\subseteq J \cup I_i \not\subseteq I_j\) which implies \(\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{I_k}\).

\(\Leftarrow\): Suppose \(\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{I_k}\), so that there exists \(J \in \mathcal{B}_{i-1}\) such that \(I_i \not\subseteq J \not\subseteq I_j\). Then \(I_i' \not\subseteq J' \not\subseteq I_j'\), and \(J' \neq I_i'\) or \(I_j'\) since \(i, j \in U_k\), so that \(I_i' \not\subseteq J' \not\subseteq I_j'\). Hence \(\{v(I_i'), v(I_j')\} \in \Gamma(\mathcal{B}/I_k)\).

(2) \(\Rightarrow\): Suppose that \(\{v(I_i'), v(I_j')\} \in \Gamma(\mathcal{B}/I_k)\), and suppose for a contradiction that \(\{v(I_i), v(I_j)\} \not\in \Gamma(\mathcal{B})|_{I_k}\). Then there exists \(J \in \mathcal{B}_{i-1}\) such that \(J \setminus I_j = I_i \setminus I_j\). Then \(J' \setminus I_j' = I_i' \setminus I_j'\) which implies the contradiction that \(\{v(I_i'), v(I_j')\} \not\in \Gamma(\mathcal{B}/I_k)\).

\(\Leftarrow\): We will prove the contrapositive that \(\{v(I_i'), v(I_j')\} \not\in \Gamma(\mathcal{B}/I_k)\) implies that \(\{v(I_i), v(I_j)\} \not\in \Gamma(\mathcal{B})|_{I_k}\). Now \(\{v(I_i'), v(I_j')\} \not\in \Gamma(\mathcal{B}/I_k)\) implies there exists \(M \in \mathcal{B}_{i-1}\) such that \(M \setminus I_j' = I_i' \setminus I_j'\).

- Assume that \(M \subseteq I_i\), and for this case refer to Figure 1. Let \(R := I_k \cap (I_i \setminus (M \cup I_j))\), and let \(J := I_i \setminus (M \cup I_k)\). Since \(M \subseteq I_i\),
by Lemma 3.1.4 there exists a decomposition of $I_i$ in $B_i$ that contains $M$. Hence $M$ is contained in a maximal component $D$ of this decomposition. Let $D'$ be the other maximal component. If $D' \cap R = \emptyset$ then $\{v(I_i), v(I_j)\} \not\in \Gamma(B)|_{U_k}$ since $D \setminus I_j = I_i \setminus I_j$, hence the desired condition holds. If $D' \cap J = \emptyset$ then $I_i \setminus I_k = D \setminus I_k$ which contradicts $i \in V_k$. If $D' \cap J \neq \emptyset$ and $D' \cap R \neq \emptyset$ then $(D' \cup I_j) \setminus I_k = I_j \setminus I_k$ which contradicts $j \in V_k$.

**Figure 1.** A picture of the sets in case (2), assuming $M \subseteq I_i$. Note that $I_i \setminus (M \cup I_j \cup I_k) = \emptyset$ by the definition of $M$.

- Assume that $M \not\subseteq I_i$. For this case refer to Figure 2. Let $H := I_i \setminus (I_j \cup I_k)$. In $(B_j/I_k)/I'_j$ both $I'_j$ and $M'$ have the same image that is given by $H$, and $H \neq \emptyset$ since $H = \emptyset$ implies $I'_j \subseteq I'_j$. Let $K := M'(I_k \cup I_i)$. Then $K \neq \emptyset$ since $K = \emptyset$ implies $I_i \setminus I_k = M \setminus I_k$, which contradicts $i \in V_k$. Let $L := M'(I_i \cup I_j)$. Now if $L = \emptyset$, then $\{v(I_i), v(I_j)\} \not\in \Gamma(B)|_{U_k}$ since $M \setminus I_j = I_i \setminus I_j$, so the desired condition holds. Suppose now $L \neq \emptyset$. Then $M$ intersects each of $H$, $K$ and $L$. Let $J$ be a minimal (for inclusion) element in in $B_{i-1}$ that intersects $H$, $K$ and $L$. Then $|J| \geq 3$ and at least one of the elements in the decomposition of $J$ (in $B_{i-1}$) must intersect exactly two of $K$, $H$ and $L$. Denote such an element by $\hat{D}$. If $\hat{D}$ intersects $K$ and $L$ then $(I_j \cup \hat{D}) \setminus I_k = I_j \setminus I_k$ which contradicts $j \in V_k$. If $\hat{D}$ intersects both $K$ and $H$ then $\{v(I_i), v(I_j)\} \not\in \Gamma(B)|_{U_k}$ since $(I_i \cup \hat{D}) \setminus I_j = I_i \setminus I_j$, so the desired condition holds. If $\hat{D}$ intersects $L$ and $H$ then $(I_i \cup \hat{D}) \setminus I_k = I_i \setminus I_k$, which contradicts $i \in V_k$. 


3.3. The Flag Simplicial Complex $\Gamma(O)$ for a Flag Ordering $O$ of $\mathcal{B}$

Figure 2. A picture of the sets in case (2), assuming $M \not\subseteq I_i$. Note that $I_i \setminus (M \cup I_j \cup I_k) = \emptyset$ by the definition of $M$.

We now consider the flag building set $\mathcal{B}|_{I_k}$. It is not necessarily true that $\mathcal{D}|_{I_k}$ is a decomposition of $I_k$. Let

$$\mathcal{D}_k := \mathcal{D}|_{I_k} \cup \{I_j \mid I_j \subseteq I_k, j \not\in V_k\}.$$  

Then $\mathcal{D}_k$ is a decomposition of $I_k$ in $\mathcal{B}$, and for any $j$ we have that $\mathcal{D}_k \cup \{I_i \mid i \leq j \text{ and } i \in V_k\}$ is a connected flag building set on $I_k$. We define $\Gamma(\mathcal{B}|_{I_k})$ to be the flag simplicial complex $\Gamma(O)$ with respect to the flag ordering $O$ of $\mathcal{B}|_{I_k}$ with decomposition $\mathcal{D}_k$ and ordering of $\mathcal{B}|_{I_k} - \mathcal{D}_k$ given by $I_{v_1}, I_{v_2}, \ldots, I_{u|V_k|}$ where $v_j$ is the $j$th element of $V_k$ listed in increasing order. We label the vertices of $\Gamma(\mathcal{B}|_{I_k})$ by $v(I_{v_1}), v(I_{v_2}), \ldots, v(I_{u|V_k|})$ rather than by their index in $V_k$. In keeping with the notation that $\mathcal{B}_j$ is the flag building set obtained after adding elements indexed up to $j$, we let $(\mathcal{B}|_{I_k})_j$ denote the flag building set $\mathcal{D}_k \cup \{I_i \mid i \leq j \text{ and } i \in V_k\}$, so that $\Gamma((\mathcal{B}|_{I_k})_j)$ is defined. Note then that for any $j$, $\mathcal{B}_j|_{I_k} \subseteq (\mathcal{B}|_{I_k})_j$.

**Proposition 3.3.5.** Let $\mathcal{B}$ be a connected flag building set with flag ordering given by $(\mathcal{D}, I_1, \ldots, I_k)$. Then $\Gamma(\mathcal{B}|_{I_k}) = \Gamma(\mathcal{B})|_{V_k}$.

**Proof.** Both $\Gamma(\mathcal{B}|_{I_k})$ and $\Gamma(\mathcal{B})|_{V_k}$ are both flag simplicial complexes with the vertex set $v(I_{v_1}), v(I_{v_2}), \ldots, v(I_{u|V_k|})$. We need to show that for any $i, j \in V_k$ where $i < j$, $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$ if and only if $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B}|_{I_k})$.

$\Rightarrow$: Suppose that $\{v(I_i), v(I_j)\} \in \Gamma(\mathcal{B})|_{V_k}$. First assume that $I_i \subseteq I_j$. Then there is some $J \in B_{i-1}$ such that $I_i \subseteq b \subseteq I_j$. Since $J \in B_{i-1}|_{I_k}$ and
\( B_{i-1} | I_k \subseteq (B|I_k)_{i-1} \) this implies that \( \{ v(I_i), v(I_j) \} \in \Gamma(B|I_k) \).

Now suppose that \( I_i \not\subseteq I_j \). Suppose for a contradiction that \( \{ v(I_i), v(I_j) \} \) is not in \( \Gamma(B|I_k) \). Then there exists some \( D \in D_k - D|I_k \), \( D \not\subseteq B_{i-1} \), such that \( D \cup I_j = I_i \cup I_j \). Since \( i \in V_k \) there exists some \( J \in B_{i-1} \) such that \( I_i \subseteq J \not\subseteq I_k \). Since \( \{ v(I_i), v(I_j) \} \in \Gamma(B)|V_k \) we have that \( J \setminus (I_i \cup I_j) \neq \emptyset \).

Since the index of \( D \) is not in \( V_k \), every element in the restriction to \( I_k \) that is earlier than \( D \) in the flag ordering is a subset of it or does not intersect it. This implies \( J \subseteq D \), so \( D \setminus (I_i \cup I_j) \neq \emptyset \), which contradicts \( D \cup I_j \not= I_i \cup I_j \).

\[ \Leftarrow: \] Suppose that \( \{ v(I_i), v(I_j) \} \in \Gamma(B|I_k) \). First assume that \( I_i \subseteq I_j \), so that there is some \( D \in (B|I_k)_{i-1} \) such that \( I_i \subseteq d \subseteq I_j \). If \( D \in B_{i-1} | I_k \) then clearly \( \{ v(I_i), v(I_j) \} \in \Gamma(B)|V_k \) as desired. If \( D \not\subseteq B_{i-1} | I_k \) then \( D \in D_k - D|I_k \). Since \( i \in V_k \) there exists some \( J \in B_{i-1} \) such that \( I_i \subseteq J \subseteq I_k \).

Since the index of \( D \) is not in \( V_k \) we have that \( I_i \subseteq J \subseteq J \). This is because \( J \) either contains or does not intersect elements that are earlier in the flag ordering and contained in \( I_k \). Then since \( D \subseteq I_j \) this implies \( J \subseteq I_j \) and since \( J \in B_{i-1} \) and \( I_i \subseteq J \subseteq I_j \) this implies \( \{ v(I_i), v(I_j) \} \in \Gamma(B)|V_k \).

Now assume that \( I_i \not\subseteq I_j \). Suppose for a contradiction that \( \{ v(I_i), v(I_j) \} \) is not in \( \Gamma(B)|V_k \). Then there exists \( J \in B_{i-1} | I_k \) such that \( J \cup I_j = I_i \cup I_j \). Since \( B_{i-1} | I_k \subseteq (B|I_k)_{i-1} \) this contradicts \( \{ v(I_i), v(I_j) \} \in \Gamma(B|I_k) \).

\[ \square \]

**Theorem 3.3.6.** Let \( B \) be a connected flag building set with flag ordering \( O \). Then \( \gamma(B) = f(\Gamma(O)) \).

**Proof.** This is a proof by induction on the number of elements of \( B - D \).

The result holds for \( k = 0 \) since \( f(\Gamma(D)) = 1 = \gamma(D) \). So we assume \( k \geq 1 \) and that the result holds for all connected flag building sets with a smaller value of \( k \).

By Propositions 3.3.4 and 3.3.5 and the inductive hypothesis we have that

\[ f(\Gamma(B)|U_k) = f(\Gamma(B/I_k)) = \gamma(B/I_k), \]

and

\[ f(\Gamma(B)|V_k) = f(\Gamma(B|I_k)) = \gamma(B|I_k). \]

Suppose that \( u \in U_k \) and \( w \in V_k \). Then \( \{ v(I_u), v(I_w) \} \in \Gamma(B) \), for suppose for a contradiction that \( \{ v(I_u), v(I_w) \} \not\subseteq \Gamma(B) \). Suppose that \( u < w \). Then there is some element \( J \in B_{u-1} \) such that \( J \cup I_w = I_u \cup I_w \). This implies that \( J \cup I_k = I_u \cup I_k \) which contradicts \( u \in U_k \). Suppose that \( w < u \). Then either \( I_u \cap I_w = \emptyset \) or \( I_w \subseteq I_u \) (otherwise \( I_u \cup I_w \) makes
Suppose that $I_w \cap I_u = \emptyset$. Then since \( \{v(I_u), v(I_w)\} \not\in \Gamma(B) \), there exists $J \in B_{w-1}$ such that $J \cup I_u = I_w \cup I_u$, and $J \cap I_u \neq \emptyset$. Then $J \cup I_u$ makes $I_u$ degenerate with respect to $I_k$, a contradiction. Suppose that $I_w \subseteq I_u$. Now $w \in V_k$ implies there is some $J \in B_{w-1}$ such that $I_w \subseteq J \subseteq I_k$. Also, $J \subseteq I_u$ else $J \cup I_u$ makes $I_u$ degenerate with respect to $I_k$. However, this implies the contradiction that $\{v(I_u), v(I_w)\} \in \Gamma(B)$ since $I_w \subseteq J \subseteq I_u$.

Hence

$$\Gamma(B)|_{U_k \cup V_k} = \Gamma(B)|_{U_k} \ast \Gamma(B)|_{V_k},$$

and therefore

$$f(\Gamma(B)|_{U_k \cup V_k}) = f(\Gamma(B)|_{U_k}) \ast f(\Gamma(B)|_{V_k}) = \gamma(B/I_k) \gamma(B|_{I_k}).$$

Since the vertex $v(I_k)$ is adjacent to the vertices indexed by elements in $U_k \cup V_k$ we have

$$f(\Gamma(B)) = f(\Gamma(B_{k-1})) + t \gamma(B/I_k) \gamma(B|_{I_k}).$$

By the induction hypothesis this implies that

$$f(\Gamma(B)) = \gamma(B_{k-1}) + t \gamma(B|_{I_k}) \gamma(B/I_k),$$

which implies that $f(\Gamma(B)) = \gamma(B)$ by Theorem 3.2.3. \qed

For two flag orderings $O_1, O_2$ of a connected flag building set $B$, it is not necessarily true that the flag simplicial complexes $\Gamma(O_1), \Gamma(O_2)$ are equivalent (up to change of labels on the vertices) even if they have the same decomposition. The following example provides a counterexample.

**Example 3.3.7.** Let $B = B(C_{\text{yc5}})$, and let

$$D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{2\}, \{3\}, \{4\}, \{5\}\}.$$

Let $O_1$ be the flag ordering with decomposition $D$ and the following ordering of $B - D$:

$$\{2, 3\}, \{2, 3, 4\}, \{2, 3, 4, 5\}, \{4, 5\}, \{3, 4, 5\}, \{3, 4\}, \{3, 4, 5, 1\}, \{4, 5, 1, 2\}, \{5, 1, 2, 3\}, \{4, 5, 1\}, \{5, 1, 2\}, \{1, 5\}.$$

Let $O_2$ be the flag ordering with decomposition $D$ and the following ordering of $B - D$:

$$\{2, 3\}, \{2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4\}, \{3, 4, 5\}, \{4, 5\}, \{3, 4, 5\}, \{3, 4\}, \{3, 4, 5, 1\}, \{4, 5, 1, 2\}, \{5, 1, 2, 3\}, \{4, 5, 1\}, \{5, 1, 2\}, \{1, 5\}.$$

Then $\Gamma(O_1)$ and $\Gamma(O_2)$ are depicted in Figure 3.
3.4. The dual simplicial complex of nestohedra

Adding building set elements in a flag ordering is equivalent to performing edge subdivisions on the dual simplicial complex (see Section 3.2). Suppose \( O \) is a flag ordering that corresponds to a subdivision sequence \( (\Sigma_{d-1}, ..., \Theta^k) \). In this section we show that \( \Gamma(O) \) is equivalent to the flag simplicial complex \( \Gamma(\Sigma_{d-1}, ..., \Theta^k) \) that we defined in Chapter 2.

Suppose that \( \Theta_{P_B} \in sd(\Sigma_{d-1}) \) is the dual simplicial complex to a flag nestohedron \( P_B \). Suppose also that \( \Theta_{P_B} \) is the result of the subdivision sequence \( (\Theta^0, ..., \Theta^k) \), (so \( \Theta^k = \Theta_{P_B} \)), that corresponds to a flag ordering \( O = (D, I_1, ..., I_k) \) of \( B \). This implies that the vertex \( w_i \in \Theta^k \) corresponds to the building set element \( I_i \) (this is also the label of the corresponding face of \( P_B \)). Again we assume that the last edge to be subdivided is \( S = \{s_a, s_b \} \). Thus, if \( J_a \) is the building set element corresponding to \( s_a \) and \( J_b \) corresponds to \( s_b \) then \( J_a \cap J_b = \emptyset \) and \( J_a \cup J_b = I_k \).

**Proposition 3.4.1.** Let \( \Theta^k \) be given as above. Then \( h \in U_k \cup V_k \) if and only if \( w_h \in K_{\Theta^k}(\{w_k\}) \).

**Proof.** Let \( \{J_{m_1}, ..., J_{m_n} \} \) be the maximal elements of the restriction to \( I_k \) in \( B_h \), and let \( J_{h1}, J_{h2} \) denote the (unique) two elements in \( B_{h-1} \) such that \( J_{h1} \cap J_{h2} = \emptyset \) and \( J_{h1} \cup J_{h2} = I_h \). First we note that \( w_h \in K_{\Theta^k}(\{w_k\}) \) is equivalent to \( w_h \in K_{\Theta^l}(\{w_{m_l}\}) \) for \( 1 \leq l \leq n \). This is true since \( w_h \in K_{\Theta^l}(\{w_{m_l}\}) \) for \( 1 \leq l \leq n \) implies that for all of the elements \( I_{l \beta} \in B_k \), \( \beta > h \) that are subsets of \( I_k \), and such that \( I_{l \beta} \) is a maximal subset of \( I_k \) in \( B_{\beta} \), we also have \( w_h \in K_{\Theta^l}(\{w_{\beta}\}) \). Conversely, \( w_h \not\in K_{\Theta^l}(\{w_{m_l}\}) \) for some \( l \in \{1, ..., n \} \) implies that \( w_h \not\in K_{\Theta^\beta}(w_{\beta}) \) for all \( \beta > h \) such that
$I_m \subseteq I_\beta \subseteq I_k$ and $I_\beta$ is a maximal subset of $I_k$ in $\mathcal{B}_\beta$.

- First we suppose that $I_h \subseteq I_k$. We show that $h \in V_k$ if and only if $w_h \in K_{\Theta^h}(\{w_k\})$.

  Suppose that $h \in V_k$, i.e. $I_h \subseteq I_k$ and there exists a building set element $J_{m_l}$ that is earlier than $I_h$ in the flag ordering such that $I_h \subseteq J_{m_l} \subseteq I_k$ (note that it is possible that $J_{m_l} = J_n$ or $I_h$). Then each of the vertices in the set $\{w_{m_1}, \ldots, w_{m_n}\}$ are adjacent to both of the vertices $w_{h1}$ and $w_{h2}$, since any pair are a nested set. Thus we have $w_h \in K_{\Theta^h}(\{w_k\})$ for $1 \leq l \leq n$, so that $w_h \in K_{\Theta^h}(\{w_k\})$.

  To show that $w_h \in K_{\Theta^h}(\{w_k\})$ implies $h \in V_k$, we show the contrapositive, that $h \notin V_k$ implies that $w_h \notin K_{\Theta^h}(\{w_k\})$. Now $h \notin V_k$ implies that $I_h = J_{m_l}$ for some $1 \leq l \leq n$, so that $w_h = w_{m_l} \notin K_{\Theta^h}(\{w_{m_l}\})$ and (by the reasoning given above) this implies that $w_h \notin K_{\Theta^h}(\{w_k\})$.

- Now suppose that $I_h \not\subseteq I_k$. We show that $h \in U_k$ if and only if $w_h \in K_{\Theta^h}(\{w_k\})$.

  Suppose that $h \in U_k$. Then $I_h \cap I_k$ is a union of maximal elements in the set $J_{m_1}, \ldots, J_{m_n}$. Also, each of the maximal elements $J_{m_1}, \ldots, J_{m_n}$ can intersect at most one of $I_{h1}$ or $I_{h2}$, and cannot be equal to one of $I_{h1}$ or $I_{h2}$ since this implies that $h \notin U_k$. We therefore have that every $J_{m_l}$, $1 \leq l \leq n$ is a nested set with either of $I_{h1}$ and $I_{h2}$ since they are a subset of it, or if not a subset of it and their union was in $\mathcal{B}_{h-1}$ then we would not have $h \in U_k$. Hence $w_{h1}$ and $w_{h2}$ are adjacent to all of the vertices $w_{m_1}, \ldots, w_{m_n}$ in $\Theta^{h-1}$, and therefore $w_h \in K_{\Theta^h}(\{w_k\})$.

  Suppose that $w_h \in K_{\Theta^h}(\{w_k\})$. Then this implies $w_h$ is in $K_{\Theta^h}(\{w_{m_l}\})$ for all $1 \leq l \leq n$, i.e. that $w_{h1}$ and $w_{h2}$ are adjacent to each of $w_{m_l}$ in $\Theta^h$. This implies that neither $I_{h1}$ or $I_{h2}$ are in $\{J_{m_1}, \ldots, J_{m_n}\}$ and neither $I_{h1}$ or $I_{h2}$ can be a union of elements in $\{J_{m_1}, \ldots, J_{m_n}\}$ (since these are the maximal elements). Since each of $I_h$, $I_{h1}$, $I_{h2}$ are a nested set with each of $J_{m_1}, \ldots, J_{m_n}$ we have that each of $I_h \cap I_k$, $I_{h1} \cap I_k$ and $I_{h2} \cap I_k$ is a union of elements of $J_{m_1}, \ldots, J_{m_n}$. This implies that neither $I_{h1}$ nor $I_{h2}$ is contained in $I_k$. Suppose for a contradiction that $h \notin U_k$, so that there is an element $I_\alpha$ that is earlier than $I_h$ in the flag ordering that has the same image
in the contraction by \( I_k \) as \( I_h \). We suppose that \( I_\alpha \) is maximal with respect to this property and will consider the following three cases for \( I_\alpha \).

- Suppose that neither \( I_\alpha \subseteq I_h \) nor \( I_h \subseteq I_\alpha \). Then (using the building set axioms) this implies the contradiction that \( I_\alpha \) is not maximal with this property.

- If \( I_\alpha \subseteq I_h \) then we have the contradiction that there is an element that is a subset of \( I_h \) earlier in the flag ordering that intersects both \( I_{h1} \) and \( I_{h2} \).

- If \( I_h \subseteq I_\alpha \) then consider a decomposition of \( I_\alpha \) in \( B_\alpha \). Note that since \( I_\alpha \) is maximal with this property that \( I_\alpha \) is the disjoint union of three sections: \( I_{h1}, I_{h2} \) and \( G := I_\alpha - (I_{h1} \cup I_{h2}) \), where \( G = \bigcup_{j=1}^n J_{ij} \) is a union of elements in \( J_{m1}, ..., J_{mn} \).

  Fix a decomposition \( \mathcal{D} \) of \( I_\alpha \) in \( B_\alpha \). There must be an element \( J \in \mathcal{D} \) that intersects exactly two elements of the set \( I_{h1}, I_{h2}, J_{i1}, ..., J_{is} \). To find such an element take the set of all elements that intersect more than one of these sets, and from this set choose an element of minimal cardinality. \( J \) cannot intersect a pair from \( J_{i1}, ..., J_{is} \) since \( J_{m1}, ..., J_{mn} \) are maximal subsets of \( I_k \) in \( B_h \). \( J \) cannot intersect \( I_{h1} \) and \( I_{h2} \) since this implies that \( I_h \in B_{h-1} \). We cannot have \( J \) intersect one of \( J_{i1}, ..., J_{is} \) and one of \( I_{h1} \) and \( I_{h2} \) since this contradicts the nested set property. Hence we have a contradiction in this case too.

\[ \square \]

**Corollary 3.4.2.** Suppose \( \Theta_{P_B} \) is the dual simplicial complex to a flag nestohedron \( P_B \), and that the subdivision sequence \((\Theta^0, ..., \Theta_{P_B})\) is equivalent to a flag ordering \( O \) of the nestohedron. Then

\[
\Gamma(\Theta^0, ..., \Theta_{P_B}) \cong \Gamma(O),
\]

where \( w_j \mapsto v(I_j) \).

**Proof.** Since Proposition 3.4.1 holds for all \( k \), we have, for any \( i, j \in 1, ..., k \) such that \( i < j \) that \( w_i \) is adjacent to \( w_j \) in \( \Gamma(\Theta_{P_B}) \) if and only if \( v(I_i) \) is adjacent to \( v(I_j) \) in \( \Gamma(O) \). \( \square \)
3.5. The flag simplicial complexes of Nevo and Petersen

In this section we compare the flag simplicial complexes that we have defined to those defined for certain graph-association by Nevo and Petersen [23]. They define flag simplicial complexes \( \Gamma(\hat{\mathcal{S}}_n), \Gamma(\hat{\mathcal{S}}_n(312)) \) and \( \Gamma(P_n) \) such that

- \( \gamma(\mathcal{B}(K_n)) = f(\Gamma(\hat{\mathcal{S}}_n)) \),
- \( \gamma(\mathcal{B}(\text{Path}_n)) = f(\Gamma(\hat{\mathcal{S}}_n(312))) \),
- \( \gamma(\mathcal{B}(\text{Cyc}_n)) = f(\Gamma(P_n)) \).

We show that there is a flag ordering for \( \mathcal{B}(\text{Path}_n) \) so that \( \Gamma(\mathcal{B}(\text{Path}_n)) \cong \Gamma(\hat{\mathcal{S}}_n(312)) \), and that the analogous statement is not true for \( \mathcal{B}(K_n) \) and \( \mathcal{B}(\text{Cyc}_n) \).

We also give combinatorial descriptions for particular flag simplicial complexes when \( \mathcal{B} \) is the building set \( \mathcal{B}(K_n) \) or \( \mathcal{B}(K_{1,n-1}) \). The best combinatorial description of \( \mathcal{B}(\text{Path}_n) \) we have found matches the description for the flag simplicial complex \( \Gamma(\hat{\mathcal{S}}_n(312)) \).

3.5.1. The flag simplicial complexes \( \Gamma(\mathcal{B}(K_n)) \) and \( \Gamma(\hat{\mathcal{S}}_n) \).

The permutohedron is the nestohedron \( P_{\mathcal{B}(K_n)} \). Note that \( \mathcal{B}(K_n) \) consists of all non-empty subsets of \( [n] \). Throughout this thesis, \( \mathcal{S}_n \) denotes the set of permutations of the set \( [n] \). A descent of a permutation \( w = w(1) \ldots w(n) \in \mathcal{S}_n \) is a position \( i \in [n-1] \) such that \( w(i) > w(i+1) \). A double descent is a position \( i \in [n-2] \) such that \( w(i) > w(i+1) > w(i+2) \), and a final descent is a descent at position \( n-1 \). \( \hat{\mathcal{S}}_n \) denotes the set of permutations in \( \mathcal{S}_n \) with no double descent or final descent. The \( \gamma \)-polynomial of \( P_{\mathcal{B}(K_n)} \) is the descent generating function of \( \hat{\mathcal{S}}_n \), (see [26, Section 11]).

We will now recall the definition of \( \Gamma(\hat{\mathcal{S}}_n) \) given by Nevo and Petersen [23, Section 4.1]. A peak of a permutation \( w = w_1 \ldots w_n \) in \( \mathcal{S}_n \) is a position \( i \in [n-1] \) such that \( w_{i-1} < w_i > w_{i+1} \), (where \( w_0 := 0 \)). We denote a peak at position \( i \) with a bar \( w_1 \ldots w_i \mid w_{i+1} \ldots w_n \). Let \( \hat{\mathcal{S}}_n \) denote the set of permutations in \( \mathcal{S}_n \) with one peak. Then \( \hat{\mathcal{S}}_n \cap \mathcal{S}_n \) consists of all permutations of the form

\[
\begin{align*}
& w_1 \ldots w_i \mid w_{i+1} \ldots w_n \\
& \text{where } 1 \leq i \leq n-2, \ w_1 < \cdots < w_i, \ w_i > w_{i+1}, \ w_{i+1} < \cdots < w_n.
\end{align*}
\]
Define the flag simplicial complex \( \Gamma(\hat{\mathcal{S}}_n) \) on the vertex set \( \hat{\mathcal{S}}_n \cap \tilde{\mathcal{S}}_n \) where two vertices 
\[
  u = u_1 | u_2
\]
and
\[
  v = v_1 | v_2
\]
with \(|u_1| < |v_1| \) are adjacent if there is a permutation \( w \in \mathfrak{S}_n \) of the form
\[
  w = u_1 | a | v_2.
\]
Equivalently, if \( v_2 \subseteq u_2, |u_2 - v_2| \geq 2, \min(u_2 - v_2) < \max(v_2) \) and \( \max(u_2 - v_2) > \min(v_2) \). (Since there must be two peaks in \( w \) this implies \(|a| \geq 2\).) The faces of \( \Gamma(\hat{\mathcal{S}}_n) \) are the cliques in this graph.

**Example 3.5.1.** Taking only the part after the peak, \( \hat{\mathcal{S}}_5 \cap \tilde{\mathcal{S}}_5 \) can be identified with the set of subsets of \([5]\) of sizes 2, 3 and 4 which are not \( \{4, 5\}, \{2, 3, 4, 5\} \). Then the edges of \( \Gamma(\hat{\mathcal{S}}_5) \) are given by:

- \( \{1, 2, 3, 4\} \) is adjacent to each of \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\} \),
- \( \{1, 2, 3, 5\} \) is adjacent to each of \( \{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 5\} \),
- \( \{1, 2, 4, 5\} \) is adjacent to each of \( \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\} \), and
- \( \{1, 3, 4, 5\} \) is adjacent to each of \( \{3, 4\}, \{3, 5\} \).

**Proposition 3.5.2.** There is no flag ordering of \( \mathcal{B}(K_5) \) so that
\[
  \Gamma(\mathcal{B}(K_5)) \cong \Gamma(\hat{\mathcal{S}}_5).
\]

**Proof.** Suppose for a contradiction that there is some flag ordering of \( \mathcal{B}(K_5) \) with decomposition \( \mathcal{D} \) such that \( \Gamma(\mathcal{B}(K_5)) \cong \Gamma(\hat{\mathcal{S}}_5) \). Then there is some vertex \( v(I_j) \in \Gamma(\mathcal{B}(K_5)) \) of degree 5. We consider the following three cases:

a) \(|I_j| = 2, \)
b) \(|I_j| = 3, \)
c) \(|I_j| = 4. \)

Note that \( \mathcal{D} \) can only be one of the following three building sets (up to permutation of \([5]\)):

- \( \{1\}, ..., \{5\}, \{2\}, \{3\}, \{4\}, \{5\} \),
- \( \{1\}, ..., \{5\}, \{1, 2\}, \{3, 4\}, \{4, 5\} \),
- \( \{1\}, ..., \{5\}, \{2\}, \{3\}, \{4, 5\}, \{5\} \).
(1) Suppose that $|I_j| = 2$. Then $V_j = \emptyset$ and $|U_j| \leq 2$ (using the fact that $D/I_j$ includes at least one 2-element subset). So there are $\geq 3 I_k$’s with $k > j$ and $j \in U_k \cup V_k$ (i.e. $v(I_j)$ is adjacent to $v(I_k)$). Such $I_k$’s must be two element sets not intersecting $I_j$ or four element sets that contain $I_j$. Without loss of generality (WLOG for short), let $I_j = \{4, 5\}$.

(1a) Suppose that no three element set containing $I_j$ occurs earlier than $I_j$. Then the case of 4-element $I_k$’s cannot occur, so $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are the $I_k$’s. Since there is a 2-element set in $D$, we have WLOG $\{3, 5\} \in D$, implying that $\{3, 4, 5\}$ is earlier than $\{4, 5\}$, a contradiction.

(1b) Suppose that exactly one 3-element set containing $I_j$, WLOG $\{3, 4, 5\}$, occurs earlier than $I_j$. Then $\{1, 3\}, \{2, 3\}, \{1, 2, 4, 5\}$ can’t occur among the $I_k$’s, so $\{1, 2\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}$ are the $I_k$’s. Hence $|U_j| = 2$, so $B_{j-1}/I_j$ consists of all non-empty subsets of $\{1, 2, 3\}$. So $\exists J \in B_{j-1}$ such that $J \cap I_j = \{2, 3\}$. But then $J \in B_{j-1}$, $\{3, 4, 5\} \in B_{j-1}$ implies $J \cup \{2, 3, 4\} = \{2, 3, 4, 5\} \in B_{j-1}$, a contradiction.

(1c) Suppose that there are at least two 3-element sets containing $I_j$, WLOG $\{2, 4, 5\}$ and $\{3, 4, 5\}$, that occur earlier than $I_j$. Then the set $\{2, 3, 4, 5\}$ occurs earlier than $I_j$ and $\{1, 2\}, \{1, 3\}, \{2, 3\}$ can’t occur among the $I_k$’s, so we have a contradiction.

(2) Suppose that $|I_j| = 3$. It is easy to see that $v(I_j)$ is not adjacent to any vertices $v(I_k)$ where $i < j$, i.e. $U_j = V_j = \emptyset$. Hence there must be 5 elements $I_k$, $k > j$, such that $j \in V_k \cup U_k$, and these elements must be of size 2. Suppose WLOG that $I_j = \{1, 2, 3\}$, and that the five elements $I_k$ are

$$\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{2, 5\}.$$

There is one two element subset of $I_j$ that is earlier than $I_j$ in the flag ordering since $I_j$ requires a decomposition, and this element must have the same image in the contraction by one of $\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{2, 5\}$ as $\{1, 2, 3\}$, hence this case cannot occur.

(3) Suppose that $|I_j| = 4$. Note that $U_j = \emptyset$.

(3a) Suppose that no three element subset of $I_j$ occurs earlier than $I_j$. Then $V_j = \emptyset$, so there are at least five $I_k$’s $j > k$ such that $j \in U_k \cup V_k$. These $I_k$’s are clearly 2-element subsets of $I_j$,
but for \( I_j \) to have a decomposition in \( \mathcal{B}_j \), two of the 2-element subsets of \( I_j \) must occur earlier than \( I_j \), a contradiction.

(3b) Suppose WLOG that \( I_j = \{1, 2, 3, 4\} \) and that \( \{1, 2, 3\} \) occurs earlier than \( I_j \). Since \( \mathcal{B}_{j-1} \) is a building set no other 3-element subset of \( I_j \) occurs before \( I_j \). If \( v(I_j) \) is adjacent to \( v(I_k) \) then either \( k < j \) which forces \( I_k \) to be a 2-element subset of \( \{1, 2, 3\} \), or \( k > j \) which also forces \( I_k \) to be a two element subset of \( \{1, 2, 3\} \) (so that \( \{1, 2, 3\} \setminus I_k \neq \{1, 2, 3, 4\} \setminus I_k \) and \( \{1, 2, 3, 4, 5\} \setminus I_k \neq \{1, 2, 3, 4\} \setminus I_k \)). So \( v(I_j) \) is adjacent to at most three vertices, a contradiction.

Since we have shown that none of the cases (1), (2) or (3) can occur we have a contradiction, as desired. \( \square \)

We will now give a combinatorial description of \( \Gamma(\mathcal{B}(K_n)) \) for a particular flag ordering. Let \( O \) be the flag ordering of \( \mathcal{B} = \mathcal{B}(K_n) \) with decomposition

\[
\mathcal{D} = \{\{1\}, \{2\}, \ldots, \{n\}, \{2\}, \{3\}, \ldots, \{n\}\}
\]

where elements \( A, B \in \mathcal{B} - \mathcal{D} \) are ordered so that \( A \) is earlier than \( B \) if:

- \( \max(A) < \max(B) \), or
- \( \max(A) = \max(B) \) and \( |A| > |B| \), or
- \( \max(A) = \max(B) \), \( |A| = |B| \) and \( \min(A \triangle B) \in A \)

where \( \triangle \) denotes the symmetric difference between two sets.

Then in \( \Gamma(O) \), vertices corresponding to elements \( A, B \in \mathcal{B} - \mathcal{D} \) are adjacent if either:

- \( A \subseteq B \), \( \max(A) < \max(B) \) and \( |B - A| \geq 2 \),
- \( \max(A) \notin B \), \( \max(A) < \max(B) \) and \( B \cap [1, \max(A)] \subseteq A \).

**Example 3.5.3.** The edges of \( \Gamma(\mathcal{B}(K_5)) \) are between the consecutive vertices in the following three sequences, which form cycles:

\[
v(\{1, 4\}), v(\{1, 2, 4, 5\}), v(\{2, 4\}), v(\{2, 3, 4, 5\}), v(\{3, 4\}), v(\{1, 3, 4, 5\}), v(\{1, 4\}), \text{ and}
\]

\[
v(\{1, 3\}), v(\{1, 2, 3, 5\}), v(\{2, 3\}), v(\{4, 5\}), v(\{1, 3\}), \text{ and}
\]

\[
v(\{1, 2, 4\}), v(\{1, 5\}), v(\{1, 3, 4\}), v(\{3, 5\}), v(\{2, 3, 4\}), v(\{2, 5\}), v(\{2, 4\}).
\]
3.5.2. The flag simplicial complexes \( \Gamma(B(\text{Path}_n)) \) and \( \Gamma(\hat{\mathcal{G}}_n(312)) \).

The associahedron is the nestohedron \( P_B(\text{Path}_n) \). Note that \( B(\text{Path}_n) \) consists of all intervals \([j, k]\) with \( 1 \leq j \leq k \leq n \). The \( \gamma \)-polynomial of the associahedron is the descent generating function of \( \hat{\mathcal{G}}_n(312) \), which denotes the set of 312-avoiding permutations with no double or final descents (see [26, Section 10.2]). We now describe the flag simplicial complex \( \Gamma(\hat{\mathcal{G}}_n(312)) \) defined by Nevo and Petersen [23, Section 4.2].

Given distinct integers \( a, b, c, d \) such that \( a < b \) and \( c < d \), the pairs \((a, b), (c, d)\) are non-crossing if either

- \( a < c < d < b \) (or \( c < a < b < d \)), or
- \( a < b < c < d \) (or \( c < d < a < b \)).

Define \( \Gamma(\hat{\mathcal{G}}_n(312)) \) to be the flag simplicial complex on the vertex set

\[ V_n := \{(a, b) \mid 1 \leq a < b \leq n - 1\} \]

with faces the sets \( S \) of \( V_n \) such that if \((a, b) \in S\) and \((c, d) \in S\) then \((a, b)\) and \((c, d)\) are non-crossing.

Let \( O \) denote the flag ordering of \( B = B(\text{Path}_n) \) with decomposition

\[ D = \{\{1\}, \{2\}, \ldots, \{n\}, [2], [3], \ldots, [n]\} \]

where elements \( A, B \in B - D \) are ordered so that \( A \) is earlier than \( B \) if:

- \( \max(A) < \max(B) \), or
- \( \max(A) = \max(B) \) and \( |A| > |B| \).

Proposition 3.5.4. For the flag ordering \( O \) of \( B = B(\text{Path}_n) \) described above, \( \Gamma(O) \cong \Gamma(\hat{\mathcal{G}}_n(312)) \) where the bijection on the vertices is given by \( v([a + 1, b + 1]) \mapsto (a, b) \).

Proof. Since \( B - D = \{[j, k] \mid 2 \leq j < k \leq n\} \), it is clear that the stated map on vertices is a bijection. Let \([l, m], [j, k]\) be distinct elements of \( B - D \) with \([l, m]\) occurring before \([j, k]\). Then \( m \leq k \), and if \( m = k \) we have \( l < j \). If \([l, m] \not\subseteq [j, k]\) then \( v([l, m]) \) is adjacent to \( v([j, k]) \) if and only if \( m < j \). If \([l, m] \subseteq [j, k]\) (which entails \( m < k \)), then \( v([l, m]) \) is adjacent to \( v([j, k]) \) if and only if \( j < l \). So in either case \( v([l, m]) \) is adjacent to \( v([j, k]) \) if and only if \((l - 1, m - 1)\) and \((j - 1, k - 1)\) are non-crossing.

\[ \square \]

3.5.3. The flag simplicial complexes \( \Gamma(B(\text{Cyc}_n)) \) and \( \Gamma(P_n) \).

The cyclohedron is the nestohedron \( P_B(\text{Cyc}_n) \). Note that \( B(\text{Cyc}_n) \) consists of all sets \( \{i, i+1, i+2, \ldots, i+s\} \) where \( i \in [n], s \in \{0, 1, \ldots, n-1\} \), and the elements are taken mod \( n \). By [26, Proposition 11.15] \( \gamma_r(B(\text{Cyc}_n)) = \binom{n}{r, r, n-2r} \). We now describe the flag simplicial complex \( \Gamma(P_n) \) defined by
Nevo and Petersen [23, Section 4.3].

Define the vertex set

\[ V_{P_n} := \{(l, r) \in [n-1] \times [n-1] \mid l \neq r\}. \]

\( \Gamma(P_n) \) is the flag simplicial complex on the vertex set \( V_{P_n} \) where vertices \((l_1, r_1), (l_2, r_2)\) are adjacent in \( \Gamma(P_n) \) if and only if \( l_1, l_2, r_1, r_2 \) are all distinct and either \( l_1 < l_2 \) and \( r_1 < r_2 \), or \( l_2 < l_1 \) and \( r_2 < r_1 \).

**Example 3.5.5.** \( \Gamma(P_5) \) is the flag simplicial complex on vertices

\[ V_{P_5} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\} \]

with edges

\[ \{(1, 3), (2, 4)\}, \{(3, 1), (4, 2)\}, \{(1, 2), (3, 4)\}, \{(1, 2), (4, 3)\}, \{(2, 1), (3, 4)\}. \]

Note that \( \Gamma(P_5) \) has exactly two vertices of degree two, and has six connected components, four of which contain more than one vertex.

**Proposition 3.5.6.** There is no flag ordering of \( B(\text{Cyc}_5) \) such that

\[ \Gamma(B(\text{Cyc}_5)) \cong \Gamma(P_5). \]

**Proof.** Suppose for a contradiction that there is some flag ordering of \( B = B(\text{Cyc}_5) \) with decomposition \( \mathcal{D} \) such that \( \Gamma(B(\text{Cyc}_5)) \cong \Gamma(P_5) \). It is not too hard to show that if vertices \( v(A) \) and \( v(B) \) are adjacent then at least one of \( A \) or \( B \) is a 2-element set. Therefore there must be at least one vertex that corresponds to a building set element of size two in each of the four non-singleton connected components of \( \Gamma(B(\text{Cyc}_5)) \). Since there must be one two element subset in \( \mathcal{D} \) this implies that there is exactly one vertex corresponding to a two element set in each non-singleton connected component, and these include the vertices of degree two.

The possibilities for \( \mathcal{D} \) (up to a cyclic permutation of [5]) are

\[ \mathcal{D}_1 = \{[5], [4], [3], [2], \{1\}, \ldots, \{5\}\}, \]
\[ \mathcal{D}_2 = \{[5], [2], \{5, 1, 2\}, \{5, 1, 2, 3\}, \{1\}, \ldots, \{5\}\}, \]
\[ \mathcal{D}_3 = \{[5], [4], \{1, 2\}, \{3, 4\}, \{1\}, \ldots, \{5\}\}, \]
\[ \mathcal{D}_4 = \{[5], [3], \{4, 5\}, \{1, 2\}, \{1\}, \ldots, \{5\}\}. \]
3.5. The flag simplicial complexes of Nevo and Petersen

The flag ordering must have decomposition $D_1$ or $D_2$ since there are four elements of size two in $B(C_{yc5}) - D$. We will show that if $D$ is $D_1$ or $D_2$ then there must be two vertices in $\Gamma(B(C_{yc5}))$ that are adjacent that correspond to building set elements of size two, a contradiction.

The size two elements in $B - D_1$ and $B - D_2$ are $\{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}$. If $I_j \in B - D_1$ is earlier in the flag ordering than every one of these size two elements, then $I_j$ must contain $\{1, 2\}$ since otherwise it would not have a decomposition in $B_j$. So the only elements of $B - D_1$ that can be earlier in the flag ordering than every element of size two are

$$S_1 = \{\{5, 1, 2, 3\}, \{5, 1, 2\}, \{4, 5, 1, 2\}\}.$$

Similarly, the elements of $B - D_2$ that can be earlier in the flag ordering than every element of size two are

$$S_2 = \{\{1, 2, 3\}, \{4, 5, 1, 2\}, \{1, 2, 3, 4\}\}.$$

Consider which of the size two elements in $B - D_1$ is earliest. Suppose that $\{1, 5\}$ is earliest in the flag order. Then $v(\{3, 4\})$ is adjacent to $v(\{1, 5\})$ since $\{1, 5\} \notin (D_1 \cup S_1)/\{3, 4\} = (D_2 \cup S_2)/\{3, 4\}$.

Suppose that $\{2, 3\}$ is earliest in the flag order. Then $v(\{4, 5\})$ is adjacent to $v(\{2, 3\})$ since $\{2, 3\} \notin (D_1 \cup S_1)/\{4, 5\} = (D_2 \cup S_2)/\{4, 5\}$.

Suppose that $\{3, 4\}$ is earliest in the flag order. Then $v(\{1, 5\})$ is adjacent to $v(\{3, 4\})$ since $\{3, 4\} \notin (D_1 \cup S_1)/\{1, 5\} = (D_2 \cup S_2)/\{1, 5\}$.

Suppose that $\{4, 5\}$ is earliest in the flag order. Then $v(\{2, 3\})$ is adjacent to $v(\{4, 5\})$ since $\{4, 5\} \notin (D_1 \cup S_1)/\{2, 3\} = (D_2 \cup S_2)/\{2, 3\}$.

3.5.4. The flag simplicial complex $\Gamma(B(K_{1,n-1}))$

Here we give a combinatorial description of $\Gamma(B(K_{1,n-1}))$ for a particular flag ordering. $B = B(K_{1,n-1})$ is the graphical building set for the graph $K_{1,n-1}$ where we assume the vertex of degree $n - 1$ is labelled 1. Then $B(K_{1,n-1})$ consists of all subsets of $[n]$ containing 1, together with $\{2\}, \{3\}, \ldots, \{n\}$. Let $O$ be the flag ordering with decomposition

$$D = \{\{1\}, \{2\}, \ldots, \{n\}, [2], [3], \ldots, [n]\},$$

where $A, B \in B - D$ are ordered so that $A$ is earlier than $B$ if:

- $\max(A) < \max(B)$, or
- $\max(A) = \max(B)$ and $|A| > |B|$, or

$\square$
The result follows. □

Example 3.5.7. The edges of $\Gamma(\mathcal{B}(K_{1,4}))$ are:

\[
\begin{align*}
\{v(\{1, 3\}), v(\{1, 2, 3, 5\})\}, \\{v(\{1, 5\}), v(\{1, 2, 4\})\}, \\{v(\{1, 5\}), v(\{1, 3, 4\})\}, \\{v(\{1, 3, 4, 5\}), v(\{1, 4\})\}, \\{v(\{1, 2, 4, 5\}), v(\{1, 4\})\}.
\end{align*}
\]

Proposition 3.5.8. The following recurrence holds for the $\gamma$-polynomial of the \((n-1)\)-dimensional stellohedron:

\[
\gamma(\mathcal{B}(K_{1,n-1})) = \gamma(\mathcal{B}(K_{1,n-2})) + t \sum_{i=0}^{n-3} \binom{n-2}{i} \gamma(\mathcal{B}(K_{1,i})) \gamma(\mathcal{B}(K_{n-2-i})).
\]

Proof. We use the flag ordering of $\mathcal{B}(K_{1,n-1})$ defined in this section. Consider an element in $\mathcal{B}(K_{1,n-1}) - \mathcal{D}$ that does not include $n$. Such elements are added before those elements that do include $n$. They are also added in the order that corresponds to the same flag ordering for $\mathcal{B}(K_{1,n-2})$, and therefore contribute the $\gamma(\mathcal{B}(K_{1,n-2}))$ term.

Now consider an element $I_k \in \mathcal{B}(K_{1,n-1}) - \mathcal{D}$ that contains $n$. Suppose that $|I_k| = i + 2$. There are $\binom{n-2}{i}$ such elements, and $i$ ranges from 0 to $n-3$. Then $\mathcal{B}(K_{1,n-1})_{k-1} | I_k = \mathcal{B}(\mathcal{B}(K_{1,i}), [1])$, where $\mathcal{B} = \{\{1\}, \{2\}, [2]\}$. Hence

\[
\gamma(\mathcal{B}(K_{1,n-1})_{k-1} | I_k) = \gamma(\mathcal{B}(K_{1,i})).
\]

Also

\[
\mathcal{B}(K_{1,n-1})_{k-1} / I_k = \mathcal{B}(K_{n-2-i})
\]

so that

\[
\gamma(\mathcal{B}(K_{1,n-1})_{k-1} / I_k) = \gamma(\mathcal{B}(K_{n-2-i})).
\]

The result follows. □

Note that Proposition 3.5.8 is useful for computing $\gamma(\mathcal{B}(K_{1,i}))$ since $\gamma(\mathcal{B}(K_n))$ is easy to compute, for example, it has a descent interpretation mentioned in Section 3.5.1.
3.5.5. **Table of values of $\gamma$-polynomials.**

In this section we include a table to values for the $\gamma$-polynomials for some graph-associahedra. The values of $\gamma(B(K_{1,i}))$ are calculated using Recurrence of Proposition 3.5.8.

**Table 1.** $\gamma$-polynomials of graph-associahedra.

<table>
<thead>
<tr>
<th>Graph</th>
<th>$\gamma$-polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>1</td>
</tr>
<tr>
<td>$K_2$</td>
<td>1</td>
</tr>
<tr>
<td>$K_3$</td>
<td>$1 + 2t$</td>
</tr>
<tr>
<td>$K_4$</td>
<td>$1 + 8t$</td>
</tr>
<tr>
<td>$K_5$</td>
<td>$1 + 22t + 16t^2$</td>
</tr>
<tr>
<td>$K_6$</td>
<td>$1 + 52t + 136t^2$</td>
</tr>
<tr>
<td>$K_{1,2}$</td>
<td>$1 + t$</td>
</tr>
<tr>
<td>$K_{1,3}$</td>
<td>$1 + 4t$</td>
</tr>
<tr>
<td>$K_{1,4}$</td>
<td>$1 + 11t + 5t^2$</td>
</tr>
<tr>
<td>$K_{1,5}$</td>
<td>$1 + 26t + 43t^2$</td>
</tr>
</tbody>
</table>
Chapter 4

Inequalities between gamma-polynomials of graph-associahedra

In this section we present the work in the paper [3], in which we prove Conjecture 0.0.4 of Postnikov, Reiner and Williams:

There exists a (nontrivial) partial order \( \leq \) on the set of (unlabelled, isomorphism classes of) trees with \( n \) vertices, with the following properties:

- \( \text{Path}_n \) is the unique \( \leq \)-minimal element,
- \( K_{1,n-1} \) is the unique \( \leq \)-maximal element,
- \( T \leq T' \) implies \( \gamma(B(T)) \leq \gamma(B(T')) \).

This conjecture implies the following lower and upper bounds for the \( \gamma \)-polynomial of a tree \( T \) with \( n \) vertices

\[
\gamma(B(\text{Path}_n)) \leq \gamma(B(T)) \leq \gamma(B(K_{1,n-1})).
\]

These upper and lower bound theorems have been proven by Buchstaber and Volodin [8, Theorem 9.4]. Moreover, they show that the lower bound is attained only for \( \text{Path}_n \) and the upper bound is attained only for \( K_{1,n-1} \). Their proof relies on some general results about \( \gamma \)-polynomials of flag nestohedra which were announced in [31] and whose proofs are included in [8]; see lemmas 3.2.2, 3.2.3, 1.4.11 and theorems 3.2.1, 3.2.4 and 3.2.5. Note that the methods of Buchstaber and Volodin require one to work with the more general class of flag nestohedra in order to deduce the results about graph-associahedra. We use the above listed theorems of [31] to prove Conjecture 0.0.4. Note that Buchstaber and Volodin had proven Conjecture 0.0.4 for the trivial partial order, that consists only of the relations \( \text{Path}_n \leq T \leq K_{1,n-1} \) for all trees \( T \) with \( n \) vertices.

We prove Conjecture 0.0.4 by defining a partial order on the set of tree graphs with \( n \) vertices that induces inequalities between the \( \gamma \)-polynomials of their associated graph-associahedra. The partial order is given by relating trees that can be obtained from one another by operations called tree shifts. We also show that tree shifts lower the \( \gamma \)-polynomials of graphs that are not trees, as do the flossing moves of Babson and Reiner originally defined in [6, Section 4.2]. It was suggested that flossing moves might lower the
4.1. TREE SHIFTS

\( \gamma \)-polynomial in [26, Section 14]. Our definition of flossing move is more general than that in [6] as it can be applied to any pair of leaves that floss a vertex, and it does not have to be applied to a tree graph.

In this chapter Section 4.1 introduces tree shifts and in Theorem 4.1.1 we show that they lower the \( \gamma \)-polynomial of the graph-associhedra. We then prove Conjecture 0.0.4, in Theorem 4.1.2. Section 4.2 introduces flossing moves and Theorem 4.2.1 shows that they lower the \( \gamma \)-polynomials.

4.1. Tree shifts

Let \( G \) be a connected graph with \( n \) vertices labelled 1 to \( n \), with the following properties and extra data (for a vertex \( v \) we also denote the set \( \{v\} \) by \( v \)):

a) \( G \) has a leaf \( l \) and the nearest vertex to \( l \) of degree greater than 2 is labelled \( c \). The vertices in the path from \( c \) to \( l \) are labelled \( c, c_1, c_2, \ldots, c_k, l \).

b) There exists a set of vertices \( F \) of \( G - \{c, c_1, \ldots, l\} \) such that \( F \cup c \) is a subgraph of \( G \) that forms a tree, and such that there is no vertex of \( G - (c \cup F) \) that is connected to a vertex in \( F \).

c) \( G - (F \cup \{c, c_1, c_2, \ldots, c_k, l\}) \neq \emptyset \), and is denoted \( E \).

A tree shift is the following move applied to a graph with the properties described. Informally, we remove \( F \) and reattach \( F \) to \( l \). More formally, we remove any edge \( (v, c) \) where \( v \in F \), and replace it with the edge \( (v, l) \) (see Figure 1).

**Figure 1.** A graph \( G \) followed by the tree shift of \( G \).
**Theorem 4.1.1.** Let $G$ be a connected graph, and let $G'$ be a resulting tree shift of $G$. Then $\gamma(\mathcal{B}(G')) \leq \gamma(\mathcal{B}(G))$.

**Proof.** We suppose that $G$ has $n$ vertices, and we label $G$ as in the definition of a tree shift. We assume by induction that for any connected graph $H$ with less than $n$ vertices, if $H'$ is a tree shift of $H$, then $\gamma(\mathcal{B}(H')) \leq \gamma(\mathcal{B}(H))$.

When $n < 4$ no tree shift is possible so the result is vacuously true. Let $v$ be a leaf of $G$ (and $G'$) contained in $F$. The set $\mathcal{B} := \mathcal{B}(G - v) \cup \{\{v\}, [n]\}$ is a flag building set contained in $\mathcal{B}(G)$ and $\overline{\mathcal{B}} = \mathcal{B}(G' - v) \cup \{\{v\}, [n]\}$ is a flag building set contained in $\mathcal{B}(G')$, hence, by Theorem 3.2.1 we can add elements to $\overline{\mathcal{B}}$ to obtain $\mathcal{B}(G)$ so that at each step the set obtained is a flag building set. Similarly, we can add elements to $\overline{\mathcal{B}}$ to obtain $\mathcal{B}(G')$ so that at each step the set we obtain is a flag building set. By Lemma 3.2.3 and Theorem 3.2.4 each time an element is added to these flag building sets the $\gamma$-polynomial of the resulting building set increases. We will construct an injection

$$\mathcal{B}(G') - \overline{\mathcal{B}} \rightarrow \mathcal{B}(G) - \overline{\mathcal{B}}$$

$I' \mapsto I,$

and show that the increase in the $\gamma$-polynomial when adding $I'$ is less than or equal to the increase when adding $I$. This shows that

$$\gamma(\mathcal{B}(G')) - \gamma(\overline{\mathcal{B}}) \leq \gamma(\mathcal{B}(G)) - \gamma(\overline{\mathcal{B}}).$$

By Lemma 1.4.11, since $\overline{\mathcal{B}} = D_2[\mathcal{B}(\{v\})], \mathcal{B}(G - v)]$, where $D_2$ is the connected building set on $\{1, 2\}$, we have

$$\gamma(\overline{\mathcal{B}}) = \gamma(\mathcal{B}(G - v)),$$

and similarly

$$\gamma(\overline{\mathcal{B}}') = \gamma(\mathcal{B}(G' - v)),$$

so that Equation 4.2 becomes

$$\gamma(\mathcal{B}(G')) - \gamma(\mathcal{B}(G' - v)) \leq \gamma(\mathcal{B}(G)) - \gamma(\mathcal{B}(G - v)).$$

By induction, since $G' - v$ is a tree shift of $G - v$, or is equal to $G - v$, we have

$$\gamma(\mathcal{B}(G' - v)) \leq \gamma(\mathcal{B}(G - v))$$

so that

$$\gamma(\mathcal{B}(G')) \leq \gamma(\mathcal{B}(G)).$$
We will now construct the injection. Suppose that \( I'_1, I'_2, \ldots, I'_k \) are the building set elements that are added to \( \overline{B} \) to obtain \( \mathcal{B}(G') \) (in order) and \( I'_j \subseteq I'_i \). Then \( j > i \), since \( I'_j \cap (I'_i \setminus \{v\}) \neq \emptyset \) and \( I'_j \cup (I'_i \setminus \{v\}) = I'_i \) which implies that when \( I'_j \) is in the building set \( I'_i \) must be too. Similarly, no subset of an element is added before it when we are adding sets to obtain \( \mathcal{B}(G) \).

Let \( \mathcal{B}'_m \) be the building set \( \mathcal{B}' \cup \{I'_1, I'_2, \ldots, I'_m\} \). By Lemma 3.2.3 we have that
\[
\gamma(\mathcal{B}'_m) - \gamma(\mathcal{B}'_{m-1}) = t \gamma(\mathcal{B}'_{m-1}|I'_m) \gamma(\mathcal{B}'_{m-1}/I'_m).
\]
Suppose that \( I'_m \cap E = \emptyset \), so that \( I'_m = D \cup \{l, c_k, \ldots, c_{k-\alpha+1}\} \) for some \( D \subseteq F \) and let \( I_m = D \cup \{c, c_1, \ldots, c_\alpha\} \), one of the elements that is added to \( \overline{B} \) to obtain \( \mathcal{B}(G) \). Note that we may have \( c_{k-\alpha+1} = c \) and \( c_\alpha = l \). Note also that \( I_m \) is not necessarily the \( m \)th element that is added to \( \overline{B} \) (see Figure 2).

\[
\begin{align*}
\text{Figure 2.} \quad &\text{The set } I_m \text{ followed by the set } I'_m. \\
&\text{We let } \mathcal{B}_m \text{ denote the building set obtained after adding the elements up to and including } I_m \text{ to } \overline{B}. \text{ Let } \mathcal{B}_{m-1} \text{ denote the building set } \mathcal{B}_m - \{I_m\} \text{ (note that } \mathcal{B}_{m-1} \text{ is not necessarily equal to } \mathcal{B}_{m-1} \text{ since } I_{m-1} \text{ is not necessarily added directly before } I_m). \text{ Then by Lemma 3.2.3} \\
&\gamma(\mathcal{B}_m) - \gamma(\mathcal{B}_{m-1}) = t \gamma(\mathcal{B}_{m-1}|I_m) \gamma(\mathcal{B}_{m-1}/I_m). \\
&\text{Since we do not add a subset of a set before adding the set, we have that} \\
&\mathcal{B}_{m-1}|I_m = \mathcal{B}(G)|I_{m-1}\{v\} \cup \{v\} \cong \mathcal{B}(G')|I_{m-1}\{v\} \cup \{v\} = \mathcal{B}'_{m-1}|I'_m. \\
&\text{We let } K' \text{ denote the set of vertices in } G' - I'_m \text{ that are adjacent in } G' \text{ to a vertex in } I'_m, \text{ and we let } K \text{ denote the set of vertices in } G - I_m \text{ that are adjacent in } G \text{ to a vertex in } I_m. \text{ Then } \mathcal{B}'_{m-1}/I'_m = \mathcal{B}(G')/I'_m. \text{ This is true since we know that } \mathcal{B}'_{m-1}/I'_m \subseteq \mathcal{B}(G')/I'_m \text{ since } \mathcal{B}'_{m-1} \subseteq \mathcal{B}(G'). \text{ To}
\end{align*}
\]
show that $B'_{m-1}/I'_m \supseteq B(G')/I'_m$, note that $B(G')/I'_m = B(\hat{G}')$ where $\hat{G}'$ is the graph $G' - I'_m$ with additional edges so that the restriction to $K'$ is a complete graph. The elements of $B(\hat{G}')$ that are the edges between elements in $K'$ are in $B'_{m-1}/I'_m$ because any two vertices in $K'$ are linked by a path of vertices contained in $I'_m - v$. By a similar argument we have that $\tilde{B}_{m-1}/I_m = B(G)/I_m$. Note that $B(G)/I_m = B(\hat{G})$ where $\hat{G}$ denotes the graph $G - I_m$ with additional edges so that the restriction to $K$ is a complete graph, (see Figure 3).

**Figure 3.** The graph $\hat{G}$ for the contraction $B_{m-1}/I_m = B(\hat{G})$ followed by the graph $\hat{G}'$ for the contraction $B'_{m-1}/I'_m = B(\hat{G}')$. The vertices $K$ and $K'$ are drawn with an additional ring around them.

We also have that $\gamma(B'_{m-1}/I'_m) \leq \gamma(\tilde{B}_{m-1}/I_m)$ because $\hat{G}'$ can be obtained from $\hat{G}$ by first removing edges (which lowers the $\gamma$-polynomial of the corresponding graphical building set by Theorem 3.2.5) and then performing a tree shift on a graph with fewer than $n$ vertices (or doing no tree shift in the case that $c_\alpha = c_k$ or $c_\alpha = l$), which we assume lowers the $\gamma$-polynomial (see Figure 4). Hence

$$\gamma(B'_{m}) - \gamma(B'_{m-1}) \leq t \gamma(B'_{m-1}/I'_m) \gamma(B'_{m-1}/I'_m) \leq t \gamma(\tilde{B}_{m-1}/I_m) \gamma(\tilde{B}_{m-1}/I_m) = \gamma(B_{m}) - \gamma(\tilde{B}_{m-1}).$$
Figure 4. The graph that is obtained after removing edges from $\hat{G}$ in Figure 3. The tree shift of this graph gives the graph $\hat{G}'$ of Figure 3.

Now suppose that $I_m' \cap E \neq \emptyset$, so that $\{c, c_1, ..., c_k, l\} \subseteq I'_m$. Let $I_m$ denote $I'_m$, which is a set that is also added to $\bar{B}$ to obtain $B(G)$ (see Figure 5). Define $\bar{B}'_{m-1}$, $\bar{B}_{m-1}$ as in the previous case.

Figure 5. The set $I_m$ followed by the set $I'_m$.

Then we have that $\tilde{B}_{m-1}|I_m = B'_{m-1}|I'_m$ and $\tilde{B}_{m-1}/I_m = B'_{m-1}/I'_m$ (which is equal to $B(G)/I_m$). This can be shown by arguments similar to those used in the case where $I'_m \cap E = \emptyset$. Hence in this case we also have

$$\gamma(B'_m) - \gamma(B'_{m-1}) = t\gamma(\bar{B}_{m-1}|I'_m)\gamma(\bar{B}_{m-1}/I'_m)$$

$$= t\gamma(\bar{B}_{m-1}|I_m)\gamma(\bar{B}_{m-1}/I_m)$$

$$= \gamma(B_m) - \gamma(\bar{B}_{m-1}).$$
Since for every element $I'_m$ that is added to $B'$ to obtain $B(G')$ there is a corresponding element $I_m$ that is added to $B$ to obtain $B(G)$ that increases the $\gamma$-polynomial by at least as much as $I'_m$, we have that
\[
\gamma(B(G')) - \gamma(B') \leq \gamma(B(G)) - \gamma(B)
\] as desired.

By applying Theorem 4.1.1 to the case where the graph is a tree we obtain the following Theorem, which was predicted by [26, Conjecture 14.1].

**Theorem 4.1.2.** Let $S$ be the set of all tree graphs on $n$ nodes. Define the relation $T' \leq T$ if $T'$ can be obtained by applying any number of tree shifts to $T$. Then $\leq$ defines a partial order on $S$ with the following properties.

- Path$_n$ is the unique $\leq$-minimum element.
- $K_{1,n-1}$ is the unique $\leq$-maximum element.
- $T' \leq T$ implies $\gamma(B(T')) \leq \gamma(B(T))$.

**Proof.** This relation is a partial order on $S$, since given any $a, b \in S$ we have that if $a \leq b$ and $b \leq a$ then $a = b$ because any tree shift decreases the number of leaves by one.

Path$_n$ is $\leq$-minimal since no tree has fewer leaves than Path$_n$. Let $T$ be a tree that is not Path$_n$. We can apply a tree shift to $T$ since given any leaf $l$, there must exist a sequence $l = a_1, ..., a_k$ of vertices such that $a_i$ is adjacent to $a_{i+1}$ for all $i = 1, ..., k-1$, and such that $a_k$ is of degree 3 or more. Hence $T$ is not $\leq$-minimal, so that Path$_n$ is the unique $\leq$-minimum element.

$K_{1,n-1}$ is $\leq$-maximal because no tree has more leaves than $K_{1,n-1}$. If $T' \neq K_{1,n-1}$, then $T'$ must contain two adjacent vertices $c$ and $l$, neither of which is a leaf. Let $C$ be the component of $T' - \{c, l\}$ that was attached to $l$ in $T'$. Let $T'$ be the tree obtained from $T'$ by attaching $C$ to $c$, so that the vertices that were adjacent to $l$ in $T'$ are adjacent to $c$ in $T$. Then $T'$ is obtained from $T$ by a tree shift. Hence $T'$ is not $\leq$-maximal, so that $K_{1,n-1}$ is the unique $\leq$-maximum element.

By Theorem 4.1.1, if $T' \leq T$ then $\gamma(B(T')) \leq \gamma(B(T))$. \qed

Theorem 4.1.1 provides a new (arguably more explicit) proof of the bounds on the $\gamma$-polynomial of trees (Equation 4.1) than that provided in [8, Theorem 9.4, (1)].
The following proposition shows that any tree shift that does not fix the underlying graph will decrease the $\gamma$-polynomial.

**Proposition 4.1.3.** Suppose that $T$ is a graph on $[n]$, and $T'$ is a tree shift of $T$. If $T'$ is not the same graph as $T$ (without considering labels), then $\gamma(B(T')) < \gamma(B(T))$.

**Proof.** We recall the proof of Theorem 4.1.1, in which an injection is constructed from $B(G') - \overline{B}$ to $B(G) - \overline{B}$. We will show that this injection is not onto, by describing an element $I$ that is not in its image. By Lemma 3.2.3, when $I$ is added to $\overline{B}$ to obtain $B(G)$, there is a strict increase in the $\gamma$-polynomial of $B(T)$. The proof of Theorem 4.1.1 shows that $\gamma(B(T')) \leq \gamma(B(T))$, without considering the contribution of adding $I$. Thus, by considering the element $I$, we have that $\gamma(B(T')) < \gamma(B(T))$.

Since $T'$ is not the same graph as $T$, this implies that (in the notation of Theorem 4.1.1) the set $E$ is non empty, and that the vertex $l$ is not equal to the vertex $c$. Let $e$ be a vertex in $E$ that is adjacent to $c$. Let $I$ be the element in $B(G)$ that is the unique path of minimal length in $G$ from $e$ to $v$. By considering the cases listed in the proof of Theorem 4.1.1, we see that $I$ is not in the image of the map. □

### 4.2. Flossing moves

Let $G$ be a graph with $n$ vertices labelled 1 to $n$. A pair of leaves $l, \hat{l}$ in $G$ floss a vertex $v \in G$ if there is a unique path in $G$ from $l$ to $\hat{l}$ of minimal length, and $v$ is the unique branched vertex (having degree $\geq 3$) on this path. [6, Proposition 4.8] shows that for any tree graph $T$ that is not $\text{Path}_n$, there exists a triple of vertices $(l, \hat{l}, v)$ in which the vertices $l, \hat{l}$ floss the vertex $v$. When $l, \hat{l}$ floss a vertex $v$, relabel so that

$$\text{dist}_G(l, v) \leq \text{dist}_G(\hat{l}, v),$$

where $\text{dist}_G(v_1, v_2)$ denotes the number of edges in a minimal path in $G$ between vertices $v_1$ and $v_2$. Flossing moves are defined in [6], and it was suggested in [26] that they might lower the $\gamma$-polynomial of the graph-associahedra. We show that this is true for flossing moves that are a generalisation of those given in [6]. Let $G$ be a graph with a triple of vertices $(l, \hat{l}, v)$ such that $l, \hat{l}$ are leaves that floss the vertex $v$ (and $\text{dist}_G(l, v) \leq \text{dist}_G(\hat{l}, v)$). A flossing move on $G$ is obtained by removing the edge $(l, w)$ and adding an edge $(\hat{l}, l)$ where $w$ is the nearest vertex (possibly $v$) to $l$. We let $r := \text{dist}_G(l, v) + 1$ (the number of vertices in the chain from $l$ to $v$), and $\hat{r} := \text{dist}_G(\hat{l}, v) + 1$ (see Figure 6).
**Theorem 4.2.1.** Let $G$ be a connected graph, and let $G'$ be the resulting flossing move of $G$. Then $\gamma(B(G')) \leq \gamma(B(G))$.

**Proof.** We suppose that $G$ has $n$ vertices, and we label $G$ by $l, \hat{l}, r, \hat{r}, v$ and $w$, as in the definition of flossing move. We assume by induction that for any graph with $< n$ vertices, that a flossing move lowers the $\gamma$-polynomial. When $n < 4$ no flossing move is possible so the result is vacuously true. $B(G)$ is a flag building set on $[n]$, and the building set $\hat{B}$ that is obtained from $B(G)$ by removing all building set elements that contain $\{l, w\}$ apart from $[n]$ is also a flag building set on $[n]$. Hence by Theorem 3.2.1, $B(G)$ can be obtained from $\hat{B}$ by successively adding building set elements so that at each step the set is a flag building set. Similarly, $B(G')$ can be obtained from $\hat{B}$ by successively adding building set elements so that at each step the set is a flag building set. Similar to the arguments used in the proof of Theorem 4.1.1, we construct an injection

$$B(G') - \hat{B} \to B(G) - \hat{B}$$

$$I' \mapsto I.$$  

We then show that the increase in the $\gamma$-polynomial when adding the element in $B(G') - \hat{B}$ is less than or equal to the increase when adding the corresponding element in $B(G) - \hat{B}$ which proves the Theorem.

Let $I_1, I_2, \ldots, I_k$ be the building set elements of $B(G') - \hat{B}$. Suppose for some $i \neq j$ that $I_j \subseteq I_i$. Then $j > i$, since $I_j \cap (I_i - \{l\}) \neq \emptyset$ and $I_j \cup (I_i - \{l\}) = I_i$ which implies that when $I_j$ is in the building set $I_i$ must be too.
4.2. LOSSING MOVES

Let $P$ be the set of vertices in the minimal path from $l$ to $\hat{l}$. Let $I'$ be an element that is added to $\hat{B}$ to obtain $B(G')$. There are three cases for $I'$ that we will consider.

- $|I'| \leq \hat{r}$,
- $|I'| \geq \hat{r} + 1$, and $I'$ does not contain all of $G - P$,
- $|I'| \geq \hat{r} + 1$, and $I'$ contains all of $G - P$.

Suppose that $|I'| \leq \hat{r}$, and let $I$ be the element of $B(G) - \hat{B}$ such that $|I \cap P| = r + \hat{r} - |I'|$, and $I$ contains all of $G - P$. In each case we let $B_1$ (respectively $B_2$) denote the building sets we have before adding $I$ (respectively $I'$). Then $B_1|_I = B_2/I' \cup \{\{l\}\}$, so that $\gamma(B_1|_I) = \gamma(B_2/I')$. Also, $B_1/I \cup \{\{l\}\} = B_2/I'$, so that $\gamma(B_1/I) = \gamma(B_2/I')$ (see Figure 7).

**Figure 7.** The graph $G$ followed by $G'$. Keeping with the values of Figure 6, we have $|I'| = 5$ and $|I \cap P| = 6$.

Suppose that $|I'| \geq \hat{r} + 1$, and suppose that $I'$ does not contain all of $G - P$. Let $I$ be the element of $B(G) - \hat{B}$ such that $|I \cap P| = |I' \cap P|$, and $I \cap (G - P) = I' \cap (G - P)$. Then we have that $B_1/I \cong B_2/I'$, and $B_1|_I = B(G_1) \cup \{\{l\}\}$, and $B_2/I' = B(G_2) \cup \{\{l\}\}$ where $G_2$ is a graph obtained from a graph $G_1$ by a flossing move (or if $\text{dist}_G(l, v) = 1$, $G_2 = G_1$). By induction on the number of vertices of the graphs involved we have that $\gamma(B(G_2)) \leq \gamma(B(G_1))$ so that $\gamma(B_2/I') \leq \gamma(B_1|_I)$ (see Figure 8).
Figure 8. The graph $B_1$ followed by $B_2$. We have $|I'| \geq 7$.

Suppose that $|I'| \geq \hat{r} + 1$ and $I'$ contains all of $G - P$. Let $I$ be the element of $B(G) - \hat{B}$ such that $|I| = r + \hat{r} - |I' \cap P|$. Then $B_1/I \cup \{\{l\}\} = B_2/I$, and $B_1|_I = B_2/I' \cup \{\{l\}\}$. Hence $\gamma(B_1/I) = \gamma(B_2/I)$ and $\gamma(B_1|_I) = \gamma(B_2/I')$ (see Figure 9).

Figure 9. The graph $B_1$ followed by $B_2$. We have $|I| = 2$ and $|I' \cap P| = 9$.

Note that no element $I \in B(G) - \hat{B}$ is used more than once, since in the first case we have that $|I| \geq r$ and $I$ contains all of $G - P$. In the second case we have that $|I| \geq \hat{r} + 1 > r$ and $I$ does not contain all of $G - P$. In the third case we have that $|I| = \hat{r} + \hat{r} - |I' \cap P| \leq \hat{r} + \hat{r} - (\hat{r} + 1) = r - 1$.

By Lemma 3.2.3 the change in the $\gamma$-polynomial when adding $I'$ is given by

$$
\gamma(B_2 \cup \{I'\}) - \gamma(B_2) = t \gamma(B_2/I') \gamma(B_2/I),
$$

and when adding $I$ it is given by

$$
\gamma(B_1 \cup \{I\}) - \gamma(B_1) = t \gamma(B_1/I) \gamma(B_1/I).
$$
4.2. FLOSSING MOVES

Since for every element $I'$ that is added to $\hat{\mathcal{B}}$ to obtain $\mathcal{B}(G')$, there is an element $I$ that is added to $\hat{\mathcal{B}}$ to obtain $\mathcal{B}(G)$ such that $\gamma(\mathcal{B}_{2}/I')\gamma(\mathcal{B}_{2}|_{I'}) \leq \gamma(\mathcal{B}_{1}/I)\gamma(\mathcal{B}_{1}|_{I})$ we have that $\gamma(\mathcal{B}(G')) \leq \gamma(\mathcal{B}(G))$.

The following proposition shows that any flossing move will strictly decrease the $\gamma$-polynomial. This is a possibility since by definition a flossing move cannot be applied to $\text{Path}_{n}$.

**Proposition 4.2.2.** Suppose that $G$ is a graph on $[n]$, and $G'$ is obtained from $G$ by a flossing move. Then $\gamma(\mathcal{B}(G')) < \gamma(\mathcal{B}(G))$.

**Proof.** We recall the proof of Theorem 4.2.1, in which an injection is constructed from $\mathcal{B}(G') - \hat{\mathcal{B}}'$ to $\mathcal{B}(G) - \hat{\mathcal{B}}$. We will show that this injection is not onto, by describing an element $I$ that is not in its image. By Lemma 3.2.3, when $I$ is added to $\hat{\mathcal{B}}$ to obtain $\mathcal{B}(G)$, there is a strict increase in the $\gamma$-polynomial of $\mathcal{B}(G)$. The proof of Theorem 4.2.1 shows that $\gamma(\mathcal{B}(G')) \leq \gamma(\mathcal{B}(G))$, without considering the contribution of adding $I$. Thus, by considering the element $I$, we have that $\gamma(\mathcal{B}(G')) < \gamma(\mathcal{B}(G))$.

Let $I$ be the element in $\mathcal{B}(G)$ that consists of the elements in the unique minimal path from $l$ to $v$ (so that $\{l, v\} \subseteq I$). By considering the cases listed in the proof of Theorem 4.2.1, we see that $I$ is not in the image of the map.

It is exactly when $\text{dist}_{G}(l, v) = 1$ that a flossing move is a kind of tree shift. This is exactly when a flossing move reduces the number of leaves. If we partition the set $S$ of all tree graphs with $n$ vertices by their number of leaves, then tree shifts send graphs between the parts, whilst flossing moves such that $\text{dist}_{T}(l, v) \neq 1$ give relations between graphs with the same number of leaves. This is illustrated in the following example for tree graphs with seven vertices.

**Example 4.2.3.** Figure 10 shows all tree graphs with 7 vertices and their tree shift and flossing move relations. Arrows are drawn between pairs of graphs with the same number of leaves when one (at the head) can be obtained from the other (at the tail) by a flossing move. Arrows are drawn from a graph with $i + 1$ leaves to one with $i$ leaves when the graph at the head can be obtained from the graph at the tail by a tree shift. Table 1 gives the $\gamma$-polynomials of the graph-associahedra corresponding to the tree graphs with 7 vertices. The graphs in Table 1 are identified by labels $G_{1}, ..., G_{11}$ specified in Figure 10.
Figure 10. Tree graphs with seven vertices.

Table 1. Tree graphs with seven vertices.

<table>
<thead>
<tr>
<th>Graph</th>
<th>$\gamma$-polynomial of graph-associahedra</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$1 + 57t + 230t^2 + 61t^3$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$1 + 42t + 142t^2 + 33t^3$</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$1 + 36t + 117t^2 + 27t^3$</td>
</tr>
<tr>
<td>$G_4$</td>
<td>$1 + 31t + 88t^2 + 18t^3$</td>
</tr>
<tr>
<td>$G_5$</td>
<td>$1 + 28t + 77t^2 + 16t^3$</td>
</tr>
<tr>
<td>$G_6$</td>
<td>$1 + 27t + 74t^2 + 15t^3$</td>
</tr>
<tr>
<td>$G_7$</td>
<td>$1 + 24t + 65t^2 + 13t^3$</td>
</tr>
<tr>
<td>$G_8$</td>
<td>$1 + 23t + 55t^2 + 10t^3$</td>
</tr>
<tr>
<td>$G_9$</td>
<td>$1 + 21t + 49t^2 + 9t^3$</td>
</tr>
<tr>
<td>$G_{10}$</td>
<td>$1 + 19t + 44t^2 + 8t^3$</td>
</tr>
<tr>
<td>$G_{11}$</td>
<td>$1 + 15t + 30t^2 + 5t^3$</td>
</tr>
</tbody>
</table>
The Wiener index of a tree graph $G$, denoted $W(G)$, is the sum of the distances $\text{dist}_G(i, j)$ for all unordered pairs of vertices $i, j$ in $G$. The Wiener index was first defined in [34], and it was suggested in [26] that a move on a tree graph that increases the Wiener index might approximately lower the $\gamma$-polynomial. The only moves that we have found that increase the Wiener index and lower the $\gamma$-polynomial are tree shifts and flossing moves.
Further research

In this section we summarize some results we obtained towards further areas of research. In Section 5.1 we prove Proposition 5.1.1 which shows that Gal’s conjecture holds for all homology spheres in \(sd(\Theta)\), where \(\Theta\) is such that for all faces \(F \in \Theta\), we have \(\gamma(lk_\Theta(F)) \geq 0\). Note that this result can be easily deduced from the work by Athanasiadis in [4], which we explain in Section 5.1.1. The results in [20] show that Gal’s conjecture holds for the order complex of a Gorenstein* complex, which we show are examples of homology spheres satisfying the condition of Proposition 5.1.1.

We also include work done towards answering Question 14.3 in [26]. We answer this question in part for interval building sets that we define in Subsection 5.2.1.

5.1. Gal’s conjecture for edge subdivisions

In this section we prove and discuss the following proposition.

**Proposition 5.1.1.** Suppose \(\Theta\) is a flag homology sphere such that

\[
\gamma(lk_\Theta(F)) \geq 0
\]

for all faces \(F \in \Theta\). If \(\tilde{\Theta} \in sd(\Theta)\), then \(\gamma(lk_{\tilde{\Theta}}(G)) \geq 0\) for all \(G \in \tilde{\Theta}\). In particular \(\tilde{\Theta}\) satisfies Gal’s conjecture.

**Proof.** We suppose that a sequence of \(k\) subdivisions have been done to obtain \(\tilde{\Theta}\) from \(\Theta\). We suppose that \(\Theta'\) is obtained after the \((k - 1)\)th subdivision, that the edge \(\{s_a, s_b\} \in \Theta'\) is subdivided to obtain \(\tilde{\Theta}\), and that the unique vertex in \(V_{\tilde{\Theta}} - V_{\Theta'}\) is labelled \(w\). We suppose by induction that after performing \(k - 1\) or less edge subdivisions that the proposition holds, so that \(\gamma(lk_{\Theta'}(G)) \geq 0\) for all \(G \in \Theta'\). We also suppose by induction the proposition holds for all flag homology spheres of smaller dimension than \(\Theta\). We consider all five cases listed in Section 1.6 for a face \(F \in \tilde{\Theta}\).

If \(F\) is in any of \(F_1, F_2\) or \(F_5\), then \(lk_{\tilde{\Theta}}(F)\) is isomorphic to \(lk_{\Theta'}(G)\) for some \(G \in \Theta'\). Hence \(\gamma(lk_{\tilde{\Theta}}(F)) \geq 0\).
If \( F \in \mathcal{F}_3 \) then \( \text{lk}_\Theta(F) = \text{lk}_\Theta(F - \{w\} \cup \{s_a, s_b\}) \ast \Sigma_0 \) so that \( \gamma(\text{lk}_\Theta(F)) = \gamma(\text{lk}_\Theta(F - \{w\} \cup \{s_a, s_b\})) \geq 0 \).

If \( F \in \mathcal{F}_4 \) then \( \text{lk}_\Theta(F) \) is the subdivision of \( \text{lk}_\Theta(F) \) in the edge \( S \). Now if \( F \neq \emptyset \) then \( \text{lk}_\Theta(F) \) is a flag homology sphere of smaller dimension than \( \Theta \) and for any face \( G \subset \text{lk}_\Theta(F) \), we have \( \text{lk}\text{lk}_\Theta(F)(G) = \text{lk}_\Theta(F \cup G) \) so that \( \gamma(\text{lk}_\Theta(F)(G)) \geq 0 \). Hence, by induction on the dimension, \( \gamma(\text{lk}_\Theta(F)) \geq 0 \). If \( F = \emptyset \), then \( \gamma(\text{lk}_\Theta(F)) = \gamma(\emptyset) \) which is non-negative by Lemma 1.6.4.

Since \( \gamma(\text{lk}_\Theta(F)) \geq 0 \) for all \( F \in \Theta \) the proposition holds by induction. \( \square \)

A corollary of this is the result of Volodin [31, Theorem 2] that the \( \gamma \)-vector of any homology sphere obtained by stellar subdividing edges of the boundary of the cross polytope (i.e. \( \text{sd}(\Sigma_{d-1}) \)) is non-negative. This is true since for any \( F \in \Sigma_{d-1} \), we have \( \text{lk}_{\Sigma_{d-1}}(F) = \Sigma_{d-1} - F \), and \( \gamma(\Sigma_{d-1}) \geq 0 \) for all \( d \). We will now consider some other classes of flag homology spheres such that the link of every face has non-negative \( \gamma \)-polynomial.

Background reading on this paragraph can be found in [17, Section 2.3]. Let \( P \) be a poset on \( S \) such that \( \hat{0} \in S \) and \( \hat{1} \in S \) (\( \hat{0} \) is an element such that \( 0 \preceq_P s \) for all \( s \in S \), and \( \hat{1} \) is an element such that \( \hat{1} \succeq_P s \) for all \( s \in S \)). The order complex \( N(P) \) is the simplicial complex with vertex set \( S - \{\hat{0}, \hat{1}\} \), such that \( C \subseteq S - \{\hat{0}, \hat{1}\} \) is in \( N(P) \) if and only if \( C \) is a chain in \( P \). The poset \( P \) is Gorenstein* if \( N(P) \) is a homology sphere. Note that \( N(P) \) is always flag. Gal’s conjecture has already been proven in [20] for order complexes of Gorenstein* posets. Note that the barycentric subdivision of any homology sphere is the order complex of a Gorenstein* poset.

**Example 5.1.2.** Consider the poset \( P \) on the set \( \{\pm \varepsilon_1, \ldots, \pm \varepsilon_n, \hat{0}, \hat{1}\} \) with covering relations

\[
\{ \varepsilon_i \succeq_P \varepsilon_{i+1}, -\varepsilon_i \succeq_P \varepsilon_{i+1}, \hat{1} \succeq_P \varepsilon_1, \pm \varepsilon_n \succeq_P \hat{0} \mid i = 1, \ldots, n - 1 \}.
\]

Then \( N(P) = \Sigma_{n-1} \).

In general, if we perform an edge subdivision on the order complex of a Gorenstein* poset, the resulting simplicial complex is not necessarily the order complex of a Gorenstein* poset, as demonstrated by the following proposition.
Proposition 5.1.3. Consider the simplicial complex $\Theta$ obtained by subdividing $\Sigma_2$ in the edge $\{\epsilon_1, \epsilon_2\}$. Then $\Theta$ is not the order complex of a Gorenstein* poset.

Proof. Suppose for a contradiction that $\Theta$ is the order complex of a Gorenstein* poset $P'$. Let $w$ be the vertex added in the edge subdivision. In $\Theta$, the vertices $\pm \epsilon_1, \pm \epsilon_2, w$ form a cycle with five vertices, and no three of these vertices are a face of $\Theta$. This implies that any vertex in this cycle is either greater than both, or less than both of its adjacent vertices (with respect to the relation on $P'$). This is not possible for a cycle with five vertices. Hence $\Theta$ cannot be the order complex of a Gorenstein* complex. \qed

Suppose that $P$ is a poset on the set $S$, and that $Q \subseteq S$. Then $P|_Q$ denotes the restriction of $P$ to $Q$, which is the poset on the set $Q$ such that for any $x, y \in Q$, we have $x \leq P|_Q y$, if and only if $x \leq_P y$.

Proposition 5.1.4. Suppose that $N(P)$ is the order complex of a Gorenstein* poset. Then $\text{lk}_{N(P)}(F)$ is the order complex of a Gorenstein* poset for all faces $F \in N(P)$.

Proof. Suppose that $P$ is a poset with underlying set $S$. Let $S_Q$ denote the vertex set of $\text{lk}_{N(P)}(F)$. Then $\text{lk}_{N(P)}(F)$ is the order complex of the restriction of $P$ to elements in $S_Q$, i.e. $\text{lk}_{N(P)}(F) = N(P|_{S_Q})$. \qed

Proposition 5.1.3 shows that the following result is not trivially true:

Corollary 5.1.5. Suppose that $N(P)$ is the order complex of a Gorenstein* poset, and that $\Theta \in \text{sd}(N(P))$. Then $\gamma(\Theta) \geq 0$.

Proof. Karu shows in [20] that if $N(P)$ is the order complex of a Gornestein* poset $P$, then $\gamma(N(P)) \geq 0$. Hence, by Proposition 5.1.4, $\gamma(\text{lk}_{N(P)}(F)) \geq 0$ for any face $F \in N(P)$. Therefore, by Proposition 5.1.1, for any $\Theta \in \text{sd}(N(P))$, we have $\gamma(\Theta) \geq 0$. \qed

5.1.1. The work of Athanasiadis on $\gamma$-vectors.

We will now show that Proposition 5.1.1 is a consequence of the work in [4]. This work extends the work of Stanley in [29], who defines a local $h$-vector for any simplicial subdivision, which he uses to show that the $h$-vector increases under quasi-geometric subdivisions. Since the local $h$-vector is symmetric, there is an associated local $\gamma$-vector introduced by Athanasiadis in [4]. He also defines homology subdivisions in [4].

We will now give the main definitions and relevant material used in [4].
A \((d - 1)\)-dimensional simplicial complex \(\Theta\) is a *homology ball* (over a field \(k\)), if there exists a subcomplex \(\partial \Theta\) of \(\Theta\), called the *boundary* of \(\Theta\) such that:

- \(\partial \Theta\) is a \((d - 2)\)-dimensional homology sphere over \(k\),
- For every \(F \in \Theta\) (including \(F = \emptyset\)) we have

\[
\tilde{H}_i(\text{lk}_\Theta(F), k) = \begin{cases} 
  k, & \text{if } F \notin \partial \Theta, \text{ and } i = \dim(\text{lk}_\Theta(F)), \\
  0, & \text{otherwise}.
\end{cases}
\]

Suppose \(\Theta\) is a simplicial complex. A *homology subdivision* of \(\Theta\) (over a field \(k\)), is a simplicial complex \(\Theta'\), and a map \(\sigma : \Theta' \to \Theta\) such that for every face \(F \in \Theta\):

- the set \(\Theta'_F := \sigma^{-1}(2F)\) is a subcomplex of \(\Theta'\) which is a homology ball (over \(k\)) of dimension \(\dim(F)\), and
- \(\sigma^{-1}(F)\) consists of the interior faces of \(\Theta'_F\).

Here \(2F\) denotes the power set of \(F\), and the definition of a homology ball can be found in [4]. Note that if \(\Theta'\) is a homology subdivision of a simplicial complex \(\Theta\), with map \(\sigma\), then the restriction of \(\sigma\) to \(\Theta'_F\) for any face \(F \in \Theta\) is a homology subdivision of the simplex \(F\).

Suppose that \(V\) is a set with \(d\) elements in it, and that \(\chi\) is a homology subdivision of the simplex \(2^V\). The *local h-polynomial* of \(\chi\) with respect to the subdivision is given by

\[
l_V(\chi)(t) := \sum_{F \subseteq V} (-1)^{d-|F|} h(\chi_F)(t) = l_0 + l_1 t + \cdots + l_d t^d.
\]

The vector of coefficients of the local h-polynomial is the *local h-vector* of \(\chi\) with respect to the subdivision. The local h-vector is symmetric (meaning \(l_i = l_{d-i}\) for all \(i = 0, ..., d\)) for any homology subdivision of a simplex (see [4, Theorem 3.3], and [29, Corollary 7.7 and Theorem 7.8]), hence it can be written

\[
l_V(\chi)(t) = (1 + t)^d \xi_V(\chi) \left(\frac{t}{(1 + t)^2}\right) = \sum_{i=0}^{\lfloor d/2 \rfloor} \xi_i t^i (1 + t)^{d-2i}.
\]

The polynomial \(\xi(\chi)(t)\) is called the *local \(\gamma\)-polynomial* with respect to the subdivision, and its vector of coefficients is called the *local \(\gamma\)-vector* with respect to the subdivision.
Homology spheres are examples of Eulerian simplicial complexes, which are defined in [4]. Therefore the following proposition may be used for proving Gal’s conjecture for edge subdivisions.

**Proposition 5.1.6.** [4, Proposition 5.3]. Let $\Theta$ be an Eulerian simplicial complex. For every homology subdivision $\Theta'$ of $\Theta$ we have

\begin{equation}
\gamma(\Theta') = \sum_{F \in \Theta} \xi_{F'}(\Theta_F') \gamma(\lk_{\Theta}(F')).
\end{equation}

Suppose a simplicial complex $\Theta'$ is obtained from a simplicial complex $\Theta$ by subdivision in an edge $\{a, b\}$, and that the new vertex is labelled $w$. Then $\Theta'$ is a homology subdivision of $\Theta$ with map $\sigma : \Theta' \to \Theta$ mapping any face $F \cup \{w\}$ to $F \cup \{a, b\}$, and any face $F$ such that $w \notin F$ maps to itself. Athanasiadis shows the following.

**Proposition 5.1.7.** [4, Proposition 6.1]. For every subdivision $\chi$ of the simplex $2^V$ that can be obtained from the trivial subdivision by successive edge subdivisions, we have $\xi_V(\chi) \geq 0$.

Then Theorem 5.1.1 is a result of Proposition 5.1.7, with the following alternative proof.

**Proof of Theorem 5.1.1.** Suppose that $\tilde{\Theta} \in sd(\Theta)$ for some homology sphere $\Theta$ such that $\gamma(\lk_{\tilde{\Theta}}(F)) \geq 0$ for all faces $F \in \Theta$. Then $\tilde{\Theta}$ is a homology subdivision of $\Theta$ with subdivision map the composition of the subdivision maps for the individual edge subdivisions (described above).

Now for any edge subdivision $\sigma : \Phi' \to \Phi$ of a simplicial complex $\Phi$, the restriction of $\sigma$ to the preimage of any subcomplex of $\Phi$ is an edge subdivision of this subcomplex. Therefore, for any face $F \in \Theta$, $\tilde{\Theta}_F$ is obtained from $F$ by successive edge subdivisions. Therefore, by Proposition 5.1.7, $
ix_{F'}(\Theta_F') \geq 0$ for every face $F \in \Theta$. Since $\gamma(\lk_{\tilde{\Theta}}(F)) \geq 0$ for all faces $F \in \Theta$, this implies that $\gamma(\tilde{\Theta}) \geq 0$, by Equation 5.1.

It would be useful to prove the Nevo and Petersen Conjecture for certain families $sd(\Theta)$, in which $\Theta$ is not in the family $sd(S_{d-1})$. The author has spent some time trying to prove this for particular families $sd(N(P))$ in which $P$ is a Gorenstein* poset (for example when $N(P)$ is the barycentric subdivision of a homology sphere).

**5.2. Question 14.3 of Postnikov, Reiner and Williams**

In this section we present some work done towards the following question posed by Postnikov, Reiner and Williams:
5.2. Generating functions

Question 5.2.1. [26, Question 14.3]. Given a (non-chordal) building set $\mathcal{B}$, is there a way to define two sets of permutations $\mathcal{G}'_n(\mathcal{B})$ and $\mathcal{\widehat{G}}'_n(\mathcal{B})$ such that:

- the $h$-polynomial of the nestohedron $P_B$ is given by the descent generating function for $\mathcal{G}'_n(\mathcal{B})$, and
- the $\gamma$-polynomial is given by the peak generating function for $\mathcal{\widehat{G}}'_n(\mathcal{B})$?

A summary of the result of Shapiro, Woan and Getu [27] is presented in [26], in which they solve Question 5.2.1 for the building set $\mathcal{B}(K_n)$. In [26], this result is generalized to include all chordal building sets. We first summarize these results, and then present our work on interval building sets.

We now recall the theory in Section 1.3 on the complete fans of cones, $\mathcal{N}(P)$, associated to any generalised permutohedra $P$, and the associated complete fan of posets $\{Q_v \mid v \in V_P\}$. We recall its description as a complete fan of rooted tree posets in Section 1.4 when $P$ is a nestohedron, and we recall Theorem 1.3.3. Since the permutohedron’s normal fan $\mathcal{N}(P_{\mathcal{B}(K_n)})$ is the braid arrangement fan, so that the complete fan of posets $\{Q_v \mid v \in V_{\mathcal{B}(K_n)}\}$ is the set of all linear orders on $[n]$, we have

\[ h(P_{\mathcal{B}(K_n)}) = \sum_{w \in \mathcal{G}_n} t^{\text{des}(w)}. \]

Recall from Chapter 3 that $\mathcal{G}_n$ denotes the set of permutations of $[n]$. The polynomial $h_{\mathcal{B}(K_n)}$ is known as the Eulerian polynomial. Shapiro, Woan and Getu show:

Theorem 5.2.2. [27, Proposition 4]. We have

\[ \gamma(P_{\mathcal{B}(K_n)}) = \sum_{w \in \mathcal{\widehat{G}}_n} t^{\text{des}(w)} \sum_{w \in \mathcal{\widehat{G}}_n} t^{\text{peak}(w)-1}. \]

Recall from Chapter 3 that $\mathcal{\widehat{G}}_n$ denotes the set of permutations in $\mathcal{G}_n$ with no double descent of final descent. Note that $\text{peak}(w) - 1 = \text{des}(w)$ for all $w \in \mathcal{\widehat{G}}_n$. Theorem 5.2.2 is proven by grouping permutations into equivalence classes such that each class contains exactly one permutation from $\mathcal{\widehat{G}}_n$. Permutations in $\mathcal{G}_n$ are equivalent if one can be obtained from another by hop operations, which are described in [26, Section 11].

For any building set $\mathcal{B}$ on $[n]$ define the set $\mathcal{G}_n(\mathcal{B}) \subseteq \mathcal{G}_n$ of permutations called $\mathcal{B}$-permutations. A permutation $w \in \mathcal{G}_n$ is a $\mathcal{B}$-permutation if for any $i \in [n]$, $w(i)$ is contained in the component of

\[ \mathcal{B}|_{[n]-\{w(n),w(n-1),...,w(i+1)\}} \]
Further research that contains the maximal element of $[n] - \{w(n), w(n-1), \ldots, w(i+1)\}$. Let $\hat{\mathcal{S}}_n(B)$ denote $\mathcal{S}_n(B) \cap \hat{\mathcal{S}}_n$, i.e., the set of $B$-permutations that contain no double descent or final descent.

A building set $B$ on $[n]$ is chordal if for any $I := \{i_1, i_2, \ldots, i_s\} \in B$ such that $i_1 < i_2 < \cdots < i_s$, and for any $i_j \in I$ the set $\{i_j, i_{j+1}, \ldots, i_s\} \subseteq I$ is an element in $B$. It follows from Proposition 1.4.7 that chordal building sets are flag. A graph $G$ is chordal if any cycle in $G$ with four or more vertices has a chord, which is an edge between two non adjacent vertices in the cycle. A perfect elimination ordering of a graph $G$ is an labelling of the vertices of $G$ by $[n]$ such that for any vertex $i$, the vertices adjacent $i$ with a higher label form a clique. Chordal graphs are exactly the graphs that can can be labelled with a perfect elimination ordering. By [26, Proposition 9.4], a graphical building set $B(G)$ is chordal exactly when $G$ is chordal (see [16]). By proving theorem 5.2.3 below, Postnikov, Reiner and Williams answer Question 5.2.1 for chordal building sets. This result is also the original proof of Gal’s conjecture for chordal building sets.

**Theorem 5.2.3.** ([26, Corollary 9, Theorem 11.6]). Suppose $B$ is a connected chordal building set on $[n]$. Then

$$h_B(t) = \sum_{w \in \hat{\mathcal{S}}_n(B)} t^{\text{des}(w)}$$

and

$$\gamma_B(t) = \sum_{w \in \hat{\mathcal{S}}_n(B)} t^{\text{des}(w)} = \sum_{w \in \hat{\mathcal{S}}_n(B)} t^{\text{peak}(w)-1}.$$

Note that $\text{peak}(w) - 1 = \text{des}(w)$ for all $w \in \hat{\mathcal{S}}_n(B)$. This is proven by grouping permutations in $\mathcal{S}_n(B)$ into equivalence classes such that each class contains exactly one permutation from $\hat{\mathcal{S}}_n(B)$. Permutations in $\mathcal{S}_n(B)$ are equivalent if one can be obtained from another by $B$-hop operations, which are described in [26, Section 11].

**5.2.1. Interval building sets.**

In this section we answer the first part of Question 5.2.1 for interval building sets, which are defined in this section.

Suppose $I \subseteq [n]$. An interval $J$ of $I$ is a subset of $I$ such that if $j_1, j_2 \in J$, with $j_1 < j_2$, then for any $i \in I$ such that $j_1 < i < j_2$, we
have \( i \in J \). A building set \( B \) on \([n]\) is interval, if for all \( S \subseteq [n] \), the maximal elements of \( B|_S \) are all intervals of \( S \). We will now describe a family of interval building sets.

A poset \( P \) on \([n]\) is natural if for any \( x, y \in [n] \) such that \( x \leq_P y \), we have \( x \leq y \) in \([n]\). Note that all posets on \([n]\) are, up to permutation, natural posets. For suppose \( P \) is a poset on \([n]\), then \( P \) may be extended to a linear order by the order-extension principle. We can then permute the labels on the vertices so that the total order is natural. This permutation on the original poset yields a natural poset. Suppose \( P \) is a poset on \([n]\). The incomparability graph of \( P \), denoted \( inc(P) \), is the graph with vertices labelled by \([n]\), such that vertices \( i, j \) are adjacent in \( inc(P) \) when \( i \) and \( j \) are incomparable in \( P \).

**Claim 5.2.4.** A graph \( G \) on \([n]\) is the incomparability graph of a natural poset \( P \) on \([n]\) if and only if for all \( i < j < k \), \( \{i, k\} \in B(G) \) implies \( \{i, j\} \in B(G) \) or \( \{j, k\} \in B(G) \).

**Proof.** \( \Longrightarrow \): Suppose \( G = inc(P) \) where \( P \) is a natural poset on \([n]\). If \( \{i, k\} \in B(G) \), and \( \{i, j\} \notin B(G) \) and \( \{j, k\} \notin B(G) \) then \( j \geq_P i \) and \( k \geq_P j \) which implies that \( k \geq_P i \), a contradiction.

\( \Longleftarrow \): Suppose \( G \) is a graph on \([n]\). Suppose that for all \( i < j < k \), \( \{i, k\} \in B(G) \Rightarrow \{i, j\} \in B(G) \) or \( \{j, k\} \in B(G) \). Let \( P \) be the natural poset defined by \( i \leq_P j \), for some \( i \leq j \), if \( i = j \) or \( \{i, j\} \notin B(G) \). This defines a poset since for any \( i < j < k \), \( i \leq_P j \) and \( j \leq_P k \) implies \( i \leq_P k \). \( G \) is clearly the incomparability graph of \( P \).

**Claim 5.2.5.** Suppose \( P \) is a natural poset on \([n]\). Then the elements of \( B(inc(P))_{\text{max}} \) are intervals of \([n]\).

**Proof.** Suppose that \( i, k \in [n] \) with \( i < k \), are in the same connected component of \( inc(P) \). We will show that any \( j \) such that \( i < j < k \) is also in this connected component of \( inc(P) \).

Since \( i \) and \( k \) are in the same connected component \( H \) of \( inc(P) \), there exists a sequence of vertices \( i = a_0, a_1, \ldots, a_{l-1}, a_l = k \) such that for all \( \alpha \in \{0, \ldots, l - 1\} \), \( a_\alpha \) is adjacent to \( a_{\alpha+1} \). By Claim 5.2.4 this implies that, if \( a_\alpha < a_{\alpha+1} \), all elements in the interval \([a_\alpha, a_{\alpha+1}]\) are in \( H \), or if \( a_{\alpha+1} < a_\alpha \), all elements in the interval \([a_{\alpha+1}, a_\alpha]\) are in \( H \). Any \( j \) such that \( i < j < k \) must be in one such interval, and so is in \( H \).
Claim 5.2.6. Suppose \( P \) is a natural poset on \([n]\). Then for any \( S \subseteq [n] \), the induced subgraph of \( \text{inc}(P) \) on the vertices in \( S \), is the incomparability graph for the poset \( P|_S \).

Proof. Clear from the definition of \( P|_S \). \( \square \)

Corollary 5.2.7. Suppose \( P \) is a natural poset on \([n]\). Then \( \mathcal{B}(\text{inc}(P)) \) is an interval building set on \([n]\).

Proof. By Claim 5.2.6, for any \( S \subseteq [n] \), \( \mathcal{B}(\text{inc}(P))|_S \) is the graphical building set for the incomparability graph of \( P|_S \), which is a natural poset on \( S \). Therefore, by Claim 5.2.5, \( (\mathcal{B}(\text{inc}(P))|_S)_{\text{max}} \) is a set of intervals on \( S \), so that \( \mathcal{B}(\text{inc}(P)) \) is interval. \( \square \)

Proposition 5.2.8. A graph \( G \) on \([n]\) is the incomparability graph of a natural poset on \([n]\) if and only if \( \mathcal{B}(G) \) is interval.

Proof. \( \implies \): The only if part was proved in Corollary 5.2.7.

\( \Leftarrow \Leftarrow \): Suppose that \( G \) is a graph on \([n]\) such that \( \mathcal{B}(G) \) is interval. Consider an arbitrary triple \( i < j < k \) such that \( \{i, k\} \in \mathcal{B}(G) \). Consider \( \mathcal{B}(G)_{\{i, j, k\}} \). Since \( \mathcal{B}(G) \) is interval, we have either \( \{i, j\} \in \mathcal{B}(G) \) or \( \{j, k\} \in \mathcal{B}(G) \). Hence by Claim 5.2.4, \( G \) is the incomparability graph of a natural poset on \([n]\). \( \square \)

Example 5.2.9. Let \( G \) be a complete \( k \)-partite graph with vertices partitioned into the the sets \( G_1, \ldots, G_k \), so that each vertex \( w \in G_i \) is adjacent to a vertex \( v \) if and only if \( v \in G_j \) for some \( j \neq i \). Suppose that the vertices of \( G \) are numbered so that for all \( i \), the vertices in \( G_i \) are labelled by the interval \( [\sum_{m=1}^{i-1} |G_m| + 1, \sum_{m=1}^i |G_m|] \). Then \( G = \text{inc}(P) \) for the poset \( P \), (on the set of vertices of \( G \)) with the relation \( x \leq_P y \) if and only if \( x \leq y \) (in \( \mathbb{Z} \)) and \( x \) and \( y \) are both vertices of \( G_i \), for some \( i \). Note that \( G \) is chordal if and only if at least \( k - 1 \) of the sets \( G_1, \ldots, G_k \) consist of exactly one vertex.

For any building set \( \mathcal{B} \) on \([n]\) define the set \( \mathcal{I}_n(\mathcal{B}) \subseteq \mathfrak{S}_n \) of permutations called \( \mathcal{B} \)-interval-permutations. A permutation \( w \in \mathfrak{S}_n \) is a \( \mathcal{B} \)-interval-permutation if for any \( i \in [n] \), \( w(i) \) is contained in the component of \( \mathcal{B}|_{[n]} \setminus \{w(n), w(n-1), \ldots, w(i+1)\} \) that contains the minimal vertex (with respect to the ordering on \([n]\)) that is greater than \( w(i + 1) \), or if there is no such vertex, then the component of \( \mathcal{B}|_{[n]} \setminus \{w(n), w(n-1), \ldots, w(i+1)\} \) that contains the maximal remaining vertex.

Theorem 5.2.10. Suppose \( \mathcal{B} \) is a connected interval building set on \([n]\). Then
\[ h(P_B)(t) = \sum_{w \in \mathcal{J}_n(B)} t^{\text{des}(w)}. \]

**Proof.** Suppose that \( B \) is an interval building set, and that \( T \) is a \( B \)-tree. Note that permutations in \( \mathcal{J}_n(B) \) biject to \( B \)-trees, where a permutation \( w = w(1), \ldots, w(n) \in \mathcal{J}_n(B) \) corresponds to the unique \( B \)-tree \( T^w \), such that for all \( i, \{w(n), w(n-1), \ldots, w(i)\} \) is an upper ideal of \( T^w \). Note that for any \( i \), the poset \( T^w|_{\{w(1), \ldots, w(i)\}} \) consists of disjoint tree posets, whose vertices are intervals of \([n] - \{w(n), \ldots, w(i+1)\}\). We will show that \( \text{des}(T^w) = \text{des}(w) \), which by Theorem 1.3.3, proves this theorem.

Suppose that for some \( i, w(i) \) is a direct descendant of \( w(i+1) \) in \( T^w \). Then \( w(i)w(i+1) \) is a descent in \( w \) exactly when the pair \( w(i), w(i+1) \) contributes a descent to \( \text{des}(T^w) \).

Suppose that for some \( i, w(i) \) is not a direct descendant of \( w(i+1) \). Note that this implies \( w(i+1) \) is maximal in \( T^w_{\leq w(i+1)} \). There are two cases to consider:

1. We have \( w(i) < w(i+1) \). Note that this occurs if and only if \( w(i+1) \) is maximal in \([n] - \{w(n), \ldots, w(i+2)\}\), and \( T^w_{\leq w(i+1)} = \{w(i+1)\} \). In this case \( w(i)w(i+1) \) does not contribute a descent to \( \text{des}(T^w) \). We claim that \( p(w(i)) > w(i) \) where \( p(w(i)) \) is the parent vertex to \( w(i) \) in \( T^w \), so that the pair \( w(i), p(w(i)) \) does not contribute a descent to \( \text{des}(w) \). Suppose for a contradiction that \( p(w(i)) < w(i) \). Then the sequence of vertices \( p(w(i)), \ldots, w(i+2), w(i+1), w(i) \) must be an increasing sequence, which contradicts \( w(i) < w(i+1) \).

2. We have \( w(i) > w(i+1) \). In this case \( w(i)w(i+1) \) contributes a descent to \( \text{des}(w) \). We claim that \( p(w(i)) < w(i) \) where \( p(w(i)) \) is the parent vertex to \( w(i) \) in \( T^w \), so that the pair \( w(i), p(w(i)) \) contributes a descent to \( \text{des}(T^w) \). Suppose for a contradiction that \( p(w(i)) > w(i) \). Then no element in the sequence of vertices \( p(w(i)), \ldots, w(i+2), w(i+1), w(i) \) can be less than \( w(i) \), which contradicts \( w(i) > w(i+1) \).

\[ \Box \]

**Example 5.2.11.** This example shows that not every interval building set is equivalent to a chordal building set. Let \( G \) be the graph illustrated in Figure 1 with six vertices.
It is not too hard to see that $G$ is the incomparability graph of the natural poset

\[ \{1 \leq 5, 1 \leq 6, 2 \leq 4, 2 \leq 6, 3 \leq 4, 3 \leq 6, \text{ and } i \leq i \text{ for all } i \in [6]\}, \]

and that $G$ is not chordal.

**Example 5.2.12.** This example shows that not every tree graph is the incomparability graph of a natural poset. Let $G$ be the graph illustrated in Figure 2 with seven vertices.

The graph $G$ cannot be the incomparability graph of a natural poset on $[7]$. For suppose by way of contradiction that $G = \text{inc}(P)$ for a natural poset $P$ on $[7]$. Then by Corollary 5.2.7, $B(G)$ is an interval building set. Then the graph obtained from $G$ by removing the vertex of degree three consists of three connected components, one with vertices labelled $\{a_1, a_2\}$, one with vertices labelled $\{a_3, a_4\}$, and one with vertices labelled $\{a_5, a_6\}$, such that $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$. Therefore, the graph obtained from $G$ by removing the vertices $a_3$ and $a_4$ contains 5 vertices, and is either labelled with both 1 and 6, or with both 2 and 7, which is not possible if $B(G)$ is an interval building set.
Note that if a graph $G$ is both the incomparability graph of a natural poset, and chordal, then it is not the case that $S_n(B)$ and $I_n(B)$ coincide. The graph of Example 5.2.9 when $k = 2$ was studied by Erokhovets in [11]. The interval permutations are in fact a generalisation of the permutations that he defines for complete 2-partite graphs.
References


