Abstract

This thesis investigates the use of asymptotic techniques and stochastic volatility models in option pricing problems. For the Heston stochastic volatility model, a fast mean reverting asymptotic approach, similar to Fouque et al. (2000) is taken. The asymptotic solution derived extends on their most recent work, with the solution presented expanded out to four terms. The worthiness and robustness of the asymptotic solution is then tested by applying it to the theory of locally risk minimizing hedges. The asymptotic approach is then further developed by applying it to a real options framework, allowing for a better understanding of what the asymptotic solution actually reflects under this model, and in particular, how it affects the optimal investment threshold, a key component in real options theory.

Asian options with general call type payoffs are then investigated and equivalency theorems derived linking them to Australian options under both a Black-Scholes model and a Heston stochastic volatility model. Examining Asian options from this ‘Australian’ perspective gives a new angle on how one can approach the pricing of Asian options under stochastic volatility. Advances are made in areas such as the PDE pricing equation, and Monte Carlo simulations. Finally, an asymptotic solution under a low volatility assumption in the Black-Scholes model for an Australian call option is derived. This extends the work of Dewynne and Shaw (2008), to cater for Australian options. It is argued that this can be used as an alternative to existing approximations under a low volatility regime, for both pricing general Australian call options and general Asian options through the equivalency theorems.

Aside from the overarching theme of asymptotic techniques and stochastic volatility, this thesis looks at how each of the newly presented solutions and/or methods, can be of benefit to the pricing of their respective option types. In particular, focus will be placed on the usage, accuracy and computational efficiency of these techniques. In all cases, the new solutions provide a high level of accuracy compared
to the true solution, and/or are much more computationally efficient than existing methodologies. The simplicity and advantages of these solutions make a valuable contribution to current option pricing techniques.

The materials presented in this thesis are the results of my original research work dating from 2009 to 2012, unless otherwise stated with full referencing and citations. These materials may also appear as published and/or working papers co-authored mainly with my supervisor Professor Christian-Oliver Ewald (Chapters 3, 4 and 7), along with Dr. Olaf Menkens (Chapter 6) and Dr. Wen-Kai Wang (Chapter 5), that been submitted to peer reviewed journals. Furthermore, part of the materials from Chapters 3 and 4 and from Chapter 7 were presented at the 2010 ANZIAM conference and 2011 Quantitative Methods in Finance conference, respectively. I maintain that the bulk of the materials (mathematical derivations and discussions) are the results of my own ideas, although I do acknowledge the guidance from my co-authors on these topics.

Published articles resulting from this thesis.

To family and friends.
Acknowledgements

I would like to thank and acknowledge all those who have contributed to my learning. It is with your help, that I know and understand what I know and understand today. In particular, I would like to mention my supervisors Christian-Oliver Ewald, Neville Weber and David Ivers, whom without, the production of this thesis would not be possible. Over the last 3-4 years, your help has immensely furthered my knowledge of mathematics and for that I thank you.

I would also like to take this opportunity to acknowledge the support of my friends. For those of us who have an interest in mathematics, the chance to discuss ideas with one another has been invaluable. Not only do I get to voice my point of views, but also take in your views. For others, the distractions (and I say this in a positive light), have provided me with an outlet to experience and take interest, in things other than mathematics.

Finally, I would like to thank my family, for giving me the opportunity, support and freedom to pursue my interests. Whilst I am unable to explain in full detail what it is I do to you, know that because of you I am able to do it. For that, I am eternally grateful.

Marten Ting
August 2012
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Abbreviations

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<th>Description</th>
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<tbody>
<tr>
<td>AAE</td>
<td>Asian-Australia equivalence</td>
</tr>
<tr>
<td>ADI</td>
<td>alternating direction implicit</td>
</tr>
<tr>
<td>ASX</td>
<td>Australian Stock Exchange</td>
</tr>
<tr>
<td>ATM</td>
<td>at-the-money</td>
</tr>
<tr>
<td>BS</td>
<td>Black-Scholes</td>
</tr>
<tr>
<td>CIR</td>
<td>Cox-Ingersoll-Ross</td>
</tr>
<tr>
<td>CN</td>
<td>Crank Nicholson</td>
</tr>
<tr>
<td>DR</td>
<td>Douglas-Rachfold</td>
</tr>
<tr>
<td>EMM</td>
<td>equivalent martingale measure</td>
</tr>
<tr>
<td>EuM</td>
<td>Euler-Maruyama</td>
</tr>
<tr>
<td>FDM</td>
<td>finite difference method</td>
</tr>
<tr>
<td>FFT</td>
<td>fast Fourier transform</td>
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<tr>
<td>GBM</td>
<td>geometric Brownian motion</td>
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<tr>
<td>GDE</td>
<td>Gamma distribution extension</td>
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<tr>
<td>GMR</td>
<td>geometric mean reverting</td>
</tr>
<tr>
<td>ITM</td>
<td>in-the-money</td>
</tr>
<tr>
<td>MC</td>
<td>Monte Carlo</td>
</tr>
<tr>
<td>MCA</td>
<td>Markov chain approach</td>
</tr>
<tr>
<td>ODE</td>
<td>ordinary differential equation</td>
</tr>
<tr>
<td>OTC</td>
<td>over-the-counter</td>
</tr>
<tr>
<td>OTM</td>
<td>out-of-the-money</td>
</tr>
<tr>
<td>OU</td>
<td>Ornstein-Uhlenback</td>
</tr>
<tr>
<td>PDE</td>
<td>partial differential equation</td>
</tr>
<tr>
<td>pdf</td>
<td>probability density function</td>
</tr>
<tr>
<td>PLR</td>
<td>profit and loss ratio</td>
</tr>
<tr>
<td>RMSE</td>
<td>root mean squared error</td>
</tr>
<tr>
<td>SDE</td>
<td>stochastic differential equation</td>
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<tr>
<td>VPO</td>
<td>variable purchase option</td>
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Chapter 1

Introduction

Mathematical finance has been a rapidly growing area of mathematics over the past 30 years. The presence of mathematics in finance should be of no surprise given its heavy dependence on numbers. However, mathematical finance, as a subsection of applied mathematics, deals with the numerical modelling aspects of finance. One of the most researched areas of mathematical finance is option pricing theory and in particular, the notion of determining a “fair” price for options.

The history of options has long been traced back to the ancient times. The first buyer of options is supposedly the Greek mathematician Thales of Miletus as told by Aristotle [4]. Thales of Miletus had purchased the rights to use a number of olive presses during the off-season, while anticipating a large olive harvest in the upcoming harvest season. As his prediction came to fruition, he rented out the olive presses he had acquired, at a higher price than the right he purchased initially, thus returning a handsome profit. This simple notion spawned the beginning of option trading.

Options, in their barest form, are essentially agreements between two parties which allow the holder of the option to exercise certain predefined rights, which the seller must oblige, if the rights are executed. In exchange for these rights, the holder must pay a premium to the seller, which is referred to as the option’s price or value. In a financial setting, options are usually traded derivatives contracts, whereby the holder usually has the right to execute a future transaction with the seller in regards to a predetermined asset, known as the underlier.

Statistics from the Bank of International Settlements states that the total notional amount outstanding of the global over-the-counter (OTC) derivatives reached 708 trillions USD, by
the end of June 2011. This figure does not include derivatives contract traded on financial exchanges around the world.

Given the large amount of money involved in the options market, it is imperative that the options be priced correctly, and above all else, “fair”. This notion of fairness is not very well defined and often subjective, but it can be thought of as the price in which a regular rational person would be indifferent about buying or not buying the option. The idea of fairness loosely relates to risk-neutrality, which is a requirement of arbitrage-free models. Arbitrage-free models are the cornerstones of mathematical modelling of financial derivatives.

Many of the mathematical models used in the pricing of options begin with the modelling of the underlying asset’s value as some functional form, with the most popular choice being to use stochastic differential equations (SDE). These assumptions allow us to infer various statistical properties about the asset’s value, and (hopefully) allow for the pricing of options. One of the most successful models is the Black-Scholes (BS) model [8], by F. Black and M. Scholes. Amongst the many assumptions, which can be found in Wilmott [79], for example, the mathematical model assumes the asset’s value follows a geometric Brownian motion (GBM). Its popularity stems from its ability to price many simple options with closed form solutions.

A step up from the GBM assumption, is one where the volatility itself is allowed to vary. There are generally two schools of thought on how to proceed; one is to use local volatility models, and the other is to use a separate stochastic process, defined via a SDE, to model the volatility. In local volatility models, the volatility is treated as a deterministic function of the asset’s value and the current time, thus creating a volatility surface. These concepts were first introduced by Dupire [23], and Derman and Kani [19]. The use of an additional SDE to model the volatility is usually referred to as stochastic volatility models. These models have been investigated by the likes of Hull and White [48], Wiggins [78], Stein and Stein [74], Heston [44], and many others. While both camps have been important to the development of volatility modelling in option pricing problems, the result of Heston [44], is the most instrumental. The Heston model assumes the instantaneous variance, the square of the volatility term in the GBM model, can be modelled by a Cox-Ingersoll-Ross (CIR) process [16]. Like the BS model, the Heston model allows the calculation of European call and put options to be priced analytically with easily implementable closed form solutions.

In general, not all pricing problems, even under the BS model, admit closed form solutions. In particular, easily computable closed form solutions for simple American type options and some types of Asian options do not exist. Many of these problems are relegated to finding
approximations to the solution instead. The approximations typically assume the solution can be written as an expansion of terms that have to be determined. Some of the most prominent works in this area under the BS model, for American put options are that of Bunch and Johnson [9], Zhu [86], and Zhu and He [89], and for arithmetic Asian options, Linetsky [56], Dufresne [22] and Dewynne and Shaw [20], to name a few. Many other expansion approximations exist for other option types, but will not be mentioned here. The appeal of using these approximation expansion techniques is that usually, the approximations are much easier to compute.

As the complexity of the problem increases with the addition of stochastic volatility, it is clear that many other problems may also not admit closed form solutions. For example, there is no guarantee that American type options and Asian options under stochastic volatility, will have closed form solutions. Furthermore, whilst real options\footnote{Essentially a perpetual call options.} under the BS framework may have closed form solutions, as calculated in Dixit and Pindyck [21], they may not under stochastic volatility. As such, it is natural to consider the use of approximating techniques for stochastic volatility models. Fouque, Papanicolaou and Sircar [31], [32], have found great success in applying asymptotic expansion techniques on fast mean reverting stochastic volatility models. These are models where the volatility process has the property that the volatility level, rapidly returns to its mean value whilst also containing a random component. The technique involves using perturbation expansion techniques on the partial differential equation (PDE) which the option’s price must satisfy. Their proposal has lead to a host of others using this technique to price more exotic options not covered in their original work. A non-exhaustive list of some of the options covered includes, arithmetic Asian options by Fouque and Han [29], geometric Asian options by Wong and Chueng [82], and most recently American call options by Souza and Zubelli [72] and perpetual put options by Zhu and Chen [87], [88].

Whilst the results of Fouque et al. [32], have been instrumental in paving the way for a unified technique to solve fast mean reverting stochastic volatility model problems with asymptotic techniques, thus far there has been very little numerical validation of the technique. The original work of Fouque et al. assumes the instantaneous variance is modelled as a function of a mean reverting Ornstein-Uhlenbeck (OU) process. Without a closed form solution to the original problem, it is difficult to ascertain the performance of the asymptotic technique. This is the issue addressed in Chapter 3 of this thesis. The contributions of this chapter will be to provide the technical justifications for applying the asymptotic technique, not to the original model studied by Fouque et al., but to the Heston model. Whilst this problem has been recently
covered by Fouque, Papanicolaou, Sircar and Sølna [33], in a revised version of their work, the material presented here has been developed independently. Furthermore, the original works of Fouque et al., derives the asymptotic solution up to the first two terms. In the more recent work of Fouque et al., this has been extended to three terms. In this chapter of the thesis, the techniques used in obtaining the third term is applied to derive a solution for the fourth term, thus adding to the original contributions of this thesis. In all, a closed form asymptotic solution for an European call option will be presented under the Heston model. The Heston model is chosen for its closed form solution as alluded to earlier, thus giving the asymptotic solution a true solution to be benchmarked against. It further provides a discussion on the conditions under which the asymptotic solution performs the best, backed by numerical examples illustrating the claims.

The asymptotic solution Heston solution is then given a practical application by applying it to locally risk minimizing hedge theory. The work in Chapter 4, follows closely to that of Poulsen, Schenk-Hoppé and Ewald [64]. From this application, it will be shown through a simulation study, that the asymptotic solution when used in locally risk minimizing hedges, is a worthwhile alternative to existing hedging methods such as a standard BS type hedge. Under a real data case study, it is shown that the asymptotic solution provides a suitable replacement to the exact Heston hedge. The advantages are that hedges are computationally more efficient to compute without sacrificing too much accuracy. Together, these results verify the worthiness and robustness of using the asymptotic solution in creating locally risk minimizing hedges.

Not only is the Fouque et al. [32], technique useful under the option pricing framework, its use has also been extended to other areas such as interest rate modelling by Stehlíková and Ševčovič [73]. In Chapter 5, the asymptotic technique is applied to a real options framework, which is a different, but related area to derivatives pricing. Real options are used by many business in their decision making processes. The option allows the holder an opportunity to be involved with a project, for an initial irretrievable outlay of money, known as the sunk cost. This opportunity has no expiration date, and thus the problem is actually a reformulation of the perpetual American call option problem. The work is similar to that of Souza and Zubelli [72], and Zhu and Chen [87], for the perpetual call and put option problem, respectively. A section is dedicated to pointing the main differences between the results presented here and that of the earlier works. The contribution of this chapter will be two fold. Firstly, the asymptotic solution for the real option problem under a fast mean reverting stochastic volatility model is presented. Two models are considered, a Heston type model and a geometric mean reverting (GMR) type
model, where the underlying project’s value is assumed to follow a GBM style and GMR style process, respectively. The application of the asymptotic technique on the latter model is new, as Fouque et al. [32], Stehlíková and Ševčovič [73] and Choi, Fouque and Kim [14], only considers GBM, CIR and CEV style processes (with modifications to add stochastic volatility), for the underliers, respectively. Both models presented in this chapter will have the volatility again being modelled by a CIR process, and an analytical expression for both the asymptotic option price, and an optimal investment threshold will be provided where possible. From a business point of view, the optimal investment threshold is of more importance than the real option price, as it dictates when a business should invest into the project. The numerical section, which encompasses the second contribution of this chapter, will indicate that the optimal asymptotic investment threshold is in fact the solution to a modified real options problem under stochastic volatility. The modification here is that the optimal investment threshold is assumed to be a constant, as oppose to one that varies with the initial variance. This is an important fact as multi-million dollar decisions are made using the optimal investment threshold, thus it is imperative that the assumptions dictating the model are well defined. The identification of the modified problem is new and could only be possible by considering the numerics of the asymptotic solutions. These results have yet to be discussed in literature to date.

The next chapter of the thesis deals with Asian options under stochastic volatility. It begins by deriving equivalencies between general Asian options and general Australian options, the latter of which is a relatively new and obscure type of option traded on the Australian Stock Exchange (ASX). The equivalencies are under the BS model, and bare many resemblances to past work such as Rogers and Shi [66], and Benhamou and Duguet [7], for example. Australian options are shown to be equivalent under appropriate changes of measure to Asian options, and as such Asian options can be investigated from this Australian perspective. The main contribution of this chapter is that the equivalencies are still valid under stochastic volatility models, and in particular the Heston model. The equivalencies to Australian options allows Asian options to be priced using an alternative PDE, with only 3 dimensions as oppose to 4, using the naive approach. Unlike Vecer and Xu [77], and Fouque and Han [29], who have also considered dimension reduction PDEs for solving Asian options under stochastic volatility, the dimension reduction presented has time independent coefficients for the partial derivatives, thus allowing easily implementable numerical schemes, such as the alternating direction implicit (ADI) method, to price Asian options. Furthermore, the Australian perspective, also allows the Asian option problem, from a Monte Carlo (MC) simulation point of view, to be transformed
from a path dependent problem to a path independent problem. The advantage gained in doing so is that path independent schemes are much easier to implement than path dependent schemes, if higher order of convergence is required. It is shown that the Order 2.0 Weak Taylor scheme, see Kloeden and Platen [52], and the standard Euler-Maruyama (EuM) scheme, applied to the Australian equivalence increases the accuracy of the Monte Carlo solutions as compared to the standard EuM scheme without the Australian equivalencies. Furthermore, in the case of the standard EuM scheme, there is the added advantage of using the same amount of computational time. The importance of these results stem from the fact that the pricing of Asian options under stochastic volatility is not well understood. Looking at Asian options from the Australian perspective at the very least, gives insight as to ways they can be approached.

Following on from the Asian-Australian options equivalency, this thesis also explores asymptotic pricing techniques for a Australian option under the BS model. The underlying value of interest for Australian options is the ratio of the time averaged asset’s value to its final value at the expiry of the option. So far, this problem has been solved (at least theoretically) by considering the process which the inverse of the variable of interest satisfies. This discovery is yet to be published, but the results will be reviewed here. It turns out that the inverse variable satisfies a GMR process which a closed form solution for the transitional probability density function (pdf) exists, as calculated by Yang and Ewald [84]. Whilst in theory, this is a good approach, however, the pdf is numerically unstable for many of the parameter sets considered. A new method is developed to obtain the option price. Following the asymptotic work of Dewynne and Shaw [20], for Asian options, this chapter will consider the method of matched asymptotics to obtain solutions to the Australian option problem. The asymptotic method assumes the constant volatility is small, and approximations get better as volatility decreases. Easily implementable solutions to pricing Australian options under low volatility are presented. Previous work on this topic is sparse with the only other notable work by Moreno and Navas [59]. In particular, their Gamma distribution approximation is extended to cater for cases where the pricing is performed after the averaging period has already started, i.e. in progress options. Further extensions to the Australian call option problem is made by considering a general Australian call option payoff. This allows the general Australian call option to be linked back to general Asian call options. It is shown that the low volatility asymptotic solution is still valid under this model, whereas the assumptions used in the Gamma distribution approximation may not be. Numerical results show that the asymptotic approach is a worthwhile alternative to the Moreno and Navas approximations. Under a low volatility regime, the
asymptotic solution performs very well, with a high degree of accuracy. These approximating solutions will be compared to a ‘true’ solution, as calculated using a Crank Nicholson (CN) finite difference method (FDM). The advantage of using the asymptotic approach, is that under the low volatility regime, several (difficult to calculate) terms can be ignored, thus leaving a closed form approximation for the Australian option.

As alluded to in the earlier paragraphs, the structure of the thesis is as follows; Chapter 2 will contain a brief review of the basic concepts used in mathematical finance. Chapter 3 presents the asymptotic solution under a Heston model. Chapter 4 investigates the application of the asymptotic Heston solution in locally risk minimizing hedges. Chapter 5 contains the work on the asymptotic technique under a real options framework. Chapter 6 focuses on the Asian and Australian options equivalency and how it can be used to price Asian options under a stochastic volatility framework. Chapter 7 develops the asymptotic solution for Australian options under a low volatility assumption and its various extensions.
Chapter 2

Review of Preliminary Work

This chapter reviews many of the basic building blocks used in mathematical finance and throughout this thesis. The materials covered include the stochastic processes, geometric Brownian motion (GBM), Cox-Ingersoll-Ross (CIR) process and geometric mean reverting (GMR) process, and option pricing techniques such as the equivalent martingale measure (EMM) and Feynman-Kac approaches and basic option types.

2.1 GBM and the Black-Scholes Model

The most basic type of options available are the European call and put options, or collectively known as just vanilla European options. A European call (put) option, gives the holder right, but not obligation to buy (sell) the underlying asset at a fixed price, known as the strike price, only at a fixed time, known as the expiry date or maturity date. In 1973 F. Black and M. Scholes [8], published a paper titled “The Pricing of Options and Corporate Liabilities.”. Known as the Black-Scholes (BS) model, the paper generated a frenzy of interest from the financial industry as it was heralded as the most important breakthrough in mathematical finance at that time. The BS model was one of the first to produce a closed form solution for the pricing of vanilla European options. To this day, the BS model is highly regarded as the cornerstone of mathematical finance.

The BS model is built upon a number of assumptions, many of which can be found in preliminary mathematical finance textbooks such as Wilmott [79]. The assumption that is to be challenged in this thesis is that the option’s underlying asset’s value process is modelled as a GBM. Mathematically, let $X_t$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. 


$X_t$ is used to model the asset’s value process at time $t$, under the risk neutral probability measure $\mathbb{P}$. Then, $X_t$ satisfies the stochastic differential equation (SDE),

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

(2.1)

where $r$ and $\sigma$ denote the constant risk-free interest rate and volatility, respectively, and $W_t$ is a Brownian motion.

The price of a European call (put) option under the BS model can be computed by either using the EMM approach or the Feynman-Kac approach. A general overview of both approaches is presented in the subsequent sections.

### 2.1.1 Equivalent Martingale Measure Approach

In essence, a theorem by Harrison and Pliska [40], showed that under arbitrage free conditions, the price of an option can be computed as the conditional discounted expectation of its terminal payoff, where the expectation is taken under the risk neutral measure $\mathbb{P}$. The expectation is conditional on the filtration $\{\mathcal{F}_t\}$, which is an increasing sequence of the $\sigma$-algebras contained in $\mathcal{F}$. Furthermore, it is assumed that $X_t$ is $\mathcal{F}_t$ measurable. Since the BS model satisfies the arbitrage free condition, the pricing of a European call option is thus,

$$V(x, t) = \mathbb{E} \left( e^{-r(T-t)} (X_T - k)^+ \mid \mathcal{F}_t; X_t = x \right),$$

(2.2)

for current asset’s value $x$, strike price $k$, current time $t$ and expiry time $T$. Further to this, the plus function is defined such that $x^+ = \max(0, x)$. A European put option has payoff $(k - X_T)^+$, and as such, a similar expression can be written for the price of a put option. A put-call parity relationship exists between the vanilla European options and thus, this thesis will largely concentrate on call options.

If $X_T$ satisfies the GBM in equation (2.1), then its closed form solution can be computed as,

$$X_T = X_t \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma (W_T - W_t) W_{T-t} \right),$$

1Note that the risk neutral probability measure is not the true probability measure of the asset’s price in the real world, but one where the investors risk preferences are irrelevant. See Wilmott [79], for more on risk neutrality.
2.1 GBM and the Black-Scholes Model

conditional on $\mathcal{F}_t$. For any fixed $T$ and $t$, this shows that the GBM is log-normally distributed, as the log of $X_T$ is normally distributed. The conditional expectation of $X_T$ is computed as,

$$ E \left( X_T \mid \mathcal{F}_t \right) = X_t \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) (T - t) \right) \mathbb{E} \left( e^{\sigma \sqrt{T-t} Z} \right) $$

$$ = X_t \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) (T - t) \right) \exp \left( \frac{1}{2} \sigma^2 (T - t) \right) $$

$$ = X_t e^{r(T-t)}, \quad (2.3) $$

where $Z$ follows a standard normal distribution, and in the second equality, moment generating function results for the standard normal distribution is used.

Using the closed form solution for the asset’s value, the expectation in equation (2.2) can be explicitly calculated as,

$$ V(x,t) = x N(d_1) - ke^{-r(T-t)} N(d_2), \quad (2.4) $$

where,

$$ d_1 = \frac{\log(x/k) + (r + \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T-t}} $$

$$ d_2 = \frac{\log(x/k) + (r - \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T-t}} $$

$$ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left( -\frac{1}{2} u^2 \right) du. \quad (2.5) $$

2.1.2 Feynman-Kac Approach

The Feynman-Kac theorem states that the expectation in equation (2.2), can be computed as the unique solution of a partial differential equation (PDE). Generally, for a SDE of the form,

$$ dX_t = \alpha(X_t) dt + \beta(X_t) dW_t, $$

and any terminal function $g(x)$, the expectation,

$$ V(x,t) = \mathbb{E} \left( e^{-\int_t^T r(X_s) ds} g(X_T) \mid \mathcal{F}_t; X_t = x \right), $$

is the unique solution of the PDE,

$$ \frac{\partial V}{\partial t} + \frac{1}{2} \beta^2(x) \frac{\partial^2 V}{\partial x^2} + \alpha(x) \frac{\partial V}{\partial x} - r(x) V = 0, $$

with boundary condition $V(x,T) = g(x)$. See Øksendal [60], for a proof. This result is also a special case of what is known as the Feynman-Kac probabilistic representation formula.
Equation (2.2) thus satisfies,
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx \frac{\partial V}{\partial x} - rV = 0,
\]  
with terminal condition \( V(x, T) = (x - k)^+ \).

It is noted that the derivation of this PDE can also be obtained using other methods such as hedging arguments. The PDE in equation (2.6) is in fact the BS PDE and is easily solvable by transforming the problem into the Heat equation, as done in Wilmott, Howison and Dewynne \[80\], for instance. The solution obtained naturally reconciles with the solution from Section 2.1.1.

### 2.2 CIR Process and the Heston Model

One of the major contributors to the pricing of vanilla European options is Steven Heston. The Heston model \[44\], replaces the GBM assumption on the asset, by adding a stochastic volatility component to the BS model. Again if \( X_t \) represents the asset’s value process, and now \( Y_t \) represents the stochastic variance process, then \( X_t \) and \( Y_t \) satisfy the following system of SDEs,
\[
dX_t = rX_t dt + \sqrt{Y_t} X_t dW_t, \tag{2.7}
\]

\[
dY_t = \alpha (m - Y_t) dt + \beta \sqrt{Y_t} dZ_t, \tag{2.8}
\]

where \( Z_t \) is another Brownian motion. \( Z_t \) is defined such that the quadratic covariation between \( W_t \) and \( Z_t \), denoted by \([W, Z]_t\), is equal to \( \rho t \). As such, \( W_t \) and \( Z_t \) are correlated with correlation \( \rho \), and,
\[
d[W, Z]_t = dW_t dZ_t
= \rho dt.
\]

This last fact is important when converting the system of SDEs to a PDE using the Feynman-Kac theorem for multidimensional problems. The system of SDEs in equations (2.7)-(2.8), will be henceforth be referred to as the Heston-GBM.

#### 2.2.1 CIR Process

The process described by the SDE in equation (2.8) is known as a Cox-Ingersoll-Ross (CIR) process, with mean reverting rate \( \alpha \), mean reverting level \( m \), and volatility of volatility \( \beta \). The CIR process has many uses in mathematical finance, mostly due to its non-negativity and its
mean reverting properties. The process has made its way into interest rate modelling problems along with stochastic volatility models.

Although there is no closed form solution for $Y_t$, it is still possible to deduce some of its basic statistical properties. For instance, Feller [27], was able to calculate its transitional probability density function conditional on a starting point. Explicitly, the transitional pdf at time $t$, associated with transitioning from $y_0$ at time 0, is given by,

$$
\Phi \left( y, t \mid y_0 \right) = \frac{2\alpha}{\beta^2 (1 - e^{-\alpha t})} \left( \frac{y}{y_0 e^{-\alpha t}} \right)^{\frac{m}{\beta^2}} \exp \left\{ -\frac{2\alpha(y_0 e^{-\alpha t} + y)}{\beta^2 (1 - e^{-\alpha t})} \right\} \times I_{\frac{m}{\beta^2} - 1} \left( \frac{4\alpha \sqrt{y_0 y e^{-\alpha t}}}{\beta^2 (1 - e^{-\alpha t})} \right),
$$

(2.9)

where $I_k(\cdot)$ is the modified Bessel function of the first kind of order $k$.

Additional calculations show that $4\alpha Y_t / \beta^2 (1 - e^{-\alpha t})$ follows a non-central $\chi^2$ distribution (conditional on $Y_0 = y_0$) with the degrees of freedom parameter, $k = 4\alpha m / \beta^2$, and the non-centrality parameter, $\lambda = 4\alpha y_0 / \beta^2 (1 - e^{-\alpha t})$, see Cairns [11]. As such, calculations of its conditional mean and variance are given by,

$$
E \left( Y_t \mid Y_0 = y_0 \right) = y_0 e^{-\alpha t} + m (1 - e^{-\alpha t}),
$$

(2.10)

$$
\text{Var} \left( Y_t \mid Y_0 = y_0 \right) = y_0 \left( \frac{\beta^2}{\alpha} \right) (e^{-\alpha t} - e^{-2\alpha t}) + m \left( \frac{\beta^2}{2\alpha} \right) (1 - e^{-\alpha t})^2.
$$

(2.11)

Furthermore, the infinitesimal generator of the CIR process is given by,

$$
\mathcal{L} = \frac{1}{2} \gamma^2 \frac{\partial^2}{\partial y^2} + \alpha (y - m) \frac{\partial}{\partial y}.
$$

(2.12)

Using equation (2.12), the invariant distribution $Y$ of a CIR process, i.e. its long run distribution or stationary distribution, can be shown to follow a Gamma distribution, with shape parameter $k = 2\alpha m / \beta^2$ and scale parameter $\theta = \beta^2 / 2\alpha$. While not shown here, the derivation shows that $k > 1$, in order for the invariant distribution to exist as a Gamma distribution. In this case, the invariant density is thus given by,

$$
\Phi_{\text{inv}}(y) = e^{-y/\theta} y^{k-1} \Gamma(k) \theta^k, \quad y \geq 0.
$$

(2.13)

### 2.2.2 The Heston Model

The appealing nature of the Heston model comes from its ability, like the BS model, to produce closed form solutions for vanilla European options. Starting with the discounted expectation of
2.2 CIR Process and the Heston Model

the terminal payoff, as in equation (2.2), but applying to the Heston-GBM model, the Feynman-
Kac theorem generalizes in two dimensions, to give a PDE for which the call option is the unique
solution. The PDE is called the Heston PDE and it is given by,

$$
\frac{\partial V}{\partial t} + \frac{1}{2} y^2 \frac{\partial^2 V}{\partial x^2} + \rho \beta y x \frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2} \beta^2 y^2 \frac{\partial^2 V}{\partial y^2} + r \left( x \frac{\partial V}{\partial x} - V \right) + \alpha (m - y) \frac{\partial V}{\partial y} = 0,
$$

(2.14)

with the terminal condition $V(x, y, T) = (x - k)^+.$

The original Heston PDE differed slightly from equation (2.14) in that it included an addi-
tional $-\lambda(x, y, t) \frac{\partial V}{\partial y}$ term. The $\lambda(x, y, t)$ function represents the market price of volatility risk
and in many cases it is assumed to be proportional to $y$, giving, $\lambda(x, y, t) = \lambda y$, for a constant
$\lambda$. However, the $\lambda$ parameter in the model can be scaled out of the PDE by defining,

$$
\alpha^* = \alpha + \lambda, \\
m^* = \frac{\alpha m}{\alpha + \lambda}.
$$

By replacing $\alpha$ and $m$ by their “starred” version in equation (2.14), the original Heston PDE is
recovered. Thus, for the remainder of this thesis, the market price of volatility risk under the
Heston model will not be considered.

The solution to the European call option problem first appeared in Heston [44]. The solution
can be written as,

$$
V(x, y, t) = x P_1(s, y, t; T, k) - ke^{-r(T-t)} P_2(s, y, t; T, k),
$$

(2.15)

where,

$$
\begin{align*}
s &= \ln(x), \\
P_j(s, y, t; T, k) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-i\phi \ln(k)} f_j(s, y, t; T, \phi)}{i \phi} \right) \, d\phi, \\
f_j(s, y, t; T, \phi) &= \exp(C_j(T - t, \phi) + y D_j(T - t, \phi) + i \phi s), \\
C_j(\tau, \phi) &= r \phi i \tau + \frac{a}{\beta^2} \left( (b_j - \rho \beta \phi i + d_j) \tau - 2 \ln \left( \frac{1 - g_j e^{d_j \tau}}{1 - g_j} \right) \right), \\
D_j(\tau, \phi) &= \frac{b_j - \rho \beta \phi i + d_j}{\beta^2} \left( \frac{1 - e^{d_j \tau}}{1 - g_j} \right), \\
g_j(\phi) &= \frac{b_j - \rho \beta \phi i + d_j}{b_j - \rho \beta \phi i - d_j}, \\
d_j(\phi) &= \sqrt{(\rho \beta \phi i - b_j)^2 - \beta^2 (2u_j \phi i - \phi^2)},
\end{align*}
$$

for $j = 1, 2$, $u_1 = 1/2$, $u_2 = -1/2$, $a = \alpha m$, $b_1 = \alpha - \rho \beta$, $b_2 = \alpha$ and $i$ denoting the imaginary unit.
2.3 GMR Process

The geometric mean reverting (GMR) process has become more popular in mathematical finance modelling problems of late. Its popularity arises from the geometric nature of the process along with its mean reverting capabilities. These desired properties have made the GMR process a successful candidate to model instruments such as interest rates derived from the marginal utility of capital and the value of a project in real option theory, see Merton [57] and Dixit and Pindyck [21], respectively, for examples. The process itself also finds its way in the evaluation of Australian type options.

If \( X_t \) satisfies the SDE,

\[
dX_t = (aX_t - bX_t^2) \, dt + cX_t \, dW_t,
\]

with \( X_0 > 0, \ a, \ b \) and \( c \) being positive constants and \( W_t \) a Brownian motion, then \( X_t \) follows a GMR process. Merton, identified that the invariant distribution of the GMR process is in fact a Gamma distribution, with shape parameter \( k = \frac{2a}{c^2} - 1 \) and scale parameter \( \theta = \frac{c}{2b} \). Its invariant density is thus similar to equation (2.13), with \( k \) and \( \theta \) adjusted accordingly for the GMR process.

While no known closed form solution for the GMR process exists, Yang and Ewald [84], are able to calculate its transitional pdf, conditional on a starting point. Similar to the CIR process, the transitional pdf at time \( t \), associated with transitioning from \( x_0 \) at time 0, is given by,

\[
\Phi(x, t \mid x_0) = \frac{c^2}{4bx^2} \exp \left( \frac{-\bar{a}^2 t}{2x^2} \right) \int_{-\infty}^{\infty} \exp \left( 2 \left( 1 - \frac{\bar{a}}{c^2} \right) z \right) f_{c^2 t/4} \left( \frac{c^2 \exp(2z) - \frac{4x}{x_0^2}}{4bx}, z \right) \, dz,
\]

where,

\[
\bar{a} = -a + \frac{c^2}{2},
\]

\[
f_t(x, y) = \exp \left( \frac{2xy + x^2 - t - t \exp(2y)}{2xt} \right) \frac{x^2}{2\pi^{3/2}} \int_0^\infty \exp \left( \frac{-z^2}{2t} - \frac{\exp(y) \cosh(z)}{x} \right) \sinh(z) \sin \left( \frac{\pi z}{t} \right) \, dz.
\]

Although the transitional pdf for the GMR process exists, it is numerically very unstable. To see this, note that in the definition for \( f_t(x, y) \), the integrand is oscillatory in nature due to the sine term and that it takes on both positive and negative values. As such, the integral is
difficult to compute making the function itself unstable. Furthermore, in the actual density function and for most practical applications, the \( t \) in \( f_t(x,y) \) is replaced by \( c^2 t/4 \), with \( c < 1 \). This in turn increases the number of oscillations in a fixed interval, which naturally means any numerical quadrature of the integrand will be very difficult. Thus, many standard methods for calculating the transitional pdf is very unstable.

The uses of the GMR process in real options theory and Australian option valuation will be explored later in the thesis.

2.4 Option Types

The review thus far has only touched upon the most basic types of options, which are the European calls and puts. Many different option types are characterized by their various attributes. This thesis will touch upon real options, Asian options and Australian options.

Real options are essentially a class of perpetual American options. American options give the holder the right to exercise the option at any point in time up to and including the expiry date. The perpetual nature of these options means they have no expiry date and the option is alive indefinitely. Real options are generally of the call type, and like many other American options have an optimal threshold level, at which the option should be exercised. Often, the difficulty with pricing American options is determining optimal threshold levels, as mathematically, they represent the free boundary of the problem.

Asian options are options where the payoffs are functions of the time averaged value of the asset. They can be characterized by either being a call or put type. The average itself can be taken in various ways, for example discretely or continuously, and the type of average, arithmetic or geometric. Also, the strike aspect of these options can be characterized as either floating or fixed. In the fixed case, the strike is fixed to a predetermined value, while for the floating case, the strike is dependent on the asset’s value at expiry. Due to the time-averaged nature, these options are usually European in nature\(^2\), that is, the option can only be exercised at the expiry date.

Australian options are similar to Asian options in that the payoff is also a function of the time averaged value of the asset, but instead of just the average, the variable of interest is the ratio of the average to the final asset’s value at expiry. Again, the average for Australian options can have any of attributes as listed for Asian options. The strikes are usually fixed, but

\[^2\text{Some studies of American Asian options do exist, but are mostly limited to numerical approaches.}\]
the idea can be generalized to include strikes which vary with the asset’s final value. This will be later explored in the chapters to follow.
Chapter 3

Fast Mean-Reverting Asymptotic Heston Solution

The addition of stochastic volatility in option pricing theory has resulted in the development of more sophisticated frameworks for pricing financial derivatives. The need to incorporate a randomly varying volatility arose from studies involving the log returns of heavily traded indices on the major stock exchanges. As previously stated, in the Black-Scholes [8], (BS) model, the asset’s value is assumed to follow a geometric Brownian motion (GBM) process, which translates to the log return of the assets following a normal distribution. Empirical studies have shown that this assumption may in fact not be valid. In general, the empirical log return distributions have heavier tails and higher peaks, which is indicative of a distribution with differing variances. For more on this, see Gatheral [34]. Further proof, in the form of implied volatilities, shows that the constant volatility assumption in the BS model is rather unrealistic. Plotting implied volatility surfaces of options prices, shows that volatility is dependent on both the time to expiry and the strike price of the option. The shapes of these surfaces are generally referred to as the volatility smile, due to the fact that looking at a fixed time to expiry, the implied volatilities as the strike price varies are sometimes reminiscent of a smile shape. Whilst other shapes such as smirks and frowns do exist, there are cases where no recognizable shapes are observed. The type of smile observed is largely dependent on the type of option class considered. These observations provide motivation for modelling volatility as a random variable.

There is no generally accepted view on which stochastic volatility model is best used to model option prices. The Heston model [44], is a favourite among practitioners due to its tractability and simplicity. The model assumes the asset’s value and variance processes follow
equations (2.7) and (2.8) respectively, that is, they satisfy the Heston-GBM model. The biggest drawcard of the Heston model is its closed form solution for vanilla European options, with the solution for the call type given in equation (2.15), along with its ability to fit observed implied volatilities for longer maturities, see Gatheral [34].

This chapter investigates the derivation of a four term asymptotic solution to the Heston call option problem. The asymptotic solution is based on the assumption that the CIR process driving the variance, possesses fast mean reverting properties. It is asymptotic in the sense that the solution becomes a better approximation as the mean reverting rate increases. The asymptotic solutions are based on modifications to the original work by Fouque, Papanicolaou and Sircar [31], [32], on stochastic volatility models with fast mean reverting Ornstein-Uhlenbeck (OU) processes. The original works only contained two terms in the asymptotic expansion. It is also noted that in Fouque and Lorig [30], the asymptotic method is applied to the Heston model in a multiscale framework1, but this is different to what is presented here.

Most recently, Fouque, Papanicolaou, Sircar and Sølna [33], have revived their work on the asymptotic solutions and have derived the third term in the asymptotic expansion. Furthermore, justifications for its application to the Heston model are also provided. However, it is noted that the derivation of the two term solution, along with the details and justifications for applying this methodology to the Heston model presented here, was developed independently of their work. In addition, this chapter will also review their methodology for obtaining the third term whilst also applying a similar technique to obtaining the fourth term. Proofs showing that the residue terms of the four term asymptotic solution have the correct order are also provided. The method used to approach the proofs are similar to the one provided by Fouque et al. [33], but modified to account for the fourth term in the expansion. As a result, the original contributions of this chapter will be the derivation and analysis of the four term asymptotic Heston solution.

Given that a closed form solution for a call option under the Heston model exists, numerical analysis on the accuracy of the asymptotic solution will be provided. In particular, some of the issues considered in determining the accuracy of the asymptotic solution are how the initial variance level and its mean reverting level affect the asymptotic solution, the accuracy of the asymptotic solution as a function of the mean reverting rate and the times to expiry. The accuracy of the two, three and four term asymptotic solutions will be investigated.

---

1 Under this framework, an additional OU process is added to the Heston model such that the volatility is now driven by the square root of the CIR process and a function of the OU process.
3.1 Derivation of the Asymptotic Solution

This section contains the derivation of the asymptotic Heston solution. The derivation of the first half is itself loosely based on that of Fouque, Papanicolaou and Sircar [31], [32], while the latter half is based on the more recent work of Fouque et al. [33]. Refer back to Section 2.2.2 for a description of the Heston model. Also note that the derivation of the asymptotic solution can be generalized to any stochastic volatility model where the volatility is a function of the CIR process. This can be done by choosing a general function \( f(Y_t) \), instead of \( \sqrt{Y_t} \), in equation (2.7), but this is omitted here as it seldom gives simple closed form solutions.

It is further noted that the derivation of the first two terms and the justifications for using the asymptotic expansion technique, were developed independently of Fouque et al. and their latest work. The authors of this new work, also considered the asymptotic solution under the Heston model, whilst deriving the third term in the asymptotic expansion. Using their technique, the fourth term in the asymptotic expansion will be derived in this chapter, and forms part of the original contributions of this thesis.

Begin by introducing the two parameters \( \epsilon \) and \( \nu^2 \), which are the inverse of the mean reverting rate \( \alpha \) and the variance of the invariant distribution \( Y \), of the CIR process. This gives,

\[
\epsilon = \frac{1}{\alpha},
\]

\[
\nu^2 = \frac{m\beta^2}{2\alpha}.
\]

In the asymptotic solution, the expansion parameter will be \( \epsilon \), and it is assumed that \( \nu \) remains fixed as \( \epsilon \) decreases.

The Heston PDE in equation (2.14) can be rewritten in a more compact form. By replacing the \( \alpha \) and \( \beta \) with \( \epsilon \) and \( \nu \) where appropriate, the compact form of the PDE is given as,

\[
\left( \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) V = 0,
\]

with the operators defined as,

\[
\mathcal{L}_0 = \frac{\nu^2 y}{m} \frac{\partial^2}{\partial y^2} + \left( m - y \right) \frac{\partial}{\partial y},
\]

\[
\mathcal{L}_1 = \rho \frac{\nu \sqrt{2}}{\sqrt{m}} \sqrt{xy} \frac{\partial^2}{\partial x \partial y},
\]

\[
\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} \frac{\nu^2 x^2}{m} \frac{\partial^2}{\partial x^2} + r \left( \frac{x}{\partial x} - \cdot \right) =: \mathcal{L}_2(\sqrt{y}),
\]

where,
3.1 Derivation of the Asymptotic Solution

1. \(\alpha L_0\) is the infinitesimal generator of the CIR process \(Y_t\);

2. \(L_1\) contains the mixed partial derivatives due to the correlation between the two Brownian motions in equations (2.7) and (2.8);

3. \(L_2\) is the BS operator with \(\sqrt{y}\) as the (constant) volatility parameter.

Assume the solution to \(V(x, y, t)\) can be expanded in the form,

\[
V = V_0 + \sqrt{\epsilon}V_1 + \epsilon V_2 + \epsilon \sqrt{\epsilon}V_3 + \epsilon^2 V_4 + \epsilon^2 \sqrt{\epsilon}V_5 + \cdots, \tag{3.5}
\]

where the \(V_i\)'s are functions of \((x, y, t)\), to be determined. In the case of a call option, the terminal conditions for the \(V_i\)'s are such that \(V_0(x, y, T) = (x - k)^+\) and \(V_1(x, y, T) = 0\).

Substituting equation (3.5) into equation (3.1) and then collecting terms of up to order \(\epsilon \sqrt{\epsilon}\) gives,

\[
\frac{1}{\epsilon} L_0 V_0 + \frac{1}{\sqrt{\epsilon}} (L_0 V_1 + L_1 V_0) + (L_0 V_2 + L_1 V_1 + L_2 V_0) + \sqrt{\epsilon} (L_0 V_3 + L_1 V_2 + L_2 V_1)
+ \epsilon (L_0 V_4 + L_1 V_3 + L_2 V_2) + \epsilon \sqrt{\epsilon} (L_0 V_5 + L_1 V_4 + L_2 V_3) = 0.
\]

By equating various orders of \(\epsilon\) to zero, the conditions which the \(V_i\)'s must satisfy can be determined.

### 3.1.1 Orders \(1/\epsilon\) and \(1/\sqrt{\epsilon}\)

Firstly, by equating the order \(1/\epsilon\) term to zero, requires,

\[
L_0 V_0 = 0.
\]

The operator \(L_0\) in equation (3.2), is made up of partial derivatives with respect to \(y\). Thus, the condition required for \(V_0\), is that it must be a function independent of \(y\).

Similarly, the condition obtained by equating the order \(1/\sqrt{\epsilon}\) term to zero, implies that \(V_1\) must also be a function independent of \(y\). This can be verified by noting that equation (3.3) takes second order partial derivatives with respect to \(x\) and \(y\), and thus when applied to \(V_0\), gives zero. Thus the condition becomes,

\[
L_0 V_1 = 0,
\]

which shows that \(V_1\) is independent of \(y\).
3.1 Derivation of the Asymptotic Solution

3.1.2 Poisson Equation Review

In going to the next order, a brief review of the Poisson equation is first provided. Consider the equation,

\[ \mathcal{L}_0 \chi + g = 0, \]

which is known as a Poisson equation for \( \chi(y) \) with respect to the operator \( \mathcal{L}_0 \). In this case, \( \mathcal{L}_0 \) is defined by equation (3.2). A solution for \( \chi(y) \) exists if and only if the function \( g(y) \) is centered with respect to the invariant distribution whose infinitesimal generator is given by \( \mathcal{L}_0 \). If a general CIR process has mean reverting rate and volatility of volatility as \( \bar{\alpha} \) and \( \bar{\beta} \), respectively, then \( \mathcal{L}_0 \) is the infinitesimal generator of a CIR process with \( \bar{\alpha} = 1 \) and \( \bar{\beta} = \nu \sqrt{2} / \sqrt{m} \). This follows directly from equations (2.12) and (3.2). Thus, the invariant distribution is Gamma distributed with shape \( k = m^2 / \nu^2 \) and scale \( \theta = \nu^2 / m \). Converting \( \nu \) back in terms of \( \alpha \) and \( \beta \) gives the parameters as \( k = 2 \alpha m / \beta^2 \) and \( \theta = \beta^2 / 2 \alpha \). The centering condition requires,

\[ \langle g \rangle := \int_0^\infty g(y) \Phi_{\text{inv}}(y) \, dy = 0, \]

where \( \Phi_{\text{inv}}(y) \) is the probability density function, as given in equation (2.13). The centering condition is shown to hold by using the Poisson equation, integration by parts, and the adjoint operators \( \mathcal{L}_0^* \) and its property that \( \mathcal{L}_0^* \Phi_{\text{inv}} = 0 \). It is further assumed that both \( g, \chi \) and all their partial derivatives with respect to \( y \), have bounded growth. The proof of the centering condition is as follows,

\[
\begin{align*}
\langle g \rangle &= - \langle \mathcal{L}_0 \chi \rangle \\
&= - \int_0^\infty (\mathcal{L}_0 \chi(y)) \Phi_{\text{inv}}(y) \, dy \\
&= - \int_0^\infty \Phi_{\text{inv}}(y) \left( (m - y) \frac{\partial \chi}{\partial y} + \frac{\beta^2}{2 \alpha} y \frac{\partial^2 \chi}{\partial y^2} \right) \, dy \\
&= - \left[ \Phi_{\text{inv}}(y) \left( (m - y) \chi(y) + \frac{\beta^2}{2 \alpha} y \frac{\partial \chi}{\partial y} \right) \right]_0^\infty \\
&\quad + \int_0^\infty \chi(y) \frac{\partial}{\partial y} \left[ (m - y) \Phi_{\text{inv}}(y) \right] \, dy + \int_0^\infty \frac{\partial \chi}{\partial y} \frac{\partial}{\partial y} \left[ \frac{\beta^2}{2 \alpha} y \Phi_{\text{inv}}(y) \right] \, dy. 
\end{align*}
\]

From the form of \( \Phi_{\text{inv}}(y) \) in equation (2.13), the boundaries at \( y = 0 \) and \( y \to \infty \) of the
3.1 Derivation of the Asymptotic Solution

The following functions are,

\[ \Phi_{\text{inv}}(0) = 0, \quad \text{if } 2\alpha m/\beta^2 > 1, \]
\[ \Phi_{\text{inv}}(y) \to 0, \quad \text{if } y \to \infty, \]
\[ y\Phi_{\text{inv}}(y) = 0, \quad \text{at } y = 0, \text{if } 2\alpha m/\beta^2 > 0, \]
\[ y\Phi_{\text{inv}}(y) \to 0, \quad \text{if } y \to \infty. \]

Thus, if the condition,

\[ \frac{2\alpha m}{\beta^2} > 1, \quad (3.7) \]

is satisfied, then the first term in equation (3.6) is zero, provided \( \chi(y) \) does not grow to infinity faster than \( \Phi_{\text{inv}}(y) \) goes to zero at the end points. Continuing on,

\[
\langle g \rangle = \int_0^\infty \chi(y) \frac{\partial}{\partial y} [(m - y)\Phi_{\text{inv}}(y)] \, dy + \int_0^\infty \frac{\partial \chi}{\partial y} \left( \frac{\beta^2}{2\alpha} y\Phi_{\text{inv}}(y) \right) \, dy \\
= - \left[ \chi(y) \frac{\partial}{\partial y} \left( \frac{\beta^2}{2\alpha} y\Phi_{\text{inv}}(y) \right) \right]_0^\infty - \int_0^\infty \chi(y)\mathcal{L}_0^*\Phi_{\text{inv}}(y) \, dy, \quad (3.8)
\]

with the adjoint operator \( \mathcal{L}_0^* \) defined as,

\[ \mathcal{L}_0^* = - \frac{\partial}{\partial y} ((m - y)\cdot) + \frac{\beta^2}{2\alpha} \frac{\partial^2}{\partial y^2} (y\cdot). \]

Using the boundary conditions at \( y = 0 \) and \( y \to \infty \) again, the first term in equation (3.8) is zero because,

\[
\frac{\partial}{\partial y} (y\Phi_{\text{inv}}(y)) = \Phi_{\text{inv}}(y) + y\Phi_{\text{inv}}'(y) \\
= \frac{2\alpha m}{\beta^2} \Phi_{\text{inv}}(y) - \frac{2\alpha}{\beta^2} y\Phi_{\text{inv}}(y).
\]

This gives,

\[
\langle g \rangle = - \int_0^\infty \chi(y) \left( \mathcal{L}_0^*\Phi_{\text{inv}}(y) \right) \, dy \\
= 0,
\]

which completes the proof. The main difference between this derivation for stochastic volatility driven by a CIR process and the derivation by Fouque et al. in [32], for stochastic volatility driven by an OU process, is the dependence on the condition \( 2\alpha m/\beta^2 > 1 \), for the asymptotic solution to be technically valid. In Fouque et al. [33], the proof showing the need for the centering condition is achieved by considering the reversibility of the CIR process, which is a more generic result, but the derivation provided here suffices for its intended purposes.
3.1 Derivation of the Asymptotic Solution

The inequality in (3.7) is in fact the same condition required for the CIR process to remain positive at all times after starting at a positive value. A discussion of this can be found in Fouque et al. [33]. Furthermore, this is also equivalent to the condition required for an invariant distribution to exist as a Gamma distribution, which is much needed here. Thus, from this point onwards, it is assumed that the condition in inequality (3.7) is satisfied at all times.

The solution for \( \chi(y) \) is formally given by,

\[
\chi(y) = \int_{0}^{\infty} \mathbb{E}\left( g(Y_t) \bigg| Y_0 = y \right) \, dt,
\]

where the expectation in equation (3.9) is calculated using the transitional pdf of the CIR process, given in equation (2.9). It is important to note that, while the invariant distribution of the infinitesimal generator \( \mathcal{L}_0 \) is exactly the same as the invariant distribution of a general CIR process, \( \mathcal{L}_0 \) actually describes a scaled CIR process\(^2\). Thus, when using equation (2.9), it is important to set \( \alpha = 1 \) and \( \beta^2 = 2\nu^2/m \). Furthermore, in Fouque et al. [33], it is argued that the solution \( \chi(y) \) can be written as an eigenfunction representation, and that the two solutions in fact coincide. The benefits of doing so is that the result will hold for processes other than the CIR process. However, given that this chapter only deals with the CIR process and that the transitional pdf of the CIR process along with various mean and variance results are available, see equations (2.10) and (2.11), respectively, equation (3.9) will be used instead.

3.1.3 Order 1

The order 1 term requires,

\[
\mathcal{L}_0 V_2 + \mathcal{L}_1 V_1 + \mathcal{L}_2 V_0 = 0,
\]

and by using the results of order \( 1/\sqrt{\tau} \), this becomes,

\[
\mathcal{L}_0 V_2 + \mathcal{L}_2 V_0 = 0.
\]

Equation (3.10) is in the form of a Poisson equation, if only the dependence from \( y \) is considered. The centering condition requires,

\[
\langle \mathcal{L}_2 V_0 \rangle = 0.
\]

\(^2\)\( \alpha \mathcal{L}_0 \) is the infinitesimal generator for regular CIR process.
3.1 Derivation of the Asymptotic Solution

Since the mean of a Gamma distribution is \( k\theta = m \), using \( \langle y \rangle = m \) and the operator given in equation (3.4), gives the PDE for \( V_0 \) as,

\[
\frac{\partial V_0}{\partial t} + \frac{1}{2} mx^2 \frac{\partial^2 V_0}{\partial x^2} + r \left( x \frac{\partial V_0}{\partial x} - V_0 \right) = 0.
\]

Together with the terminal condition, it is easy to conclude that \( V_0 \) is the BS solution with \( \sqrt{m} \) as the constant volatility parameter.

From equation (3.10) and using equation (3.11), it is possible to calculate the following,

\[
\mathcal{L}_0 V_2 = -\mathcal{L}_2 V_0
\]
\[
= -(\mathcal{L}_2 V_0 - \langle \mathcal{L}_2 V_0 \rangle)
\]
\[
= -\frac{1}{2} (y - m) x^2 \frac{\partial^2 V_0}{\partial x^2}.
\]

This gives \( V_2 \) as,

\[
V_2(t, x, y) = -\frac{1}{2} \mathcal{L}_0^{-1} (y - m) x^2 \frac{\partial^2 V_0}{\partial x^2}
\]
\[
= -\frac{1}{2} (\phi(y) + c(t, x)) x^2 \frac{\partial^2 V_0}{\partial x^2},
\]

where \( c(t, x) \) is an arbitrary function independent of \( y \) and \( \phi(y) \) is the solution to the Poisson equation,

\[
\mathcal{L}_0 \phi = y - m.
\]

Using equation (3.9) and knowledge from equation (2.10), the solution for \( \phi(y) \) is given as,

\[
\phi(y) = \int_0^\infty E \left( m - Y_t \left| Y_0 = y \right. \right) dt
\]
\[
= \int_0^\infty (m - y) e^{-t} dt
\]
\[
= m - y.
\]

### 3.1.4 Order \( \sqrt{\epsilon} \)

The condition derived from the order \( \sqrt{\epsilon} \) term is,

\[
\mathcal{L}_0 V_3 + \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 = 0,
\]

which again requires a centering condition, this time of the form,

\[
\langle \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 \rangle = 0.
\]
Using this and the form for $V_2$ in equation (3.12), gives

$$\mathcal{L}_2 \left( \sqrt{m} \right) V_1 = \left( \mathcal{L}_2 V_1 \right)$$
$$= - \left( \mathcal{L}_1 V_2 \right)$$
$$= \frac{1}{2} \left( \mathcal{L}_1 \phi(y) x^2 \frac{\partial^2 V_0}{\partial x^2} \right),$$

where it is noted that $\mathcal{L}_1$ takes derivatives with respect to $y$, and that $c(x,t)$ is independent of $y$. It can be shown that, for any function $u(x,t)$,

$$\langle \mathcal{L}_1 \phi(y) u(x,t) \rangle = \rho \nu \sqrt{\frac{2}{m}} \left( 2x^2 \frac{\partial^2 V_0}{\partial x^2} + x^3 \frac{\partial^3 V_0}{\partial x^3} \right).$$

Applying this to $u(x,t) = x^2 \frac{\partial^2 V_0}{\partial x^2}$ gives,

$$\langle \mathcal{L}_2 V_1 \rangle = \frac{1}{\sqrt{2}} \rho \nu \sqrt{m} \left( 2x^2 \frac{\partial^2 V_0}{\partial x^2} + x^3 \frac{\partial^3 V_0}{\partial x^3} \right).$$

Multiply the above by $\sqrt{\epsilon}$ and define $\tilde{V}_1 = \sqrt{\epsilon} V_1$. Then, convert the parameters $\nu$ and $\epsilon$ back in terms of $\alpha$ and $\beta$. This sequence of calculations result in,

$$\mathcal{L}_2 \left( \sqrt{m} \right) \tilde{V}_1 = \frac{\rho \beta m}{2 \alpha} \left( 2x^2 \frac{\partial^2 V_0}{\partial x^2} + x^3 \frac{\partial^3 V_0}{\partial x^3} \right).$$

The solution for $\tilde{V}_1$ is given by,

$$\tilde{V}_1(x,t) = \frac{\rho \beta m}{2 \alpha} \left( T - t \right) \left( 2x^2 \frac{\partial^2 V_0}{\partial x^2} + x^3 \frac{\partial^3 V_0}{\partial x^3} \right).$$

This can be verified by noting that for any positive integer $n$,

$$\mathcal{L}_2(\sqrt{m}) \left( x^n \frac{\partial^n V_0}{\partial x^n} \right) = x^n \frac{\partial^n}{\partial x^n} \left( \mathcal{L}_2 \left( \sqrt{m} \right) V_0 \right) = 0. \tag{3.14}$$

To prove the equality in (3.14), it is sufficient to show that,

$$x^k \frac{\partial^k}{\partial x^k} \left( x^n \frac{\partial^n V_0}{\partial x^n} \right) = x^n \frac{\partial^n}{\partial x^n} \left( x^k \frac{\partial^k V_0}{\partial x^k} \right),$$

for $k = 1, 2$. Define the differential operator,

$$\mathcal{L}^{(n)} = x^n \frac{\partial^n}{\partial x^n}. \tag{3.15}$$
Using the transformation \( s = \log(x) \), it can be shown that \( \mathcal{L}^{(n)} \) becomes a \( n \)-th order linear differential operator with constant coefficients. For \( n = 1 \),

\[
\frac{\partial}{\partial x} = \frac{ds}{dx} \frac{\partial}{\partial s} = \frac{1}{x} \frac{\partial}{\partial s},
\]

\[
x \frac{\partial}{\partial x} = \frac{\partial}{\partial s}. \tag{3.16}
\]

Thus, it is true for \( n = 1 \). Assume this is true for some positive integer \( n > 1 \), and write,

\[
\mathcal{L}^{(n)} = x^n \frac{\partial^n}{\partial x^n} = a_n \frac{\partial^n}{\partial s^n} + a_{n-1} \frac{\partial^{n-1}}{\partial s^{n-1}} + \cdots + a_1 \frac{\partial}{\partial s}, \tag{3.17}
\]

where \( a_n, a_{n-1}, \ldots, a_1 \) are constants. Now,

\[
\mathcal{L}^{(n+1)} = x^{n+1} \frac{\partial^{n+1}}{\partial x^{n+1}} = x \frac{\partial}{\partial x} \left( x^n \frac{\partial^n}{\partial x^n} \right) - nx^n \frac{\partial^n}{\partial x^n}
\]

\[
= a_n \frac{\partial^{n+1}}{\partial s^{n+1}} + (a_{n-1} - na_n) \frac{\partial^n}{\partial s^n} + (a_{n-2} - na_{n-1}) \frac{\partial^{n-1}}{\partial s^{n-1}}
\]

\[
+ \cdots + (a_1 - na_2) \frac{\partial^2}{\partial s^2} - na_1 \frac{\partial}{\partial s}
\]

\[
= b_{n+1} \frac{\partial^{n+1}}{\partial s^{n+1}} + \cdots + b_1 \frac{\partial}{\partial s},
\]

by using equations (3.16) and (3.17) and renaming the constants to \( b_{n+1}, b_n, \ldots, b_1 \). Therefore, by induction, the assumption is true for all \( n \). Now, for \( k = 1 \),

\[
x \frac{\partial}{\partial x} \left( x^n \frac{\partial^n V_0}{\partial x^n} \right) = \mathcal{L}^{(1)} \left( \mathcal{L}^{(n)} V_0 \right)
\]

\[
= \mathcal{L}^{(n)} \left( \mathcal{L}^{(1)} V_0 \right)
\]

\[
= x^n \frac{\partial^n}{\partial x^n} \left( x \frac{\partial V_0}{\partial x} \right),
\]

by using the commutative property of linear differential operators with constant coefficients. Similarly, this holds for \( k = 2 \) and in fact for any positive integer.

### 3.1.5 Order \( \epsilon \)

The following derivation for the third term in the asymptotic solution follows Fouque et al. [33]. It is repeated here for continuity, as the fourth term will be subsequently derived using similar techniques.
3.1 Derivation of the Asymptotic Solution

The solution for \( V_2(x, y, t) \) is rewritten as,

\[
V_2(x, y, t) = \frac{1}{2} (y - m) \mathcal{L}^{(2)} V_0 + C(x, t),
\]

where \( C(x, t) \) is a function independent of \( y \). Note that \( C(x, t) \) has absorbed \( c(x, t) \) and other functions dependent only on \( (x, t) \) from equation (3.12). As done in Fouque et al., it is assumed that \( C(x, T) = 0 \). Furthermore, the solution for \( V_2(x, y, t) \) now makes use of the differential operator defined in equation (3.15). This operator will now feature heavily in the subsequent sections for tractability purposes.

For order \( \epsilon \), it is required that,

\[
\mathcal{L}_0 V_4 + \mathcal{L}_1 V_3 + \mathcal{L}_2 V_2 = 0.
\]

This is a Poisson equation for \( V_4 \), with the centering condition given by,

\[
\langle \mathcal{L}_1 V_3 + \mathcal{L}_2 V_2 \rangle = 0. \tag{3.19}
\]

Note that \( \mathcal{L}_0 V_3 \) can be written as,

\[
\mathcal{L}_0 V_3 = -\mathcal{L}_1 V_2 - \mathcal{L}_2 V_1 \\
= -\mathcal{L}_1 V_2 - \mathcal{L}_2 V_1 + \langle \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 \rangle \\
= - (\langle \mathcal{L}_1 V_2 \rangle - \langle \mathcal{L}_2 V_1 \rangle - \langle \mathcal{L}_2 V_1 \rangle),
\]

where the order \( \sqrt{\epsilon} \) equation, and centering condition is used. Furthermore, \( \mathcal{L}_1 V_2 \) can be calculated as,

\[
\mathcal{L}_1 V_2 \equiv \rho \beta y \frac{\partial^2}{\partial x \partial y} \left( \frac{1}{2} (y - m) \mathcal{L}^{(2)} V_0 + C(t, x) \right) \\
= \frac{\rho \beta y \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0}{2 \sqrt{\alpha}}.
\]

Therefore,

\[
\mathcal{L}_1 V_2 - \langle \mathcal{L}_1 V_2 \rangle = \frac{\rho \beta}{2 \sqrt{\alpha}} (y - m) \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0. \tag{3.20}
\]

For \( \mathcal{L}_2 V_1 \), it can be easily shown that,

\[
\mathcal{L}_2 V_1 - \langle \mathcal{L}_2 V_1 \rangle = \frac{1}{2} (y - m) \mathcal{L}^{(2)} V_1, \tag{3.21}
\]

Justifications for this choice can be found in [33], and will be recited later.
3.1 Derivation of the Asymptotic Solution

by noting that $V_1$ is independent of $y$. Equations (3.20) and (3.21) show that $\mathcal{L}_0 V_3$ can be calculated as,

$$\mathcal{L}_0 V_3 = -(y - m) \left( \frac{\rho \beta}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} \mathcal{L}^{(1)} V_1 \right),$$

and thus,

$$V_3 = -\mathcal{L}_0^{-1} \left( (y - m) \left( \frac{\rho \beta}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} \mathcal{L}^{(1)} V_1 \right) \right)$$

$$= -\phi(y) \left( \frac{\rho \beta}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} \mathcal{L}^{(1)} V_1 \right) + F(x,t)$$

$$= (y - m) \left( \frac{\rho \beta}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} \mathcal{L}^{(1)} V_1 \right) + F(x,t),$$

where $F(x,t)$ is a function independent of $y$, much like the role $C(x,t)$ plays in $V_2$. It is assumed that this function has terminal value $F(x,T) = 0$, which will be later justified.

The centering condition in equation (3.19) gives,

$$\langle \mathcal{L}_2 V_2 \rangle = -\langle \mathcal{L}_1 V_3 \rangle.$$

First consider $\mathcal{L}_1 V_3$. This expression and its averaged version can be calculated as,

$$\mathcal{L}_1 V_3 = \frac{\rho \beta}{\sqrt{\alpha}} y \left( \frac{\rho \beta}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_1 \right),$$

$$\langle \mathcal{L}_1 V_3 \rangle = \frac{\rho^2 m \beta^2}{2\alpha} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{\rho m \beta^2}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_1,$$

respectively. Next consider $\langle \mathcal{L}_2 V_2 \rangle$. This can be calculated as,

$$\langle \mathcal{L}_2 V_2 \rangle = \left( \mathcal{L}_2 \left( \frac{1}{2} (y - m) \mathcal{L}^{(2)} V_0 \right) \right) + \langle \mathcal{L}_2 C \rangle$$

$$= \frac{1}{2} \langle (y - m) \mathcal{L}_2 \mathcal{L}^{(2)} V_0 + \langle \mathcal{L}_2 C \rangle.$$

Furthermore, it is noted that,

$$\langle (y - m) \mathcal{L}_2 \rangle = \frac{1}{2} \left( \langle y^2 \rangle - m^2 \right) \mathcal{L}^{(2)}$$

$$= \frac{m \beta^2}{4\alpha} \mathcal{L}^{(2)},$$

which uses the fact, $\langle y^2 \rangle = m \beta^2 / 2\alpha + m^2$.

In combining all this, the PDE which $C(x,t)$ satisfies is thus,

$$\langle \mathcal{L}_2 C \rangle = -\frac{m \beta^2}{8\alpha} \mathcal{L}^{(2)} V_0 - \frac{\rho^2 m \beta^2}{2\alpha} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 - \frac{\rho m \beta^2}{2\sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_1$$

$$= -\frac{m \beta^2}{8\alpha} \mathcal{L}^{(2)} V_0 - \frac{\rho^2 m \beta^2}{2\alpha} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0$$

$$- \frac{\rho^2 m^2 \beta^2}{4\alpha} (T - t) \mathcal{L}^{(1)} \mathcal{L}^{(2)} \mathcal{L}^{(2)} V_0,$$
where $V_1(x,t)$ is written as,

$$V_1(x,t) = \frac{\rho m \beta}{2\sqrt{\alpha}} (T - t) L(1) L(2) V_0.$$  

This is just the BS PDE with constant volatility $\sqrt{m}$ and source terms as given. The solution to $C(x,t)$ is thus given as,

$$C(x,t) = (T - t) \left( \frac{m \beta^2}{8\alpha} L(1) L(2) V_0 + \frac{\rho^2 m \beta^2}{2\alpha} L(1) L(2) V_0 \right)$$

$$+ \frac{(T - t)^2 \rho^2 m^2 \beta^2}{4\alpha} L(1) L(2) L(1) L(2) V_0.$$  

(3.24)

This solution can be verified by direct substitution and noting the result in equation (3.14).

3.1.6 Order $\epsilon \sqrt{\epsilon}$

The methodology for the fourth term in the asymptotic expansion follows closely to that of the previous section. Recall that the solution for $V_3(x,y,t)$ is given as,

$$V_3(x,y,t) = (y - m) \left( \frac{\rho \beta}{2\sqrt{\alpha}} L(2) V_0 + \frac{1}{2} L(1) V_1 \right) + F(x,t),$$  

(3.25)

where $F(x,t)$ is a function independent of $y$, and has boundary condition $F(x,T) = 0$. This choice of boundary condition will be justified later when proving the accuracy of the asymptotic solution. Proceed similarly to the previous section, by deriving a PDE which $F(x,t)$ satisfies and then solve it to obtain the complete solution for $V_3(x,y,t)$.

The order $\epsilon \sqrt{\epsilon}$ equation is given by,

$$\mathcal{L}_0 V_5 + \mathcal{L}_1 V_4 + \mathcal{L}_2 V_3 = 0,$$

which is a Poisson equation for $V_5$. The centering condition required is,

$$\langle \mathcal{L}_1 V_4 + \mathcal{L}_2 V_3 \rangle.$$

From the order $\epsilon$ equation, $\mathcal{L}_0 V_4$ can be written as,

$$\mathcal{L}_0 V_4 = -\mathcal{L}_1 V_3 - \mathcal{L}_2 V_2$$

$$= -\mathcal{L}_1 V_3 - \mathcal{L}_2 V_2 + \langle \mathcal{L}_1 V_3 + \mathcal{L}_2 V_2 \rangle$$

$$= - (\mathcal{L}_1 V_3 - \langle \mathcal{L}_1 V_3 \rangle) - (\mathcal{L}_2 V_2 - \langle \mathcal{L}_2 V_2 \rangle),$$
where the order $\epsilon$ centering condition is also used. From equations (3.22) and (3.23), it can be deduced that,

$$\mathcal{L}_1 V_3 - \langle \mathcal{L}_1 V_3 \rangle = (y - m) \left( \frac{\rho^2 \beta^2}{2 \alpha} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{\rho \beta}{2 \sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_1 \right). \quad (3.26)$$

Next, by definition,

$$\mathcal{L}_2 V_2 = \frac{1}{2} (y - m) \mathcal{L}_2 \mathcal{L}^{(2)} V_0 + \mathcal{L}_2 C.$$

Starting with the $y$ dependent terms in the first term on the right hand side of the equality, they can be written as,

$$(y - m) \mathcal{L}_2 = \mathcal{L}_2 - m \mathcal{L}_2$$

$$= y \langle \mathcal{L}_2 \rangle + \frac{1}{2} y^2 \mathcal{L}^{(2)} - \frac{1}{2} y m \mathcal{L}^{(2)}$$

$$- m \langle \mathcal{L}_2 \rangle - \frac{1}{2} y m \mathcal{L}^{(2)} + \frac{1}{2} m^2 \mathcal{L}^{(2)}$$

$$= (y - m) \langle \mathcal{L}_2 \rangle + \frac{1}{2} (y - m)^2 \mathcal{L}^{(2)}.$$

Thus,

$$\frac{1}{2} (y - m) \mathcal{L}_2 \mathcal{L}^{(2)} V_0 = \frac{1}{4} (y - m)^2 \mathcal{L}^{(2)} \mathcal{L}^{(2)} V_0,$$

using $\langle \mathcal{L}_2 \rangle \mathcal{L}_2 V_0 = \mathcal{L}_2 \langle \mathcal{L}_2 \rangle V_0 = 0$, which can be shown through several applications of equation (3.14).

From here, it is clear that,

$$\mathcal{L}_2 V_2 - \langle \mathcal{L}_2 V_2 \rangle = \frac{1}{4} (y - m)^2 \mathcal{L}^{(2)} \mathcal{L}^{(2)} V_0 - \frac{1}{4} \langle (y - m)^2 \rangle \mathcal{L}^{(2)} V_0 + \frac{1}{2} (y - m) \mathcal{L}^{(2)} C$$

$$= \frac{1}{4} \left( (y - m)^2 - \frac{m \beta^2}{2 \alpha} \right) \mathcal{L}^{(2)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} (y - m) \mathcal{L}^{(2)} C, \quad (3.27)$$

using the fact that $\langle (y - m)^2 \rangle = m \beta^2 / 2 \alpha$, and $C(x,t)$ being independent of $y$. Equations (3.26) and (3.27) thus show,

$$\mathcal{L}_0 V_4 = -(y - m) \left( \frac{\rho^2 \beta^2}{2 \alpha} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{\rho \beta}{2 \sqrt{\alpha}} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_1 + \frac{1}{2} \mathcal{L}^{(2)} C \right)$$

$$- \frac{1}{4} \left( (y - m)^2 - \frac{m \beta^2}{2 \alpha} \right) \mathcal{L}^{(2)} \mathcal{L}^{(2)} V_0.$$
Continuing,

\[ V_4 = -L_0^{-1}(y - m) \left( \frac{\rho \beta^2}{2\alpha} L^{(1)} L^{(2)} V_0 + \frac{\rho \beta}{2\sqrt{\alpha}} L^{(1)} L^{(2)} V_1 + \frac{1}{2} L^{(2)} C \right) \]

\[ - \frac{1}{4} \phi_C^{-1} \left( (y - m)^2 - \frac{m \beta^2}{2\alpha} \right) L^{(2)} V_0 + G(x, t) \]

\[ = (y - m) \left( \frac{\rho \beta^2}{2\alpha} L^{(1)} L^{(2)} V_0 + \frac{\rho \beta}{2\sqrt{\alpha}} L^{(1)} L^{(2)} V_1 + \frac{1}{2} L^{(2)} C \right) \]

\[ - \frac{1}{4} \phi(y) L^{(2)} V_0 + G(x, t), \]

which uses the fact that \( L_0^{-1}(y - m) = \phi(y) = m - y, L_0^{-1} \left( (y - m)^2 - \frac{m \beta^2}{2\alpha} \right) = \tilde{\phi}(y), \)

for some function \( \tilde{\phi}(y), \) and \( G(x, t) \) being a function independent of \( y. \) The function \( G(x, t) \)

absorbs all the constants that are independent of \( y, \) much like \( C(x, t) \) and \( F(x, t) \) for \( V_2 \) and \( V_3, \) respectively. The goal is now to determine the solution of \( \tilde{\phi}(y). \) Using equation (3.9), the

solution for \( \tilde{\phi}(y) \) is given as,

\[ \tilde{\phi}(y) = \int_0^\infty E \left( \frac{m \beta^2}{2\alpha} - (Y_t - m)^2 \right | Y_0 = y \) dt

\[ = \int_0^\infty e^{-t} \left( \frac{y \beta^2}{\alpha} - \frac{\beta^2 m}{2\alpha} - m^2 + 2my - y^2 \right) + e^{-t} \left( \frac{\beta^2 m}{\alpha} - \frac{\beta^2 y}{\alpha} \right) dt

\[ = \frac{\beta^2 (3m - 2y) - 2m^2 + 2my - y^2}{4\alpha}, \]

where mean and variance results from equations (2.11) and (2.10) are used.

Now,

\[ L_1 V_4 = \frac{\rho \beta}{\sqrt{\alpha}} y L^{(1)} \left( \frac{\rho \beta^2}{2\alpha} L^{(1)} L^{(2)} V_0 + \frac{\rho \beta}{2\sqrt{\alpha}} L^{(1)} L^{(2)} V_1 + \frac{1}{2} L^{(2)} C \right) \]

\[ - \frac{\rho \beta}{4\sqrt{\alpha}} y \tilde{\phi}(y) L^{(2)} V_0, \]

and thus,

\[ \langle L_1 V_4 \rangle = \frac{\rho \beta m}{\sqrt{\alpha}} L^{(1)} \left( \frac{\rho \beta^2}{2\alpha} L^{(1)} L^{(2)} V_0 + \frac{\rho \beta}{2\sqrt{\alpha}} L^{(1)} L^{(2)} V_1 + \frac{1}{2} L^{(2)} C \right) \]

\[ - \frac{\rho \beta}{4\sqrt{\alpha}} \langle y \tilde{\phi}(y) \rangle L^{(2)} V_0. \]

The calculation for \( \langle y \tilde{\phi}(y) \rangle, \) is straightforward.

\[ \langle y \tilde{\phi}(y) \rangle = \left\{ -y^2 + my - \frac{\beta^2 y}{2\alpha} \right\}

\[ = -\frac{m \beta^2}{\alpha}. \]
3.1 Derivation of the Asymptotic Solution

Next,
\[
\langle \mathcal{L}_2 V_3 \rangle = \left( \mathcal{L}_2 (y - m) \left( \frac{\rho \beta}{2\sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0 - \frac{1}{2} \mathcal{L}^{(2)}(1) V_1 \right) \right) + \langle \mathcal{L}_2 F \rangle 
\]
\[
= \frac{\rho \beta}{2\sqrt{\alpha}} \langle (y - m) \mathcal{L}_2 \rangle \langle \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0 + \frac{1}{2} \langle (y - m) \mathcal{L}_2 \rangle \mathcal{L}^{(2)}(1) V_1 + \langle \mathcal{L}_2 F \rangle 
\]
\[
= \frac{pm \beta^3}{8\alpha \sqrt{\alpha}} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 + \frac{m \beta^2}{8\alpha} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_1 + \langle \mathcal{L}_2 F \rangle .
\]

It can thus be shown that the PDE which \( F(t, x) \) satisfies is,
\[
\langle \mathcal{L}_2 F \rangle = \frac{pm \beta^3}{8\alpha \sqrt{\alpha}} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 - \frac{m \beta^2}{8\alpha} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_1 - \frac{pm \beta^3}{4\alpha \sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0 
\]
\[
- \frac{\rho \beta}{\sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 + \frac{\rho \beta}{\sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_1 + \frac{1}{2} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) C
\]
\[
= \frac{3pm \beta^3}{8\alpha \sqrt{\alpha}} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 - \frac{m \rho \beta^3}{2\alpha \sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0 
\]
\[
- \frac{\rho \beta}{\sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 - \frac{\rho \beta}{\sqrt{\alpha}} \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_1 + \frac{1}{2} \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) C
\]
\[
= A_1 \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 + A_2 \mathcal{L}^{(1)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 
\]
\[
+ (T - t) \left( A_3 \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 + A_4 \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0 \right)
\]
\[
+ (T - t)^2 A_5 \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0,
\]

where,
\[
A_1 = \frac{3pm \beta^3}{8\alpha \sqrt{\alpha}}, \quad A_2 = \frac{m \rho \beta^3}{2\alpha \sqrt{\alpha}}, 
\]
\[
A_3 = -\frac{pm \beta^3}{8\alpha \sqrt{\alpha}}, \quad A_4 = -\frac{m \rho \beta^3}{2\alpha \sqrt{\alpha}}, 
\]
\[
A_5 = -\frac{\rho \beta}{\sqrt{\alpha}}, 
\]

This is again the BS PDE with constant volatility \( \sqrt{m} \) and source terms as given. The solution for \( F(x, t) \) is thus,
\[
F(x, t) = -(T - t) \left( A_1 \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 + A_2 \mathcal{L}^{(1)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 \right) 
\]
\[
- \frac{(T - t)^2}{2} \left( A_3 \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(2)}(1) V_0 + A_4 \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0 \right) 
\]
\[
- \frac{(T - t)^3}{3} A_5 \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) \mathcal{L}^{(1)}(1) \mathcal{L}^{(2)}(1) V_0.
\] (3.28)

Substitution of this solution into the PDE and using the result in equation (3.14), verifies this claim.
3.1 Derivation of the Asymptotic Solution

3.1.7 Asymptotic Solution and the Greeks

To summarize, the first four terms in the expansion of $V$, makes up the asymptotic Heston solution. For a European call option, this is explicitly given as,

$$V_{\text{asymp}} = V_0 + \frac{\beta \rho \sqrt{(T-t)m}}{2\alpha \sqrt{2\pi}} \left( 1 - \frac{d_1}{\sqrt{(T-t)m}} \right) + \frac{V_2(x,y,t)}{\alpha} + \frac{V_3(x,y,t)}{\alpha \sqrt{\alpha}}, \quad (3.29)$$

where $V_0$ is given in equation (2.4), with $\sigma^2 = m$, the expression in equation (3.13) being simplified using the following partial derivatives,

$$\frac{\partial^2 V_0}{\partial x^2} = \frac{e^{-\frac{1}{2}d_1^2}}{x \sqrt{2\pi (T-t)m}},$$

$$\frac{\partial^3 V_0}{\partial x^3} = -\frac{e^{-\frac{1}{2}d_1^2}}{x^2 \sqrt{2\pi (T-t)m}} \left( \frac{d_1}{\sqrt{(T-t)m}} + 1 \right),$$

and $V_2(x,y,t)$ and $V_3(x,y,t)$ as given in equations (3.18) and (3.25), respectively, with $C(x,t)$ and $F(x,t)$ defined in equations (3.24) and (3.28), respectively. Note that the last two terms are not written out explicitly as a function of the model’s original parameters. This is because the solution itself is quite long and complicated, involving up to the 9th order partial derivative of the BS solution, with respect to $x$. Whilst the solution is convoluted in nature, many symbolic capable mathematical software such as Mathematica are easily able to compute its functional form, in terms of elementary functions.

In addition to the closed form asymptotic solution, it is also possible to provide the partial derivatives of the asymptotic solution. The Greeks, as they are known\(^4\), are particularly useful for areas such as hedging. The main Greeks of interest are Delta, which is the partial derivative with respect to the asset value $x$, and a Vega like Greek, which is the partial derivative with respect to the instantaneous variance $y$\(^5\).

The approximation of the partial derivative of the Heston call option with respect to $x$ is given as,

$$\frac{\partial V}{\partial x} \approx \frac{\partial V_{\text{asymp}}}{\partial x} = N(d_1) + \frac{\beta \rho \sqrt{(T-t)m}e^{-\frac{1}{2}d_1^2}}{2\alpha \sqrt{2\pi}} \left( 1 - \frac{2d_1}{\sqrt{(T-t)m}} + \frac{d_1^2 - 1}{(T-t)m} \right) + \frac{1}{\alpha} \frac{\partial V_2}{\partial x} + \frac{1}{\alpha \sqrt{\alpha}} \frac{\partial V_3}{\partial x} \quad (3.30)$$

\(^4\)The Greeks are partial derivatives of the option price with respect to various parameters, such as the current asset value and volatility for example.

\(^5\)The true Vega is in fact the partial derivatives with respect to the volatility.
3.1 Derivation of the Asymptotic Solution

which can be shown by one of two ways, either differentiating the asymptotic solution with respect to \(x\), or differentiating the Heston PDE with respect to \(x\), then applying the perturbation technique for \(\partial V/\partial x\). Both methods result in the same solution as given in equation (3.30).

Taking partial derivatives with respect to \(y\) of equation (3.29) gives the approximation to the Vega like Greek as,

\[
\frac{\partial V}{\partial y} \approx \frac{\partial V_{\text{asymp}}}{\partial y} = \frac{1}{2\alpha} \mathcal{L}^{(2)} V_0 + \frac{1}{\sqrt{\alpha}} \left( \frac{\rho \beta}{2\alpha} \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0 + \frac{1}{2} \mathcal{L}^{(2)} V_1 \right). \tag{3.31}
\]

Unlike many of the previous papers on this topic, the asymptotic solution is presented with parameters from the CIR process. Generally, when solving for the order \(\sqrt{\epsilon}\) term, many papers\(^6\) present the solution using constants, which are made up of the stochastic volatility model’s parameters and the other constants, but are not calculated explicitly, see Fouque et al.\(^{\text{[32]}}\), for example. It is argued that by grouping these parameters together allows for calibration of the summarized constants as a whole, and thus knowledge of the parameters in the stochastic volatility model is not required, resulting in a reduction in the number of parameters. While this approach is advantageous when the volatility is any arbitrary function of the \(Y_t\) stochastic process\(^7\), it does not show how each of the stochastic volatility’s parameters affect the asymptotic solution, whereas the approach presented here does.

3.1.8 Accuracy of the Asymptotic Solution

Given that the recent work of Fouque et al.\(^{[33]}\), has shown how to incorporate the Heston model’s asymptotic expansion, the proof of accuracy for the two and three term approximations follows directly and thus omitted. However, using the generalization of their proof, it can be shown that the four term asymptotic solution is also of correct order.

Begin by defining,

\[
\tilde{V} = V_0 + \sqrt{\epsilon} V_1 + \epsilon V_2 + \epsilon \sqrt{\epsilon} V_3,
\]

\[
\hat{V} = V_0 + \sqrt{\epsilon} V_1 + \epsilon V_2 + \epsilon \sqrt{\epsilon} V_3 + \epsilon^2 V_4 + \epsilon^2 \sqrt{\epsilon} V_5,
\]

\[
R = V - \tilde{V},
\]

where \(V\) is the true value of option’s price. Further define the Heston operator as,

\[
\mathcal{L} = \left( \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right)
\]

\(^6\)Aside from the recent work of Fouque et al.\(^{[33]}\).

\(^7\)That is, the volatility is not necessarily just \(\sqrt{Y_t}\), and closed form solutions may not be obtainable.
Given the definitions for $V$ and $V_i$, for $i = 0, \ldots, 5$, it can be deduced that,

$$\mathcal{L}R + \epsilon^2 (\mathcal{L}_1 V_0 + \mathcal{L}_2 V_4 + \sqrt{\epsilon} \mathcal{L}_3 V_5) = 0.$$  

Furthermore, the terminal condition reveals that,

$$R(x, y, T) = -\epsilon V_2(x, y, T) - \epsilon \sqrt{\epsilon} V_3(x, y, T) - \epsilon^2 \left(V_4(x, y, T) + \sqrt{\epsilon} V_5(x, y, T)\right)$$

The Feynman-Kac probabilistic representation formula for the solution $R(x, y, t)$ is then given as,

$$R(x, y, t) = -\epsilon \mathbb{E} \left(e^{-r(T-t)}V_2(X_T, Y_T, T) \mid X_t = x, Y_t = y\right) - \epsilon \sqrt{\epsilon} \mathbb{E} \left(e^{-r(T-t)}V_3(X_T, Y_T, T) \mid X_t = x, Y_t = y\right) + \mathcal{O}(\epsilon^2)$$

$$= -\frac{\epsilon}{2} e^{-r(T-t)} \mathbb{E} \left((Y_T - m) \mathcal{L}^{(2)} V_0(X_T, Y_T, T) \mid X_t = x, Y_t = y\right)$$

$$- \frac{\rho \sqrt{\epsilon}}{2} e^{-r(T-t)} \mathbb{E} \left((Y_T - m) \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0(X_T, Y_T, T) \mid X_t = x, Y_t = y\right)$$

$$- \frac{\epsilon \sqrt{\epsilon}}{2} e^{-r(T-t)} \mathbb{E} \left((Y_T - m) \mathcal{L}^{(2)} V_1(X_T, Y_T, T) \mid X_t = x, Y_t = y\right) + \mathcal{O}(\epsilon^2),$$

(3.32)

where it is noted that $C(x, t)$ and $F(x, t)$ are 0 at $t = T$. This justifies the boundary choices made for $C(x, t)$ and $F(x, t)$. Following the arguments of Fouque et al. [33], when $\epsilon \to 0$, the process $X_t$ converges to a GBM with constant volatility $\sqrt{m}$. In the limit, $X_t$ is independent of $Y_t$ and thus the three expectations in equation (3.32) is given as,

$$\mathbb{E} \left((Y_T - m) \mathcal{L}^{(2)} V_0(X_T, Y_T, T) \mid X_t = x, Y_t = y\right) = C_1 \mathbb{E} \left(Y_T - m \mid Y_t = y\right),$$

$$\mathbb{E} \left((Y_T - m) \mathcal{L}^{(1)} \mathcal{L}^{(2)} V_0(X_T, Y_T, T) \mid X_t = x, Y_t = y\right) = C_2 \mathbb{E} \left(Y_T - m \mid Y_t = y\right),$$

$$\mathbb{E} \left((Y_T - m) \mathcal{L}^{(2)} V_1(X_T, Y_T, T) \mid X_t = x, Y_t = y\right) = C_3 \mathbb{E} \left(Y_T - m \mid Y_t = y\right),$$

respectively. Note that the expectations involving the $X_t$ process is given as some constant $C_1$, $C_2$ and $C_3$, respectively. Since $Y_t$ has $\epsilon^{-1} \mathcal{L}_0$ as its infinitesimal generator, as argued in Fouque et al. $\mathbb{E} \left(Y_T - m \mid Y_t = y\right)$ converges exponentially fast as $\epsilon \to 0$, since $(y - m) = 0$. Thus, $R(x, y, t)$ is of order $\mathcal{O}(\epsilon^2)$. 

3.1 Derivation of the Asymptotic Solution
Now the residue of the 4 term asymptotic solution is given as,

\[ V - \tilde{V} = V - \hat{V} + \epsilon^2 V_4 + \epsilon^2 \sqrt{\epsilon} V_5, \]

\[ = R + \epsilon^2 (V_4 + \sqrt{\epsilon} V_5) \]

\[ = O(\epsilon^2). \]

This shows that the residue of the 4 term asymptotic solution has order \( O(\epsilon^2) \), as required, thus completing the proof.

## 3.2 Numerical Analysis

Before the numerical analysis of the asymptotic solution begins, a short discussion on some of the numerical aspects of the asymptotic and exact Heston solution is provided.

The exact Heston solution provides an analytic solution for the pricing of call options under the CIR stochastic volatility model. However, the solution is in the form of integrals with complicated integrands, see equation (2.15). The most common techniques used in evaluating these integrals are numerical quadrature or fast Fourier transforms (FFT). When using either techniques, there is usually a trade off between its accuracy and its efficiency in terms of speed.

The numerical results that follow for the exact Heston solution will be calculated using code provided by Janek and Weron [50].

The accuracy of extremely out-of-the-money (OTM) call options using the asymptotic solution can be quite poor. In particular, the asymptotic solution may become negative, due to the addition of \( V_1, V_2 \) and \( V_3 \) to \( V_0 \). This is problematic for extremely OTM call options, because the BS solution is already quite small in value in this region, and if the sum of these extra terms is negative and has magnitude greater than the BS solution, then the asymptotic solution will also be negative. A workaround proposed for this issue is that for all negative option prices, the asymptotic solution will be replaced by the value of zero\(^9\). In general, this should not have much impact unless the options being considered are extremely OTM.

Following from this discussion, the numerical analysis on the asymptotic Heston solution, as compared to the true closed form solution, is presented. Some of the areas covered include how

\(^8\)The code provided both solutions using the Gauss-Kronrod quadrature and FFT. Speed tests showed that the Gauss-Kronrod quadrature solution was quicker than the FFT method. Both solutions were tested against other freely available codes and it was found that the Gauss-Kronrod quadrature provided better results. The Gauss-Kronrod quadrature method is implemented using the MATLAB quadgk routine.

\(^9\)While it is obvious that by setting these values to zero, the option will be mispriced, practically the asymptotic solution should not be used in these cases.
the parameters such as the initial variance level, its mean reverting level, the mean reverting rate and the time to expiry, affect the accuracy of the solution. These four parameters are in fact very integral to the accuracy of the asymptotic solution. Firstly, the two term asymptotic solution is independent of the initial variance level, so it will be interesting to test the accuracy of the solution, when using only its mean reverting level as a proxy. The asymptotic solution is expanded in terms of the inverse of the mean reverting rate, thus it should be possible to test the asymptotic behaviour of the solution by varying this parameter. Lastly, the assumption that the stochastic process modelling the variance is fast mean reverting, requires that the option is alive for a substantial amount of time, enough for the mean reverting property to take affect. This is directly related to the time to expiry of an option, as the option is only alive when the option has yet to expire. Furthermore, in these analyses, the comparisons will be made using the two, three and four term asymptotic solutions to evaluate the performance of these additional terms in the asymptotic expansion.

3.2.1 Initial Variance and its Mean Reverting Level

The first analysis presented shows the effects of varying the initial variance level and its mean reverting level. From the closed form solution of the asymptotic solution, it is clear that the two term solution is independent of $y$, the initial variance level. Furthermore, the three and four term solutions are linear in $y - m$, thus, it is expected that the two term solution should perform moderately well when $y \approx m$, and that the three and four term solutions excel when $y$ is far from $m$.

To test this, 4 parameter sets are chosen to be representative of various conditions. The problems considered are all at-the-money (ATM) call options using parameters $x_0 = 100$, the initial asset’s value, and $\tau = T - t = 1$, the time to expiry of the option, with the other parameters listed in in Table 3.1.

<table>
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<th>$m$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\rho$</th>
<th>$r$</th>
<th>$y_0$</th>
</tr>
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<td>0.1</td>
<td>2</td>
<td>0.3</td>
<td>-0.7</td>
<td>0.05</td>
<td></td>
<td>[0.0,0.2]</td>
</tr>
<tr>
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<td>5</td>
<td>0.2</td>
<td>0.4</td>
<td>0.1</td>
<td></td>
<td>[0.1,0.3]</td>
</tr>
<tr>
<td>0.15</td>
<td>10</td>
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<td>0.7</td>
<td>0.15</td>
<td></td>
<td>[0.05,0.25]</td>
</tr>
<tr>
<td>0.25</td>
<td>100</td>
<td>0.1</td>
<td>-0.4</td>
<td>0.2</td>
<td></td>
<td>[0.15,0.35]</td>
</tr>
</tbody>
</table>

Table 3.1: Parameters for the Sensitivity Analysis of the Initial Variance

The relative errors of these test cases as compared to the exact Heston solution, can be
found in Figure 3.1. It can be seen from this figure that the two term solution performs well when \( y \approx m \), and that for \( y \) much smaller or larger than \( m \), the two term asymptotic solution over and under prices the option, respectively. One reason for the moderately good performance of the two term solution when \( y \approx m \) is that the mean reverting level of the variance is used as a proxy for the true value of the initial variance, so when the initial variance is close to its mean reverting level, then the proxy will be a good choice.

![Graphs showing sensitivity analysis of initial variance and its mean reverting level](image)

**Figure 3.1: Sensitivity Analysis of the Initial Variance and its Mean Reverting Level**

- The asymptotic solution has the smallest absolute relative error when the initial variance is near its mean reverting level. The three and four term solution with their dependence on \( y \), performs better than the BS and two term solution.

Furthermore, the performance of the three and four term solution in these test cases is very good. These two solutions are able to provide prices which are quite close to the exact Heston solution across all the values of \( y \) considered. However, this result is not surprising given that these solutions have a dependency on \( y \), thus by nature it should give a better fit than the two term solution.
3.2.2 Mean Reverting Rate

The second analysis presented shows the effects of varying the mean reverting rate. The asymptotic solution is derived by expanding the solution in terms of powers of the inverse of the mean reverting rate. As such, the accuracy of the asymptotic solution should increase as the mean reverting rate approaches infinity.

From equation (3.29), it can be observed that as $\alpha$ increases to infinity, the asymptotic solution tends towards the BS solution. Under this limit, the BS solution is a good proxy for the Heston solution. This can be reasoned by the fact that for an extremely large mean reverting rate, the $Y_t$ process will hover about its mean level indefinitely. As such, this effectively makes $Y_t$ constant, which implies the BS solution should be close to the exact Heston solution.

To test the asymptotic behaviour of the asymptotic solutions, the following parameters are used. The mean reverting rate is set at 2, 5, 10, 15 and 25, while the other parameters are $m = 0.1$, $\beta = 0.3$, $\rho = 0.7$, $r = 0.05$, $y_0 = 0.1$, $\tau = 1$ and $x_0 \in [80, 120]$. Importantly, the initial variance equals the mean reverting rate in these cases. Figure 3.2 shows relative errors of this test. It can be seen from this figure that the asymptotic behaviour of the solutions are performing as expected. For each of the two, three and four term solution, as the mean reverting rate increases, the asymptotic solutions approach the exact Heston solution. In particular, for a typical value of $\alpha$, set at 5, the absolute relative errors for the two, three and four term asymptotic solution is no larger than 2.35%, 1.32% and 1.19%, respectively.

Figure 3.3 shows the absolute relative errors of the above results, but for $\alpha = 5$, and across the two, three and four term asymptotic solutions. In this case, the two term asymptotic solution performs better than the three and four term asymptotic solution, when the option is slightly in-the-money (ITM). However, although not shown, when the negative of $\rho$ is taken, the observation is reversed. It must be noted that these are tests where $y = m$, and thus the most ideal condition for the two term solution to excel. In fact, when $y = m + 0.02$ for example, the three and four term asymptotic solutions, outperform the two term solution for a large majority of the initial asset’s values, regardlessly of whether it is a positive or negative $\rho$. These issues will be revisited in a later section when comparing the two, three and four term solution amongst themselves.

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10Conlon and Sullivan [15], showed that within an OU variance driven model, as in the original work of Fouque et al. [32], the exact option price, does not converge to the BS solution unless $\rho = 0$. As such, it is not expected that the exact Heston solution converges (in a mathematical sense) to the BS solution. However, the asymptotic solution still presents a good approximation of the exact solution over the range of $\alpha$’s considered.
Figure 3.2: Sensitivity Analysis of the Mean Reverting Rate - Relative error of the asymptotic solution decreases as mean reverting rate increases. This shows the asymptotic behaviour of the two, three and four term asymptotic solutions.
3.2 Numerical Analysis

Figure 3.3: Comparisons between the different Asymptotic Solutions 1 - Absolute relative error of the asymptotic solution at $\alpha = 5$, for the two, three and four term solutions. The two term solution provides a lower absolute relative error when the initial asset’s value is greater than 103.

3.2.3 Time to Expiry

The third analysis presented shows the effects of varying the time to expiry of the option. The asymptotic solution is derived from time averaging arguments, assuming that on average, the instantaneous variance stays at or around its mean reverting level more often than not. The issue of whether the initial variance starts at its mean reverting level or not, has been addressed. What needs to be considered is the length of time given for this assumption to be realized. This length of time is in fact the time to expiry of an option, since pricing only takes place while the option is still alive. Up until this point, all the graphs have been calculated using times to expiry set at 1 year.

To test the effects of the time to expiry of the option, the following parameters are used. The times to expiry are set to 0.2, 0.4, 0.6, 0.8 and 1, mean reverting rate at 5, with the other parameters being the same as the mean reverting rate examples. Figure 3.4 shows the result of these tests. These results show that as the time to expiry increases, the asymptotic approximation becomes better with the best result in Figure 3.4 being $\tau = 1$. Whilst it is known that the asymptotic solution is not asymptotic as a function of the time to expiry, it is seen here that implicitly, the time to expiry plays a role in the converging nature of the asymptotics solution. This is due to allowing the variance process more time to undergo its mean reverting
3.2 Numerical Analysis

properties. For smaller times to expiry, the approximations can be quite poor. This may be quite problematic if the asymptotic solutions are applied to other areas such as hedging for example, where calculations of the option price and its Greeks, are required right up until the expiry of the option.

![Graphs showing sensitivity analysis of the time to expiry](image)

**Figure 3.4: Sensitivity Analysis of the Time to Expiry** - Relative errors of the asymptotic solution decreases as time to expiry increases. This is observed for the two, three and four term asymptotic solutions.

### 3.2.4 Two, Three and Four Term Solutions

Thus far, the numerical investigations and discussions have all been performed on the two, three and four term asymptotic solution independently of each other. This section will now look at, and discuss the advantages of, using either the two, three or four term solution over the others.

It is clear that when $y$ is not near $m$, and the other parameters are all relatively standard\(^\text{11}\), the three and four term asymptotic solutions perform better than the two term solution. However, it is not clear as to whether the four term solution is superior to the three term solution or not. In the four test cases presented in the initial variance and its mean reverting level section, only the test case for $m = 0.1$ seems to show any significant differences in the accuracy

\(^{11}\text{Mean reverting rate is reasonably large (≈ 3 or larger), and not a short time to expiry.}\)
of the solution. This is because the other test cases have a larger mean reverting rate, and thus the accuracy of both the three and four term solutions are very high. In particular, for the other three test cases, the absolute relative error is no larger than 0.1% across the parameters considered. For the \( m = 0.1 \) case, the absolute relative error is shown in Figure 3.5. This figure shows for a wider range of initial variance levels, the four term solution excels over the three term solution.

![Figure 3.5: Three and Four Term Solutions](image)

**Figure 3.5: Three and Four Term Solutions** - Absolute relative errors of the three and four term asymptotic solution. For a large range of starting initial variances, the four term solution is superior to the three term solution. For starting values where this is not true, the differences in the errors are quite smaller.

In the mean reverting rate tests, it was shown that for some larger initial asset values, the two term asymptotic solution performs better than the three and four term solutions. However, this statement neglects the fact that the tests were performed when \( y = m \). In Figure 3.6, the same parameters are used as in Figure 3.3, but with \( y = m + 0.02 \). These results show that in general, when \( y \neq m \), the three and four term solutions outperform the two term solution most of the time. Furthermore, the four term solution is shown to exhibit a lower absolute relative error than the three term solution. Figure 3.7 shows the same parameters but with \( \alpha = 15 \). It is evident from this figure that when the mean reverting rate increases, the four term solution still outperforms the three term solution. Although not shown for the other \( \alpha \) values, the results are similar with the four term solution performing better than the three term solution for the majority of initial asset values considered.
3.2 Numerical Analysis

Figure 3.6: Comparisons between the different Asymptotic Solutions 2 - Absolute relative error of the asymptotic solution at $\alpha = 5$, for the two, three and four term solution. The initial variance level is set to $m + 0.02$, in this case. The four term solution produces a lower absolute relative error across the range of initial asset values, than the two and three term solutions.

Figure 3.7: Three and Four Term Solutions 2 - Absolute relative error of the asymptotic solution at $\alpha = 15$, for the three and four term solution. For this larger value of $\alpha$, the four term solution still outperforms the three term solution.
3.2 Numerical Analysis

In the time to expiry examples, it was noted that as \( \tau \) increases the asymptotic solutions provide better approximations to the exact Heston solution. However, for a small \( \tau \) the asymptotic solutions are quite poor, particularly for OTM options. Figure 3.8 shows the result of taking \( \tau = 0.2 \) from time to expiry test section. These results show that for \( y = m \) the two, three and four term solutions provide quite poor estimates when the option is OTM. When \( y = m + 0.02 \) as in Figure 3.9, the two term solution surprisingly performs better for OTM options than the three and four term solutions, but this observation is reversed for when a negative \( \rho \) is used, as shown in Figure 3.10. The problems here lies in the fact with a short time to expiry, the fast mean reverting properties do not have enough time to ‘kick’ in. Furthermore, for OTM options, which would have very little value to begin with, any small mispricing will result in a large relative error, which is what is being observed here. It seems that any advantage in using the two, three or four term solutions over the others, in the case of a short time to expiry and OTM options, are more coincidental than there being a deeper underlying reason.

![Figure 3.8: Three and Four Term Solutions](image)

**Figure 3.8: Three and Four Term Solutions** - Absolute relative error of the asymptotic solution at \( \tau = 0.2 \), for the two, three and four term solutions. All three solutions do not perform well for OTM options.

These numerical results show the superiority of the four term solution over the two and three term solutions, particularly when the time to expiry is large. The four and three term solutions can cater for cases where \( y \neq m \), which the two term solution cannot, with the four term solution performing better than the three term solution. However, for short times to expiry, depending on the application, it may not be worthwhile to implement the asymptotic
3.2 Numerical Analysis

Figure 3.9: **Three and Four Term Solutions**
- Absolute relative error of the asymptotic solution at $\tau = 0.2$ and $y = m + 0.02$, for the two, three and four term solutions. For a positive $\rho$, OTM options are better priced with the two term solution.

Figure 3.10: **Three and Four Term Solutions**
- Absolute relative error of the asymptotic solution at $\tau = 0.2$ and $y = m + 0.02$, for the two, three and four term solutions. For a negative $\rho$, OTM options are better priced with the four or three term solution.
3.3 Concluding Remarks

In this chapter, the asymptotic Heston solution for a European call option has been derived up to the fourth term in the expansion. The derivation is based on the original work by Fouque et al. [32], but applied to the Heston model. The two term solution was derived independently of the recent work by Fouque et al. [33], where they also considered the Heston model. Following their work, the third term in the expansion is also provided as a review. Using a similar methodology, this chapter presented the derivation of the fourth term in the expansion, along with proving that the solutions are of the correct order as an asymptotic solution.

Using the closed form solution for the Heston call option, numerical analysis was performed on the accuracy of the asymptotic solutions. In particular, the parameters with the most impact are the initial variance level, its mean reverting level, the mean reverting rate, and the time to expiry of the option. It was shown that generally, the two term solution only performs well when $y \approx m$, and even then, it is hard to distinguish whether it is superior to the three and four term solutions. However, for $y \neq m$, the three and four term solutions perform better, with the four term solution often outperforming the three term solution. Problem arises for OTM options with short times to expiry. Firstly, they are low in value, thus any mispricing using the asymptotic solutions would ultimately have a large impact on the relative error, and secondly, because of the shorter times to expiry there is less time for the fast mean reverting assumption of the model to run its course. In these cases, it may be worthwhile to use a combination of the two and four term solution\textsuperscript{12}, as the two term solution was found to be quite robust in this setting.

While the computational efficiency of the asymptotic solution was not considered in this chapter, it can be reasoned that these approximations are faster to compute than the exact calculations of the Heston call price. This is due to the simplicity of the asymptotic solution, which can be written in terms of elementary functions. Without a doubt, the speed in which the asymptotic solution can be obtained is dependent on the number of terms being used. However, the trade off here would be the accuracy. The computational efficiency of the asymptotic solution as compared to calculating the exact Heston solution using standard techniques will

\textsuperscript{12}This idea is investigated in the next chapter.
3.3 Concluding Remarks

be considered in the next chapter as the asymptotic Heston solution will be applied to locally risk minimizing hedges.
Chapter 4

Locally Risk Minimizing Hedges using Asymptotic Heston Solution

The issue of pricing options under stochastic volatility has been long explored, most notably by Heston [44], in deriving the closed form solution. However, it is still unclear as to how to hedge such options efficiently. It is well known that under a stochastic volatility model, a vanilla option cannot be perfectly hedged with just a combination of the assets and money from a risk-free bank account. This is due to the randomness of the volatility in the model, which unlike the underlying asset’s value, is not explicitly observable or tradable. Thus, stochastic volatility models are said to be incomplete market models.

The conducting of risk minimizing hedges in incomplete markets have intrigued many. This is mainly due to the fact that self-financing hedges, which are hedges that attract no risk, do not exist in incomplete markets. Papers such as Föllmer and Schweizer [28], and Schweizer [69], have provided general results on the nature of risk minimizing hedges in incomplete markets. Risk minimizing hedges involve the use of trading strategies such that the risk, as measured by predefined criteria, is minimized. The concept of locally risk minimizing hedges, as introduced by Schweizer [69], requires the minimization of a quadratic risk function at each time step. It was shown that this problem is indeed solvable and in fact is related to variance-optimal hedging under a martingale measure. In Heath, Platen and Schweizer [42], locally risk minimizing hedging is compared to mean-variance hedging. Note however that the latter is performed using self-financing strategies, which is a conceptually very different approach. For more infor-
mation on mean variance and variance-optimal hedging and how they are related to locally risk minimizing hedging refer to Schweizer [70], [71], and Pham, Rheinlander and Schweizer [61]. Other related literature on the aspects of hedging in incomplete markets includes Alexander and Nogueria [2], [3], as well as Bakshi, Cao and Chen [5].

El Karoui, Peng and Quenez [24], showed how it is possible to explicitly calculate the locally risk minimizing strategy, and in Poulsen, Schenk-Hoppé and Ewald [64], the hedges were derived for a general class of stochastic volatility models. Note that the locally risk minimizing framework not only determines a hedging strategy, but also fixates the pricing measure as the so-called minimal martingale measure. Poulsen et al. also performed an empirical analysis under the Heston model to evaluate its effectiveness over traditional hedging methods. As in most of the risk minimizing hedging literature, the goal of this chapter will be to hedge derivatives with only primary assets (the underlying asset and bonds). In fact, by adding an additional option as a hedging instrument to hedge other types of options would, in the case of stochastic volatility, complete the market, thus allowing for perfect self-financing hedges and hence rendering the locally risk minimizing approach as meaningless.

Up until now, very few practical applications of the asymptotic solutions derived from the Fouque et al. [32] techniques have been investigated. Whilst the pricing of European call options with the asymptotic solution has been thoroughly examined in the previous chapter, the attention now turns to using this for the creation of hedging strategies. In particular, this chapter will investigate the use of the asymptotic solution in creating locally risk minimizing hedges for European call options. Whilst some aspects of hedging has been discussed by Fouque et al., they often lack numerical examples and a comparable model to compare against. This chapter will provide these missing discussion by following the development in Poulsen et al. but with the asymptotic solution as an alternative hedging method. The performance of the locally risk minimizing hedges created using the asymptotic solutions will be assessed against the hedges created using the exact Heston solution. In the following, hedges are created using both a simulation study, and historical data. To evaluate the worthiness of applying the asymptotic solutions the accuracy, simplicity and computational speed will considered. As an addition, in the case of the simulation study, the hedges will also be compared against that of a traditional (but outdated) Black-Scholes Delta hedge.
4.1 Locally Risk Minimizing Hedging

The concept of reducing risk in option hedging has been explored for a long time. Self-financing perfect hedges are hedges whose cost process is constant and replicates the derivative perfectly, thus attracting no risk. However, such hedges are only available in complete market models. As real world markets show clear signs of incompleteness\(^1\), the need to develop trading strategies for incomplete market models, to minimize the risk involved in hedging strategies, arose. One method of minimizing the risk involves a criterion proposed by Föllmer and Schweizer [28], which is to minimize a risk function, defined as the conditional variance process of the cost process involved in conducting the hedge. However, this leads to a dynamic optimization problem which may not have any solutions.

Schweizer [69], explored the concept of locally risk minimizing hedges for incomplete markets. The general idea is to minimize the conditional variance of instantaneous cost increments sequentially over time. While this problem is solved in theory, many computational aspects of practical implementation still deserve attention. El Karoui, Peng and Quenez [24], showed how it is possible to obtain the locally risk minimizing hedges by first completing the market by introducing a new tradable asset, then calculating a hedging strategy for this complete market and finally, projecting the hedging strategy back onto the original incomplete market. In Poulsen, Schenk-Hoppé and Ewald [64], the locally risk minimizing hedges for a general class of stochastic volatility model is derived in explicit form. A brief review of their results is outlined below.

4.1.1 Cost Function of a Trading Strategy

Define a trading strategy \( \varphi(t) = (\varphi^0(t), \varphi^1(t)) \), such that the components indicate the holding amounts (in units of) risk-free asset \( B_t \) (e.g. bank account or bonds) and risky asset \( X_t \) (e.g. stock), respectively. The cost function is defined as the difference between the holdings of the trading strategy at time \( t \), and the cumulative gains or losses up to time \( t \). Mathematically, the cost associated with a trading strategy \( \varphi(t) \) at time \( t \) is calculated as,

\[
\text{Cost}_\varphi(t) = V_\varphi(t) - \int_0^t \varphi^0(s) \, dB_s - \int_0^t \varphi^1(s) \, dX_s,
\]

where \( V_\varphi(t) = \varphi^0(t)B_t + \varphi^1(t)X_t \) denotes the value of the trading strategy at time \( t \). If the cost associated with a trading strategy is constant, the trading strategy is said to be self-financing. Stochastic volatility models describe incomplete markets; the volatility cannot be

\(^1\)The presence of stochastic volatility makes the model incomplete.
4.1 Locally Risk Minimizing Hedging

This means that not all contingent claims can be perfectly hedged using self-financing hedging strategies and thus the need to develop other forms of hedging evolves.

4.1.2 Locally Risk Minimizing Strategy

Poulsen, Schenk-Hoppé and Ewald [64], derive the locally risk minimizing strategy for a general class of stochastic volatility models using the three step procedure by El Karoui et al. The focus of the rest of this chapter, will be the locally risk minimizing strategy applied to the Heston model under a minimal martingale measure and in particular, the use of the asymptotic Heston solutions.

Using the notation for the Heston model as introduced in Section 2.2.2, define the value of a European call option at time \( t \), with terminal payoff \( (X_T - k)^+ \) as \( V(X_t, Y_t, t) \). It follows from Poulsen et al. that the locally risk minimizing hedging strategy for the Heston model is given by,

\[
\varphi^0_{\min}(t) = e^{-rt} \left[ V(X_t, Y_t, t) - \varphi^1_{\min}(t)X_t \right],
\]

\[
\varphi^1_{\min}(t) = V_X + \rho\beta V_Y X_t,
\]

where \( V_X \) and \( V_Y \) denote the partial derivatives of the option price with respect to asset’s value and variance, respectively.

In using the asymptotic solution, the price and partial derivatives with respect to \( x \) and \( y \) will come from equations (3.29), (3.30) and (3.31), respectively, that is the four term asymptotic solution, unless otherwise stated. It is worthy to note that the asymptotic solutions are written as functions of the CIR parameters. This is an important point because in the original work of Fouque et al. [32], and many others, the asymptotic solution is written in terms of parameters that are functions of the stochastic volatility model’s parameters. It is argued that the stochastic volatility model’s parameters are not important, only the final transformed parameter, which is often calibrated to market data, see Section 3.2. In essence, the meaning of the model parameters is thus lost. Whilst this is sufficient for many of the applications, it is not here, as the hedging strategy presented here requires knowledge of the correlation \( \rho \) and the volatility of volatility \( \beta \). It is noted that most recently in Fouque et al. [33], the three term asymptotic Heston solution has been written in terms of the CIR model’s parameters, though in what follows, the four term solution is used.
4.2 Asymptotic Hedge on Simulated Data

This section contains the results of using the asymptotic solution in locally risk minimizing hedging on simulated data. The design of this analysis is similar to Poulsen et al. [64]. Asset values and instantaneous variance paths are simulated according to set parameters. Different trading strategies are applied to each of the simulation paths and the costs associated are recorded. This is repeated 10000 times in order to compute the hedging error which is defined to be the standard deviation of the cost process at expiry divided by the initial cost of the option, as calculated using the exact Heston solution, as a percentage. This is calculated as,

$$\text{Hedging Error} = 100 \times \frac{\sqrt{\text{Var}(\text{Cost}_\phi(T))}}{V(X_0, Y_0, 0)}.$$ 

Smaller hedging errors indicate better performance since less variance associated with the final cost of the trading strategy means the hedges will be more manageable in terms of risk.

4.2.1 Hedger Types

In this analysis, 4 different types of hedgers are considered:

- **Hedger 1**: The exact Heston locally risk minimizing hedger who uses the full parameter set, the Heston solution and its partial derivatives.

- **Hedger 2**: The asymptotic Heston locally risk minimizing hedger who uses the full parameter set, the asymptotic Heston solutions and its partial derivatives.

- **Hedger 3**: The BS locally risk minimizing hedger who uses the BS solution and its partial derivatives in place of the exact Heston solution and its partial derivatives. This hedger also takes the square root of the instantaneous variance to be the BS volatility parameter.

- **Hedger 4**: The BS Delta hedger, who uses the BS solution and its Delta, to create a standard Delta hedge. This hedger takes the square root of the instantaneous variance to be the BS volatility parameter and does not use any stochastic volatility model.

The main difference between this simulation study and that of Poulsen et al., is the inclusion of Hedger 2, the asymptotic Heston hedger. The options to be hedged against are all 1 year European call options. The portfolio is to be re-hedged daily, assuming a 250 day per year calendar.
4.2 Asymptotic Hedge on Simulated Data

4.2.2 Simulation Parameters

The simulated data are generated using 2 different parameter sets listed in Table 4.1.

<table>
<thead>
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<th>Set No.</th>
<th>$\alpha$</th>
<th>$m$</th>
<th>$\beta$</th>
<th>$\rho$</th>
<th>$r$</th>
<th>$\tau$</th>
<th>$k$</th>
<th>$x_0$</th>
<th>$y_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.75</td>
<td>0.0483</td>
<td>0.550</td>
<td>[-0.575,0.575]</td>
<td>0.04</td>
<td>1</td>
<td>100e^r</td>
<td>100</td>
<td>0.0483</td>
</tr>
<tr>
<td>2</td>
<td>5.00</td>
<td>0.0500</td>
<td>0.300</td>
<td>[-0.575,0.575]</td>
<td>0.10</td>
<td>1</td>
<td>100</td>
<td>100</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 4.1: Parameter Sets for the Simulation

The hedging errors within each parameter set will be compared across different values of $\rho$, in the interval [-0.575,0.575], spaced apart by units of 0.05.

The reasoning behind choosing these parameter sets is as follows; Set 1 is the same parameter set used by Poulsen et al. [64], but with $y = m$ here. The parameters are in fact calibrated from historical data by Eraker [25]. Using this parameter set allows for a direct comparison of these results and the ones previously obtained. Set 2 contains parameters with a larger $\alpha$ and a smaller $\beta$. This signifies that the variance process should be less volatile and exhibit faster mean reverting properties.

In both cases, the initial variance level is set to its mean reverting level. There are two reasons for this; firstly, the main reason is to give less bias to the exact and asymptotic Heston hedges, as the BS hedger only uses the mean reverting level as a proxy. Thus if the initial variance was away from its mean reverting level, then the simulated variance paths will always have to revert back to its mean level, thus inducing more errors for the BS hedgers at the start of the hedges by default. The second reason is that whilst it has been shown that the four term solution at times may not perform as well as the two term solution for $y \approx m$, on average the approximations are quite good. Furthermore, because of the nature of this experiment, $y$ will fluctuate and not always be at $m$, and thus the four term solution should perform very well on average, whilst giving the BS hedges the best possible conditions for it to perform.

The interest rate parameters are chosen as a plausible value, while the strike prices $k$ are chosen so that the options are either ATM or forward ATM. This choice of strikes is usually popular in the markets as the options will rarely finish extremely OTM or ITM. This is advantageous as the numerical sections from the previous chapter show that the asymptotic solution performs the best near the money.

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2Historical option data are calibrated by fitting a joint posterior density, then Markov Chain Monte Carlo sampling is used to obtain a sample of the model parameters.
Figure 4.1 shows the option price and partial derivatives with respect to asset and instantaneous variance, for a particular simulated path using parameter set 1. The plots are generated with a $\rho$ value of -0.575. These figures show that the asymptotic solution, as it tracks a simulated asset’s value over time, performs reasonably well. As expected, most of the inaccuracies arise from when the times to expiry are short, see Section 3.2 for an explanation. In most cases, with large times to expiry, the asymptotic solution outperforms the pure BS solution. From the third graph of Figure 4.1, it is expected that Hedger 3 will perform very poorly due to the bad estimates of the partial derivative with respect to the instantaneous variance. A plot for the other parameter set is similar to the one in Figure 4.1, and is thus omitted.

Figure 4.1: Tracking the Option Price and Greeks for Parameter Set 1 - The asymptotic solution and its partial derivatives track the exact Heston solution closely over time. The difference is smaller when time to expiry is large. The BS solution and its derivatives does not do as well.

As noted in Section 3.2 and the above, for short times to expiry, the four term solution can
be quite poor. In particular, as evident in the second graph of Figure 4.1, there is a large and sudden increase in the Delta of the option when the time to expiry is less than 0.2. Whilst the price seems to track well in this simulation, the error in the Delta, for when there is a short time to expiry, may completely ruin the whole hedge, by increasing or decreasing the final cost associated with the hedge. One way of combatting this problem may be to use the exact Heston solution for times to expiry that are less than 0.2 (years). This would of course perform well, given that one-fifth of the hedge (assuming a 1 year option) will be the same as the exact Heston hedge, but it will not showcase the robustness of the asymptotic solution. Instead, for Hedger 2, the two term solution is opted for when the time to expiry is less than 0.2 and the four term solution remains for other times to expiry. To justify this, numerical results will show that overall, the hybrid hedging scheme provides lower hedging errors than using just the four or two term solutions, whilst also performing better than the BS hedgers.

4.2.3 Hedging Errors and Costs for Different Correlations

The hedging errors across different correlations are shown in Figures 4.2 and 4.3, for parameter sets 1 and 2, respectively. The general observations that can be made about the performance of these hedges are that, Hedger 3 performs the worst in all situations as expected and Hedger 1 performs strongly when the magnitude of the correlation is high. The asymptotic hedge performs better than the standard Delta hedge for negative $\rho$. For positive $\rho$ and parameter set 1, the performance is roughly the same as the standard Delta hedges, while for parameter set 2, the performance is better throughout.

Since parameter set 1 is the same as the one used in Poulsen et al. [64], a direct comparison can be made. The asymptotic hedging errors are found to lie in between that of the exact Heston hedging errors and the standard Delta hedging errors for negative $\rho$. For positive $\rho$, the asymptotic hedge seems to return a hedging error that is slightly higher than the standard Delta hedge at times, but it does not seem too significant. Given that the parameter set is actually derived from calibrated data, and in the calibrated parameter set, $\rho$ was -0.569, this gives hope in using the asymptotic solution as a replacement to the Heston solution at least in locally risk minimizing hedges.

For parameter set 2, the hedging errors of the asymptotic hedge are between that of the exact Heston hedges and the BS Delta hedges, for all values of $\rho$ considered. The performance is slightly better for negative $\rho$, as evident by the larger difference in the hedging errors, between itself and Hedger 4, but still good for positive $\rho$. Even for the exact Heston hedges, the hedging
4.2 Asymptotic Hedge on Simulated Data

Figure 4.2: Hedging Error for Parameter Set 1 - Performance of the asymptotic hedge is better than the standard BS hedges for negative $\rho$, while for positive $\rho$, the performance is similar to standard BS Delta hedge. Note that the ‘bump’ for Hedger 1 at $\rho = -0.425$ is the result of one large outlier in the cost price simulations, thus resulting in this anomaly.

Figure 4.3: Hedging Error for Parameter Set 2 - Performance of the asymptotic hedge is better than the standard BS hedges across both positive and negative $\rho$. 
errors are better for negative $\rho$ than positive $\rho$, when the magnitudes of $\rho$ are equal. This shows promise in using the asymptotic hedge at the very least, as an alternative to the BS type hedges.

It must be reiterated that the hedging errors are only indicative of the variances involved in the cost associated with performing these hedges and by no means relay the actual cost associated with these hedges. Whilst it may be useful to know the variances involved in performing these hedges, from a practical point of view, the expected cost involved is equally as important. Since the hedges are supposed to replicate the call option, on average, the final costs associated with these hedges should in fact be close to the initial Heston price. For the exact Heston locally risk minimizing hedge it can be concluded from Föllmer and Schweizer [28], that the hedge is in fact mean self-financing, i.e. subtracting the initial cost (option price) from the expectation of the cost process is zero at all times. This at least is the theory, however in practice, due to the discretization, the hedges may not be mean self-financing.

Figures 4.4 and 4.5 show the average final costs subtracting the initial Heston option price, associated with these hedges. These figures show that Hedgers 1, 2 and 3 have average final costs which are roughly near the initial Heston price. Furthermore, these average final costs are roughly equal to one another. The absolute difference between the average final cost of Hedger 4 and the initial Heston price increases as the absolute value of $\rho$ increases.

Figure 4.6 shows result of Hedger 2 using both the hybrid scheme\(^3\) and just the two term solution for parameter set 2. The four term only hedge has been omitted because its performance is very poor, much worse than Hedger 3 and 4. Most of the issues surrounding the four term only hedge is due to the errors when the times to expiry is very small. The errors lead to several simulations having significantly greater cost processes, which has a large impact on the hedging errors. The figure shows the hybrid scheme outperforming the two term only scheme in returning lower hedging errors across all values of $\rho$. This shows the benefit of using the hybrid scheme over the pure two term only scheme.

In summary, Hedger 1 performs the best in terms of having a lower hedging error, and being able to maintain an average final cost close to the initial Heston price. Hedger 3, while having an average final cost close to the initial Heston price, also has a much greater hedging error. Hedger 4, has a stable hedging error across various values of $\rho$ but the average final cost of the hedges are much greater (in absolute value) than the initial Heston price, i.e. not mean self-financing. Hedger 2, has a lower hedging error than the two BS type hedges for negative $\rho$.

\(^3\)A combination of the four and two term solution is used.
4.2 Asymptotic Hedge on Simulated Data

Figure 4.4: Average Final Cost for Parameter Set 1 - Average final costs shows that the asymptotic hedging strategy costs roughly the same as the Exact Heston hedging strategy. Hedger 1, 2 and 3 also show mean self-financing properties.

Figure 4.5: Average Final Cost for Parameter Set 2 - Average final costs shows that the asymptotic hedging strategy costs roughly the same as the Exact Heston hedging strategy. The results are similar to parameter set 1.
4.2 Asymptotic Hedge on Simulated Data

![Graph showing hybrid and two term scheme comparison]

**Figure 4.6: Hybrid and Two Term Scheme** - The hybrid scheme is shown to provide a lower hedging error than the two term scheme.

while for positive $\rho$, the hedging error is lower, if not roughly the same\(^4\) as Hedger 4. Hedger 2 also exhibits an average final cost that is close to the initial Heston price. These facts support the conclusion that Hedger 1 performs the best in all scenarios, which is expected. Hedger 2 outperforms Hedger 3, in terms of having smaller hedging errors, even though they are both mean self-financing. Hedger 2 also outperforms Hedger 4 in the sense that for negative $\rho$, the hedging error is smaller than Hedger 4, while for positive $\rho$, the hedging errors are comparable, if not better. Hedger 2 also has the added advantage of being mean self-financing. For these reasons, it can be concluded that the asymptotic hedge is a viable alternative to the two BS Delta hedgers.

These results show that the asymptotic hedge is a viable alternative to traditional BS methods. However, it must be noted that in practice, model parameters are recalibrated to actual option data at every re-hedge, instead of assuming they are fixed over the life of the hedge. Of course, in these simulations, only the asset’s value and variance paths are simulated, so there is no option data to recalibrate against. A study that is more in line with industry practices, i.e. re-calibration to option data at every re-hedge, is undertaken in the next section.

\(^4\)The difference between the hedging errors in parameter set 1 is not too significant.
4.3 Asymptotic Hedge on Real Data

In the following analysis, the asymptotic Heston hedge is compared to that of the exact Heston hedge using real historical data. In this comparison the calibration, the accuracy, the performance, and the computational time of the hedge will be considered.

The setup of the hedges are slightly different to the setup used in the simulation study of Section 4.2. When the portfolio is re-hedged, the model parameters are re-calibrated using option price data available at that time. The motivation for this is so that at each re-hedge, the trading strategy is more in line with market data than past data. Furthermore, it cannot be expected that the option being hedged against, will be priced in the market, by the parameters calibrated when initializing the hedge, but more so by the parameters calibrated at the time of re-hedge. As such, the hedges are more like a Heston hedge that has been set in motion, in the spirit of Carmona and Nadtochiy [12], than a traditional Heston hedge, with fixed parameters.

4.3.1 Dataset

The asymptotic Heston hedge is applied to two datasets. They are the S&P 500 and EUROSTOXX 50 index spanning from 07 January 2004 to 04 June 2008. The data are collected weekly with 231 weeks in total. For each week, there are 15 implied volatility values corresponding to the 3 lengths of expiry times, those being 1 year, 6 months and 3 months. Furthermore, each expiry date has a strike price at 110%, 105%, 100%, 95% and 90% of the current spot price. In addition, there are 3 interest rates, that is the one year, the six months and the three months rate.

4.3.2 Calibration

Calibration was performed using both the exact Heston solution and the asymptotic Heston solution on the two datasets. For each week’s implied volatilities, the calibration process yields a set of model parameters. These parameters are $\alpha$, $\beta$, $m$, $\rho$ and $y_0$, for the both the asymptotic and exact Heston solution. The calibration process is performed by using the minimization of least squares method through the MATLAB routine $\text{lsqnonlin}$.

Figures 4.7 and 4.8 show the root mean square errors (RMSE) of the calibrated implied volatilities of both the exact and asymptotic solutions. The exact Heston calibration is shown to be a better overall fit than the asymptotic Heston calibration. However, the calibration using the asymptotic solution seems to provide a reasonable approximation.
4.3 Asymptotic Hedge on Real Data

Figure 4.7: RMSE for the S&P 500 Calibration - The RMSE is generally higher for the calibration using the asymptotic solution than the exact Heston solution.

Figure 4.8: RMSE for the EUROSTOXX 50 Calibration - Similar to the S&P 500 calibration, the RMSE is generally higher for the asymptotic solution than the exact Heston solution.
4.3 Asymptotic Hedge on Real Data

The calibration was performed by selecting the initial parameter estimates to be 4, 0.3, 0.05, -0.8, and 0.05, for $\alpha$, $\beta$, $m$, $\rho$, and $y_0$, respectively. A constraint required on the calibrated parameters is that it must satisfy the inequality in (3.7). The calibration for each subsequent week uses the previous weeks’ calibrated parameters as the starting points. Other calibration methods tested include using the calibrated parameters from the asymptotic solution as a starting point for the exact Heston solution calibration. However, it was found that there were no benefits, in terms of accuracy or computational time, in doing so. The calibrated parameters using the exact Heston and asymptotic solution are in fact quite different from each other. This observation can be explained by the fact that the calibrated parameters from the asymptotic solution minimizes the sum of the squared distance from the observed implied volatilities to the implied volatilities calculated from the asymptotic solution, and likewise for the exact Heston solution. As such, the calibrated parameters may not be a true representation of the model’s parameters, but a means of best fitting the implied volatility curves to the one observed. Due to the differences in the functional form of the exact and asymptotic solutions, the calibrated parameters may exhibit very different values, even though together with their respective solution, they provide a very good fit to the data.

4.3.3 Hedge Portfolio

The performance of the hedge is tested on 1 year call options with forward ATM strikes. The hedging is performed as follows: At time $t$, $\varphi^1(t)$ units of the index are held and $\varphi^0(t)$ units of money are invested in the money market. The trading strategy with $\varphi^1$ and $\varphi^0$ are calculated using both the exact and asymptotic Heston solutions, by using their respective calibrated parameters. The quality of the hedges are measured via the weekly profit and loss ratios (PLR) defined as,

$$P&L(t + dt) = \frac{\varphi^1(t)X_{t+dt} + \varphi^0(t)e^{rdt} - C(t + dt)}{C(t)}$$

where $dt = 1/52$, $r$ is the one year interest rate, and $C(t)$ is the call option price at time $t$, calculated using the exact Heston solution and its calibrated parameters. Ideally, $C(t)$ would be the actual observed option prices, however they may not exist in the market, with the parameters as specified, especially at the specific strikes and times to expiry as required. Also, when calculating the PLR for the asymptotic Heston hedge, the exact Heston option price will...
still be used\(^5\). With 231 weeks of data, a total of 179 different hedges can be started, with the first hedge starting in the first week, and the last starting in week 179. This is because each hedge takes 52 weeks to complete, and thus for hedges starting on or after week 180, there is not enough data to complete the hedges.

A problem that has been briefly mentioned before, is when the option price is extremely OTM and short times to expiry, see Section 3.2. Whilst in simulated cases, treating these prices and their partial derivatives as zero or swapping the four term solution out for the two term solution may not have that much of an impact, in the real world, this has much greater ramifications. One way to deal with this is by introducing a hybrid hedging process, which will be different to the one introduced in the simulation study. To implement such a scheme, the asymptotic solution and its calibrated parameters will be designated as the default pricing formula used in calculating the trading strategy. If whenever any of the asymptotic option prices or their partial derivatives are zero or return unrealistic results due to the option being extremely OTM, then all calculations for that point in time, are replaced by the calculations using the exact Heston solution. This is justified because, even though the option may be of very low value, it is not exactly zero either. By treating it as zero, it may in fact be ruining the whole hedge as the hedging strategy is a function of the option’s price and partial derivatives. Suppose the approximations for these quantities, at some point before expiry, are replaced with a zero value, then the trading strategy dictates that you hold neither bonds nor asset, from equations (4.1) and (4.2). Realistically, this cannot occur as the option’s value will always be positive prior to expiry, thus at a minimum either the holdings in the bond or asset, must be positive. In transitioning from the asymptotic to the exact Heston solution, the parameters to be used would be the parameters calibrated using the asymptotic solution. This is because the calibrated parameters from the asymptotic solution are able to fit the general shape of the observed market prices, just not the theoretical price of extremely OTM options. In terms of usage, it is not expected that the exact Heston solution will be invoked much, unless the option is nearing expiry and at the same time being quite OTM, such that the option has very little value. At worse, the hedge will just default back to using the exact Heston hedges with parameters calibrated from the asymptotic solution. From here onwards, references to the asymptotic hedging process will actually be referring to this hybrid hedging process.

\(^{5}\)The assumption is that other market participants would only be calculating their Heston prices using the exact Heston solution.
4.3.4 Results

The performance of the PLR for the exact and asymptotic hedges will be compared over the life of the hedges, and also for hedges starting on different dates. Figure 4.9 shows the PLR over time, of hedges starting on the dates 18/02/2004, 20/10/2004 and 15/06/2005, labeled Week Index 7, 42 and 76, respectively, for the S&P 500 dataset. The figure shows the PLR for the asymptotic hedge has similar performance to the exact Heston hedge for Week Index 42 and 76 (bottom two graphs), while better performance for Week Index 7 (top graph). These results already show promise in using the asymptotic hedge as an alternative.

Figure 4.9: 3 Random Samples of PLRs over the life of the Hedge - They correspond to Week Index 7, 42 and 76, respectively from the top to the bottom. The PLR are very similar for both hedging methods.

Figure 4.10 shows in more detail the performances of the 179 hedges for the S&P 500 dataset. The top part of the figure shows the difference in the mean PLR, while the bottom part shows the difference in the standard deviation of the PLR, for hedges starting on weeks given in

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4.3 Asymptotic Hedge on Real Data

the Week Index. For example, the hedge starting on 18/02/2004, has Week Index 7, and the corresponding differences in the mean and standard derivations for the life of that hedge, can be found in Figure 4.10, by looking at Week Index 7. The asymptotic hedge does not only track the exact Heston hedge well, but for Week Index 1 to 160, the asymptotic hedges produces, on average, a higher mean PLR. This is again a good result for the asymptotic solution, and further justifies its use as an alternative to the exact Heston solution. A discussion on the results for Week Index 160 onwards is provided below.

After Week Index 160, the asymptotic and exact Heston hedge seem to experience some difficulties relative to each other and in general. This can be explained by noting that at around Week Index 211, the spot price of the S&P 500 index, began falling. This meant that many of the options starting on and after Week Index 160, finished OTM. This poses two problems, the first of which as discussed above, only affects the asymptotic hedge, and has been dealt with using the hybrid scheme. The second problem is that as these options approach the expiry date, they begin to lose a large portion of their value. This translates to the PLRs (for both the asymptotic and exact Heston hedges) close to the expiry date, being less meaningful, as a decreasing denominator in the ratio makes the ratio unnecessarily large in absolute value. This in turn, distorts the mean and standard deviations of the PLR over the life of that hedge. Figure 4.11 shows the mean for Week Index 160 to 179, by taking the first 40 PLRs for each hedge, instead of the full 52 PLRs. It is clear that for Week Index 160 to 172, the asymptotic hedge still tracks well with exact Heston hedge, for the first 40 weeks of their respective hedges. The effects of the S&P 500 index falling, can still be seen in Week Index 173 to 179, even though they are restricted to the first 40 PLRs.

Figure 4.12 shows the difference in the mean and standard derivation of the PLR for the EUROSTOXX 50 dataset. Many of the phenomenon observed in the S&P 500 dataset are again observed here. The asymptotic hedge tracks the exact Heston hedge well until about Week Index 160. A decrease in the spot price of the EUROSTOXX 50 index explains the results observed.

It is also noted that the differences in the standard deviations of the PLR is not as important, as a measure of performance, as the differences in the mean. For example, consider the top graph in Figure 4.9, and suppose that for one of the weeks since the start of the hedge, a PLR of 1.0 is observed for the asymptotic hedge. Given the mean of the PLRs are roughly about

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\(^6\)Except for the fact that the asymptotic hedge does not always provide a higher PLR than the exact Heston solution. However, on average, the mean PLR of the asymptotic hedge is lower than the exact Heston hedge.
4.3 Asymptotic Hedge on Real Data

Figure 4.10: Differences of the Mean and Standard Deviations of the PLRs for the S&P 500 Index - Each Week Index corresponds to one of the 179 hedges performed. They show that the asymptotic hedge performs as well as exact Heston hedge. Note the exceedances from Week Index 160 onwards are explained in Section 4.3.4.

Figure 4.11: Mean PLRs for Week Index 160 to 179 - The means are calculated using the first 40 PLRs for each of the hedges. They show that for the early stages of the hedges, the asymptotic solution still tracks the exact Heston hedge, when there is a downturn in the markets.
0.3, a PLR of 1.0, would have a large impact on the standard derivation of the PLRs for the asymptotic hedge. In this scenario, the standard derivation for the asymptotic hedge would be greater than that of the exact Heston hedge. However, the asymptotic hedge would also be returning higher PLRs, than the exact Heston hedges. In Figures 4.10 and in particularly 4.12, even though the differences in the standard deviations of the PLRs are quite large, more often than not, the mean PLRs seems to favour the asymptotic solution, at least for the first 160 hedges. Thus, as a measure of performance, the differences in the mean PLRs are more important, and in this sense, the asymptotic hedges are worthy as an alternative.

In the hybrid hedging process, the exact Heston solution was only invoked for a maximum of two times for each of the first 160 weeks in the S&P 500 and EUROSTOXX 50 datasets. This highlights that when all is “well” in the market, then the asymptotic hedge is a good replacement for the exact Heston hedge.

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7In this case, “well” refers to the options finishing in the money, and since the strike is forward ATM, then this means the underlier has increase its value in the last year.
4.3 Asymptotic Hedge on Real Data

4.3.5 Computational Time

In accessing the worthiness of using the asymptotic hedge, the most important factor to consider is the computational time associated with the whole process. While the trading strategy in these case studies are only re-hedged on a weekly basis, it stands to reason that the time taken to calibrate the model parameters and to compute the trading strategies, scales with the number of re-hedgings taking place.

Table 4.2 lists the computational time required for each step of the process in one particular run. It is important to note that these run times differ depending on the computer used. Furthermore, the calibration is also dependent on the initial guess of the parameter.

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P 500</th>
<th>EURSTOCK 50</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Asymptotic Hedge</td>
<td>Exact Hedge</td>
</tr>
<tr>
<td>Calibration</td>
<td>1 minute 34 seconds</td>
<td>6 minutes 22 seconds</td>
</tr>
<tr>
<td>Profit and Loss Ratio</td>
<td>41.07 seconds</td>
<td>58.26 seconds</td>
</tr>
</tbody>
</table>

Table 4.2: Computational Time Required for the Calculations

The calibration process involves all 231 weeks of data and the profit and loss ratio process involves the calculation of the ratio and the trading strategies of the 179 weeks. The run times show that the calibration process and the profit and loss calculations using the asymptotic solution has reduced the computational time by a factor of about 4 and 1.4, respectively. The reason for the differences between run times in the calibration is due to the fact that, computationally the asymptotic solution is a much simpler expression to evaluate than the exact solution. This was largely covered in Section 3.2. Thus, in general, it is expected that any computations involving the asymptotic solution will be much faster compared to the exact solution.

The PLR run times do not differ as significantly, only because the PLR for the asymptotic hedge required the calculation of the exact Heston call price. This is due to assuming that observed market price, used in the calculations of the PLRs, were based on the exact Heston calculations, which should be used by other market participants. When the Heston prices are given, the run time for the PLR for the asymptotic hedge is only 9.18 and 10.77 seconds, respectively for the two datasets, which is a reduction in computational time by a factor of about 5.3 to 6.3. Further note that the calculations of the asymptotic Heston trading strategy
4.4 Concluding Remarks

This chapter has investigated the use of the asymptotic Heston solution in locally risk minimizing hedging. The asymptotic solution as derived in Chapter 3 replaces the exact Heston solution in the locally risk minimizing hedging formulae derived by Poulsen et al. [64]. It provides a thorough investigation into the possible practical applications of the asymptotic Heston solution, whilst proving a competitor, the exact Heston solution, to compare against. In particular, a simulation study and a real data study are provided to assess the worthiness of using the asymptotic solution.

The asymptotic hedges created using simulated data, results in hedging errors that do not differ too much from the exact Heston hedges, whilst also being mean self-financing. The scheme shown here uses a combination of the four and two term asymptotic solution, and is found to be superior to just the two term solution. This increase in performance comes about due to the fact that the four term solution is quite accurate for long times to expiry, while the two term solution is more robust for shorter times to expiry. Furthermore, in many cases, the hybrid scheme of using the two and four term solutions in the hedges, outperforms the other two BS type hedges considered in this section. These results show that the asymptotic Heston hedge is a viable alternative to existing BS hedging methods.

Under real historical data, the asymptotic hedge is found to require less computational time in calibration than the exact Heston solution. However the trade off is the accuracy, with the exact Heston solution returning a lower RMSE error than that of the asymptotic solution. The average PLR of the asymptotic hedges are quite similar to the average PLR of the exact Heston hedge, and at times return a higher value. The computational times involved in calculating trading strategy for the asymptotic hedges are less than those of the exact Heston hedges. These advantages provide enough evidence to show that the asymptotic Heston solution is a viable alternative to the exact Heston solution in the context of locally risk minimization hedges.

The calculations (including those from the simulation study) were all performed using an Intel Core 2 Quad 3.6Ghz PC with 8Gb of RAM. As such, run times may differ from PC to PC, but the magnitude of the differences in the run times should remain.
Chapter 5

Investment-Uncertainty Relationship in a Real Options Model with Stochastic Volatility

The discussion on the investment-uncertainty relationship in the context of irreversible investment drawing back to Hartman [41], Pindyck [62], Craine [17] and Caballero [10], has recently been revived with a series of important contributions by Wong [83], Gutiérrez [36], and Sarkar [68]. The canon of these recent articles is that if measures other than the investment threshold are imposed to quantify the relationship between investment and uncertainty, for example the expected time until investment is undertaken, the question of whether a higher level of uncertainty delays or accelerates investment is highly non-trivial. In this chapter, the focus is placed back on to the investment threshold, but with an increase in the complexity of the uncertainty.

Starting from the classical models discussed in Dixit and Pindyck [21], which feature geometric Brownian motion and geometric mean reverting processes modelling the investment project’s value, this chapter will replace the constant volatility assumption by allowing the volatility process itself to be stochastic. More specifically, the instantaneous variance of the project’s value will be assumed to follow a Cox-Ingersoll-Ross process. In the theory of option pricing under stochastic volatility, these models are often also referred to as Heston stochastic volatility models. In particular, this chapter will focus on using the Heston-GBM and Heston-GMR models to model the project’s value.

The real option problem under a stochastic volatility framework has received very little attention, however the pricing of American options with finite time maturity has. Due to the
complexity of having two state variables and possibly one time variable (for the finite time problem), it is not realistic to assume that either the real option problem or the American derivatives problem admit closed form solutions, unless under trivial cases. Most literature on the pricing of American derivatives under stochastic volatility has been focusing on the numerical methods to solve the problem. Generally, the two methodologies can be divided into either a rather complex numerical scheme that targets the exact option price, or a rather simple analytical and tractable formula that approximates the option price reasonably well. These two methodologies are also applicable to the real option problem under stochastic volatility.

One of the aims of this chapter is to understand at least qualitatively, but in the case of the Heston-GBM model also quantitatively in approximation, how the stochastic volatility assumption affects the investment threshold. In order to do so, an analytical and tractable formula is required, and hence this chapter will follow the latter methodology as mentioned above. A suitable approach for American derivatives under a stochastic volatility assumption, for the Ornstein-Uhlenbeck process, has been presented in Fouque, Papanicolaou and Sircar \cite{32}, and also work by Zhu and Chen \cite{87}, on perpetual American put options are related to this area. It turns out that work by Souza and Zubelli \cite{72}, briefly touches on the subject of perpetual real options with some results available. A discussion on how the material presented here differs and provides new insight to that of Zhu and Chen and Souza and Zubelli will be provided. Further to this, material presented in Chapter 3 will also be used. The idea behind this approach, which has been proven to be a powerful approximation for other types of options, eg European, Asian, etc., is to expand the PDE that governs its real option’s price, in orders of the inverse of the square-root of the mean reversion rate, and subsequently derive PDEs which govern the solution at each order.

This chapter will follow the approach by Fouque et al., but focusing on the Heston-GBM and Heston-GMR models, and considering a real option framework instead of the classical option pricing framework. It is assumed that the variance process exhibits fast mean reverting properties thus allowing the asymptotic technique to be applied. In using the asymptotic expansion method, not only a value function for the option is derived, but also an expansion for the optimal investment threshold. This allows relationships between the parameters of the CIR process and the investment threshold to be deduced, both quantitatively, where closed form solutions exist, and qualitatively, where they may not. It will be shown that the zero order term in the expansions of both the value function and the investment threshold are given by the corresponding values in the classical, constant volatility models, where the deterministic
volatility equals the square root of the mean reverting level of the CIR process. However, the first order terms presented are new. In the case of the Heston-GBM model, the computed first order terms for both the value function and the investment threshold are in compact closed form, whereas for the Heston-GMR model, an analytical expression is derived. This expression is dependent on an integral which consists of a combination of Kummer-M functions, which to the best of our knowledge can not be computed in closed form\(^1\). In the following, the first order terms are referred to as the stochastic volatility correction terms, or simply the correction terms.

In the case of the Heston-GBM model, the closed form expression of the correction terms shows that the investment threshold is higher (lower) than in the classical model with corresponding deterministic volatility depending on whether the correlation \(\rho\) between volatility and project value is positive (negative). Even though a closed form expression for the correction term of the investment threshold in the Heston-GMR model can not be derived, it is possible to conclude from the maximum principle for ordinary differential equations (ODE), that the same relationship holds true for the case of Heston-GMR under certain conditions.

To justify the approach, a number of numerical experiments will be presented, in which the superiority of the asymptotic approach compared to the classical model, is demonstrated. The numerical section will also show that the asymptotic solution, is not a solution to the original problem, but a modified problem. The original problem assumes the optimal investment threshold varies with the initial variances level, whereas the asymptotic solution solves the problem for when the optimal investment threshold is independent of the variance level. This is an important fact as the fundamental assumptions on the optimal investment threshold in the two problems are quite different and has not be identified as yet. Similar to the European call option case, the asymptotic solution for the real option problem also uses the mean reverting level of the variance as a proxy for the variance, and thus leading to the asymptotic optimal investment threshold being independent of the variance. However, this is justifiable due to the fast mean reverting assumption and the time independent nature of the real option problem.

\section{Real Options with Heston-GBM}

The motivation for the real option valuation problem comes from the idea of applying option valuation techniques to capital budgeting decisions. Dixit and Pindyck [21], provides a compre-

\(^1\)The numerical computation of this integral however is efficient and stable.
5.1 Real Options with Heston-GBM

In its simplest form, the real option problem can be formulated as follows; Suppose the value of an investment project $X_t$, changes dynamically over time. At what point is it optimal to pay a sunk (irretrievable) cost $k$ in order to be involved in the project (and benefit from its profits)?

Denote the value of the option to invest by $V(X)$ and assuming rational behavior from the investor, $V(X)$ can be valued as,

$$V(X) = \sup_\tau \mathbb{E} \left( e^{-\tau r} (X_\tau - k) \right)$$

$$\tau = \inf \{ t | X_t \geq X^* \},$$

where $\tau$ runs through all admissible stopping times, and $X^*$ is the optimal stopping boundary. The solution to this problem depends on the assumptions about the dynamics of the project’s value. Closed form solutions have been derived where $X_t$ follows GBM (Section 5.2, Dixit and Pindyck [21]), GMR (Section 5.5A, Dixit and Pindyck [21]) and CIR process (Ewald and Wang [26]).

In this section, the GBM is taken as a starting point, but the constant volatility assumption, is replaced with the assumption that the variance of the project value is itself modelled using a stochastic process. The form for the stochastic variance process will be the CIR process, and thus the model will be of a Heston-GBM form. The SDE for $X_t$ is,

$$dX_t = \kappa X_t dt + \sqrt{Y_t} X_t dW_t$$

where $\kappa$ is drift rate of the project’s value. The SDE for $Y_t$ is given by equation (2.8), where $W_t$ and $Z_t$ are correlated Brownian motions with correlation $\rho$ and the parameters for the CIR process as previously given in Section 2.2.1. Furthermore, define $r$ to be the risk-free interest rate and $\delta = r - \kappa$, the implied dividend rate, which is assumed to be positive.

Given the formulation of the model, the goal is to compute the value of the investment opportunity (option price), which will be denoted by $V(x, y)$. In the proceeding sections, the asymptotic techniques from Section 3.1 will be applied to derive an asymptotic solution for the real option problem under the Heston-GBM framework.

5.1.1 PDE for Real Option under Heston-GBM

As in the classical case without stochastic volatility, using dynamic programming principles to solve for the real option price, leads to a free boundary problem. The problem in the Heston-GBM framework has an increase in the complexity, as two spatial variables, $x$ and $y$, need to be
considered. The space of the spatial variables can be broken up into two regions, the hold- and exercise-region. The boundary of the two regions, in the context of real options, is regarded as the investment threshold.

In the hold-region, the PDE associated with the real option value is given by,
\[
\frac{1}{2} y x^2 \frac{\partial^2 V}{\partial x^2} + (r - \delta) x \frac{\partial V}{\partial x} - r V + x y \beta \rho \frac{\partial^2 V}{\partial x \partial y} + \alpha (m - y) \frac{\partial V}{\partial y} + \frac{1}{2} y \beta^2 \frac{\partial^2 V}{\partial y^2} = 0, \quad (5.2)
\]
for \(x < x_{fb}(y)\), while in the exercise-region \(x > x_{fb}(y)\), the value of the real option is given by its payoff,
\[
V(x, y) = x - k. \quad (5.3)
\]

Here, \(x_{fb}(y)\) denotes the investment threshold, and is in fact the free boundary of the problem.

In conjunction with equations (5.2) and (5.3), the following boundary conditions are required;
\[
\begin{align*}
V(x_{fb}(y), y) &= x_{fb}(y) - k, \\ V_x(x_{fb}(y), y) &= 1, \\ V_y(x_{fb}(y), y) &= 0, \\ V(0, y) &= 0. \quad (5.4, 5.5, 5.6, 5.7)
\end{align*}
\]

Here the sub-indices in \(V_x\) and \(V_y\) denote the corresponding partial derivatives with respect to each subscript. Equation (5.4) denotes the value matching condition at the free boundary, while equations (5.5) and (5.6) are two smooth pasting conditions for each of the spatial variables \(x\) and \(y\) respectively. Equation (5.5) is analogous to the classical case without stochastic volatility, while the intuition behind equation (5.6) is that at the optimal investment time, the actual payoff does not depend on the current variance \(y\). These first three boundary conditions ensure that the solution is smooth across the investment threshold. Equation (5.7) is analogous to the classical case, requiring equation (5.2) to have a fixed point at \(x = 0\).

The derivation for equation (5.2) is analogous to Dixit and Pindyck [21], but with two spatial variables. In the constant volatility problem, the option price is only dependent on the project’s value, thus it is a function of one variable and the free boundary problem is one of an ODE. In the stochastic volatility case, the addition of the extra variable \(y\) forces the problem to become a PDE. For completeness the derivation of the PDE is given in the next section.
5.1 Real Options with Heston-GBM

PDE Derivation

In the hold-region where it is not optimal to invest, the Bellman equation is given by,

\[ rV(x, y)dt = E(dV). \]

The Bellman equation states that over a small interval of time \( dt \), the total expected return from the investment opportunity \( rV(x, y)dt \) is equal to its expected rate of capital appreciation. Using Itô’s formula, \( dV \) can be expanded by,

\[ dV = \frac{\partial V}{\partial x}dX_t + \frac{\partial V}{\partial y}dY_t + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} d[X, X]_t + \frac{1}{2} \frac{\partial^2 V}{\partial y^2} d[Y, Y]_t + \frac{\partial^2 V}{\partial x \partial y} d[X, Y]_t. \]

Taking expectations of \( dV \) and dividing through by \( dt \), the Bellman equation is thus equivalent to

\[ \frac{1}{2} yx^2 \frac{\partial^2 V}{\partial x^2} + \kappa x \frac{\partial V}{\partial x} - rV + xy\beta \rho \frac{\partial^2 V}{\partial x \partial y} + \alpha (m - y) \frac{\partial V}{\partial y} + \frac{1}{2} y \beta^2 \frac{\partial^2 V}{\partial y^2} = 0. \]

Much like in Dixit and Pindyck [21], to reconcile with the contingent claim analysis approach, the substitution of \( \kappa = r - \delta \) completes the proof.

5.1.2 Stochastic Volatility Asymptotics

The techniques used to derive an asymptotic solution for the real option problem will be loosely based on Fouque et al. [32], on their work for American options, and that of Section 3.1. It is noted that Zhu and Chen [87], have also investigated the use of this asymptotic technique in deriving asymptotic solutions for perpetual American put options but under an OU stochastic volatility model. While the material presented in this chapter was developed independently to their work, there are some similarities. Furthermore, the asymptotic solution to be presented is derived up to the first two terms. A short discussion on this issue and those previously listed, will be provided in subsequent sections.

In deriving the asymptotic solution, begin by defining the parameters \( \nu^2 = m\beta^2/2\alpha \) and \( \epsilon = 1/\alpha \), and replacing the \( \alpha \) and \( \beta \) by \( \nu \) and \( \epsilon \) in equation (5.2). Further define the operators,

\[ \mathcal{L}_0 = \frac{\nu^2 y}{m} \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \]

\[ \mathcal{L}_1 = \frac{\rho \nu \sqrt{2}}{m} xy \frac{\partial^2}{\partial x \partial y}, \]

\[ \mathcal{L}_2 = \frac{1}{2} yx^2 \frac{\partial^2}{\partial x^2} + (r - \delta)x \frac{\partial}{\partial x} - r, \]

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where $\mathcal{L}_0$ and $\mathcal{L}_1$ are as defined previously in Section 3.1, and $\mathcal{L}_2$ is the Dixit and Pindyck GBM real option operator, with volatility parameter $\sqrt{y}$. These operators enable the PDE to be written in its compact form as,

$$
\left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) V = 0. 
$$

(5.11)

Assume the solution $V(x, y)$ and the free boundary $x_{\text{fb}}(y)$ can be expanded in the form,

$$
V(x, y) = V_0(x, y) + \sqrt{\epsilon} V_1(x, y) + \epsilon V_2(x, y) + \cdots ,
$$

$$
x_{\text{fb}}(y) = x_0(y) + \sqrt{\epsilon} x_1(y) + \epsilon x_2(y) + \cdots ,
$$

respectively. The PDE and boundary conditions can then be rewritten in powers of $\sqrt{\epsilon}$ as well. Keeping terms of up to order $\sqrt{\epsilon}$ only, the PDEs and boundary conditions are,

$$
\frac{1}{\epsilon} \mathcal{L}_0 V_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_0 V_1 + \mathcal{L}_1 V_0) + (\mathcal{L}_0 V_2 + \mathcal{L}_1 V_1 + \mathcal{L}_2 V_0) + \sqrt{\epsilon} (\mathcal{L}_0 V_3 + \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1) = 0,
$$

(5.12)

and,

$$
V_0(x_0(y), y) + \sqrt{\epsilon} \left( x_1(y) \frac{\partial V_0}{\partial x} \bigg|_{x_0} + V_1(x_0(y), y) \right) = x_0(y) + \sqrt{\epsilon} x_1(y) - k,
$$

(5.13)

$$
\left. \frac{\partial V_0}{\partial x} \right|_{x_0} + \sqrt{\epsilon} \left( x_1(y) \frac{\partial^2 V_0}{\partial x^2} \bigg|_{x_0} + \frac{\partial V_1}{\partial x} \bigg|_{x_0} \right) = 1,
$$

(5.14)

$$
\left. \frac{\partial V_0}{\partial y} \right|_{x_0} + \sqrt{\epsilon} \left( x_1(y) \frac{\partial^2 V_0}{\partial x \partial y} \bigg|_{x_0} + \frac{\partial V_1}{\partial y} \bigg|_{x_0} \right) = 0.
$$

(5.15)

The left hand side of equations (5.13) to (5.15) are calculated by using Taylor’s expansion of $V(x, y)$ at $x_0(y)$. Note that the derivatives evaluated at $x_0(y)$ are the one sided derivatives from the $x < x_0(y)$ region. Following Fouque et al. [32], the task is now to equate various orders of $\sqrt{\epsilon}$ to identify equations that determine the functions $V_i$.

**Zero Order Term**

The largest order, $1/\epsilon$, gives the following set of equations and boundary conditions,

$$
\mathcal{L}_0 V_0(x, y) = 0, \quad x < x_0(y),
$$

$$
V_0(x, y) = x - k, \quad x \geq x_0(y),
$$

(5.16)

$$
\left. \frac{\partial V_0}{\partial x} \right|_{x_0(y)} = 1.
$$

(5.17)
5.1 Real Options with Heston-GBM

Since \( \mathcal{L}_0 \) takes derivatives with respect to \( y \) only, the first equation implies that \( V_0 \) is independent of \( y \). In addition, since \( V_0 \) is independent of \( y \) on either side of the \( x_0(y) \) frontier, it can be concluded that \( x_0 \) is also independent of \( y \). Furthermore, the asymptotic expansion for the price will only be considered in the hold-region, thus \( V_1 = 0 \) in the \( x \geq x_0 \) region.

The PDE and boundary conditions for the next order, \( 1/\sqrt{\epsilon} \), are given by,

\[
L_0 V_1(x, y) = 0, \quad x < x_0,
\]
\[
V_1(x, y) = 0, \quad x \geq x_0,
\]
\[
x_1(y) \left. \frac{d^2 V_0}{dx^2} \right|_{x_0} + \left. \frac{\partial V_1}{\partial x} \right|_{x_0} = 0.
\]

The first equation is the result of \( V_0 \) being independent of \( y \), and thus \( L_1 V_0 = 0 \). Similarly as with \( V_0 \), it can be concluded that \( V_1 \) is also independent of \( y \).

The subsequent order, \( (1/\epsilon)^0 = 1 \), provides the condition for \( V_0 \) as,

\[
L_0 V_2 + L_2 V_0 = 0, \quad x < x_0,
\]
\[
V_2(x, y) = 0, \quad x \geq x_0,
\]

where it is used that \( L_1 V_1 = 0 \). The first equation is a Poisson equation for \( V_2 \) with respect to the operator \( L_0 \), see Section 3.1.2. A solution for \( V_2 \) exists if and only if \( L_2 V_0 \) is centered with respect to the invariant distribution of the diffusion whose infinitesimal generator is \( L_0 \). From Section 3.1.2 it is clear that the centering condition requires,

\[
\langle L_2 V_0 \rangle = 0.
\]

Since \( V_0 \) is independent of \( y \), the angled brackets acting on \( L_2 V_0 \) is interchangeable with the angled brackets acting on the operator \( L_2 \), then operating on \( V_0 \). As such, the centering condition becomes \( \langle L_2 V_0 \rangle = 0 \) and using \( \langle y \rangle = m^2 \), this is equivalent to,

\[
\frac{1}{2} m x^2 \left. \frac{d^2 V_0}{dx^2} \right|_x + (r - \delta) x \left. \frac{dV_0}{dx} \right|_x - r V_0 = 0,
\]

(5.19)

with boundary conditions given in equations (5.16) and (5.17). For more details on the Poisson equation and various technical justifications, see Fouque et al. [32], (for OU variance process) and Section 3.1.2 (for CIR variance process).

Equation (5.19) can now be identified with the ODE determining the value of a real option where the project’s value follows a GBM with constant volatility \( \sqrt{m} \). As such, the closed form

\[\text{See Section 3.1.2 on the invariant distribution is a Gamma distribution}\]
solution for this problem exists, and is given by,

\[
V_0 = \begin{cases} \\
\frac{x_0 - k}{x_0} x^{\beta_1}, & \text{if } x < x_0, \\
x - k, & \text{if } x \geq x_0,
\end{cases}
\]

where the threshold and the constant \( \beta_1 \), not to be confused with \( \beta \) from the CIR process, are given by,

\[
x_0 = \frac{\beta_1}{\beta_1 - 1} k, \\
\beta_1 = \frac{1}{2} - (r - \delta)/m + \sqrt{\left( (r - \delta)/m - \frac{1}{2} \right)^2 + 2r/m},
\]  
(5.20)

respectively. The derivation of this solution can be found in Dixit and Pindyck [21].

**Correction Term**

For the next order, \( \sqrt{\epsilon} \), the equations of concern are,

\[
\mathcal{L}_0 V_3 + \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 = 0 \\
V_3 = 0
\]

\( x < x_0 \)

\( x \geq x_0 \)

This leads to a Poisson equation for \( V_3 \), with the corresponding centering condition being,

\[
\langle \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 \rangle = 0.
\]

Following similar arguments to Section 3.1.3, it can be shown that if \( \tilde{V}_1 = \sqrt{\epsilon} V_1 \), then \( \tilde{V}_1 \) satisfies the following ODE with source term,

\[
\left( \mathcal{L}_2 \tilde{V}_1 \right) = -\frac{\rho \beta m}{2\alpha} \left( 2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3} \right).
\]

The latter ODE is equivalent to,

\[
\frac{1}{2} m x^3 \frac{d^2 \tilde{V}_1}{dx^2} + (r - \delta) x \frac{d \tilde{V}_1}{dx} - r \tilde{V}_1 = c_1 x^{\beta_1},
\]

where,

\[
c_1 = -\frac{\rho \beta m}{2\alpha} A \beta_1^2 (\beta_1 - 1),
\]

\[
A = \frac{x_0 - k}{x_0^{\beta_1}},
\]

are constants.
5.1 Real Options with Heston-GBM

To solve this ODE, the method of variation of parameters technique is used, see Section 9.3.2 in Haberman [37]. Proceed by making the substitution $x = \exp(s)$, and then dividing by $m/2$. The ODE then becomes,

$$\frac{d^2\tilde{V}_1}{ds^2} + \left(\frac{2(r - \delta)}{m} - 1\right) \frac{d\tilde{V}_1}{ds} - \frac{2r}{m} \tilde{V}_1 = \frac{2c_1}{m} \exp(\beta_1 s).$$

(5.21)

The two linearly independent solutions to the homogenous part of equation (5.21) are,

$$u_1(s) = \exp(\beta_1 s),$$

$$u_2(s) = \exp(\beta_2 s),$$

where $\beta_2$, again not to be confused with $\beta$ from the CIR process or $\beta_1$ as defined earlier, is given as,

$$\beta_2 = \frac{1}{2} - (r - \delta)/m - \sqrt{\left(\frac{r - \delta}{m} - \frac{1}{2}\right)^2 + 2r/m}.$$

The values $\beta_1$ and $\beta_2$ are in fact the roots of the quadratic function,

$$Q(\lambda) = \lambda^2 + \left(\frac{2(r - \delta)}{m} - 1\right) \lambda - \frac{2r}{m},$$

which is well known in real option literature. Following Dixit and Pindyck [21], it is easy to verify that $\beta_1 > 1 > 0 > \beta_2$, by considering the concavity of $Q(\lambda)$, and where its roots lie in relation to 0 and 1.

The Wronskian of the two linearly independent solutions is given as,

$$W(s) = (\beta_2 - \beta_1) \exp((\beta_1 + \beta_2)s),$$

and the general solution for the ODE in equation (5.21) can be calculated as $A(s)u_1(s) + B(s)u_2(s)$, where $A(s)$ and $B(s)$ are the functions defined by,

$$A(s) = \int -\frac{1}{W(s)} f(s)u_2(s) \, ds,$$

$$B(s) = \int \frac{1}{W(s)} f(s)u_1(s) \, ds,$$

and the source term $f(s) = 2c_1 \exp(\beta_1 s)/m$ in equation (5.21). These terms are calculated as,

$$A(s) = -\frac{2c_1}{m(\beta_2 - \beta_1)} s + \tilde{c}_2,$$

$$B(s) = -\frac{2c_1}{m(\beta_2 - \beta_1)^2} \exp((\beta_1 - \beta_2) s) + c_3,$$

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for arbitrary constants \( \tilde{c}_2 \) and \( c_3 \).

Converting back to the \( x \) variable, the general solution for the correction term can be computed as,

\[
\tilde{V}_1(x) = c_2 x^\beta_1 + c_3 x^\beta_2 - \frac{2c_1}{m(\beta_2 - \beta_1)} \log(x)x^\beta_1,
\]

where \( c_2 \) is an arbitrary constant. The boundary conditions for \( \tilde{V}_1 \) requires that at \( x = 0 \), \( \tilde{V}_1(0) = 0 \), leading to \( c_3 = 0 \). At \( x = x_0 \), the boundary conditions require \( \tilde{V}_1(x_0) = 0 \), which leads to \( c_2 = 2c_1 \log(x_0)/m(\beta_2 - \beta_1) \). Hence the correction term is given by,

\[
\tilde{V}_1 = -\frac{\beta \rho A \beta^2 (\beta_1 - 1)}{(\beta_2 - \beta_1)\alpha} \log \left( \frac{x_0}{x} \right) x^\beta.
\]

### Threshold Expansion

In the previous section, the zero order term for the threshold expansion was derived as \( x_0 \) in equation (5.20). Using the boundary conditions in equation (5.18), it is possible to isolate the \( x_1(y) \) term to retrieve the next term in the threshold expansion. This idea is used by Zhu and Chen [87], in their work for perpetual put options with fast mean reverting stochastic volatility models driven by an OU process, but identified independently here. The observation is that \( x_1(y) \) can be written as,

\[
x_1 = \frac{dV_1}{dx} \bigg|_{x_0} \frac{d^2 V_0}{dx^2} \bigg|_{x_0}.
\]

Given that the right hand side of equation (5.22) is independent of \( y \), the correction term for the threshold expansion is thus also independent of \( y \). Again, note that the derivatives in equation (5.22) are the one sided derivatives from the \( x < x_0 \) region. Using the closed form expressions for \( V_0 \) and \( \tilde{V}_1 \), it is possible to obtained a closed form solution for the correction term of the threshold.

The second order derivative of \( V_0 \) is given as,

\[
\frac{d^2 V_0}{dx^2} = A \beta_1 (\beta_1 - 1)x^\beta_1 - 2.
\]

Instead of focusing on \( V_1 \), turn the attention to \( \tilde{V}_1 \), since the latter is just a scaled version of the former. \( \tilde{V}_1 \) can be rewritten as,

\[
\tilde{V}_1 = -\frac{\beta \rho A \beta^2 (\beta_1 - 1)}{(\beta_2 - \beta_1)\alpha} \log \left( \frac{x_0}{x} \right) x^\beta,
\]

\[
= B \rho \log \left( \frac{x_0}{x} \right) x^\beta,
\]
where $B$ is a constant defined by grouping all the other non-essential constants, except for $\rho$, together. It is easy to verify that $B$ is in fact positive, given that $\beta_1 > 1 > 0 > \beta_2$, and $A > 0$.

The first order derivative of $\tilde{V}_1$ is given by,

$$\frac{d\tilde{V}_1}{dx} = B\rho \left( \beta_1 \log \left( \frac{x_0}{x} \right) - 1 \right) x^{\beta_1 - 1},$$

and thus at $x_0$, the first order derivative is evaluated as,

$$\left. \frac{d\tilde{V}_1}{dx} \right|_{x=x_0} = -B\rho x_0^{\beta_1 - 1}. \quad (5.24)$$

Dividing equation (5.24) by $1/\sqrt{\alpha}$ and then substituting this into equation (5.22) along with using equation (5.23), gives the correction term for the threshold as,

$$x_1 = \frac{\rho \beta \beta_1 x_0}{(\beta_1 - \beta_2) \sqrt{\alpha}}.$$ 

### 5.1.3 Asymptotic Solution

Using the first two terms in the expansions for the real option price and investment threshold, the asymptotic solution is presented as,

$$V(x,y) \approx \begin{cases} 
Ax^{\beta_1} - \frac{A^2 \rho^2 \beta_2 (\beta_1 - 1)}{(\beta_2 - \beta_1)^2} \log \left( \frac{x_0}{x} \right) x^{\beta_1} , & \text{if } x < x_0, \\
 x - k , & \text{if } x \geq x_0, 
\end{cases}$$

and

$$x_{0b} \approx x_0 + \frac{\rho \beta \beta_1 x_0}{(\beta_1 - \beta_2) \sqrt{\alpha}},$$

respectively, where all constants appearing are as given earlier.

It is of interest to note that in the case where the project’s value and its variance process are uncorrelated, the asymptotic solutions (both the real option price, and its investment threshold) coincides with the classical Dixit and Pindyck solution for the standard GBM case, with constant volatility parameter $\sqrt{\alpha}$. For positive correlation $\rho$, the investment threshold under the Heston-GBM model is slightly higher than compared to the standard GBM model, while for negative $\rho$, the reverse is observed. These features, along with the real option’s price, will be investigated numerically to confirm the results in a latter section of this chapter.
5.2 Real Options with Heston-GMR Process

In this section, the model assumptions of Section 5.1 are modified to accommodate a new model. Instead of the Heston-GBM model, the project’s value is modeled via the SDE,
\[ dX_t = \eta(\bar{x} - X_t)X_t dt + \sqrt{Y_t}X_t dW_t, \]
while the variance process \( Y_t \) remains the same as in equation (2.8). Again, \( W_t \) and \( Z_t \) are Brownian motions, with correlation \( \rho \), while the CIR process retains the same parameters. For the project’s value, the new parameters \( \eta \) and \( \bar{x} \) are the mean reverting rate and mean reverting level, respectively.

Given the formulation of the model, the goal is now to price the value of this real option which will again be denoted by \( V(x, y) \).

5.2.1 PDE for Real Option under Heston-GMR

As in the case for the Heston-GBM model, the space of the spatial variables can be split into the hold- and exercise-region. In the hold-region, the PDE associated with the real option’s value is given by,
\[ \frac{1}{2} y x^2 \frac{\partial^2 V}{\partial x^2} + \eta(\bar{x} - x) x \frac{\partial V}{\partial x} - r V + x y \beta \rho \frac{\partial^2 V}{\partial x \partial y} + \alpha (m - y) \frac{\partial V}{\partial y} + \frac{1}{2} x^2 \beta^2 y^2 \frac{\partial^2 V}{\partial y^2} = 0, \quad (5.25) \]
for \( x < x_{fb}(y) \), while the value of the real option in the exercise-region, \( x > x_{fb}(y) \), is given by,
\[ V(x, y) = x - k. \]

As before, \( x_{fb}(y) \) denotes the free boundary of the problem, which is again the investment threshold. The derivation of this PDE is very similar to the Heston-GBM case in 5.1.1. In particular, for the derivation, let \( \kappa = \eta(\bar{x} - X_t) \), and the result follows analogously.

The boundary conditions are identical (in form) to those of the Heston-GBM model, equations (5.4) to (5.7), and have the same intuition behind their nature.

5.2.2 Stochastic Volatility Asymptotics

As with the previous Heston-GBM case, define the parameters, \( \nu^2 = m \beta^2 / 2 \alpha \) and \( \epsilon = 1 / \alpha \), and replace the \( \alpha \) and \( \beta \) by \( \nu \) and \( \epsilon \) in equation (5.25). This is done similarly in Fouque et al. [32], and that of Section 3.1. Further define the operators \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) as in equations (5.8) and (5.9), respectively, while \( \mathcal{L}_2 \) is modified to be defined as,
\[ \mathcal{L}_2 = \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + \eta(\bar{x} - x) x \frac{\partial}{\partial x} - r \cdot . \quad (5.26) \]
With these changes, the PDE in equation (5.25) can be written in its compact form, analogous to equation (5.11).

Expand the solution for $V(x, y)$ in the hold-region as well as the free boundary $x_{fb}(y)$ in powers of $\sqrt{\epsilon}$. The expanded PDE and boundary conditions, keeping up to order $\sqrt{\epsilon}$, has the same format as in the Heston-GMR case and are given by equations (5.12) to (5.15), but with $\mathcal{L}_2$ defined via equation (5.26). Following the same approach, various equations defining the functions $V_i$ can be determined by equating various orders of $\sqrt{\epsilon}$ to zero.

Zero Order Term

To largest order, $1/\epsilon$, gives the following set of equations and boundary conditions,

\[
\mathcal{L}_0 V_0(x, y) = 0, \quad x < x_0(y),
\]
\[
V_0(x, y) = x - k, \quad x \geq x_0(y),
\]
\[
\frac{\partial V_0}{\partial x} \bigg|_{x_0(y)} = 1.
\]  

(5.27)  

(5.28)

Using similar arguments to Section 5.1.2, it can be deduced that $V_0$ and $x_0$ are both independent of $y$.

The next order, $1/\sqrt{\epsilon}$, provides the following boundary conditions,

\[
\mathcal{L}_0 V_1(x, y) = 0, \quad x < x_0,
\]
\[
V_1(x, y) = 0, \quad x \geq x_0,
\]
\[
x_1(y) \frac{d^2 V_0}{dx^2} \bigg|_{x_0} + \frac{\partial V_1}{\partial x} \bigg|_{x_0} = 0,
\]

where, the first equation uses the fact that $\mathcal{L}_1 V_0 = 0$. It can then be concluded that $V_1$ is also independent of $y$.

The subsequent order, $(1/\epsilon)^0 = 1$ shows that,

\[
\mathcal{L}_0 V_2 + \mathcal{L}_2 V_0 = 0, \quad x < x_0,
\]
\[
V_2(x, y) = 0, \quad x \geq x_0,
\]

as a consequence of $\mathcal{L}_1 V_1 = 0$. The first equation can be identified as the Poisson equation for $V_2$ with respect to the operator $\mathcal{L}_0$. The corresponding centering condition, which $V_0$ needs to satisfy, is given by,

\[
\frac{1}{2} m x^2 \frac{d^2 V_0}{dx^2} + \eta (x - x) x \frac{dV_0}{dx} - r V_0 = 0,
\]  

(5.29)
along with boundary conditions given in equations (5.27) and (5.28). Equation (5.29) together with the boundary conditions can be identified as the problem of valuing a real option under a GMR process with constant volatility $\sqrt{m}$. This is a well known problem, and its solution is given in Dixit and Pindyck [21] as,

$$V_0(x) = Ax^\theta M(\theta, b, 2\eta x/m),$$

where,

$$\theta = 1/2 - \eta \bar{x}/m + \sqrt{(\eta \bar{x}/m - 1/2)^2 + 2r/m},$$

(5.30)

$$b = 2\theta + 2\eta \bar{x}/m,$$

(5.31)

and $M(a, b, z)$ is the confluent hyper-geometric function, see Chapter 13 of Abramowitz and Stegun [1]. Unfortunately, no closed form solutions are available for the constants $A$ and $x_0$, and thus need to be computed numerically.

**Correction Term**

For the next order, $\sqrt{\epsilon}$, the equations of concern are,

$$L_0 V_3 + L_1 V_2 + L_2 V_1 = 0 \quad x < x_0,$$

$$V_3 = 0 \quad x \geq x_0.$$

The first equation is again a Poisson equation for $V_3$, which leads to the centering condition,

$$\langle L_1 V_2 + L_2 V_1 \rangle = 0.$$

For notational convenience, again define $\tilde{V}_1 = \sqrt{\epsilon} V_1$, and it can be shown, using arguments similar to Section 3.1.3, that $\tilde{V}_1$ satisfies the ODE,

$$\left\langle L_2 \tilde{V}_1 \right\rangle = -\frac{\rho \beta m}{2\alpha} \left( 2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3} \right).$$

(5.32)

Equation (5.32) is equivalent to

$$\frac{1}{2} m x^2 \frac{d^2 \tilde{V}_1}{dx^2} + \eta (\bar{x} - x) x \frac{d \tilde{V}_1}{dx} - r \tilde{V}_1 = -\frac{\rho \beta m}{2\alpha} \left( 2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3} \right).$$

(5.33)

The method of variation of parameters is again used to attempt to solve this ODE. Begin by dividing equation (5.33) by $mx^2/2$ to obtain,

$$\frac{d^2 \tilde{V}_1}{dx^2} + \eta (\bar{x} - x) \frac{d \tilde{V}_1}{dx} - \frac{2r}{m x^2} \tilde{V}_1 = -\frac{\rho \beta}{\alpha} \left( 2 \frac{d^2 V_0}{dx^2} + x \frac{d^3 V_0}{dx^3} \right) =: f(x),$$

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where $f(x)$ represents the source term.

Solving the homogeneous part of the ODE in equation (5.33) is equivalent to solving the Kummer Differential Equation, and thus the two linearly independent solutions are given as,

$$u_1(x) = x^{\theta} M(\theta, b, 2\eta x/m),$$

$$u_2(x) = x^{\theta} \left( \frac{2\eta x}{m} \right)^{1-b} M(\theta - b + 1, 2 - b, 2\eta x/m),$$

where $\theta$ and $b$ are defined in equations (5.30) and (5.31), respectively. The Wronskian of these two functions is given as,

$$w(x) = \left( \frac{2\eta}{m} \right)^{1-b} (1-b) x^{2\theta - b} \exp (2\eta x/m).$$

The $x$ dependent part of the source term $f(x)$, can be simplified as follows,

$$2 \frac{d^2 V_0}{dx^2} + x \frac{d^3 V_0}{dx^3} = A \theta x^{\theta-2} \left( (\theta - 1) M(\theta, b, 2\eta x/m) \right.$$  
\begin{align*}
+ & \frac{2\eta x (3\theta + 1)}{b} M(\theta + 1, b + 1, 2\eta x/m) \\
+ & \left( \frac{2\eta x}{m} \right)^{2} \frac{(\theta + 1)(2 + 3\theta)}{b(b + 1)} M(\theta + 2, b + 2, 2\eta x/m) \\
+ & \left( \frac{2\eta x}{m} \right)^{3} \frac{(\theta + 1)(2 + \theta)}{b(b + 1)(b + 2)} M(\theta + 3, b + 3, 2\eta x/m) \\
= & A \theta x^{\theta-2} (\theta - 1) M(\theta, b, 2\eta x/m) \\
+ & M(\theta + 1, b + 1, 2\eta x/m) \frac{2\eta x}{mb} \left( \theta(3\theta + 1) + \frac{2\eta x}{m} \theta(\theta + 1) \right) \\
+ & M(\theta + 2, b + 2, 2\eta x/m) \left( \frac{2\eta x}{m} \right)^{2} \frac{\theta + 1}{b(b + 1)} \left( 3\theta + 1 - b + \frac{2\eta x}{m} \right) \right).
\end{align*}

where,

\begin{align*}
f_1(x; \eta, m, \theta, b) &= \frac{2\eta x}{mb} \left( 3\theta + 1 - b + \frac{2\eta x}{m} \right) \theta (\theta - 1), \\
f_2(x; \eta, m, \theta, b) &= \frac{2\eta x}{mb} \left( 3\theta + 1 - b + \frac{2\eta x}{m} \right) \left( \frac{2\eta x}{m} - b \right) + \theta (3\theta + 1) + \frac{2\eta x}{m} (\theta + 1).
\end{align*}

Note that in Dixit and Pindyck [21], the parameter $\theta$ is chosen as the positive square root of a characteristic equation. However, if the negative square root was chosen, then one would again arrive at solving the Kummer Differential Equation. The two solutions resulting from that are in fact linearly dependent to the ones given above. Of course, this must be true since the original ODE is of order 2, and thus can only have two linearly independent solutions.
The simplification makes use of the recurrence relation,
\[ azM(a + 1, b + 1, z) = b(1 - b + z)M(a, b, z) + b(b - 1)M(a - 1, b - 1, z), \]
which can be found in Chapter 13 of Abramowitz and Stegun [1].

The functions \( A(x) \) and \( B(x) \) used in the variation of parameter technique involve the calculation of the following integrals,

\[
A(x) = -\int \frac{1}{w(x)} u_2(x)f(x) \, dx
= \tilde{A} \int \frac{1}{x} \exp \left( -\frac{2\eta x}{m} \right) M(\theta - b + 1, 2 - b, 2\eta x/m)(M_1 f_1 + M_2 f_2) \, dx,
\]

\[
B(x) = \int \frac{1}{w(x)} u_1(x)f(x) \, dx
= -\tilde{A} \left( \frac{2\eta}{m} \right)^{b-1} \int x^{b-2} \exp \left( -\frac{2\eta x}{m} \right) M_1 (M_1 f_1 + M_2 f_2) \, dx,
\]

where \( \tilde{A} = \frac{e^{2\eta \theta}}{\alpha(1 - b)} \) is a constant\(^4\) and \( M_1 \) and \( M_2 \) are abbreviations for the functions,

\[
M_1 = M(\theta, b, 2\eta x/m),
M_2 = M(\theta + 1, b + 1, 2\eta x/m).
\]

Unfortunately, these integrals are very difficult to compute and it is not clear whether closed form solutions exist. Furthermore, the value of the constants of integration must also be determined using the boundary conditions of the ODE. Due to the difficulty involved with obtaining closed form solutions, in the numerical section of this chapter, the correction term will be computed by solving the corresponding ODE in equation (5.33) numerically, instead of using the method of variation of parameters technique. In doing so, it bypasses numerically calculating the two integrals \( A(x) \) and \( B(x) \), in exchange of solving one ODE numerically, and thus minimizing the potential sources of errors.

The Sign of the Correction Term

Although no closed form solution for \( \tilde{V}_1 \) is available, it is possible to determine the sign of the solution under certain conditions. This is important, as it will be used to identify the sign of

\[^4\text{Note the difference between the constant } A \text{ defined in the context of the zero order term, and the function } A(x) \text{ defined here.}\]
5.2 Real Options with Heston-GMR Process

the correction term in the investment threshold. In what follows, assume that $r - \eta \tilde{x} > 0^5$. As a consequence, this results in $\theta > 1$, and by definition, $b > 0$. Consider the ODE,

$$x^2 \frac{d^2 \tilde{V}_1}{dx^2} + \frac{2 \eta (\bar{x} - x)}{m} x \frac{d \tilde{V}_1}{dx} - \frac{2r}{m} \tilde{V}_1 = -\frac{\rho \beta}{\alpha} \left( 2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3} \right),$$

which $\tilde{V}_1$ must satisfy. Transform the ODE by using the substitution, $s = \log(x)$, but for the time being, keeping the source terms as functions of $x$. This results in,

$$\frac{d^2 \tilde{V}_1}{ds^2} + \frac{2 (\eta (\bar{x} - \exp(s)) - 1)}{m} \frac{d \tilde{V}_1}{ds} - \frac{2r}{m} \tilde{V}_1 = -\frac{\rho \beta}{\alpha} \left( 2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3} \right),$$

with the domain for the $s$ variable being $(-\infty, \log(x_0)]$.

Define the following functions,

$$g(s) = \frac{2 (\eta (\bar{x} - \exp(s)) - 1)}{m},$$

$$h(s) = -\frac{2r}{m},$$

$$f(x) = -\frac{\rho \beta}{\alpha} \left( 2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3} \right),$$

$$\bar{f}(s) = f(\exp(s)).$$

It is clear that the functions $g(s)$ and $h(s)$ are bounded in the $s$ domain and $h(s) < 0$. It can be verified that the term $2x^2 \frac{d^2 V_0}{dx^2} + x^3 \frac{d^3 V_0}{dx^3}$ is always positive from equation (5.34), by noting that all the other relevant parameters and the confluent hyper-geometric function are positive. Hence the function $f(x)$ is positive (negative) whenever $\rho$ is negative (positive). Furthermore $\bar{f}(x)$ is bounded and thus, after transforming to the $s$ variable, $\bar{f}(s)$ must too be bounded and positive (negative) whenever $\rho$ is negative (positive).

Assuming that $\rho$ is negative, an upper bound for $\tilde{V}_1$ can be deduced by using an appropriate type of Maximum Principle for ODE’s as for example can be found in Section 1.5 of Protter and Weinberger [65]. In particular, using Protter and Weinberger’s notation, by choosing $z_1(s) = 0$ one obtains $\tilde{V}_1 \leq 0$. Similarly, assuming $\rho$ is positive a lower bound can be found by choosing $z_2(s) = 0$, and that results in $\tilde{V}_1 \geq 0$. This result shows that a negative (positive) $\rho$ will result in a lower (higher) real option price than the constant volatility solution$^6$.

$^5$It may be possible that the results to follow may also be true when this inequality is not satisfied as well. However this cannot be formally proven as yet. Furthermore, there is some intuition behind assuming that this inequality is satisfied. The expression $\eta \bar{x}$ represents the growth rate of the project at $X_t = 0$, when the local dynamic is effectively deterministic. The assumption it thus, at this state, the growth rate of the project cannot exceed the risk-free interest rate, which is a realistic assumption from an arbitrage point of view.

$^6$A similar approach has been used in Carr, Ewald and Xiao [13], to determine the sign of certain Greek’s of Asian Options.
5.3 General Notes on the Asymptotic Solutions

Threshold Expansion

The threshold expansion cannot be calculated analytically due to the lack of a closed form solution for the option’s value. However, qualitative relationships between the expansion terms and the CIR model parameters are available. As in Section 5.1, the zero order term $x_0$ coincides with the constant volatility, set at $\sqrt{m}$, threshold, which occurs in Dixit and Pindyck [21]. The correction term can again be determined using the boundary condition,

$$x_1 = -\frac{dV_1}{dx}\bigg|_{x_0} \left/ \frac{d^2V_0}{dx^2}\bigg|_{x_0} \right..$$

The analysis of Section 5.2.2, allows the relationship between the sign of $\rho$ and the sign of $x_1$ to be determined. Again, assume the case $r - \eta \bar{x} > 0$, i.e. $\theta > 1$. It is then easy to check that, $d^2V_0/dx^2 > 0$. Assuming the case $\rho < 0$, the bounds on $\tilde{V}_1$, shows that $\tilde{V}_1 \leq 0$, and hence $V_1 \leq 0$. Since $V_1$ at $x_0$ is zero, the bounds show that the left hand derivative of $V_1$ at $x_0$ must be greater than or equal to zero, thus $x_1 \leq 0$. On the other hand, if $\rho > 0$, then one can deduce that $x_1 \geq 0$.

5.2.3 Asymptotic Solution

As in Section 5.1, the asymptotic solution will contain the first two terms in the expansion. The constant $A$ and free boundary $x_0$, in $V_0$ will need to be determined numerically, while the solution $\tilde{V}_1$ is obtained by numerically solving the ODE in equation (5.33). Without a closed form solution for $\tilde{V}_1$, the correction term in the threshold expansion cannot be calculated analytically. However, it is possible to obtain the asymptotic threshold numerically.

In Ewald and Wang [26], the real option problem under the assumption that the project’s value follows a CIR process. The solution, like Dixit and Pindyck GMR problem, is in terms of confluent hypergeometric functions. The results obtained in this section, can easily be modified to cover the case of a Heston-CIR model, but it is not done here.

5.3 General Notes on the Asymptotic Solutions

In this section, a brief overview of some of the issues associated with the asymptotic expansion of both the solution and threshold is provided. It further provides a means of interpreting the analytical results obtained above.

\footnote{Closed form solutions for the correction terms in a Heston-CIR model, like the Heston-GMR case, is not easily obtainable, but can be solved numerically.}
5.3 General Notes on the Asymptotic Solutions

5.3.1 The CIR Process

As in Chapter 3, the CIR process is used to drive the stochastic volatility model. The original work of Fouque et al. [32] in deriving asymptotic solutions under a fast mean reverting stochastic volatility model is based on the OU process. In Chapter 3 and more recently in Fouque et al. [33], it was shown that the technique is in fact valid, under certain conditions, inequality in (3.7) for example, when using the CIR process.

Many of the technical calculations in deriving the asymptotic solutions, for example in Sections 5.1.2 and 5.2.2 have been omitted since they follow closely to that of Section 3.1. The main differences are the absence of a time dependent partial derivative, and the need to calculate a threshold point. The former has very little bearing on the actual derivation, while the latter has been covered at great length.

5.3.2 The Asymptotic Solution

Due to the free boundary nature of the problem, the asymptotic solution obtained in the previous sections expands the real option’s value and the threshold point. The first term in both expansions is given by the classical constant volatility solution, with $\sqrt{m}$ playing the constant volatility role. By adding stochastic volatility, it is possible to use the asymptotic solutions to determine how the main parameters of the CIR model, mean reversion rate and correlation between the project’s value and the variance process, affects the second term in both the expansions. In particular, the relationship between these parameters and the real options price and threshold point, can be better used to understand their effects in the physical world.

Whilst for the Heston call option, a four term asymptotic solution is derived, only the two term asymptotic solution is derived in this chapter for the real option problem. This is mainly due to existence of the optimal threshold, which induces additional boundary conditions. In more detail, suppose the boundary condition in equation (5.13), is extended to the next term using Taylor’s expansion. This results in,

$$
V_0(x_0(y), y) + \sqrt{\epsilon} \left( x_1(y) \frac{\partial V_0}{\partial x} \bigg|_{x_0} + V_1(x_0(y), y) \right) + \epsilon \left( \frac{\partial V_0}{\partial x} \bigg|_{x_0} x_2(y) + \frac{\partial^2 V_0}{\partial x^2} \bigg|_{x_0} \frac{x_1}{2} + \frac{\partial V_1}{\partial x} \bigg|_{x_0} x_1 + V_2(x_0, y) \right) = x_0 + \sqrt{\epsilon} x_1 + \epsilon x_2(y) - k.
$$

By equating the $\epsilon$ term on either sides of the equation, it can be shown that,

$$
V_2(x_0, y) = -\frac{\partial^2 V_0}{\partial x^2} \bigg|_{x_0} \frac{x_1^2}{2} - \frac{\partial V_1}{\partial x} \bigg|_{x_0} x_1,
$$

(5.35)
by noting that $\partial V_0 / \partial x$ at $x_0$ is 1. On the other hand, $V_2$ would have a functional form much like that of equation (3.18), but with $C(x)$ a function independent of $y^8$. This would be written as,

$$V_2(x, y) = \frac{1}{2} (y - m) x^2 \frac{d^2 V_0}{dx^2} + C(x). \quad (5.36)$$

Much like $V_1$, the function $C(x)$ would satisfy the Dixit and Pindyck real options ODE (for either the GBM or GMR case), with source terms that are independent of $y$, analogous to the PDE $C(x,t)$ satisfies in Section 3.1.5. Since this ODE is of order 2, there are two constants of integration to be accounted for by using the boundary conditions at $x = 0$ and $x = x_0$.

Given that the functional forms of $V_0$, $V_1$, $x_0$ and $x_1$ are independent of $y$, it is clear that the boundary condition in (5.35) is independent of $y$. On the other hand, in equation (5.36), at $x = x_0$, the quantity $x^2 \frac{d^2 V_0}{dx^2}$ is non zero, meaning the first term has a dependency on $y$. For $V_2(x_0, y)$ to be independent of $y$, requires the constants of integration of $C(x)$ to contain terms involving $y$, such that it negates the $y$ dependency from the first term. However, this is a contradiction as $C(x)$ is a function strictly independent of $y$, by design. As such, the third term of the expansion cannot be made to consistently satisfy the known constraints.

The Heston-GBM model admits closed form asymptotic solutions for both option value and threshold. From the expressions for $\tilde{V}_1$ and $x_1$ it is evident that both decrease as the mean reversion rate $\alpha$ increases. The situation where $\alpha$ is very large corresponds to having the variance process revert back to its mean level $m$ very rapidly. Thus, the mean reverting level $m$ can be used as a proxy for the variance, (square of the volatility) in a constant volatility setting.

In the case of the Heston-GMR model, no closed form expressions for the correction terms in the asymptotic expansions are available. However, following the intuition of the arguments for the Heston-GBM model, it is expected that there will only be a small contribution from the correction terms in the expansions, when the mean reversion rate is high. These contributions will get smaller and smaller as the mean reverting level increases. This is confirmed by our numerical analysis to follow.

The other parameter of interest is $\rho$, or more importantly the sign of $\rho$, and how it affects the threshold. In case of the Heston-GBM model, its effect can be derived directly from the closed form solution for $x_1$. For the Heston-GMR model, the analysis on the bounds of $\tilde{V}_1$ leads to the same result, in a qualitative sense, to the Heston-GBM model. Explicitly, when $\rho$ is negative

---

* Naturally, $C(x)$ will also be independent of $t$ given the real option problem is independent of time, unlike the Heston call option problem.
5.3 General Notes on the Asymptotic Solutions

(positive), $x_1$ is negative (positive), thus the two term asymptotic expansion of the threshold is lower (higher) than the one term classical (constant volatility) threshold. Intuitively, this can be explained as follows: First, suppose that $\rho$ is negative. As the project’s value increases to the threshold (which is always greater than its current value, otherwise one would have already exercised the option), the variance process decreases. Less variance in the project’s value means that there is less uncertainty in the project’s value, thus one would not insist on as high a threshold as one would under a constant volatility. Similar arguments can be applied in the case of a positive $\rho$.

Finally, in the case that $\rho = 0$, the correction term for the threshold point is zero and the two term expansion coincides with the classical solution under constant volatility. Even for $\rho \neq 0$, under realistic parameters choices, see numerical section to follow, $x_1$ is small in magnitude, so the results can also be interpreted in a way that even under stochastic volatility, the classical solution provides a rather good choice for the threshold point. On the other side, taking $x_1$ into account will lead to a slightly better choice and at least in the case of Heston-GBM, it is not more difficult to compute.

However, it is not immediately clear if the asymptotic solution is superior to the constant volatility solution, and if so, by what means. In the numerical section to follow, it is indicated that the asymptotic solution is in fact the solution to a modified problem, one where the investment threshold is assumed to be constant, rather than one that varies with the initial variance level. Under this new problem, the asymptotic threshold is shown to be a superior boundary point to the constant volatility threshold. Using this observation, it is easier to understand which problem the asymptotic solution solves. This contribution is new, and the numerics has yet be considered in literature.

5.3.3 Consolidation with Published Literature

As stated, both Zhu and Chen [87], and Souza and Zubelli [72], have used this asymptotic technique in pricing perpetual put and call options, respectively. Although the material in this chapter was developed independently to those authors, a discussion on how the works differ and as a result what new insight has been provided, is presented below.

Perpetual Put Options

Zhu and Chen [87], has derived solutions to the perpetual put option problem using a similar asymptotic technique to the one used in this chapter. While their results are for perpetual
5.3 General Notes on the Asymptotic Solutions

put options, inherently they are similar, as the real option problem is essentially a perpetual call option. One small point of difference is that their applications is to the model where the variance is driven by the OU process. In particular, if the asymptotic technique is applied to the Heston-GBM model for American perpetual put options, then the asymptotic solution for the option price is given by,

\[ \bar{V}(x, y) \approx \begin{cases} 
B x^{\beta_2} - \frac{B \beta_2 (2 \beta_2 - 1)}{(\beta_2 - \beta_1) \alpha} \log \left( \frac{x}{\bar{x}_0} \right) x^{\beta_2}, & \text{if } x > \bar{x}_0, \\
\frac{k - x}{x_0^{\beta_2}}, & \text{if } x < \bar{x}_0,
\end{cases} \]

with constants,

\[ \bar{x}_0 = \frac{\beta_2}{\beta_2 - 1} k, \]
\[ B = \frac{k - \bar{x}_0}{x_0^{\beta_2}}, \]

and others as defined in Section 5.1.1. Here, \( \bar{x}_0 \) is the constant volatility threshold. The asymptotic expansion for the optimal threshold boundary is,

\[ \bar{x}_{fb} \approx \bar{x}_0 - \frac{\rho \beta_2 \bar{x}_0}{(\beta_1 - \beta_2) \alpha}. \]

It is of interest to note that if \( \rho < 0 \) (\( \rho > 0 \)), then the asymptotic solution for the option price is greater (less) than the constant volatility equivalent. Furthermore, if \( \rho < 0 \) (\( \rho > 0 \)), then the two term asymptotic threshold is less (greater) than the constant volatility threshold. These facts coincide with the observations made by Zhu and Chen [87], which stated that “when \( \rho < 0 \), the presence of the volatility tends to add value to a perpetual put option, and to postpone its early exercise time, had the underlying prices been assumed to be falling”. Thus, the asymptotic solutions presented in this chapter is consistent with their findings.

Given the availability of the Zhu and Chen results, the question of whether the asymptotic solution for the real option (perpetual call option) problem under stochastic volatility, is of any value. It is noted that under the Heston model, a put-call parity relationship exists between European call and put options. In a working paper by Barone [6], a put-call parity type relationship is derived for perpetual American put and call options under the BS model. Specifically,

\[ C - P = (X_u - k) \mathcal{F}(X_0; X_u, \infty) - (k - X_d) \mathcal{F}(X_0; X_d, \infty), \]

where \( C \) and \( P \) are the prices for the perpetual call and put options, respectively, \( X_u \) and \( X_d \) their optimal thresholds, respectively, and \( \mathcal{F}(X_0; H, \infty) \) is the price of a perpetual binary option for some barrier level \( H \), which has a different formulation depending on whether \( X_0 \geq H \).
5.3 General Notes on the Asymptotic Solutions

or $X_0 \leq H$. However, it must be pointed that this put-call parity relationship is essentially reformulating the solution to a perpetual call and put option in terms of perpetual binary options and scaling them appropriately. Assuming that this relationship holds under a stochastic volatility model, a few questions still needs to be addressed. Firstly, it is not clear as to how one obtains solutions to the perpetual binary options problem under stochastic volatility. Secondly, supposing that these solutions do exist and is known, and that one has the solution to the perpetual call option problem\(^9\), then the optimal threshold for a perpetual call option is still required in order to invoke this put-call parity relationship. Thus, unlike the put-call parity relationship for vanilla European options where knowing the call (put) option price is enough to price the put (call) option, in the perpetual framework, this is not sufficient. As such, the results presented in this chapter for the real option problem is definitely of value.

Perpetual Real Options

Souza and Zubelli\(^72\), have derived asymptotic solutions to the perpetual real option problem using the same techniques as in this chapter. While their initial focus was the non perpetual real options, i.e. American call option, they also extended the work to the perpetual case. Although the material here was developed independently of their studies, it is thus natural to ask, what has been done in this chapter that is different and most importantly, what new insight has this provided?

Firstly, like the work of Zhu and Chen, the Souza and Zubelli problem uses the OU process to drive the variance. Furthermore, the underlying asset’s value is modelled as a GBM type process but with stochastic volatility. This differs slightly from the material in this chapter, as the focus here is on using the CIR process for the variance and both GBM and GMR style processes with stochastic volatility for the underlier. In particular, while both are able to derive closed form solutions to the real option’s price under GBM style models, in this chapter, some qualitative results are also derived for the Heston-GMR model.

Secondly, a main point of difference is the final formulation of the asymptotic solution. As pointed out in Section 3.1.7, many papers write the asymptotic solution in terms of parameters which lump the stochastic volatility model’s parameters together. Souza and Zubelli also do this and are thus unable to determine some of the critical relationships the parameters have on the asymptotic solution. Furthermore, only the first two terms in the expansion for the real option’s price are derived, and not the optimal investment threshold. This makes their investigation

\(^9\)Or an approximation like in Zhu and Chen\(^87\).
5.4 Numerical Results

severely lacking when looking at the problem from this point of view. As a consequence of the above, they are unable to derive the relationship between the sign of $\rho$ and the asymptotic solutions for both the price and the optimal investment threshold, which is an important result.

Thirdly, while neither solutions are able to go beyond the second term in the expansions, only the material in this chapter sufficiently explains why the current methodology is not adaptable to higher order terms. Given that the focus of the article was for the non perpetual case, the article lacks any in depth discussions and numerical results for the perpetual case. Lastly, although there are some typographical errors in their proof\textsuperscript{10}, it seems that this did not impact the final result.

In all, the materials presented in this chapter provides valuable contributions to understanding the perpetual real option problem, in addition to what has been derived by Souza and Zubelli.

5.4 Numerical Results

Unlike classical option pricing frameworks, where the main focus is to determine a fair option price, in real option theory, the optimal threshold point is of more interest as it determines when to partake in the project. This numerical section investigates the use of the asymptotic threshold, as compared to the constant volatility threshold and the true optimal threshold. The true optimal threshold will be computed numerically, with more details found in Appendix A.1.

This section will be divided into three parts; The first, considers the asymptotic threshold with $\rho = 0^{11}$ and compares it to the true optimal threshold. This is in fact, the original problem of having a threshold varying with $y$. The second, considers the modified problem of having a constant threshold assumption, and compares the asymptotic threshold to the exact optimal constant volatility threshold for non-zero $\rho$. The third part is a sensitivity analysis of the relationship between the mean reverting rate, and the sign of $\rho$ using the asymptotic threshold, under the modified problem.

5.4.1 Original Problem

In this section, the Markov chain approximation (MCA) developed by Kushner and Dupuis [54], is applied to solve the original problem given in equation (5.2) with boundary conditions in

\textsuperscript{10}A factor of $x^2$ for the second derivative term in both the ODE for the first and second order term of the option’s price and the boundary conditions with appropriate subscripts are missing.

\textsuperscript{11}In this case the asymptotic threshold and constant volatility threshold coincides.
5.4 Numerical Results

The idea behind the MCA is to approximate the continuous stochastic optimization problem by some discrete stochastic optimization problems. In other words, the PDE in equation (5.2) for the continuous problem is replaced by some Bellman equations to which algorithms have been developed to solve the problem numerically. Further details of the numerical scheme can be found in the Appendix A.1.

The goal will be to evaluate the optimal investment threshold and compare this to the asymptotic threshold. In this example, the case where $\rho = 0$ is only considered, and thus the asymptotic threshold is equal to the Dixit and Pindyck constant volatility threshold. The reason for this is not only to show that the asymptotic threshold is a good approximation to the true optimal threshold, but the Dixit and Pindyck threshold is as well, under the zero correlation assumption. Furthermore, the scheme described in Appendix A.1, is slow converging and requires certain conditions on the parameters. In particular, for non-zero $\rho$, the state space for $x$ is restricted and may in fact not contain the true optimal threshold.

The parameters considered in this example will be given in the Table 5.1. The $\alpha$ parameter will vary from 0.5 to 4, in increments of 0.5.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\beta$</th>
<th>$r$</th>
<th>$\kappa$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>0.08</td>
<td>0.04</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 5.1: Real Option Parameters for the Heston-GBM; Original Problem

Upon extraction of $x^*_{fb}(y; \alpha)$\(^{12}\) from the MCA solution, a Monte Carlo simulation is then applied to simulate a range of starting initial variance values. Due to the time independent nature of the problem, it can be assumed that the starting distribution for $y$, can be approximated by its long run (invariant) distribution. Ideally, the simulation should be performed by sampling from a distribution of the starting values $y$, however this is not possible. The invariant distribution of the CIR process $Y$, is a Gamma distribution with shape and scale parameters given by $k = 2\alpha m/\beta^2$ and $\theta = \beta^2/2\alpha$, respectively, see Section 2.2.1. Thus, it is then possible to compute,

$$
\mathbb{E}^Y(x^*_{fb}(Y; \alpha)),
$$

where the expectation is over the Gamma distribution. The expectation can be thought of as the mean optimal threshold, and is calculated from 500,000 samples of $Y$ for each $\alpha$. The

\(^{12}\)The numerical solution for $x^*_{fb}(y; \alpha)$. Note that the mean reversion rate $\alpha$ of the volatility process $Y_t$ is included in the notation to emphasize the dependency of the optimal investment threshold on this parameter.
reason for using this metric over $x^*_fb(y)$ itself is that the constant optimal threshold will always be inferior to a boundary that varies with $y$, especially when $y$ is far from $m$. Only when $y \approx m$, does the constant thresholds become a good approximation. Furthermore, due to the time independent nature of the problem and the fast mean reverting assumption, the use of the Gamma distribution as a proxy for the starting variance distribution does not seem too farfetched.

The numerical result of this test is shown in Figure 5.1. It can be seen that the asymptotic threshold is not far from the expectations. It can concluded that the asymptotic free boundary is reliable, at least in the case of $\rho = 0$. When the same parameter set is used for $\alpha = 5, 10, 25$ and 50, the resulting expectations in these four cases are all around 1.5479, which may imply that the expectation is convergent as $\alpha$ tends to infinity.

A comparison between the exact free boundary with the asymptotic one is now given. In Figure 5.2, it is found that:

1. The exact free boundary, $x^*_fb(y; \alpha)$, is concave for each $\alpha$.

2. Given $\alpha_1 < \alpha_2$,

   \[
   x^*_fb(y; \alpha_1) < x^*_fb(y; \alpha_2), \text{ if } y \text{ is sufficiently small,} \\
   x^*_fb(y; \alpha_1) > x^*_fb(y; \alpha_2), \text{ otherwise.}
   \]

Figure 5.1: $E^Y(x_{fb}(y))$ for different $\alpha$ and the asymptotic free boundary - The expected value of the free boundary lies roughly near the asymptotic free boundary.
3. \( x_{fb}^\alpha(y; \alpha) \) tends to the asymptotic free boundary for \( \alpha \to \infty \) and the curves becomes much flatter. This phenomenon supports the convergence of \( \mathbb{E}^Y (x_{fb}^\alpha(Y; \alpha)) \) with respect to \( \alpha \).

4. It also indicates that the asymptotic free boundary is a nice approximation for the exact free boundary, particularly if \( \alpha \) is large or \( y \approx m \).

![Figure 5.2: The numerical exact free boundaries for different \( \alpha \) and the asymptotic free boundary](image)

> The free boundary becomes flatter as \( \alpha \) increases and points to evidence of converging towards the asymptotic free boundary.

Given the analysis, it must be emphasized again that in this case \( \rho \) is zero and hence the Dixit and Pindyck threshold is also a good option to replace the exact free boundary if the value of \( Y_t \) cannot be measured.

### 5.4.2 Modified Problem

This analysis considers the modified problem of having a constant threshold, which is independent of \( y \). This is done because the asymptotic threshold is independent of \( y \) and only uses information from the invariant distribution. These facts indicate that the asymptotic solution is in fact, not an asymptotic solution to the original problem, but one where the optimal threshold point is assumed to be a constant. This point has yet to be brought up in the existing literature. In particular, the modified problem becomes,

\[
\bar{V}(X) = \sup_z \mathbb{E}^Y \left( e^{-r \tau_z} (X_{\tau_z} - K) \right),
\]
with \( \tau_z = \inf\{t|X_t \geq z\} \). Here, \( z \) is the optimal constant threshold point. The optimization problem above is obtained in a more rigorous way by setting,

\[
V(x, y, z) = \mathbb{E}\left( e^{-r \tau_z} (X_{\tau_z} - K) \bigg| X_0 = x, Y_0 = y \right),
\]

and then computing,

\[
\max_z \mathbb{E}^Y (V(x, Y, z)),
\]

where \( Y \) is the invariant distribution of the CIR process.

In order to compute the above, the following methodology is adopted:

1. Discretization of the constant threshold is first required. Using a fixed set of parameters, calculate the asymptotic threshold and the constant volatility threshold. An interval which uses these two points as the end points is then established. Divide the interval such that there are 20 equidistant partitions between the asymptotic threshold and the constant volatility threshold. Extend the interval’s endpoints at both ends by a further 5 units of the partition. In total, there are 31 points to test for the optimal constant threshold.

2. Using each of the 31 threshold test points as the fixed threshold, solve equation (5.2), in the \( x < x_{th}(y) \) region, with \( x_{th}(y) \) set as the fixed threshold test point. To do this, use the projection approach to solve the PDE numerically, which is outlined in Appendix A.2. Each of the 31 solution surfaces will be denoted as \( V^i(x, y) \), where \( i = 1, \ldots, 31 \).

3. For each starting value of \( x \), a MC simulation is performed to calculate,

\[
\mathbb{E}^Y (V^i(x, Y)).
\]

Note that at each fixed \( x \), the same generated samples are used for each \( i \). To conduct the MC simulation, first note that the invariant distribution of the CIR process, see Section 5.4.1 or 2.2.1. Simulate 100000 samples of \( Y \), and then compute,

\[
\max_i \left( \mathbb{E}^Y (V^i(x, Y)) \right).
\]

4. For each \( x \), it is possible to obtain determine which of the constant threshold boundary values produced the maximal real option price, and thus should be closest to the true optimal constant threshold.
5.4 Numerical Results

The first two tests will focus on the Heston-GBM model using the parameters listed in Table 5.2, with $\rho$ set to -0.569 and 0.569. These parameters are chosen as typical values which satisfy the fast mean reverting assumption and the inequality in (3.7) which allows the use of the asymptotic expansion technique.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$m$</th>
<th>$\beta$</th>
<th>$r$</th>
<th>$\kappa$</th>
<th>$K$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.75</td>
<td>0.04</td>
<td>0.1</td>
<td>0.08</td>
<td>0.04</td>
<td>1</td>
<td>[0.5, 2]</td>
</tr>
</tbody>
</table>

Table 5.2: Real Option Parameters for the Heston-GBM; Modified Problem

The asymptotic threshold for the negative and positive value of $\rho$ are 2.7682 and 2.7927, respectively, while the constant volatility threshold is 2.7808. Figure 5.3 and 5.4 show the results of the analysis for negative and positive $\rho$, respectively. Firstly, it is evident from these figures that the optimal constant threshold is in fact closer to the asymptotic threshold than the constant volatility threshold. Secondly, for many starting values of $x$, the asymptotic threshold is in fact equal to the optimal constant threshold.

![Figure 5.3: Optimal Constant Threshold for Heston-GBM and Negative Correlation](image)
- The asymptotic threshold is close to the optimal constant threshold for many starting values of $x$.

The second pair of tests will be for the Heston-GMR model using the parameters listed in Table 5.3, with $\rho$ set to -0.569 and 0.569. Again, the parameters are chosen for their fast mean reverting properties along with satisfying the inequality in (3.7).
Figure 5.4: Optimal Constant Threshold for Heston-GBM and Positive Correlation

- The results are similar to negative $\rho$, but in this case the asymptotic threshold is equal to the optimal constant threshold for many more starting values of $x$.

Table 5.3: Real Option Parameters for the Heston-GMR; Modified Problem

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$m$</th>
<th>$\beta$</th>
<th>$r$</th>
<th>$\eta$</th>
<th>$\bar{x}$</th>
<th>$K$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.75</td>
<td>0.04</td>
<td>0.2</td>
<td>0.08</td>
<td>0.05</td>
<td>1.5</td>
<td>1</td>
<td>[0.2,1.5]</td>
</tr>
</tbody>
</table>
5.4 Numerical Results

In the Heston-GMR tests, the asymptotic threshold for the negative and positive value of $\rho$ is 1.69194 and 1.75538, respectively, while the constant volatility threshold is 1.72366. Figure 5.5 and 5.6 show the results of the analysis for negative and positive $\rho$, respectively. Much like the Heston-GBM case, the optimal constant threshold is closer to the asymptotic threshold than the constant volatility threshold. However, the accuracy is not as good as the Heston-GBM model. The reason for this may be due to the correction term for the Heston-GMR being solved numerically. This may lead to possible inaccuracies in the calculation of the asymptotic threshold, but nevertheless, it appears to be an improvement over the constant volatility threshold.

![Threshold Comparison](image)

**Figure 5.5: Optimal Constant Threshold for Heston-GMR and Negative Correlation**
- Optimal constant threshold chosen is closer to the asymptotic threshold for many starting values of $x$.

In all 4 examples, as the starting value of $x$ increases, the asymptotic threshold approximation becomes worse. This is explained by the fact that as you get closer to the true optimal constant threshold, the closer you are to exercising the option. Thus, in the same spirit as for a European call option, when the time to expiry is very small, there is less time for the mean reverting properties of the CIR process to take effect. This same rationale is used by Fouque et al. [32], in their asymptotic calculations for an American put option. Ultimately, this results in a poorer approximation close to optimal constant threshold.
5.4 Numerical Results

Figure 5.6: Optimal Constant Threshold for Heston-GMR and Positive Correlation
- The results are similar to negative $\rho$ in that the asymptotic threshold is closer to the optimal constant threshold.

The results of the modified problem show that the asymptotic threshold is an improvement on the constant volatility threshold, when one assumes that the threshold is a constant value. For the Heston-GBM model, the asymptotic threshold is very close to the true optimal constant threshold, whereas for the Heston-GMR case, there are still some errors associated with it.

5.4.3 Sensitivity Analysis

This subsection investigates the effects on the two term threshold expansion, when the mean reversion rate $\alpha$, is varied over a specified range, while also using a positive and a negative value for $\rho$. This will confirm the effects derived analytically in the previous sections.

Figure 5.7 demonstrates the effect of varying $\alpha$ over the range from 2 to 53 using the Heston-GMR model. The parameters are the same as in the Modified Problem examples, with the blue (solid) line representing the case of a negative $\rho$, while the red (dashed) line represents the case of a positive $\rho$. The black (horizontal) line shows the one term, constant volatility threshold.

From this figure, it is evident that as $\alpha$ increases, the two term threshold approaches the one term threshold. This is exactly what was hypothesized earlier through reasoning with the Heston-GBM model. The sign of $\rho$ also confirms what was analytically derived earlier in that a
negative (positive) correlation, results in a lower (higher) threshold than the constant volatility threshold.

Figure 5.8 demonstrates the same relationship, but for the Heston-GBM model. Qualitatively, the results are the same as the Heston-GMR model, but this can also be deduced by examining the closed form solution for the asymptotic threshold.

**Figure 5.7: Sensitivity Analysis for the Heston-GMR Model** - As mean reverting rate increases, the asymptotic threshold converges to the constant volatility threshold. The sign of $\rho$ also determines whether the asymptotic threshold is above or below the constant volatility threshold.

### 5.5 Concluding Remarks

In this chapter, asymptotic solutions for real options in the presence of stochastic volatility were derived. The models considered assume that the project’s value and variance process can be modelled as either a Heston-GBM or a Heston-GMR process. Both examples lead to a free boundary problem in two dimensions. The asymptotic solution for the real option’s value is made up of the first two terms of an asymptotic expansion, which are independent of the initial variance levels. In the Heston-GBM model, a closed form expression for the asymptotic solution can be derived, while for the Heston-GMR model, the second term in the asymptotic solution needs to be determined numerically by solving an ODE.

In addition to the asymptotic solution for the real option’s value, an asymptotic expansion for the investment threshold, which is again made up of the first two terms in the expansion,
is derived. The first term of the expansion is simply the classical threshold obtained under a constant volatility assumption, while the second term is dependent on the parameters that characterize the stochastic volatility process. The asymptotic threshold is in fact independent of $y$, and numerical examples indicate that the asymptotic solution, is in fact, the solution to the problem when one assumes the optimal threshold is a constant.

It is shown that under the modified problem, as the mean reverting rate of the CIR process increases, the correction term decreases in magnitude with the two term threshold converging to the one term threshold. The other result, which is arguably of more significance, is that in both the Heston-GBM and Heston-GMR models, a negative (positive) correlation between the project’s value and its variance process, leads to the two term threshold being lower (higher) than the classical one term threshold. The methodology and results used to derive this conclusion is found to be consistent with those used by Zhu and Chen [87], for the perpetual put option problem. This result, which in perspective of the classical literatures, clearly contributes to the understanding of the investment uncertainty relationship in the context of irreversible investments.

Figure 5.8: Sensitivity Analysis for the Heston-GBM Model - Same qualitative results as the Heston-GMR model.
Asian options are options whose payoffs are dependent on the time average of the underlying asset’s value over a specified time interval. The average can be taken in various ways, with the most prominent being either the arithmetic or geometric average. Another issue relating to the specifications of the averaging type, is the frequency of the sampling, which is either taken continuously or discretely. For tractability purposes in this chapter, the type of averaging considered will be the continuously sampled arithmetic average, unless otherwise stated.

Under the Black-Scholes model, simple types of geometric averaged Asian call and put options are well understood with closed form solutions for their pricing available. These can be readily found in undergraduate textbooks such as Wilmott, Howison and Dewynne [80]. One of the main reasons for geometric Asian options being well understood, is the fact that a product of independently distributed log-normal random variables is also log-normally distributed. However, the distribution of the sum of independently distributed log-normal random variables is not so simple, and thus the difficulty in pricing arithmetic Asian options. Some of the most prominent results for pricing arithmetic Asian options are Geman and Yor using the Laplace transform approach [35], Linetsky’s spectral expansion approach [56], Dufresne’s Laguerre Series expansions for Asian options [22], Vecer’s PDE approach involving dimension reduction techniques [76], and Dewynne and Shaw’s method of matched asymptotics [20]. The focus of this chapter will be the pricing of Asian options using an Australian options perspective.

Australian options are like Asian options in that the payoffs are dependent on the average of the underlying asset value over a specified time interval. However, it is also dependent on the
The actual variable of interest is the ratio of the time averaged asset’s value to its final value at expiry. Again, it must be stated that the averaging nature considered in this chapter for Australian options, is also the continuously sampled arithmetic average.

Australian options occur as special types of variable purchase options (VPOs) and have been traded on the Australian Stock Exchange (ASX) since 1992. See Handley [38], [39], for more information about VPOs and their existence on the ASX. Dependent on averages, these options have an attached Asian feature, and are in fact one of the very few examples of Asian like options which are traded on institutionalized option markets. Almost all Asian and Asian like options are traded over-the-counter, which makes empirical analysis of their prices almost impossible. For financial research, the existence of exchange traded Australian options is extremely valuable, as “academic” pricing formulas can then be verified and models calibrated at market prices.

Obviously, Australian options formally differ from Asian options, where the underlying is simply the average, and not a ratio involving to average. It is also unclear, whether the prices of exchange traded Australian options can be used for research on Asian options as well. This chapter will show that Australian options are in fact equivalent to Asian options, after the drift rate is adjusted appropriately under the BS model. Virtually all data available for Australian options, can theoretically be applied to the study of Asian options.

In order to reconcile the differences between Asian and Australian options, this chapter presents equivalency theorems between the two. In the BS case, equivalency theorems for Asian options are not a new concept. Henderson and Wojakowski [43], have investigated the equivalencies between floating and fixed strike options by also considering expressions similar to the ones presented in this chapter. However their use of time reversal arguments for Brownian motion, are not applicable to models with stochastic volatility, whereas the approach undertaken in this chapter is. Furthermore, the consideration of expressions similar to Australian options has also been touched upon by others, Rogers and Shi [66], for example, in pricing Asian options, but have not extended the results to cater for stochastic volatility. The contributions of this chapter will be to show that the links established under a BS model, still exist under a Heston stochastic volatility model.

The main point of difference between this chapter and previously published work is the presentation of the Asian and Australian option equivalency, whilst also focusing on the pricing of Asian options under stochastic volatility. The topic of stochastic volatility in Asian options
6.1 Equivalence of Asian and Australian Options

has been studied to some degree, and in particular the asymptotic expansion method, similar to that of Section 3.1, has been considered by Wong and Chueng [82], and Fouque and Han [29], for geometric and arithmetic averaged Asian options, respectively. Whilst it is also possible to consider the asymptotic expansion under the Asian-Australian equivalence (AAE), this chapter will be mainly focusing on using the equivalence to price Asian options using finite difference methods (FDM) and Monte Carlo simulations.

6.1 Equivalence of Asian and Australian Options

This section presents the equivalency theorems between Asian and Australian options. Two cases are considered, the BS model and the Heston stochastic volatility model. The ideas underpinning the development relating to the Heston stochastic volatility model can be applied to many more general stochastic volatility models.

6.1.1 Black-Scholes Equivalence

Assume, under a risk neutral measure $\mathbb{P}$, that the underlying asset’s value $X_t$, to be averaged satisfies the geometric Brownian motion in equation (2.1). Define the Australian state process as,

$$ AU_t = \frac{\int_0^t X_u \, du - k_1 T}{X_t} - k_2 T X_t. $$

(6.1)

The payoff functions for a general Asian call and a general Australian call option are given by,

$$ \left( \frac{1}{T} \int_0^T X_u \, du - k_1 - k_2 X_T \right)^+, $$

$$ \left( \frac{\int_0^T X_u \, du - k_1 T}{T X_t} - k_2 \right)^+, $$

respectively, for some strike price $k_1$ and $k_2$. Thus, price of a general Asian call option can be computed as,

$$ \text{AsianOption}(r) = e^{-r(T-t)} \mathbb{E}^F \left( \left( \frac{1}{T} \int_0^T X_u \, du - k_1 - k_2 X_T \right)^+ \bigg| \mathcal{F}_t \right), \quad (6.2) $$

$^1$Both articles use a function of the OU process to drive the variance of the asset’s value, whereas Section 3.1 uses the CIR process.
and that the price of a general Australian call option can be computed as,

$$\text{AusOption}(r) = e^{-r(T-t)} \mathbb{E}^\mathbb{P} \left( \left( \frac{AU_T}{T} - k_2 \right) ^+ \bigg| \mathcal{F}_t \right)$$

$$= e^{-r(T-t)} \mathbb{E}^\mathbb{P} \left( \frac{1}{TX_T} \left( \int_0^T X_u du - k_1 T - k_2 T X_T \right) ^+ \bigg| \mathcal{F}_t \right),$$  \hspace{1cm} (6.3)

where for Asian options, the strikes $k_1$ and $k_2$ are the fixed and floating strike parameters, respectively, and for Australian options, it is vice versa. Note that the definition for a floating strike Asian call option in this thesis will be different to other published work. For consistency, the time averaged asset’s value will always be considered as the underlier, and thus the floating strike Asian call option is defined to have payoff given by,

$$\left( \frac{1}{T} \int_0^T X_u du - k_2 X_T \right)^+ .$$

Furthermore, both options are written as a function of $r$, the drift rate in the GBM and that the expectations are taken under the measure $\mathbb{P}$. The reason for writing the options as a function of the drift rate will soon be made clear.

Introduce an equivalent probability measure $\mathbb{Q}$, defined via the Radon-Nikodym derivative,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{X_0 e^{(r-\sigma^2)T}}{X_T} = \exp \left( -\frac{1}{2} \int_0^T \sigma^2 du - \int_0^T \sigma dW_u \right).$$  \hspace{1cm} (6.4)

It follows from the Girsanov theorem that,

$$W_t^\mathbb{Q} = W_t + \sigma t,$$

is a Brownian motion under the measure $\mathbb{Q}$. Given $\sigma$ is a constant, Novikov’s condition is thus satisfied, and the Radon-Nikodym derivative in equation (6.4) is valid. Using Lemma 8.9.2 in Kuo [53], the Australian option price at time $t$ can then be computed as,

$$\text{AusOption}(r) = \frac{e^{-r(T-t)}}{X_0 e^{(r-\sigma^2)T}} \mathbb{E}^\mathbb{Q} \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \left( \frac{1}{T} \int_0^T X_u du - k_1 - k_2 X_T \right)^+ \bigg| \mathcal{F}_t \right)$$

$$= \frac{e^{-r(T-t)}}{X_0 e^{(r-\sigma^2)T}} \mathbb{E}^\mathbb{Q} \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg| \mathcal{F}_t \right) \mathbb{E}^\mathbb{Q} \left( \left( \frac{1}{T} \int_0^T X_u du - k_1 - k_2 X_T \right)^+ \bigg| \mathcal{F}_t \right)$$

$$= \frac{e^{-\left(2r-\frac{1}{2}\sigma^2\right)(T-t)}}{X_t} \mathbb{E}^\mathbb{P} \left( e^{-\sigma W_{t-1}} \right) \mathbb{E}^\mathbb{Q} \left( \left( \frac{1}{T} \int_0^T X_u du - k_1 - k_2 X_T \right)^+ \bigg| \mathcal{F}_t \right).$$
6.1 Equivalence of Asian and Australian Options

\[
\frac{e^{-r(T-t)}}{X_t} e^{-(r-\sigma^2)(T-t)} \mathbb{E}_Q^{\mathcal{F}_t} \left( \left( \frac{1}{T} \int_0^T X_u \, du - k_1 - k_2 X_T \right)^+ \right) \\
= \frac{e^{-r(T-t)}}{X_t} \text{AsianOption}(r - \sigma^2),
\]

(6.5)

where under the measure $Q$, the process $X_t$ satisfies,

\[
dX_t = \left( r - \sigma^2 \right) X_t \, dt + \sigma X_t \, dW_t^Q.
\]

From this, it is quite evident that the price of a general Australian call option can be computed as the price for a general Asian call option.

From equation (6.5), replacing $r$ by $r + \sigma^2$ and rearranging the terms gives,

\[
\text{AsianOption}(r) = X_t e^{(r+\sigma^2)(T-t)} \text{AusOption}(r + \sigma^2),
\]

(6.6)

which shows that the price of a general Asian call option can be computed as the price for a general Australian call option. Alternatively, this can be derived from first principles by introducing an equivalent probability measure, again $Q$, defined via the Radon-Nikodym derivative,

\[
\frac{dQ}{dP} = \frac{X_T}{X_0 e^{rT}}.
\]

Following similar arguments, one can derive the result in equation (6.6). The equivalency theorem shows that when $k_2 = 0$, the fixed strike Asian option is equivalent to a pure floating strike Australian option, while for $k_1 = 0$, the floating strike Asian option is equivalent to a pure fixed strike Australian option. Furthermore, while not done here, the equivalency theorems can be extended to general Asian and Australian put option.

As demonstrated, Asian and Australian options are hence equivalent after the drift rate is adjusted appropriately. This of course is very relevant for the pricing of Australian options, but beyond this, the following sections will show that a lot can be learned about Asian options, from looking at them from this so called “Australian Perspective”.

6.1.2 Stochastic Volatility Equivalence

This section considers the addition of a stochastic volatility model in the equivalency of Asian and Australian options. The Heston model will be the specific functional form of the stochastic volatility model, but it will be evident that the results to be presented may carry over to many more general stochastic volatility models. The following result is based on the generalization of the proceeding section.
6.1 Equivalence of Asian and Australian Options

Assume the asset’s value process $X_t$, and variance process $Y_t$, follow the SDEs outlined in equations (2.7) and (2.8), respectively, under a risk neutral measure $\mathbb{P}$. Furthermore, since $W_t$ and $Z_t$ are Brownian motions with correlation $\rho$, $Z_t$ can be written as,

$$Z_t = \rho W_t + \sqrt{1 - \rho^2} \tilde{Z}_t,$$

where $\tilde{Z}_t$ is a Brownian motion independent of $W_t$.

The solution to the asset’s value process, conditional on the filtration $\mathcal{F}_t$, can be written in integral form as,

$$X_T = X_t \exp \left( r(T - t) - \frac{1}{2} \int_t^T Y_u du + \int_t^T \sqrt{Y_u} dW_u \right),$$

and the price of a general Asian call option with fixed strike $k_1$ and floating strike $k_2$ is given by,

$$\text{AsianOption}(r) = e^{-r(T-t)} \mathbb{E} \left( \left( \frac{1}{T} \int_0^T X_u du - k_1 - k_2 X_T \right)^+ \Bigg| \mathcal{F}_t \right). \quad (6.7)$$

Similarly to Section 6.1.1, define an equivalent measure $\mathbb{Q}$, via the Radon-Nikodym derivative,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{X_T}{X_0 e^{rT}} = \exp \left( -\frac{1}{2} \int_0^T Y_u du + \int_0^T \sqrt{Y_u} dW_u \right).$$

The process defined by,

$$W^Q_t = W_t - \int_0^t \sqrt{Y_u} du,$

is then a Brownian motion under the measure $\mathbb{Q}$, by the Girsanov theorem. Novikov’s condition requires,

$$\mathbb{E}^\mathbb{P} \left( \exp \left( \frac{1}{2} \int_0^T Y_u du \right) \right) < \infty,$

which is true, provided $\alpha > \beta$. The proof of this can be found in Wong and Heyde [81]. From here onwards, the case where $\alpha$ and $\beta$ satisfy the inequality given earlier is only considered. Furthermore, the processes $X_t$ and $Y_t$, under the measure $\mathbb{Q}$, satisfy the following SDEs,

$$dX_t = (r + Y_t) X_t dt + \sqrt{Y_t} X_t dW^Q_t,$$

$$dY_t = \alpha^* (m^* - Y_t) dt + \beta \sqrt{Y_t} \left( \rho dW^Q_t + \sqrt{1 - \rho^2} d\tilde{Z}_t \right), \quad (6.8)$$
where $\alpha^* = \alpha - \beta \rho$ and $m^* = a m/\alpha^*$. Using equation (6.7) and again Lemma 8.9.2 in Kuo [53], the price of an Asian option can be obtained as,

$$AsianOption(r) = X_0 e^{rt} \mathbb{E}^Q \left( \frac{dQ}{dP} \left| _{\mathcal{F}_t} \right. \right) \mathbb{E}^P \left( \int_0^T X_u du - k_1 T - k_2 T X_T \right)^+$$

$$= X_0 e^{rt} \mathbb{E}^Q \left( \frac{dQ}{dP} \left| _{\mathcal{F}_t} \right. \right) \mathbb{E}^Q \left( \frac{1}{TX_T} \left( \int_0^T X_u du - k_1 T - k_2 T X_T \right)^+ \right| $$

$$= X_t \mathbb{E}^Q \left( \frac{1}{TX_T} \left( \int_0^T X_u du - k_1 T - k_2 T X_T \right)^+ \right| $$

where the equality,

$$X_0 e^{rt} \mathbb{E}^Q \left( \frac{dQ}{dP} \left| _{\mathcal{F}_t} \right. \right) = X_t,$$

is used. The proof of this equality is as follows,

$$\mathbb{E}^P \left( \frac{dQ}{dP} \left| _{\mathcal{F}_t} \right. \right) = \mathbb{E}^P \left( \frac{X_T}{X_0 e^{rT}} \left| _{\mathcal{F}_t} \right. \right)$$

$$= \frac{X_t e^{r(T-t)}}{X_0 e^{rT}} \mathbb{E}^P \left( \exp \left( -\frac{1}{2} \int_t^T Y_u du + \int_t^T \sqrt{Y_u} dW_u \right) \right)$$

$$= \frac{X_t e^{-rt}}{X_0}.$$

From this form of the solution, it can be seen that the expectation part of equation (6.9), is related to the expectation part of an Australian option under the measure $Q$, with stochastic volatility. In Section 6.2, it will be shown how it is possible to use this result in the pricing of Asian options with stochastic volatility.

### 6.2 Asian Options with Stochastic Volatility

This section shows how Australian option pricing techniques can be used to price Asian options under stochastic volatility. The methods to be presented are those of solving an Australian option related PDE through FDM and using MC simulations of a process related to the Australian state process. Throughout, assume the form for the Asian option model will be that of Section 6.1.2.
6.2 Asian Options with Stochastic Volatility

6.2.1 Derivation

Begin by considering equation (6.9), and in particular the expectation,

\[ \mathbb{E}^Q \left( \frac{1}{X_T} \left( \int_0^T X_u \, du - k_1 T - k_2 T X_T \right)^+ \right| \mathcal{F}_t \). \tag{6.10} \]

Define the Australian state process, as in equation (6.1), but noting that \( X_t \) is now from the Heston-GBM model. Using stochastic partial integration and the Itô formula, it can be shown that the process \( AU_t \), satisfies the SDE,

\[
dAU_t = d \left( \int_0^t X_u \, du - k_1 T \right) \frac{1}{X_t} + d \left( \int_0^t X_u \, du - k_1 T \right) \left( \frac{1}{X_t^2} dX_t + \frac{2}{X_t^3} [X, X]_t \right) \]
\[
= dt - 0 + \left( \int_0^t X_u \, du - k_1 T \right) \frac{1}{X_t} \left( - \frac{1}{X_t} dX_t + \frac{2}{X_t^2} Y_t X_t^2 dt \right)
\]
\[
= dt + AU_t \left( - (r + Y_t) dt - \sqrt{Y_t} dW_t^Q + Y_t dt \right) \]
\[
= dt - r AU_t dt - \sqrt{Y_t} AU_t dW_t^Q, \tag{6.11} \]

The derivation makes use of,

\[ d \left( \frac{1}{X_t} \right) = - \frac{1}{X_t^2} dX_t + \frac{2}{X_t^3} [X, X]_t, \tag{6.12} \]

where \([X, X]_t\) is the quadratic variation, and that \([X, X]_t = dX_t dX_t\). Equation (6.12) can be shown to hold by considering the Itô formula for \( f(t, X_t) = 1/X_t \), and then deriving the SDE for \( df(t, X_t) \).

Equation (6.10) can then be defined as some function \( U(t, x, y) \), and written as,

\[ \mathbb{E}^Q \left( (AU_T - k_2 T)^+ \right| \mathcal{F}_t \) = \mathbb{E}^Q \left( (AU_T - k_2 T)^+ \right| AU_t = x, Y_t = y \] =: U(t, x, y). \]

Using the two dimensional form of the Feynman-Kac theorem, \( U(t, x, y) \) satisfies the following PDE,

\[
\frac{\partial U}{\partial t} + \frac{1}{2} x^2 y \frac{\partial^2 U}{\partial x^2} + (1 - xr) \frac{\partial U}{\partial x} + \frac{1}{2} \beta y \frac{\partial^2 U}{\partial y^2} + \alpha^* (m^* - y) \frac{\partial U}{\partial y} - \rho \beta x y \frac{\partial^2 U}{\partial x \partial y} = 0, \tag{6.13} \]

with terminal condition,

\[ U(T, x, y) = (x - k_2 T)^+. \]

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Currently, there is no closed form solution for equation (6.13), and thus in subsequent sections, the PDE will be solved using a FDM. Furthermore, the system of SDEs in equations (6.11) and (6.8) can be used together to produce solutions to the Asian option problem via MC simulations.

### 6.2.2 Finite Difference Method

The PDE in equation (6.13) is a three dimensional PDE with two spatial variables and one time variable. The coefficients of the partial derivatives and the spatial variables are also time independent constants. Alternatively, it is possible to derive a PDE to price Asian options naively. This PDE would have four dimensions, with three spatial variables and one time variable. The first two spatial variables are to accommodate the asset’s value and variance, while the last one is reserved for the running sum of the asset’s value. From this, it is already clear that the PDE derived through the Australian perspective, has an advantage over the naively obtained PDE, namely a dimension reduction.

The dimension reduction in equation (6.13) is not an entirely new concept. Vecer and Xu [77], also presented a non path dependent method, as is the Australian perspective method, for pricing Asian options under a more general semi-martingale model. Their method is based on the dimension reduction results first presented by Vecer [76], and extended to stochastic volatility by Fouque and Han [29]. The variable of interest in this case is again a ratio of the time average asset’s value to the asset’s final value at expiry. However, their ratio also contains some time dependent functions and as such, the associated SDE and the resulting PDE also have time dependent coefficients. The addition of these time dependent coefficients make numerically solving the three dimensional PDE much more difficult.

To solve equation (6.13) using a FDM, boundary conditions around the domain of interest are required. The following equations are the boundary conditions for $x$ and $y$,

\[
U(t, -\infty, y) = 0,
\]

\[
\frac{\partial U(t, \infty, y)}{\partial x} = 1,
\]

\[
U(t, x, \infty) = x,
\]

and

\[
\frac{\partial U(t, x, 0)}{\partial t} + (1 - rx) \frac{\partial U(t, x, 0)}{\partial x} + \alpha^* m^* \frac{\partial U(t, x, 0)}{\partial y} = 0.
\]
The two boundary conditions for $x$ are straightforward, when $x$ approaches infinity, it is reasonable to assume that the option will finish extremely in-the-money such that its partial derivative with respect to $x$ approaches 1. Similarly, when $x$ approaches negative infinity, the option will finish extremely out-of-the-money such that it will be worthless. The two boundary conditions for $y$ are not as straightforward. For the $y = 0$ boundary, it makes intuitive sense to consider the PDE at $y = 0$ as its boundary condition. For when $y$ approaches infinity the boundary is chosen to be the value $x$ (for a call type option). The latter boundary condition is similar to the choice made when solving the Heston model using FDM, see In’t Hout and Foulon [49].

The FDM used to solve equation (6.13) will be that of the alternating direction implicit (ADI) method. ADI methods are methods that reduce multi-dimensional PDEs into a series of one-dimensional steps, which explains the origins of its name. In’t Hout and Foulon [49], covers this topic quite thoroughly for the Heston model, while a thesis by Lin [55], provides working MATLAB codes implementing the ADI scheme for a call option under the Heston model. Given the similarity between the call option PDE and the Asian-Australian option PDE, it is easy to modify the code to solve for the Asian option problem, using the AAE.

6.2.3 Monte Carlo Simulation

Monte Carlo simulations are a good way to obtain option pricing results when closed form solutions of the problem do not exist. For Asian options, two ways of performing the simulation are presented. The first and naive method is to simulate the sample paths of the asset’s value and variance process (under the measure $\mathbb{P}$), while the alternative is the AAE method, which is to simulate the sample paths of the Australian state process and variance process (under the measure $\mathbb{Q}$). A discussion on the performance of these simulation methods is provided.

Naive (Classical) Method

The naive method requires the simulation of the asset and variance process in equations (2.7) and (2.8). The easiest MC method to use is the Euler-Maruyama scheme. For this two dimensional problem, the scheme becomes,

$$X_{i+1} = X_i + rX_i \Delta t + X_i \sqrt{Y_i \Delta t} W_i,$$

$$Y_{i+1} = Y_i + \alpha (m - Y_i) \Delta t + \beta \sqrt{Y_i \Delta t} \left( \sqrt{1 - \rho^2} Z_i + \rho W_i \right),$$

---

2It is possible for $x$ to be negative if $k_1 > \frac{1}{\mu} \int_0^1 X_u \, du$. 

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where $X_i$ and $Y_i$ are the values simulated at timestep $i$, $\Delta t$ is the timestep size, $W_i$ and $Z_i$ are independent simulated values from a standard normal distribution.

Due to the path dependent nature of Asian options and the discretization in using MC simulations, the average of the asset’s value process is taken by using the discrete arithmetic average of the simulated asset’s value, as a proxy for the continuously sampled arithmetic average. Having simulated many sample paths, an expectation like that of equation (6.2) is then calculated to price the Asian option.

**Asian-Australian Equivalence Method (AAE Method)**

The Asian-Australian equivalence method requires the simulation of the Australian state process and variance process under the measure $Q$, by using equations (6.11) and (6.8). Again the EuM scheme is used to simulate the required processes. The scheme in this case becomes,

$$AU_{i+1} = AU_i - rAU_i \Delta t + \Delta t - AU_i \sqrt{Y_i} \Delta t W_i,$$

$$Y_{i+1} = Y_i + \alpha^* (m^* - Y_i) \Delta t + \beta \sqrt{Y_i} \Delta t \left( \sqrt{1 - \rho^2} Z_i + \rho W_i \right),$$

where $AU_i$ and $Y_i$ are the values simulated at timestep $i$, and the others as before.

Due to the path independent nature of the AAE, only the terminal value of the Australian state process is required in the calculations of the expectation. Furthermore, again due to its path independence, the simplified Order 2.0 Weak Taylor scheme is also considered.

The general formulation of the simplified Order 2.0 Weak Taylor scheme can be found in Section 14.2 of Kloeden and Platen [52]. For the AAE method, the scheme is given as follows; the pair of SDEs in equations (6.11) and (6.8) can be written in matrix form as,

$$d\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 1 - rX_{1,t} \\ \alpha^* (m^* - X_{2,t}) \end{pmatrix} dt + \begin{pmatrix} -X_{1,t} \sqrt{X_{2,t}} \\ \beta \rho \sqrt{X_{2,t}} \sqrt{1 - \rho^2} \sqrt{X_{2,t}} \end{pmatrix} d\begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix},$$

where $AU_t$ and $Y_t$ are replaced with the notations $X_{1,t}$ and $X_{2,t}$, respectively, and $W_t^Q$ and $Z_t$ replaced by $W_{1,t}$ and $W_{2,t}$, respectively. Define the drift vector and volatility matrix of the pair of SDE as,

$$a = \begin{pmatrix} a_1(x_1) \\ a_2(x_2) \end{pmatrix} = \begin{pmatrix} 1 - r x_1 \\ \alpha^* (m^* - x_2) \end{pmatrix},$$

$$b = \begin{pmatrix} b_{11}(x_1, x_2) & b_{12} \\ b_{21}(x_2) & b_{22}(x_2) \end{pmatrix} = \begin{pmatrix} -x_1 \sqrt{x_2} \\ \beta \rho \sqrt{x_2} \beta \sqrt{1 - \rho^2} \sqrt{x_2} \end{pmatrix},$$

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respectively. Further, define the following operators,

\[ L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{2} a_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l=1}^{2} b_{kl} \frac{\partial^2}{\partial x_k \partial x_l}, \]

\[ L^1 = \sum_{k=1}^{2} b_{k1} \frac{\partial}{\partial x_k}, \]

\[ L^2 = \sum_{k=1}^{2} b_{k2} \frac{\partial}{\partial x_k}. \]

Introduce \( \Delta W^j \), for \( j = 1, 2 \), to be independent three point distributed random variables satisfying,

\[ P(\Delta W^j = -\sqrt{3} \Delta t) = 1/6, \quad P(\Delta W^j = 0) = 2/3, \quad P(\Delta W^j = \sqrt{3} \Delta t) = 1/6, \]

for some time discretization \( \Delta t \). Also define \( V_{1,2} \), to be an independent two point distributed random variable satisfying,

\[ P(V_{1,2} = \pm \Delta t) = 1/2, \]

and have \( V_{2,1} = -V_{1,2} \), while \( V_{1,1} \) and \( V_{2,2} \) equal to \(-\Delta t\).

The simplified Order 2.0 Weak Taylor scheme for the AAE method, is thus given by,

\[ X_{k,i+1} = X_{k,i} + a_k \Delta t + \frac{1}{2} L^0 a_k (\Delta t)^2 + \sum_{j=1}^{2} \left( b_{kj} + \frac{1}{2} \Delta t (L^0 b_{kj} + L^j a_k) \right) \Delta W^j \]

\[ + \frac{1}{2} \sum_{j_1,j_2=1}^{2} L^{j_1} b_{j_1 j_2} (\Delta W^{j_1} \Delta W^{j_2} + V_{j_1,j_2}). \]

for \( k = 1, 2 \). Note that on the left hand side of the equation, \( a_k \), \( b_{kj} \), \( L^j a_k \), and \( L^j b_{kl} \), for all possible subscript combinations, are evaluated at \( x_1 = X_{1,i} \) and \( x_2 = X_{2,i} \). Also, it goes without saying that for each \( i \), \( \Delta W^j \) and \( V_{j_1,j_2} \) are re-simulated.

**Discussion on the Different MC Methods**

In assessing the performance of the different MC methods, a measure of the order of convergence is required. The two types of convergence associated with MC methods are the weak and strong convergence. The definitions for these convergence type is found in Kloeden and Platen [52], and is as follows; Suppose \( X_T \) is the process being approximated by \( X(T) \), and that the timestep are of size \( \Delta t \). \( X(T) \) is said to converge weakly to \( X_T \), with order \( \gamma \), if there exists a positive constant \( C \), independent of \( \Delta t \), such that,

\[ \left| \mathbb{E}(g(X_T)) - \mathbb{E}(g(X(T))) \right| \leq C \Delta t^\gamma, \]
for all functions $g$, in some class. Typically, $g$ must satisfy smoothness and polynomial growth conditions. The approximation $X(T)$ converges strongly to $X_T$ with order $\gamma$, if there exists a positive constant $\bar{C}$, independent of $\Delta t$, such that,

$$
\mathbb{E} \left( \left| X_T - X(T) \right| \right) \leq \bar{C} \Delta t^\gamma.
$$

From the convergence definitions, it is evident that the weak order of convergence mainly deals with distributional properties at time $T$, while the strong order of convergence deals with the pathwise approximation at time $T$. Thus, in calculating path independent expectations, such as using AAE method, the weak order of convergence is of importance, while for path dependent expectations, such as the naive method for Asian options, it is the strong order of convergence.

It is well understood that in one dimensional problems, the EuM scheme has a weak and strong convergence of order 1 and $1/2$, respectively, see Kloeden and Platen for a proof and Higham, for numerical results. Another scheme yet to be discussed is that of Milstein. This scheme has weak and strong convergence of orders 1 and 1, respectively. The simplified Order 2.0 Weak Taylor scheme has a weak convergence of order 2, but because the scheme is only designed for its terminal distributional properties, and not for pathwise approximations, it should not be used for path dependent expectations.

The Milstein scheme is often advocated due when dealing with Asian options, especially for the constant volatility case. This is due to the scheme possessing strong convergence of order 1. However, the disadvantage in using the Milstein scheme in a multi-dimensional problem, as in a stochastic volatility model, is the difficulty in its implementation. The difficulty lies in the computation of a double integral known as the Levy Area, which is related to the existence of multiple Brownian motions in the problem. MATLAB implementations of the Milstein schemes are rather slow, while Higham, also cites the difficulty in applying the Milstein scheme to a system of SDEs. Other commentary by Poklewsiki-Koziell, states that the Milstein scheme does not perform well for the Heston model because the drift and diffusion coefficients are not ‘sufficiently smooth, real-values functions satisfying a linear growth bound’. Furthermore, there is no guarantee that the Milstein strong convergence of order 1 may hold, when drift and diffusion coefficient conditions are not satisfied. Due to this fact, the Milstein scheme will not be considered as an appropriate scheme for pricing Asian options with stochastic volatility. However, a few numerical results using this scheme will be presented.

The EuM scheme, while simple in its implementation, was found by Poklewsiki-Koziell to be quite robust in handling the pricing of Asian options under the Heston model. Applying the
6.2 Asian Options with Stochastic Volatility

EJM scheme to the naive method of pricing Asian options may not necessarily be optimal, due to the path dependent nature of the problem, however it may be worthwhile to do so using the AAE. In using the Australian perspective to price Asian options, the expectation of interest only contains the terminal value of Australian state process, and not its path, thus the main type of convergence of interest is the weak convergence.

Furthermore, because the analysis is essentially testing the viability of using the AAE method, the weak order of convergence is what really matters. As such, the Order 2.0 Weak Taylor scheme will also be considered. The scheme is in essence an extension of the EJM scheme, by considering the double stochastic integrals, in the Ito-Taylor expansion\(^4\), see Kloeden and Platen. The simplified version of the scheme replaces various randomly generated variables with much more simple random variables (three- and two-point distributed random variables), provided that they satisfy some moment conditions, as outlined in Kloeden and Platen. As previously mentioned, the scheme is only designed for its terminal distributional properties, and not for pathwise approximations, and thus the simplified Order 2.0 Weak Taylor scheme is used only for the AAE, and not the naive method.

6.2.4 Numerical Results

This section contains the numerical results obtained from using the methods, as previously listed, in pricing Asian options. The parameters for the Heston model will be similar to set 1 from Section 4.2.2, but with some minor changes. The \(\rho\) parameter is fixed at -0.569 and the strikes \(k_1\) and \(k_2\) are set to 100 and 0, respectively. The parameters are listed below for convenience and are used due to the fact that they have been calibrated from real historical time series data on the S&P 500 index prices, see Eraker\(^25\).

<table>
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<th>(\alpha)</th>
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Table 6.1: Simulation Parameters for Asian Options

ADI Method

The PDE in equation (6.13) is solved using the ADI method together with the boundary conditions from Section 6.2.2. The domain for the spatial variables \(x\) and \(y\) are chosen to be [-3,3] and [0,3], respectively. This domain is chosen because it is large enough for the

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\(^4\)EJM scheme only considers the single stochastic integral terms in the Ito-Taylor expansion.
boundary condition to have negligible effects on solution in the region of interest. An equidistant partitioning of the \( x, y \) and \( t \) variables with 1000 intervals was used. Furthermore, the choice of ADI method used is the Douglas-Rachford (DR) method\(^4\), to which more details can be found in Lin [55].

Using the above parameters and settings, the ADI method produced an Asian option price of 5.852839. This took approximately 17 minutes to compute on a standard home computer\(^5\). In testing the MC methods and their solution, the prices they produce will be ultimately compared to the one obtained from the ADI method.

**MC Methods**

This section provides the numerical results from comparing the aforementioned MC methods to the ADI method. The various tests will include looking at the convergence as a function of number of simulation paths and as a function of number of timesteps used. For convenience, in the proceeding section and figures, the Asian Euler, Aus Euler and the Aus Taylor 2.0 refers to the EuM scheme applied to the naive method, EuM scheme applied to the AAE method and the simplified Order 2.0 Weak Taylor scheme applied to the AAE method, respectively.

The first MC experiment will be to test the rate at which the option price converges to the ADI produced price as a function of the number of paths used. For this test, the number of timesteps is fixed to 1000, while the number of simulation paths used varies. Figures 6.1 and 6.2 show the convergence of the price and the RMSE of those prices as a function of number of simulation paths respectively. The convergence of the price in using the Asian Euler and Aus Euler methods both follow a similar pattern, with the Aus Euler method being closer to the ADI solution with less simulations. The Aus Taylor 2.0 method reaches the ADI solution much quicker than the other two methods, and then hovers near the ADI solution.

The RMSE results as a function of \( n \), the number of simulation paths, is shown in Figure 6.2. The RMSE is calculated by,

\[
RMSE(n) = \sqrt{\frac{\sum_{i=1}^{n} (Price_i - ADIPrice)^2}{n}} \quad (6.14)
\]

where \( Price_i \) is the value obtained by each simulation path \( i \), and \( ADIPrice \) is the price obtained from the ADI method. The result in Figure 6.2 shows that using the AAE method produces simulated paths which when used to compute Asian option prices, are on average closer to the real price than using the naive method.

---

\(^4\)The \( \theta \) parameter that determines the weighting of the implicit and explicit scheme is set to 0.5.

\(^5\)The same PC listed in Chapter 3 was used to perform the calculations.
6.2 Asian Options with Stochastic Volatility

Figure 6.1: Convergence of Price as a Function of the Number of Simulation Paths - The Aus Taylor 2.0 method approaches the ADI solution more rapidly in this simulation run.

Figure 6.2: RMSE as a Function of Number Simulation Paths - The naive method produces a higher RMSE than the Asian-Australian Equivalency method.
For added reference, the Milstein method applied to the naive method produced a price and RMSE of 5.8298 and 7.4649 respectively. This simulation used 100000 paths, with 1000 timesteps each. This took approximately 27 minutes to compute, which is significantly more than even the ADI method. The code for this scheme can be found in Pokleowski-Koziell [63]. The Aus Taylor 2.0 and the Euler solutions, with the same number of simulation paths and timesteps took roughly, 30 and 8 seconds, respectively. It is interesting to note that a simulation with 25000 paths using the Aus Taylor 2.0 method also roughly takes 8 seconds. The computational efficiency of these methods will be revisited at the end of this section.

The second MC experiment considers the rate of convergence as a function of the number of timesteps, while fixing the total number of simulated paths. This test consists of simulating 100000 paths at each number of timesteps, chosen to be 100, 200, . . . , 1000. Figures 6.3 and 6.4 show the convergence of price and the RMSE of this test respectively. From Figure 6.3, it is evident that the Aus Taylor 2.0 method increases in performance, as the number of timesteps per simulation is increased. This result is not observed for the Euler solutions, however it is pointed out that for a low number of timesteps, the Euler solutions perform better than the Aus Taylor 2.0. Figure 6.4 shows the Aus Taylor 2.0 simulations have a lower RMSE associated with each simulation path than the Euler solutions, across all the numbers of timesteps considered. As with the first experiment, the figure also shows that the AAE methods, provide a lower RMSE than the naive method.

So far, the results demonstrate that the AAE method does seem to have advantages over the classical naive method in dealing with MC methods. However, the results thus far have not shown whether it is worthwhile to implement the simplified Order 2.0 Weak Taylor scheme and whether the results are due to particular randomization seeds used in the simulation. To answer this last question, the two previous tests are repeated 100 times, with different seeds at each run. The methodology for testing this is outlined below. The repetition of the first test is carried out as follows;

1. Simulate 100000 paths using 1000 timesteps and calculate the corresponding price.
2. Repeat step 1, 100 times, with new seeds for each run.
3. Using the 100 prices obtained from step 2, calculate the RMSE of the mean prices.
4. Repeat all the above steps with 200000, 300000, . . . , 1000000 paths.

The repetition of the second test is carried out as follows;
6.2 Asian Options with Stochastic Volatility

Figure 6.3: Option Price as a Function of the Number of Timesteps - Timestep size does not influence the convergence to the option price.

Figure 6.4: RMSE as a Function of Number Timesteps - The naive method produces a higher RMSE than the Asian-Australian Equivalency method.
6.2 Asian Options with Stochastic Volatility

1. Simulate 100000 paths using 100 timesteps and calculate the corresponding price.

2. Repeat step 1, 100 times, with new seeds for each run.

3. Using the 100 prices obtained from step 2, calculate the RMSE of the mean prices.

4. Repeat all the above steps with 200, 300, ..., 1000 timesteps.

Figures 6.5 and 6.4 show the results of the two repetition tests. The RMSE calculated in step 3 (of both repetition tests) are done using the formula in equation (6.14), with \( Price_i \) being the mean price calculated from each of the 100 repetitions in step 2\(^6\). From both figures, the EuM scheme applied to both the naive and AAE method shows a very similar result. It is hard to distinguish with certainty which of the two methods is better. In the first repetition test, the Aus Taylor 2.0 method shows a lower RMSE across all the numbers of simulation path considered, but in the second repetition test, only a lower RMSE is observed when the number of timesteps are equal or greater than 400. While the latter observation may seem like the EuM scheme has a better performance in some cases, one must also take into account that the accuracy levels, at those lower number of timesteps, are quite poor. Overall, when taking the simplified Order 2.0 Weak Taylor scheme into account, the results show that the AAE method provides a superior result to that of the naive method.

As previously stated, if the number of timesteps are fixed, and only the number of simulations is varied, then the Aus Taylor 2.0 method increases computational time by approximately a factor of 4 as compared to the EuM schemes. Thus, for a fair comparison in repetition test 1, the computational time required to perform the calculations must also be considered. From Figure 6.5, the results for the 400000 and 100000 simulation paths (for each of the 100 runs) for the Aus Euler and Aus Taylor 2.0 methods, respectively, show that the Aus Taylor 2.0 method produces a slightly lower RMSE than the Euler method. In particular, if one considers the results from the 800000 and 200000 pair, then the Aus Taylor 2.0 method returns an RMSE that is 0.006 less than the Euler methods. Given that the Aus Euler method uses 4 times as many simulation paths, then computational time for both solutions should be roughly the same.

The two repetition tests were then performed on two additional parameter sets to showcase the AAE method in other scenarios. The parameter sets for these two tests are listed in Table 6.2.

\(^6\)Previously, each \( i \) represents a sample path, but now each \( i \) represent each of the 100 repetitions.
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Figure 6.5: Repetition Test 1; Number of Simulation Paths - When the first test is repeated 100 times, the Aus Taylor 2.0 method returns a lower RMSE than the other two methods as the number of simulation paths increase.

Figure 6.6: Repetition Test 2; Number of Timesteps - When the second test is repeated 100 times, the Aus Taylor 2.0 method returns a lower RMSE than the other two methods, when the number of timesteps are more than 400.
6.3 Concluding Remarks

The option price produced from the ADI method for these parameter sets 2 and 3 are 9.323261 and 7.122597, respectively. The results of the repetition tests can be found in Figure 6.7. The two graphs at the top are repetition test 1 and 2, for parameter set 2, while the bottom two are for parameter set 3. The repetition tests show that the AAE method returns a lower RMSE than the naive method across all number of simulation paths and most of the number of timesteps considered. For the first repetition test, the computational efficiency outlined above, in using the Aus Taylor 2.0 method was not observed for parameter set 2. In parameter set 3, if one considers the RMSE from the 200000 and 800000 simulation paths, for the Aus Taylor 2.0 and Aus Euler method, respectively, then the RMSE are quite similar. Nevertheless, if computational time is not of concern, then the Aus Taylor 2.0 method returns a lower RMSE than the EuM methods when using the same number of simulation paths. If computational time is of concern, then there are still advantages in using the Aus Euler method over the Asian Euler method. The results for repetition test 2 are similar to the results obtained earlier.

In all, the numerical results show promise in using the path independent nature of the AAE to price Asian options under stochastic volatility. Whilst the only methods tested using MC simulations are that of the Euler and Weak Taylor 2.0 schemes, it is possible to look at higher order schemes. However, higher order path dependent schemes are not computationally efficient compared to their equivalent order path independent counterparts. Furthermore, easily implementable numerical algorithms such as the ADI method, can be applied to the Australian equivalent PDE with the time independent coefficients to produce solutions for Asian option prices. These advantages make it worthwhile to consider Asian options using the Australian perspective.

6.3 Concluding Remarks

This chapter has investigated the use of Australian options in the pricing of Asian options. It is shown that under the BS model, the pricing of a general Asian option is equivalent to that of pricing a general Australian option under an equivalent probability measure. These

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Table 6.2: Additional Simulation Parameters for Asian Options
Figure 6.7: Repetition Tests for Additional Parameters - The top and bottom graphs are for parameter set 2 and 3 respectively.
6.3 Concluding Remarks

equivalencies extend the work of Henderson and Wojakowski [43], to encompass Australian options. The results presented are similar to Rogers and Shi [66], but have been extended to cater for the Heston stochastic volatility model. It is easy to see that, the method can in fact be generalized to many other stochastic volatility models.

In using the equivalency of these two option types, this chapter also outlined ways of pricing Asian options under the Heston model. Unlike previously derived PDEs such as those found in Vecer and Xu [77], the PDE presented in this chapter has partial derivatives, whose coefficients are time independent, whilst also having the advantage of containing one less dimension than the naive Asian option PDE. Due to these properties, the use of the ADI method to solve an Asian-Australian equivalent PDE is relatively straightforward, and thus the pricing of Asian options under stochastic volatility is quite simple.

Furthermore, MC methods have also been explored in pricing these Asian options. Numerous tests were conducted to compare various MC methods when applied to the naive method and the AAE method of simulating Asian option prices. Due to the path independent nature of the AAE method, it is found that the option prices obtained have a lower RMSE than the naive method. When comparing the simplified Order 2.0 Weak Taylor scheme on the AAE method, and the EuM scheme on the AAE and naive method, at a fixed number of simulations and a reasonably large number of time steps, if computational time is of no importance, than the former method produces a lower RMSE. However, if computational time is of importance, then the EuM scheme applied to the AAE method, produces better results. In all, there are advantages in using the AAE method in MC simulations for Asian option prices.

It is quite evident from this chapter that looking at Asian options from an Australian perspective is not only worthwhile, but also valuable in being able to better understand their relationship.
Chapter 7

Low Volatility Asymptotics for Australian Options

The subject of Australian options has been briefly touched on in Chapter 6, mostly in using equivalencies to price Asian options. However the study of Australian options themselves as an exotic derivative, has received very little attention. The most notable work in this area is that of Handley [38], [39] and Moreno and Navas [59]. As previously, this chapter will only be concerned with the continuously sampled arithmetic averaged Australian option and in addition, the floating strike parameter $k_1$ from Chapter 6 will be assumed to be zero. This is purely due to the fact that Australian options are rarely traded in the general form given in Chapter 6, and that most analysis of Australian options only considers fixed strikes. Furthermore, unlike the previous chapters where stochastic volatility is considered, this chapter will only focus on the Black-Scholes model, that is, the underlying asset’s value is assumed to follow a geometric Brownian motion. Lastly, an additional parameter to denote the continuous dividend yield will be introduced to reconcile with other previously established techniques found in Moreno and Navas. A closed form solution for the pricing of an Australian call option is currently available and will be reviewed here. However, the instability of numerical methods in evaluating this formula makes it difficult to obtain a robust pricing technique.

Whilst the equivalencies derived in Chapter 6 provide ways of pricing Australian options as Asian options, it must be noted that no closed form solution for Asian options, even in the BS model, currently exist. Despite the various equivalences, some pricing techniques used for Asian options do not directly carry over to Australian options. In some cases the conversion

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1This statement does not apply to geometrically averaged Asian options with a fixed strike.
between Australian and Asian options hides potential problems that can occur with either of them but also opportunities that may be missed out, when focusing on the case of classical Asian options only. This applies particularly to the various approximations that have been obtained in the past, for example, the low volatility asymptotics of Dewynne and Shaw [20], where the expansion is performed on a scaled volatility parameter. From Chapter 6, equivalency theorems between Asian and Australian options showed that the drift rate is adjusted via the addition of a volatility related term, thus making the results of Dewynne and Shaw not directly applicable.

The focus of this chapter will be to develop a low volatility asymptotic solution for Australian options, using the method of matched asymptotics, similar to Dewynne and Shaw. The method of matched asymptotics is a widely used perturbation expansion technique in applied mathematics. References such as Hinch [46], and Van Dyke [75], provide a good overview of the subject while in Howison [47], the technique is applied to many other mathematical finance problems. As previously stated, the results of Dewynne and Shaw, are not directly applicable to Australian options. In particular, the reason for this is that the Australian options PDE has two terms involving the constant volatility parameter, and thus any expansion of the solution in terms of a scaled volatility parameter must take this additional term into account. The asymptotic solution to be presented will be in closed form, when the risk-free interest rate equals the continuous dividend yield. When the risk-free interest rate does not equal the continuous dividend yield, procedures for obtaining the asymptotic solution numerically will be outlined.

In addition, two extensions to the Australian options pricing problem are made. The first extends the work of Moreno and Navas, and their Gamma distribution approximation, to cater for in progress options. This is done by considering the conditional means and variances of the Australian state process in moment matching to the reciprocal Gamma distribution. The second extension is to show that the low volatility asymptotic solution is still technically valid under the general Australian call option problem, whereas some of the other existing work in literature may not be. Together with the equivalency theorems from Chapter 6, the asymptotic solution provides a simple formula to unifying both Asian and Australian options.

Numerical tests are performed to compare the asymptotic solution to established methods of approximating Australian option prices. In particular, the methods to be referenced against are the Gamma distribution approximations and Edgeworth approximations by Moreno and Navas, along with FDM on the Australian options PDE and MC simulations. The results will show that the asymptotic solution excels in performance when the low volatility assumption is
satisfied, and that in this low volatility regime, the general Australian call option can be used to price general Asian call options.

7.1 Review of the Australian Option Model

This section will briefly review some of the model assumptions behind the Australian options model, put-call parity relationships, its closed form pricing formula and the PDE associated with its pricing.

7.1.1 Model Assumptions

Unlike previous BS models considered in this thesis, the asset’s value process will be assumed to follow the GBM described by the SDE,

\[ dX_t = (r - q) X_t dt + \sigma X_t dW_t, \]

where all the parameters are as previously defined in equation (2.1), with the addition of \( q \) as the continuous dividend yield. Its inclusion in this analysis is purely so that the low volatility asymptotic solution can be reconciled with other previously published approximating solutions.

Define the Australian state process as,

\[ AU_t = \frac{\int_{t_0}^t X_u \, du}{TX_t}. \] (7.1)

where \( t_0 = 0 \) and \( T \) are the start and end times of the averaging period of the Australian state process and \( t \) the current time. In pricing problems, it is necessary to only consider \( 0 \leq t < T \).

The noticeable differences between this definition and that of equation (6.1) is that \( k_1 = 0 \) and the presence of the \( 1/T \) factor. Setting the \( k_1 \) strike to zero makes comparisons to previously published work easier since a non-zero \( k_1 \) is rarely considered. The inclusion of the \( 1/T \) factor will result in a PDE that has a similar structure to that of the Asian options PDE as derived in Dewynne and Shaw [20]. This makes applying the method of matched asymptotics much easier to Australian options. Using the Itô formula, similarly to equation (6.11), it can be shown that \( AU_t \) satisfies the SDE,

\[ dAU_t = \left( (\sigma^2 - (r - q)) AU_t + \frac{1}{T} \right) dt - \sigma AU_t dW_t. \] (7.2)

The price of an Australian call option is thus defined as,

\[ V(\eta, t) = e^{-r(T-t)} E \left( (AU_T - k)^+ \bigg| AU_t = \eta \right), \] (7.3)
where the ‘2’ subscript for $k_2$ as in equation (6.3), is dropped since there will only be one strike parameter. Similarly, for an Australian put option, its price is given by,

$$e^{-r(T-t)}E \left( (k - AU_T)^+ \mid AU_t = \eta \right).$$

(7.4)

### 7.1.2 Put-Call Parity

A put-call parity exists for Australian options, and is derived as follows; Firstly, from equations (7.3) and (7.4), it is easy to show that,

$$e^{-r(T-t)}E \left( (k - AU_T)^+ \mid AU_t = \eta \right) = e^{-r(T-t)}E \left( (AU_T - k)^+ \mid AU_t = \eta \right) + ke^{-r(T-t)} - e^{-r(T-t)}E \left( AU_T \mid AU_t = \eta \right).$$

Thus it suffices to show that a solution for $E( AU_T \mid \mathcal{F}_t)$ exists, and then Australian put options can be priced using Australian call options.

The expectations for the Australian call and put options are conditional on the filtration $\mathcal{F}_t$. In particular, at time $t$, the information required is simply the value of $AU_t$, assumed to be given by $\eta$. The expectation of $AU_T$ conditional on $AU_t = \eta$ can be calculated by manipulating the SDE in equation (7.2). To see this, integrate the SDE from $t$ to $T_1$ to give,

$$AU_{T_1} - AU_t = \int_t^{T_1} \left( \sigma^2 - r + q \right) AU_s + \frac{1}{T} ds - \int_t^{T_1} \sigma AU_s dW_s.$$

Taking the expectation conditional on the filtration $\mathcal{F}_t$, results in,

$$E \left( AU_{T_1} - AU_t \mid \mathcal{F}_t \right) = E \left( \int_t^{T_1} \left( \sigma^2 - r + q \right) AU_s + \frac{1}{T} ds - \int_t^{T_1} \sigma AU_s dW_s \mid \mathcal{F}_t \right)$$

$$= E \left( \int_t^{T_1} \left( \sigma^2 - r + q \right) AU_s + \frac{1}{T} ds \mid \mathcal{F}_t \right)$$

$$- E \left( \int_t^{T_1} \sigma AU_s dW_s \mid \mathcal{F}_t \right)$$

$$= E \left( \int_t^{T_1} \left( \sigma^2 - r + q \right) AU_s + \frac{1}{T} ds \mid \mathcal{F}_t \right).$$

(7.5)

Note that the second expectation term in the second equality is equal to zero because the Brownian motion increments are zero mean martingale increments$^2$. This can be shown by

$^2$Strictly speaking, the condition required is $E \left( \int_t^{T_1} AU_s^2 ds \right) < \infty$, but since $AU_t$ is a linear SDE, this is condition is satisfied. The condition is also shown to hold by considering an application of Fubini’s theorem and the proof of the condition in (7.44), which is to come.
7.1 Review of the Australian Option Model

considering the standard definition of an Itô integral,

\[
\int_t^{T_1} \sigma AU_s \, dW_s = \lim_{n \to \infty} \sum_{[t_{i-1}, t_i) \in \pi_n} \sigma AU_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}),
\]

(7.6)

where \( \pi_n \) is a partitioning of \([t, T_1]\). Taking the conditional expectation of equation (7.6), and interchanging the limit and expectation by using the Lebesgue dominated convergence theorem, see Royden [67], for each \( t_{i-1} \), the term in the summation is,

\[
E \left( \sigma AU_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \mid \mathcal{F}_{t_i} \right) = E \left( \sigma AU_{t_{i-1}} \left( W_{t_i} - W_{t_{i-1}} \right) \mid \mathcal{F}_{t_{i-1}} \right) \mid \mathcal{F}_{t_i}
\]

\[
= E \left( \sigma AU_{t_{i-1}} \mathbb{E} \left( \left( W_{t_i} - W_{t_{i-1}} \right) \mid \mathcal{F}_{t_{i-1}} \right) \mid \mathcal{F}_{t_i} \right)
\]

\[
= 0,
\]

Thus showing that the conditional expectation of equation (7.6) is equal to zero.

Define \( Y(T_1; t) \) to be a deterministic function given by,

\[
Y(T_1; t) = E \left( AU_{T_1} \mid AU_t = \eta \right).
\]

On the right hand side of equation (7.5), the expectation and integration can be interchanged by an application of Fubini’s theorem, see Royden [67]. To check that this application is valid, all that is required is to show that,

\[
\int_t^{T_1} \mathbb{E} \left( |AU_s| \mid \mathcal{F}_t \right) \, ds < \infty.
\]

(7.7)

Firstly, \( AU_s \) is always positive given its definition in equation (7.1). Secondly, \( Y(s; t) \) can be shown to be a bounded function of \( s \) by considering the following,

\[
\mathbb{E} \left( |AU_s| \mid \mathcal{F}_t \right) = \mathbb{E} \left( \frac{\int_0^t X_u \, du}{TX_s} \mid \mathcal{F}_t \right)
\]

\[
= \mathbb{E} \left( \frac{\int_0^t X_u \, du}{TX_s} + \frac{\int_t^s X_u \, du}{TX_s} \mid \mathcal{F}_t \right)
\]

\[
= \frac{\int_0^t X_u \, du}{T} \mathbb{E} \left( \frac{1}{X_s} \mathcal{F}_t \right) + \frac{1}{T} \mathbb{E} \left( \int_t^s X_u \, du \mid \mathcal{F}_t \right).
\]

(7.8)

Given that \( X_s \) is a GBM, i.e. log-normally distributed, \( 1/X_s \) is also log-normally distributed. Using the fact that the conditional expectation of a log-normal is a known continuous function (similar to equation (2.3)), it can be concluded that the first term of equation (7.8) is finite. Furthermore, the expectation and integration of the second term in equation (7.8) can be interchanged through another application of Fubini’s theorem. Again, this is justified as
7.1 Review of the Australian Option Model

$X_u/X_s$ is also log-normally distributed\(^3\), thus ensuring its positivity, and that its conditional expectation, also a known continuous function. Thus the definite integral of this conditional expectation must be bounded and so the second term in equation (7.8) is also finite. In all, this shows that $Y(s; t)$ must be bounded, and that the definite integral of $Y(s; t)$ must also be bounded, proving the inequality in (7.7) is satisfied.

After interchanging the expectation and integration in equation (7.5), differentiating this equation with respect to $T_1$ gives,

$$
\frac{dY(T_1; t)}{dT_1} = (\sigma^2 - r + q) Y(T_1; t) + \frac{1}{T}.
$$

This ODE can be solved by considering the equivalent ODE,

$$
\frac{d\left(Y(T_1; t)e^{-(\sigma^2-r+q)T_1}\right)}{dT_1} = \frac{1}{T}e^{-(\sigma^2-r+q)T_1},
$$

and then integrating this with respect to $T_1$. Assuming $\sigma^2 - r + q \neq 0$, the result of the integration is given by,

$$
Y(T_1; t)e^{-(\sigma^2-r+q)T_1} = -\frac{1}{T(\sigma^2 - r + q)}e^{-(\sigma^2-r+q)T_1} + c_1,
$$

for some constant $c_1$. At time $T_1 = t$, $Y(t; t)$ is given by the known value of $AU_t$, which is assumed to equal $\eta$. The constant $c_1$ is then calculated as,

$$
c_1 = \eta e^{-(\sigma^2-r+q)t} + \frac{1}{T(\sigma^2 - r + q)}e^{-(\sigma^2-r+q)t},
$$

and thus,

$$
E\left( AU_{T_1} \mid AU_t = \eta \right) = Y(T_1; t)
= \eta e^{(\sigma^2-r+q)(T_1-t)} + \frac{1}{T(\sigma^2 - r + q)} \left( e^{(\sigma^2-r+q)(T_1-t)} - 1 \right).
$$

For the case where $\sigma^2 - r + q = 0$, the conditional expectation of $AU_{T_1}$ is given by,

$$
E\left( AU_{T_1} \mid AU_t = \eta \right) = \frac{1}{T}(T_1 - t) + \eta.
$$

Given the solution for the conditional expectation, the put-call parity is established, and thus the pricing of Australian put options will not be considered.

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\(^3\)This will be shown in more detail in Section 7.3.1. However, all that is currently required is to note that the reciprocals and products of a log-normal is log-normal.
7.1 Review of the Australian Option Model

7.1.3 Closed Form Solution

The closed form solution for pricing the Australian call option in equation (7.3) is presented below. Begin by defining a New Zealand state process as the inverse of the Australian state process. Thus, \((NZ)_t\) is given by,

\[
(NZ)_t = \frac{TX_t}{\int_{t_0}^t X_u \, du}.
\]

Note that this is only defined for \(t > t_0\). Using the Itô formula on \((NZ)_t\), it can be shown that,

\[
d(NZ)_t = \frac{TX_t}{\int_{t_0}^t X_u \, du} \left( (r - q) \, dt + \sigma dW_t - TX_t \left( \frac{X_t}{\int_{t_0}^t X_u \, du} \right)^2 \right) dt
\]

\[
= \left( (r - q) (NZ)_t - \frac{1}{T} (NZ)^2_t \right) dt + \sigma (NZ)_t dW_t,
\]

where,

\[
d \left( \frac{1}{\int_{t_0}^t X_u \, du} \right) = - \frac{X_t}{\left( \int_{t_0}^t X_u \, du \right)^2} dt.
\]

From equation (7.9), it can be seen that \((NZ)_t\) follows a GMR process, with \(a = r - q\), \(b = 1/T\) and \(c = \sigma\), as defined in Section 2.3. Note, for practical applications, it is assumed that \(r > q\), and thus \(a\) is positive.

The price of an Australian call option can thus be priced as,

\[
\phi(\eta, t) = e^{-r(T-t)} \mathbb{E} \left( (AU_T - k)^+ \left| AU_t = \eta \right. \right)
\]

\[
= e^{-r(T-t)} \mathbb{E} \left( \frac{1}{(NZ)_T} - k \right)^+ \left| \frac{1}{(NZ)_t} = \eta \right.
\]

\[
= e^{-r(T-t)} \int_0^{1/k} \left( \frac{1}{x} - k \right) \Phi (x, t|1/\eta) \, dx,
\]

with \(\Phi(x, t|1/\eta)\) being the transitional probability density function of a GMR process, as defined in equation (2.17). The transitional pdf has parameters \(a\), \(b\) and \(c\) as previously mentioned.

Unfortunately, the density in equation (2.17) is difficult to evaluate numerically, see Section 2.3 for more details. Further evidence of this can be found in Yang, Ewald and Menkens [85], where a similar density is used for the pricing of Asian options. As such, the closed form solution for the pricing of Australian call options cannot be used to provide meaningful results.
7.2 Asymptotic Expansion

7.1.4 The Pricing PDE

The Australian option pricing PDE is derived by considering equations (7.2) and (7.3). Define,

\[ \psi(\eta, t) = \mathbb{E}(AU_T - k)^+ | AU_t = \eta), \]

which is the expectation part of the Australian call option in equation (7.3). Invoking the Feynman-Kac theorem shows that \( \psi(\eta, t) \) satisfies the PDE,

\[
\frac{\partial \psi}{\partial t} + \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( \frac{1}{T} + (\sigma^2 - (r - q)) \eta \right) \frac{\partial \psi}{\partial \eta} = 0, \tag{7.10}
\]

\[ \psi(\eta, T) = (\eta - k)^+. \]

This PDE is then non-dimensionalized by introducing the following variables,

\[
\tau = 1 - \frac{t}{T},
\]

\[
\epsilon^2 = \sigma^2 T,
\]

\[
\theta = (r - q) T,
\]

which results in,

\[
\frac{\partial \psi}{\partial \tau} = \frac{1}{2} \epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( 1 + (\epsilon^2 - \theta) \eta \right) \frac{\partial \psi}{\partial \eta}, \tag{7.11}
\]

\[
\psi(\eta, 0) = (\eta - k)^+. \]

The variables \( \tau, \epsilon \) and \( \theta \) can be regarded as the scaled time, volatility and drift rate variables, respectively. The form of this PDE is similar to the one studied in Dewynne and Shaw, but with the notable difference of having an additional \( \epsilon^2 \eta \frac{\partial \psi}{\partial \eta} \) term and the one-half factor in the diffusion term. The additional \( \epsilon \) related advection term in the PDE needs to be taken care of, since the asymptotic solution will be expanded in terms of \( \epsilon \). In fact, this is what prevents a direct application of the Dewynne and Shaw result.

7.2 Asymptotic Expansion

The approach taken to derive asymptotic expansions for an Australian call option will be that of the method of matched asymptotics. This methodology has been used to great effect by Dewynne and Shaw [20], for Asian options in finding analytical approximations to Asian fixed strike call options in a low volatility regime. The derivation for the Australian case follows similarly, however due to the additional volatility related advection term in the Australian PDE as in equation (7.11), additional steps must be taken.
The method of matched asymptotic expansion assumes a solution for an outer region and an inner region. The outer region can be loosely thought of as the region in which the volatility parameter has negligible effect on the option, and the inner region where its effect cannot be ignored. Intuitively, the outer region corresponds to deeply ITM or deeply OTM options, whereas the inner region corresponds to near-the-money options. In many pricing problems, only the inner region is of interest, but in order to determine the inner region, one must solve for the outer region first.

7.2.1 Outer Region

The derivation of the solution in the outer region in the Australian option case follows similarly to that of the Asian option case. Assume that the solution \( \psi(\eta, \tau) \) can be expanded in the form,

\[
\psi(\eta, \tau) = \Phi_0(\eta, \tau) + \epsilon^2 \Phi_1(\eta, \tau) + \cdots.
\]

To leading order, \( \Phi_0(\eta, \tau) \) satisfies the following linear first-order hyperbolic PDE,

\[
\frac{\partial \Phi_0}{\partial \tau} = (1 - \theta \eta) \frac{\partial \Phi_0}{\partial \eta},
\]

where the assumption is that,

\[
\epsilon^2 \frac{\partial^2 \Phi_0}{\partial \eta^2} \ll 1, \quad (7.13)
\]

\[
\epsilon^2 \frac{\partial \Phi_0}{\partial \eta} \ll 1. \quad (7.14)
\]

The characteristic projection of equation (7.12) is given by,

\[
\frac{\partial \eta}{\partial \tau} = (\theta \eta - 1).
\]

Let \( \eta = \eta_0 \) at \( \tau = 0 \), and then the solution to this characteristic projection is,

\[
\eta(\tau) = \begin{cases} 
\eta_0 e^{\theta \tau} + \frac{1}{\theta} \left( 1 - e^{\theta \tau} \right), & \theta \neq 0, \\
\eta_0 - \tau, & \theta = 0.
\end{cases}
\]

Along these characteristic projections, \( \Phi_0 \) is constant, and so if,

\[
\Phi_0(\eta, 0) = F(\eta),
\]

for some function \( F \), then,

\[
\Phi_0(\eta, \tau) = \begin{cases} 
F \left( \eta e^{-\theta \tau} + \frac{1}{\theta} \left( 1 - e^{-\theta \tau} \right) \right), & \theta \neq 0, \\
F(\eta + \tau), & \theta = 0.
\end{cases} \quad (7.15)
\]
The assumptions in (7.13) and (7.14) are not necessarily true along certain characteristic projections. In particular, for the payoff function considered,

\[ F(\eta) = (\eta - k)^+, \quad (7.16) \]

the second partial derivative, \( \partial^2 \psi / \partial \eta^2 \) is in fact a delta function at \( \eta = k \) and \( \tau = 0 \). This characteristic projection is given by,

\[ \eta^*(\tau) = \begin{cases} 
ke^{\theta \tau} + \frac{1}{\theta} (1 - e^{\theta \tau}), & \theta \neq 0, \\
\frac{k}{\tau}, & \theta = 0,
\end{cases} \quad (7.17) \]

which is called the critical characteristic. Since the assumptions do not hold around this critical characteristic, the asymptotic expansion solution fails, which leads to the need for new solution around this characteristic. The region around this characteristic is dubbed the inner region, and will be the subject of the next sections. Although the method of matched asymptotics is used, it is actually not necessary to impose the matching conditions explicitly, as it can be shown that the leading order solutions for both the outer and inner regions automatically agree. This will be shown in Section 7.2.4 after the inner regions have been calculated.

### 7.2.2 Inner Region for \( r = q \)

When solving the solution for the inner region, there are two cases to consider; when \( \theta = 0 \) and \( \theta \neq 0 \), which correspond to \( r = q \) and \( r \neq q \), respectively. This section derives the asymptotic solution for the inner region when \( \theta = 0 \), with the subsequent section dealing with \( \theta \neq 0 \). As before, the critical characteristic is given by equation (7.17) and the resulting PDE of interest is,

\[ \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( 1 + \epsilon^2 \eta \right) \frac{\partial \psi}{\partial \eta}, \]

\[ \psi(\eta, 0) = (\eta - k)^+. \]

Define an inner variable via,

\[ \zeta = \frac{1}{\epsilon} (\eta - \eta^*(\tau)), \]

and by applying the chain rule, the resulting PDE of interest becomes,

\[ \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \left( \epsilon \zeta + \eta^*(\tau) \right)^2 \frac{\partial^2 \psi}{\partial \zeta^2} + \epsilon \left( \epsilon \zeta + \eta^*(\tau) \right) \frac{\partial \psi}{\partial \zeta}, \]

\[ \psi(\zeta, 0) = \epsilon \zeta^+. \]
7.2 Asymptotic Expansion

Assume that the solution for $\psi(\zeta, \tau)$, in the inner region, can be written in terms of a power series given by,

$$
\psi(\zeta, \tau) = \sum_{j=1}^{\infty} \epsilon^j \psi_j(\zeta, \tau).
$$

This solution gives rise to a sequence of PDEs which the $\psi_j$’s must satisfy. These PDEs are given as follows;

$$
\frac{\partial \psi_1}{\partial \tau} - \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_1}{\partial \zeta^2} = 0,
$$

$$
\frac{\partial \psi_2}{\partial \tau} + \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_2}{\partial \zeta^2} = \eta^*(\tau) \frac{\partial^2 \psi_1}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_1}{\partial \zeta},
$$

$$
\frac{\partial \psi_j}{\partial \tau} + \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_j}{\partial \zeta^2} = \eta^*(\tau) \frac{\partial^2 \psi_{j-1}}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_{j-1}}{\partial \zeta} + \frac{\partial^2 \psi_{j-2}}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_{j-2}}{\partial \zeta}, \quad j \geq 3,
$$

with initial conditions,

$$
\psi_j(\zeta, 0) = \begin{cases} 
\zeta^+, & \text{if } j = 1, \\
0, & \text{if } j > 1.
\end{cases}
$$

Note that for $j = 2$ and $j \geq 3$, each of these PDEs feature one and two additional source terms, respectively, when compared to the corresponding sequence of PDEs in Dewynne and Shaw [20]. This is a direct result of the additional advection term in the original PDE. Furthermore, whilst not as important, there is an additional $1/2$ factor in the diffusion term on the left hand side.

**First Order Solution**

The next step is to solve the sequence of PDEs from Section 7.2.2. The first order solution satisfies the PDE given by,

$$
\frac{\partial \psi_1}{\partial \tau} - \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_1}{\partial \zeta^2} = 0,
$$

$$
\psi(\zeta, 0) = \zeta^+.
$$

Let $\dot{t}(\tau)$ be defined as,

$$
\dot{t}(\tau) = \int_0^\tau \frac{1}{2} \eta^*(\tau')^2 d\tau',
$$

$$
= \frac{1}{2} \int_0^\tau (k - \tau')^2 d\tau',
$$

$$
= \frac{k^2 \tau}{2} - \frac{k \tau^2}{2} + \frac{\tau^3}{6}.
$$
By transforming the problem to the \((\zeta, \hat{t})\) variables, the PDE for \(\psi(\zeta, \hat{t})\) becomes,

\[
\frac{\partial \psi_1}{\partial \hat{t}} = \frac{\partial^2 \psi_1}{\partial \zeta^2}, \tag{7.18}
\]

which is the standard diffusion equation. The solution to this PDE with the initial condition is given by,

\[
\psi_1(\zeta, \hat{t}) = \zeta N\left(\frac{\zeta}{\sqrt{2\hat{t}}}\right) + \sqrt{\frac{\hat{t}}{\pi}} \exp\left(-\frac{\zeta^2}{4\hat{t}}\right), \tag{7.19}
\]

with \(N(x)\) denoting the cumulative standard normal distribution function, as in equation (2.5). This can be shown by invoking the convolution theorem to solve equation (7.18), see Section 10.4.3 in Haberman [37], for example. The solution can be calculated by computing,

\[
\psi_1(\zeta, \hat{t}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \hat{t}}} \exp\left(-\frac{(\zeta - y)^2}{4\hat{t}}\right) y^+ dy.
\]

Further, define the following partial derivatives as they will be required in calculations for the higher order solutions;

\[
\mathcal{G}_1(\zeta, \hat{t}) := \frac{\partial^2 \psi_1}{\partial \zeta^2} = \sqrt{\frac{1}{4\pi \hat{t}}} \exp\left(-\frac{\zeta^2}{4\hat{t}}\right),
\]

\[
\mathcal{G}_2(\zeta, \hat{t}) := \frac{\partial \psi_1}{\partial \zeta} = N\left(\frac{\zeta}{\sqrt{2\hat{t}}}\right).
\]

Note that the subscripts of \(\mathcal{G}\) and the orders of the partial derivatives are switched. In Dewynne and Shaw, only \(\mathcal{G}_1(\zeta, \hat{t})\) is required, while for Australian options \(\mathcal{G}_2(\zeta, \hat{t})\) is also needed.

**Second Order Solution**

The second order solution satisfies the PDE given by,

\[
\frac{\partial \psi_2}{\partial \tau} - \frac{1}{2} \eta^*(\tau) \frac{\partial^2 \psi_2}{\partial \zeta^2} = \zeta \eta^*(\tau) \frac{\partial^2 \psi_1}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_1}{\partial \zeta}. \tag{7.20}
\]

Using the functions \(\mathcal{G}_1(\zeta, \hat{t})\) and \(\mathcal{G}_2(\zeta, \hat{t})\) and transforming the PDE to the \((\zeta, \hat{t})\) variables gives,

\[
\frac{\partial \psi_2}{\partial \hat{t}} - \frac{\partial^2 \psi_2}{\partial \zeta^2} = \zeta \eta^*(\tau) \mathcal{G}_1(\zeta, \hat{t}) + \eta^*(\tau) \mathcal{G}_2(\zeta, \hat{t}) \frac{\partial \psi_1}{\partial \zeta}. \tag{7.21}
\]

Seek a solution for \(\psi_2(\zeta, \hat{t})\) in the form of,

\[
\psi_2(\zeta, \hat{t}) = \zeta g_1(\hat{t}) \mathcal{G}_1 + g_2(\hat{t}) \mathcal{G}_2,
\]
7.2 Asymptotic Expansion

and substitute this into equation (7.20). This results in the following PDE,

\[ \frac{\partial \psi_2}{\partial t} - \frac{\partial^2 \psi_2}{\partial \zeta^2} = \zeta \left( \frac{g_1}{\hat{t}} + \frac{dg_1}{dt} \right) \partial_1 + \frac{dg_2}{dt} \partial_2, \]

which, when equated with equation (7.21), gives a series of ordinary differential equations that determines the functions \( g_1(\hat{t}) \) and \( g_2(\hat{t}) \). Also, the boundary conditions of these functions are fixed such that \( g_i(0) = 0 \), for \( i = 1, 2 \). Note that the derivation of this PDE makes use of the following identities,

\[ \frac{\partial G_1}{\partial \hat{t}} = \frac{\zeta^2}{4\hat{t}^2} G_1 - \frac{1}{2\hat{t}} G_1; \quad \frac{\partial G_1}{\partial \zeta} = -\frac{\zeta}{2\hat{t}} G_1; \quad \frac{\partial^2 G_1}{\partial \zeta^2} = \frac{\zeta^2}{4\hat{t}^2} G_1 - \frac{1}{2\hat{t}} G_1; \]

\[ \frac{\partial G_2}{\partial \hat{t}} = -\frac{\zeta}{2\hat{t}} G_1; \quad \frac{\partial G_2}{\partial \zeta} = \frac{G_1}{2 \zeta}; \quad \frac{\partial^2 G_2}{\partial \zeta^2} = -\frac{\zeta}{2\hat{t}} G_1. \]

The function \( g_2(\hat{t}) \) is defined by the ODE,

\[ \frac{dg_2}{dt} = \frac{2}{\eta^*(\tau)}. \]

When converting back to the \( \tau \) variable, the ODE for \( g_2(\hat{t}(\tau)) \) becomes,

\[ \frac{dg_2}{d\tau} = \eta^*(\tau) = k - \tau. \]

The solution to this ODE is,

\[ g_2(\hat{t}(\tau)) = k\tau - \frac{\tau^2}{2}. \]

For the function \( g_1(\hat{t}) \), consider the ODE,

\[ \frac{d(\hat{t}g_1)}{dt} = \frac{dg_1}{dt} + \hat{t} \frac{dg_1}{dt} = \frac{2\hat{t}}{\eta^*(\tau)}. \]

Again, converting back to the \( \tau \) variable, the ODE for \( g_1(\hat{t}(\tau)) \) becomes,

\[ \frac{d(\hat{t}g_1)}{d\tau} = \hat{t} \eta^*(\tau), \]

which upon integration by \( \tau \) and then division by \( \hat{t} \) gives the solution for \( g_1(\tau) \) as,

\[ g_1(\hat{t}(\tau)) = \frac{\tau \left( 15k^3 - 20k^2\tau + 10k\tau^2 - 2\tau^3 \right)}{10(3k^2 - 3k\tau + \tau^2)}. \]
Third Order Solution

To calculate the third order solution, the following partial derivatives are required:

\[
\frac{\partial \psi}{\partial \zeta} = \left( g_1 + g_2 - \frac{g_1}{2t} \right) G_1,
\]

\[
\frac{\partial^2 \psi}{\partial \zeta^2} = \left( -\left( \frac{3g_1}{2t} + \frac{g_2}{2t} \right) \zeta + \frac{g_1}{4t^2} \zeta^3 \right) G_1.
\]

This is similar to the second order solution requiring the partial derivatives of the first order solution. Using these partial derivatives, the PDE for the third order solution in terms of the \((\zeta, \hat{t})\) variables becomes,

\[
\frac{\partial \psi_3}{\partial \hat{t}} - \frac{\partial^2 \psi_3}{\partial \zeta^2} = H(\zeta, \hat{t}) G_1 + \zeta G_2 + \left( \frac{1}{2} - \frac{g_2 \eta^*(\tau)}{2t} - \frac{2g_1 \eta^*(\tau)}{t} \right) \xi^2 + \frac{g_1 \eta^*(\tau)}{4t^2} \xi^4.
\]

where the function \(H(\zeta, \hat{t})\) is defined as,

\[
H(\zeta, \hat{t}) = \eta^*(\tau) (g_1 + g_2) + \left( \frac{1}{2} - \frac{g_2 \eta^*(\tau)}{2t} - \frac{2g_1 \eta^*(\tau)}{t} \right) \xi^2 + \frac{g_1 \eta^*(\tau)}{4t^2} \xi^4.
\]

Seek a solution for \(\psi_3(\zeta, \hat{t})\) in the form of,

\[
\psi_3(\zeta, \hat{t}) = (f_1(\hat{t}) + f_2(\hat{t}) \zeta^2 + f_3(\hat{t}) \zeta^4) G_1 + f_4(\hat{t}) \zeta G_2,
\]

and similarly, upon substitution into the left hand side of equation (7.25) results in the following PDE,

\[
\frac{\partial \psi_3}{\partial \hat{t}} - \frac{\partial^2 \psi_3}{\partial \zeta^2} = \left[ \frac{df_1}{dt} - 2f_2 - 2f_4 \right] + \left( \frac{df_2}{dt} + \frac{2f_2}{t} - 12f_3 \right) \xi^2 + \left( \frac{df_3}{dt} + \frac{4f_1}{t} \right) \xi^4 \right] G_1 + \frac{df_4}{dt} \zeta G_2,
\]

for \(\psi_3(\zeta, \hat{t})\). Equating equation (7.26) to equation (7.25) leads to another series of ODEs which the \(f_i(\hat{t})\)'s must satisfy. Again, the boundary conditions are such that \(f_i(0) = 0\), for \(i = 1, 2, 3\) and 4.

The ODE for \(f_4\) is converted from the \(\hat{t}\) variable to the \(\tau\) by the following,

\[
\frac{df_4}{d\hat{t}} = \frac{2}{\eta^*(\tau)^2},
\]

\[
\Rightarrow \frac{df_4}{d\tau} = 1.
\]

This yields the solution \(f_4(\hat{t}(\tau)) = \tau\).
For $f_3$, consider the ODE,
\[
\frac{d(i^4 f_3)}{dt} = i^4 \left( \frac{df_3}{dt} + \frac{4f_3}{t} \right)
= g i^2 \eta^*(\tau) \frac{2}{\eta^*(\tau)^2},
\]
\[
\Rightarrow \frac{d(i^4 f_3)}{d\tau} = \frac{g i^2 \eta^*(\tau)}{4}.
\]
Upon integration by $\tau$ and then division by $i^4$, the resulting solution for $f_3(\hat{t}(\tau))$ is given as,
\[
f_3(\hat{t}(\tau)) = \frac{9 (15k^3 - 20k^2\tau + 10k\tau^2 - 2\tau^3)^2}{200 (3k^2 - 3k\tau + \tau^2)^3}.
\]

For $f_2$, consider the ODE,
\[
\frac{d(i^2 f_2)}{dt} = \frac{1}{2} - \frac{\eta^*(\tau)}{i^2} \left( \frac{g_2}{2} + 2g_1 \right)^2 + 12f_3 i^2,
\]
\[
\Rightarrow \frac{d(i^2 f_2)}{d\tau} = \left( \frac{1}{2} - \frac{\eta^*(\tau)}{i^2} \left( \frac{g_2}{2} + 2g_1 \right) \right) i^2 + 6f_3 i^2 \eta^*(\tau)^2,
\]
This has solution in terms of $\tau$ as,
\[
f_2(\hat{t}(\tau)) = -\frac{\tau \left( \sum_{j=0}^6 f_2^j k^{6-j} \tau^j \right)}{1400 (3k^2 - 3k\tau + \tau^2)^3},
\]
where the constants $f_2^j$’s are given in Table 7.1.

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_2^j$</td>
<td>22050</td>
<td>-61425</td>
<td>74445</td>
<td>-50715</td>
<td>20460</td>
<td>-4640</td>
<td>464</td>
</tr>
</tbody>
</table>

**Table 7.1:** $f_2^j$’s Constants in Third Order Solution ($r = q$)

Finally, for $f_1$ the ODE in terms of $\hat{t}$ and $\tau$ is given by,
\[
\frac{df_1}{d\hat{t}} = \frac{2 (g_1 + g_2)}{\eta^*(\tau)} + 2 (f_2 + f_4),
\]
\[
\Rightarrow \frac{df_1}{d\tau} = \eta^*(\tau) (g_1 + g_2) + \eta^*(\tau)^2 (f_2 + f_4),
\]
respectively. The solution for $f_1(\hat{t}(\tau))$ is given as,
\[
f_1(\hat{t}(\tau)) = \frac{\tau^2 \left( \sum_{j=0}^6 f_1^j k^{6-j} \tau^j \right)}{4200 (3k^2 - 3k\tau + \tau^2)^2},
\]
where the constants $f_1^j$’s are given in Table 7.2.

Note that the calculations leading to the last two integral solutions can be implemented using symbolic software such as Mathematica. When using Mathematica, it is important to make use of the `Assuming` function and making the assumptions that $\tau > 0$ and $k \geq 0$, which are indeed satisfied here.
7.2 Asymptotic Expansion

Table 7.2: $f_j$’s Constants in Third Order Solution ($r = q$)

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_j$</td>
<td>36225</td>
<td>-111825</td>
<td>151620</td>
<td>-115185</td>
<td>51645</td>
<td>-12970</td>
<td>1437</td>
</tr>
</tbody>
</table>

Higher Order Solutions

In principle, it is possible to obtain higher order solutions by considering solutions whose forms follow a similar pattern to that of $\psi_2$ and $\psi_3$. However, it will be shown that for the case of low volatility, the asymptotic solution, up to the third order, provides a very good approximation to the true solution as calculated via FDM. As such, higher order solutions are not considered here.

7.2.3 Inner Region for $r \neq q$

This section derives the asymptotic expansion when $\theta \neq 0$, i.e. $r \neq q$. The corresponding PDE of interest is given as,

$$
\frac{\partial \psi}{\partial \tau} = \frac{1}{2} \epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + (1 + (\epsilon^2 - \theta) \eta) \frac{\partial \psi}{\partial \eta},
$$

(7.27)

$\psi(\eta, 0) = (\eta - k)^+.$

Again, define an inner variable by,

$$
\zeta = \frac{1}{\epsilon} (\eta - \eta^*(\tau)),
$$

where $\eta^*(\tau)$ is defined as in equation (7.17).

Through the application of the chain rule, the PDE in equation (7.27) becomes,

$$
\frac{\partial \psi}{\partial \tau} = \frac{1}{2} \left( \epsilon \zeta + \eta^*(\tau) \right)^2 \frac{\partial^2 \psi}{\partial \zeta^2} - \theta \zeta \frac{\partial \psi}{\partial \zeta} + \epsilon \left( \epsilon \zeta + \eta^*(\tau) \right) \frac{\partial \psi}{\partial \zeta},
$$

$\psi(\zeta, 0) = \epsilon \zeta^+. $

As in the $\theta = 0$ case, assume that the solution for $\psi(\zeta, \tau)$ can be written as a power series in the form of,

$$
\psi(\zeta, \tau) = \sum_{j=1}^{\infty} \epsilon^j \psi_j(\zeta, \tau).
$$
This solution gives rise to the following sequence of PDEs, which the $\psi_j$ must satisfy:

\[
\frac{\partial \psi_1}{\partial \tau} - \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_1}{\partial \zeta^2} + \theta \zeta \frac{\partial \psi_1}{\partial \zeta} = 0,
\]

\[
\frac{\partial \psi_2}{\partial \tau} - \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_2}{\partial \zeta^2} + \theta \zeta \frac{\partial \psi_2}{\partial \zeta} = \zeta \eta^*(\tau) \frac{\partial^2 \psi_1}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_1}{\partial \zeta},
\]

\[
\frac{\partial \psi_j}{\partial \tau} - \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_j}{\partial \zeta^2} + \theta \zeta \frac{\partial \psi_j}{\partial \zeta} = \zeta \eta^*(\tau) \frac{\partial^2 \psi_{j-1}}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_{j-1}}{\partial \zeta} + \frac{1}{2} \zeta^2 \frac{\partial^2 \psi_{j-2}}{\partial \zeta^2} + \zeta \frac{\partial \psi_{j-2}}{\partial \zeta},
\]

\[j \geq 3,
\]

with initial conditions,

\[
\psi_j(\zeta, 0) = \begin{cases} 
\zeta^+ & \text{if } j = 1, \\
0 & \text{if } j > 1.
\end{cases}
\]

Again, as in Section 7.2.2, the Australian options sequences of PDEs have additional terms in source terms, as compared to the Dewynne and Shaw [20], equivalent series of PDEs. This is due to the additional advection term in the original PDE.

**First Order Solution**

The first order solution (and subsequent order solutions) is solved similarly as in Dewynne and Shaw for the $r \neq q$ case. Define the new variables, $\hat{t} = \hat{t}(\tau)$ and $x = \hat{f}(\tau) \zeta$, and further note that the function $\hat{t}(\tau)$ is different to the one defined in the $r = q$ case. Under this transformation, the PDE becomes,

\[
\frac{d \hat{t}}{d \tau} \frac{\partial \psi_1}{\partial \hat{t}} = \frac{1}{2} \eta^*(\tau)^2 \hat{f}(\tau)^2 \frac{\partial^2 \psi_1}{\partial x^2} - \zeta \left( \frac{\partial \hat{f}}{\partial \tau} + \theta \hat{f} \right) \frac{\partial \psi_1}{\partial x}.
\]

Choose the functions $\hat{t}(\tau)$ and $\hat{f}(\tau)$ such that,

\[
\frac{\partial \hat{f}}{\partial \tau} + \theta \hat{f} = 0, \quad (7.28)
\]

\[
\frac{d \hat{t}}{d \tau} = \frac{1}{2} \eta^*(\tau)^2 \hat{f}(\tau)^2 = 0, \quad (7.29)
\]

with initial conditions $\hat{f}(0) = 1$ and $\hat{t}(0) = 0$. The first order solution is then equivalent to solving the standard diffusion equation given by,

\[
\frac{\partial \psi_1}{\partial t} = \frac{\partial^2 \psi_1}{\partial x^2}, \quad \psi_1(x, 0) = x^+.
\]
The solutions for \( \hat{f}(\tau) \) and \( \hat{t}(\tau) \) can be easily obtained by solving equations (7.28) and (7.29), respectively, together with their initial conditions. Their solutions are given as,
\[
\hat{f}(\tau) = e^{-\theta \tau}, \\
\hat{t}(\tau) = e^{-2\theta \tau} \left( 4e^{\theta \tau} \left( 1 - k\theta \right) - 1 + e^{2\theta \tau} \left( -3 + 2\theta \tau + 2k^2\theta^2 \tau - 4k\theta (\theta \tau - 1) \right) \right).
\]

Having computed \( \hat{f}(\tau) \) and \( \hat{t}(\tau) \), the solution for \( \psi_1(x, \hat{t}) \) can then be written as,
\[
\psi_1(x, \hat{t}) = xN \left( \frac{x}{\sqrt{2\hat{t}}} \right) + \sqrt{\frac{t}{\pi}} \exp \left( -\frac{x^2}{4\hat{t}} \right),
\]
which is derived similarly to the solution given in equation (7.19).

As with the \( r = q \) case, the following partial derivatives are required when solving for higher order terms. Define them as follows;
\[
G_1(x, \hat{t}) := \frac{\partial^2 \psi_1}{\partial x^2} = \sqrt{\frac{1}{4\pi t}} \exp \left( -\frac{x^2}{4t} \right), \\
G_2(x, \hat{t}) := \frac{\partial \psi_1}{\partial x} = N \left( \frac{x}{\sqrt{2\hat{t}}} \right).
\]

Note that the solution and partial derivatives for the first order solution look familiar to the first order solutions for the \( r = q \) case. However, due to the difference in the transformed variables, \((x, \hat{t})\) and \((\zeta, \hat{t})\), for the cases \( r \neq q \) and \( r = q \), respectively, they are in fact different.

### Second Order Solution

For the second order solution, the PDE that governs its solution is given by,
\[
\frac{\partial \psi_2}{\partial \tau} - \frac{1}{2} \eta^*(\tau)^2 \frac{\partial^2 \psi_2}{\partial \zeta^2} + \theta_\zeta \frac{\partial \psi_2}{\partial \zeta} = \zeta \eta^*(\tau) \frac{\partial^2 \psi_1}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_1}{\partial \zeta}.
\]

Transforming this PDE to the \((x, \hat{t})\) variables results in,
\[
\frac{\partial \psi_2}{\partial \hat{t}} - \frac{\partial^2 \psi_2}{\partial x^2} = \zeta \eta^*(\tau) \frac{\partial^2 \psi_1}{\partial \zeta^2} + \eta^*(\tau) \frac{\partial \psi_1}{\partial \zeta} \\
= \frac{2}{\eta^*(\tau) f(\tau)} x G_1(x, \hat{t}) + \frac{2}{\eta^*(\tau) f(\tau)} G_2(x, \hat{t}), \quad (7.30)
\]
which again makes use of the similar identities like those found in equations (7.22), (7.23) and (7.24).

Similarly to the \( r = q \) case, seek a solution for \( \psi_2(x, \hat{t}) \) in the form of,
\[
\psi_2(x, \hat{t}) = g_1(\hat{t}) x G_1(x, \hat{t}) + g_2(\hat{t}) G_2(x, \hat{t}).
\]
7.2 Asymptotic Expansion

Substituting this into the left hand side of equation (7.30) gives,

\[
\frac{\partial \psi_2}{\partial t} - \frac{\partial^2 \psi_2}{\partial x^2} = \left( \frac{g_1(\hat{t})}{\hat{t}} + \frac{dg_1(\hat{t})}{dt} \right) x g_1(x, \hat{t}) + \frac{dg_2(\hat{t})}{dt} g_2(x, \hat{t}),
\]

which can then be equated to the right hand side of equation (7.30). This then identifies a series of ODEs which the \( g_i \)'s must satisfy. These are given as,

\[
\begin{align*}
g_1(\hat{t}) \frac{\hat{t}}{\hat{t}} + \frac{dg_1(\hat{t})}{dt} &= \frac{2}{\eta^*(\tau) f(\tau)}, \\
g_2(\hat{t}) \frac{d\hat{t}}{dt} &= \frac{2}{\eta^*(\tau) f(\tau)},
\end{align*}
\]

where the boundary condition is \( g_i(0) = 0 \), for \( i = 1, 2 \).

For the function \( g_1(\hat{t}) \), a similar technique to the one used in Dewynne and Shaw, is employed. Define a function \( h(\tau) = \hat{t}(\tau) g_1(\hat{t}(\tau)) \), where the independent variable is now back to the \( \tau \) variable. Consider the ODE,

\[
\frac{dh}{d\tau} = \frac{dh}{d\hat{t}} \frac{d\hat{t}}{d\tau} = \frac{dh}{d\hat{t}} \frac{\eta^*(\tau)^2 f(\tau)^2}{2} = \frac{2\hat{t}(\tau)}{\eta^*(\tau) f(\tau)} \frac{\eta^*(\tau)^2 f(\tau)^2}{2} = \hat{t}(\tau) \eta^*(\tau) f(\tau).
\]

The solution for \( h(\tau) \) can be obtained via integration of equation (7.31), together with its initial condition \( h(0) = 0 \). Using Mathematica, the solution for \( h(\tau) \) is given as,

\[
\begin{align*}
h(\tau) &= \frac{e^{-3\tau \theta}}{24\theta^5} \left( 2 + h_{11}(\tau) + h_{12}(\tau) + h_{13}(\tau) \right), \\
h_{11}(\tau) &= 15e^{\tau \theta} (k\theta - 1), \\
h_{12}(\tau) &= -6e^{2\tau \theta} \left( -5 + 2\tau \theta - 4k\theta (\tau \theta - 2) + 2k^2\theta^2 (\tau \theta - 1) \right), \\
h_{13}(\tau) &= e^{3\tau \theta} \left( -17 + 18\tau \theta - 6\tau^2 \theta^2 + 6k^3\tau^2 \theta^5 - 6k^2\theta^2 (2 - 4\tau \theta + 3\tau^2 \theta^2) + 3k\theta (11 - 14\tau \theta + 6\tau^2 \theta^2) \right).
\end{align*}
\]

Division of \( h(\tau) \) by \( \hat{t}(\tau) \) easily gives the solution to \( g_1(\hat{t}(\tau)) \). Given closed form solutions for \( h(\tau) \) and \( \hat{t}(\tau) \), \( g_1(\hat{t}(\tau)) \) can be written in closed form, but due to complexity, this will be omitted here.
For the function $g_2(\hat{t})$, consider the ODEs,

$$\frac{dg_2(\hat{t})}{d\hat{t}} = \frac{2}{\eta^*(\tau)\hat{f}(\tau)},$$

$$\Rightarrow \frac{dg_2}{d\tau} = \eta^*(\tau)\hat{f}(\tau),$$ \hspace{1cm} (7.33)

where in equation (7.33), the problem is converted back to the $\tau$ variable. Again, $g_2(\hat{t}(\tau))$ can be obtained via integration, together with the boundary condition $g_2(0) = 0$. Its solution is given as,

$$g_2(\hat{t}(\tau)) = 1 - e^{-\tau\theta} + \tau\theta(k\theta - 1)\theta^2.$$

### Third Order Solution

The third order solution is given by solving the PDE,

$$\frac{\partial \psi_3}{\partial \tau} - \frac{1}{2}\eta^*(\tau)^2 \frac{\partial^2 \psi_3}{\partial \xi^2} + \theta \frac{\partial \psi_3}{\partial \xi} = \zeta \eta^*(\tau) \frac{\partial^2 \psi_2}{\partial \xi^2} + \eta^*(\tau) \frac{\partial \psi_2}{\partial \xi} + \frac{1}{2} \zeta^2 \frac{\partial^2 \psi_1}{\partial \xi^2} + \zeta \frac{\partial \psi_1}{\partial \xi}. \hspace{1cm} (7.34)$$

Before converting to the $(x, \hat{t})$ variable, some preliminary identities for the source terms on the right hand side of equation (7.34) are required. These are given as,

$$\zeta^2 \frac{\partial^2 \psi_1}{\partial \xi^2} = x^2 \mathcal{G}_1(x, \hat{t}),$$

$$\zeta \frac{\partial \psi_1}{\partial \xi} = x \mathcal{G}_2(x, \hat{t}),$$

$$\frac{\partial \psi_2}{\partial \xi} = \hat{f} \left ( g_1 + g_2 - \frac{x^2g_1}{2\hat{t}} \right ) \mathcal{G}_1(x, \hat{t}),$$

$$\frac{\partial^2 \psi_2}{\partial \xi^2} = \frac{\hat{f}^2x}{2\hat{t}} \left ( \frac{g_1x^2}{2\hat{t}} - 3g_1 - g_2 \right ) \mathcal{G}_1(x, \hat{t}).$$

The transformed PDE for the third order solution is now given by,

$$\frac{\partial \psi_3}{\partial \hat{t}} - \frac{\partial^2 \psi_3}{\partial x^2} = \frac{3\mathcal{H}(x, \hat{t})\mathcal{G}_1(x, \hat{t}) + x\mathcal{G}_2(x, \hat{t})}{2\eta^*(\tau)^2\hat{f}(\tau)^2}, \hspace{1cm} (7.35)$$

where,

$$\mathcal{H}(x, \hat{t}) = (g_1 + g_2) \eta^* \hat{f} + \left ( \frac{1}{2} - \frac{\eta^* \hat{f}}{\hat{t}} \left ( \frac{g_1 + g_2}{2} \right ) \right ) x^2 + \frac{g_1\eta^* \hat{f}}{4\hat{t}^2} x^4.$$

To solve for $\psi_3(x, \hat{t})$, assume that the solution can be written in the form of,

$$\psi_3(x, \hat{t}) = \left ( f_1(\hat{t}) + f_2(\hat{t})x^2 + f_3(\hat{t})x^4 \right ) \mathcal{G}_1(x, \hat{t}) + x f_4(\hat{t}) \mathcal{G}_2.$$
Substitution of this solution into left hand side of equation (7.35) results in,
\[
\frac{\partial \psi_3}{\partial t} - \frac{\partial^2 \psi_3}{\partial x^2} = \left( \left( \frac{df_4}{dt} - 2f_4 - 2f_2 \right) + \left( \frac{2f_3}{t} - 12f_3 + \frac{df_2}{dt} \right)x^2 \right)
\]
\[
+ \left( \frac{4f_3}{t} + \frac{df_3}{dt} \right)x^4 \right) \mathcal{G}_1 + x \frac{df_4}{dt} \mathcal{G}_2,
\]
which can then be equated to the right hand side of equation (7.35). This results in a series of ODEs which the \(f_i\)’s satisfy. Again, the boundary condition is set so that \(f_i(0) = 0\), for \(i = 1, 2, 3\) and 4.

For the function \(f_4\), the ODE is given as,
\[
\frac{df_4}{dt} = \frac{2}{(\eta^* f)^2}.
\]

When converting back to the \(\tau\) variable, \(f_4(\hat{t}(\tau))\) satisfies the following ODE,
\[
\frac{df_4}{d\tau} = 1,
\]
and together with its boundary condition \(f_4(0) = 0\), the solution is given as \(f_4(\hat{t}(\tau)) = \tau\).

The solution for the function \(f_3\) is derived in a similar fashion to Dewynne and Shaw [20]. Consider the ODE,
\[
\frac{d(\hat{t}^4 f_3)}{d\tau} = \frac{d(\hat{t}^4 f_3)}{dt} \frac{dt}{d\tau} = \hat{t}^4 (\eta^* \hat{f})^2 \frac{g_1 \eta^* \hat{f}}{4t^2 \frac{1}{2}(\eta^* f)^2}
\]
\[
= \frac{h \eta^* \hat{f}}{4}
\]
\[
= \frac{1}{4} \frac{dh}{d\tau}
\]
(7.36)
where \(h(\tau)\) is given in equation (7.32). Setting \(F_3(\tau) = \hat{t}^4(\tau)f_3(\hat{t}(\tau))\), the solutions for \(F_3(\tau)\) and \(f_3(\hat{t}(\tau))\) can be obtained as,
\[
F_3(\tau) = \frac{h^2(\tau)}{8},
\]
\[
f_3(\hat{t}(\tau)) = \frac{h^2(\tau)}{8\hat{t}^4},
\]
respectively.
For the function \( f_2(\hat{t}(\tau)) \), consider the ODE,
\[
\frac{d}{d\tau} \left( \hat{t}^2 f_2 \right) = \frac{d}{dt} \left( \hat{t}^2 f_2 \right) \frac{d\hat{t}}{d\tau} = \left( 2\hat{t} f_2 + \hat{t}^2 \frac{df_2}{d\hat{t}} \right) \frac{d\hat{t}}{d\tau} = \frac{1}{2} \hat{t}^2 + 6 f_3 (\eta^* \hat{f})^2 \hat{t}^2 - (\eta^* \hat{f}) \left( 2h + \frac{g_2 \hat{t}}{2} \right). \tag{7.37}
\]
Again, the solution for \( \hat{t}^2 f_2(\hat{t}(\tau)) \) can be obtained via integration of equation (7.37), and thus \( f_2(\hat{t}(\tau)) \) can be obtained through division by \( \hat{t}^2(\tau) \).

Finally, observe that the following ODE,
\[
\frac{df_1}{d\tau} = \frac{df_1}{d\hat{t}} \frac{d\hat{t}}{d\tau} = \frac{1}{2} (\eta^* \hat{f})^2 (2f_2 + 2f_4) + (g_1 + g_2) (\eta^* \hat{f}) = (\eta^* \hat{f}) \left( g_1 + g_2 + (\eta^* \hat{f})(f_2 + f_4) \right), \tag{7.38}
\]
determines the function \( f_1(\hat{t}(\tau)) \). Through integration by \( \tau \), the function \( f_1(\hat{t}(\tau)) \) can be obtained.

It must be noted that the solutions for \( f_1(\hat{t}(\tau)) \) and \( f_2(\hat{t}(\tau)) \) have not been written out explicitly. The reason for this is that the solutions are in fact very difficult to compute for general parameters \( \theta \) and \( k \). The next section will discuss some of the details and suggestions for implementing these solutions.

Notes on Implementation

Implementation of the asymptotic solution presented in the last section can be quite difficult, especially for the third order solution. Compared to the Dewynne and Shaw [20], asymptotic solution for Asian options, the Australian option asymptotic solution has an additional parameter in its functions. Specifically, the fixed strike parameter for an Asian call option is absorbed in the spatial variable of the corresponding PDE, but for the Australian call option, this is not the case. As such, many of the derived functions in the asymptotic solution are dependent on \( k \), the Australian call strike parameter as well. Along with a non-zero \( \theta \), closed form solutions for the asymptotic solution are quite difficult to obtain. Technically, software with symbolic capabilities are able to make many of the above calculations. However, this may be quite time consuming, as well as producing unstable solutions. The lack of stability comes from the convoluted nature of the integrands, namely those in equation (7.37) and (7.38). Below are some suggestions on how to implement the asymptotic solution in Mathematica.
7.2 Asymptotic Expansion

So far, the functions \( \eta^*(\tau), \hat{f}(\tau), \hat{i}(\tau), \Phi_1(x, \hat{i}), g_1(\hat{i}(\tau)), g_2(\hat{i}(\tau)), g_3(x, \hat{i}), \) \( h(\tau), f_4(\hat{i}(\tau)) \) and \( f_5(\hat{i}(\tau)) \) have all either been presented in closed form or can be calculated using closed form functions. The functions \( f_1(\hat{i}(\tau)) \) and \( f_2(\hat{i}(\tau)) \) have been left in their differential form because they are in fact quite difficult to compute symbolically. In terms of implementation, it is simply easier to input the actual values for the parameters \( \theta \) and \( k \), occurring in the pricing problem, into the ODEs rather than trying to symbolically calculate the solution for a general \( \theta \) and \( k \).

A method for obtaining the solution for \( f_2(\hat{i}(\tau)) \) is now presented. Define \( F_2(\tau) \) to be \( \hat{i}^2(\tau)f_2(\hat{i}(\tau)) \) and integrating equation (7.37) gives the solution to \( F_2(\tau) \) as,

\[
F_2(\tau) = \int_0^\tau \left[ \frac{1}{2} \hat{i}(\tau')^2 + 6 f_3(\hat{i}(\tau'))(\eta^*(\tau')\hat{f}(\tau'))^2 \hat{i}(\tau')^2 \right] d\tau' - \int_0^\tau (\eta^*(\tau')\hat{f}(\tau')) \left( 2 h(\tau') + \frac{g_2(\hat{i}(\tau'))\hat{i}(\tau')}{2} \right) d\tau' .
\]

(7.39)

Observe that,

\[
I_1(\tau) = \int_0^\tau 6 f_3(\hat{i}(\tau'))(\eta^*(\tau')\hat{f}(\tau'))^2 \hat{i}(\tau')^2 d\tau' \\
= \int_0^\tau 3 h(\tau')^2 d\tau' \frac{d}{d\tau'} \left( \frac{h(\tau')}{2(\tau')} \right) d\tau' \\
= \left[ \frac{3 h(\tau')^2}{2(\tau')} \right]_0^\tau - 3 \int_0^\tau h(\tau') \frac{h(\tau')}{\tau'} d\tau' + 2 I_1(\tau) \\
= 3 \int_0^\tau h(\tau') \eta^*(\tau') \hat{f}(\tau') d\tau' - \frac{3 h(\tau')^2}{2(\tau')},
\]

by using equation (7.29) and (7.36), integration by parts, some elementary calculations, and equation (7.31), for the second, third, fourth and fifth equalities, respectively. Furthermore, note that,

\[
I_2(\tau) = \int_0^\tau \eta^*(\tau') \hat{f}(\tau') g_2(\hat{i}(\tau')) \hat{i}(\tau') d\tau' \\
= \int_0^\tau g_2(\hat{i}(\tau')) \frac{d}{d\tau'} \left( \frac{h(\tau')}{\tau'} \right) d\tau' \\
= g_2(\hat{i}(\tau)) h(\tau) - \int_0^\tau h(\tau') \frac{g_2(\hat{i}(\tau'))}{d\tau'} d\tau' \\
= g_2(\hat{i}(\tau)) h(\tau) - \int_0^\tau h(\tau') \eta^*(\tau') \hat{f}(\tau') d\tau',
\]
also by integration by parts. Thus, \( F_2(\tau) \) can be simplified as,

\[
F_2(\tau) = \int_0^\tau \left( \frac{1}{2} \hat{t}^2(\tau') - 2h(\tau')\eta^* (\tau') \hat{f}(\tau') \right) d\tau' + I_1(\tau) - \frac{1}{2} I_2(\tau)
\]

\[
= \int_0^\tau \left( \frac{1}{2} \hat{t}^2(\tau') + \frac{3}{2} h(\tau')\eta^*(\tau') \hat{f}(\tau') \right) d\tau' - \frac{3}{2} \frac{h(\tau)^2}{\hat{t}(\tau)} - \frac{1}{2} g_2(\hat{t}(\tau)) h(\tau).
\] (7.40)

It is evident that the integral part in equation (7.40) is much simpler to evaluate than the complete integral in equation (7.39).

The `Integrate` function in Mathematica can be utilized to evaluate the integral part of \( F_2(\tau) \), and together with the other terms, provide a closed form solution. However, for typical values of parameters considered, it is found that \( F_2(\tau) \) exhibits highly oscillatory tendencies around the origin. When divided by \( \hat{t}(\tau)^2 \), to obtain \( f_2(\hat{t}(\tau)) \), these oscillatory tendencies are further amplified due to \( \hat{t}(\tau) \) being a function that approaches zero, when \( \tau \) approaches zero. One way to rectify this problem is to do the following:

1. Obtain the closed form solution to \( F_2(\tau) \) and divide through by \( \hat{t}(\tau)^2 \). Define this as \( \bar{f}_2(\tau) \).

2. Set the explicit values for \( k \) and \( \theta \) in the pricing problem, and expand \( \bar{f}_2(\tau) \) using the `Series` function in Mathematica, around the point 0. Only keep terms of up to powers of 20 in \( \tau \).

3. Collect terms with only non-negative powers in the series expansion and set this as \( f_2(\hat{t}(\tau)) \). This is so that \( f_2(\hat{t}(\tau)) \) does not approach plus or minus infinity as \( \tau \) approaches zero, when terms of negative powers of \( \tau \) are taken. Generally, the coefficients of the negative powers are very small in magnitude, thus the omission will not impact on the accuracy of the solution.

Figure 7.1 shows a plot of the exact solution and its series expansion for \( f_2(\hat{t}(\tau)) \) using parameters from test case no. 3\(^4\). The solution and its approximation are indicated with blue solid and red dashed lines, respectively. The figure shows that the series expansion approximation is very good for large values of \( \tau \), whilst also smoothing out the oscillatory nature for smaller \( \tau \). As such, the series expansion will be used, instead of the exact solution, in subsequent calculations involving \( f_2(\hat{t}(\tau)) \).

From equation (7.38), it is evident that the solution for \( f_1(\hat{t}(\tau)) \) requires an integral where the integrand is a function of \( f_2(\hat{t}(\tau)) \). Given the oscillatory nature of the exact solution for

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\(^4\)To be presented in the numerical results section of this chapter.
Asymptotic Expansion

Figure 7.1: Approximating $f_2(\hat{t}(\tau))$ - The series expansion using up to 21 terms provides a good approximation to the exact solution.

Given that $f_2(\hat{t}(\tau))$, it is clear that the curve in equation (7.38) would also be quite unstable around the origin. Figure 7.2 shows a plot of this curve using both the exact solution and the series expansion for $f_2(\hat{t}(\tau))$, with the same parameter set. Again, the curve with the exact solution and its approximation are indicated with blue solid and red dashed lines, respectively. Given that the solution to $f_1(\hat{t}(\tau))$ requires calculating the integral of the curve in Figure 7.2, using the series expansion for $f_2(\hat{t}(\tau))$ would result in a more stable solution. To see this, note that the series expansion for $f_2(\hat{t}(\tau))$ is sufficiently smooth such that when choosing any small partitioning of the $\tau$ space, for numerical quadrature, that the resulting integral solution is consistent. On the other hand, the oscillatory nature of the exact solution for $f_2(\hat{t}(\tau))$ would make the integral solution highly dependent on which particular partitioning intervals are used, hence making the integral solution unstable.

Given that $f_2(\hat{t}(\tau))$ is calculated for a fixed $k$ and $\theta$, the solution for $f_1(\hat{t}(\tau))$ must also be fixed at those parameter values. Furthermore, since no other function, besides higher order solutions, requires the use of $f_1(\hat{t}(\tau))$, its calculation can be performed using \texttt{NIntegrate}, where the integral is numerically evaluated for the value of $\tau$ in the pricing problem. Together with the series expansion for $f_2(\hat{t}(\tau))$, the numerical integration is computationally much easier to implement than to integrate for a general value of $\tau$. The only downside to doing this is that higher order solutions cannot be calculated.
7.2 Asymptotic Expansion

![Figure 7.2: Approximating \( df_1/d\tau \)- The series expansion for \( f_2 \) makes the curve much more smooth. Ultimately, integration of this curve results in much more stable results.]

**Low Volatility Solution**

Under specific circumstances, the asymptotic solution under a low volatility regime, can be further simplified. For some parameter sets, the contribution from the value \( G_1(x, \hat{t}) \) is negligible, and thus can be omitted from the solution. This is certainly the case when the absolute value of \( x \) is large, or when \( \hat{t}(\tau) \) is small. A general parameter space (in terms of the pricing problem’s original parameters) for which the value \( G_1(x, \hat{t}) \) can be ignored is difficult to obtain. However, if all parameters are kept constant, and only the volatility, \( \sigma \) is varied, then a smaller \( \sigma \) implies a smaller \( \epsilon \), which further implies a greater absolute value for \( x \). Of course, it is prudent to check that \( G_1(x, \hat{t}) \) can indeed be ignored before proceeding. In cases where \( G_1(x, \hat{t}) \) can be ignored, calculations for \( g_1(\hat{t}(\tau)), f_1(\hat{t}(\tau)), f_2(\hat{t}(\tau)) \) and \( f_3(\hat{t}(\tau)) \) are no longer required and thus makes the asymptotic solution much easier to obtain because closed form solutions for the other functions are readily available.

### 7.2.4 Matching Leading Order Solutions

As stated in Section 7.2.1, the leading order solutions automatically agree when moving between the outer and inner regions. Since the proof follows similarly, only the \( r = q \) case will be
7.3 Extensions to the Model

In this section, some extensions to the current work in literature is provided along with extensions to the low volatility asymptotic solution. In particular, it is shown how the Gamma distribution approximation by Moreno and Navas \[59\], can be extended to cater for in progress options, and how the low volatility asymptotic solution can be extended to cater for a general Australian call option as defined in Chapter 6.

### 7.3.1 Gamma Distribution Approximation

This section focuses on extending the Gamma distribution approximation to cater for in progress options. These are cases where the pricing is considered after the averaging period has already started, i.e. \( t > t_0 \). In progress options are rarely considered in literature, even for Asian options. In some cases, like European call and put options, a simple shift in time suffices, and the option’s price can be easily price. However, this is not sufficient for in progress Australian options, as the running ‘sum’ of the asset’s value is required in the averaging at maturity.

Before the Gamma distribution extension (GDE) is made, a bit of background information is provided. Moreno and Navas, point out that the infinite sum of log-normal distributions, follows a reciprocal Gamma distribution. If one were to discretize the integral in the Australian
Extensions to the Model

state process, given in equation (7.1), then the resulting process is a sum of log-normals\(^5\). As the discretization gets smaller and smaller, the number of terms in the summation gets larger and larger, thus justifying the approximation with an infinite sum of log-normals.

Similar to work done by Milevsky and Posner \(^5\), in using the reciprocal Gamma distribution to price Asian options, Moreno and Navas, propose to match the mean and variance of the Australian state process to that of a reciprocal Gamma distribution. More importantly, if \(X\) follows a Gamma distribution with \(\alpha\) and \(\beta\) as the shape and scale parameters, respectively, then \(1/X\) follows a reciprocal Gamma distribution with \(\alpha\) and \(1/\beta\) as the shape and scale parameters, respectively\(^6\). The moments are matched via the equations,

\[
\mathbb{E}(AU_T) = \frac{1}{\beta(\alpha - 1)}, \tag{7.41}
\]

\[
\text{Var}(AU_T) = \frac{1}{\beta^2 (\alpha - 1)^2(k-2)}, \tag{7.42}
\]

and then solved simultaneously to determine \(\alpha\) and \(\beta\). The approximation of the Australian call option can then be computed as,

\[
e^{-rT}\mathbb{E}\left((AU_T - k)^+\right) \approx e^{-rT} \left( \frac{\Gamma(\alpha - 1, \beta, \frac{1}{k})}{\beta(\alpha - 1)} - k\Gamma(\alpha, \beta, \frac{1}{k}) \right),
\]

where \(\Gamma(\alpha, \beta, x)\) is the cumulative distribution function for a Gamma distribution with shape and scale parameters given by \(\alpha\) and \(\beta\), respectively. The cumulative Gamma distribution function is defined as,

\[
\Gamma(\alpha, \beta, x) = \int_0^x \frac{e^{-y/\beta}y^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} dy.
\]

In Moreno and Navas, the formulas for \(\mathbb{E}(AU_T)\) and \(\text{Var}(AU_T)\) are provided. Their derivation relies on working with the GBM process directly, and it is not clear as to how these calculations are performed when taking the conditional expectations, as required in the case of in progress options. The extension proposed here is to consider the conditional mean and variance instead of just their unconditional versions. The benefit of doing so is that the conditional expectations also cater for the unconditional case. In particular, from a pricing point of view, using the conditional version of the expectations, allows for the pricing of Australian options when the option has yet to start averaging (unconditional case), as well as in progress options (conditional case).

\(^5\)\(X_u/X_t\) is still log-normally distributed.

\(^6\)The shape and scale parameters for the Gamma and reciprocal Gamma distribution have different definitions.
In what follows, the conditional mean and variances will be considered directly from the Australian option SDE. In Section 7.1.2, the conditional mean of the Australian state process was derived, so all that remains is to derive its conditional variance. Using the SDE in equation (7.2), one can derive the SDE for $AU_t^2$ via the Itô formula,

$$d \left( AU_t^2 \right) = 2AU_t dAU_t + \frac{1}{2} d [AU, AU]_t$$

$$= 2AU_t^2 (\sigma^2 - r + q) ds + \frac{2}{T} AU_t ds - 2\sigma AU_t dW_t + \sigma^2 AU_t^2 ds$$

$$= (3\sigma^2 - 2(r - q)) AU_t^2 ds + \frac{2}{T} AU_t ds - 2\sigma AU_t^2 dW_t,$$

where $[AU, AU]_t$ is the quadratic variation of $AU_t$. Taking the integral from $t$ to $T_1$ of this SDE, and then the conditional expectation results in,

$$E \left( AU_{T_1}^2 - AU_t^2 \mid \mathcal{F}_t \right) = E \left( \int_t^{T_1} \left( (3\sigma^2 - 2(r - q)) AU_s^2 + \frac{2}{T} AU_s \right) ds \mid \mathcal{F}_t \right)$$

$$- E \left( \int_t^{T_1} 2\sigma AU_s^2 dW_s \mid \mathcal{F}_t \right)$$

$$= E \left( \int_t^{T_1} \left( (3\sigma^2 - 2(r - q)) AU_s^2 + \frac{2}{T} AU_s \right) ds \mid \mathcal{F}_t \right)$$

(7.43)

Note that the last term in the first equality is zero, arguing as in Section 7.1.2.

Define $X(T_1; t)$ to be the deterministic function given by,

$$X(T_1; t) = E \left( AU_{T_1}^2 \mid AU_t = \eta \right).$$

The integration and expectation on the right hand side of equation (7.43) can be interchanged through an application of Fubini’s theorem provided,

$$\int_t^{T_1} E \left( \mid AU_s^2 \mid \mathcal{F}_t \right) ds < \infty.$$  

(7.44)

The goal is then to show that $X(s; t)$ is a bounded function of $s$, which can be seen by consid-
7.3 Extensions to the Model

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\[

ering,

\[
\mathbb{E}\left( AU_s^2 \mid \mathcal{F}_t\right) = \frac{1}{T^2} \mathbb{E}\left( \left( \frac{\int_0^s X_u du}{X_s} \right)^2 \mid \mathcal{F}_t\right)
\]

\[
= \frac{1}{T^2} \mathbb{E}\left( \left( \frac{\int_0^t X_u du + \int_t^s X_u du}{X_s} \right)^2 \mid \mathcal{F}_t\right)
\]

\[
= \frac{1}{T^2} \left[ \left( \int_0^t X_u du \right)^2 \mathbb{E}\left( \frac{1}{X_s^2} \mid \mathcal{F}_t\right) + 2 \left( \int_0^t X_u du \right) \mathbb{E}\left( \int_t^s \frac{X_u}{X_s} du \mid \mathcal{F}_t\right) \right]
\]

\[
+ \frac{1}{T^2} \mathbb{E}\left( \left( \int_t^s X_u du \right)^2 \mid \mathcal{F}_t\right)
\]

(7.45)

The terms in the square brackets on the right hand side of equation (7.45) are finite as \(1/X_s^2\) and \(X_u/X_s^2\) are log-normally distributed, thus the conditional expectations are known continuous function (similar to equation (2.3)). For the last term in equation (7.45), firstly consider \(X_u/X_s\), conditional on \(\mathcal{F}_t\). This process has closed form solution given by,

\[
\frac{X_u}{X_s} = X_t e^{(r-q-t \sigma^2)(s-t) + \sigma W_{s-t}}
\]

\[
= e^{(\sigma^2 + q-r)(s-u) - \sigma W_{s-u}}.
\]

This shows that \(X_u/X_s\) is log-normally distributed and is independent of the filtration \(\mathcal{F}_t\). Applying the transformation \(t' = s - u\), the expectation part of the last term in equation (7.45) becomes,

\[
\mathbb{E}\left( \left( \int_t^s \frac{X_u}{X_s} du \right)^2 \mid \mathcal{F}_t\right) = \mathbb{E}\left( \left( \int_t^{s-t} e^{(\sigma^2 + q-r)t' - \sigma W_{t'}} dt' \right)^2 \right).
\]

Using the Minkowski integral inequality, see Royden [67], results in,

\[
\left( \mathbb{E}\left( \left( \int_0^{s-t} e^{(\sigma^2 + q-r)t' - \sigma W_{t'}} ds \right)^2 \right) \right)^{1/2} \leq \int_0^{s-t} \left( \mathbb{E}\left( e^{(\sigma^2 + q-r)t' - \sigma W_{t'}} \right)^2 \right)^{1/2} dt'
\]

(7.46)

Consider the term,

\[
\mathbb{E}\left( e^{(\sigma^2 + q-r)t' - \sigma W_{t'}} \right)^2 = \mathbb{E}\left( e^{2(\sigma^2 + q-r)t' - 2\sigma W_{t'}} \right),
\]

(7.47)

which is just the expectation of a log-normal random variable. This is a known continuous function in \(t'\), thus on the right hand side of the inequality in (7.46), the square root of the
term in equation (7.47), must also be continuous in \( t' \), meaning the definite integral is also finite. Thus if,

\[
\left( \mathbb{E} \left( \left( \int_0^{s-t} e^{\left( \frac{1}{2} \sigma^2 + q - r \right) t' - \sigma W_{t'} \ ds \right)^2 \right) \right)^{1/2} < 1,
\]

then,

\[
\mathbb{E} \left( \left( \int_0^{s-t} e^{\left( \frac{1}{2} \sigma^2 + q - r \right) t' - \sigma W_{t'} \ ds \right)^2 \right) < \infty.
\]

Otherwise, by squaring the inequality in (7.46), it is shown that,

\[
\mathbb{E} \left( \left( \int_0^{s-t} e^{\left( \frac{1}{2} \sigma^2 + q - r \right) t' - \sigma W_{t'} \ ds \right)^2 \right) < \infty,
\]
given that the right hand side of the inequality in (7.46) is finite.

Together, this shows that \( X(s; t) \) is finite and bounded, and thus, by extension the inequality in (7.44) is satisfied.

After interchanging the expectation and integration in equation (7.43), differentiate this with respect to \( T_1 \). The resulting ODE for \( X(T_1; t) \) is,

\[
\frac{dX(T_1; t)}{dT_1} = (3\sigma^2 - 2(r - q))X(T_1; t) + \frac{2}{T} Y(T_1; t),
\]

which can be solved by considering the equivalent ODE given by,

\[
\frac{d \left( X(T_1; t)e^{-\left(3\sigma^2-2(r-q)\right)T_1} \right)}{dT_1} = \frac{2}{T} Y(T_1; t)e^{-\left(3\sigma^2-2(r-q)\right)T_1}.
\]

Assuming neither \( a := -\sigma^2 + r - q, b := -2\sigma^2 + r - q \) or \( c := -3\sigma^2 + 2(r - q) \) are zero, integrating this with respect to \( T_1 \) gives,

\[
X(T_1; t)e^{-\left(3\sigma^2-2(r-q)\right)T_1} = \frac{2}{T} \left( \eta + \frac{1}{T (\sigma^2 - r + q)} \right) e^{-\left(\sigma^2-r+q\right)t} \int e^{\left(-2\sigma^2+r-q\right)t_1} \ dt_1
- \frac{2}{T^2 (\sigma^2 - r + q)} \int e^{-\left(3\sigma^2-2(r-q)\right)t_1} \ dt_1,
\]

\[
= \frac{2}{T} \left( \eta + \frac{1}{T (\sigma^2 - r + q)} \right) e^{-\left(\sigma^2-r+q\right)t} \frac{e^{\left(-2\sigma^2+r-q\right)T_1}}{\left(-2\sigma^2 + r - q\right)}
- \frac{2}{T^2 (\sigma^2 - r + q)} \left(-3\sigma^2 + 2(r - q)\right) + c_2,
\]

for some constant \( c_2 \). The special case where any of \( a, b \) or \( c \) equal zero will be treated separately.

At \( T_1 = t \), \( X(t; t) = AU_t^2 = \eta^2 \), and thus the constant \( c_2 \) can be calculated as,

\[
c_2 = \eta^2 e^{\left(-3\sigma^2+2(r-q)\right)t} - \frac{2}{T} \left( \eta + \frac{1}{T (\sigma^2 - r + q)} \right) \frac{e^{\left(-3\sigma^2+2(r-q)\right)t}}{\left(-2\sigma^2 + r - q\right)}
+ \frac{2}{T^2 (\sigma^2 - r + q)} \left(-3\sigma^2 + 2(r - q)\right).
\]
Therefore,
\[
\mathbb{E}\left( AU_{T_1}^2 \middle| AU_t = \eta \right) = \eta^2 e^{(2\sigma^2-2(r-q))(T_1-t)} + \frac{2}{T} \left( \eta + \frac{1}{T} - \frac{2}{T^2} e^{(3\sigma^2-2(r-q))(T_1-t)} \right) + 2 \frac{2}{T^2} e^{(3\sigma^2-2(r-q))(T_1-t)} - 1.
\]

For the special cases, note that if \( \eta = a + b \) and \( \beta = a - \sigma^2 \). Thus,

- if \( \eta = 0 \), then \( c = -\sigma^2 \neq 0 \),
- if \( \beta = 0 \), then \( c = \sigma^2 \neq 0 \),
- if \( c = 0 \), then \( a = \sigma^2/2 \neq 0 \) and \( b = -\sigma^2/2 \neq 0 \).

The solutions to \( X(T_1;t) \) in these cases are given as,

\[
X(T_1;t) = \begin{cases} 
\frac{1}{\sigma^2 T} e^{\sigma^2 (T_1-t)} \left( 1 + (1 + \eta \sigma^2 T)^2 \right) + \frac{2}{\sigma^2 + \eta \sigma^2 T} (\eta^2 + 2 \eta \sigma^2 - 1), & \text{if } \eta = 0, \\
\eta^2 e^{-\sigma^2 (T_1-t)} + \frac{2}{T} (\eta - \frac{1}{T} \sigma^2 ) (T_1-t) e^{-\sigma^2 (T_1-t)} + \frac{2}{\sigma^2 + \eta \sigma^2 T} (1 - e^{-\sigma^2 (T_1-t)}) - 1, & \text{if } \eta = 0, \\
\eta^2 + \frac{8}{T \sigma^2 T} \left( e^{-\frac{1}{2} \sigma^2 (T_1-t)} - 1 \right) + \frac{4}{\sigma^2 T} \left( \eta \left( 1 - e^{-\frac{1}{2} \sigma^2 (T_1-t)} \right) + \frac{T_1-t}{2} \right), & \text{if } \eta = 0.
\end{cases}
\]

Using \( X(T_1;t) \) and \( Y(T_1;t) \), the conditional variance can be obtained as,

\[
\text{Var} \left( AU_{T_1} \middle| F_t \right) = X(T_1;t) - \left( Y(T_1;t) \right)^2.
\]

The extension to the Gamma distribution approximation is thus to replace the unconditional mean and variance in equations (7.41) and (7.42), by their conditional counterparts. After solving for \( \alpha \) and \( \beta \), these can then be used to approximate the price of an Australian call option by using,

\[
e^{-r(T-t)} \mathbb{E} \left( (AU_T - k)^+ \middle| F_t \right) \approx e^{-r(T-t)} \left( \frac{\Gamma(\alpha - 1, \beta, \frac{1}{k})}{\beta (\alpha - 1)} - k \Gamma \left( \alpha, \beta, \frac{1}{k} \right) \right).
\]

### 7.3.2 General Australian Call Options

In Chapter 6, the general formulation of the Australian call option problem was presented. In particular, the two strike parameters \( k_1 \) and \( k_2 \) were introduced. At the start of this chapter, it was made clear that the material to be presented, only considers \( k_1 = 0 \). This is due to the
fact that many studies on Australian options (thus far), only consider this case. However, the addition of a non-zero \( k_1 \) strike will now be considered and its affect on the Australian option investigated.

From the beginning, define a (new) general Australian state process as,

\[
\hat{AU}_t = \frac{\int_{t_0}^t X_u \, du - k_1 T}{TX_t} = AU_t - \frac{k_1}{X_t}. \tag{7.48}
\]

It follows that \( \hat{AU}_t \) satisfies a SDE with the same form as equation (7.2), with \( \hat{AU}_t \) replacing \( AU_t \). Furthermore, the general Australian call and put options have form similar to equations (7.3) and (7.4), respectively, with again \( \hat{AU}_T \) replacing \( AU_T \). In particular, the put-call parity holds, and the conditional expectation of \( \hat{AU}_T \), given the filtration \( \mathcal{F}_t \), again has the same form as \( Y(T_1; t) \), but with \( \eta = \hat{AU}_t \), instead of \( AU_t \). To check the validity of using Fubini’s theorem, it is easy to see that,

\[
\left| \hat{AU}_t \right| \leq \left| AU_t \right| + \left| \frac{k_1}{X_t} \right|,
\]

by the triangle inequality, and thus, the conditional expectation of \( \left| \hat{AU}_t \right| \) is also finite.

The closed form solution of a general Australian option does not exist. Note that the equivalent definition of a (new) general New Zealand state process, as the inverse of the general Australian state process, can at times be negative for \( t > t_0 \). However, the New Zealand state process, as a GMR process, must be positive at all times. Thus, the general New Zealand state process cannot be identified as a GMR process, and so the closed form solution for pricing an general Australian call option cannot be derived using a similar methodology.

The pricing PDE and the low volatility asymptotic solution all remains valid under a general Australian call option. To see this, again note that the SDE for a general Australian state process, follows a similar form to the (old) Australian state process. Thus, the resulting PDE for the general Australian call option remains the same and as such, the low volatility asymptotic solution is still valid. The only point of difference is that when using this pricing method, the initial value is that of \( \hat{AU}_t \) and not \( AU_t \).

The general Australian call option cannot technically be priced with the Gamma distribution approximation. Recall that the assumption in using the Gamma distribution approximation,

\footnote{The strike \( k_1 \) is strictly positive.}
is that the (old) Australian state process can be approximated by a reciprocal Gamma distribution. By definition, a Gamma distribution is a non-negative random variable. However the general Australian state process can take on negative values, in particular at \( t = t_0 \), for any non-zero \( k_1 \). Furthermore, looking at this from a more probabilistic point of view, it is possible to use Cantelli’s inequality, see Theorem 4.8 in DasGupta [18], to determine an upper or lower bound on the probability that \( \overline{AU}_T \) is negative. Note that the condition for \( \overline{AU}_T \leq 0 \) is equivalent to,

\[
\int_{t_0}^{T} X_u \, du - k_1 T \leq 0.
\]

Let,

\[
\bar{\mu} = \mathbb{E} \left( \int_{t_0}^{T} X_u \, du \right), \quad \bar{\sigma}^2 = \text{Var} \left( \int_{t_0}^{T} X_u \, du \right), \quad a = \bar{\mu} - k_1 T.
\]

Then,

\[
\mathbb{P} \left( \int_{t_0}^{T} X_u \, du - k_1 T \leq 0 \right) = \mathbb{P} \left( \int_{t_0}^{T} X_u \, du - \bar{\mu} \leq k_1 T - \bar{\mu} \right)
= \mathbb{P} \left( \int_{t_0}^{T} X_u \, du - \bar{\mu} \leq -a \right)
\begin{cases}
\leq \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + a^2}, & \text{if } a > 0, \\
\geq \frac{a^2}{\bar{\sigma}^2 + a^2}, & \text{if } a < 0.
\end{cases}
\]

The calculations for \( \bar{\mu} \) and \( \bar{\sigma}^2 \) are easily derived from the first two moments of an Asian option’s underlier, which can be readily found in Milevsky and Posner [58]. For any specific pricing problem \( \bar{\mu} \) and \( \bar{\sigma}^2 \) are fixed. Assuming \( a > 0 \), the upper bounds on the probability that \( \overline{AU}_T \leq 0 \) increases with \( k_1 \). For \( a < 0 \), the lower bound increases with \( k_1 \). In both cases, for an increasing \( k_1 \), the chance that \( \overline{AU}_T \leq 0 \) also increases. This explains why the assumptions used in approximating the Australian call option, is not valid under a general Australian call option.

The importance in being able to price general Australian call options cannot be stressed enough. Recall in Section 6.1.1, that an equivalency between general Asian and Australian call options was presented. Along with the low volatility asymptotic solution for a general Australian call option, these two results are able to unify the pricing of Asian and Australian call options across different strike types under a low volatility regime.
Further to this, recall that Dewynne and Shaw [20], has derived closed form, low volatility asymptotic solutions for fixed strike Asian options. Their work also claims that a similar technique can be used for the floating strike cases. However, in their brief treatment of floating strike options, they only consider the case where $k_2 = 1$, and thus lack generality. Of course, the technique can be modified to cater for a $k_2$ parameter that is allowed to vary, but at the very least under the Australian options setting, it was shown that with this additional parameter, closed form solutions for the higher orders, are difficult to obtain. Assuming that a closed form solution for the asymptotic solution can be derived, there is still the issue of modifying the problem to cater for a general Asian call option. The low volatility asymptotic solution for general Australian call options essentially by-passes and solves all these issues. In fact, it is akin to looking at the general Asian call option problem from a different perspective, which was motivated by the general Australian call option problem. As such, the asymptotic solution is able to solve the general Asian call option problem with ease.

7.4 Numerical Results

This section compares the numerical results of the asymptotic solution of Australian call options by considering various test cases. The test cases include those from published literature and low volatility regimes.

7.4.1 Comparisons to the Literature

This section contains the comparisons between the numerical results obtained from the asymptotic solution to that of published results. As noted in the chapter introduction, literature on Australian options is generally very sparse, with the only notable publication containing numerical results being that of Moreno and Navas [59]. The test cases used in Moreno and Navas, do not meet the low volatility assumption, with $\sigma$ being set at 0.2 and 0.4. However comparisons between the two results are still made. Furthermore, the solutions will be compared to those obtained using a Crank Nicholson (CN) FDM to solve the PDE in equation (7.10), as well as using a Monte Carlo simulation method. The latter MC simulation, differs from Moreno and Navas, in the sense that it uses more timesteps and simulation paths.

The 4 test cases taken from Moreno and Navas, have parameters $X_t = 1$, $r = 0.1$, $q = 0.03$ and $t = 0$, with further parameters given in Table 7.3. The relative errors of these test cases are

\[9\] This is largely due to the presence of both $\theta$ and $k_2$ as free parameters, thus increasing the complexity of the problem.
7.4 Numerical Results

given in Table 7.4, where it is assumed that the true solution is given by the CN FDM solution.

<table>
<thead>
<tr>
<th>No.</th>
<th>$\sigma$</th>
<th>$T$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.5</td>
<td>1.1</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>1.0</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 7.3: Moreno and Navas Test Case; Parameters

<table>
<thead>
<tr>
<th>No.</th>
<th>Asymp-20</th>
<th>MC</th>
<th>MN-MC</th>
<th>MN-W</th>
<th>MN-GD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3982</td>
<td>0.0839</td>
<td>-0.1509</td>
<td>0.2432</td>
<td>-0.1509</td>
</tr>
<tr>
<td>2</td>
<td>-0.2298</td>
<td>-1.6425</td>
<td>-0.8426</td>
<td>-3.7237</td>
<td>1.7983</td>
</tr>
<tr>
<td>3</td>
<td>-0.0345</td>
<td>0.0620</td>
<td>-0.0134</td>
<td>0.0139</td>
<td>-0.0025</td>
</tr>
<tr>
<td>4</td>
<td>-0.1024</td>
<td>0.0787</td>
<td>0.1939</td>
<td>0.0990</td>
<td>-0.0552</td>
</tr>
</tbody>
</table>

Table 7.4: Moreno and Navas Test Case; Relative Errors in percentages, as compared to CN FDM

Asymp-20 denotes the asymptotic solution using power series expansions of up to powers of 20 in $\tau$ in calculating the functions $f_2(\hat{t}(\tau))$ and subsequently $f_1(\hat{t}(\tau))$, while MC refers to the MC simulations with 4096 number of timesteps and 200000 sample paths.

The MN prefix in Table 7.4 indicates that the results are taken from Moreno and Navas, with MN-MC, MN-W and MN-GD representing their MC, Wilkinson approximation and the Gamma distribution approximation, respectively. Their MC simulation is performed using 10000 paths and antithetic variables, with 1000 timesteps in their averaging period. The Wilkinson approximation is essentially using the first two cumulants in the generalized Edgeworth series expansion to approximate the true density function of the Australian state variable. In Moreno and Navas, the log-normal distribution is used as the approximating distribution. The Gamma distribution approximation is the result of those discussed in Section 7.3.1.

These results show that even for relatively large volatility, the asymptotic solution does a reasonable job at approximating the true solution, with the absolute relative error being no larger than 0.40%. However, it is noted that high accuracy is not achieved. Furthermore, the relative errors, as compared to the CN FDM solution shows that in all 4 test cases, none of the MN approximating solutions nor the asymptotic solution outright performs better than the
7.4 Numerical Results

others. While Moreno and Navas do not favour any of their approximating solutions over the others, it is clear from the results that the asymptotic solution can be a suitable substitute.

7.4.2 Low Volatility Regime

In this section, the numerical solutions for the low volatility regime test cases will be presented. Generally, define a low volatility regime to be when the volatility is less than or equal to 0.1, which is the same definition used in Dewynne and Shaw. Furthermore, the test cases to follow will include cases where \( r \neq q \), \( r = q \), in progress options, and general Australian and Asian call options.

Case: \( r \neq q \)

The following test cases have \( r \neq q \). The test cases are designated as test case 5 to 8, with the parameters \( X_t = 1 \), \( r = 0.1 \), \( q = 0.03 \), \( T = 1 \), \( t = 0 \) and \( k = 0.8 \) while volatility varies in values of 0.1, 0.05, 0.02 and 0.01. Asymp-LV denotes the low volatility solution that is computed by assuming the contribution from \( \mathcal{G}_1(x, \hat{t}) \) is negligible. The MN-GD solution is also provided as a reference. The result of these tests are found in Table 7.5.

<table>
<thead>
<tr>
<th>No.</th>
<th>Asymp-20</th>
<th>Asymp-LV</th>
<th>CN FDM</th>
<th>MC</th>
<th>MN-GD</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.154345</td>
<td>0.154343</td>
<td>0.154360</td>
<td>0.154361</td>
<td>0.154360</td>
</tr>
<tr>
<td>6</td>
<td>0.151104</td>
<td>0.151104</td>
<td>0.151105</td>
<td>0.151112</td>
<td>0.151105</td>
</tr>
<tr>
<td>7</td>
<td>0.150197</td>
<td>0.150197</td>
<td>0.150197</td>
<td>0.150207</td>
<td>0.150197</td>
</tr>
<tr>
<td>8</td>
<td>0.150068</td>
<td>0.150068</td>
<td>0.150068</td>
<td>0.150079</td>
<td>0.150068</td>
</tr>
</tbody>
</table>

Table 7.5: Low Volatility Solution for \( r \neq q \)

Both the Asymp-20 and Asymp-LV solutions match the CN FDM to a very high degree of accuracy. In particular, for the lowest volatility test case, the asymptotic solutions are accurate to at least 6 significant figures, while for the highest volatility, the solutions are still accurate to 4 significant figures. For the parameters tested, it can be concluded that if \( \mathcal{G}_1(x, \hat{t}) \) is negligible, which is indeed the case here, the difference between the two asymptotic solutions is quite small and match well to the CN FDM solutions. It is noted that the MN-GD solution performs well in these test cases.

Case: \( r = q \)

The following test cases are for the \( r = q \) case. The same parameter sets are used as in test cases 5 to 8, but with \( r = q = 0.03 \), and will be referred to as test cases 9 to 12. The closed
form asymptotic solution, using up to the third order solution, will be listed under the Asymp heading, in Table 7.6.

<table>
<thead>
<tr>
<th>No.</th>
<th>Asymp</th>
<th>CN FDM</th>
<th>MC</th>
<th>MN-GD</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.198941</td>
<td>0.198958</td>
<td>0.198947</td>
<td>0.198958</td>
</tr>
<tr>
<td>10</td>
<td>0.195302</td>
<td>0.195303</td>
<td>0.195300</td>
<td>0.195303</td>
</tr>
<tr>
<td>11</td>
<td>0.194283</td>
<td>0.194283</td>
<td>0.194283</td>
<td>0.194283</td>
</tr>
<tr>
<td>12</td>
<td>0.194138</td>
<td>0.194138</td>
<td>0.194139</td>
<td>0.194138</td>
</tr>
</tbody>
</table>

Table 7.6: Low Volatility Solution for $r = q$

The asymptotic solution performs as well as its $\theta \neq 0$ counterpart, as there is a high level of agreement between the three solutions. The Asymp solution is accurate to at least 4 significant figures for all test cases, with some achieving agreement to 6 significant figures. Again, the Mn-GD solution performs well.

**Case: In Progress Options**

Test cases 13 and 14 involve looking at in progress options, when $r \neq q$ and $r = q$, respectively. Furthermore, results using the GDE\(^{10}\) will be provided. Test case 13 has parameters $X_t = 100$, $\sigma = 0.05$, $r = 0.1$, $q = 0.03$, $T = 1$, $t = 0.5$, $\int_{t_0}^{t} X_u \, du = 50$, while test case 14 has the same set, but with $r = 0.03$.

In test case 13, only the Asymp-LV solution is presented, due to $\mathcal{G}_1$ being negligible for this parameter set. The results of test case 13 and 14 can be found in Tables 7.7 and 7.8, respectively. In both cases, the relative errors for the low volatility asymptotic solution are calculated assuming that the CN FDM solution is the true solution.

The relative errors in Table 7.7 and 7.8 show that low volatility asymptotic solution has quite a high degree of accuracy compared to the CN FDM solution. Both are within 5, and 6 significant figures of the CN FDM solution, for the $r \neq q$ and $r = q$ cases, respectively. The GDE approximation provides an even greater level of accuracy, with 8 or more significant figures in agreement when $k \leq 0.8$. The reason as to why the performance of the asymptotic solution increases as $k$ decreases can be explained as follows; Recall that in deriving the asymptotic solution, an outer and inner region was formed. Furthermore, the outer region corresponds to the case where the option is either deeply ITM or deeply OTM, and that in this region the volatility has negligible effect on the option’s price. Now, as $k$ decreases, the option begins to

\(^{10}\)Note that the Gamma distribution approximation is now referred to as the GDE, as the original MN-GD form of the solution cannot price in progress options, or general Australia and Asian call options to come.
7.4 Numerical Results

<table>
<thead>
<tr>
<th>$k$</th>
<th>Asymp-LV</th>
<th>CN FDM</th>
<th>GDE</th>
<th>Relative Errors for Asymp-LV (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.071406333</td>
<td>0.071408436</td>
<td>0.071410754</td>
<td>-0.0029</td>
</tr>
<tr>
<td>0.8</td>
<td>0.166524802</td>
<td>0.166525282</td>
<td>0.166525281</td>
<td>-0.0003</td>
</tr>
<tr>
<td>0.7</td>
<td>0.261647744</td>
<td>0.261648224</td>
<td>0.261648224</td>
<td>-0.0002</td>
</tr>
<tr>
<td>0.6</td>
<td>0.356770687</td>
<td>0.356771166</td>
<td>0.356771166</td>
<td>-0.0001</td>
</tr>
<tr>
<td>0.5</td>
<td>0.451893629</td>
<td>0.451894109</td>
<td>0.451894109</td>
<td>-0.0001</td>
</tr>
<tr>
<td>0.4</td>
<td>0.547016572</td>
<td>0.547017051</td>
<td>0.547017051</td>
<td>-0.0001</td>
</tr>
<tr>
<td>0.3</td>
<td>0.642139514</td>
<td>0.642139994</td>
<td>0.642139994</td>
<td>-0.0001</td>
</tr>
<tr>
<td>0.2</td>
<td>0.737262457</td>
<td>0.737262936</td>
<td>0.737262936</td>
<td>-0.0001</td>
</tr>
<tr>
<td>0.1</td>
<td>0.832385399</td>
<td>0.832385879</td>
<td>0.832385879</td>
<td>-0.0001</td>
</tr>
</tbody>
</table>

Table 7.7: Low Volatility Solution for $r \neq q$, for In Progress Option, Test Case 13

<table>
<thead>
<tr>
<th>$k$</th>
<th>Asymp</th>
<th>CN FDM</th>
<th>GDE</th>
<th>Relative Errors for Asymp (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.099434925</td>
<td>0.099435274</td>
<td>0.099435421</td>
<td>-0.00035</td>
</tr>
<tr>
<td>0.8</td>
<td>0.197945930</td>
<td>0.197946444</td>
<td>0.197946444</td>
<td>-0.00026</td>
</tr>
<tr>
<td>0.7</td>
<td>0.296457124</td>
<td>0.296457638</td>
<td>0.296457638</td>
<td>-0.00017</td>
</tr>
<tr>
<td>0.6</td>
<td>0.394968318</td>
<td>0.394968832</td>
<td>0.394968832</td>
<td>-0.00013</td>
</tr>
<tr>
<td>0.5</td>
<td>0.493479512</td>
<td>0.493480026</td>
<td>0.493480026</td>
<td>-0.00010</td>
</tr>
<tr>
<td>0.4</td>
<td>0.591990706</td>
<td>0.591991219</td>
<td>0.591991219</td>
<td>-0.00009</td>
</tr>
<tr>
<td>0.3</td>
<td>0.690501900</td>
<td>0.690502413</td>
<td>0.690502413</td>
<td>-0.00007</td>
</tr>
<tr>
<td>0.2</td>
<td>0.789013094</td>
<td>0.789013607</td>
<td>0.789013607</td>
<td>-0.00007</td>
</tr>
<tr>
<td>0.1</td>
<td>0.887524288</td>
<td>0.887524801</td>
<td>0.887524801</td>
<td>-0.00006</td>
</tr>
</tbody>
</table>

Table 7.8: Low Volatility Solution for $r = q$, for In Progress Option, Test Case 14
move from being nearer to the money to being more ITM, and as it does, the effects of the volatility diminishes. As the asymptotic solution is expanded in the scaled volatility parameter, higher order solutions thus provide negligible contributions to the accuracy as $k$ decreases.

**Case: General Australian Call Options**

The following test cases are for general Australian call options. The parameters are $X_t = 100$, $\sigma = 0.05$, $r = 0.04$, $q = 0.03$, $T = 1$, $t = 0$, $k = k_2 = 0.1$, with $k_1$ varying from 90 to 80. The result of this test case can be found in Table 7.9. For added reference, the GDE solutions are also listed.

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>Asymp-LV</th>
<th>CN FDM</th>
<th>GDE</th>
<th>Relative Errors for Asymp-LV (%)</th>
<th>Relative Errors for GDE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>0.010908</td>
<td>0.010882</td>
<td>0.010421</td>
<td>0.2550</td>
<td>-4.2236</td>
</tr>
<tr>
<td>89</td>
<td>0.016880</td>
<td>0.016712</td>
<td>0.015935</td>
<td>1.0058</td>
<td>-4.6468</td>
</tr>
<tr>
<td>88</td>
<td>0.024178</td>
<td>0.023941</td>
<td>0.023202</td>
<td>0.9899</td>
<td>-3.0869</td>
</tr>
<tr>
<td>87</td>
<td>0.032473</td>
<td>0.032258</td>
<td>0.031790</td>
<td>0.6645</td>
<td>-1.4530</td>
</tr>
<tr>
<td>86</td>
<td>0.041404</td>
<td>0.041259</td>
<td>0.041060</td>
<td>0.3529</td>
<td>-0.4820</td>
</tr>
<tr>
<td>85</td>
<td>0.050684</td>
<td>0.050608</td>
<td>0.050552</td>
<td>0.1512</td>
<td>-0.1106</td>
</tr>
<tr>
<td>84</td>
<td>0.060126</td>
<td>0.060094</td>
<td>0.060084</td>
<td>0.0524</td>
<td>-0.0175</td>
</tr>
<tr>
<td>83</td>
<td>0.069632</td>
<td>0.069621</td>
<td>0.069620</td>
<td>0.0156</td>
<td>-0.0018</td>
</tr>
<tr>
<td>82</td>
<td>0.079160</td>
<td>0.079156</td>
<td>0.079156</td>
<td>0.0048</td>
<td>-0.0001</td>
</tr>
<tr>
<td>81</td>
<td>0.088694</td>
<td>0.088692</td>
<td>0.088692</td>
<td>0.0021</td>
<td>0.0000</td>
</tr>
<tr>
<td>80</td>
<td>0.098230</td>
<td>0.098228</td>
<td>0.098228</td>
<td>0.0015</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

**Table 7.9:** Low Volatility Solution for General Australian Call Options, Test Case 15

The results for the general Australian call option test is quite surprising. Firstly, for a low $k_1$ strike, the GDE solution seems to approximate the true solution to a high degree of accuracy. To explain this, recall that the $k_1$ strike is embedded into the general Australian state process, see equation (7.48). For lower $k_1$ strikes, the general Australian state process, may still be positive at its terminal value, and thus the reciprocal Gamma approximation may still be robust enough to provide accurate solutions. The upper bound of the probability that $\tilde{AU}_T \leq 0$ as a function of $k_1$, is shown in Figure 7.3. This figure shows that $\tilde{AU}_T$ has a greater probability of obtaining a negative terminal value, as $k_1$ increases. Thus, as $k_1$ increases, the probability that the Gamma distribution approximation remaining valid decreases. This explains the GDE results shown in Table 7.9. For the Asymp-LV result, a decreasing $k_1$ again
signifies the options moving to a more ITM region, thus its approximation of the true solution increases in performance.

To further explain the observed phenomenon, consider the extreme case where \( k = k_2 = 0 \).

In this scenario, the general Australian call option essentially reduces to computing,

\[
e^{-r(T-t)} \mathbb{E} \left( \left( \overline{AU}_T \right)^+ \bigg| \mathcal{F}_t \right).
\]

If \( k_1 \) is small such that, \( \overline{AU}_T \) is rarely negative, then

\[
e^{-r(T-t)} \mathbb{E} \left( \left( \overline{AU}_T \right)^+ \bigg| \mathcal{F}_t \right) \approx e^{-r(T-t)} \mathbb{E} \left( \overline{AU}_T \bigg| \mathcal{F}_t \right),
\]

which is the GDE approximation. However, if \( \overline{AU}_T \) is sometimes negative, then the approximation breaks down. Regardless, the GDE solution will always be a lower bound to the true solution, i.e. \( \mathbb{E} \left( \overline{AU}_T \bigg| \mathcal{F}_t \right) < \mathbb{E} \left( \left( \overline{AU}_T \right)^+ \bigg| \mathcal{F}_t \right) \).

Table 7.10 shows the result of test case 16, which has \( k_2 = 0 \), and all other parameters as set in test case 15. It is evident in this test case that as \( k_1 \) increases, such that the GDE approximation breaks down, the low volatility asymptotic solution still provides reasonable approximations. Again, the explanation of the increase in performance as \( k_1 \) decreases is similar to test case 15, for both the GDE and Asymp-LV. Of interest, the GDE is also shown to underestimate the true value of the solution as predicted.
7.4 Numerical Results

Table 7.10: Low Volatility Solution for General Australian Call Options, Test Case 16

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>Asymp-LV</th>
<th>CN FDM</th>
<th>GDE</th>
<th>Relative Errors for Asymp-LV (%)</th>
<th>Relative Errors for GDE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.012900</td>
<td>0.012869</td>
<td>0.003585</td>
<td>0.2420</td>
<td>-72.1415</td>
</tr>
<tr>
<td>97.5</td>
<td>0.029757</td>
<td>0.029542</td>
<td>0.027425</td>
<td>0.7273</td>
<td>-7.16424</td>
</tr>
<tr>
<td>95</td>
<td>0.051609</td>
<td>0.051495</td>
<td>0.051266</td>
<td>0.2207</td>
<td>-0.4457</td>
</tr>
<tr>
<td>92.5</td>
<td>0.075135</td>
<td>0.075115</td>
<td>0.075106</td>
<td>0.0269</td>
<td>-0.0120</td>
</tr>
<tr>
<td>90</td>
<td>0.098949</td>
<td>0.098946</td>
<td>0.098946</td>
<td>0.0028</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

In both test cases 15 and 16, the performance of the low volatility asymptotic solution is quite reasonable. For higher strikes $k_1$ strikes, the low volatility asymptotic solution outperforms the GDE solution as evident by the relative errors, while for lower $k_1$ strikes, the relative errors are within acceptable levels. These results show that the low volatility asymptotic solution generally performs quite well in general Australian call option problems.

Case: General Asian Call Options

This section showcases the universality of the low volatility asymptotic solution for a general Australian call option, by using its solutions to price general Asian call options.

The first example uses results taken from Dewynne and Shaw [20], and their low volatility asymptotic solution for a fixed strike Asian call option. The parameters are as follows; $X_t = 2$, $r = 0.02$, $q = 0$, $T = 1$, $t = 0$, $k_2 = 0$ and $k_1 = 2$. Five different volatility parameters are tested, with the values being 0.1, 0.05, 0.01, 0.005 and 0.001. These examples are designated as test cases 17 to 21, respectively, and are labelled in Dewynne and Shaw as 4, 4A, 4B, 4C and 4D, respectively. The results of these test cases are found in Table 7.11, with DS representing the Dewynne and Shaw low volatility asymptotic solution.

Table 7.11: Low Volatility Solution for Fixed Strike Asian Call Options

<table>
<thead>
<tr>
<th>No.</th>
<th>Asymp-LV</th>
<th>DS</th>
<th>GDE</th>
<th>Relative Errors for Asymp-LV (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>0.0559781</td>
<td>0.0559860</td>
<td>0.0199337</td>
<td>-0.0141</td>
</tr>
<tr>
<td>18</td>
<td>0.0340347</td>
<td>0.0339412</td>
<td>0.0197847</td>
<td>0.2755</td>
</tr>
<tr>
<td>19</td>
<td>0.0199339</td>
<td>0.0199278</td>
<td>0.0197373</td>
<td>0.0306</td>
</tr>
<tr>
<td>20</td>
<td>0.0197357</td>
<td>0.0197357</td>
<td>0.0197358</td>
<td>0.0000</td>
</tr>
<tr>
<td>21</td>
<td>0.0197353</td>
<td>0.0197353</td>
<td>0.0197353</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The relative errors are calculated assuming the DS solution as the exact solution. This is a
valid assumption because Dewynne and Shaw found their low volatility asymptotic solution to be quite accurate compared to other techniques. While the DS solution excels at pricing fixed strike Asian call options, the relative errors show that the Asymp-LV solution does reasonably well, especially as volatility decreases. Furthermore, the GDE approximations are very poor for test cases 17 and 18. This is reasoned by the fact that not only does the Gamma distribution approximation assumption break down, but also the volatility is not as negligible as in test cases 19 to 21.

In the next test cases, general Asian call options with non-zero $k_1$ and $k_2$, are considered. The test cases are designated 22 to 28, with parameters similar to test case 18, but with $k_1$ and $k_2$ varying. The strike parameters are chosen such that $k_1 + k_2 X_t = X_t$, i.e. the strike combinations are equal to the current asset’s value. As stated in the introduction of Chapter 6, Vecer [76], has derived a PDE which the price of a general Asian call option must satisfy. However, as no closed form solutions are available, the only way to approach the problem is through numerical techniques11. The solutions to test cases 22 to 28 are calculated using the Asymp-LV, GDE and Vecer PDE method, with the results found in Table 7.12.

<table>
<thead>
<tr>
<th>No.</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>Asymp-LV</th>
<th>GDE</th>
<th>Vecer</th>
<th>Relative Errors for Asymp-LV (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>2.00</td>
<td>0.000</td>
<td>0.0340347</td>
<td>0.0197847</td>
<td>0.0339462</td>
<td>0.2605</td>
</tr>
<tr>
<td>23</td>
<td>1.75</td>
<td>0.125</td>
<td>0.0269880</td>
<td>0.0253055</td>
<td>0.0269185</td>
<td>0.2584</td>
</tr>
<tr>
<td>24</td>
<td>1.50</td>
<td>0.250</td>
<td>0.0204606</td>
<td>0.0201979</td>
<td>0.0204370</td>
<td>0.1154</td>
</tr>
<tr>
<td>25</td>
<td>1.25</td>
<td>0.375</td>
<td>0.0149670</td>
<td>0.0149355</td>
<td>0.0149666</td>
<td>0.0027</td>
</tr>
<tr>
<td>26</td>
<td>1.00</td>
<td>0.500</td>
<td>0.0113563</td>
<td>0.0114004</td>
<td>0.0113499</td>
<td>0.0570</td>
</tr>
<tr>
<td>27</td>
<td>0.75</td>
<td>0.625</td>
<td>0.0101416</td>
<td>0.0101683</td>
<td>0.0101453</td>
<td>-0.0360</td>
</tr>
<tr>
<td>28</td>
<td>0.50</td>
<td>0.750</td>
<td>0.0107609</td>
<td>0.0107320</td>
<td>0.0107051</td>
<td>0.5213</td>
</tr>
</tbody>
</table>

Table 7.12: Low Volatility Solution for General Asian Call Options

Even though the Vecer PDE can be solved numerically, questions about its accuracy and efficiency come into question. In particular, for these particular runs, each option price took about 10 seconds to compute, with 2000 grid spacings for the space and time variable. For the Asymp-LV and GDE solutions, 10000 calculations of each took roughly 0.66 and 0.59 seconds, respectively, in total. The closed form nature of the Asymp-LV and GDE solutions, leads to the

---

11A freely available copy of MATLAB codes that numerically solves the Vecer PDE is available at http://en.literateprograms.org/Asian_Options_Pricing_(MATLAB). Slight modifications are required to cater for continuous dividends, and the floating strike parameter. The code uses the MATLAB routine pdepe to numerically solve the PDE.
computational efficiency of these approximations, whereas numerically solving PDEs is always going to be comparatively slower. The trade off for this computational efficiency, is the accuracy of the solution. The Asymp-LV solution performs quite well, with the absolute relative error no larger than 0.53%, in these test cases. The GDE relative errors are omitted, but it is noted that the performance is poor for greater $k_1$ strikes, which is explained as previously. Furthermore, unlike previous test cases where for a fixed volatility, there was a general trend of the accuracy increasing significantly as the strikes were decreased, this is not observed here. This is reasoned by the fact that in all these test cases, the strike combinations are chosen to be fixed to $X_t$ at time $t$. Thus, unlike previous examples, there is no general trend of the test cases moving to a more ITM region, which translate to no trend in decreasing volatility effects, leading to no trend in greater accuracy. Out of interest, if one takes the same test cases with the strikes being 95% or 90% of what they currently are, then the absolute relative errors are no larger than 0.08% or 0.002%, respectively, thus showing greater accuracy when the combinations of the strikes are lower. None-the-less, these results show that the low volatility asymptotic solution for general Australian call options can also be used to price general Asian call options, under low volatility regimes.

Summary

To summarise, in the low volatility asymptotic regime the asymptotic solution provides a high level of accuracy for the cases $r \neq q$, $r = q$ and in progress options. However, in these cases, the Gamma distribution approximation also performs very well. For general Australian and Asian call options, the asymptotic solution provides a reasonable level of accuracy, while there are instances where the GDE approximation can be very poor. Furthermore, there are computational advantages for using the asymptotic solution to price generalAsian call options with non-zero strike types. These results all show the robustness of using the asymptotic solution in the low volatility regime, to price a wide range of general Australian and Asian call options.

7.5 Concluding Remarks

This chapter has investigated the pricing of Australian options in the case of low volatility. Unlike Chapter 6, the Australian option was treated as its own separate entity and not used for any equivalency theorems. A brief review of pricing Australian options using both the
expectation method and the PDE method was provided, with the expectation method giving a closed form solution. However, because of the numerical instability of the integrals in the Australian option solution, the closed form solution has limited practical use.

An asymptotic solution for Australian options in the case of low volatility was then presented. The methodology is an adaptation of the Dewynne and Shaw [20], approach to computing asymptotic solutions for Asian options. The solution is written as a power series in $\epsilon$ which can be regarded as a scaled volatility parameter. Although the pricing of Australian options is equivalent to that of pricing an Asian option under an appropriate transformation, see Chapter 6, a direct application of the Dewynne and Shaw result is not possible. Applications of other numerical approximations to Asian options is infeasible.

The asymptotic solutions for the cases when the risk-free interest rate and continuous dividend yield are equal and not equal are derived up to the third order, with the former having an easy to implement closed form solution. In the second case, it is shown how the asymptotic solution can be simplified to have a closed form solution, if the value of the function $G_1(x, \hat{t})$ is negligible. This is generally true in the low volatility regime, which has been defined as cases where volatility is less than 0.1. However, it is easy to check whether $G_1(x, \hat{t})$ is negligible, by numerically calculating its value, and thus a decision on omitting it can be checked on a case by case basis.

Extensions to the Australian options model were provided. The two extensions considered were advancing the Gamma distribution approximation by Moreno and Navas [59], to cater for in progress options, and also incorporating the general Australian call option problem, see Chapter 6. For the general Australian call option problem, the assumptions of the closed form solution and Gamma distribution approximations are not valid. However, the derivation of the pricing PDE, and subsequently the low volatility asymptotic solution still is valid. This result is quite powerful as it now allows for the low volatility asymptotic solution to be used in the pricing of general Asian call options by invoking the equivalency theorems from Chapter 6.

Numerical solutions for Australian options in published literature are sparse. However, for the test cases in Moreno and Navas, the numerics of the asymptotic solutions perform moderately well. This is even when the volatility is not deemed to be in a low volatility regime. In the low volatility regime, the asymptotic solution performs very well with high levels of accuracy. The solutions are tested against results obtained using FDM. Test cases involved options where the averaging period and the current time are equal and also where the averaging period has already begun. Tests cases with various strike prices and reasonable
values of interest and dividend rates are also included. General Australian call options were also considered and it was found that generally, the low volatility asymptotic solution performs quite well. By invoking the equivalency theorems, numerical results showed that general Asian call options can also be priced quite effectively.
Appendix A

Real Options Numerical Schemes

This appendix details the numerical scheme used in Chapter 5, in computing the numerical solutions for the real option problem with stochastic volatility. These contributions are made by Dr. Wen-Kai Wang who suggested these approaches.

A.1 Original Problem

In this section, the details on applying the implicit method to derive the free boundary under an infinite time problem under the Heston-GBM are presented. A similar method is used for the Heston-GMR case, and thus omitted. Suppose that $h_x$ and $h_y$ are the lengths of a small fixed interval in the state spaces, $x$ and $y$, respectively. Define $Q((x, y), (x', y'))$ to be the probability of $(x, y)$ moving to $(x', y')$ in the time interval $[t, t + \Delta t]$, i.e. a transitional probability. Let the transitional probabilities and the time interval interpolation $\Delta t$ be given as follows:

\begin{align*}
Q((x, y), (x + h_x, y + h_y)) &= \frac{\Delta t(x, y) \max \{0, \rho\} \beta xy}{2h_x h_y}, \quad \text{(A.1)} \\
Q((x, y), (x - h_x, y - h_y)) &= Q((x, y), (x + h_x, y + h_y)), \quad \text{(A.2)} \\
Q((x, y), (x + h_x, y - h_y)) &= \frac{\Delta t(x, y) \max \{0, -\rho\} \beta xy}{2h_x h_y}, \quad \text{(A.3)} \\
Q((x, y), (x - h_x, y + h_y)) &= Q((x, y), (x + h_x, y - h_y)), \quad \text{(A.4)}
\end{align*}
\[ Q ((x, y), (x + h_x, y)) = \frac{\Delta t(x, y) \kappa_x}{h_x} + \frac{\Delta t(x, y) x^2 y}{2h_x^2} - Q ((x, y), (x + h_x, y + h_y)) - Q ((x, y), (x + h_x, y - h_y)), \]  
\[ Q ((x, y), (x - h_x, y)) = \frac{\Delta t(x, y) x^2 y}{2h_x^2} - Q ((x, y), (x - h_x, y + h_y)) - Q ((x, y), (x - h_x, y - h_y)), \]  
\[ Q ((x, y), (x, y + h_y)) = \frac{\Delta t(x, y) \max \{0, m - y\}}{h_y} + \frac{\Delta t(x, y) \beta^2 y}{2h_y^2} - Q ((x, y), (x + h_x, y + h_y)) - Q ((x, y), (x - h_x, y + h_y)), \]  
\[ Q ((x, y), (x, y - h_y)) = \frac{\Delta t(x, y) \max \{0, y - m\}}{h_y} + \frac{\Delta t(x, y) \beta^2 y}{2h_y^2} - Q ((x, y), (x + h_x, y - h_y)) - Q ((x, y), (x - h_x, y - h_y)), \]  
and
\[ Q ((x, y), (x, y)) = 1 - Q ((x, y), (x + h_x, y)) - Q ((x, y), (x - h_x, y)) - Q ((x, y), (x, y + h_y)) - Q ((x, y), (x, y - h_y)) - 2(Q ((x, y), (x + h_x, y + h_y)) + Q ((x, y), (x + h_x, y - h_y))), \]  
where
\[ \Delta t(x, y) = \left[ \frac{\kappa_x}{h_x} + \frac{x^2 y}{h_x^2} + \frac{\alpha |m - y|}{h_y} + \frac{\beta^2 y}{h_y} - \frac{|\rho| \beta x y}{h_x h_y} \right]^{-1}. \]

These equations convert the continuous time problem to a discrete time problem given as,
\[ W(x, y) = \max_r \mathbb{E} \left\{ e^{-\tau r} (x_r - K) \mid (X_1, Y_1) = (x, y) \right\}, \]
where the \( \mathbb{E} \{ \} \) is the expectation under the transitional probability. The processes in (5.1) and (2.8) are also replaced by two discrete random processes according to the transitional probabilities. Due to the principle of dynamic programming, \( W(x, y) \) solves,
\[ W(x, y) = e^{-\tau \Delta t \mathbb{E}} \{ W(x', y') \mid (X_1, Y_1) = (x, y) \}, \]
with \( W(0, y) = 0 \) as a boundary condition, and the free boundary \( x_{B_1}(y) \) satisfying,
\[ W(x, y) = x - K, \quad \text{if } x \geq x_{B_1}(y), \]
\[ W(x, y) < x - K, \quad \text{if } x < x_{B_1}(y). \]
Partition the state space by the following,

\[ X = \{ x_0 < x_0 + h_x < x_0 + 2h_x < \cdots < x_0 + ih_x < \cdots < x_f \} , \]
\[ Y = \{ 0 < h_y < 2h_y < \cdots < jh_y < \cdots < y_{max} \} . \]

Since the transitional probabilities in (A.1)-(A.9) are non-negative, the following conditions are required,

\[ x_0 > \frac{h_x |p| \beta}{h_y} , \]
\[ x_f < \frac{h_x \beta}{h_y |p|} . \]

### A.2 Modified Problem

In this section, the details for solving the modified problem using the projection method under the Heston-GBM model is presented. A similar method is used for the Heston-GMR model, and thus omitted. See Judd [51], for further details on the numerical method presented below.

Consider the PDE in (5.2), with the boundary conditions,

\[ V(0, y) = 0 , \quad (A.10) \]
\[ V(x^*, y) = x^* - K , \quad (A.11) \]

for all \( y \geq 0 \) and \( x^* \) being any constant threshold test point.

Let \( V(x, y) \) be defined on \( S = [0, x^*] \times [0, y_f] \). Choose some grid points in \( S \) which are called Chebyshev nodes:

\[ X \times Y , \]

where,

\[ X = \left\{ x_i = \frac{x_f}{2} \left[ 1 - \sec \left( \frac{\pi}{2n_x} \right) \cos \left( \frac{2i - 1}{2n_x} \pi \right) \right] \mid i = 1, 2, \ldots, n_x \right\} , \]
\[ Y = \left\{ y_i = \frac{y_f}{2} \left[ 1 - \sec \left( \frac{\pi}{2n_y} \right) \cos \left( \frac{2i - 1}{2n_y} \pi \right) \right] \mid i = 1, 2, \ldots, n_y \right\} . \]

Note that 0 and \( x_f (y_f) \) are included in \( X (Y) \). Chebyshev nodes are widely used in polynomial interpolation since they minimize the interpolation error.
Consider the following two Bernstein polynomials:

\[
P(x) = \sum_{i=0}^{n_x-1} a_i \frac{(n_x-1)!}{i!(n_x-1-i)!} x^i (x_f - x)^{n_x-1-i},
\]

\[
Q(y) = \sum_{i=0}^{n_y-1} b_i \frac{(n_y-1)!}{i!(n_y-1-i)!} y^i (y_f - y)^{n_y-1-i},
\]

where \(a_i\) and \(b_i\) are unknown coefficients. Substitute all points in \(X\) and \(Y\) into \(P(x)\) and \(Q(y)\), respectively, to obtain two linear systems \(Aa\) and \(Bb\), where,

\[
A = \left( \frac{(n_x-1)!}{j!(n_x-1-j)!} x^j (x_f - x_i)^{n_x-1-j} \right)_{i,j}, \quad \text{for } i, j = 1, \ldots, n_x
\]

\[
B = \left( \frac{(n_y-1)!}{j!(n_y-1-j)!} y^j (y_f - y_i)^{n_y-1-j} \right)_{i,j}, \quad \text{for } i, j = 1, \ldots, n_y
\]

\[
a = (a_1 \ a_2 \ \cdots \ a_{n_x})^T,
\]

\[
b = (b_1 \ b_2 \ \cdots \ b_{n_y})^T,
\]

and \((\cdot)^T\) denotes the transpose of a vector. Applying a similar idea, one can find the linear systems \(A_x a, B_y b, A_x A_x a\) and \(B_y B_y a\) to represent \(P'(x), Q'(y), P''(x)\) and \(Q''(y)\). Define,

\[
C = A \otimes B, \quad \text{(A.12)}
\]

\[
C_x = A_x \otimes B,
\]

\[
C_y = A \otimes B_y,
\]

\[
C_{xx} = A_{xx} \otimes B_x,
\]

\[
C_{yy} = A \otimes B_{yy},
\]

\[
C_{xy} = A_x \otimes B_y,
\]

\[
c = a \otimes b, \quad \text{(A.18)}
\]

where \(\otimes\) denotes the tensor product of two matrices. Note that the size of the matrices in equations (A.12)-(A.17) are \(n_x n_y \times n_x n_y\) and equation (A.18) is a \(n_x n_y\)-dimensional vector.

Let \(z_i = x_k y_l\), where \(k\) is the smallest positive integer greater than \(i/n\) and,

\[
l - 1 = (i - 1) \mod n.
\]
Define the following diagonal matrices:

\[
\begin{align*}
R &= (r)_{i,i}, \\
R_x &= (\kappa x_k)_{i,i}, \\
R_y &= (\alpha (m - y_l))_{i,i}, \\
R_{xx} &= \left(\frac{x_k^2 y_l}{2}\right)_{i,i}, \\
R_{yy} &= \left(\frac{\beta^2 y_l}{2}\right)_{i,i} , \\
R_{xy} &= (\rho \beta x_k y_l)_{i,i}
\end{align*}
\]

where \( i = 1, 2, \ldots, n_x n_y \) and \( k \) and \( l \) as defined previously. Then the PDE in equation (5.2) can be approached by solving the linear system, \( Dc = 0 \), where,

\[
D = R_{xx}C_{xx} + R_{yy}C_{yy} + R_{xy}C_{xy} + R_x C_x + R_y C_y - RC.
\]

Note that to take the boundary conditions (A.10) and (A.11) into account, if \( k = 1 \) or \( k = m \), the corresponding rows in \( D \) are replaced by the corresponding rows in \( C \). We denote such a matrix by \( \tilde{D} \). Define a \( n_x n_y \)-dimensional vector \( d \) such that the last entries are \( x_f - K \) and 0 otherwise. Then equation (5.2) with boundaries (A.10) and (A.11), can be approached by solving the linear system \( \tilde{D}c = d \).
Appendix B

Australian Options Proof

B.1 Moment Condition Requirements

It is required to show that,

$$\mathbb{E}\left( \int_t^{T_1} AU_s^4 \, ds \mid \mathcal{F}_t \right) < \infty,$$

for some fixed $T_1$ and $t$. By Fubini’s theorem, this is equivalent to proving,

$$\int_t^{T_1} \mathbb{E}\left( AU_s^4 \mid \mathcal{F}_t \right) \, ds < \infty,$$

which is equivalent to,

$$\mathbb{E}\left( AU_s^4 \mid \mathcal{F}_t \right) < M < \infty,$$  \hspace{1cm} (B.1)

for all $s \in [t, T_1]$ and some constant $M$. That is, the left hand side of the inequality in (B.1) must be bounded. Now expanding the left hand side of the inequality in (B.1), gives,

$$\mathbb{E}\left( AU_s^4 \mid \mathcal{F}_t \right) = \frac{1}{T^4} \mathbb{E}\left( \left( \frac{\int_{t_0}^{t} X_u \, du}{X_s} \right)^4 \mid \mathcal{F}_t \right)$$

$$= \frac{1}{T^4} \mathbb{E}\left( \left( \frac{\int_{t_0}^{t} X_u \, du + \int_{t}^{s} X_u \, du}{X_s} \right)^4 \mid \mathcal{F}_t \right)$$

$$= \frac{1}{T^4} \left[ \left( \int_{t_0}^{t} X_u \, du \right)^4 \mathbb{E}\left( \frac{1}{X_s^4} \mid \mathcal{F}_t \right) + 4 \left( \int_{t_0}^{t} X_u \, du \right)^3 \mathbb{E}\left( \int_{t}^{s} \frac{X_u}{X_s^2} \, du \mid \mathcal{F}_t \right) ight.$$

$$+ 6 \left( \int_{t_0}^{t} X_u \, du \right)^2 \mathbb{E}\left( \left( \int_{t}^{s} \frac{X_u}{X_s^2} \, du \right)^2 \mid \mathcal{F}_t \right)$$

$$+ 6 \left( \int_{t}^{s} X_u \, du \right)^2 \mathbb{E}\left( \left( \int_{t}^{s} \frac{X_u}{X_s^2} \, du \right)^2 \mid \mathcal{F}_t \right)$$

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B.1 Moment Condition Requirements

\[ + 4 \left( \int_{t_0}^{t} X_u \, du \right) \mathbb{E} \left( \left( \int_{t}^{s} \frac{X_u}{X_s^{4/3}} \, du \right)^3 \bigg| \mathcal{F}_t \right) + \mathbb{E} \left( \left( \int_{t}^{s} \frac{X_u}{X_s} \, du \right)^4 \bigg| \mathcal{F}_t \right) \]  

(B.2)

Firstly, given the filtration \( \mathcal{F}_t \), it is noted that the quantity \( \int_{t_0}^{t} X_u \, du \), is known and finite. To prove that inequality in (B.1), is satisfied, it is sufficient to show that each expectation term in equation (B.2) is a bounded continuous function in \( s \).

The first term in the square brackets of equation (B.2) is a bounded continuous function in \( s \). This is shown by observing that \( X_s \) is log-normally distributed, and thus so are \( 1/X_s \) and \( 1/X_s^{4} \). Hence, \( \mathbb{E} \left( 1/X_s^{4} \big| \mathcal{F}_t \right) \) is the expectation of a log-normally distributed random variable whose solution is a known continuous function, and thus can be bounded for all \( s \in [t, T_1] \).

For the second term, the expectation and integration can be interchanged through application of Fubini’s theorem, see Royden \[67\], so long as \( \int_{t}^{s} \mathbb{E} \left( X_u/X_s^{4} \big| \mathcal{F}_t \right) \, du \) is finite. The random variable \( X_u/X_s^{4} \) can be shown to be log-normally distributed given the filtration \( \mathcal{F}_t \), and thus \( \mathbb{E} \left( X_u/X_s^{4} \big| \mathcal{F}_t \right) \) is continuous and finite. Hence \( \int_{t}^{s} \mathbb{E} \left( X_u/X_s^{4} \big| \mathcal{F}_t \right) \, du \), must too be finite as well, therefore the application of Fubini’s theorem is valid, and that the second term is finite and bounded for all \( s \in [t, T_1] \).

For the third, fourth and fifth term, a generalized proof to show,

\[ \mathbb{E} \left( \left( \int_{t}^{s} \frac{X_u}{X_s^{m}} \, du \right)^p \bigg| \mathcal{F}_t \right) < \infty, \]

for any positive \( m \geq 1 \) and positive integer \( p \), is provided. Furthermore, it is argued that the left hand side of this inequality is bounded for all \( s \in [t, T_1] \). Firstly, note that \( X_u/X_s^{m} \) given the filtration \( \mathcal{F}_t \), is log-normally distributed and a non negative quantity,

\[ \frac{X_u}{X_s^{m}} \overset{d}{=} \frac{1}{X_s^{m-1}} \exp \left[ \left( r - q - \frac{1}{2} \sigma^2 \right) (u - t - m (s - t)) - \sigma \left( mW_{s-u} + (m - 1) W_{u-t} \right) \right], \]

(B.3)

where the \( = \overset{d}{\text{notation, represents equal to in distribution. Thus } \int_{t}^{s} \frac{X_u}{X_s^{m}} \, du \text{ is also a positive quantity. Now,}} \]

\[ \left[ \mathbb{E} \left( \left( \int_{t}^{s} \frac{X_u}{X_s^{m}} \, du \right)^p \bigg| \mathcal{F}_t \right) \right]^{1/p} = \left( \mathbb{E} \left( \left| \int_{t}^{s} \frac{X_u}{X_s^{m}} \, du \right|^p \bigg| \mathcal{F}_t \right) \right)^{1/p} \]

\[ \leq \int_{t}^{s} \left( \mathbb{E} \left( \frac{X_u}{X_s^{m}} \bigg| \mathcal{F}_t \right) \right)^{1/p} \, du \]

(B.4)
where the inequality comes from the Minkowski integral inequality, see Royden [67]. If,

\[
\left[ \mathbb{E} \left( \left( \int_t^s \frac{X_u}{X^m_s} \, du \right)^p \bigg| \mathcal{F}_t \right) \right]^{1/p} > 1,
\]

then the inequality in (B.4), to the power of \( p \), would preserve the direction of the inequality sign\(^1\) and also show that \( \mathbb{E} \left( \left( \int_t^s \frac{X_u}{X^m_s} \, du \right)^p \bigg| \mathcal{F}_t \right) \) is finite and bounded. To see this last point, note that,

\[
\mathbb{E} \left( \left( \frac{X_u}{X^m_s} \right)^p \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \left( \frac{X_u}{X^m_t} \right)^p \bigg| \mathcal{F}_t \right) = \mathbb{E} \left( \left( \frac{X^p_u}{X^{pm}_t} \right) \bigg| \mathcal{F}_t \right). \tag{B.5}
\]

Given the result in equation (B.3), it is easy to see that the expectation of \( \frac{X^p_u}{X^{pm}_s} \) must take on an exponential function form (as the expectation of a log normal random variable takes an exponential form). The expression in equation (B.5) is thus continuous in \( u \in [t, s] \) and \( s \), and as such is bounded for some fixed \( s \). Taking equation (B.5), to the power of \( 1/p \) would still result in a continuous and bounded expression. The definite integral on the right hand side of equation (B.4) can thus be bounded, by some finite constant. From this the required results follows.

For the third, fourth and fifth terms of equation (B.2), use the above result for \( m = 2, 4/3, 1 \) and \( p = 2, 3, 4 \), respectively. Together these results show that the inequality in (B.1), is satisfied, and concludes the proof.

\(^1\)Trivially, if the inequality is not satisfied, then the left hand side of the inequality in (B.4), to the power of \( p \), would be bounded by 1.
References


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