Under HARA preferences and a standard opportunity set, constant-proportion portfolio insurance is optimum if and only if the investor has declining absolute and relative risk aversion. The optimum floor is the capitalized value of the investor's subsistence consumption rate if and only if the investor's family size is constant.
1 Introduction

Constant-proportion portfolio insurance (CPPI) has been remarkably successful in capturing the imagination of numerous practically-minded finance economists. Black and Jones (1987) summarize this dynamic portfolio allocation principle by means of the catchy formula

\[ e = mc \]

where \( e \) is the investor's exposure to a risky asset, \( m \) is a fixed multiple, and \( c \) is the investor's "cushion" -- the maximum amount he or she is prepared to lose. Put another way, \( c \) represents the difference between the investor's actual level of wealth and "floor" level of wealth -- the minimum level he or she is prepared to accept. What CPPI offers, then, is a combination of upside capture potential and downside damage limitation. It assumes that the price of the risky asset shows some continuity. It absolutely rules out further venturesome investments -- no matter how tempting -- if ever one's cushion were dissipated by a series of low returns on the risky asset.

The existing literature contains several brief mentions of Merton's (1971) classic article as providing (in some unspecified sense) the theoretical foundations of CPPI. Yet nowhere does the existing literature spell out what particular special case of Merton's wide-ranging analysis actually serves to justify CPPI. Nor does it spell out the implications of such a case for the design of CPPI policies. The purpose of the present contribution is to fill these two gaps.
The discussion is organized as follows. Section 2 poses an investor's optimum problem that will be recognized as a minor variation of the problem posed and solved in Section 6 of Merton's article. Only when the two parameters of the instantaneous utility function are appropriately restricted in sign does this set-up justify CPPI. Section 3 solves the investor's optimum problem, and deduces what it implies for the preferences of an investor demanding CPPI. It is at this stage that the appropriate sign restrictions are introduced. Section 4 deduces what the solution implies for optimum multiples, cushions and floors. Building on Black and Jones' article, new suggestions are offered for the design of CPPI policies.

2. The Investor's Optimum Problem

Consider the following optimum problem facing an investor acting on behalf of an infinitely-lived family of individuals with identical preferences:

$$\max_{q, w} \int_{t=0}^{\infty} \exp\left[-(\rho-n)t\right] \left[ \frac{q(t)-b}{1-a} \right] \frac{1}{1-a}$$

subject to

$$dx(t) = [(\alpha-r)w(t)x(t)+(r-n)x(t)-q(t)]dt+w(t)x(t)\sigma dz$$

$$0 \leq n < \rho$$

$$0 \leq n < r < \alpha$$

$$x(0) = x_{o}$$

In the objective (1), $q$, $w$, $E$, $\rho$, $n$, $a$ and $b$ stand for consumption per head, the fraction of wealth allocated to the risky
asset, the expectations operator, the rate of time preference, the rate of growth of household membership, the curvature parameter of the instantaneous utility function, and the origin-displacement parameter of the instantaneous utility function. That function is evidently of HARA form. Following a tradition going back to Ramsey, the investor's lifetime utility function is total discounted family utils rather than discounted utils per head. In the standard constraint (2), \( x, \alpha, r \) and \( \sigma \) stand for wealth per head, the mean rate of growth of the price of the risky asset, the rate of interest on the safe asset, and the standard deviation of the return on the risky asset. The boundary condition (3) is standard as well.

The problem (1), (2) and (3) matches up perfectly with the problem posed and solved in Section 6 of Merton's article, once the following two steps are taken. First, specialize Merton's analysis to the infinite-horizon case. Second, replace his rate of time preference \( \rho \) and rate of interest \( r \) by their "growth-adjusted" counterparts \( \rho - n \) and \( r - n \), everywhere except where Merton subtracts the interest rate from the mean rate of return on the risky asset (so that the risk premium \( \alpha - r \) is left alone).

The rate of growth of the investor's family (or, more accurately, dynasty) seldom features in theoretical contributions to finance. Merton's article is no exception. Yet it does feature here. The reason is that it serves to motivate a policy of modest growth in the investor's floor, as has been advocated by most proponents of CPPI.
3. The Solution

Having noted the match-up with Merton's analysis, the portfolio allocation rule entailed by problem (1), (2) and (3) is easily seen to be the following minor variation of Merton's eq. (49):

\[ w(t)x(t) = \frac{(\alpha - r)x(t)}{a\sigma^2} - \frac{b(\alpha - r)}{a(r-n)\sigma^2} \]  

(4)

Let \( N(t) = N_0 \exp(nt) \) be the number of members of the investor's family, and let \( f(t) \) stand for the investor's floor. Upon multiplying (4) through by \( N(t) \) we get

\[ e(t) = \frac{(\alpha - r)}{a\sigma^2} c(t) + \frac{(\alpha - r)}{a\sigma^2} \left( f(t) - \frac{bN(t)}{r-n} \right) \]  

(5)

where \( e = Nwx \) and \( c = Nx - f \). This is a prescription for CPPI if and only if

\[ m = \frac{(\alpha - r)}{a\sigma^2} \quad \text{with} \quad a > 0 \]  

(6)

\[ f(t) = \frac{bN_0 \exp(nt)}{(r-n)} \quad \text{with} \quad b > 0 \]  

(7)

That is, the portfolio allocation rule (5) specializes to the formula \( e = mc \) if and only if the utility parameters \( a \) and \( b \) are positive.

Because \( b \) is now positive, one can now follow the literature on the Stone-Geary utility function and the linear expenditure system in interpreting \( b \) as the "subsistence" rate of consumption.
Inspection of Table 1 of Merton's article reveals that a and b are both positive if and only if the coefficients of absolute and relative risk aversion, A(q) and B(q), both have a negative first derivative. Within the overall setting (1), (2) and (3), then, a rational investor will demand CPPI if and only if his or her preferences manifest declining absolute and relative risk aversion.

4. Implications for Optimum Multiples, Cushions and Floors

The CPPI investor's multiple should be directly proportional to the risk premium. It should be inversely proportional to the curvature of the utility function and to the variance of the return on the risky asset. ³

The CPPI investor's floor in the case of zero family growth should equal the capitalized value of his or her subsistence rate of consumption. On the other hand, in the case of positive family growth, the floor should be larger to begin with, and should then rise through time, at the family growth rate -- a policy that is moderately at odds with the usual one of equating the floor growth rate with the interest rate. Irrespective of the floor growth rate, the rational floor level will be independent of (i) the risk-return characteristics of the risky asset, and (ii) the investor's initial level of wealth.

Because the optimum floor is independent of initial wealth, the optimum initial cushion -- how much the CPPI investor is initially prepared to lose -- will vary dollar-for-dollar with the investor's initial wealth.
Footnotes

1. See. e.g. Ferold and Sharpe 1988.

2. Until further notice the parameters $a$ and $b$ are restricted only to the extent necessary to ensure concavity of the instantaneous utility function.

3. Black and Jones make this point (that the multiple should be negatively related to volatility).
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