The Valuation of Options on Traded Accounts: Continuous and Discrete Time Models

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Discipline of Finance
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Statement of Originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that the assistance received in preparing this thesis and sources have been acknowledged.

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Hamish Malloch
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Abstract

In this thesis we are concerned with valuing options on traded accounts using both continuous and discrete time models. An option on a traded account is a zero strike call on the balance of a trading account which consists of a position of size $\theta$ in a risky asset (which we refer to as a stock) and the remaining wealth in a risk-free account. The choice of trading positions throughout the life of the option are made by the buyer, subject to constraints specified in the contract at the time of purchase. The specification of these trading constraints gives rise to some of the more well known examples including passport options and vacation options. At maturity, the option buyer is entitled to any positive wealth accumulated in the trading account whilst any losses are covered by the option seller.

First, we examine the problem of valuing these options in continuous time. A review of some existing methods is presented, including a complete derivation of the pricing formula for the passport option and the option on a traded account following the methods proposed by Hyer et al. (1997) and Shreve and Vecer (2000), though we often use different techniques to those authors. We also present an alternative derivation for the value of a passport option using our own methodology which we believe is simpler than those currently available.

Secondly, we consider the valuation problem in a discrete time setting by looking at one specific discrete time model, the binomial tree. This is a new contribution to the literature as binomial models for these options have not been previously examined. Using this approach, the greatest difficulty is the determination of an optimal trading strategy which is required to price this class of option. We show that in general, binomial models and continuous time models do not have the same trading strategy, and in fact that the analytic determination of the trading strategy for an option on a traded account may in fact be impossible to obtain. We then turn to passport options, where we are able to derive an analytic optimal strategy which in this case is identical to that used in the continuous time models, thus the problem of valuing
passport options is reduced to the same computational burdens as a binomial valuation without recombining branches.

Lastly, we examine some numerical methods which could be used to value options on traded accounts with binomial models. Our problem is shown to be an NP-hard convex maximisation which we convert into both an $l_1$-norm convex maximisation and an indefinite quadratic program. Whilst we present algorithms which are guaranteed to obtain the optimal solution, they are also known to be inefficient and thus inappropriate for any likely application beyond a few time steps. We conclude by summarising our results and give directions for future research in this area.
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CHAPTER 1

Introduction

The theory of options pricing was pioneered in the seminal papers of Black and Scholes (1973) and Merton (1973). In their work, the authors considered what the fair, arbitrage free price for a call option should be under a number of assumptions. The main assumption in this framework, commonly referred to as the Black-Scholes model, is that the underlying financial asset (commonly called a stock, though exchange rates and other financial assets are also valid) follows a continuous time geometric Brownian motion (g.B.m.). Coupling this assumption with a cleverly chosen hedging arrangement, these authors showed that the value of an option satisfies a certain second order partial differential equation (PDE) which is today referred to as the Black-Scholes equation. Solving this equation subject to boundary conditions implied by the options payoff yields the arbitrage free value of that option.

An alternative to this method was proposed by Cox et al. (1979) (hereafter called the CRR model). As is clear by the title of their paper, the aim was to make the process of option pricing simpler and more intuitive. To this end, the g.B.m. assumption was replaced with a discrete time binomial tree. Using a similar hedging argument to that used in the Black-Scholes framework, the authors showed that a riskless portfolio could be created with a cleverly chosen position in both the underlying and its option which provided the risk-neutral framework required to price options. A numerical method is then implemented which computes the value of the option by progressing backwards from maturity until the initial price is finally obtained.

Whilst the Black-Scholes model became extremely popular within industry, and many extensions have been made to it such that it can price a much wider class of options, the binomial model has also gained
widespread popularity. This is not only due to its simplicity but also due to the flexibility that the model provides. Well known examples where the CRR model outperforms the Black-Scholes framework are American options and the inclusion of discrete dividends. Whilst the Black-Scholes model struggles to value options with these criteria, it is a comparatively straightforward process to adjust the binomial model to incorporate these features.

Traditionally, options have existed on an underlying financial asset such as a stock price, interest rate or exchange rate, to name but a few. In these standard cases the option holder has no control over the evolution of the underlying. In principle however, an option may be written on any financially measurable quantity. In this thesis, we will examine how to value an option when the underlying is the value of a trading account consisting of a risk-free position in cash and a risky position in stock. Hence the option holder has some degree of control over the underlying asset dynamics. Essentially, the option holder makes trades on a selected stock subject to trading restrictions agreed upon by the option seller, and determines the performance of his trades via the value of this trading account. At maturity, any nett gain on this trading account is kept by the option holder and any nett loss is covered by the option seller. Special cases of this type of option include the passport option where the allowed trading positions in the stock are within the interval \([-1, 1]\) and the vacation call and put which have allowed trading position intervals given by \([0, 1]\) and \([-1, 0]\) respectively.

The passport option was first introduced by Hyer et al. (1997) where an analytical solution for a particular case (which the authors call the symmetric case) was derived using results from stochastic control theory, while solutions to other cases were obtained via numerical methods. Since that paper, work has been done by others who have used a variety of numerical and analytical techniques to price the extended class of options on traded accounts using both probabilistic and PDE methods.
1.1. Contributions

What all of the current methods have in common is that they assume that the underlying stock price dynamics are given by a continuous time process, usually a g.B.m. or some derivative of it, such as the stochastic volatility model examined by Henderson and Hobson (2001). What is missing from the literature is an examination of options on traded accounts assuming that the evolution of stock prices follows a discrete time process such as the simple binomial model of Cox et al. (1979). While Andersen et al. (1998) does mention the possibility of using a discrete time process to value these options, they specifically state that they are unsuited to the problem and instead focus on solving the associated PDE via finite difference techniques.

The reason binomial trees have been thought to be unsuited to the problem of valuing options on traded accounts is due to the fact that even though the binomial tree governing the stock price may recombine, resulting in \((N + 1)\) terminal nodes for an \(N\)-step tree, the tree which describes the trading account evolution will in general not recombine and thus potentially has \(2^N\) terminal nodes. As this is essentially a restriction imposed only by computing power, we still believe that an investigation into using these binomial models could prove useful.

1.1. Contributions

The purpose of this thesis is to determine the value of an option on a traded account, and as such that is where our contributions to the literature are made. Since the value obtained depends upon the model used, we will present several different methods to obtain this valuation. First, we consider a continuous time setting which has the same assumptions as the Black-Scholes framework. Valuation within this framework is the focus of chapter 3. Pricing the option in this manner has already received a reasonably thorough treatment in the literature. After briefly reviewing the current research pertaining to these options, we provide a detailed analysis of two of the more prominent papers in the literature. The first of these papers is the seminal paper by Hyer et al. (1997) which was the first to value passport options (trading interval given by \([-1, 1]\)). The second paper to be examined is by Shreve
and Vecer (2000) and is the first to consider the general option on a traded account (trading interval given by \([\alpha, \beta]\) where \(\alpha < \beta\)).

In both cases, closed form solutions for the option value is possible only under specific conditions which are referred to as the symmetric case. Since much focus in the literature has already been paid to numerical methods for solving the PDE which the option value must satisfy, we will not attempt to cover this aspect of the problem. Rather, our focus in continuous time is to find an analytic solution. To this end we follow the approach of Hyer et al. (1997) and Shreve and Vecer (2000) though we adopt several variations to make solving the problem simpler and more transparent.

In chapter 4, we tackle the problem of valuing options on traded accounts under the assumption that the stock evolves according to a discrete time binomial lattice. The specific form of this lattice is one which closely resembles a g.B.m. in presentation. This process, developed by He (1990), was chosen as it is a special case of a generalised multi-nomial tree which converges to a multi-variate g.B.m. and has become somewhat of a standard in financial mathematics. Initially, it might be thought that the use of a discrete time process may simplify the task of pricing such options. Even if the computations involved were to become impractical due to non-recombining branches, it might be expected that this simpler model could provide some insight and intuition which is difficult to obtain from the complicated mathematical structure of the problem posed in continuous time. Unfortunately, the opposite turns out to be true. In general the optimal trading strategy for continuous time valuation and discrete time valuation are not the same. In fact, whilst continuous time optimal trading strategies require knowledge of only the current state variables and trading interval in a rather simple way, the discrete time version requires the same information as well as the drift and volatility in a highly non-linear and extremely complicated way. This makes valuation in this framework an enormously complicated task, which transforms to a global optimisation problem in a very high dimensional space. As it turns out, the
optimisation for this particular problem is known to be NP-hard.\footnote{A problem is NP-hard (nondeterministic polynomial time hard) if an algorithm for solving it can be translated into one which can solve any NP (nondeterministic polynomial time) problem. Informally, an NP-hard problem is at least as hard as any NP problem, though it may in fact be harder.} A technique guaranteed to find the optimal solution is presented, though due to the intractability of this approach, we spend some time discussing the problem structure which we try to exploit to find a more efficient method of solving the optimisation problem, and hence value the option. At the time of writing this thesis, the general problem for finding the optimal trading strategy for options on traded accounts using a binomial model remains unsolved and is still therefore an open question.

Since valuing this class of options requires a reasonably high degree of mathematical sophistication, chapter 2 presents some of the required mathematical techniques and theorems which will be used throughout this thesis. The content covered is not exhaustive and is aimed principally at the problem of valuing options on traded accounts. For example, while we cover some techniques in solving PDE’s, we restrict our attention to those used in valuing this class of options and not on general solution methods. Finally, we conclude the thesis with a summary of results and directions for future research.
CHAPTER 2

Mathematical Preliminaries

In this chapter we will present some of the mathematical techniques and theorems which will be utilised in this thesis. Whilst we cover quite an extended range of topics from basic stochastic calculus to advanced PDE solution techniques, we do not intend to cover these topics in any great detail. Rather, these theorems are included to ensure that this thesis is reasonably self contained. For greater detail, references are provided in the relevant sections.

2.1. Gaussian Random Variables

Financial mathematics, and in particular Black-Scholes option valuation, makes extensive use of Gaussian random variables in the description of asset returns. For this reason we cover the notation and theorems associated with Gaussian random variables in this section. This discussion is not intended to be exhaustive but rather to outline the notation that will be used throughout the thesis and present one of the less known, but more useful theorems on Gaussian random variables, the Gaussian shift theorem.

Firstly, we say that \( Z \) is a standard Gaussian random variable and write \( Z \sim N(0, 1) \) if \( Z \) is normally distributed with zero mean and unit variance. The probability density function (PDF) of \( Z \) is given by the function \( \phi(z) \) (\( z \in \mathbb{R} \))

\[
\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad (2.1)
\]

and the corresponding cumulative density function (CDF) is given by

\[
\mathcal{N}(z) = \mathbb{P}\{Z \leq z\} = \int_{-\infty}^{z} \phi(y)dy. \quad (2.2)
\]
2.1. GAUSSIAN RANDOM VARIABLES

We now list some useful properties of Gaussian random variables which will be used throughout the thesis. For complete proofs of these properties we refer the reader to any good book on statistics such as Jacod and Protter (2004)

1. $\phi(z)$ is even symmetric, that is $\phi(z) = \phi(-z)$. It also asymptotically approaches zero, i.e. $\phi(\pm \infty) = 0$.

2. $N(z)$ is monotonic increasing, with symmetry $N(-z) = 1 - N(z)$. We also note that the extreme values of this function are $N(-\infty) = 0$ and $N(\infty) = 1$.

Whilst there are many other useful properties for Gaussian random variables, we only emphasise those which have a direct use in computations within this thesis.

We now derive a very useful theorem involving Gaussian random variables which simplifies many calculations encountered in Black-Scholes type option valuation. This is known as the Gaussian shift theorem (GST) which is presented below.

**Theorem 2.1 (Gaussian shift theorem (GST)).** Let $Z \sim N(0, 1)$, $c$ be any constant and $F(Z)$ be any measurable function of $Z$ with finite expectation. Then the following relation holds:

$$
\mathbb{E}\{e^{cZ}F(Z)\} = e^{\frac{1}{2}c^2}\mathbb{E}\{F(Z + c)\}
$$

(2.3)

**Proof.** First, consider the left hand side,

$$
\mathbb{E}\{e^{cZ}F(Z)\} = \int_{-\infty}^{\infty} e^{cy}F(y)\phi(y)dy.
$$

Now, consider the term $e^{cy}\phi(y)$. This may be written as

$$
e^{cy}\phi(y) = \frac{1}{\sqrt{2\pi}} e^{cy} e^{-\frac{1}{2}y^2} = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(2cy-y^2)} = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}[-(c-y)^2+c^2]} = e^{\frac{1}{2}c^2}\phi(c-y).
$$

(2.4)
Using equation (2.4), we can write
\[ \int_{-\infty}^{\infty} e^{cy} F(y) \phi(y) dy = e^{\frac{1}{2}c^2} \int_{-\infty}^{\infty} F(y) \phi(y - c) dy = e^{\frac{1}{2}c^2} \int_{-\infty}^{\infty} F(z + c) \phi(z) dy \quad (z = y - c) = e^{\frac{1}{2}c^2} E \{ F(Z + c) \} . \]
which is the required result. □

Theorem 2.1 is useful because these types of expectations frequently occur in Black-Scholes options valuation. In fact this theorem often allows one to calculate options prices without recourse to Girsanov’s theorem and without the necessity of doing formal intergations.

2.2. The $l_p$-norm

In this section we introduce the concept of the $l_p$-norm. This is an important concept in many optimisation problems as typically we are interested in either maximising or minimising some sort of distance metric. The $l_p$-norm generalises this notion of distance. Consider an $(n \times 1)$ vector $\tilde{x}$ with components $x_i$. We define the $l_p$-norm of this vector, $||\tilde{x}||_p$ as
\[ ||\tilde{x}||_p = \left[ \sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}}. \tag{2.5} \]
Frequently used examples of this norm are the $p = (1, 2$ and $\infty$) cases.\footnote{$p = \infty$ corresponds to $||\tilde{x}||_\infty = \max(x_i)$.}
A common application of the $l_2$-norm is in least squares regression where this norm is minimised in order to find the best statistical estimators. The $l_1$-norm has similarly been used in statistics in a method referred to as robust estimation. In this thesis we will relate the valuation of options on traded accounts to the problem of maximising an $l_1$-norm. Often the optimisation of such norms must be performed numerically as the simple approach of finding stationary points may not be applicable.
2.3. Stochastic Processes and Ito Calculus

We present here some results and theorems relating to stochastic processes and the associated Ito calculus. While we assume the reader has a familiarity with stochastic calculus, we briefly outline some results which are utilised in this thesis. These results are mainly associated with specific forms of Ito’s lemma. We begin with a recap of Ito’s lemma.

\textbf{Theorem 2.2 (Ito’s lemma).} Let $X_t$ satisfy the SDE given by
\[ dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dW_t \]
where $\alpha(X_t, t)$ and $\beta(X_t, t)$ are arbitrary functions and $W_t$ is a standard Weiner process. If $F(X_t, t)$ is a $C_{2,1}$ function, then the process $F(X_t, t)$ is a random process and satisfies the following SDE
\[ dF = \left(F_t + \alpha F_x + \frac{1}{2} \beta^2 F_{xx}\right)dt + \beta F_x dW_t \]
where the subscripts on the function $F$ denote partial derivatives.

\textbf{Proof.} First, using Taylor’s theorem for a function of two variables,
\[ dF = F(X_t + DX_t, t + dt) - F(X_t, t) \]
\[ = F_t(X_t, t)dt + F_x(X_t, t)dX_t + \frac{1}{2} F_{xx}(X_t, t)(dX_t)^2 + h.o.t. \]
The expression \textit{h.o.t} represents the higher order terms. Since $\mathbb{E}\{(dX_t)^2\} = \beta^2 dt$ to order $dt$, if we make this substitution, expand the term $dX_t$ and take the limit as $dt \to 0$ we obtain the desired result. \hfill \Box

We will now derive a special case of Ito’s lemma when we wish to find the differential of the product of two given functions.

\textbf{Corollary 2.3 (Ito’s product rule).} Let $X_t$ satisfy the SDE
\[ dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dW_t \]
and let $F(X_t, t)$ and $G(X_t, t)$ be two given functions, then we have that
\[ d(FG) = FdG + GdF + \beta^2(F_x G_x)dt \]
where the subscripts on $F$ and $G$ denote partial derivatives.

**Proof.** By Ito’s lemma,

$$d(FG) = [(FG)_t + \alpha (FG)_x + \frac{1}{2} \beta^2 (FG)_{xx}] dt + \beta (FG)_x dW_t$$

and since

$$(FG)_t = GF_t + FG_t; \quad (FG)_x = FG_x + GF_x;$$

and

$$(FG)_{xx} = FG_{xx} + GF_{xx} + 2F_xG_x,$$

substituting yields

$$d(FG) = F[(G_t + \alpha G_x + \frac{1}{2} \beta^2 G_{xx}) dt + \beta G_x dW_t]
+ G[(F_t + \alpha F_x + \frac{1}{2} \beta^2 F_{xx}) dt + \beta F_x dW_t]
+ \beta^2 F_x G_x dt
= FdG + GdF + \beta^2 (F_x G_x) dt
\text{as required.}$$

We now derive a similar result for the quotient of two functions.

**Corollary 2.4 (Ito’s quotient rule).** Let $X_t$ satisfy the SDE

$$dX_t = \alpha(X_t, t) dt + \beta(X_t, t) dW_t$$

and let $F(X_t, t)$ and $G(X_t, t)$ be two given functions, then we have that

$$d\left( \frac{F}{G} \right) = \frac{GdF - FdG}{G^2} + \frac{\beta^2 G_x}{G^3} (FG_x - GF_x) dt$$

where the subscripts on $F$ and $G$ denote partial derivatives.

**Proof.** Again, applying Ito’s lemma, we have

$$d\left( \frac{F}{G} \right) = \left[ \left( \frac{F}{G} \right)_t + \alpha \left( \frac{F}{G} \right)_x + \frac{1}{2} \beta^2 \left( \frac{F}{G} \right)_{xx} \right] dt + \beta \left( \frac{F}{G} \right)_x dW_t$$

and since

$$\left( \frac{F}{G} \right)_t = \frac{F_t G - FG_t}{G^2}; \quad \left( \frac{F}{G} \right)_x = \frac{F_x G - FG_x}{G^2};$$
and
\[
\left( \frac{F}{G} \right)_{xx} = \frac{F_{xx}G^3 - 2F_xGxG^2 - FG_{xx}G^2 + 2FG(G_x)^2}{G^4},
\]
we may substitute to obtain
\[
d\left( \frac{F}{G} \right) = \frac{G[(F_t + \alpha F_x + \frac{1}{2} \beta^2 F_{xx}) dt + \beta F_x dW_t]}{G^2}
- \frac{F[(G_t + \alpha G_x + \frac{1}{2} \beta^2 G_{xx}) dt + \beta G_x dW_t]}{G^2}
+ \frac{1}{2} \beta^2 \left( \frac{-2F_xG_xG^2 + 2FG(G_x)^2}{G^4} \right)
= \frac{GdF - FdG}{G^2} + \frac{\beta^2 G_x}{G^3} (FG_x - GF_x)
\]
as required. \[\square\]

2.4. Dynamic Programming and the HJB Equation

The technique of dynamic programming (DP) was pioneered by Richard Bellman as a general method for solving sequential decision problems. In order to illustrate this technique, we will briefly introduce the type of problem we are considering and then describe a general procedure for solving that class of problems. We note that this section is largely drawn from Bellman (1957) and Bertsekas (2005).

Assume that we have a discrete time system which evolves according to the equation
\[
x_{k+1} = f_k(x_k, u_k, w_k)
\]
where \(x_k\) is the current system state, \(u_k\) is the control and \(w_k\) is some stochastic disturbance term. Often the control \(u_k\) is a function of the current state, that is \(u_k = \mu_k(x_k)\) where \(\mu_k\) is a strategy function. If this is the case we say that the problem is a closed-loop problem. The aim is to determine a policy \(\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}\) which is a set of control functions, which optimises the function
\[
J_0^*(x_0) = \min_{u_k \in U_k} \mathbb{E} \left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k) \right\}. \tag{2.6}
\]
In this case \( g_k \) is a cost function at each step, \( g_N \) is a terminal cost and \( U_k \) is the set of allowable strategies. Thus in this case we are attempting to choose a policy which will minimise the expected cost over a series of decisions.\(^2\) This type of problem may be solved by using dynamic programming. Since dynamic programming rests on the principle of optimality, we will first outline this principle so it may be used to develop the DP method.

### 2.4.1. Principle of Optimality and the Bellman Equation.

In this section, we introduce the principle of optimality and from this we develop the technique of dynamic programming. We begin with a derivation of the famous Bellman equation.

**Proposition 2.5 (Principle of optimality).** An optimal policy has the property that whatever the initial decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the original decision. Mathematically, this may be described as follows:

Let \( \pi^* = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\} \) be an optimal policy (set of decisions) for a particular discrete time problem. Now, imagine that at some time \( i \) a state \( x_i \) occurs with a positive probability. Then the truncated policy \( \{\mu_i, \mu_{i+1}, \ldots, \mu_{N-1}\} \) is optimal for the problem which starts at state \( x_i \).

The principle of optimality provides a foundation upon which the DP technique is built. Essentially the application of DP solves the optimisation problem by recursively solving larger tail problems until eventually the entire problem is solved. For example, initially the problem at one-step from maturity is solved, then this solution is used to solve the problem at two-steps from maturity and so on until a final solution is reached. We now apply this DP algorithm to equation (2.6) and show that it indeed does solve the required optimisation problem.

**Proposition 2.6 (DP algorithm).** The optimal cost \( J_0^*(x_0) \) is equal to \( J_0(x_0) \) which is given by the final step in the following algorithm:

\[
J_N(x_N) = g_N(x_N),
\]

\(^2\)If the functions \( g \) and \( h \) represent a profit function then we would wish to maximise rather than minimise the function \( J_0^* \).
2.4. DYNAMIC PROGRAMMING AND THE HJB EQUATION

\[ J_k(x_k) = \min_{u_k \in U_k} \mathbb{E}_k \left\{ g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)) \right\}, \quad (2.7) \]

\[ k = N - 1, N - 2, \ldots, 1 \]

where the expectation is taken with respect to the probability distribution of \( w_k \). Equation (2.7) is also known as the Bellman equation.

**Proof.** For any admissible policy \( \pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\} \) and each \( k = 0, 1, \ldots, N - 1 \), denote \( \pi^k = \{\mu_k, \mu_{k+1}, \ldots, \mu_{N-1}\} \). For \( k = 0, 1, \ldots, N - 1 \), let \( J_k^*(x_k) \) be the optimal cost for the \((N - k)\)-stage problem that starts at state \( x_k \) and time \( k \) and ends at time \( N \), that is

\[ J_k^*(x_k) = \min_{u_k \in U_k} \mathbb{E}_{[k,N-1]} \left\{ g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, \mu_i(x_i), w_i) \right\}, \]

where \( \mathbb{E}_{[k,N-1]} \) denotes the expectation taken with respect to the set of random variables \( \{w_k, w_{k+1}, \ldots, w_{N-1}\} \). The aim is to now show via induction that the functions \( J_k^* \) are equivalent to \( J_k \) generated by the DP algorithm. By definition, for \( k = N \), we have that \( J_N^*(x_N) = g_N(x_N) \), and we assume that for some \( k \) and all \( x_{k+1} \), we have that \( J_{k+1}^*(x_{k+1}) = J_{k+1}(x_{k+1}) \). Then, since \( \pi^k = \{\mu_k, \pi^{k+1}\} \), we have for all \( x_k \) that

\[
J_k^*(x_k) = \min_{\{\mu_k, \pi^{k+1}\}} \mathbb{E}_{[k,N-1]} \left\{ g_k(x_k, \mu_k(x_k), w_k) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_i(x_i), w_i) \right\} \\
= \min_{\mu_k} \mathbb{E}_k \left\{ g_k(x_k, \mu_k(x_k), w_k) \right\} + \min_{\pi^{k+1}} \left[ \mathbb{E}_{[k+1,N-1]} \left\{ g_N(x_N) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_i(x_i), w_i) \right\} \right] \\
= \min_{\mu_k} \mathbb{E}_k \left\{ g_k(x_k, \mu_k(x_k), w_k) + J_{k+1}^*(f_k(x_k, \mu_k(x_k), w_k)) \right\} \\
= \min_{u_k \in U_k(x_k)} \mathbb{E}_k \left\{ g_k(x_k, \mu_k(x_k), w_k) + J_{k+1}^*(f_k(x_k, \mu_k(x_k), w_k)) \right\} \\
= J_k(x_k).
\]
Note that we have used the principle of optimality in the second step to separate out the expectations, and the inductive assumption in the fourth step completes the proof.

The Bellman equation just derived has been the cornerstone of dynamic programming in discrete time. However, a continuous time counterpart is required when the underlying dynamics of the system are also continuous. Such a counterpart is given by the Hamilton-Jacobi-Bellman (HJB) equation. We derive this equation in the following section.

2.4.2. The Hamilton-Jacobi-Bellman Equation. The Bellman equation derived in the previous section applies when the evolution of the underlying follows a discrete time stochastic process. However, it is often more convenient, and sometimes more mathematically tractable, to define the state evolution in terms of a continuous time stochastic process. This is particularly evident in the options pricing literature where the Black-Scholes framework (a continuous time description) is the de facto method of option valuation. For this reason it would be useful to have a continuous time counterpart to the Bellman equation given by (2.7). This continuous time counterpart is referred to as the Hamilton-Jacobi-Bellman (HJB) equation as it was discovered to be an extension of the already known Hamilton-Jacobi equation from classical physics. We will now derive the HJB equation.

\textbf{Theorem 2.7.} Consider the optimisation problem

\[ J^*(x_t, t) = \min_{u_t \in U_t} E \left\{ h(x_T, T) + \int_t^T g(x_s, \mu_s(x_s), w_s) ds \bigg| \mathcal{F}_t \right\}, \]

where \( \mathcal{F}_t \) denotes the filtration over the interval \([0, t]\) and the evolution of \( x_t \) is given by the controlled SDE

\[ dx_t = a(x_t, \mu_t(x_t), t) dt + b(x_t, \mu_t(x_t), t) dW_t. \]
Then the function $J^*(x_t, t)$ satisfies the following PDE with boundary condition given at $T$:

$$
\begin{align*}
0 &= \frac{\partial J^*}{\partial t} + \min_{u_t \in U_t} \left[ g(x_t, u_t, t) + a \frac{\partial J^*}{\partial x_t} + \frac{1}{2} b^2 \frac{\partial^2 J^*}{\partial x_t^2} \right] \\
J^*(x_T, T) &= h(x_T, T)
\end{align*}
$$

**Proof.** To prove this theorem, we will start with the discrete time case and then use a convergence argument to obtain the continuous time counterpart. To simplify the process, we will introduce a new notation which will lend itself to the convergence more easily. Define the problem on the time interval $[0, T]$, let $t \in [0, T]$ and divide this interval into $N$ equal parts of size $dt$. This implies that $dt = \frac{T}{N}$.

Denote the current time $t$ and the time one step in the future as $t + dt$. This notation will replace $k$ and $k+1$ used previously to denote successive times. We also define a stochastic process for the evolution of the state variable

$$
dx_t = a(x_t, \mu_t(x_t), t)dt + b(x_t, \mu_t(x_t), t)dW_t
$$

where $W_t \overset{d}{=} \sqrt{dt}Z_t$, $Z_t \sim N(0, 1)$. Let us now rewrite the objective function (2.6) in a more suggestive manner. First, to avoid confusion replace the terminal cost function $g_N$ with $h$. Thus $g_N(x_N) = h(x_T, T)$. We may now rewrite equation (2.7) as

$$
J^*(x_t, t) = \min_{u_t \in U_t} \mathbb{E}\left\{ h(x_T, T) + \sum_{i=0}^{N-1} g(x_{t+idt}, u_{t+idt}, t + idt)dt \right\}
$$

Now, applying the DP algorithm, we have that the following equation must hold for $J^*$,

$$
J^*(x_t, t) = \min_{u_t \in U_t} \mathbb{E}\{g(x_t, u_t, t)dt + J^*(x_{t+dt}, t + dt)\}
$$

= $\min_{u_t \in U_t} \mathbb{E}\{g(x_t, u_t, t)dt + J^*(x_t + dx_t, t + dt)\}$. (2.9)

Applying Taylor’s theorem to the term $J^*(x_t + dx_t, t + dt)$, we have that

$$
J^*(x_t + dx_t, t + dt) = \left\{ J^*(x_t, t) + \frac{\partial J^*}{\partial t}dt + \frac{\partial J^*}{\partial x_t}dx_t \\
+ \frac{1}{2} \left[ \frac{\partial^2 J^*}{\partial t^2}(dt)^2 + \frac{\partial^2 J^*}{\partial x_t^2}(dx_t)^2 + \frac{\partial^2 J^*}{\partial t \partial x_t} dtdx_t \right] \\
+ h.o.t. \right\}
$$
Substituting the Taylor expanded expression for $J^*(x_t + dx_t, t + dt)$ into (2.9) and cancelling $J^*(x_t, t)$ gives

$$0 = \min_{u_t \in U_t} E \left\{ g(x_t, u_t, t) + \frac{\partial J^*}{\partial t} dt + \frac{\partial J^*}{\partial x_t} dx_t + \frac{1}{2} \left[ \frac{\partial^2 J^*}{\partial x_t^2} (dt)^2 + \frac{\partial^2 J^*}{\partial x_t^2} (dx_t)^2 + \frac{\partial^2 J^*}{\partial x_t^2} (dx_t) \cdot dt \right] \right\}. $$

We now take the limit $dt \to 0$ which means that any terms of order higher than $dt$ becomes negligible and may be set equal to 0. This gives

$$0 = \min_{u_t \in U_t} E \left\{ g(x_t, u_t, t) dt + \frac{\partial J^*}{\partial t} dt + \frac{\partial J^*}{\partial x_t} dx_t + \frac{1}{2} \frac{\partial^2 J^*}{\partial x_t^2} b^2 dt \right\},$$

the objective function now becomes

$$J^*(x_t, t) = \min_{u_t \in U_t} E \left\{ h(x_T, T) + \int_t^T g(x_s, u_s, W_s) ds \right\},$$

and the dynamics in equation (2.8) converge to a Brownian motion. Equation (2.10) may then be simplified to

$$0 = \frac{\partial J^*}{\partial t} dt + \min_{u_t \in U_t} E \left\{ g(x_t, u_t, t) dt + \frac{\partial J^*}{\partial x_t} (adt + bdW_t) + \frac{1}{2} \frac{\partial^2 J^*}{\partial x_t^2} b^2 dt \right\},$$

(2.11)

where we have used the fact that $\frac{\partial J^*}{\partial t}$ is non-stochastic at $t$ and independent of $u_t$ and $E \{dW_t\} = 0$ to perform the simplifications. Equation (2.11) is the desired Hamilton-Jacobi-Bellman equation and has the boundary condition $J^*(x_T, T) = h(x_T, T)$. This completes the proof. \[\square\]

In this thesis we will use the HJB equation to ascertain the value of an option on a traded account. Due to the discounting inherent in option valuation, the HJB equation used for the pricing of options on traded accounts will take a slightly different form than that presented here. We will cover this difference when the equation is presented in the following chapter.
2.5. The Laplace Transform

The Laplace transform is an integral transform that is used extensively in solving linear partial differential equations. The reason for this is that the Laplace transform converts operations of calculus to algebraic operations. Thus a difficult problem may often be simplified by considering the Laplace transformed version, applying algebraic manipulations to simplify the problem, then converting back via an inverse Laplace transform to obtain the solution to the original problem. We now present the mathematical definition of a Laplace transform and some properties which will prove useful later in the thesis. We will present only results in this section. For full derivations, we refer the reader to a reference such as Kreyszig (1999).

**Definition 2.8.** Let \( f(t) \) be a given function that is defined for all \( t \geq 0 \). Then the Laplace transform of this function denoted \( \bar{f}(s) \) is given by

\[
\bar{f}(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt.
\]

Furthermore, the original function \( f \) may be retrieved from the Laplace transformed function \( \bar{f} \) via the inverse Laplace transform,

\[
f(t) = \mathcal{L}^{-1}(\bar{f}) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \bar{f}(s) ds
\]

The Laplace transform has the following properties:

1. **Linearity of \( \mathcal{L}(f) \):**
   \[
   \mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
   \]

2. **Shift theorem:**
   \[
   \mathcal{L}\{e^{at}f(t)\} = \bar{f}(s - a)
   \]

3. **Laplace transform of a derivative:**
   \[
   \mathcal{L}\{(f'(t))\} = s\bar{f}(s) - f(0)
   \]
(4) Laplace transform of an integral:
\[ \mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} \bar{f}(s); \quad s > 0 \]

(5) Laplace transform of a Dirac delta function:
\[ \mathcal{L}\{\delta(t-a)\} = e^{-as}; \quad a > 0 \]

(6) Convolution theorem: Consider Two functions \( f(t) \) and \( g(t) \).
Define their convolution \( (f * g)(t) \) as
\[ h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau) d\tau. \]
It can be shown that
\[ \mathcal{L}\{h(t)\} = \mathcal{L}\{(f * g)(t)\} = \bar{f}(s)\bar{g}(s). \]

We note that in practice the inverse Laplace transform very rarely needs to be explicitly computed. Rather, it is usually sufficient to use the relations we have just outlined in order to infer the inverse Laplace transform from a table of base inverse transforms such as those found in Abramowitz and Stegun (1965).

2.6. Green’s Function Solution to the Heat Equation

In this section we will introduce a method of solving linear PDE’s through the use of Green’s functions. Specifically, we will derive the Green’s function for the 1-dimensional heat equation and later the results will be used in the valuation of the option on a traded account. Again, this section is somewhat brief in its development. For a full treatment of Green’s functions we refer the reader to Haberman (2004).

First, let us define a Green’s function.

**Theorem 2.9.** Consider the linear differential equation in general form
\[ L(x)u(x) = f(x) \quad (2.12) \]
where $L$ is a linear differential operator and $f$ is a known non-homogeneous term. Then the Green’s function $G(x, x_0)$ which satisfies

$$L(x)G(x, x_0) = \delta(x - x_0) \tag{2.13}$$

where $\delta$ is the Dirac delta function, can be used to solve (2.12) via

$$u(x) = \int_{-\infty}^{\infty} G(x, x_0)f(x_0)dx_0 \tag{2.14}$$

**Proof.** Substitute (2.14) into (2.12) and show that we obtain the appropriate solution.

$$L(x)u(x) = L \int_{-\infty}^{\infty} G(x, x_0)f(x_0)dx_0$$

$$= \int_{-\infty}^{\infty} L(x)G(x, x_0)f(x_0)dx_0$$

$$= \int_{-\infty}^{\infty} \delta(x - x_0)f(x_0)dx_0$$

$$= f(x).$$

We note that in the second step the operator $L$ may be brought inside the integral as a result of Fubini’s theorem. □

Clearly the form of the Green’s function will depend on the operator $L$. Thus in the following section we will derive the Green’s function relevant to the heat equation in one dimension.

### 2.6.1. Green’s Function for the 1-D Heat Equation

We will now derive the appropriate Green’s function for the heat equation.

**Theorem 2.10.** Consider the one dimensional heat equation

$$\begin{cases} u_t - ku_{xx} = f(x) \\ u(x, 0) = h(x). \end{cases} \tag{2.15}$$

The corresponding Green’s function is given by

$$G(x, x_0, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{|x-x_0|^2}{4kt}}. \tag{2.16}$$

**Proof.** It is known that if we consider a homogeneous heat equation with no source terms ($f(x) = 0$) and an initial condition given by
the Dirac delta function \( h(x) = \delta(x - x_0) \), then the solution is in fact the Green’s function itself. Thus the Green’s function \( G \) may be found by solving

\[
\begin{cases}
G_t = kG_{xx} \\
G(x, x_0, 0) = \delta(x - x_0)
\end{cases}
\]

where \( x \in \mathbb{R}, \ k > 0 \) and \( t > 0 \). To solve this equation, we take the Laplace transform of (2.15) in \( t \), denoting the resulting function by \( \bar{G} \). The resulting ODE is given by

\[ s\bar{G} - \delta(x - x_0) = k\bar{G}_{xx} \]

and rearranging this equation into the standard form, we have

\[ \bar{G}_{xx} - \frac{s}{k}\bar{G} = -\frac{1}{k}\delta(x - x_0) \] (2.17)

which is a second order, nonhomogeneous ODE with solution of the form

\[ \bar{G}(x, x_0, s) = Ae^{-\sqrt{s/k}|x-x_0|} \]

where \( A \) is a constant to be determined. To this end, we will compute the first and second derivatives, substitute into equation (2.17) and solve for \( A \). We have that

\[ \bar{G}_x = -A\sqrt{\frac{s}{k}}\text{sgn}(x - x_0)e^{-\sqrt{s/k}|x-x_0|} \]

\[ \bar{G}_{xx} = A\frac{s}{k}e^{-\sqrt{s/k}|x-x_0|} - 2\sqrt{\frac{s}{k}}A\delta(x - x_0), \]

hence

\[ \bar{G}_{xx} - \frac{s}{k}\bar{G} = -2\sqrt{\frac{s}{k}}A\delta(x - x_0) \]

and from (2.17), we have that \(-2\sqrt{\frac{s}{k}}A = -\frac{1}{k}\). This means \( A = \frac{1}{2\sqrt{ks}} \)

and thus we have that

\[ \bar{G}(x, x_0, s) = \frac{1}{2\sqrt{ks}}e^{-\sqrt{s/k}|x-x_0|}. \]

Now, to obtain the desired function \( G \), we must invert our Laplace transform. Using a known transform from Abramowitz and Stegun
(1965) and using the list of properties given earlier, we have that
\[ L^{-1} \left\{ \frac{1}{2\sqrt{ks}} e^{-\sqrt{k|x-x_0|}} \right\} = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{|x-x_0|^2}{4kt}} \]
and thus we have our desired result, equation (2.16).

Equation (2.16) is often called the fundamental solution to the heat equation. What needs to be determined now is how to use that solution to represent the solution of an inhomogeneous heat equation with initial condition given by (2.15). That is, how can we represent the solution of (2.15) using (2.16)? The appropriate representation is given in theorem 2.11 below.

**Theorem 2.11.** Consider the inhomogeneous heat equation with initial condition \( h(x) \) and \( k = 1 \),
\[
\begin{align*}
&\left\{ u_t - u_{xx} = f(x,t) \\
&u(x,0) = h(x)
\right. \quad (2.18)
\end{align*}
\]
Define the Green’s function with \( k = 1 \) and \( x_0 = 0 \) as
\[
\hat{g}(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}.
\]
Then the function \( u \) which satisfies (2.18) is given by
\[
u(x,t) = h(x) \ast \hat{g}(x,t) + \int_0^t f(x,\tau) \ast \hat{g}(x,t-\tau)d\tau \quad (2.19)
\]
where \( \ast \) denotes convolution, that is
\[
h(x) \ast \hat{g}(x,t) = \int_{-\infty}^{\infty} h(\xi)\hat{g}(x-\xi,t)d\xi.
\]

**Proof.** Computing the first \( t \) derivative and second \( x \) derivative of (2.19) and using the property that \( \frac{\partial}{\partial x}[f(x) \ast g(x)] = f' \ast g = f \ast g' \), we have that
\[
u_t - u_{xx} = h \ast \hat{g}_t + f(x,t) \ast \hat{g}(x,0) + \int_0^t f(x,\tau) \ast \hat{g}_t(x,t-\tau)d\tau
\]
\[- h \ast \hat{g}_{xx} - \int_0^t f(x,\tau) \ast \hat{g}_{xx}(x,t-\tau)d\tau.
\]

\(^3\text{We set } k = 1 \text{ as this is the form we require for solving the HJB equation in the following chapter.}\)
First, we can simplify the above expression to obtain
\[ u_t - u_{xx} = h \ast (\hat{g}_t - \hat{g}_{xx}) + f(x, t) \ast \delta(x) \]
\[ + \int_0^t f(x, \tau) \ast [\hat{g}_t - \hat{g}_{xx}](x, t - \tau) d\tau \]
\[ = f(x, t) \]
where we have used the fact that \( \hat{g}_t - \hat{g}_{xx} = 0 \) and \( \hat{g}(x, 0) = \delta(x) \), and secondly by examining (2.19) with \( t = 0 \), we have that
\[ u(x, 0) = h(x) \ast \delta(x) + 0 \]
\[ = h(x). \]

We now have the results, related to using Green’s functions to solve inhomogeneous heat equations with initial conditions, that will be required in the valuation of passport options.

2.7. Concluding Remarks

This chapter has covered a number of mathematical requirements to undertake the task of valuing options on traded accounts. Essentially most of the focus in this chapter was on the HJB equation, Laplace transform and Green’s functions techniques for solving ODE’s and PDE’s. The reasons for this will become clear in the following chapter where we examine the problem of valuing options on traded accounts in a continuous time setting. As we will show, the valuation problem may be thought of as a problem of stochastic optimal control and as such the HJB equation may be used to describe the value function. Solving this equation will require the techniques developed in sections 2.5 and 2.6. Alternatively the problem may be set up in a probabilistic manner. In this case, much of the theory covered in section 2.3 will be also be required.
CHAPTER 3

Continuous Time Valuation of Options on Traded Accounts

3.1. Introduction

In this chapter we examine the problem of valuing an option on a traded account using continuous time models. This is a problem which has already been addressed in the literature, and as such, this chapter addresses current solutions to this problem, but also provides some new contributions in the form of alternative techniques which enhance clarity and transparency.

Specifically, this chapter will proceed in the following manner. First, we construct what we call the basic problem. This section will serve to define the valuation problem at hand, thus providing an overview of the problem and the notation which will be used throughout the remainder of this chapter. Once the problem is defined, we then proceed to provide a review of the literature pertaining to this problem. Following this, we go into more depth by providing detailed derivations of the option’s value function for the special cases of the constant strategy option, the passport option and the more general option on a traded account. For the latter two, we will follow the general approach of two of the reviewed papers. However, we do not identically follow those authors’ methods. Instead we prefer to use our own techniques, many of which have been described in chapter 2, as we believe that these techniques serve to simplify the task. Finally, we make our most significant contribution to the literature by providing a new derivation for the value function for a passport option.
3.2. The Basic Problem

Since all the valuation methods in this chapter rely on the same methodology to derive the dynamics of the trading account, we will first present some basic relations that will be generally applicable to all the papers reviewed.

Let us assume that the dynamics of the stock\(^1\) under the risk neutral measure \(\mathbb{P}\) are given by the g.B.m.

\[
dS_t = S_t((r - \delta)dt + \sigma dW_t),
\]

where \(\delta\) is the dividend yield and \(W_t\) is a standard one-dimensional Brownian motion. Consider now a trading account which consists of a position in the stock of size \(\theta\) with the remaining wealth accruing at the rate \(r\). We naturally require that \(\theta\) is constrained to an interval \([\alpha, \beta]\) as described earlier. Then, under \(\mathbb{P}\), the dynamics of the trading account \(X\) are given by

\[
dX_t = rX_t dt + \theta_t(dS_t - \nu S_t dt) \\
= [rX_t + \theta_t(r - \delta - \nu)]dt + \theta_t S_t \sigma dW_t
\]

(3.2)

where \(\nu\) is the cost-of-carry which is subtracted from the trading account when a long position is taken and added to the account when taking a short position. This is simply a cost for maintaining a position in the risky asset. In this case we assume that a long position requires an expense of \(\nu\) to hold, while a short position provides this rate as income.

It should be noted that we will often refer to the “symmetric case”. This essentially simplifies the problem by setting the parameters \(\delta\) and \(\nu\) equal to 0 and the return on the account \(r\) equal to the risk-free rate \(r\). If this is done, the equations (3.1) and (3.2) become

\[
dS_t = S_t(r dt + \sigma dW_t)
\]

(3.3)

\[
dX_t = rX_t dt + \theta_t \sigma dW_t
\]

(3.4)

\(^1\)Technically this could be any traded risky financial asset.
3.2. THE BASIC PROBLEM

respectively. It is important to note that equations (3.3) and (3.4) are coupled SDE’s which are driven by a single Brownian motion. Consider now a payoff function, \( \Pi \), which is given by a zero strike call on the value of the trading account at a fixed time \( T \). This payoff is expressed mathematically as

\[
\Pi(X_T) = \max(X_T, 0) = (X_T)^+.
\] (3.5)

The problem is to use (3.3) and (3.4) to price an option with payoff function given by (3.5) at any time \( t \) prior to the time of maturity, \( T \).

Now, for some arbitrary trading strategy \( \theta \), we may derive a PDE which is analogous to the Black-Scholes PDE via a continuous hedging strategy. This equation is derived in theorem 3.1 below.

**Theorem 3.1.** The value of an option on a traded account, \( V(s, x, t) \) where \( s = S_t \) and \( x = X_t \), under some arbitrary trading strategy \( \theta \) and the assumptions of the symmetric case satisfies the following PDE:

\[
-V_t = -rV + rsV_s + rxV_x + \frac{1}{2}\sigma^2 s^2 (V_{ss} + 2\theta V_{sx} + \theta^2 V_{xx})
\] (3.6)

where the subscripts on \( V \) denote partial derivatives.

**Proof.** Given that the option value is a function of the trading account \( x \), which itself is a function of the stock price \( s \), it is clear that the option value will be a function of these two state variables. Using the two dimensional version of Ito’s lemma, we have to second order that

\[
dV = V_t dt + V_s ds + V_x dx + \frac{1}{2} [V_{ss}(ds)^2 + 2V_{sx}dsdx + V_{xx}(dx)^2].
\] (3.7)

Using some simple rules of stochastic calculus, it is straightforward to show that the following relations hold

\[
(ds)^2 = \sigma^2 s^2 dt
\]
\[
dsdx = \theta \sigma^2 s^2 dt
\]
\[
(dx)^2 = \theta^2 \sigma^2 s^2 dt.
\]
3.2. THE BASIC PROBLEM

Substituting these into equation (3.7) gives

\[ dV = V_t dt + V_s ds + V_x dx + \frac{1}{2} \sigma^2 s^2 (V_{ss} + 2\theta V_{sx} + \theta^2 V_{xx}) dt. \]

Now, let us construct a portfolio, \( P \), which consists of one long passport option and \( h \) units short of the stock. Such a portfolio will have an initial value given by \( P = V - hS \) and dynamics

\[ dP = dV - hdS \\
= [V_t + r s V_s + r x V_x + \frac{1}{2} \sigma^2 s^2 (V_{ss} + 2\theta V_{sx} + \theta^2 V_{xx})] dt \\
+ \sigma s (V_s + \theta V_x) dW - h s (r dt + \sigma dW). \] (3.8)

Since there is only one stochastic term, it can be completely eliminated by choosing \( h = V_s + \theta V_x \). Choosing \( h \) in this manner implies that the portfolio has no risk and thus must earn the risk-free rate. This implies that \( dP = rP dt = r(V - hs) dt \). Substituting this into equation (3.8) and rearranging yields equation (3.6).

\[ \square \]

From the results of Harrison and Pliska (1981) the function which satisfies (3.6) subject to the terminal condition \( V(s, x, T) = (x)^+ \) is given by

\[ V(s, x, t) = e^{-r(T-t)} \mathbb{E}_P \{(X_T)^+ | \mathcal{F}_t \}, \] (3.9)

where \( \mathcal{F}_t \) is a filtration induced by \( W_t \) satisfying the usual properties.

By inspection, we can see that the value of the option is dependent on the trading strategy chosen by the option holder. This could give rise to a potentially infinite number of option values for the infinite number of trading strategies available. To overcome this problem, and value the option unambiguously, it is assumed that the option holder will attempt to maximise their terminal wealth. This is also consistent with the option seller charging as much as possible for the option, the value of which can be used to hedge the potential payout should the option holder make a loss on their trading account. Under these assumptions, the value of the option on a traded account is given by
the optimisation problem

\[ V^*(s, x, t) = \max_{\theta_t \in [\alpha, \beta]} e^{-r(T-t)} \mathbb{E}_t \{(X_T)^+ | \mathcal{F}_t\}. \quad (3.10) \]

Using the HJB equation, we will show later that the PDE which corresponds to equation (3.10) is in fact almost identical to (3.6), differing only by the inclusion of a max function.

Before presenting a derivation of the option value function, we will briefly discuss some of the previous work which has been done on this class of options. While we restrict ourselves to the symmetric case for analytic tractability, the work we will examine has either analysed the problem just presented, or some extension of it.

### 3.3. Literature Review

The solution methods for this problem can essentially be split into two distinct groups. The first are those that use the theory of stochastic control and thus consider the Hamilton-Jacobi-Bellman PDE and its properties to ascertain the optimal trading strategy and solve for the option value. The second class uses a variety of techniques, which can be broadly described as probabilistic, starting with equation (3.10) and then using a series of transformations to simplify the pricing problem to obtain the optimal trading strategy and option value. We now examine the existing literature for both these approaches in turn.

#### 3.3.1. Partial Differential Equation Methods.

The use of the Hamilton-Jacobi-Bellman (HJB) equation has been the traditional way to solve the basic problem given by equations (3.1), (3.2) and (3.10) as it is a classic problem in the area of stochastic optimal control.

As stated earlier, the problem of pricing passport options was first examined by Hyer et al. (1997). Later, we will give a detailed account of the approach these authors used, however we will briefly present their methodology here so as to place it in context with the rest of the literature on this topic. The setup used by these authors is similar to the basic problem presented in equations (3.1) and (3.2) in that they allow for a rate other than the risk-free rate \( r \) for funds in the
3.3. LITERATURE REVIEW

trading account, and the inclusion of a dividend yield and cost-of-carry. The derivation of an analytic solution however, requires the specific form of the symmetric case and thus account returns must be set to equal to the risk-free rate and the dividend yield and cost of carry are set equal to zero. Under this scenario the discounted trading account is a martingale under the risk neutral measure for any given trading strategy. Whilst numerical results are provided for the general case, analytic solutions are available for the symmetric case only.

Considering the account dynamics, the authors then provide an argument that only the set of Markovian strategies need be examined since the payout under an arbitrary strategy is also Markovian. They then use Ito’s lemma to derive the dynamics of the option value under an arbitrary trading strategy and then through the formation of a risk-free portfolio derive a PDE for the value of a passport option under some arbitrary trading strategy. This equation is equivalent to our equation (3.6). The existence of this hedging strategy implies that a unique risk neutral measure exists and thus the option may be valued via equation (3.9). The assumption that option holders maximise their expected wealth implies that the value of the passport option in obtained by solving equation (3.10). To this end the authors introduce the Hamilton-Jacobi-Bellman (HJB) equation for the trading account but note that it is an equation in two state variables, the stock and trading account value. This equation, applied to the symmetric case of (3.3), (3.4) and which satisfies (3.10) is given by

\[
\begin{cases}
-rV^* + V_t^* + rsV_s^* + rxV_x^* + \max_{|\theta|\leq 1} \left( \frac{1}{2} \sigma^2 s^2 (V_{ss}^* + 2\theta V_{sx}^* + \theta^2 V_{xx}^*) \right) = 0 \\
V^*(s, x, T) = (x)^++
\end{cases}
\]

(3.11)

where the subscripts denote partial derivatives, \( S_t = s \) and \( X_t = x \) for \( t \in [0, T] \). By showing that the equation for the value of the passport option value is homogenous of degree 1 in \( s \) and \( x \), a transformation is proposed which simplifies the analysis of the HJB equation. The key is to rescale the problem via the transformation \( Z_t = \frac{x}{S_t} \) and then
express the HJB equation in terms of $Z_t$. Using this transformation, (3.11) reduces to

$$U_t^* + \frac{1}{2} \sigma^2 U_{zz}^* \max_{|\theta| \leq 1} \{(z - \theta)^2\} = 0$$

(3.12)

where $Z_t = z$ and $V^*(s, x, t) = sU^*(z, t)$ due to the homogeneity of (3.10). Given that the term $U_{zz}^* \geq 0$ everywhere due to convexity, it is clear from (3.12) that the optimal strategy for this symmetric case is given by

$$\theta^* = \begin{cases} 
-1 & \text{if } z \geq 0 \\
1 & \text{if } z \leq 0
\end{cases}$$

(3.13)

As the authors point out, this strategy is intuitively consistent with a convex payout function since such functions achieve their maximum at their extrema. We point out that optimal strategies which occur only at the boundaries are called “bang-bang” strategies. Substituting (3.13) into (3.12) yields

$$U_t^* + \frac{1}{2} \sigma^2 (|z| + 1)^2 U_{zz}^* = 0$$

(3.14)

with boundary condition $U^*(z, T) = (z)^+$. Now, in order to solve (3.14), the authors make some transformations to express the problem in a manner that is as close as possible to the standard heat equation and then derive the relevant Green’s function. As Hyer et al. (1997) point out, this is the most difficult task, as once the Green’s function is known, solving the problem is a simply a matter of integration. Whilst some numerical results are presented by the authors, the algorithm used to obtain them is not discussed.

Numerical methods to solve the HJB equation for the purpose of pricing passport options were further analysed by Andersen et al. (1998) who also presented a special case which could be solved analytically. Whilst the main points examined are essentially the same as those covered by Hyer et al. (1997), the setup they use is slightly different. Rather than considering the underlying as a trading account

---

2For the case $z = 0$, both $\theta = 1$ and $\theta = -1$ yield the same optimal result.
which can accrue at some rate of interest \((r\) in the symmetric case),
the authors instead opt to examine the gains process from a particular
sequence of trades. That is, the underlying in this case is given by
\[
\hat{X}_t = \int_0^t \theta_u dS_u, \quad \hat{X}_0 = 0
\] (3.15)
where we have used \(\hat{X}\) for the gains process to keep it distinct from
the previously defined trading account \(X\), and \(\theta_t \in [-1, 1]\). The payoff
for this passport option is given by \(\hat{V}(s, x, T) = (\hat{X}_T)^+\) and since the
discounted value of the option is a martingale under \(\mathbb{P}\), we have that
\[
\hat{V}^*(s, x, t) = \max_{|\theta_t| \leq 1} e^{-r(T-t)} \mathbb{E}_\mathbb{P}\{ (\hat{X}_T)^+ | \mathcal{F}_t \}.
\]
Noting that (3.15) may be written in the differential form
\[
d\hat{X}_t = \theta_t dS_t
= \theta_t S_t r dt + \theta_t S_t \sigma dW_t
\] (3.16)
allows the authors to formulate a stochastic control problem which
may be solved by analysing the associated HJB equation. These au-
thors use the transformation provided by Hyer et al. to reduce the
dimensionality of the problem, though this is introduced as a mea-
sure change rather than being a result of homogeneity. Following this
tansformation, the authors use convexity properties of the payoff func-
tion together with the HJB equation to show that the optimal strategy
is of the “bang-bang” type. When dealing with the symmetric case, the
authors note that the optimal strategy for their problem is identical to
that derived by Hyer et al. (1997), namely to be long when behind and
short when ahead. Under the assumptions of the symmetric case, the
authors derive an analytic solution to the pricing problem and provide
some intuition as to why the optimal strategy is as derived. Consid-
ering the problem at the final trading date, the authors represent the
pricing problem as choosing between a standard European put or call
option with specially set strikes and then use the Black-Scholes for-
mula to show that switching gains some value. As the authors point
out this is in fact directly related to the skew of the log-normal distribution. The authors then turn their attention to solving the more general problem (without the simplifications required to achieve the symmetric case) which they solve numerically. Their method of solution is a mixed finite difference scheme for the pricing PDE. Essentially this is a scheme which can switch between implicit, explicit and Crank-Nicholson methods through the value of a mixing parameter. With this numerical technique, some extensions are examined, these being discrete switching times, alternative payoffs functions and American exercise.

The generalisation of American exercise also formed the basis for the work of Chan (1999). After rederiving many of the results obtained in Andersen et al. (1998), and thus presenting the HJB equation, the author then turns to the problem involved with exercising the passport option at any time, that is the option holder decides at which time the account value will be claimed. In this case we have an optimal stopping problem and the relevant equation to solve is given by

$$V^*_{Amer}(s, x, t) = \max_{\tau \in [t, T]} V^*(s, x, \tau)$$

$$= \max_{\tau \in [t, T]} \left\{ \max_{|\theta| \leq 1} e^{-r(T-t)} \mathbb{E}_{\mathbb{F}}\{ (\hat{X}_{\tau})^+ | \mathcal{F}_t \} \right\}.$$  

Andersen et al. (1998) presented a method of solving such a problem by formulating it as a free boundary problem. Chan (1999) however, transforms the problem into a system of linear variational inequalities which define the option value function. This system of equations is then solved numerically using a finite difference scheme, specifically with a solver called PATH, the details of which we will not expand upon in this thesis. The reason for this effort is because the author (Chan) believes that the PATH solver provides superior efficiency and accuracy over traditional methods used to solve the HJB equation.

Other exotic features beyond American exercise have been included in the analysis of passport options pricing. Exotic variants such as the chooser, barrier, smooth trader, reset, double stake, magic potion and
switch passport option were examined in Penaud et al. (1999) while Ahn et al. (1999) examined multi-asset passport options with discrete trading features. In both cases the authors derive the relevant HJB equation and then use finite difference techniques to solve the pricing PDE. The exotic features are accounted for by altering the boundary conditions associated with the pricing equation.

### 3.3.2. Probabilistic Methods.

Following the original research by Hyer et al. (1997), presented in the previous section, much work has been done to price the passport option and its extension, the option on a traded account, using probabilistic methods rather than the PDE methods that are a result of following the clear path of using stochastic optimal control techniques. The main reasons for this departure is to both gain a greater level of understanding about the value function and to address limitations of the PDE methods.

The work of Henderson and Hobson (2000) deviated from the traditional PDE methods to provide a more concise way of pricing the passport option. Working within the bounds of the symmetric case, their technique involves using Tanaka’s formula to rewrite the expression for the option value by introducing a local time process, then use Skorohod’s theorem to write the option pricing problem in a simpler form than given in PDE methods. The simplicity of this form relies upon knowledge of the optimal strategy, and to this end the authors show that the standard optimal strategy, namely that it is optimal to be long when behind and short when ahead, applies to a variety of diffusion processes and not just for g.B.m. The proof of this is based on stochastic coupling arguments. The authors also link the pricing of passport options to that of lookback options and provide examples under the assumption of a g.B.m. Additional results regarding the valuation of passport options under stochastic volatility were examined by Henderson and Hobson (2001). They found that under a range of stochastic volatility models, the usual optimal strategy derived for g.B.m. with constant volatility in fact remains optimal. The Hull and White (1987) and Stein and Stein (1991) models are specifically examined.
The work of these authors imply that the optimal trading strategy associated with passport options is in fact very robust to model mis-specifications. Nagayama (1998) also uses similar ideas from stochastic analysis to price a passport option in the symmetric case, and provides a numerical approximation for the general case of a different funding rate and a dividend paying stock. The author also derives the value of the passport option as a limit of a discrete time process for the trading account.

Further work on the valuation of passport options was undertaken by Delbaen and Yor (2002) which provided a link between the valuation of such options and the properties of certain martingales. Specifically, the finiteness of the option value is shown to be related to the properties of \( H^1 \) semi-martingales. The authors also examine a variety of different possible contracts which include classes which vary whether the option holder receives interest on the account, dividends from the stock and what trading frequency is allowed. This essentially gathers both the SDE for the account process and the gains process defined in equations (3.2) and (3.16) respectively into a single framework. The tools used to develop the pricing theory include Skorohod’s lemma and concepts of local time. The authors also develop a proof that under the assumption of g.B.m., the optimal strategy for a passport option with discrete trading constraints is the same as that for continuous trading constraints, namely to be long when the account value is negative and short when positive. The proof of this is built around a dynamic programming argument, though the actual proof is quite technical and requires that the authors in fact derive many properties associated with the option value functions. Previously, the inclusion of this discrete trading feature had only been included in numerical solutions and the optimal strategy had not been proved analytically. At this stage we would like to emphasise that this result regarding the optimal trading strategy under discrete trading restrictions has much in common with our discrete time analysis of passport options which will be undertaken in chapter 4.
While these works added to the exposition of the valuation formulas and have provided a depth of understanding regarding the optimal trading strategy for passport options, they nonetheless still valued a contract with the same trading constraint, that is $\theta \in [-1, 1]$. Shreve and Vecer (2000) introduced an extension to the passport option which they called an option on a traded account, and provided an analytic solution, in the symmetric case, for the value of such an option as well as the optimal trading strategy. We will provide a detailed analysis of this paper in section 3.4.3, but for now we briefly outline its results.

This extension permitted a far wider class of trading possibilities by allowing the positions in the stock to be within some arbitrary interval. Mathematically speaking, the trading constraint is now written as $\theta \in [\alpha, \beta]$ where $\alpha < \beta$. With this extension, two new options were introduced, the vacation call and vacation put which have trading constraints $\theta_{vc} \in [0, 1]$ and $\theta_{vp} \in [-1, 0]$ respectively. The authors use a transformation similar to that first used by Hyer et al. (1997) though an additional term is included to account for the lack of symmetry in the trading interval. A measure change allows them to express the transformed account value as a martingale, then the mean comparison theorem of Hajek (1985) is used to provide a means of identifying the optimal strategy by maximising the absolute value of the volatility term. The optimal strategy for the option on a traded account is

$$\theta_t = \alpha \mathbb{1}(X_t \geq \frac{\alpha + \beta}{2} S_t) + \beta \mathbb{1}(X_t < \frac{\alpha + \beta}{2} S_t) \quad (3.17)$$

which reduces to the standard optimal strategy when $\alpha = -1$ and $\beta = 1$, that is to go long when behind and short when ahead. Substituting this strategy into the pricing equation and using probabilistic methods, the authors derive a formula for the value of an option on a traded account. They also derive a put-call parity relationship for options on traded accounts and pay special attention to the value of vacation options. They also show that if the option holder does not act optimally, then the seller may benefit by taking the appropriate hedge.
3.4. Derivation of the Option Pricing Formula

Having provided a brief overview of the literature pertaining to the valuation of options on traded accounts, we will now provide a detailed analysis of two of the more prominent methods outlined in the previous section. The first will detail the PDE approach of solving the HJB equation through the use of Green’s functions. This essentially follows the work of Hyer et al. (1997) but provides more detail and some innovations. The second approach will be probabilistic and is based on the work of Shreve and Vecer (2000), again with more detail. It should be noted that while we may apply the general methodology used by the authors mentioned above, we often use different techniques in solving the specific steps of the problem. These special methods that we use simplify the overall approach of this highly non-trivial valuation problem, and thus we consider it to be an important contribution to the existing literature.

Before we begin with these two methods of valuing passport options and options on traded accounts, we consider a simplified version of the option on a traded account. In this case, we assume that the option holder sets their position at the start of the contract and may not change it for the duration of the option. This is referred to as constant strategy option on a traded account, and as we will show, these options are in fact directly related to vanilla put and call options. This result was first presented by Shreve and Vecer (2000). In the following section we re-derive this result by directly computing the value of such an option.

3.4.1. Value of Constant Strategy Options on Traded Accounts. Following from section 3.2, we wish to find the value of the option $V$ under the conditions

$$dS_t = S_t(rdt + \sigma dW_t)$$
$$dX_t = rX_t dt + \theta \sigma S_t dW_t$$
$$V(s, x, t) = e^{-r(T-t)} \mathbb{E}\{(X_T)^+|\mathcal{F}_t\}$$
where \( s = S_t, \ x = X_t \) and \( \theta \) is a constant. To simplify exposition, we will also make the substitution \( \tau = T - t \) and note that all expectations are with respect to the filtration \( \mathcal{F}_t \) induced by \( W_t \), though we often omit this from the expression to simplify notation. First, we notice that the dynamics of \( X_t \) require two state variables, so we propose a transformation which reduces the dimensionality of the problem. This transformation was first used by Hyer et al. (1997). Define \( Z_t = \frac{X_t}{S_t} \). Now, using Ito’s quotient rule (theorem 2.4), we have that the SDE for \( Z_t \) is given by

\[
d\left( \frac{X_t}{S_t} \right) = dZ_t = (Z_t - \theta)[\sigma^2 dt - \sigma dW_t]
\]

and since \( \theta \) is a constant, this is equivalent to

\[
d(Z_t - \theta) = (Z_t - \theta)[\sigma^2 dt - \sigma dW_t]
\]

which has the form of a geometric Brownian motion. This SDE has a well known solution given by

\[
Z_T = \theta + (Z_t - \theta)e^{\frac{1}{2}\sigma^2 \tau - \sigma \sqrt{T} z}
\]  

(3.18)

where\(^3 \) \( z \sim N(0,1) \). Now, since \( S_T > 0 \) always, we may write

\[
V(s, x, t) = e^{-rt} \mathbb{E}\{S_T(Z_T)^+ | \mathcal{F}_t\}
\]  

(3.19)

and since \( S_t \) follows a geometric Brownian motion, the expression for \( S_T \) is known to be

\[
S_T = se^{(r - \frac{1}{2}\sigma^2) \tau + \sigma \sqrt{T} z}
\]

where the random variable \( z \) is the same in \( S_T \) as it is in \( Z_T \). Substituting this expression for \( S_T \) and equation (3.18) into (3.19) gives

\[
V(s, x, t) = s \mathbb{E}\left\{ \left(G + \theta e^{\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{T} z}\right)^+ | \mathcal{F}_t \right\}.
\]

\(^3\)We point out that ordinarily the symbol \( Z \) would be used in place of \( z \), however we have used this notation to keep the standard normal random variable distinct from the ratio \( Z_t \).
First consider the term $e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z}$. This is an exponential martingale with initial value 1. Hence, $E\left\{e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z}\right\} = 1$ and obviously $e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z} > 0$.

We now consider the option value for different values of $\theta$. To simplify the notation, we omit the filtration $\mathcal{F}_t$ from the expectations, though it should be understood that all future expectations are taken with respect to $\mathcal{F}_t$ unless otherwise specified.

**Case 1a: $\theta > 0$.** First, let us assume that $Z_t - \theta \geq 0$. Then the option value is

$$
V(s, x, t) = sE\left\{(Z_T - \theta) + \theta e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z}\right\}^+ \\
= sE\left\{(Z_T - \theta) + \theta e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z}\right\} \\
= s(Z_t - \theta + \theta) \\
= X_t
$$

**Case 1b: Again let $\theta > 0$, but $Z_t - \theta < 0$.** To determine the option value we now need to know when the argument of the plus function is positive. This occurs when

$$(Z_T - \theta) + \theta e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z} > 0 \quad \Rightarrow \quad z > \frac{\log(1 - \frac{Z_t}{\theta}) + \frac{1}{2}\sigma^2 \tau}{\sigma \sqrt{\tau}}$$

Let us define the term

$$
\delta_1 = \frac{\log(1 - \frac{Z_t}{\theta}) + \frac{1}{2}\sigma^2 \tau}{\sigma \sqrt{\tau}}.
$$

Then the value of the option can be written as

$$
V(s, x, t) = sE\left\{(Z_T - \theta) + \theta e^{-\frac{1}{2}\sigma^2 \tau + \sigma \sqrt{\tau} z}\right\} \mathbb{1}(z > \delta_1) \\
= s(Z_t - \theta)E\{\mathbb{1}(z > \delta_1)\} + s\theta E\left\{e^{\sigma \sqrt{\tau} z} \mathbb{1}(z > \delta_1)\right\} \\
= s(Z_t - \theta)E\{\mathbb{1}(z < -\delta_1)\} + s\theta E\left\{\mathbb{1}(z + \sigma \sqrt{\tau} > \delta_1)\right\} \\
= s(Z_t - \theta)\mathcal{N}(-\delta_1) + s\theta\mathcal{N}(\delta_2) \quad (3.20)
$$
where we have defined $\delta_2 = \delta_1 - \sigma \sqrt{\tau}$ and have used the Gaussian shift theorem and the symmetry property of Gaussian random variables to arrive at the final solution. Thus the complete solution for $\theta > 0$ is

$$V(s, x, t) = \begin{cases} x & \text{if } x \geq \theta s \\ (x - \theta s)N(-\delta_1) + s\theta N(-\delta_2) & \text{if } x < \theta s \end{cases} \quad (3.21)$$

We now examine the case where $\theta < 0$.

**Case 2a:** $\theta < 0$. For simplicity, let us set $\theta = -\alpha$ where $\alpha > 0$. First, assume $Z_t + \alpha \leq 0$. In this case all the arguments of the plus function are negative and hence the value of the option is $V(s, x, t) = 0$.

**Case 2b:** Assume now that $Z_t + \alpha > 0$. We thus need to determine when the argument of the plus function is positive. This occurs when

$$(Z_T + \alpha) - \alpha e^{-\frac{1}{2} \sigma^2 \tau + \sigma \sqrt{\tau} z} > 0 \quad \Rightarrow \quad z < \frac{\log(1 + \frac{Z_t}{\alpha}) + \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}}$$

or, $z < \delta_1$ with $\theta = -\alpha$. The value of the option in this case is given by

$$V(s, x, t) = sE \left\{ \left[ (Z_T + \alpha) - \alpha e^{-\frac{1}{2} \sigma^2 \tau + \sigma \sqrt{\tau} z} \right] I(z < \delta_1) \right\} = s(Z_t + \alpha)N(\delta_1) - s\alpha N(\delta_2)$$

where a similar approach was taken as that for deriving equation (3.20).

Thus the general formula for when $\theta = -\alpha$, $\alpha > 0$ is

$$V(s, x, t) = \begin{cases} 0 & \text{if } x \leq -\alpha s \\ (x + \alpha s)N(\delta_1) - s\alpha N(\delta_2) & \text{if } x > -\alpha s \end{cases} \quad (3.22)$$

We now consider two special cases of the constant strategy option on a traded account.

**3.4.1.1. $V(s, x, t)$ when $\theta = 1$.** In this scenario, we will assume that the option holder must hold a long position in the stock for the duration of the contract. In this case the option value is

$$V(s, x, t) = \begin{cases} x & \text{if } x \geq s \\ (x - s)N(-\delta_1) + sN(-\delta_2) & \text{if } x < s \end{cases}$$
Recall the value for a European call option under the Black-Scholes framework is given by

\[ C(s, t) = sN(d_1) - ke^{-r\tau}N(d_2) \]

where

\[ d_1 = \frac{\log\left(\frac{s}{k}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}. \]

Now, if we set the strike \( k \) of this call to \( k = e^{r\tau}(s - x) \) with \( s - x > 0 \), then \( d_1 \) is written as\(^4\)

\[ d_1 = \frac{\log\left(\frac{s}{s-x}e^{-r\tau}\right) + r\tau + \frac{1}{2}\sigma^2\tau}{\sigma \sqrt{\tau}} \]
\[ = \frac{\log\left(\frac{1}{1-Z_t}\right) + \frac{1}{2}\sigma^2\tau}{\sigma \sqrt{\tau}} \]
\[ = \frac{-\log(1 - Z_t) + \frac{1}{2}\sigma^2\tau}{\sigma \sqrt{\tau}} \]
\[ = -\delta_2 \]

and a similar procedure may be used to show that with this particular strike, \( d_2 = -\delta_1 \). Thus we have that an option on a traded account with constant strategy \( \theta = 1 \) is equivalent to a European call option with strike \( k = e^{r\tau}(s - x) \).

3.4.1.2. \( V(s, x, t) \) when \( \theta = -1 \). In this case the option value is given by

\[ V(s, x, t) = \begin{cases} 
0 & \text{if } x + s \leq 0 \\
(x + s)N(\delta_1) - sN(\delta_2) & \text{if } x + s > 0
\end{cases} \]

Consider now the value of a European put option. Under the Black-Scholes framework this has a value \( P \) given by

\[ P(s, t) = ke^{-r\tau}N(-d_2) - sN(-d_1) \]

\(^4\)If \( s > x \) is not satisfied, then we are considering a call option with negative strike which is not economically reasonable.
where \( d_1 \) and \( d_2 \) are as previously defined. Now, consider this option with a strike \( k = e^{r\tau}(x + s) \). The term \(-d_1\) in this case is given by

\[
-d_1 = \frac{-\log\left(\frac{s}{x+s}e^{-r\tau}\right) - r\tau - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}
\]

\[
= \log(1 + Z_t) - \frac{1}{2}\sigma^2\tau
\]

\[
= \delta_2.
\]

Similarly, it may be shown that \(-d_2 = \delta_1\). Thus the option on a traded account with constant strategy \( \theta = -1 \) is equivalent to a European put option with strike \( k = e^{r\tau}(x + s) \).

What this analysis shows is just how flexible the option on a traded account framework is. With no changes to the position allowed, the option is equivalent to a vanilla European option. With a single position change allowed, we would be dealing with an option which is equivalent to a standard American option. Obviously with other conditions placed on how frequently position changes are made, other types of exotic options may be replicated. While we will not detail these requirements and their corresponding options here, we note that Vecer (2001) showed that arithmetic average Asian options are in fact a special case of options on traded accounts. It is well known that such options have no closed form solutions.

In the following sections we will drop the restriction of \( \theta \) being constant and solve the problem of pricing an option on a traded account, in which \( \theta \) may have any value in \([\alpha, \beta]\). We will begin with the special case of passport options and then cover the extension of general options on traded accounts.

### 3.4.2. Passport Option Value: Solving the HJB Equation.

In this section, we will derive the formula for pricing a passport option by solving the HJB equation via a Green’s function. This essentially follows the procedure used by Hyer et al. (1997). Since an analytic solution is available only for the symmetric case, we will use that as our starting point. The relevant equations were outlined in section 3.2,
however we restate them here for clarity. Consider the problem

\[ dS_t = S_t(rdt + \sigma dW_t); \quad S_t = s \]  \tag{3.23}

\[ dX_t = X_t rdt + \theta_t S_t \sigma dW_t; \quad X_t = x \]  \tag{3.24}

\[ V^*(s, x, t) = \max_{\theta_t \in [-1, 1]} e^{-r(T-t)} \mathbb{E}_F \{(X_T)^+ | \mathcal{F}_t\} \]  \tag{3.25}

where \( S_t \) denotes the stock price and \( X_t \) the account value at time \( t \).\(^5\) As was outlined in section 2.4 of chapter 2, this is a problem of stochastic optimal control and as such can be solved via the corresponding HJB equation. Due to the slight differences in model specification between this problem and the general problem described in section 2.4, we will now derive the appropriate HJB equation for the function \( V^* \). Consider the optimisation problem \textit{without discounting} given by

\[ v(s, x, t) = \max_{\theta_t \in [-1, 1]} \mathbb{E}\{(X_T)^+ | \mathcal{F}_t\} \]

Since \( X_T \) depends on both \( S_t \) and \( X_t \), we require the two dimensional version of the HJB equation given by

\[ v_t + \alpha_s v_s + \alpha_x v_x + \max_{\theta_t \in [-1, 1]} \left\{ \frac{1}{2} \beta_s^2 v_{ss} + \frac{1}{2} \beta_x^2 v_{xx} + \beta_s \beta_x v_{sx} \right\} = 0 \]

where \( \alpha_{s/x} \) and \( \beta_{s/x} \) are the drift and volatility terms for the stock and account processes defined by equations (3.23) and (3.24) and the subscripts on \( v \) denote partial differentiation. Now, the value of the passport option is given by the function \( V^* = e^{-r(T-t)}v \). From this definition, it is easy to compute

\[ v_t = e^{r(T-t)} V^* - r V^*, \quad v_{s/x} = e^{r(T-t)} V_{s/x}^* \]

and

\[ v_{ss/xx/sx} = e^{r(T-t)} V_{ss/xx/sx}^*. \]

\(^5\)Again, all future expectations will be with respect to the filtration \( \mathcal{F}_t \), though we often omit this to simplify notation.
Making the substitutions for the drift, volatility and new derivatives, we have that the HJB equation satisfied by the function $V^*$ is

\[
\begin{cases}
V_t^* = rV^* - rsV_s^* - rxV_x^* - \max_{|\theta| \leq 1} \left\{ \frac{1}{2} \sigma^2 s^2 \left( V_{ss}^* + 2\theta V_{sx}^* + \theta^2 V_{xx}^* \right) \right\} \\
V^*(s, x, T) = (X_T)^+
\end{cases}
\]

(3.26)

Now, to simplify equation (3.26), we note $V^*(s, x, t)$ is homogeneous of degree 1 in $x$ and $s$. This means that rescaling the initial stock price and initial account value by the same value will simply rescale the option value by that amount also. This implies the existence of a function $U$ such that $V^*(s, x, t) = sU(\frac{x}{s}, t)$. By introducing the variable $z = \frac{x}{s}$, we may reduce (3.26) to the problem

\[
\begin{cases}
-U_t = \frac{1}{2} \sigma^2 \max_{\theta} (z - \theta)^2 U_{zz} \\
U(z, T) = (z)^+
\end{cases}
\]

(3.27)

which is in fact the one factor HJB for this problem. The details of this transformation are contained in section C.1 of appendix C.

Now since the payoff for this option is convex, the term $U_{zz} > 0$. This implies that the term to be optimised is convex in $\theta$ and thus its optimal value is at the boundary of its allowable range, that is the optimal strategy is to be either long or short to the maximum extent allowed, i.e. a “bang-bang” strategy. Since $U_{zz} > 0$, it is also clear that the coefficient is maximised when

\[
\theta = \theta^* = -\text{sgn}(z)
\]

where we define $\text{sgn}(0) = 1$. Substituting the optimal strategy $\theta^*$ into (3.27) yields the PDE

\[
\begin{cases}
-U_t = \frac{1}{2} \sigma^2 (|z| + 1)^2 U_{zz} \\
U(z, T) = (z)^+
\end{cases}
\]

(3.28)

Here we follow Hyer et al. (1997) and transform (3.28) into an equation which is as closely related to the heat equation as possible, then use a Green’s function to solve the resulting equation. Consider the
transformations

\[ \tau = \frac{\sigma^2}{2}(T - t) \]
\[ y = \text{sgn}(z) \log(|z| + 1) \]
\[ f(y, \tau) = e^{\frac{-|y|}{\tau}}U(y, \tau). \]

With these substitutions, the problem of (3.28) transforms into

\[
\begin{aligned}
    f_{\tau} - f_{yy} &= \delta(y)f(y, \tau) \\
    f(y, 0) &= \left(e^{\frac{y}{2}} - e^{-\frac{y}{2}}\right)^+. 
\end{aligned}
\]

(3.29)

This is not a simple calculation and as such we provide details in section C.2 of appendix C. At this stage we believe it important to point out that these transformations are essentially the heart of the solution to this problem. As will be shown later, all other methods we use to solve the problem of valuing this class of options rely in some manner upon these cleverly chosen transformations outlined above.

3.4.2.1. Derivation of the Green’s Function. To solve this problem, we must first find the corresponding Green’s function. First, let us define

\[ \hat{g}(y, \tau) = \frac{1}{\sqrt{4\pi \tau}}e^{-\frac{|y|^2}{4\tau}} \]

which is the Green’s function for the classical heat equation. Denoting the Green’s function for (3.29) by \( G(y, y_0, \tau) \), this function will be governed by the problem

\[
\begin{aligned}
    G_{\tau} - G_{yy} &= \delta(y)g(y_0, \tau) \\
    G(y, y_0, 0) &= \delta(y - y_0) 
\end{aligned}
\]

where \( g(y_0, \tau) = G(0, y_0, \tau) \). By Theorem 2.11, we may represent \( G(y, y_0, \tau) \) by

\[
G(y, y_0, \tau) = \delta(y - y_0) * \hat{g}(y, \tau) + \int_0^\tau \delta(y)g(y_0, \tau') * \hat{g}(y, \tau - \tau')d\tau' \\
= \hat{g}(y - y_0, \tau) + \int_0^\tau g(y_0, \tau')\hat{g}(y, \tau - \tau')d\tau'.
\]

(3.30)
Now, to find a solution to \( g(y_0, \tau) \), we set \( y = 0 \) and obtain from (3.30)

\[
g(y_0, \tau) = \hat{g}(y_0, \tau) + \int_{0}^{\tau} g(y_0, \tau') \hat{g}(0, \tau - \tau') d\tau'
\]

\[
= \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{y_0^2}{4\tau}} + \int_{0}^{\tau} \frac{g(y_0, \tau')}{\sqrt{4\pi(\tau - \tau')}} d\tau' \tag{3.31}
\]

and thus we have an integral equation for \( g(y_0, \tau) \). The solution of this equation is given in Theorem 3.2.

**Theorem 3.2.** The integral equation (3.31) has solution given by

\[
g(y_0, \tau) = \hat{g}(y_0, \tau) + \frac{1}{2} e^{-\frac{1}{2}y_0 + \frac{1}{4}\tau} \mathcal{N} \left( \sqrt{\frac{\tau}{2}} - \frac{y_0}{\sqrt{2\tau}} \right) \tag{3.32}
\]

**Proof.** To solve this equation, we first simplify by taking the Laplace transform in \( \tau \). Using results from Abramowitz and Stegun (1965) and our list of Laplace Transform properties, this gives

\[
\bar{g}(y_0, s) = \frac{1}{2\sqrt{s}} e^{-|y_0|\sqrt{s}} + \frac{\bar{g}(y_0, s)}{2\sqrt{s}}
\]

and solving for \( \bar{g}(y_0, s) \), we obtain

\[
\bar{g}(y_0, s) = \frac{1}{2} \left( e^{-|y_0|\sqrt{s}} \right)
\]

Now, we take the inverse Laplace transform, again using a known result in Abramowitz and Stegun (1965)\(^6\) to obtain

\[
\mathcal{L}^{-1}\{\bar{g}(y_0, s)\} = \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{y_0^2}{4\tau}} + \frac{1}{4} e^{-\frac{1}{2}y_0 + \frac{1}{4}\tau} \text{erfc} \left( -\frac{1}{2} \sqrt{\tau} + \frac{y_0}{2\sqrt{\tau}} \right) \tag{3.33}
\]

where \( \text{erfc} \) is the complimentary error function and is related to the normal cumulative density as follows:

\[
\text{erfc}(z) = 1 - \text{erf}(z)
\]

\[
= 1 - [2\mathcal{N}(\sqrt{2}z) - 1]
\]

\[
= 2[1 - \mathcal{N}(\sqrt{2}z)]
\]

\[
= 2\mathcal{N}(-\sqrt{2}z).
\]

\(^6\)This inverse Laplace transform is presented in appendix B
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We can now replace the term containing erfc in equation (3.33) to obtain (3.32).

Now, since we have an expression for $g(y_0, \tau)$ from theorem 3.2, and equation (3.30) provides an expression for $G(y, y_0, \tau)$ in terms $g(y_0, \tau)$ and $\hat{g}(y, \tau)$, we can substitute to obtain the following expression

$$G(y, y_0, \tau) = \hat{g}(y - y_0, \tau) + \int_{0}^{\tau} \left[ \frac{\hat{g}(y_0, \tau')} {\sqrt{\frac{4}{\pi}}} e^{-\frac{1}{2}y_0 + \frac{1}{4}\tau'} N\left(\sqrt{\frac{\tau'}{2}} - \frac{y_0}{\sqrt{2\tau'}}\right) \right] \hat{g}(y, \tau - \tau') d\tau'.$$

To evaluate the above equation, we first take a Laplace transform in $\tau$. The resulting expression for $\bar{G}(y, y_0, s)$ is

$$\bar{G}(y, y_0, s) = \frac{1}{2} e^{-\sqrt{s}|y-y_0|} + \frac{\left(\frac{e^{-|y_0|\sqrt{s}}}{2\sqrt{s}-1}\right)}{2\sqrt{s}} e^{-\sqrt{s}|y|}$$

$$= \frac{1}{2} e^{-\sqrt{s}|y-y_0|} + \frac{e^{-\sqrt{s}(|y|+|y_0|)}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})}.$$  

Taking the inverse Laplace transform of (3.34) through known transformations in Abramowitz and Stegun (1965)$^7$, and using the same connection between the complimentary error function and the cumulative Gaussian density used earlier, we may write the Green’s function for problem (3.29) as

$$G(y, y_0, \tau) = \hat{g}(y - y_0, \tau) + \frac{1}{2} e^{-\frac{1}{2}(|y|+|y_0|)+\frac{1}{4}\tau} N\left(\sqrt{\frac{\tau}{2}} - \frac{|y|+|y_0|}{\sqrt{2\tau}}\right).$$  

(3.35)

3.4.2. Derivation of the Option Pricing Formula. Now that we have derived the Green’s function for problem (3.29), we may write the solution to the problem in terms of an integration of the product of $G(y, y_0, \tau)$ and $f(y, 0)$. Thus we have

$$f(y, \tau) = \int_{-\infty}^{\infty} G(y, y_0, \tau) f(y_0, 0) dy_0$$

$$= \int_{-\infty}^{\infty} G(y, y_0, \tau) \left( e^{\frac{y_0}{2}} - e^{-\frac{y_0}{2}} \right)^+ dy_0$$

$^7$Again the reader is directed to appendix B for these inverse Laplace transforms.
and since \( e^{\frac{y_0}{2}} - e^{-\frac{y_0}{2}} = 2 \sinh(\frac{y_0}{2}) \), it is known that \( e^{\frac{y_0}{2}} - e^{-\frac{y_0}{2}} > 0 \) when \( y_0 > 0 \), and as a result we can simplify the solution by changing the integration limits and removing the plus function. This gives

\[
f(y, \tau) = \int_{0}^{\infty} G(y, y_0, \tau) \left(e^{\frac{y_0}{2}} - e^{-\frac{y_0}{2}}\right) dy_0 \tag{3.36}
\]

which is the result obtained by Hyer et al. (1997). To solve this integral, we use the fact that the integral is taken over the domain \( y_0 \in \mathbb{R}^+ \) which means that \(|y_0| = y_0\), and thus we can expand (3.36) in the following way,

\[
f(y, \tau) = I_1 - I_2 + I_3 - I_4 \tag{3.37}
\]

where

\[
I_1 = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{y_0}{2}} \phi \left(\frac{y - y_0}{\sqrt{2\tau}}\right) dy_0
\]

\[
I_2 = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{y_0}{2}} \phi \left(\frac{y - y_0}{\sqrt{2\tau}}\right) dy_0
\]

\[
I_3 = \frac{1}{2} e^{-\frac{1}{2}|y| + \frac{1}{4} \tau} \int_{0}^{\infty} \mathcal{N} \left(\frac{\tau - |y| - y_0}{\sqrt{\tau}}\right) dy_0
\]

\[
I_4 = \frac{1}{2} e^{-\frac{1}{2}|y| + \frac{1}{4} \tau} \int_{0}^{\infty} e^{-y_0} \mathcal{N} \left(\frac{\tau - |y| - y_0}{\sqrt{\tau}}\right) dy_0.
\]

To solve these integrals we will use theorem 3.3 presented below.

**Theorem 3.3.** Consider the integrals,

\[
\hat{I}_1 = \int_{k}^{\infty} e^{-cx} \mathcal{N}(a - bx) dx \tag{3.38}
\]

\[
\hat{I}_2 = \int_{k}^{\infty} e^{-cx} \phi(a - bx) dx. \tag{3.39}
\]

Then

\[
\hat{I}_1 = \frac{1}{c} e^{-ck} \mathcal{N}(a - bk) - \frac{1}{c} e^{-\frac{ca}{b} + \frac{c^2}{2b^2}} \mathcal{N}(a - bk - \frac{c}{b}) \tag{3.40}
\]

\[
\hat{I}_2 = \frac{1}{b} e^{-\frac{ca}{b} + \frac{c^2}{2b^2}} \mathcal{N}(a - bk - \frac{c}{b}). \tag{3.41}
\]

**Proof.** To prove the above theorem, let us initially consider the integral \( \hat{I}_1 \). First, we will write this integral in a form which makes the
application of integration by parts straightforward. Notice that
\[ \hat{I}_1 = \int_k^\infty N(a - bx) \frac{d}{dx} (-\frac{1}{c} e^{-cx}) \, dx \]
and applying integration by parts,
\[ \hat{I}_1 = \frac{1}{c} e^{-ck} N(a - bk) - \frac{b}{c} \int_k^\infty e^{-cx} \phi(a - bx) \, dx. \quad (3.42) \]
To complete the integration we need an expression for the second term in the above equation. This is our integral \( \hat{I}_2 \). Making the substitution \( u = a - bx \), we have that
\[
\hat{I}_2 = \frac{1}{b} \int_{-\infty}^{a-bk} e^{-\frac{c}{b} (a-u)} \phi(u) \, du \\
= \frac{1}{b} e^{-\frac{ca}{b}} E \left[ e^{\frac{c}{b} z} (z < a - bk) \right] \\
= \frac{1}{b} e^{-\frac{ca}{b} + \frac{1}{2} c^2 b^2} E \left[ \mathbb{I}(z + \frac{c}{b} < a - bk) \right] \\
= \frac{1}{b} e^{-\frac{ca}{b} + \frac{1}{2} c^2 b^2} N(a - bk - \frac{c}{b})
\]
where the third line is obtained via the Gaussian shift theorem. This gives us our final expression for \( \hat{I}_2 \) and when this is substituted into (3.42), we obtain our expression for \( \hat{I}_1 \). \( \square \)

Thus, we may rewrite the terms \( I_i \) where \( i = 1, 2, 3, 4 \) as
\[
\begin{align*}
I_1 &= \frac{1}{\sqrt{2\pi}} \hat{I}_2(a, b, c, k); \quad \left\{ a = \frac{y}{\sqrt{2\pi}}, b = \frac{1}{\sqrt{2\pi}}, c = -\frac{1}{2}, k = 0 \right\} \\
I_2 &= \frac{1}{\sqrt{2\pi}} \hat{I}_2(a, b, c, k); \quad \left\{ a = \frac{y}{\sqrt{2\pi}}, b = \frac{1}{\sqrt{2\pi}}, c = \frac{1}{2}, k = 0 \right\} \\
I_3 &= \frac{1}{2} e^{-\frac{1}{2} |y| + \frac{1}{4} r} \hat{I}_1(a, b, c, k); \quad \left\{ a = \frac{r-|y|}{\sqrt{2\pi}}, b = \frac{1}{\sqrt{2\pi}}, c = 0, k = 0 \right\} \\
I_4 &= \frac{1}{2} e^{-\frac{1}{2} |y| + \frac{1}{4} r} \hat{I}_1(a, b, c, k); \quad \left\{ a = \frac{r-|y|}{\sqrt{2\pi}}, b = \frac{1}{\sqrt{2\pi}}, c = 1, k = 0 \right\}.
\end{align*}
\]
We note that for \( I_3 \) we require an expression for \( \lim_{c \to 0} \hat{I}_1 \). Using L'Hôpital's rule, this is given by
\[
\lim_{c \to 0} \hat{I}_1 = \left( \frac{a}{b} - k \right) N(a - bk) + \frac{1}{b} \phi(a - bk).
\]
Performing the required substitutions, we obtain the following results for the integrals,

\[ I_1 = e^{\frac{1}{2}y + \frac{1}{4}\tau} \mathcal{N}\left(\frac{y + \tau}{\sqrt{2\tau}}\right) \]

\[ I_2 = e^{\frac{1}{2}y - \frac{1}{4}\tau} \mathcal{N}\left(\frac{y - \tau}{\sqrt{2\tau}}\right) \]

\[ I_3 = \frac{1}{2} e^{-\frac{1}{2}|y| + \frac{1}{4}\tau} \left[ (\tau - |y|) \mathcal{N}\left(\frac{\tau - |y|}{\sqrt{2\tau}}\right) + \sqrt{2\tau} \phi\left(\frac{\tau - |y|}{\sqrt{2\tau}}\right) \right] \]

\[ I_4 = \frac{1}{2} e^{-\frac{1}{2}|y| + \frac{1}{4}\tau} \left[ \mathcal{N}\left(\frac{\tau - |y|}{\sqrt{2\tau}}\right) - e^{|y|} \mathcal{N}\left(\frac{-\tau - |y|}{\sqrt{2\tau}}\right) \right]. \]

Recovery of the function \( U(z,t) \) is obtained via

\[ U(z,t) = e^{\frac{1}{8}\sigma^2 (T-t)} \sqrt{1 + |z|} \times f \left( \text{sgn}(z) \log(1 + |z|), \frac{1}{2}\sigma^2 (T-t) \right) \]

where we have made use of the relation \( e^{a \log(b)} = b^a \). Due to the presence of the absolute value function applied to the variable \( z \), we will have two separate cases to examine, these being \( z > 0 \) (case 1) and \( z < 0 \) (case 2).\(^8\)

Case 1: \( z > 0 \). This implies that \( |z| = z \) and thus \( y = \log(1+z) > 0 \) and thus \( |y| = y \). Let us now define the variables

\[ d_\pm = \frac{1}{\sigma \sqrt{2\tau}} \log(1+z) \pm \frac{1}{2} \sigma \sqrt{T-t} \]

which will be used to simplify the expression for the solution to \( U(z,t) \). Making the appropriate substitutions, we have that \( U(z,t) \) may be written as

\[ U(z,t) = \left\{ \begin{array}{l} (1+z) \mathcal{N}(d_+) - \mathcal{N}(d_-) - \frac{1}{2} \sigma \sqrt{T-t} d_- \mathcal{N}(-d_-) \\ + \frac{1}{2} \sigma \sqrt{T-t} \phi(-d_-) - \frac{1}{2} \mathcal{N}(-d_-) + \frac{1}{2} (1+z) \mathcal{N}(-d_+) \end{array} \right\} \]

which may be simplified using the relation \( \mathcal{N}(-x) = 1 - \mathcal{N}(x) \) to give

\[ U(z,t) = \frac{1}{2} \left\{ \begin{array}{l} z + (1+z) \mathcal{N}(d_+) - \mathcal{N}(d_-) \\ - \sigma \sqrt{T-t} d_- \mathcal{N}(-d_-) + \sigma \sqrt{T-t} \phi(-d_-) \end{array} \right\} \]

and the final solution \( V^*(s,x,t) \) is given by

\[ V^*(s,x,t) = s U(z,t) \]

\(^8\)We point out that the case \( z = 0 \) can be placed in either case 1 or case 2.
This formula for the passport option price is identical to that derived by Shreve and Vecer (2000). We now move to the second case.

Case 2: \( z < 0 \). In this case we have that \( y = -\log(1 - z) \) and \(|y| = \log(1 - z)\). We also define the variables

\[
\epsilon_\pm = \frac{1}{\sqrt{T-t}} \log(1 - z) \pm \frac{1}{2} \sigma \sqrt{T-t}.
\]

Now, solving for \( U(z,t) \) via equation (3.37) and the integrals \( I_1 \) through \( I_4 \), the function \( U(z,t) \) can be written as

\[
U(z,t) = \frac{1}{2} \left\{ N(-\epsilon_\pm) - (1 - z)N(-\epsilon_+) - \frac{1}{2} \sigma \sqrt{T-t} \epsilon_\pm N(-\epsilon_+) + \frac{1}{2} \sigma \sqrt{T-t} \phi(\epsilon_\pm) \right\}
\]

which may be simplified similarly to case 1, and using the property \( \phi(x) = \phi(-x) \),

\[
U(z,t) = \frac{1}{2} \left\{ z - N(-\epsilon_-) + (1 - z)N(\epsilon_+) - \sigma \sqrt{T-t} \epsilon_- N(-\epsilon_-) \right\}.
\]

Thus the final solution for the value of a passport option in this case is given by

\[
V^*(s,x,t) = sU(z,t)
= \frac{1}{2} \left\{ x - sN(\epsilon_-) + (s - x)N(\epsilon_+) - s\sigma \sqrt{T-t} \epsilon_- N(-\epsilon_-) + s\sigma \sqrt{T-t} \phi(\epsilon_-) \right\}. \tag{3.44}
\]

We point out that this solution is not identical to that presented in Shreve and Vecer (2000). The difference lies with the final term, namely the argument of the function \( \phi \). In our case the argument is \( \epsilon_- \), whilst Shreve and Vecer have the argument as \( \epsilon_+ \). It is unknown if this is simply a typographical error or a larger mistake, however since the remainder of the formula is identical to ours we would suggest that their error is most likely typographical.

Having derived the value of a passport option under the conditions of the symmetric case, we now attempt to achieve the same outcome for the more general option on a traded account. To recap, the extension
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here is in regards to the trading position interval. Whilst in the previous case the position limit was restricted to the interval $\theta \in [-1, 1]$, that is a maximum position of long one unit and minimum position of short one unit was allowed. In the following case we allow the trading positions to lie within an arbitrary interval, that is $\theta \in [\alpha, \beta]$ where $\alpha < \beta$.

3.4.3. The Value of an Option on a Traded Account: A Probabilistic Approach. In this section, we will value an option on a traded account utilising probabilistic techniques. This will differ from the previous section in two distinct ways. First, in this case we examine a more general class of derivative where the trading strategy $\theta$ is restricted to some arbitrary interval $[\alpha, \beta]$, and secondly we will use probabilistic techniques to solve for the expectation which corresponds to the option value.

Recapping the problem outline for the symmetric case from section 3.2, we assume a g.B.m. for the evolution of the stock price $S_t$ and this implies the continuous time dynamics for our account $X_t$. The precise form of the dynamics for these two assets is given by

$$dS_t = S_t (rdt + \sigma dW_t)$$
$$dX_t = rX_t dt + \theta_t \sigma S_t dW_t.$$  

Now, as has been discussed earlier, it is known that the value of an option on the traded account $X$ is given by

$$V^*(s, x, t) = \max_{\theta \in [\alpha, \beta]} \mathbb{E}_P\{ (X_T)^+ | \mathcal{F}_t \}$$  

(3.45)

where $\beta > \alpha$, $t \in [0, T]$, $S_t = s$, $X_t = x$ and $\mathcal{F}_t$ is a filtration. As stated, solving equation (3.45) is difficult due to it being a function of two state variables. To simplify the problem, we introduce a transformation which will reduce the dimensionality of the problem. Following Shreve and Vecer (2000), let us define

$$Z_t = \frac{X_t}{S_t} - \frac{1}{2}(\alpha + \beta).$$  

(3.46)

This transformation corresponds to a change in numeraire from the risk free asset to the risky asset $S_t$. In other words, rather than expressing
the value of the trading account in terms of dollars invested in a risk free bank account, we are now expressing the value of the trading account in terms of the stock price, to within the constant $\frac{1}{2}(\alpha + \beta)$. Now, since the new state variable $Z_t$ is given by the quotient of $X_t$ and $S_t$ plus a constant, the dynamics of $Z_t$ will be given by the quotient of the dynamics of $X_t$ and $S_t$. Expressed mathematically, this implies that $dZ_t = d\left(\frac{X_t}{S_t}\right)$. Using corollary 2.4, Ito’s quotient rule, we have that the dynamics of $Z_t$ are given by

$$dZ_t = \sigma^2 \left[Z_t + \frac{1}{2}(\alpha + \beta) - \theta_t\right] dt - \sigma \left[Z_t + \frac{1}{2}(\alpha + \beta) - \theta_t\right] dW_t.$$  \hfill (3.47)

Now, clearly (3.47) is not a martingale under the measure $\mathbb{P}$. Thus we introduce the new measure

$$\tilde{\mathbb{P}}(\Omega) = \int_{\Omega} e^{-\frac{1}{2}\sigma^2(T-t) + \sigma W(T-t)} d\mathbb{P},$$

or, expressed as a Radon-Nikodym derivative,

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{1}{2}\sigma^2(T-t) + \sigma W(T-t)}.$$

Then, according to Girsanov’s theorem, $d\tilde{W}_t = dW_t - \sigma dt$ is a Brownian motion under the measure $\tilde{\mathbb{P}}$ and under this measure, $Z_t$ is a martingale\(^9\) given by

$$dZ_t = -\sigma (Z_t + \frac{1}{2}(\alpha + \beta) - \theta_t) d\tilde{W}_t.$$  \hfill (3.48)

At this stage, it is important to note that

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{1}{2}\sigma^2(T-t) + \sigma W(T-t)} = e^{-\kappa(T-t) \frac{S_T}{S_t}},$$

and thus we may write

$$V(s, x, t) = e^{-\kappa(T-t)} \mathbb{E}\{ (X_T)^+ | \mathcal{F}_t \} = e^{-\kappa(T-t)} \tilde{\mathbb{E}}\left\{ (X_T)^+ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} | \mathcal{F}_t \right\} = s \tilde{\mathbb{E}}\{ \left( \frac{X_T}{S_T} \right)^+ | \mathcal{F}_t \} = s \tilde{\mathbb{E}}\{ (Z_T + \frac{1}{2}(\alpha + \beta))^+ | \mathcal{F}_t \}.$$  

\(^9\)Technically, $Z_t$ is actually only a local martingale.
Thus the problem given by (3.45) is reduced to that of finding \( \theta \) which satisfies

\[
u(Z_t, t) = \max_{\theta \in [\alpha, \beta]} \mathbb{E}\{(Z_T + \frac{1}{2}(\alpha + \beta))^+ | \mathcal{F}_t\} \tag{3.49}\]

and then computing the option value as

\[V^*(s, x, t) = su(Z_t, t). \tag{3.50}\]

3.4.3.1. Determination of the optimal strategy. First we tackle the problem of finding the optimal strategy, and later we will compute expectations. It was shown by Shreve and Vecer (2000) that the optimal strategy may be obtained by using the mean comparison theorem of Hajek (1985). What is implied by this mean comparison theorem is that (3.49) is in fact maximised if the absolute value of the volatility term in (3.48) is maximised. Formally, we wish to find \( \theta_t \) which satisfies

\[
\max_{\theta \in [\alpha, \beta]} |Z_t + \frac{\alpha + \beta}{2} - \theta_t|.
\]

It is easy to verify that the optimal strategy \( \theta^*_t \) is given by

\[
\theta^*_t = \alpha I(Z_t \geq 0) + \beta I(Z_t < 0). \tag{3.51}\]

The strategy defined by equation (3.51) is not unexpected given the optimal strategy derived earlier for passport options. It is consistent with convex payoff functions in that it only prescribes boundary (“bang-bang”) positions and is similar to the long when behind, short when ahead strategy, though this is altered slightly to now be \( \beta \) when the account value is below the trading interval midpoint, and \( \alpha \) when the account is above that midpoint.

We now wish to apply this optimal strategy to equation (3.48) to obtain the dynamics of \( Z_t \) under the optimal trading strategy. Consider the term

\[
\frac{1}{2}(\alpha + \beta) - \theta^*_t = \begin{cases} 
\frac{1}{2}(\alpha + \beta) - \alpha & \text{if } Z_t \geq 0 \\
\frac{1}{2}(\alpha + \beta) - \beta & \text{if } Z_t < 0 
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2}(\beta - \alpha) & \text{if } Z_t \geq 0 \\
\frac{1}{2}(\alpha - \beta) & \text{if } Z_t < 0 
\end{cases}
\]
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\[ = \frac{1}{2}(\beta - \alpha)\text{sgn}(Z_t). \]

This implies that the dynamics of \( Z_t \) under the optimal trading strategy are given by

\[ dZ_t = -\sigma[Z_t + \frac{1}{2}(\beta - \alpha)\text{sgn}(Z_t)]d\tilde{W}. \] (3.52)

Since we have used the optimal strategy, we may remove the max function from (3.49) and simply evaluate

\[ u(Z_t, t) = \tilde{E}\left\{ (Z_T + \frac{\alpha + \beta}{2})^+ |\mathcal{F}_t \right\} \] (3.53)

where the evolution of \( Z_t \) is given by (3.52).

Computing the expectation in (3.53) in its present form is not a simple task. To make the computation tractable, we will instead introduce a new variable \( Y_t = f(Z_t) \) and compute its transition density \( \tilde{P}(Y_{t+\tau} \in x|Y_t = y) \), or more compactly \( \tilde{P}(x, y, \tau) \), and use this transition density to compute the expectation and hence the option value. We begin by defining the function \( Y_t \) and derive its dynamics. This is carried out in the following theorem.

**Theorem 3.4.** Define the function

\[ Y_t = f(Z_t) = \frac{1}{\sigma}\text{sgn}(Z_t) \log\left(\frac{1}{\gamma}|Z_t| + 1\right) \] (3.54)

where \( \gamma = \frac{\beta - \alpha}{2} \). Then this random process has dynamics given by

\[ dY_t = -\text{sgn}(Y_t)(\frac{1}{2}\sigma dt + d\tilde{W}_t). \] (3.55)

**Proof.** To apply Ito’s lemma, we must first compute the first and second derivatives of \( f(Z_t) \). We note that \( \frac{d}{dZ_t}|Z_t| = \text{sgn}(Z_t) \), \( \frac{d}{dZ_t}\text{sgn}(Z_t) = 2\delta(Z_t) \) and \( \text{sgn}(Z_t)^2 = 1 \). Using these relations, we have that\(^{10}\)

\[ f'(Z_t) = \log\left(\frac{1}{\gamma}|Z_t| + 1\right) \cdot \frac{1}{\sigma}2\delta(Z_t) + \frac{1}{\sigma}\text{sgn}(Z_t) \cdot \left[ \frac{1}{\gamma}|Z_t| + 1 \right] \]

\[ = \frac{1}{\sigma} \frac{1}{|Z_t| + \gamma} \]

\(^{10}\)Technically we require \( f \) to be \( C_2 \). This is not the case in this instance, however in the extension to “generalised functions” this doesn’t affect our result.
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\[ f''(Z_t) = \frac{-\frac{1}{\sigma} \text{sgn}(Z_t) + (|Z_t| + \gamma)(0)}{(|Z_t| + \gamma)^2} \]
\[ = \frac{\frac{1}{\sigma} \text{sgn}(Z_t)}{(|Z_t| + \gamma)^2}. \]

A simple application of Ito’s lemma coupled with the relation \( Z_t = \text{sgn}(Z_t)|Z_t| \) yields
\[ dY_t = \frac{1}{2}[-\sigma \text{sgn}(Z_t)(|Z_t| + \gamma)]^2 \left[ \frac{-\frac{1}{\sigma} \text{sgn}(Z_t)}{(|Z_t| + \gamma)^2} \right] dt \]
\[ - \sigma \text{sgn}(Z_t)(|Z_t| + \gamma) \left[ \frac{1}{\sigma} \right] d\bar{W}_t \]
\[ = -\text{sgn}(Z_t)(\frac{1}{2}\sigma dt + d\bar{W}_t) \]

and since the sign of \( Y_t \) is determined by the term \( \left( \frac{1}{\sigma} \text{sgn}(Z_t) \right) \), we have that \( \text{sgn}(Z_t) = \text{sgn}(Y_t) \). Making this substitution in the above equation yields (3.55).

\[ \square \]

3.4.3.2. Derivation of the transition density. We now wish to derive the transition density \( \tilde{P}(x, y, \tau) \) for the process (3.55). This is rather involved as it requires solving the forward Kolmogorov equation. The result is detailed in the following theorem.

**Theorem 3.5.** The transition density \( \tilde{P}(x, y, \tau) \) for the process (3.55) where \( y \geq 0 \) is given by

\[
\tilde{P}(x, y, \tau) = \begin{cases} 
\frac{1}{\sqrt{\tau}} \phi \left( \frac{y - x - \frac{\sigma}{2} \tau}{\sqrt{\tau}} \right) + \frac{\sigma}{2} e^{-\sigma x} \mathcal{N} \left( \frac{\frac{\sigma}{2} \tau - (x + y)}{\sqrt{\tau}} \right) & (x \geq 0) \\
\frac{1}{\sqrt{\tau}} e^{\sigma y} \phi \left( \frac{x - y - \frac{\sigma}{2} \tau}{\sqrt{\tau}} \right) + \frac{\sigma}{2} e^{\sigma x} \left[ \mathcal{N} \left( \frac{\frac{\sigma}{2} \tau - y + x}{\sqrt{\tau}} \right) \right] & (x < 0)
\end{cases}
\]

(3.56)

**Proof.** In order to compute this transition density, we must solve the forward Kolmogorov equation. To simplify notation, let us define \( \kappa = \frac{1}{2} \sigma \). The forward Kolmogorov equation applied to the SDE (3.55)
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yields
\[
\frac{\partial \tilde{P}}{\partial \tau} = -\frac{\partial}{\partial x}(-\kappa \text{sgn}(x)\tilde{P}) + \frac{1}{2} \frac{\partial^2}{\partial x^2}((-\text{sgn}(x))^2\tilde{P})
\]
\[
= \kappa \left[ \frac{\partial}{\partial x} \text{sgn}(x)\tilde{P} \right] + \frac{1}{2} \frac{\partial^2 \tilde{P}}{\partial x^2}
\] (3.57)
and the initial condition for this PDE is
\[
\tilde{P}(x, y, 0) = \delta(x - y) \quad (y \geq 0).
\] (3.58)

Depending on the sign of \(x\), there are two possible resulting equations for the transition density. These are given by
\[
\frac{\partial \tilde{P}}{\partial \tau} = \begin{cases} 
\kappa \frac{\partial \tilde{P}}{\partial x} + \frac{1}{2} \frac{\partial^2 \tilde{P}}{\partial x^2} & (x \geq 0) \\
-\kappa \frac{\partial \tilde{P}}{\partial x} + \frac{1}{2} \frac{\partial^2 \tilde{P}}{\partial x^2} & (x < 0)
\end{cases}
\] (3.59)

Now in order to solve (3.59), we take a Laplace transform in \(\tau\). This will give us a pair of ODE’s, the solution to which will be the Laplace transformed transition density \(\tilde{p}(x, y, s)\). Once these transformed densities are known, it will simply be a matter of inverting the transformation and applying boundary conditions at \(x = 0\) to recover the desired transition density.

Denoting the Laplace transform operator \(^{11}\) by \(\mathcal{L}\), we have that
\[
\mathcal{L}\left(\frac{\partial \tilde{P}}{\partial \tau}\right) = s\tilde{p} - \tilde{P}(x, y, 0).
\] Applying this to equation (3.59) gives
\[
s\tilde{p} - \delta(x - y) = \begin{cases} 
\kappa \tilde{p}_x + \frac{1}{2} \tilde{p}_{xx} & \text{when } x \geq 0 \\
-\kappa \tilde{p}_x + \frac{1}{2} \tilde{p}_{xx} & \text{when } x < 0
\end{cases}
\]
where the subscripts attached to \(\tilde{p}\) denote partial derivatives. Rearranging yields the two second order ODE’s
\[
\frac{1}{2} \tilde{p}_{xx} + \kappa \tilde{p}_x - s\tilde{p} = -\delta(x - y) \quad (x \geq 0) \tag{3.60}
\]
\[
\frac{1}{2} \tilde{p}_{xx} - \kappa \tilde{p}_x - s\tilde{p} = 0 \quad (x < 0).
\] (3.61)

Let us consider equation (3.61) first, as it is the simpler of the two equations. It is well known (see for example Kreyszig (1999)) that

\(^{11}\)We direct the reader to section 2.5 of chapter 2 for results regarding Laplace transforms.
(3.61) has a solution of the form $\bar{p} = e^{kx}$ where $k$ satisfies the quadratic equation

$$\frac{1}{2}k^2 - \kappa k - s = 0. \quad (3.62)$$

Defining $\lambda = \sqrt{\kappa^2 + 2s}$, we can represent the two solutions of (3.62), $k_1$ and $k_2$ as

$$k_{1,2} = \lambda \mp \kappa$$

and thus the general solution to (3.61) is given by $\bar{p} = A_1 e^{-k_1x} + A_2 e^{k_2x}$ where $A_{1,2}$ are constants which are yet to be determined. Examining this solution more closely, we see that the as $x \to -\infty$, the term containing $e^{-k_1x}$ grows without bound. Since $\bar{p}$ must be bounded, require that $A_1 = 0$ and conclude that

$$\bar{p}(x, s)_{x<0} = A_2 e^{k_2x}. \quad (3.63)$$

Now, turning our attention to (3.60), we must solve a second order non-homogeneous ODE. We do this using the method of variation of parameters which states that the general solution of (3.60) is of the form $\bar{p}(x, y, s) = \bar{p}_h(x, y, s) + \bar{p}_p(x, y, s)$ where $\bar{p}_h$ is the corresponding homogeneous solution, and $\bar{p}_p$ is a particular solution to the ODE. Working in a similar fashion to that above, we have that general solution for $\bar{p}_h$ is a sum of the two basis solutions $\bar{p}_{h1}$ and $\bar{p}_{h2}$. In this case, $\bar{p}_h = B_1 \bar{p}_{h1} + B_2 \bar{p}_{h2} = B_1 e^{k_1x} + B_2 e^{-k_2x}$. Now, removing unbounded solutions as $x \to \infty$ implies that the required homogeneous solution is

$$\bar{p}_h(x, s) = B_2 \bar{p}_{h2}(x, s) = B_2 e^{-k_2x}. \quad (3.64)$$

Now we must obtain the particular solution $\bar{p}_p$. This will require both $e^{k_1x}$ and $e^{-k_2x}$. If we are to use known results for the particular solution, then we must have the equation in a standard form. The standard form of equation (3.60) (leading coefficient of the highest derivative $= 0$) is

$$\bar{p}_{xx} + 2\kappa \bar{p}_x - 2s\bar{p} = -2\delta(x - y)$$
and the particular solution to this equation is given by (see for example Kreyszig (1999))

\[
\tilde{p}_p(x, y) = \tilde{p}_{h1}(x) \int_x^\infty \frac{-2\tilde{p}_{h2}(z)\delta(z-y)}{W(z)} \, dz + \tilde{p}_{h2} \int_{-\infty}^x \frac{-2\tilde{p}_{h1}(z)\delta(z-y)}{W(z)} \, dz
\]

where \( W(x) \) is the Wronskian, defined by

\[
W(x) = \det \begin{bmatrix} \tilde{p}_{h1} & \tilde{p}_{h2} \\ \tilde{p}'_{h1} & \tilde{p}'_{h2} \end{bmatrix}
= -2\lambda e^{(k_1-k_2)x}.
\]

Substituting in the functions \( \tilde{p}_{h1} \) and \( \tilde{p}_{h2} \) yields

\[
\tilde{p}_p(x, y, s) = e^{k_1x} \int_x^\infty \frac{-2e^{-k_2z}\delta(z-y)}{-2\lambda e^{(k_1-k_2)z}} \, dz + e^{-k_2x} \int_{-\infty}^x \frac{-2e^{k_1z}\delta(z-y)}{-2\lambda e^{(k_1-k_2)z}} \, dz
\]

\[
= \frac{1}{\lambda} e^{k_1x} \int_{-\infty}^x e^{-k_1z}\delta(z-y) \, dz + \frac{1}{\lambda} e^{-k_2x} \int_x^\infty e^{k_2z}\delta(z-y) \, dz
\]

and combining this solution with the homogeneous solution \( \tilde{p}_h \) gives the desired general solution

\[
\tilde{p}(x, y, s)_{s \geq 0} = \frac{1}{\lambda} e^{k_1(x-y)} \mathbb{I}(x < y) + \frac{1}{\lambda} e^{-k_2(x-y)} \mathbb{I}(x > y) + B_2 e^{-k_2x}.
\]

(3.65)

Now, to complete our solutions for (3.60) and (3.61), we must find expressions for the constants \( A_2 \) and \( B_2 \). To this end we require some boundary conditions on the function \( \tilde{p}(x, y, s) \). The conditions we will use are the facts that the total probability under any distribution is equal to 1, and that the distribution must be continuous at the point where the form of the function changes, at \( x = 0 \). The implications of
the first condition on the function $\bar{p}$ is that since
\[ \int_{-\infty}^{\infty} \bar{p}(x, y, \tau) dx = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \bar{p}(x, y, s) dx = \frac{1}{s}. \]
This implies that
\[ \int_{-\infty}^{0} \bar{p}_{x<0} dx + \int_{0}^{\infty} \bar{p}_{x>0} dx = \frac{1}{s}. \quad (3.66) \]
Substituting equation (3.63) and (3.65) into the above equation and changing the integration limits as dictated by the indicator functions, we have that
\[ \frac{1}{s} = \left[ \frac{1}{\lambda} \int_{y}^{\infty} e^{-k_{2}x} dx + \frac{1}{\lambda} \int_{0}^{y} e^{-k_{1}x} dx \right] \\
+ B_{2} \int_{0}^{\infty} e^{-k_{2}x} dx + A_{2} \int_{-\infty}^{0} e^{k_{2}x} dx \\
= \frac{1}{\lambda} \left( \frac{1}{k_{2}} + \frac{1}{k_{1}} \right) (1 - e^{-k_{1}y}) + \frac{B_{2}}{k_{2}} + \frac{A_{2}}{k_{2}}. \quad (3.67) \]
To simplify this expression, we note that \( \frac{1}{\lambda} \left( \frac{1}{k_{2}} + \frac{1}{k_{1}} \right) = \frac{1}{s} \) and thus (3.67) reduces to
\[ A_{2} + B_{2} = \frac{k_{2}}{\lambda k_{1}} e^{-k_{1}y}. \quad (3.68) \]
Now, the requirement that $\bar{p}$ is continuous at $x = 0$ implies that the expression for $\bar{p}_{x>0}$ and $\bar{p}_{x<0}$ are equal when $x = 0$, i.e. $\bar{p}(0^{-}, y, s) = \bar{p}(0^{+}, y, s)$. This means that
\[ \frac{1}{\lambda} e^{-k_{1}y} + B_{2} = A_{2}. \]
Taking this equation together with equation (3.68) provides us with two equations and two unknowns which may be solved simultaneously to give
\[ A_{2} = \frac{e^{-k_{1}y}}{k_{1}}, \quad B_{2} = \frac{ke^{-k_{1}y}}{\lambda k_{1}}. \]
and thus we arrive at our final solution for the Laplace transformed transition density

\[
\bar{p}(x, y, s) = \begin{cases} 
\frac{1}{\lambda} e^{-k_2 (x-y)} I(x > y) + \frac{1}{\lambda} e^{k_1 (x-y)} I(x < y) \\
+ \frac{\kappa}{\lambda k_3} e^{-k_2 x - k_1 y} \\
\frac{1}{k_1} e^{k_2 x - k_1 y}
\end{cases}
\]  
\quad (x \geq 0) 
\]  
\quad (x < 0)

(3.69)

Now, to compute the actual transition density, we must take the inverse Laplace transform of (3.69). Rather than computing this inverse Laplace transform directly, we will present some known Laplace transforms which will be used to invert (3.69). To keep notation as simple as possible, we present the required transforms as a series of lemmas. We note that these transforms may be found in Abramowitz and Stegun (1965), though we have adjusted them slightly to better suit the exposition.

**Lemma 3.6.**

\[
\mathcal{L} \left\{ \frac{1}{\sqrt{T}} \phi \left( \frac{y - \kappa T}{\sqrt{T}} \right) \right\} = \frac{1}{\lambda} e^{-k_1 y} I(y > 0) + \frac{1}{\lambda} e^{k_2 y} I(y < 0) \quad (3.70)
\]

**Proof.** We have from Abramowitz and Stegun (1965) the standard Laplace transform

\[
\mathcal{L} \left\{ \frac{1}{\sqrt{\pi T}} e^{-\frac{y^2}{4T}} \right\} = \frac{1}{\sqrt{s}} e^{-k\sqrt{s}}, \quad k \geq 0. \quad (3.71)
\]

Multiplying through by \(\frac{1}{\sqrt{2}}\) and defining \(|y| = \frac{k}{\sqrt{2}}\), we may rewrite (3.71) as

\[
\mathcal{L} \left\{ \frac{1}{\sqrt{T}} \phi \left( \frac{y}{\sqrt{T}} \right) \right\} = \frac{1}{\sqrt{2s}} e^{-|y|\sqrt{s}}.
\]

We also have that

\[
\phi \left( \frac{y - \kappa T}{\sqrt{T}} \right) = e^{(-\frac{1}{2}k^2 T + \kappa y)} \phi \left( \frac{y}{\sqrt{T}} \right)
\]

and thus

\[
\mathcal{L} \left\{ \frac{1}{\sqrt{T}} \phi \left( \frac{y - \kappa T}{\sqrt{T}} \right) \right\} = \mathcal{L} \left\{ e^{(-\frac{1}{2}k^2 T + \kappa y)} \phi \left( \frac{y}{\sqrt{T}} \right) \right\}
\]
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\[ e^{\kappa y} \frac{1}{\sqrt{2s + \kappa^2}} e^{-|y|\sqrt{2s + \kappa^2}} \]

\[ = \frac{1}{\lambda} e^{\kappa y - \lambda|y|} \]

\[ = \left\{ \begin{array}{ll}
\frac{1}{\lambda} e^{\kappa y - \lambda y} & (y > 0) \\
\frac{1}{\lambda} e^{\kappa y + \lambda y} & (y < 0)
\end{array} \right. \]

\[ = \frac{1}{\lambda} e^{-k_1 y} \mathbb{I}(y > 0) + \frac{1}{\lambda} e^{k_2 y} \mathbb{I}(y < 0) \]

as required. \qed

**Lemma 3.7.**

\[ \mathcal{L} \left\{ \mathcal{N} \left( \frac{\kappa \tau - \alpha}{\sqrt{\tau}} \right) \right\} = \frac{1}{\lambda k_1} e^{-k_1 \alpha} \mathbb{I}(\alpha > 0) + \left[ \frac{1}{s} - \frac{1}{\lambda k_2} e^{k_2 \alpha} \right] \mathbb{I}(\alpha < 0) \]

(3.72)

**Proof.** Consider that

\[ \mathcal{L} \left\{ \mathcal{N} \left( \frac{\kappa \tau - \alpha}{\sqrt{\tau}} \right) \right\} = \mathcal{L} \left\{ \int_{-\infty}^{\alpha} \phi(\nu) d\nu \right\}. \]

Now, applying the substitution \( \nu = \frac{\kappa \tau - u}{\sqrt{\tau}} \), we have

\[ \mathcal{L} \left\{ \int_{-\infty}^{\alpha} \phi(\nu) d\nu \right\} = \mathcal{L} \left\{ \int_{\alpha}^{\frac{\alpha}{\sqrt{\tau}}} \frac{1}{\sqrt{\tau}} \phi \left( \frac{\kappa \tau - u}{\sqrt{\tau}} \right) du \right\} \]

\[ = \int_{\alpha}^{\infty} \mathcal{L} \left\{ \frac{1}{\sqrt{\tau}} \phi \left( \frac{\kappa \tau - u}{\sqrt{\tau}} \right) \right\} du \]

\[ = \int_{\alpha}^{\infty} \left[ \frac{1}{\lambda} e^{-k_1 u} \mathbb{I}(u > 0) + \frac{1}{\lambda} e^{k_2 u} \mathbb{I}(u < 0) \right] du. \]

We now have two cases to consider:

**Case 1: \( \alpha > 0 \)**

\[ \mathcal{L} \left\{ \mathcal{N} \left( \frac{\kappa \tau - \alpha}{\sqrt{\tau}} \right) \right\} = \int_{\alpha}^{\infty} \frac{1}{\lambda} e^{-k_1 u} du \]

\[ = \frac{1}{\lambda} \left[ \frac{1}{-k_1} e^{-k_1 u} \right]_{\alpha}^{\infty} \]

\[ = \frac{1}{\lambda k_1} e^{-k_1 \alpha} \]
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Case 2: $\alpha < 0$

$$\mathcal{L}\left\{ \mathcal{N}\left( \frac{\kappa \tau - \alpha}{\sqrt{\tau}} \right) \right\} = \int_{0}^{\infty} \frac{1}{\lambda} e^{-k_1 u} du + \int_{\alpha}^{0} \frac{1}{\lambda} e^{k_2 u} du$$

$$= \frac{1}{\lambda k_1} + \frac{1}{\lambda} \left[ \frac{1}{k_2} e^{k_2 u} \right]_{\alpha}^{0}$$

$$= \frac{1}{\lambda} \left( \frac{1}{k_1} + \frac{1}{k_2} \right) - \frac{1}{\lambda k_2} e^{k_2 \alpha}$$

$$= \frac{1}{\sqrt{\tau}} \phi \left( \frac{y - x}{\sqrt{\tau}} \right) + \kappa e^{-2\kappa x} N \left( \frac{\kappa \tau - (x + y)}{\sqrt{\tau}} \right),$$

Combining the two cases into a single equation yields the desired result.

□

Using lemmas 3.6 and 3.7, we can now compute the transition densities. Consider the case where $x \geq 0$ first. Using the fact that $k_2 = k_1 + 2\kappa$ and lemma 3.7, we have that

$$\mathcal{L}^{-1} \left\{ \frac{\kappa}{\lambda k_1} e^{-k_2 x - k_1 y} \right\} = \mathcal{L}^{-1} \left\{ \frac{\kappa}{\lambda k_1} e^{-(k_1 + 2\kappa) x - k_1 y} \right\}$$

$$= \kappa e^{-2\kappa x} \mathcal{L}^{-1} \left\{ \frac{1}{\lambda k_1} e^{-k_1 (x + y)} \right\}$$

$$= \kappa e^{-2\kappa x} \mathcal{N} \left( \frac{\kappa \tau - (x + y)}{\sqrt{\tau}} \right),$$

and from lemma 3.6,

$$\mathcal{L}^{-1} \left\{ \frac{1}{\lambda} e^{k_2 (y - x)} \mathbb{I}(y < x) + \frac{1}{\lambda} e^{-k_1 (y - x)} \mathbb{I}(y > x) \right\} = \frac{1}{\sqrt{\tau}} \phi \left( \frac{(y - x) - \kappa \tau}{\sqrt{\tau}} \right).$$

Combining these results, we have that the transition density when $x \geq 0$ is given by

$$\tilde{p}_{(x \geq 0)}(x, y, \tau) = \frac{1}{\sqrt{\tau}} \phi \left( \frac{(y - x) - \kappa \tau}{\sqrt{\tau}} \right) + \kappa e^{-2\kappa x} \mathcal{N} \left( \frac{\kappa \tau - (x + y)}{\sqrt{\tau}} \right),$$

as required.

Now, turning to the case where $x < 0$, we will employ some relations to simplify the treatment. Via a little algebra, it can be shown that
\[
\frac{1}{k_1} = \frac{1}{\lambda} [1 + \frac{\kappa}{k_1}].
\]
Substituting this into (3.69) gives
\[
\bar{p}(x<0) = \frac{1}{\lambda} \left[ 1 + \frac{\kappa}{k_1} \right] e^{k_2 x - k_1 y}
\]
\[
= \frac{1}{\lambda} e^{k_2 x - k_1 y} + \frac{\kappa}{\lambda k_1} e^{k_2 x - k_1 y}
\]
and using \(k_2 = k_1 + 2\kappa\) and \(k_1 = k_2 - 2\kappa\), this reduces further to
\[
\bar{p}(x<0) = \frac{1}{\lambda} e^{k_2 x - (k_2 - 2\kappa)y} + \frac{\kappa}{\lambda k_1} e^{(k_1 + 2\kappa)x - k_1 y}
\]
\[
= \frac{e^{2\kappa y}}{\lambda} e^{k_2 (x-y)} + \frac{\kappa e^{2\kappa x}}{\lambda k_1} e^{k_1 (x-y)},
\]
and since \(x < 0\) and \(y > 0\), it is clear that \(x - y < 0\). Thus, employing lemmas 3.6 and 3.7, the transition density can be obtained by
\[
\bar{p}_{(x<0)} = \mathcal{L}^{-1} \left\{ \frac{e^{2\kappa y}}{\lambda} e^{k_2 (x-y)} + \frac{\kappa e^{2\kappa x}}{\lambda k_1} e^{k_1 (x-y)} \right\}
\]
\[
= \frac{e^{2\kappa y}}{\lambda} \mathcal{L}^{-1} \left\{ \frac{1}{e^{k_2 (x-y)}} \right\} + \kappa e^{2\kappa x} \mathcal{L}^{-1} \left\{ \frac{1}{\lambda k_1} e^{k_1 (x-y)} \right\}
\]
\[
= \frac{e^{2\kappa y}}{\lambda} \left[ \frac{1}{\sqrt{\tau}} \phi \left( \frac{\kappa \tau - (x - y)}{\sqrt{\tau}} \right) \right] + \kappa e^{2\kappa x} \left[ \mathcal{N} \left( \frac{\kappa \tau - (y - x)}{\sqrt{\tau}} \right) \right]
\]
\[
= \frac{e^{2\kappa y}}{\lambda} \left[ \frac{1}{\sqrt{\tau}} \phi \left( \frac{x - y - \kappa \tau}{\sqrt{\tau}} \right) \right] + \kappa e^{2\kappa x} \left[ \mathcal{N} \left( \frac{\kappa \tau - y + x}{\sqrt{\tau}} \right) \right].
\]
This completes the proof of Theorem 3.5. \(\square\)

\[3.4.3.3.\text{Derivation of the Option Value.}\] Now that we have the transition density for this problem, computation of the expectation which gives the option value requires knowledge of the possible outcomes of \(Z_T\). To obtain these outcomes we will invert the function \(f(Z_T) = Y_T\). As stated previously, the function \(Y_T\), which is monotonic increasing in \(Z_T\), is defined as
\[
Y_T = \frac{1}{\sigma \text{sgn}(Z_T)} \log \left( \frac{1}{\gamma} |Z_T| + 1 \right).
\]
The inversion of this function is somewhat cumbersome, so we will proceed in a few steps. Notice that, with \( \gamma = \frac{1}{2}(\beta - \alpha) \),

\[
\begin{align*}
\log \left( \frac{1}{\gamma} |Z_T| + 1 \right) &= \sigma Y_T \quad (Z_T > 0) \\
\log \left( \frac{1}{\gamma} |Z_T| + 1 \right) &= -\sigma Y_T \quad (Z_T < 0)
\end{align*}
\]

and rearranging to make \( Z_T \) the subject yields

\[
\begin{align*}
Z_T &= \gamma \left[ e^{\sigma Y_T} - 1 \right] \quad (Z_T > 0) \\
Z_T &= -\gamma \left[ e^{-\sigma Y_T} - 1 \right] \quad (Z_T < 0)
\end{align*}
\]

which we may write more succinctly as

\[ Z_T = \gamma \left[ e^{\sigma |Y_T|} - 1 \right] \text{sgn}(Y_T). \]

Now, since the variable of interest is actually \( Z_T + \frac{1}{2}(\alpha + \beta) \), we compute

\[
Z_T + \frac{1}{2}(\alpha + \beta) = \begin{cases} 
\frac{1}{2}(\beta - \alpha)e^{\sigma Y_T} + \alpha & (Y_T > 0) \\
\beta - \frac{1}{2}(\beta - \alpha)e^{-\sigma Y_T} & (Y_T < 0) \\
0 & \text{if } Y_T < -\frac{1}{\sigma} \log \left( \frac{2\beta}{\beta - \alpha} \right)
\end{cases}
\]

and the value of these functions will depend on the relative values of the parameters \( \alpha \) and \( \beta \). Whilst we always assume that \( \beta > \alpha \), the manner in which the magnitudes of \( \alpha \) and \( \beta \) are related gives rise to two cases which we detail below.

**Case 1: Assume \( \beta > 0 \) and \( |\alpha| < \beta \).** In this case the outcomes for \( Z_T + \frac{1}{2}(\alpha + \beta) \) are given by

\[
Z_T + \frac{1}{2}(\alpha + \beta) = \begin{cases} 
\frac{1}{2}(\beta - \alpha)e^{\sigma Y_T} + \alpha & (Y_T > 0) \\
\beta - \frac{1}{2}(\beta - \alpha)e^{-\sigma Y_T} & (Y_T < 0) \\
0 & \text{if } Y_T < -\frac{1}{\sigma} \log \left( \frac{2\beta}{\beta - \alpha} \right)
\end{cases}
\]

**Case 2: Assume \( |\alpha| > |\beta| \).** The outcomes for \( Z_T \) in this case are given by

\[
Z_T + \frac{1}{2}(\alpha + \beta) = \begin{cases} 
\frac{1}{2}(\beta - \alpha)e^{\sigma Y_T} + \alpha & (Y_T > \frac{1}{\sigma} \log \left( \frac{-2\alpha}{\beta - \alpha} \right)) \\
0 & \text{if } Y_T < \frac{1}{\sigma} \log \left( \frac{-2\alpha}{\beta - \alpha} \right)
\end{cases}
\]

Now, setting \( \tau = T - t \) and using the transition density defined by theorem 3.5, we can compute expectations by a simple integration.
We note that in both case 1 and 2 there are only three possible outcomes for \( (Z_T + \frac{1}{2}(\alpha + \beta)) \), and as such it is possible to incorporate the expectation into two functions. Define the two functions \( g \) and \( h \) as

\[
g(y, w, \tau) = \int_{w}^{\infty} \left( \frac{1}{2}(\beta - \alpha)e^{\sigma x} + \alpha \right) \tilde{P}(x, y, \tau) dx \\
h(y, w, \tau) = \int_{-w}^{0} \left( \beta - \frac{1}{2}(\beta - \alpha)e^{-\sigma x} \right) \tilde{P}(x, y, \tau) dx.
\]

Then we may write the expectation in case 1 as

\[
\tilde{E}[Z_T + \frac{1}{2}(\alpha + \beta)] = g(y, 0, T - t) + h(y, w_1, T - t) \tag{3.77}
\]

where

\[
w_1 = \frac{1}{\sigma} \log \left( \frac{2\beta}{\beta - \alpha} \right)
\]

and the expectation in case 2 can be written as

\[
\tilde{E}[Z_T + \frac{1}{2}(\alpha + \beta)] = g(y, w_2, T - t) \tag{3.78}
\]

where

\[
w_2 = \frac{1}{\sigma} \log \left( \frac{-2\alpha}{\beta - \alpha} \right)
\]

Expanding the functions \( g \) and \( h \), we may write them as a weighted sum of integrals,

\[
g(y, w, \tau) = \frac{\beta - \alpha}{2} I_1 + \frac{\alpha \sigma}{4} I_2 + \alpha I_3 + \frac{\alpha \sigma}{2} I_4 \tag{3.79}
\]

\[
h(y, w, \tau) = \beta J_1 + \frac{\beta \sigma}{2} J_2 - \frac{\beta - \alpha}{2} J_3 - \frac{(\beta - \alpha)\sigma}{4} J_4. \tag{3.80}
\]

Before we explicitly write and solve the integrals \( I_i, J_i \) for \( i = \{1, 2, 3, 4\} \), we will present a theorem which will simplify the computation of the integrations.

**Theorem 3.8.** Consider the integrals

\[
\hat{I}_1(a, b, c, k) = \int_{k}^{\infty} e^{-ax} N(a - bx) dx
\]

\footnote{The form of the integrals was chosen to be identical to those used by Shreve and Vecer (2000) so that we may directly compare our results.}
\[ \hat{I}_2(a, b, c, k) = \int_k^\infty e^{-cx} \phi(a - bx) dx \]
\[ \hat{I}_3(a, b, c, k) = \int_{-k}^0 e^{-cx} \mathcal{N}(a - bx) dx \]
\[ \hat{I}_4(a, b, c, k) = \int_{-k}^0 e^{-cx} \phi(a - bx) dx \]

where \( a, \ b, \ c \) and \( k \) are constants with \( k > 0 \) and \( c \geq 0 \). Then, the value of these integrals is given by

\[ \hat{I}_1 = \frac{1}{c} e^{-ck} \mathcal{N}(a - bk) - \frac{1}{c} e^{-\frac{ca}{b} + \frac{c^2}{2b^2}} \mathcal{N}(a - bk - \frac{c}{b}) \]
\[ \hat{I}_2 = \frac{1}{b} e^{-\frac{ca}{b} + \frac{c^2}{2b^2}} \mathcal{N}(a - bk - \frac{c}{b}) \]
\[ \hat{I}_3 = \frac{1}{c} [e^{ck} \mathcal{N}(a + bk) - \mathcal{N}(a)] - \frac{1}{c} e^{-\frac{ca}{b} + \frac{c^2}{2b^2}} [\mathcal{N}(a + bk - \frac{c}{a}) - \mathcal{N}(a - \frac{c}{a})] \]
\[ \hat{I}_4 = \frac{1}{b} e^{-\frac{ca}{b} + \frac{c^2}{2b^2}} [\mathcal{N}(a + bk - \frac{c}{a}) - \mathcal{N}(a - \frac{c}{a})] \]

**Proof.** We note that the integrals \( \hat{I}_1 \) and \( \hat{I}_2 \) have already been derived in theorem 3.3, thus we begin from \( \hat{I}_3 \). In a similar manner to how we treated \( \hat{I}_1 \) in theorem 3.3, we can write \( \hat{I}_3 \) as

\[ \hat{I}_3 = \int_{-k}^0 \mathcal{N}(a - bx) \frac{d}{dx}(-\frac{1}{c} e^{-cx}) dx. \]

Integrating by parts, we obtain

\[ \hat{I}_3 = \frac{1}{c} [e^{ck} \mathcal{N}(a + bk) - \mathcal{N}(a)] - \frac{b}{c} \int_{-k}^0 e^{-cx} \phi(a - bx) dx. \quad (3.81) \]

As before, we need an expression for the integral in the second term, \( \hat{I}_4 \). Making the substitution \( u = a - bx \), we have that

\[ \hat{I}_4 = \frac{1}{b} \int_a^{a+bk} e^{-\frac{c}{b}(a-u)} \phi(u) du \]
\[ = \frac{1}{b} e^{-\frac{ca}{b}} E\left(e^{\frac{c}{b} z}(a < z < a + bk)\right) \]
\[ = \frac{1}{b} e^{-\frac{ca}{b}} + \frac{1}{2} e^{\frac{c^2}{2b^2}} E\left(\mathbb{I}(a\frac{c}{b} < z < a + bk - \frac{c}{b})\right) \]
\[ = \frac{1}{b} e^{-\frac{ca}{b}} + \frac{1}{2} e^{\frac{c^2}{2b^2}} [\mathcal{N}(a + bk - \frac{c}{b}) - \mathcal{N}(a - \frac{c}{b})]. \]
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Substituting the result for \( \hat{I}_4 \) into (3.81) gives the required expression for \( \hat{I}_3 \). This completes the proof. □

Using the results of theorem 3.8, we will now compute the integrals \( I_i \) and \( J_i \) for \( i = \{1, 2, 3, 4\} \). Consider \( I_1 \),

\[
I_1 = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \phi \left( \frac{y - x - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \, dx
\]

where \( a = \frac{y - \frac{1}{2} \sigma \tau}{\sqrt{\tau}}, \) \( b = \frac{1}{\sqrt{\tau}}, \) \( c = -\sigma \) and \( k = w \). Making these substitutions,

\[
I_1 = e^{\sigma y} N \left( \frac{y - w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right)
\]

which is identical to the result derived by Shreve and Vecer (2000). Turning to \( I_2 \),

\[
I_2 = \int_{\infty}^{\infty} e^{\sigma x} e^{-\sigma x} N \left( -\frac{y - x + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \, dx
\]

where \( a = \frac{1}{2} \sigma \tau - y, \) \( b = \frac{1}{\sqrt{\tau}}, \) \( c = 0 \) and \( k = w \). When \( c = 0 \), we require an expression for \( \lim_{c \to 0} \hat{I}_1 \). By L'Hôpital's rule,

\[
\lim_{c \to 0} \hat{I}_1 = (\frac{a}{b} - k) N(a - bk) + \frac{1}{b} \phi(a - bk)
\]

and thus

\[
I_2 = (-y - w + \frac{1}{2} \sigma \tau) N \left( \frac{-y - w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) + \sqrt{\tau} \phi \left( \frac{-y - w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right).
\]

For \( I_3 \), we have

\[
I_3 = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} \sigma^2} \phi \left( \frac{y - x + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \, dx
\]

where \( a = \frac{y + \frac{1}{2} \sigma \tau}{\sqrt{\tau}}, \) \( b = \frac{1}{\sqrt{\tau}}, \) \( c = \sigma \) and \( k = 0 \). Making these substitutions,

\[
I_3 = N \left( \frac{y - w - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right)
\]
where \( a = \frac{y - \frac{1}{2} \sigma \tau}{\sqrt{\tau}}, \ b = \frac{1}{\sqrt{\tau}}, \ c = 0 \) and \( k = w \). For \( I_4 \),

\[
I_4 = \int_k^\infty e^{-\sigma x} N\left( \frac{-y - x + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) dx
\]

\[
= \hat{I}_1(a, b, c, k)
\]

\[
= \frac{1}{\sigma} e^{-\sigma w} N\left( \frac{-y - w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) - \frac{1}{\sigma} e^{\sigma y} N\left( \frac{-y - w - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right)
\]

where \( a = \frac{\frac{1}{2} \sigma \tau - y}{\sqrt{\tau}}, \ b = \frac{1}{\sqrt{\tau}}, \ c = \sigma \) and \( k = w \). It should be noted that this result is not exactly the same as that presented by Shreve and Vecer (2000). Rather, in the second term they have \( e^{-\sigma y} \) in place of \( e^{\sigma y} \). Their error appears to be simply typographical.

We now turn to the \( J_i \) integrals. Define

\[
J_1 = \int_{-w}^0 \frac{1}{\sqrt{\pi}} e^{\sigma y} \phi \left( \frac{-y + x - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) dx
\]

\[
= e^{\sigma y} \frac{1}{\sqrt{\pi}} \hat{I}_4(a, b, c, k)
\]

\[
= e^{\sigma y} \left[ N\left( \frac{y + w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) - N\left( \frac{y + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \right]
\]

(3.82)

where \( a = \frac{\frac{1}{2} \sigma \tau - y}{\sqrt{\tau}}, \ b = \frac{1}{\sqrt{\tau}}, \ c = 0 \) and \( k = w \). For \( J_2 \), define

\[
J_2 = \int_{-w}^0 e^{\sigma x} N\left( \frac{-y + x + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) dx
\]

\[
= \hat{I}_3(a, b, c, k)
\]

\[
= \frac{1}{\sigma} \left\{ \left[ N\left( \frac{-y + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) - N\left( \frac{-y - w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \right] e^{-\sigma w} N\left( \frac{-y - w + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \right. \\
\left. + e^{\sigma y} \left[ N\left( \frac{-y - w - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) - N\left( \frac{-y - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \right] \right\}
\]

(3.83)

where \( a = \frac{\frac{1}{2} \sigma \tau - y}{\sqrt{\tau}}, \ b = -\frac{1}{\sqrt{\tau}}, \ c = -\sigma \) and \( k = w \). Again our result differs slightly from that in Shreve and Vecer (2000) in that the second term of the above result in their paper is missing the term \( \frac{1}{\sigma} \). Again
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This error is most likely typographical. For $J_3$, we have

$$J_3 = \int_{-\infty}^{\infty} e^{-\sigma x} \frac{e^{\sigma y}}{\sqrt{\pi}} \phi \left( \frac{-y + x - \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \, dx$$

$$= \frac{e^{\sigma y}}{\sqrt{\tau}} \hat{I}_4(a, b, c, k)$$

$$= \mathcal{N} \left( y + w - \frac{1}{2} \sigma \tau \right) - \mathcal{N} \left( y - \frac{1}{2} \sigma \tau \right)$$

where $a = \frac{1}{2} \sigma \tau + y$, $b = \frac{1}{\sqrt{\tau}}$, $c = \sigma$ and $k = w$. Lastly, for $J_4$,

$$J_4 = \int_{-\infty}^{\infty} e^{-\sigma x} e^{\sigma x} \mathcal{N} \left( \frac{-y + x + \frac{1}{2} \sigma \tau}{\sqrt{\tau}} \right) \, dx$$

$$= \hat{I}_3(a, b, c, k)$$

where $a = \frac{1}{2} \sigma \tau - y$, $b = -\frac{1}{\sqrt{\tau}}$, $c = 0$ and $k = w$. Since $c = 0$, we require an expression for $\lim_{c \to 0} \hat{I}_3$. Using L'Hôpital's rule, this is given by

$$\lim_{c \to 0} \hat{I}_3 = (k + \frac{2}{5}) \mathcal{N}(a + bk) - \frac{2}{5} \mathcal{N}(a) + \frac{1}{5} \phi(a + bk) - \phi(a)$$

and thus

$$J_4 = \begin{cases} 
(y + w - \frac{1}{2} \sigma \tau) \mathcal{N} \left( \frac{-y - w + 1}{\sqrt{\tau}} \right) + (-y + \frac{1}{2} \sigma \tau) \mathcal{N} \left( \frac{-y + 1}{\sqrt{\tau}} \right) \\
-\sqrt{\tau} \left[ \phi \left( \frac{-y - w + 1}{\sqrt{\tau}} \right) - \phi \left( \frac{-y + 1}{\sqrt{\tau}} \right) \right]
\end{cases}$$

which completes the evaluation of the integrals required to compute the expectation. Now what remains is to use these integrals to compute the formula for the option value. We present this as theorem 3.9.

**Theorem 3.9.** The value of an option on a traded account is given by the following functions, where $s = S_t$, $x = X_t$, $z = Z_t$ and $\tau = T - t$:

If $\beta > 0$ and $|\alpha| \leq \beta$, then when $x \geq \frac{1}{2}(\alpha + \beta)s$

$$V^s(s, x, t) = \begin{cases} 
\frac{(\alpha + \beta)^2 s}{4(\beta - \alpha)} - \frac{\alpha x}{(\beta - \alpha)} - \frac{1}{3} (\beta - \alpha) \sigma \sqrt{\tau} s \mathcal{N}(d_-) \\
+ \frac{\beta}{(\beta - \alpha)} (x - \alpha s) \mathcal{N}(d_+) \\
- \frac{1}{4} (\beta - \alpha) [1 - \sigma \sqrt{\tau} d_] s \mathcal{N}(d_-) \\
+ \frac{1}{4} (\beta - \alpha) \sigma \sqrt{\tau} s \phi(d_-)
\end{cases}$$

(3.84)
where
\[ d_{\pm} = \frac{1}{\sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \frac{x}{s} - \alpha \right) \right) + \frac{1}{\sqrt{\tau}} \log \left( \frac{2\beta}{\beta - \alpha} \right) \pm \frac{1}{2}\sigma \sqrt{\tau}. \]
and when \( x < \frac{1}{2}(\alpha + \beta)s \),
\[ V^*(s, x, t) = \begin{cases} x - (x - \beta s)N(e_-) - \beta s N(e_-) \\ -\frac{\alpha}{\beta - \alpha} (x - \alpha s)N(-e_+) \\ -\frac{1}{4}(\beta - \alpha)[1 + \sigma \sqrt{\tau} e_-]sN(-e_-) \\ +\frac{1}{4}(\beta - \alpha)\sigma \sqrt{\tau} s \phi(e_-) \end{cases} \] (3.85)
where
\[ e_{\pm} = \frac{1}{\sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \beta - \frac{x}{s} \right) \right) \pm \frac{1}{\sqrt{\tau}} \log \left( \frac{2\beta}{\beta - \alpha} \right) \pm \frac{1}{2}\sigma \sqrt{\tau}. \]
When \(|\alpha| > \beta \) and \( x \geq \frac{1}{2}(\alpha + \beta)s \), then
\[ V^*(s, x, t) = \begin{cases} (x - \alpha s)N(d_+) + \alpha sN(d_-) \\ -\frac{\alpha}{\beta - \alpha} (x - \alpha s)N(-d_+) \\ -\frac{1}{4}(\beta - \alpha)[1 + \sigma \sqrt{\tau} d_-]sN(-d_-) \\ +\frac{1}{4}(\beta - \alpha)\sigma \sqrt{\tau} s \phi(d_-) \end{cases} \] (3.86)
where
\[ d_{\pm} = \frac{1}{\sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \frac{x}{s} - \alpha \right) \right) \pm \frac{1}{\sqrt{\tau}} \log \left( \frac{-2\alpha}{\beta - \alpha} \right) \pm \frac{1}{2}\sigma \sqrt{\tau}, \]
and when \( x < \frac{1}{2}(\alpha + \beta)s \),
\[ V^*(s, x, t) = \begin{cases} \frac{(\alpha + \beta)^2 s}{4(\beta - \alpha)} - \frac{\alpha x}{\beta - \alpha} - \frac{1}{4}(\beta - \alpha)\sigma \sqrt{\tau} s e_- \\ +\frac{\alpha}{\beta - \alpha} (x - \beta s)N(e_+) \\ -\frac{1}{4}(\beta - \alpha)[1 - \sigma \sqrt{\tau} e_-]sN(e_-) \\ +\frac{1}{4}(\beta - \alpha)\sigma \sqrt{\tau} s \phi(e_-) \end{cases} \] (3.87)
where
\[ e_{\pm} = \frac{1}{\sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \beta - \frac{x}{s} \right) \right) + \frac{1}{\sqrt{\tau}} \log \left( \frac{-2\alpha}{\beta - \alpha} \right) \pm \frac{1}{2}\sigma \sqrt{\tau}. \]

We note that these results are the same as those obtained by Shreve and Vecer (2000) with the exception of (3.85). In their paper, the fifth term has the factor of \( \frac{1}{2} \) rather than the factor of \( \frac{1}{4} \) which we have. It is believed that this is simply a typographical error, and not the result
of carrying through the previously identified errors, as the remainder of the formulas are identical to ours.

**Proof.** We will split this computation into two cases. These cases were alluded to earlier in the thesis, though we complete the formula derivation here.

**Case 1:** Assume that $\beta > 0$ and $|\alpha| < \beta$. Also recall that $z = Z_t - \frac{1}{2}(\alpha + \beta) \geq 0$. This implies that $x \geq \frac{1}{2}(\alpha + \beta)s$. As we showed in equation (3.77), the value of the function $u(z, t)$ is given by

$$u(z, t) = \tilde{E}\{Z_T + \frac{1}{2}(\alpha + \beta)\} = g(y, 0, \tau) + h(y, w, \tau)$$

where

$$y = \frac{1}{\sigma} \text{sgn}(z) \log \left( \frac{1}{2}|z| + 1 \right) = \frac{1}{\sigma} \log \left( \frac{2}{\beta - \alpha}(\frac{x}{s} - \alpha) \right)$$

when $z$ is defined as above. We also have that

$$w = \frac{1}{\sigma} \log \left( \frac{2\beta}{\beta - \alpha} \right)$$

which of course comes from equation (3.73) where it is the point at which the outcome function switches form. The functions $g$ and $h$ are defined previously in equations (3.79) and (3.80) respectively. Now, after some rather tedious algebra, the functions $g$ and $h$ can be written as

$$g(y, 0, \tau) = \left\{ \begin{array}{l} \frac{1}{2}\alpha - \frac{1}{2}\alpha e^{\sigma y} - \frac{1}{4}\beta \sigma \sqrt{\tau} \delta_- + \frac{1}{2}\beta e^{\sigma y} N(\delta_+) \\ + \left[ \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} \delta_- + \frac{1}{2}\alpha \right] N(\delta_-) \\ + \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} \phi(\delta_-) \end{array} \right\}$$

$$h(y, w, \tau) = \left\{ \begin{array}{l} \frac{1}{2}\beta - \frac{1}{2}\beta e^{-\sigma w} + \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} \delta_- - \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} d_- \\ - \frac{1}{2}\beta e^{\sigma w} N(\delta_+) - \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} \delta_- N(\delta_-) \\ - \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} \phi(\delta_-) + \frac{1}{2}\beta e^{\sigma w} N(d_+) \\ + \left[ \frac{1}{2}\beta e^{-\sigma w} - \frac{1}{2}(\beta - \alpha) + \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} d_- \right] N(d_-) \\ + \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} \phi(d_-) \end{array} \right\}$$
where
\[ \delta_\pm = \frac{1}{\sqrt{\tau}} (y \pm \frac{1}{2} \sigma \tau) \]
\[ = \frac{1}{\sigma \sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \frac{\xi}{s} - \alpha \right) \right) \pm \frac{1}{2} \sigma \sqrt{\tau} \]
and
\[ d_\pm = \frac{1}{\sqrt{\tau}} (y + w \pm \frac{1}{2} \sigma \tau) \]
\[ = \frac{1}{\sigma \sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \frac{\xi}{s} - \alpha \right) \right) \pm \frac{1}{2} \sigma \sqrt{\tau}. \]

We point out that the terms in \( g \) and \( h \) which contain \( \delta_\pm \) cancel when added together, thus the final solution doesn’t actually contain that parameter. Thus, after more algebra, the function \( u \) can be written as
\[ u \left( \frac{\xi}{s} - \frac{1}{2} (\alpha + \beta), t \right) = \left\{ \begin{array}{l}
\frac{(\alpha + \beta)^2}{4(\beta - \alpha)} - \frac{\alpha x}{(\beta - \alpha)s} - \frac{1}{4}(\beta - \alpha) \sigma \sqrt{\tau} d_- \\
+ \frac{\beta}{(\beta - \alpha)} \left( \frac{x}{s} - \alpha \right) \mathcal{N}(d_+) \\
- \frac{1}{2}(\beta - \alpha) [1 - \sigma \sqrt{\tau} d_-] \mathcal{N}(d_-) \\
+ \frac{1}{2}(\beta - \alpha) \sigma \sqrt{\tau} \phi(d_-)
\end{array} \right. \]

Given the function \( u \) above, we can easily obtain equation (3.84) by simply computing
\[ V^*(s, x, t) = s u \left( \frac{x}{s} - \frac{1}{2} (\alpha + \beta), t \right). \]

We now consider the second possibility for relative values of \( \alpha \) and \( \beta \).

**Case 2:** Assume that \( |\alpha| > |\beta| \) and that \( z = \frac{x}{s} - \frac{1}{2} (\alpha + \beta) \geq 0 \).

This implies that \( x \geq \frac{1}{2} (\alpha + \beta)s \). It is important to note that while \( x \) may be negative in this scenario, the value of \( y \) which is given by equation (3.88) is still positive meaning the transition density given by theorem 3.5 is still valid. In this case, we have that the expectation which provides the option value may be written as
\[ u(z, t) = \mathbb{E}\{Z_T + \frac{1}{2} (\alpha + \beta)\} \]
\[ = g(y, w, \tau) \]
where,
\[ w = \frac{1}{\sigma} \log \left( \frac{-2\alpha}{\beta - \alpha} \right). \]
3.4. THE OPTION PRICING FORMULA

After a little algebra we have that

\[
g(y, w, \tau) = \begin{cases} 
(x_s - \alpha) \mathcal{N}(d_-) + \alpha \mathcal{N}(d_-) \\
-\frac{\alpha}{\beta - \alpha} (x_s - \alpha) \mathcal{N}(-d_+) \\
-\frac{1}{4} (\beta - \alpha)[1 + \sigma \sqrt{\tau}] \mathcal{N}(-d_-) \\
+\frac{1}{4} (\beta - \alpha) \sigma \sqrt{\tau} \phi(d_-) 
\end{cases}
\]

where

\[
d_{\pm} = \frac{1}{\sqrt{\tau}} (y \pm w \pm \frac{1}{2} \sigma \tau)
\]

\[
= \frac{1}{\sigma \sqrt{\tau}} \log \left( \frac{2}{\beta - \alpha} \left( \frac{x_s}{s} - \alpha \right) \right) \pm \frac{1}{\sigma \sqrt{\tau}} \log \left( \frac{-2\alpha}{\beta - \alpha} \right) \pm \frac{1}{2} \sigma \sqrt{\tau}.
\]

Again, using \( V^* = s u(z, t) \), we arrive at our formula for the value of an option on a traded account under this scenario which is given by (3.86).

What we have computed so far are option values when \( z \geq 0 \). If we were to apply the strategy used above when \( z < 0 \), we would require a form for the transition density which allows \( y < 0 \). As has been seen, deriving the required transition density for one scenario was rather complicated, so to avoid having to repeat that exercise, we will use a generalised put-call parity relationship first presented by Shreve and Vecer (2000). For completeness, we present Shreve and Vecer’s generalised put-call parity in theorem 3.10 below.

**Theorem 3.10.** Options on traded accounts satisfy a certain parity relationship which we refer to as a generalised put-call parity. Given trading limits of \([\alpha, \beta]\), the option values satisfy the relationship

\[
V^*_{[\alpha, \beta]}(s, x, t) - V^*_{[-\beta, -\alpha]}(s, -x, t) = x
\]

where \( s = S_t \) and \( x = X_t \).

**Proof.** First, denote the variable \( Z_t = \frac{X_t}{S_t} - \frac{1}{2}(\mu + \beta) \) with \( \theta_t \in [\alpha, \beta] \) as \( Z_t^{[\alpha, \beta]} \). We have then by definition that

\[
Z_t^{[\alpha, \beta]} = \frac{X_t}{S_t} - \frac{1}{2}(\mu + \beta)
\]
which has dynamics under the measure $\mathbb{P}$ given by equation (3.47) and under $\tilde{\mathbb{P}}$ by (3.48). Applying the optimal strategy to (3.48) yields (3.52).

Consider the same variable but with trading limits given by $\theta_t \in [-\beta, -\alpha]$. Then this variable is given by

$$Z_t^{[-\beta,-\alpha]} = \frac{X_t}{S_t} + \frac{1}{2} (\alpha + \beta).$$

(3.90)

Using this equation and Itô’s quotient rule, we may derive the dynamics for $Z_t^{[-\beta,-\alpha]}$ in a similar fashion to the process described above. Using this procedure, it is straightforward to show that under its respective optimal strategy, both $Z_t^{[\alpha,\beta]}$ and $Z_t^{[-\beta,-\alpha]}$ have the same equation under the measure $\tilde{\mathbb{P}}$ which is given by (3.52) and repeated below

$$dZ_t = -\sigma \left( Z_t + \frac{1}{2} (\beta - \alpha) \text{sgn}(Z_t) \right) d\tilde{W}_t.$$

From equation (3.50), we have that

$$V^*_{[\alpha,\beta]}(s, x, t) = s\tilde{\mathbb{E}} \left\{ (Z_t^{[\alpha,\beta]} + \frac{\alpha + \beta}{2})^+ \left| Z_t^{[\alpha,\beta]} = \frac{x}{s} - \frac{\alpha + \beta}{2} \right. \right\},$$

and

$$V^*_{[-\beta,-\alpha]}(s, x, t) = s\tilde{\mathbb{E}} \left\{ (Z_t^{[-\beta,-\alpha]} - \frac{\alpha + \beta}{2})^+ \left| Z_t^{[-\beta,-\alpha]} = \frac{x}{s} + \frac{\alpha + \beta}{2} \right. \right\}.$$

Now, for the problem with trading limits $[\alpha, \beta]$, set initial conditions $S_t = s$, $X_t = x$ and for the problem with trading interval $[-\beta, -\alpha]$ we set $S_t = s$ and $X_t = -x$. Then we have that

$$-Z_t^{[-\alpha,-\beta]} = \frac{x}{s} - \frac{1}{2} (\alpha + \beta) = Z_t^{[\alpha,\beta]}$$

and thus

$$d \left( -Z_t^{[-\alpha,-\beta]} \right) = -\sigma \left( \left( -Z_t^{[-\alpha,-\beta]} \right) + \frac{1}{2} (\beta - \alpha) \text{sgn} \left( -Z_t^{[-\alpha,-\beta]} \right) \right) d\tilde{W}_t$$

$$= d \left( Z_t^{[\alpha,\beta]} \right)$$

which means that $-Z_t^{[-\alpha,-\beta]}$ and $Z_t^{[\alpha,\beta]}$ have the same distribution. Thus we may write $V^*_{[-\beta,-\alpha]}(s, -x, t)$ as

$$V^*_{[-\beta,-\alpha]}(s, -x, t) = s\tilde{\mathbb{E}} \left\{ (Z_t^{[-\beta,-\alpha]} - \frac{\alpha + \beta}{2})^+ \left| Z_t^{[-\beta,-\alpha]} = \frac{x}{s} + \frac{\alpha + \beta}{2} \right. \right\}.$$
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\[ s \tilde{E} \left\{ \left(-Z_{t}^{[\alpha,\beta]} - \frac{\alpha + \beta}{2}\right)^{+} \left| Z_{t}^{[\alpha,\beta]} = \frac{x}{s} - \frac{\alpha + \beta}{2}\right\{ \right. \]

\[ = s \tilde{E} \left\{ \left(Z_{t}^{[\alpha,\beta]} + \frac{\alpha + \beta}{2}\right)^{-} \left| Z_{t}^{[\alpha,\beta]} = \frac{x}{s} - \frac{\alpha + \beta}{2}\right\{ \right. \]

So, using the relation \( x^+ - x^- = x \), we can write

\[ V_{[\alpha,\beta]}^{*}(s, x, t) - V_{[-\beta,-\alpha]}^{*}(s, -x, t) = s \tilde{E} \left\{ Z_{t}^{[\alpha,\beta]} + \frac{\alpha + \beta}{2} \left| Z_{t}^{[\alpha,\beta]} = \frac{x}{s} - \frac{\alpha + \beta}{2}\right\{ \right. \]

\[ = x, \]

which is a direct result of the martingale property of the account value \( X_t \).

Using theorem 3.10 we may continue with the derivation. First consider case 1, that is \( \beta > 0 \) and \( |\alpha| < \beta \), though this time we assume that \( x < \frac{1}{2}(\alpha + \beta) s \). To utilise the put-call parity relationship of theorem 3.10, we define the following parameters. Let \( \beta' = -\alpha \), \( \alpha' = -\beta \) and \( x' = -x \). We note that using these new parameters we have that \( x' > \frac{1}{2}(\alpha' + \beta') \), but also that \( |\alpha'| > |\beta'| \) and thus the formula for \( V_{[\alpha',\beta']}^{*}(s, x', t) \) is given by equation (3.86). Thus we can write the value of the option under the conditions described above as

\[ V_{[\alpha,\beta]}^{*}(s, x, t) = x + V_{[\alpha',\beta']}^{*}(s, x, t). \] (3.91)

Evaluation of the last term in the above equation essentially involves using equation (3.86) and replacing \( \alpha \) with \( -\beta \), \( \beta \) with \( -\alpha \) and \( x \) with \( -x \). Making these substitutions in (3.91) and simplifying yields equation (3.85). We point out that the expression for \( e_{\pm} \) is obtained from \( d_{\pm} \) where the same replacements, namely \( \alpha' = -\beta \), \( \beta' = -\alpha \) and \( x' = -x \), as above are used.

The expression for case 2, where \( |\alpha| > \beta \) but with \( x < \frac{1}{2}(\alpha + \beta) s \) is obtained from (3.84) using the put-call parity relationship in a similar manner to that described above to derive equation (3.87).

This completes our derivation of the value of an option on a traded account. As we can see, this is a highly complicated valuation which results in four distinct formulas which describe this option’s value under different initial conditions. Whilst we believe that our treatment of this
valuation has simplified the task of obtaining the appropriate formulas, we nonetheless have followed fairly closely the path set by Shreve and Vecer (2000). Obviously the general formulas obtained could be used to value passport options also, though in their present form they are somewhat unwieldy. To overcome this, in the following section, we will provide a new approach to deriving the value function for passport options. The advantages of our method are that the Green’s function doesn’t need to be explicitly derived and we require only a single formula for the option’s value.

3.5. A New Approach to the Valuation of Passport Options

In this section we introduce a new method for deriving the value of the passport option. As shown in section 3.4.2 of this chapter, Hyer et al. (1997) solved this problem by specifically deriving the Green’s function and used this to solve the relevant PDE. In the method we present, we don’t actually need to specifically derive the Green’s function, but rather implicitly use it as a result of a series of Laplace transforms. We believe this method is therefore a simplified version of that used by Hyer et al. (1997).

Given that the problem we are considering is identical to that in section 3.4.2 in this chapter, we begin our analysis from equation (3.29). Rather than solving this equation via a Greens function as Hyer et al. (1997) did, we instead proceed by using Laplace transforms to simplify the problem which we then solve. Finally, we invert those transforms to obtain the desired solution. Recall that the problem described by equation (3.29) is given by

\[
\begin{align*}
\left\{ \begin{array}{l}
f_y = f_{yy} + \delta(y)f(0, \tau) \\
f(y, 0) = \left( e^{\frac{1}{2} y} - e^{-\frac{1}{2} y} \right)^+ = g^+(y)
\end{array} \right.
\end{align*}
\]

(3.92)

where we have transformed from the one factor HJB\(^{13}\). To solve this equation, we first take a Laplace transform in \(\tau\) and denote the Laplace transform of \(f\) by \(F(s, \tau)\). Then we solve the transformed equation in \(s\) and \(\tau\), and finally invert the Laplace transforms to obtain the solution for \(f\).
transform of this variable\(^\text{14}\) as \(s\). As before, we denote the Laplace transformed function with a bar, namely \(L\{f(\tau)\} = \bar{f}(s)\). The Laplace transform of (3.92) is given by
\[
s\bar{f} - g^+(y) = \bar{f}_{yy} + h(s)\delta(y)
\]
where \(h(s) = \bar{f}(0,s)\). Rearranging into the more standard form for a linear ODE gives
\[
\bar{f}_{yy} - s\bar{f} = -h(s)\delta(y) - g^+(y)
\]
which is a second order non-homogeneous ODE. This equation has the solution\(^\text{15}\)
\[
\bar{f}(y,s) = \frac{h(s)}{2\sqrt{s}} e^{-|y|\sqrt{s}} + \int_{-\infty}^{\infty} g^+(\xi) \frac{e^{-|y-\xi|\sqrt{s}}}{2\sqrt{s}} d\xi. \tag{3.94}
\]
First, we determine the expression for \(h(s)\). This requires setting \(y = 0\) in equation (3.94). Thus we have
\[
h(s) = \frac{h(s)}{2\sqrt{s}} + \int_{-\infty}^{\infty} g^+(\xi) \frac{e^{-|\xi|\sqrt{s}}}{2\sqrt{s}} d\xi
\]
\[
\Rightarrow h(s) = \frac{1}{2(\sqrt{s} - \frac{1}{2})} \int_{-\infty}^{\infty} g^+(\xi) \frac{e^{-|\xi|\sqrt{s}}}{2\sqrt{s}} d\xi. \tag{3.95}
\]
Substituting (3.95) into (3.94) gives
\[
\bar{f}(y,s) = \int_{-\infty}^{\infty} g^+(\xi) \left[ \frac{e^{-(|y|+|\xi|)\sqrt{s}}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})} + \frac{e^{-|y-\xi|\sqrt{s}}}{2\sqrt{s}} \right] d\xi
\]
\[
= \int_{0}^{\infty} \frac{g(\xi)}{2\sqrt{s}} \left[ \frac{e^{-(|y|+|\xi|)\sqrt{s}}}{2(\sqrt{s} - \frac{1}{2})} + e^{-|y-\xi|\sqrt{s}} \right] d\xi.
\]
Noting that \(g(\xi) = \left( e^{\frac{1}{2}y} - e^{-\frac{1}{2}y} \right)\), we may write
\[
\bar{f}(y,s) = (I_1 - I_2) + (I_3 - I_4)
\]
\(^\text{14}\)Note that this is not the stock price. We use this notation simply because it is standard practice in the field.
\(^\text{15}\)The derivation of this solution may be found in appendix D.
where

\[ I_1 = \int_0^\infty \frac{e^{\frac{1}{2}\xi}}{2\sqrt{s}} \times \frac{e^{-(|y|+\xi)\sqrt{s}}}{2(\sqrt{s} - \frac{1}{2})} d\xi \]

\[ = \frac{e^{-|y|\sqrt{s}}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})} \int_0^\infty e^{-(\sqrt{s} - \frac{1}{2})\xi} d\xi \]

\[ = \frac{e^{-|y|\sqrt{s}}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})^2}, \]

\[ I_2 = \int_0^\infty \frac{e^{-\frac{1}{2}\xi}}{2\sqrt{s}} \times \frac{e^{-(|y|+\xi)\sqrt{s}}}{2(\sqrt{s} - \frac{1}{2})} d\xi \]

\[ = \frac{e^{-|y|\sqrt{s}}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})} \int_0^\infty e^{-(\sqrt{s} + \frac{1}{2})\xi} d\xi \]

\[ = \frac{e^{-|y|\sqrt{s}}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})(\sqrt{s} + \frac{1}{2})}, \]

\[ I_3 = \frac{1}{2\sqrt{s}} \int_0^y e^{\frac{1}{2}\xi - |y-\xi|\sqrt{s}} d\xi, \]

\[ I_4 = \frac{1}{2\sqrt{s}} \int_0^\infty e^{-\frac{1}{2}\xi - |y-\xi|\sqrt{s}} d\xi. \]

In order to determine the values of the integrals \( I_3 \) and \( I_4 \), we need to consider different cases for the value of \( y \). First, let us consider the outcome when \( y > 0 \). For \( I_3 \) we then have

\[ I_3 = \frac{1}{2\sqrt{s}} \int_0^y e^{-|y-\xi|\sqrt{s} + \frac{1}{2}\xi} d\xi + \frac{1}{2\sqrt{s}} \int_y^\infty e^{-|y-\xi|\sqrt{s} + \frac{1}{2}\xi} d\xi. \]

Noting that when \( \xi < y \), \( e^{-|y-\xi|\sqrt{s}} = e^{-y\sqrt{s}} e^{\xi\sqrt{s}} \) and when \( \xi > y \), \( e^{-|y-\xi|\sqrt{s}} = e^{y\sqrt{s}} e^{-\xi\sqrt{s}} \),

\[ I_3 = \frac{e^{-y\sqrt{s}}}{2\sqrt{s}} \int_0^y e^{\left(\frac{1}{2} + \sqrt{s}\right)\xi} d\xi + \frac{e^{y\sqrt{s}}}{2\sqrt{s}} \int_y^\infty e^{\left(-\frac{1}{2} - \sqrt{s}\right)\xi} d\xi \]

\[ = \frac{e^{\frac{1}{2}y} - e^{-y\sqrt{s}}}{2\sqrt{s}(\sqrt{s} + \frac{1}{2})} + \frac{e^{\frac{1}{2}y}}{2\sqrt{s}(\sqrt{s} - \frac{1}{2})}. \]
For $I_4$, we take a similar approach to that used for $I_3$ and find that for $y > 0$,

$$I_4 = \frac{e^{-\frac{1}{2}y} - e^{-y\sqrt{s}}}{2\sqrt{s}(\sqrt{s} - \frac{1}{2})} + \frac{e^{-\frac{1}{2}y}}{2\sqrt{s}(\sqrt{s} + \frac{1}{2})}.$$

Taking these results for $I_3$ and $I_4$ together, we have that when $y > 0$, the term $(I_3 - I_4)$ is given by

$$\left(I_3 - I_4\right) = \frac{e^{\frac{1}{2}y} - e^{-\frac{1}{2}y} - e^{-y\sqrt{s}}}{2\sqrt{s}(\sqrt{s} + \frac{1}{2})} + \frac{e^{\frac{1}{2}y} - e^{-\frac{1}{2}y} + e^{-y\sqrt{s}}}{2\sqrt{s}(\sqrt{s} - \frac{1}{2})}$$

$$= \frac{g(y)}{s - \frac{1}{4}} + \frac{e^{-y\sqrt{s}}}{2\sqrt{s}} \left(\frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}}\right). \quad (3.96)$$

Now, let us consider the alternative case, $y < 0$. To treat this case we define a new variable $\hat{y} > 0$ and let $y = -\hat{y}$. We note that since $|\hat{y} - \xi| = \hat{y} + \xi$, we can write $I_3$ as

$$I_3 = \frac{1}{2\sqrt{s}} \int_{0}^{\infty} e^{\frac{1}{2}\xi - \hat{y}\sqrt{s} - \xi\sqrt{s}} d\xi$$

$$= \frac{e^{-\hat{y}\sqrt{s}}}{2\sqrt{s}(\sqrt{s} - \frac{1}{2})}$$

and similarly for $I_4$,

$$I_4 = \frac{1}{2\sqrt{s}} \int_{0}^{\infty} e^{-\frac{1}{2}\xi - \hat{y}\sqrt{s} - \xi\sqrt{s}} d\xi$$

$$= \frac{e^{-\hat{y}\sqrt{s}}}{2\sqrt{s}(\sqrt{s} + \frac{1}{2})},$$

so in this case the term $(I_3 - I_4)$ is given by

$$\left(I_3 - I_4\right) = \frac{e^{-\hat{y}\sqrt{s}}}{2\sqrt{s}} \left(\frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}}\right).$$

Comparing this expression with equation (3.96) which is the equivalent term for the case $y > 0$, we see that they differ only by the term $g(y)/(s - \frac{1}{4})$. Thus, we may write for all $y$,

$$\hat{f}(y, s) = (I_1 - I_2) + (I_3 - I_4)$$
3.5. NEW APPROACH TO PASSPORT OPTIONS

\[
\begin{align*}
\bar{f}(y,s) & = \frac{e^{-|y|\sqrt{s}}}{4\sqrt{s}(\sqrt{s} - \frac{1}{2})} \left[ \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}} \right] + \frac{g(y)\mathbb{1}(y > 0)}{s - \frac{1}{4}} \\
& + \frac{e^{-|y|\sqrt{s}}}{2\sqrt{s}} \left[ \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}} \right] \\
& = \frac{g^+(y)}{s - \frac{1}{4}} + \frac{e^{-|y|\sqrt{s}}}{4\sqrt{s}} \left[ \left( \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}} \right) \left( \frac{1}{\sqrt{s} - \frac{1}{2}} + 2 \right) \right].
\end{align*}
\]

Considering the term in square brackets and labelling it \( A \), a little algebra can show that

\[
A = \left[ \left( \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}} \right) \left( \frac{1}{\sqrt{s} - \frac{1}{2}} + 2 \right) \right]
\]

\[
= \frac{2\sqrt{s}}{\sqrt{s} - \frac{1}{2}} \left( \frac{1}{(\sqrt{s} - \frac{1}{2})(\sqrt{s} + \frac{1}{2})} \right)
\]

and utilising the relation

\[
\frac{1}{(\sqrt{s} - \frac{1}{2})(\sqrt{s} + \frac{1}{2})} = \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}}
\]

we arrive at our final expression for \( \bar{f}(y,s) \),

\[
\bar{f}(y,s) = \frac{g^+(y)}{s - \frac{1}{4}} + \frac{e^{-|y|\sqrt{s}}}{2(\sqrt{s} - \frac{1}{2})^2} - \frac{e^{-|y|\sqrt{s}}}{2} \left( \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}} \right). 
\]

(3.97)

To simplify, let us write the above expression as

\[
\bar{f}(y,s) = \bar{f}_1(y,s) + \bar{f}_2(y,s) - \bar{f}_3(y,s)
\]

(3.98)

where it is obvious that

\[
\begin{align*}
\bar{f}_1(y,s) & = \frac{g^+(y)}{s - \frac{1}{4}} \\
\bar{f}_2(y,s) & = \frac{e^{-|y|\sqrt{s}}}{2(\sqrt{s} - \frac{1}{2})^2} \\
\bar{f}_3(y,s) & = \frac{e^{-|y|\sqrt{s}}}{2} \left( \frac{1}{\sqrt{s} - \frac{1}{2}} - \frac{1}{\sqrt{s} + \frac{1}{2}} \right)
\end{align*}
\]

Inverting each of these functions via \( f_i(y,\tau) = \mathcal{L}^{-1}\{\bar{f}_i(y,s)\} \), we can write the desired function \( f(y,\tau) = f_1(y,\tau) + f_2(y,\tau) - f_3(y,\tau) \). Using
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the results of appendix B to invert these functions, we have by applying (B.1), (B.3) and (B.2) respectively that

\[ f_1(y, \tau) = g^+(y)e^{\frac{1}{4} \tau} \]

\[ f_2(y, \tau) = e^{-\frac{1}{2} |y| + \frac{1}{2} \tau} \left[ \left(1 - \frac{1}{2} |y| + \frac{1}{2} \tau\right) \mathcal{N}\left(\frac{1}{2} \sqrt{2\tau} - \frac{|y|}{\sqrt{2\tau}}\right) \right. \]

\[ \left. + \frac{1}{2} \sqrt{2\tau} \phi\left(-\frac{1}{2} \sqrt{2\tau} + \frac{|y|}{\sqrt{2\tau}}\right) \right] \]

\[ f_3(y, \tau) = \frac{e^{\frac{1}{4} \tau}}{2} \left[ e^{-\frac{1}{2} |y|} \mathcal{N}\left(\frac{1}{2} \sqrt{2\tau} - \frac{|y|}{\sqrt{2\tau}}\right) \right. \]

\[ \left. + e^{\frac{1}{2} |y|} \mathcal{N}\left(-\frac{1}{2} \sqrt{2\tau} - \frac{|y|}{\sqrt{2\tau}}\right) \right] \]

and thus we have our final expression for \( f(y, \tau) \) by substituting these expressions back into (3.98). What remains is to back transform our variables so that we may express our solution in terms of financial variables. First we note that \( \sqrt{2\tau} = \sigma \sqrt{T - t} \). Also, since \( \text{sgn}(y) = \text{sgn}(z) \), we have that \( |y| = \log(1 + |z|) = \log(1 + \frac{|z|}{s}) \). We also point out that \( U(y, \tau) = f(y, \tau)e^{\frac{1}{2} |y| - \frac{1}{4} \tau} \). This means that we will be able to write the value of the option \( V^* \) in the following way,

\[ V^*(s, x, t) = V_1^*(s, x, t) + V_2^*(s, x, t) - V_3^*(s, x, t) \] (3.99)

where \( V_i^*(s, x, t) = sf_i(y, \tau)e^{\frac{1}{2} |y| - \frac{1}{4} \tau} \). Starting with \( V_1^* \),

\[ V_1^*(s, x, t) = sf_1(y, \tau)e^{\frac{1}{2} |y| - \frac{1}{4} \tau} \]

\[ = sg^+(y)e^{\frac{1}{2} |y|} \]

\[ = s \left( e^{\frac{1}{2} y} - e^{-\frac{1}{2} y} \right) + e^{\frac{1}{2} |y|} \]

\[ = s(e^y - 1)I(y > 0) \]

\[ = xI(x > 0). \]

Now, moving to \( V_2^* \) we introduce some notation to simplify the exposition. Let

\[ d_\pm = \frac{|y|}{\sqrt{2\tau}} \pm \frac{1}{2} \sqrt{2\tau} \]
3.6. CONCLUDING REMARKS

\[ \log \left( 1 + \frac{|x|}{\sigma} \right) = \frac{1}{2} \sigma \sqrt{T-t} \pm \frac{1}{2} \sigma \sqrt{T-t} \]

then

\[ V_2^*(s, x, t) = sf_2(y, \tau) e^{\frac{1}{2}|y| - \frac{1}{4} \tau} \]

\[ = s \left( 1 - \frac{1}{2} d_- \sigma \sqrt{T-t} \right) \mathcal{N}(-d_-) + \frac{1}{2} s \sigma \sqrt{T-t} \phi(d_-), \]

and for \( V_3^* \),

\[ V_3^*(s, x, t) = sf_3(y, \tau) e^{\frac{1}{2}|y| - \frac{1}{4} \tau} \]

\[ = \frac{1}{2} [s \mathcal{N}(-d_-) + (s + |x|)\mathcal{N}(-d_+)]. \]

Using the property that \( \mathcal{N}(-x) = 1 - \mathcal{N}(x) \) to simplify, it is straightforward to show that

\[ V^*(s, x, t) = \begin{cases} 
-\frac{1}{2} sd_- \sigma \sqrt{T-t} \mathcal{N}(-d_-) + \frac{1}{2} s \sigma \sqrt{T-t} \phi(d_-) \\
+xI(x > 0) - \frac{1}{2} [(s + |x|)\mathcal{N}(-d_+) - s \mathcal{N}(-d_-)] 
\end{cases} \]

Now, using the identities

\[ xI(x > 0) - \frac{1}{2} |x| = \frac{1}{2} x \]

\[ (s + |x|)\mathcal{N}(-d_+) - s \mathcal{N}(-d_-) = |x| - (s + |x|)\mathcal{N}(d_+) + s \mathcal{N}(d_-), \]

we can reduce the above expression for \( V^* \) to

\[ V^*(s, x, t) = \frac{1}{2} \begin{cases} 
 x + (s + |x|)\mathcal{N}(d_+) - s \mathcal{N}(d_-) \\
-sd_- \sigma \sqrt{T-t} \mathcal{N}(-d_-) + s \sigma \sqrt{T-t} \phi(d_-)
\end{cases} \]

This expression is valid for all \( x \geq 0 \) and \( x < 0 \). This formula is also in agreement with equations (3.43) and (3.44) which were derived in section 3.4.2 of this chapter and are also the same solutions as those derived by Shreve and Vecer (2000), taking into account the typographical error indicated earlier.

3.6. Concluding Remarks

In this chapter we have closely examined the problem of valuing an option on a traded account under the continuous time setting provided by the Black-Scholes framework. This has involved deriving the option
value formulas for the special cases of the constant strategy option, which was shown to be identical to a vanilla European option, and the passport option. The value of the passport option followed the path of solving the associated HJB equation which was achieved by utilising a Green’s function in a manner very close to the seminal paper by Hyer et al. (1997). We have then derived the value of the more general option on a traded account using a framework based on the work of Shreve and Vecer (2000), though we have provided significant detail in the derivation of the transition density, and utilised some unique theorems to evaluate the integrals required to compute the expectations necessary for the value function. Specifically, the Gaussian shift theorem played an important role in this analysis and was seen to simplify the computation of the integrals and thus the option value function considerably.

In the following chapter we will remove ourselves from the continuous time setting and analyse the option on a traded account in a binomial framework. This will require a completely new set of mathematical techniques. We will also require some sophisticated numerical techniques which will be required to tackle the optimisation problem inherent in the valuation of this class of options.
CHAPTER 4

Binomial Valuation of Options on Traded Accounts

In this chapter we will examine the problem of valuing the option on a traded account under the binomial framework. As far as the author is aware, this is something that has yet to be attempted in the literature. While there have previously been no results concerning the use of binomial trees, or other lattice methods, to value this class of options, they have been mentioned in the literature. Andersen et al. (1998) stated that lattice techniques were unsuited to the valuation of this class of options since the lattice describing the account value is generally unlikely to possess the recombining property. This property is essential if option prices are to be computed quickly as the number of nodes to be examined increases dramatically when this property cannot be exploited. For example, a non-recombining binomial tree has potentially $2^N$ terminal nodes to consider whereas the recombining tree has only $(N + 1)$ nodes present for a tree consisting of $N$ steps.

Whilst the computational problems associated with a non recombining tree are well understood\(^1\), in principle this is not the largest challenge in valuing options of this nature with a binomial tree. Rather, the bigger challenge is determining the optimal trading strategy. As we saw in the continuous time models, this was a relatively simple part of the pricing problem and in fact the optimal strategy could be determined simply by writing the appropriate HJB equation and using convexity arguments. It was observed that the optimal strategy was a function of only two state variables, the current stock price and trading account value. As we will show, this is not the case for the binomial counterpart. Instead, the general optimal strategy requires knowledge

\(^1\)This is often referred to as the curse of dimensionality.
of the drift and volatility values in a highly non-linear way. This may make finding an analytic optimal strategy for the option on a traded account an impossible task, and at present, ascertaining the optimal strategy for an option on a traded account under a binomial model remains an unsolved problem. For this reason we will examine the use of numerical methods which may be applied to solve the problem.

This chapter will proceed in several stages. First we will introduce the binomial lattice we will use to model the stock price evolution. We will also provide some comparisons between our choice and other well known choices for binomial trees. We also provide some convergence results to show that our specification converges to the Black-Scholes counterpart. Secondly, we will examine the one and two step models analytically to show how in principle this option may be valued. This will also serve to highlight the intractability of such a procedure for a general specification with \( N \) steps. To tackle these more general specifications, we must turn to numerical methods, thus we will examine how the problem of valuing these options may be transformed into a number of different problem types including \( l_1 \)-norm maximisation, indefinite quadratic programming and pseudo-Boolean maximisation. Each of these problem classes will be examined and we will also consider some algorithms specifically designed to handle them.

4.1. A Binomial Model for the Stock and Trading Account

The first use of a binomial model to value options was due to Cox et al. (1979). This work followed from that of Black and Scholes and was developed to provide a simpler method of valuing options which did not require the complicated mathematics associated with continuous time valuation. To this day this model has remained popular, particularly among practitioners as it is easy to understand and computational efficiency is almost on par with closed form solutions due to the speed of modern computers. Other reasons for its popularity include its ability to handle features of exercise which the Black-Scholes framework is not well suited to. The best known example of this is the case of American options which can easily be valued by a binomial
model, but which are extremely difficult to handle in a continuous time setting.

The binomial model as set out by Cox et al. (1979) is as follows. Let $k$ be the current node, then we assume that the stock price one period in the future, denoted $S_{k+1}$, may take on one of two possible values. It may either go up in value such that $S_{k+1} = S_k u$ where $u$ represents an up movement factor, or it may go down in value such that $S_{k+1} = S_k d$ where $d$ represents a down factor. Each of these factors occurs with probabilities $p$ and $q$ respectively and since they are the only possible outcomes, we have that $p + q = 1$. The question remains, what are appropriate values for $u$, $d$, $p$ and $q$? Cox et al. showed that in order for the market to be free of arbitrage, one such choice is to set the parameters in the following manner:

$$u = e^{\sigma \sqrt{\Delta t}}; \quad d = e^{-\sigma \sqrt{\Delta t}}$$

$$p = \frac{e^{r \Delta t} - d}{u - d}; \quad q = 1 - p$$

where $r$ is the risk-free rate, $\sigma$ is the volatility and $\Delta t$ is the time between nodes. This specification for the binomial tree is derived by assuming the appropriate discount factor over the time $\Delta t$ is $e^{-r \Delta t}$ and by using a hedging argument similar to that used by Black-Scholes to determine the arbitrage free parameters. In fact the above specification may be considered to be the stock dynamics under the martingale measure.

However, this is not the only available form for a binomial lattice. There are in fact many different variations of the binomial model. Some examples include the Jarrow and Rudd (1983), Tian (1993) and the Trigeorgis (1991) specifications. The reason for this variety is that binomial models in general have three parameters ($u$, $d$, $p$) which are related by two conditions, i) the martingale restriction and ii) the matching of the volatility of the binomial jumps to the stock price volatility, thus there may in fact be many specifications. In this thesis we derive a binomial tree which is more suitable for our purpose of valuing options on traded accounts. The method we use has in fact been extended
by He (1990) who constructed an \((n + 1)\)-nomial tree for an economy consisting of \(n\)-risky and 1 risk-free asset. Our approach differs to He’s however in that we allow for different probabilities of the up and down states, whereas He maintained a probability of \(\frac{1}{2}\) for each state. We also focus on the single risky asset case only.

First, we provide a brief description of the model proposed by He (1990). This was developed by He to be a more natural way to discretise a geometric Brownian motion. For example, consider an economy with only two assets, a risky stock \(S\) and and a risk-free bond \(B\). In continuous time, the equations describing the evolution of two such assets would be given by

\[
\begin{align*}
    dB_t &= rB_t dt \\
    dS_t &= rS_t dt + \sigma S_t dW_t
\end{align*}
\]

where the usual notation applies. He suggested that we may model the Weiner term \(W\) with a binomial random variable \(\epsilon\) which has the properties \(P[\epsilon = 1] = P[\epsilon = -1] = \frac{1}{2}\). We note that this implies that \(E[\epsilon] = 0\) and \(\text{Var}[\epsilon] = 1\). Thus we could write a discretised version of (4.1) and (4.2) in the following way. Consider the time interval \([0, T]\) and let us break this interval into \(N\) equally sized intervals of size \(\Delta t = \frac{T}{N}\). Let \(k\) denote an arbitrary node and let \(\Delta\) be the forward difference operator. Then we may write

\[
\begin{align*}
    \Delta B_k &= rB_k \Delta t \\
    \Delta S_k &= rS_k \Delta t + \sigma S_k \sqrt{\Delta t} \epsilon_k
\end{align*}
\]

Expanding the difference operator, we obtain

\[
S_{k+1} = \begin{cases} 
S_k(1 + r \Delta t + \sigma \sqrt{\Delta t}) = S_k u \\
S_k(1 + r \Delta t - \sigma \sqrt{\Delta t}) = S_k d
\end{cases}
\]
and since each outcome has equal probability, it is clear that $\mathbb{E}[S_{k+1}] = S_k(1 + r\Delta t)$ and thus satisfies the martingale restriction implying that we are working under the risk neutral measure.

He’s method is actually designed to handle multi-dimensional cases in a manner which is both mathematically appropriate in that the multinomial lognormal is achieved in the limit as $\Delta t \to 0$ while also ensuring a complete market in the sense of Arrow-Debreu state price processes. While we will not detail this model in general, as we will be modelling only one risky asset, we note that the model employed does indeed have this multi asset extension so that we may discuss a generalised version of the standard option on a traded account without altering the model setup to any great extent. To explain briefly, the key to He’s method is to model the stochastic terms of $n$ variates with an $(n+1)$-nomial process. In other words, each of the $n$ assets is modelled as an $(n+1)$-nomial tree. The question now arises as to how does one choose the value of the stochastic factors and the probabilities assigned to them? To answer this problem, He used entries from a real orthogonal matrix which is restricted to have the last column equal to $(n + 1)^{-\frac{1}{2}}$. It can be shown that such a matrix always exists and may be constructed via the Gram-Schmidt algorithm. For the case of a single risky asset ($n = 1$) the matrix which satisfies this requirement is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (4.3)$$

While the matrix $A$ for the case $n = 1$ is unique, this is not the case in general. Now, specify the matrix of disturbance terms, $\mathcal{E}$, as $\sqrt{n+1}A$, that is

$$\mathcal{E} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \quad (4.4)$$

Applying equal probability to each of the matrix rows provides the required stochastic terms for the binomial tree. He further showed that
this scheme in general converges to a lognormal process in the limit and that option prices converge to their Black-Scholes counterparts.

However, a more general version of this binomial tree is available. The key difference is that in the method proposed by He, each state has an equal and fixed probability. A more general solution would be one which allows an arbitrary, though consistent, choice of probabilities. We derive the appropriate dynamics in the following proposition.

**Proposition 4.1.** Assume that a stock price $S_{k+1}$ has the following form

$$S_{k+1} = \begin{cases} S_k u \\ S_k d \end{cases}$$

where the state $u$ occurs with probability $p$, and $d$ with probability $q = (1 - p)$. Then an appropriate choice for $u$ and $d$ are

$$u = 1 + r \Delta t + \sigma \sqrt{\frac{q \Delta t}{p}}$$

$$d = 1 + r \Delta t - \sigma \sqrt{\frac{p \Delta t}{q}}.$$

**Proof.** The key to this result is to utilise the martingale restriction and then match the volatility of a g.B.m over the period $\Delta t$. Firstly, the martingale restriction implies that under the risk neutral measure

$$\mathbb{E}\{S_{k+1}\} = S_k (1 + r \Delta t) \Rightarrow p S_k u + q S_k d = S_k (1 + r \Delta t)$$

$$\Rightarrow pu + qd = (1 + r \Delta t). \quad (4.5)$$

Also, the fact that the volatility of our discrete time process must match that of a g.B.m means that

$$\text{Var}(S_{k+1}) = S_k^2 \sigma^2 \Delta t \Rightarrow \mathbb{E}\{(S_{k+1} - \mathbb{E}\{S_{k+1}\})^2\} = S_k^2 \sigma^2 \Delta t.$$  

Using the well known relation

$$\mathbb{E}\{(S_{k+1} - \mathbb{E}\{S_{k+1}\})^2\} = \mathbb{E}\{S_{k+1}^2\} - (\mathbb{E}\{S_{k+1}\})^2,$$

we can write

$$S_k^2 \sigma^2 \Delta t = ps_k u^2 + qs_k d^2 - (ps_k u + ds_k q)^2$$

$$= ps_k u^2 + qs_k d^2 - ps_k^2 u^2 - q^2 s_k^2 d^2 - 2pq s_k u d$$
A simple rearrangement of the above equation yields
\[ u - d = \sigma \sqrt{\Delta t}. \] (4.6)

Equations (4.5) and (4.6) provide two equations in two unknowns. Solving these simultaneously yields the desired solution. □

4.1.1. Convergence to the Balck-Scholes Model. A natural question to ask when dealing with binomial models is whether the proposed specifications will converge to the Black-Scholes model as the time step is reduced to zero. In the following corollary, we show that our proposed model not only converges to the Black-Scholes model for a standard option, but also converges to the HJB equation which describes the value of an option on a traded account. Before we do this however, we first need the discrete time description for the trading account \( X \) assuming the stock evolves according to theorem 4.1. Since this trading account consists of a position of \( \theta_k \) units at node \( k \) in the risky asset, and the remaining wealth is invested at the risk free rate \( r \) which grows by \( r\Delta t \) over the period \( \Delta t \), we have that
\[
X_{k+1} = (X_k - \theta_k S_k)(1 + r\Delta t) + \theta_k S_{k+1} \\
= \begin{cases} 
(X_k - \theta_k S_k)(1 + r\Delta t) + \theta_k S_k u \\
(X_k - \theta_k S_k)(1 + r\Delta t) + \theta_k S_k d 
\end{cases} \\
= \begin{cases} 
X_k(1 + r\Delta t) + \theta_k S_k \sigma \sqrt{\frac{p}{q}} \Delta t \\
X_k(1 + r\Delta t) - \theta_k S_k \sigma \sqrt{\frac{q}{p}} \Delta t 
\end{cases}
\] (4.7)

**Corollary 4.2.** The binomial model specified in proposition 4.1 converges in distribution to the Black-Scholes model.

**Proof.** First we show convergence in distribution to the Black-Scholes model. We note that the stock dynamics may be expressed in
an additive form in the following way

\[ su = s + \Delta s_u \quad \Rightarrow \quad \Delta s_u = sr\Delta t + s\sigma \sqrt{\frac{p}{q}} \Delta t \]

\[ sd = s + \Delta s_d \quad \Rightarrow \quad \Delta s_d = sr\Delta t - s\sigma \sqrt{\frac{p}{q}} \Delta t \]

where \( s = S_k \). We also know that if \( p \) and \( q \) represent the risk neutral probabilities, then the following equation holds for \( V \), the value of an option\(^2\).

\[
(1 + r\Delta t)V(s, t) = pV(su, t + \Delta t) + qV(sd, t + \Delta t)
\]

\[
= pV(s + \Delta s_u, t + \Delta t) + qV(s + \Delta s_d, t + \Delta t).
\]

Applying Taylor’s theorem to the right hand side, we may express the option value in the up and down states as

\[
V(s + \Delta s_u, t + \Delta t) = \begin{cases} 
V(s, t) + \Delta s_u V_s + \Delta t V_t \\
+ \frac{1}{2} [(\Delta s_u)^2 V_{ss} + \Delta t^2 V_{tt} + 2\Delta s_u \Delta t V_{st}] \\
+ \text{h.o.t}
\end{cases}
\]

\[
V(s + \Delta s_d, t + \Delta t) = \begin{cases} 
V(s, t) + \Delta s_d V_s + \Delta t V_t \\
+ \frac{1}{2} [(\Delta s_d)^2 V_{ss} + \Delta t^2 V_{tt} + 2\Delta s_d \Delta t V_{st}] \\
+ \text{h.o.t}
\end{cases}
\]

where \text{h.o.t} represents the higher order terms and the subscripts on \( V \) denote the partial derivatives. Also, all partial derivatives are evaluated at \((s, t)\). Taking the limit as \( \Delta t \to 0 \), we may eliminate any terms which contain \( \Delta t \) to order greater than 1. This simplifies the expression for \( V \) in the up and down states to

\[
V(s + \Delta s_u, t + \Delta t) = \begin{cases} 
V(s, t) + s(r\Delta t + \sigma \sqrt{\frac{p}{q}} \Delta t)V_s \\
+ \Delta t V_t + \frac{1}{2} s^2 \sigma^2 \frac{p}{q} \Delta t V_{ss}
\end{cases}
\]

\[
V(s + \Delta s_d, t + \Delta t) = \begin{cases} 
V(s, t) + s(r\Delta t - \sigma \sqrt{\frac{p}{q}} \Delta t)V_s \\
+ \Delta t V_t + \frac{1}{2} s^2 \sigma^2 \frac{q}{p} \Delta t V_{ss}
\end{cases}.
\]

\(^2\)This is a well known result of hedging a portfolio containing a long option and short stock such that risk is eliminated.
Using the property that $p + q = 1$, we may simplify the above equation to

$$
\begin{align*}
V(s + \Delta s_u, t + \Delta t) + qV(s + \Delta s_d, t + \Delta t)
\end{align*}
$$

and by equating the right hand side of the above equation with $(1 + r\Delta t)V(s, t)$ and simplifying, we obtain

$$-V_t = -rV + rsV_s + \frac{1}{2}s^2\sigma^2V_{ss}\Delta t,$$

which is the celebrated the Black-Scholes equation, thus proving our convergence in distribution.

We now consider the convergence argument for an option on a traded account. In this case the option value is not governed by the Black-Scholes equation, but rather by the HJB equation. This is outlined in the following corollary which proceeds in a similar manner to corollary 4.2

**Corollary 4.3.** Using the binomial description of proposition 4.1 for the evolution of the stock and hence equation (4.7) for the evolution of the trading account, the option value will converge to the Black-Scholes counterpart, namely the HJB equation.

**Proof.** We proceed in a similar manner to that carried out in corollary 4.2, however as we will be dealing with a second state variable, namely the trading account, we extend our notation to include movements in that variable. Let $x = X_k$ and define

$$
\begin{align*}
\Delta x_u &= xr\Delta t + \theta s\sigma \sqrt{\frac{2 \Delta t}{p}} \\
\Delta x_d &= xr\Delta t - \theta s\sigma \sqrt{\frac{2 \Delta t}{q}}
\end{align*}
$$

which implies that $X_{k+1}^u = x + \Delta x_u$ and $X_{k+1}^d = x + \Delta x_d$ in accordance with equation (4.7). Again if $p$ and $q$ represent risk neutral probabilities
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we have that the option value $V$ may be expressed as

$$(1 + r\Delta t)V(s, x, t) = \max_{\theta \in [\alpha, \beta]} \left\{ pV(s + \Delta s_u, x + \Delta x_u, t + \Delta t) + qV(s + \Delta s_d, x + \Delta x_d, t + \Delta t) \right\}.$$  

(4.8)

Taking a Taylor expansion in three dimensions, we have that

$$V(s + \Delta s_u, x + \Delta x_u, t + \Delta t) = \max_{\theta \in [\alpha, \beta]} \left\{ V(s, x, t) + \Delta s_u V_s + \Delta x_u V_x + \frac{1}{2}(\Delta s_u)^2 V_{ss} + \frac{1}{2}(\Delta x_u)^2 V_{xx} + \frac{1}{2}(\Delta t)^2 V_{tt} + \Delta s_u \Delta x_u V_{sx} + \Delta s_u \Delta t V_{st} + \Delta x_u \Delta t V_{xt} + h.o.t \right\}.$$  

(4.9)

As before, we take the limit as $\Delta t \to 0$ which means we may eliminate terms containing $\Delta t$ of order greater than 1. So, to order $\Delta t$, we have the following expressions for the terms contained in the equation above

$$(\Delta s_u)^2 = s^2 \sigma^2 \frac{q}{p} \Delta t$$

$$(\Delta x_u)^2 = \theta^2 s^2 \sigma^2 \frac{q}{p} \Delta t$$

$$(\Delta s_u \Delta x_u) = \theta s^2 \sigma^2 \frac{q}{p} \Delta t$$

$$\Delta s_u \Delta t = 0$$

$$\Delta x_u \Delta t = 0$$

so, substituting these expressions into equation (4.9), we obtain

$$V(s + \Delta s_u, x + \Delta x_u, t + \Delta t) = \max_{\theta \in [\alpha, \beta]} \left\{ V(s, x, t) + \Delta t V_t + s(r \Delta t + \sigma \sqrt{\frac{q}{p} \Delta t}) V_s + (x \Delta t + \theta \sigma \sqrt{\frac{2}{p} \Delta t}) V_x + \frac{1}{2} s^2 \sigma^2 \frac{q}{p} \Delta t V_{ss} + \frac{1}{2} \theta^2 s^2 \sigma^2 \frac{q}{p} \Delta t V_{xx} + \theta \sigma^2 \frac{q}{p} \Delta t V_{sx} \right\}.$$  

(4.10)
Taking a similar approach for the option value in the down state, we have that

\[
V(s + \Delta s, x + \Delta x, t + \Delta t) = \max_{\theta \in [\alpha, \beta]} \begin{cases} 
V(s, x, t) + \Delta t V_t \\
+ s(r \Delta t - \sigma \sqrt{p \Delta t}) V_s \\
+ (\theta s \sigma \sqrt{p \Delta t}) V_x \\
+ \frac{1}{2} s^2 \sigma^2 \frac{p}{q} \Delta t V_{ss} \\
+ \frac{1}{2} \theta^2 s^2 \sigma^2 \frac{p}{q} \Delta t V_{xx} \\
+ \theta s^2 \sigma^2 \frac{p}{q} \Delta t V_{sx} 
\end{cases} 
\]

(4.11)

Now, substituting (4.10) and (4.11) into (4.8) and applying some simple algebra yields

\[
V_t = r V - r s V_s - r x V_x - \frac{1}{2} s^2 \sigma^2 \max_{\theta \in [\alpha, \beta]} [V_{ss} + \theta^2 V_{xx} + 2 \theta V_{sx}] 
\]

which is the required HJB equation given by equation (3.26) and presented by Shreve and Vecer (2000). Thus we have shown that under our discrete time dynamics, as the time step becomes vanishingly small, our option value converges in distribution to the HJB equation, and hence to the model examined in the previous chapter which was derived under the Black-Scholes assumptions.

Now, in order to define a specific value for the factors \(u\) and \(d\), we must choose values for the probabilities \(p\), and thus \(q\). Any reasonable choice of \(p\) and \(q\) will do, provided they are not too close to 0 or 1. The simplest such choice is to select \(p = q = \frac{1}{2}\). This is the choice that will be used in the remainder of this thesis unless otherwise specified. When the probabilities are set in this way, we may express the stock and account lattice in the following way

\[
S_{k+1} = S_k (1 + r \Delta t + \sigma \sqrt{\Delta t} \epsilon) \\
X_{k+1} = (1 + r \Delta t) X_k + \theta \sigma S_k \sqrt{\Delta t} \epsilon
\]

(4.12) (4.13)

where \(\epsilon\) is a binomial random variable with the property \(\mathbb{P}[\epsilon = 1] = \mathbb{P}[\epsilon = -1] = \frac{1}{2}\). It is important to realise that while the tree describing \(S\) is recombining, the tree for \(X\) in general will not be.
Whilst the notation we have used moves forward in time, for example expressing $X_{k+1}$ in terms of its previous value $X_k$, this will often not be the best notation to use for our purposes. The reason for this is that we will employ the technique of dynamic programming frequently in this thesis and as such a notation which goes backwards in time is more appropriate for this purpose. A function which will prove useful in this respect is one which expresses the value of the terminal account value $X_N$ in terms of the account value some steps $k$ prior. The appropriate formula for this is given in the following theorem where we define

$$\rho = 1 + r\Delta t$$  \hfill (4.14) $$\bar{\sigma} = \sigma \sqrt{\Delta t}$$  \hfill (4.15)$$
to simplify the notation.

**Theorem 4.4.** The terminal account value along an arbitrary path, $X_N$, may be expressed as a function of the account value $k$ steps earlier via the formula

$$X_N = \rho^k X_{N-k} + \bar{\sigma} \sum_{i=0}^{k-1} \rho^{k-1-i} \theta_{N-k+i} S_{N-k+i} \epsilon_{N-k+i}$$  \hfill (4.16)$$
for all $k = 1, \ldots, N$.

**Proof.** This formula is easily seen to hold by recursively substituting. However, to be more technical we will proceed by induction. First, let $k = 1$. Then (4.16) reduces to (4.13), though we write it using backward time notation as

$$X_N = \rho X_{N-1} + \bar{\sigma} \theta_{N-1} S_{N-1} \epsilon_{N-1}.$$  

Now, assume that

$$X_N = \rho^k X_{N-k} + \bar{\sigma} \sum_{i=0}^{k-1} \rho^{k-1-i} \theta_{N-k+i} S_{N-k+i} \epsilon_{N-k+i}.$$
and note that we can write \( X_{N-k} = \rho X_{N-k-1} + \bar{\sigma} \theta_{N-k-1} S_{N-k-1} \epsilon_{N-k-1} \).

Substituting the inductive assumption into the above equation gives (4.16).

\[ \square \]

4.2. The Valuation Problem

Now that we have the discrete time dynamics for the stock and trading account, we will outline the problem that needs to be solved in order to price an option on a traded account. From the work of Harrison and Pliska (1981), we have that the value of an option on a traded account under some arbitrary trading strategy is given by

\[ V_{N-k}(S_{N-k}, X_{N-k}) = \rho^{-k} E_{N-k} \{(X_N)^+\}. \] (4.17)

where \( X_N \) is the terminal account value at node \( N \), \( \rho^{-k} \) is the discount factor applicable over \( k \)-steps and the expectation is taken with respect to the risk neutral measure. We also use the notation that \( E_{N-k}\{x\} = \mathbb{E}\{x|\mathcal{F}_{N-k}\} \) where \( \mathcal{F}_{N-k} \) is a filtration containing information up to and including the node \( (N-k) \). As before, we assume that the option holder will try to maximise their wealth, or from another perspective, that the option seller will charge as much as possible for the option. This means that the actual value of an option on a traded account is given by

\[ V^*_{N-k}(S_{N-k}, X_{N-k}) = \max_{\{\theta\}_{N-k}} \rho^{-k} E_{N-k} \{(X_N)^+\} \] (4.18)

where \( \{\theta\}_{N-k} \) represents the set of all trading positions during the life of the option. The problem now consists of determining the optimal value for each trading position.

We note that for the discrete time valuation we are performing, working with the plus function is not always the most straightforward way of proceeding. Thus, we will decompose \( V^* \) to make the mathematics more tractable.

**Theorem 4.5.** The plus function can be expressed with the absolute value function in the following way, for any \( x \in \mathbb{R} \):

\[ (x)^+ = \frac{1}{2}(x + |x|) \] (4.19)
4.2. THE VALUATION PROBLEM

Proof. The proof of this theorem is trivial. Consider the case $x \geq 0$. In this case $(x)^+ = x$ and $\frac{1}{2}(x + |x|) = \frac{1}{2}(x + x) = x$ as required. Now consider the case $x < 0$, we have that $(x)^+ = 0$ and $\frac{1}{2}(x + |x|) = \frac{1}{2}(x - x) = 0$. □

Using theorem 4.5, we have the following corollary:

Corollary 4.6. The value function for an option on a traded account can be expressed in terms of the absolute value function in the following manner:

$$V^*_N(S_{N-k}, X_{N-k}) = \frac{1}{2\rho^k} \left[ X_{N-k}\rho^k + U(X_{N-k}, S_{N-k}) \right]$$  \hspace{1cm} (4.20)

where

$$U(S_{N-k}, X_{N-k}) = \max_{\{\theta\}_{N-k}} \mathbb{E}_{N-k} \{|X_N|\}$$  \hspace{1cm} (4.21)

Proof.

$$V^*_N(S_{N-k}, X_{N-k}) = \frac{1}{\rho^k} \max_{\{\theta\}_{N-k}} \mathbb{E}_{N-k} \{ (X_N)^+ \}$$
$$= \frac{1}{\rho^k} \max_{\{\theta\}_{N-k}} \mathbb{E}_{N-k} \{ \frac{1}{2} (X_N + |X_N|) \}.$$  \hspace{1cm} (4.20)

Now, separating out the expectation, we have that by equation (4.16) $\mathbb{E}_{N-k}\{X_N\} = \rho^k X_{N-k}$. Using this result, we then have

$$V^*_N(S_{N-k}, X_{N-k}) = \frac{1}{2\rho^k} \left[ X_{N-k}\rho^k + \max_{\{\theta\}_{N-k}} \mathbb{E}_{N-k}\{|X_N|\} \right]$$
$$= \frac{1}{2\rho^k} \left[ X_{N-k}\rho^k + U(X_{N-k}, S_{N-k}) \right].$$

where clearly $U(S_{N-k}, X_{N-k}) = \max_{\{\theta\}_{N-k}} \mathbb{E}_{N-k}\{|X_N|\}$. □

What corollary 4.6 tells us is that if we can find the optimal strategy for the function $U$, then this will also be the optimal strategy for the value function $V^*$. This allows us to focus on an optimisation of the absolute value function instead of the plus function. This places our problem in the realm of $l_1$-norm optimisation which provides us a more general setting for our problem allowing us to use results from convex analysis and, more specifically, allows us to draw from the field of convex maximisation.
4.3. The Analytic Value of an Option on a Traded Account

As we have seen in section 2.4 of chapter 2 on dynamic programming, the most general method to solve the optimisation problem that is encountered in the valuation of options on traded accounts is to work through the decisions backwards in time. This involves first determining the optimal strategy at one step from maturity, then using this solution we determine the optimal strategy at two steps from maturity. The solution to the two step problem is then used to solve the three step problem and so on until the entire set of optimal strategies is known. Sometimes the objective function has the property of time separability, that is, if we have an objective function $J$, which is a function of many variables at different times, it may be written in the form

$$J(x_0, x_1, \ldots, x_{N-1}) = \sum_{i=0}^{N-1} J_i(x_i)$$

and thus the problem of optimising a multivariate objective function reduces to the much simpler problem of optimising a sum of univariate functions. Unfortunately, in practice not many functions satisfy this property and thus dynamic programming cannot provide this simplification. Our case is no exception and as such there is no simple one-step representation for the entire problem. Nonetheless, dynamic programming is still the most general method of solving this type of problem and if the dimension is sufficiently low it can still be used to perform optimisation. This section will focus on performing this task. Due to the exponentially increasing complexity associated with binomial trees, we will examine only the one and two-step cases. Whilst problems of this size are of little practical value, they are of sufficient size to highlight the problems associated with the valuation of these contracts and demonstrate why, in general, an optimal strategy for an $N$-step problem cannot be obtained analytically. In this section we will specifically examine the option on a traded account which has limits on the trading strategy $\theta$ given by the interval $[\alpha, \beta]$ where $\beta > \alpha$. 
4.3.1. The One-Step Problem. To conform with the application of dynamic programming to be used in the next section, we will use notation that describes the problem at one step from maturity, namely we start at node \((N - 1)\) and the contract terminates at node \(N\). The mathematics involved and conclusions derived are identical to a model which uses only one binomial step to describe the entire evolution of the stock price. Mathematically, the evolution of the stock price is given by

\[
S_N^{\gamma_1} = S_{N-1}(\rho + \bar{\sigma} \epsilon_{N-1})
\]

where \(\gamma_1\) represents a path of length one (in this case \(\gamma_1 \in \{u, d\}\)). This is represented diagrammatically in figure 4.1.

![Figure 4.1: The one step binomial tree for the stock price.](image)

Turning to the account value evolution which is derived from the stock price dynamics, we have that

\[
X_N^{\gamma_1} = \rho X_{N-1} + \bar{\sigma} \theta_{N-1} S_{N-1} \epsilon_{N-1}
\]

which is also displayed diagrammatically in figure 4.2. The variables \(X_N^u\) and \(X_N^d\) must be interpreted as the account values given that the stock has proceeded to move up or down respectively.

The value of an option on a traded account in this framework is given by the function

\[
V_{N-1}^\ast(S_{N-1}, X_{N-1}) = \max_{\theta_{N-1} \in \alpha, \beta} \rho^{-1} \mathbb{E}_{N-1} \{(X_N)^+\}
\]

\[
= \frac{1}{2} \rho^{-1} \left[ X_{N-1} \rho + U_{N-1}(X_{N-1}, S_{N-1}) \right]
\]
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\[ X_N^u = X_{N-1} \rho + \theta_{N-1} S_{N-1} \bar{\sigma} \]
\[ X_N^d = X_{N-1} \rho - \theta_{N-1} S_{N-1} \bar{\sigma} \]

\[ \text{Figure 4.2: The one step binomial tree for the trading account.} \]

where

\[ U_{N-1}(S_{N-1}, X_{N-1}) = \max_{\theta_{N-1} \in [a, b]} E_{N-1}\{ |X_N| \} \]

by corollary 4.6.

We will now proceed to determine the strategy (choice of \( \theta_{N-1} \)) which satisfies \( U_{N-1} \). First, let us ascertain the regions over which we will evaluate the objective function \( U_{N-1} \). To simplify the notation, define the variable\(^3\)

\[ \phi = \frac{\sigma \sqrt{\Delta t}}{1 + r \Delta t} = \frac{\bar{\sigma}}{\rho}. \tag{4.22} \]

It turns out that expressing the account values in terms of \( \phi \) is quite convenient as the requirement that \( d > 0 \) ensures that \( \phi \in (0, 1) \). Using this notation, we note that

\[ X_N^u > 0 \Rightarrow \rho X_{N-1}^u > -\theta_{N-1} S_{N-1} \bar{\sigma} \]
\[ \Rightarrow X_{N-1} > -\theta_{N-1} S_{N-1} \phi \]
\[ X_N^d > 0 \Rightarrow \rho X_{N-1} > \theta_{N-1} S_{N-1} \bar{\sigma} \]
\[ \Rightarrow X_{N-1} > \theta_{N-1} S_{N-1} \phi \]

which allows us to split the analysis into certain regions of \( X_{N-1} \) as shown in figure 4.3. However, the regions will be different depending on which value of \( \theta_{N-1} \) is chosen. To simplify the task, we make use

\(^3\)We point out that this variable is not the standard Normal PDF function used in the previous chapter.
of the fact that the maximum of a convex function is at one of its end points, thus we know that the optimal value of $\theta_{N-1}$ is $\alpha$ or $\beta$ and so we evaluate the objective function $U_{N-1}$ at these two values, then choose whichever is greater. This gives rise to two distinct cases depending on the relative magnitudes of $\alpha$ and $\beta$. We will consider each case in turn.

4.3.1.1. Case 1: $|\alpha| < |\beta|$. If $|\alpha| < |\beta|$, then figure 4.3 may be can be represented by figure 4.4 where we have evaluated the regions at $\theta_{N-1} = \alpha$ and $\theta_{N-1} = \beta$. As we can see, the figure splits into more regions due to the different magnitudes of $\alpha$ and $\beta$. We have numbered these regions 1 to 5 and we will now compute the function $U_{N-1}$ and ascertain the optimal strategy.

4Note that throughout this analysis we assume, as in figure 4.4 that $\alpha < 0$ and $\beta > 0$. These conditions are not required (they are simply used for illustration) and altering them will simply switch some of the signs without altering any of the outcomes.
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Region 1: $X_{N-1} > \beta \phi S_{N-1}$. First, we compute the value of $U_{N-1}$ given that $\theta_{N-1} = \beta$.

$$(U_{N-1}|\theta_{N-1} = \beta) = \frac{1}{2}[|\rho X_{N-1} + \beta \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \beta \bar{\sigma} S_{N-1}|]$$

$$= \frac{1}{2}[\rho X_{N-1} + \beta \bar{\sigma} S_{N-1} + \rho X_{N-1} - \beta \bar{\sigma} S_{N-1}]$$

$$= \rho X_{N-1}.$$ Comparing this with $(U_{N-1}|\theta_{N-1} = \alpha)$,

$$(U_{N-1}|\theta_{N-1} = \alpha) = \frac{1}{2}[|\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}|]$$

$$= \frac{1}{2}[\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1} + \rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}]$$

$$= \rho X_{N-1},$$

we see that both strategies yield the same outcome and thus in this region we are indifferent about which strategy is chosen. This is a direct result of the martingale property of the trading account which is preserved when the initial account value is sufficiently large such that the absolute value function (or equivalently the plus function) is not invoked.

Region 2: $-\alpha \phi S_{N-1} < X_{N-1} < \beta \phi S_{N-1}$. In this region, we have

$$(U_{N-1}|\theta_{N-1} = \beta) = \frac{1}{2}[|\rho X_{N-1} + \beta \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \beta \bar{\sigma} S_{N-1}|]$$

$$= \frac{1}{2}[\rho X_{N-1} + \beta \bar{\sigma} S_{N-1} - \rho X_{N-1} + \beta \bar{\sigma} S_{N-1}]$$

$$= \beta \bar{\sigma} S_{N-1}$$

and

$$(U_{N-1}|\theta_{N-1} = \alpha) = \frac{1}{2}[|\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}|]$$

$$= \frac{1}{2}[\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1} + \rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}]$$

$$= \rho X_{N-1},$$

thus $(U_{N-1}|\theta_{N-1} = \beta) > (U_{N-1}|\theta_{N-1} = \alpha)$ and hence $\theta_{N-1} = \beta$ is the optimal strategy in this region.

Region 3: $\alpha \phi S_{N-1} < X_{N-1} < -\alpha \phi S_{N-1}$. In this region, we have

$$(U_{N-1}|\theta_{N-1} = \beta) = \frac{1}{2}[|\rho X_{N-1} + \beta \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \beta \bar{\sigma} S_{N-1}|]$$
\[ U_{N-1}|\theta_{N-1} = \alpha \] = \frac{1}{2}[|\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}|]
\[ = \frac{1}{2}[-\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1} + \rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}] \]
\[ = -\alpha \bar{\sigma} S_{N-1}, \]

thus \((U_{N-1}|\theta_{N-1} = \beta) > (U_{N-1}|\theta_{N-1} = \alpha)\) and hence \(\theta_{N-1} = \beta\) is the optimal strategy in this region.

\textit{Region 4:} \(-\beta \phi S_{N-1} < X_{N-1} < \alpha \phi S_{N-1}\). In this region, we have

\[ (U_{N-1}|\theta_{N-1} = \beta) = \frac{1}{2}[|\rho X_{N-1} + \beta \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \beta \bar{\sigma} S_{N-1}|] \]
\[ = \frac{1}{2}[-\rho X_{N-1} - \beta \bar{\sigma} S_{N-1} + \rho X_{N-1} - \beta \bar{\sigma} S_{N-1}] \]
\[ = \beta \bar{\sigma} S_{N-1} \]

and

\[ (U_{N-1}|\theta_{N-1} = \alpha) = \frac{1}{2}[|\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}|] \]
\[ = \frac{1}{2}[-\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1} - \rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}] \]
\[ = -\rho X_{N-1}, \]

thus \((U_{N-1}|\theta_{N-1} = \beta) > (U_{N-1}|\theta_{N-1} = \alpha)\) and hence \(\theta_{N-1} = \beta\) is the optimal strategy in this region.

\textit{Region 5:} \(X_{N-1} < -\beta \phi S_{N-1}\). In this region, we have

\[ (U_{N-1}|\theta_{N-1} = \beta) = \frac{1}{2}[|\rho X_{N-1} + \beta \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \beta \bar{\sigma} S_{N-1}|] \]
\[ = \frac{1}{2}[-\rho X_{N-1} - \beta \bar{\sigma} S_{N-1} - \rho X_{N-1} + \beta \bar{\sigma} S_{N-1}] \]
\[ = -\rho X_{N-1} \]

and

\[ (U_{N-1}|\theta_{N-1} = \alpha) = \frac{1}{2}[|\rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}| + |\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1}|] \]
\[ = \frac{1}{2}[-\rho X_{N-1} - \alpha \bar{\sigma} S_{N-1} - \rho X_{N-1} + \alpha \bar{\sigma} S_{N-1}] \]
\[ = -\rho X_{N-1}, \]
thus \((U_{N-1}|\theta_{N-1} = \beta) = (U_{N-1}|\theta_{N-1} = \alpha)\) and hence we are indifferent about the choice of \(\theta_{N-1}\) in this region.

Combining the optimal strategies derived from all these regions, we have that for this case it is optimal to set \(\theta_{N-1} = \beta\) always. The resulting outcomes for \(U_{N-1}\) are

\[
U_{N-1} = \begin{cases} 
\beta \sigma S_{N-1} & \text{if } |X_{N-1}| < \beta \phi S_{N-1} \\
|\rho X_{N-1}| & \text{if } |X_{N-1}| > \beta \phi S_{N-1}
\end{cases}
\]

which provides option values, \(V^*_N\), of

\[
V^*_N = \begin{cases} 
\frac{1}{2} (X_{N-1} + \beta \phi S_{N-1}) & \text{if } |X_{N-1}| < \beta \phi S_{N-1} \\
(X_{N-1})^+ & \text{if } |X_{N-1}| > \beta \phi S_{N-1}
\end{cases}
\]

Given that we have solved the problem of valuing an option on a traded account on a one step binomial tree when \(|\beta| > |\alpha|\), we now turn to the second case when \(|\alpha| > |\beta|\).

4.3.1.2. Case 2: \(|\alpha| > |\beta|\). The analysis undertaken in the previous section for case 1 is identical to what is to be undertaken for case 2. Examining figure 4.4, it is clear that the only difference is that the \(\alpha\) and \(\beta\)'s will switch place.\(^5\) This means that in terms of the analysis performed previously, the terms containing \(\alpha\) will play the role that the terms involving \(\beta\) did. For this reason we omit the complete analysis and note that regions where we were indifferent remain so under this new scenario, and regions where \(\theta_{N-1} = \beta\) was optimal will now have \(\theta_{N-1} = \alpha\) as the optimal strategy. Thus the function \(U_{N-1}\) is given by

\[
U_{N-1} = \begin{cases} 
-\alpha \sigma S_{N-1} & \text{if } |X_{N-1}| < -\alpha \phi S_{N-1} \\
|\rho X_{N-1}| & \text{if } |X_{N-1}| > -\alpha \phi S_{N-1}
\end{cases}
\]

which produces option values

\[
V^*_N = \begin{cases} 
\frac{1}{2} (X_{N-1} - \alpha \phi S_{N-1}) & \text{if } |X_{N-1}| < -\alpha \phi S_{N-1} \\
(X_{N-1})^+ & \text{if } |X_{N-1}| > -\alpha \phi S_{N-1}
\end{cases}
\]

Perhaps the most important aspect to take from this analysis of the one step model is the optimal strategy as it will be utilised in the

\(^5\)Note that in this case \(|\alpha| > |\beta|\) naturally implies that \(\alpha < 0\) since we always have the condition that \(\beta > \alpha\).
two step model and, in principle, is required for the analytic solution of a model consisting of an arbitrary number of steps. This optimal strategy may be summarised by the statement that it is optimal to take a position of the highest magnitude possible when one step from maturity, or expressed mathematically

\[
\theta_{N-1}^* = \begin{cases} 
\alpha & \text{if } |\alpha| > |\beta| \\
\beta & \text{if } |\alpha| < |\beta|
\end{cases}
\]  

(4.23)

where \(\theta_{N-1}^*\) is the optimal strategy at one step from maturity. Comparing this strategy with that obtained for the continuous time model, we notice that these strategies are already different from one another.\(^6\) We point out that while this strategy is quite simple, the analysis required to obtain it was somewhat complicated requiring the “brute-force” evaluation of the option value function over all possible regions of the initial account value. We therefore expect that as we add complexity by increasing the number of steps in the model, the effort required to obtain the optimal strategy will also increase substantially. As we show in the two step case in the following section, this is indeed the case.

### 4.3.2. The Two-Step Problem

In this section we will increase the number of binomial steps used to model the evolution of the stock price from one to two. As will be shown, this seemingly small increase in complexity of the model will in fact have a significant impact on the complexity of the optimal strategy function. The two-step model for the stock evolution is depicted in figure 4.5 and the two-step model for the account value is depicted in figure 4.6.

As stated previously, the value of the option on a traded account, \(V_{N-2}^*\), under this framework is given by

\[
V_{N-2}^*(S_{N-2}, X_{N-2}) = \max_{\{\theta_{N-2}, \theta_{N-1}^u, \theta_{N-1}^d\}} \rho^{-2}\mathbb{E}_{N-2}\{(X_N)^+\}
\]

\(^6\)Recall that the optimal strategy for this option in continuous time is \(\theta_t = \alpha I(x > \frac{1}{2}(\alpha + \beta)s) + \beta I(x \leq \frac{1}{2}(\alpha + \beta)s)\) where \(s = S_t\) and \(x = X_t\).
and using corollary 4.6, we can find the optimal strategy by considering the function

$$U_{N-2}(S_{N-2}, X_{N-2}) = \max_{\{\theta_{N-2}, \theta^u_{N-1}, \theta_d^{d_{N-1}}\}} \mathbb{E}_{N-2}\{|X_N|\}$$  \hspace{1cm} (4.24)$$

To find the value of $\theta_{N-2}, \theta^u_{N-1}$ and $\theta_d^{d_{N-1}}$ which satisfies $U_{N-2}$, we will employ the principle of optimality by using the results derived from the previous section for the one-step model. This will give rise to two separate cases depending on the values of $\alpha$ and $\beta$. First, we will consider the case when $|\beta| > |\alpha|$. Using result (4.23), we have that
regardless of the value of $X_{N-1}$, the optimal strategy is to set $\theta_{N-1} = \beta$. This implies that $\theta_{N-1} = \theta_{N-1}^d = \beta$ and hence problem 4.24 reduces to

$$U_{N-2} = \max_{\theta_{N-2}} \frac{1}{4} \left[ |\rho^2 X_{N-2} + \theta_{N-2} \rho \sigma S_{N-2} + \beta (\rho + \sigma) \sigma S_{N-2}| + |\rho^2 X_{N-2} + \theta_{N-2} \rho \sigma S_{N-2} - \beta (\rho + \sigma) \sigma S_{N-2}| + |\rho^2 X_{N-2} - \theta_{N-2} \rho \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}| + |\rho^2 X_{N-2} - \theta_{N-2} \rho \sigma S_{N-2} - \beta (\rho - \sigma) \sigma S_{N-2}| \right]$$

$$= \rho^2 S_{N-2} \max_{\theta_{N-2}} \frac{1}{4} \left[ |Z_{N-2} + \theta_{N-2} \phi + \beta (\phi + \phi^2)| + |Z_{N-2} + \theta_{N-2} \phi - \beta (\phi + \phi^2)| + |Z_{N-2} - \theta_{N-2} \phi + \beta (\phi - \phi^2)| + |Z_{N-2} - \theta_{N-2} \phi - \beta (\phi - \phi^2)| \right]$$

$$= \rho^2 S_{N-2} \max_{\theta_{N-2}} \frac{1}{4} \left[ |Z_{N-2}^{uu}(Z_{N-2}, \theta_{N-2})| + |Z_{N-2}^{ud}(Z_{N-2}, \theta_{N-2})| + |Z_{N-2}^{du}(Z_{N-2}, \theta_{N-2})| + |Z_{N-2}^{dd}(Z_{N-2}, \theta_{N-2})| \right]$$

where $Z_{N-2} = \frac{X_{N-2}}{S_{N-2}}$ and $Z_N^{\gamma}$ is the function describing the terminal account value reduced by the factor $\rho^2 S_{N-2}$. Splitting the variable $Z_{N-2}$ into regions which we can directly evaluate $U_{N-2}$ at $\theta_{N-2} = \alpha$ and $\beta$ now involves the parameter $\phi$ in a quadratic manner. Figure 4.7 depicts the values of $Z_{N-2}$ for which each path is equal to zero for the special case of a vacation call option ($\alpha = 0$, $\beta = 1$), and hence this will define the regions over which we need to consider the evaluation of the function $U_{N-2}$ for this special case. As we can see, the rank of the paths depends on the value of $\phi$ and thus the optimal strategy will also depend on this parameter. We now present the general case of the two-step option on a traded account to make the ideas presented via the special case in figure 4.7 more concrete.

Firstly, given $\theta_{N-1}^u = \theta_{N-1}^d = \beta$, we define each individual path for $\theta_{N-2} = \alpha$ or $\beta$. When $\theta_{N-2} = \beta$, we have

$$Z_{N}^{uu}(Z_{N-2}, \beta) = Z_{N-2} + 2 \beta \phi + \beta \phi^2$$
$$Z_{N}^{ud}(Z_{N-2}, \beta) = Z_{N-2} - \beta \phi^2$$
$$Z_{N}^{du}(Z_{N-2}, \beta) = Z_{N-2} - \beta \phi^2$$
$$Z_{N}^{dd}(Z_{N-2}, \beta) = Z_{N-2} - \beta \phi^2$$
4.3. ANALYTIC VALUATION

Figure 4.7: The values for $Z_{N-2}$ for which the paths $Z_{N}^{\gamma}$ are equal to zero. Note that in this case the values vary with the volatility parameter $\phi$ which leads to paths crossing each other. In this case $\alpha = 0$ and $\beta = 1$. We also point out that $Z_{ud}^{dd}(\beta) = Z_{du}^{dd}(\beta)$.

$Z_{N}^{dd}(Z_{N-2}, \beta) = Z_{N-2} - 2\beta \phi + \beta \phi^2$

and when $\theta_{N-2} = \alpha$,

$Z_{N}^{uu}(Z_{N-2}, \alpha) = Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2$

$Z_{N}^{ud}(Z_{N-2}, \alpha) = Z_{N-2} + \alpha \phi - \beta \phi - \beta \phi^2$

$Z_{N}^{du}(Z_{N-2}, \alpha) = Z_{N-2} - \alpha \phi + \beta \phi - \beta \phi^2$

$Z_{N}^{dd}(Z_{N-2}, \alpha) = Z_{N-2} - \alpha \phi - \beta \phi + \beta \phi^2$.

Whilst it is rather simple to identify the rank of some of the paths (for example it is clear that $Z_{N}^{uu}(Z_{N-2}, \beta)$ is always the largest) some of the other paths are more difficult to rank. For example, consider the comparison of the paths $Z_{N}^{dd}(Z_{N-2}, \beta)$ and $Z_{N}^{ud}(Z_{N-2}, \alpha)$:

$Z_{N}^{dd}(Z_{N-2}, \beta) > Z_{N}^{ud}(Z_{N-2}, \alpha) \Rightarrow -\beta \phi + \beta \phi^2 > \alpha \phi - \beta \phi^2$
\[ \phi(-\alpha - \beta + 2\beta\phi) > 0 \]
\[ \Rightarrow \phi > \frac{\alpha + \beta}{2\beta} \]

since \( \phi > 0 \). Thus the rank of these two paths is dependent on the values of \( \alpha \) and \( \beta \) and how they compare with the value of the parameter \( \phi \). Also consider the comparison of the paths \( Z_{N-2}^{uu}(Z_{N-2}, \alpha) \) and \( Z_{N-2}^{du}(Z_{N-2}, \alpha) \). This results in the following criteria:

\[ Z_{N}^{uu}(Z_{N-2}, \alpha) > Z_{N}^{du}(Z_{N-2}, \alpha) \Rightarrow \alpha\phi + \beta\phi^2 > -\alpha\phi - \beta\phi^2 \]
\[ \Rightarrow 2\phi(\alpha + \beta\phi) > 0 \]
\[ \Rightarrow \phi > \frac{-\alpha}{\beta}. \]

Now, the value of \( \frac{-\alpha}{\beta} \) is valid for the parameter \( \phi \) only if \( \alpha < 0 \) and \( |\alpha| < |\beta| \), however it is invalid if \( \alpha > 0 \) since \( \phi \in (0, 1) \). Given these specific examples of cases where the rank of the paths switches, we compute the pair-wise comparisons between each of the paths for \( \theta_{N-2} = \alpha \) and \( \beta \) so that we may find all cases where the paths ranks switch. The results are summarised in table 1. We note that in order to save space, we have omitted \( Z_{N-2} \) from the arguments of these functions, though it is to be understood that it is still included.

<table>
<thead>
<tr>
<th>( \text{LE} )</th>
<th>( \text{TE} )</th>
<th>( Z_{N}^{uu}(\beta) )</th>
<th>( Z_{N}^{ud}(\beta) )</th>
<th>( Z_{N}^{du}(\beta) )</th>
<th>( Z_{N}^{dd}(\beta) )</th>
<th>( Z_{N}^{uu}(\alpha) )</th>
<th>( Z_{N}^{ud}(\alpha) )</th>
<th>( Z_{N}^{du}(\alpha) )</th>
<th>( Z_{N}^{dd}(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_{N}^{uu}(\beta) )</td>
<td>=</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>( Z_{N}^{ud}(\beta) )</td>
<td>=</td>
<td>=</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>( \frac{\alpha + \beta}{2\beta} &gt; \phi )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Z_{N}^{du}(\beta) )</td>
<td>=</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>( \frac{\alpha + \beta}{2\beta} &gt; \phi )</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( Z_{N}^{dd}(\beta) )</td>
<td>=</td>
<td>F</td>
<td>( \frac{\alpha + \beta}{2\beta} &lt; \phi )</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Z_{N}^{uu}(\alpha) )</td>
<td>=</td>
<td>T</td>
<td>( -\frac{\alpha}{\beta} &lt; \phi )</td>
<td>T</td>
<td></td>
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<tr>
<td>( Z_{N}^{ud}(\alpha) )</td>
<td>=</td>
<td>( \frac{\alpha}{\beta} &gt; \phi )</td>
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<tr>
<td>( Z_{N}^{du}(\alpha) )</td>
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<tr>
<td>( Z_{N}^{dd}(\alpha) )</td>
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</tbody>
</table>

**Table 1:** Pairwise comparisons for the order of the paths for the two-step model. The table is to be read by first choosing an entry from the left column (LE) and then considering the inequality with an entry from the top row (TE) of the form \( LE > TE \). The labels T, F, = and some inequality relating \( \alpha \), \( \beta \) and \( \phi \) indicate that the inequality is true always, false always, equality holds always and the inequality holds true if the stated inequality is satisfied.
Due to the added complexity provided by the fact that the paths may change their rank depending on the value of $\phi$, we need to consider several different situations which we examine below.

4.3.2.1. **Case 1a:** $\alpha > 0$, $|\beta| > |\alpha|$. Using Table 1, we can show that when $\phi \in (0, \frac{\alpha}{\beta})$ the paths have the ranking stated below:\(^7\)

\[ Z_N^u(\beta) > Z_N^u(\alpha) > Z_N^{du}(\alpha) = Z_N^{ud}(\beta) > Z_N^{ud}(\alpha) > Z_N^{dd}(\alpha) > Z_N^{dd}(\beta). \]

Thus, we may easily split the analysis into regions for $Z_{N-2}$ and compute the option values to obtain the optimal strategy.

**Region 1:** $Z_{N-2} > 2\beta \phi - \beta \phi^2$. In this case, the outcome for the account value over any path is positive, thus we have

\[ (U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \right. \]

\[ + Z_{N-2} + \beta \phi - \beta (\phi + \phi^2) \]

\[ + Z_{N-2} - \beta \phi + \beta (\phi - \phi^2) \]

\[ + Z_{N-2} - \beta \phi - \beta (\phi - \phi^2) \]

\[ = \rho^2 X_{N-2}, \]

\[ (U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \right. \]

\[ + Z_{N-2} + \alpha \phi - \beta (\phi + \phi^2) \]

\[ + Z_{N-2} - \alpha \phi + \beta (\phi - \phi^2) \]

\[ + Z_{N-2} - \alpha \phi - \beta (\phi - \phi^2) \]

\[ = \rho^2 X_{N-2} \]

and thus we conclude that in this region, we are indifferent about the choice of $\theta_{N-2}$.

**Region 2:** $(\alpha + \beta) \phi - \beta \phi^2 < Z_{N-2} < 2\beta \phi - \beta \phi^2$. In this case, we have the following outcomes

\[ (U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \right. \]

\[ + Z_{N-2} + \beta \phi - \beta (\phi + \phi^2) \]

\[ + Z_{N-2} - \beta \phi + \beta (\phi - \phi^2) \]

\[ - Z_{N-2} + \beta \phi + \beta (\phi - \phi^2) \]

\[ = \frac{1}{2} [\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta (\rho - \bar{\sigma}) \bar{\sigma} S_{N-2}], \]

\(^7\)Note that because $\alpha > 0$, the condition $\phi < \frac{-\alpha}{\beta}$ is invalid. Also note that $\frac{\alpha}{\beta} < \frac{\alpha + \beta}{2\beta}$.\]
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\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \\
+Z_{N-2} + \alpha \phi - \beta (\phi + \phi^2) \\
+Z_{N-2} - \alpha \phi + \beta (\phi - \phi^2) \\
+Z_{N-2} - \alpha \phi - \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \rho^2 X_{N-2}.
\]

Now, since \(\frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta (\rho - \bar{\sigma}) \bar{\sigma} S_{N-2}] > \rho^2 X_{N-2}\) implies that \(Z_{N-2} < 2\beta \phi - \beta \phi^2\) which is known to be true, then it follows that in this region, \(\theta_{N-2} = \beta\) is optimal.

**Region 3:** \((-\alpha + \beta) \phi + \beta \phi^2 < Z_{N-2} < (\alpha + \beta) \phi - \beta \phi^2\). In this region, we have that

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \\
+Z_{N-2} + \beta \phi - \beta (\phi + \phi^2) \\
+Z_{N-2} - \beta \phi + \beta (\phi - \phi^2) \\
-Z_{N-2} + \beta \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta (\rho - \bar{\sigma}) \bar{\sigma} S_{N-2}],
\]

and thus again it is optimal to take \(\theta_{N-2} = \beta\) in this region.

**Region 4:** \(\beta \phi^2 < Z_{N-2} < (-\alpha + \beta) \phi + \beta \phi^2\).

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \\
+Z_{N-2} + \beta \phi - \beta (\phi + \phi^2) \\
+Z_{N-2} - \beta \phi + \beta (\phi - \phi^2) \\
-Z_{N-2} + \beta \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta (\rho - \bar{\sigma}) \bar{\sigma} S_{N-2}],
\]
\[ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \alpha \phi + \beta(\phi + \phi^2) \\ -Z_{N-2} - \alpha \phi + \beta(\phi + \phi^2) \\ +Z_{N-2} - \alpha \phi + \beta(\phi - \phi^2) \\ -Z_{N-2} + \alpha \phi + \beta(\phi - \phi^2) \end{bmatrix} \]

\[ = \frac{1}{2} [\beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma})\bar{\sigma} S_{N-2}], \]

and since \( \frac{1}{2} [\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma})\bar{\sigma} S_{N-2}] > \frac{1}{2} [\beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma})\bar{\sigma} S_{N-2}] \)
implies that \( Z_{N-2} > \beta \phi^2 \) which is known to hold true, then \( \theta_{N-2} = \beta \) is optimal in this region.

**Region 5:** \( (\alpha - \beta)\phi + \beta \phi^2 < Z_{N-2} < \beta \phi^2. \)

\[ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \beta \phi + \beta(\phi + \phi^2) \\ -Z_{N-2} - \beta \phi + \beta(\phi + \phi^2) \\ -Z_{N-2} + \beta \phi + \beta(\phi - \phi^2) \end{bmatrix} \]

\[ = \frac{1}{2} [-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2}], \]

\[ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \alpha \phi + \beta(\phi + \phi^2) \\ -Z_{N-2} - \alpha \phi + \beta(\phi + \phi^2) \\ +Z_{N-2} - \alpha \phi + \beta(\phi - \phi^2) \end{bmatrix} \]

\[ = \frac{1}{2} [\beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma})\bar{\sigma} S_{N-2}], \]

and \( \frac{1}{2} [-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2}] > \frac{1}{2} [\beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma})\bar{\sigma} S_{N-2}] \Rightarrow Z_{N-2} < \beta \phi^2 \) which is true in this region and hence \( \theta_{N-2} = \beta \) is optimal in this region also.

**Region 6:** \( -(\alpha + \beta)\phi - \beta \phi^2 < Z_{N-2} < (\alpha - \beta)\phi + \beta \phi^2. \)

\[ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \beta \phi + \beta(\phi + \phi^2) \\ -Z_{N-2} - \beta \phi + \beta(\phi + \phi^2) \\ -Z_{N-2} + \beta \phi + \beta(\phi - \phi^2) \end{bmatrix} \]

\[ = \frac{1}{2} [-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2}], \]
(\(U_{N-2} | \theta_{N-2} = \alpha\)) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} + \alpha \phi - \beta (\phi - \phi^2) \\ -Z_{N-2} + \alpha \phi + \beta (\phi - \phi^2) \end{bmatrix}

= \frac{1}{2} [-\rho^2 X_{N-2} + \alpha \rho \bar{\sigma} S_{N-2} + \beta (\rho + \bar{\sigma}) \bar{\sigma} S_{N-2}],

and in this case it is clear that since \(\beta > \alpha\), it is optimal to set \(\theta_{N-2} = \beta\).

Region 7: \(-2 \beta \phi - \beta \phi^2 < Z_{N-2} < -(\alpha + \beta) \phi - \beta \phi^2\).

(\(U_{N-2} | \theta_{N-2} = \beta\)) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} - \beta \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} + \beta \phi - \beta (\phi - \phi^2) \\ -Z_{N-2} + \beta \phi + \beta (\phi - \phi^2) \end{bmatrix}

= \frac{1}{2} [-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta (\rho + \bar{\sigma}) \bar{\sigma} S_{N-2}],

(\(U_{N-2} | \theta_{N-2} = \alpha\)) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \alpha \phi - \beta (\phi + \phi^2) \\ -Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} + \alpha \phi - \beta (\phi - \phi^2) \\ -Z_{N-2} + \alpha \phi + \beta (\phi - \phi^2) \end{bmatrix}

= -\rho^2 X_{N-2},

and \(\frac{1}{2} [-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta (\rho + \bar{\sigma}) \bar{\sigma} S_{N-2}] > -\rho^2 X_{N-2} \Rightarrow -2 \beta \phi - \beta \phi^2 < Z_{N-2}\) which is true and thus \(\theta_{N-2} = \beta\) is again optimal.

Region 8: \(Z_{N-2} < -2 \beta \phi - \beta \phi^2\). In this case all paths are negative, so we have

(\(U_{N-2} | \theta_{N-2} = \beta\)) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \beta (\phi + \phi^2) \\ -Z_{N-2} - \beta \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} + \beta \phi - \beta (\phi - \phi^2) \\ -Z_{N-2} + \beta \phi + \beta (\phi - \phi^2) \end{bmatrix}

= -\rho^2 X_{N-2},

(\(U_{N-2} | \theta_{N-2} = \alpha\)) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \alpha \phi - \beta (\phi + \phi^2) \\ -Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} + \alpha \phi - \beta (\phi - \phi^2) \\ -Z_{N-2} + \alpha \phi + \beta (\phi - \phi^2) \end{bmatrix}
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= −ρ^2 X_{N-2},

and thus again we are indifferent about the choice of θ_{N-2}. Thus we can say that when \( \phi \in \left(0, \frac{\alpha}{\beta}\right) \), the optimal strategy is to take \( \theta_{N-2} = \beta \) always.

Now we will consider the problem of determining the optimal strategy for the domain \( \phi \in \left(\frac{\alpha+\beta}{2\beta}, \infty\right) \). Using table 1, we can again rank the paths, and for this new range of \( \phi \) the order is:

\[
Z^u_N(\beta) > Z^u_N(\alpha) > Z^{du}_N(\alpha) = Z^{dd}_N(\beta) > Z^{ud}_N(\alpha) > Z^{dd}_N(\beta).
\]

From this order, we may now define our regions for computing the objective function.

**Region 1:** \( 2\beta \phi - \beta \phi^2 < Z_{N-2} \). In this region all paths are positive regardless of the choice of \( \theta_{N-2} \), thus

\[
(U_{N-2}|\theta_{N-2} = \beta) = \rho^2 X_{N-2}
\]

and thus we are indifferent about the choice of \( \theta_{N-2} \).

**Region 2:** \( (\alpha + \beta) \phi + \beta \phi^2 < Z_{N-2} < 2\beta \phi - \beta \phi^2 \). In this region, the function \( U_{N-2} \) is given by

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ Z_{N-2} + \beta \phi + \beta(\phi + \phi^2) \right. \\
+ Z_{N-2} - \beta \phi - \beta(\phi + \phi^2) \\
\left. + Z_{N-2} - \beta \phi + \beta(\phi - \phi^2) \\
- Z_{N-2} + \beta \phi + \beta(\phi - \phi^2) \right]
\]

\[
= \frac{1}{2} \left[ \rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma}) \bar{\sigma} S_{N-2} \right],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \rho^2 X_{N-2}.
\]

Since \( \frac{1}{2} \left[ \rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho - \bar{\sigma}) \bar{\sigma} S_{N-2} \right] > \rho^2 X_{N-2} \Rightarrow Z_{N-2} < 2\beta \phi - \beta \phi^2 \), we have that \( \theta_{N-2} = \beta \) is optimal in this region.
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Region 3: \((\alpha + \beta) \phi - \beta \phi^2 < Z_{N-2} < (-\alpha + \beta) \phi + \beta \phi^2\). In this region we have

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \\
+ Z_{N-2} + \beta \phi - \beta (\phi + \phi^2) \\
+ Z_{N-2} - \beta \phi + \beta (\phi - \phi^2) \\
- Z_{N-2} + \beta \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2} [\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \\
- Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\
+ Z_{N-2} - \alpha \phi + \beta (\phi - \phi^2) \\
+ Z_{N-2} - \alpha \phi - \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2} [\rho^2 X_{N-2} - \alpha \rho \sigma S_{N-2} + \beta (\rho + \sigma) \sigma S_{N-2}],
\]

and since \(\frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}] > \frac{1}{2}[\rho^2 X_{N-2} - \alpha \rho \sigma S_{N-2} + \beta (\rho + \sigma) \sigma S_{N-2}]\) implies that \(\phi < \frac{\alpha + \beta}{2\beta}\), which is true in this case, we have that \(\theta_{N-2} = \beta\) is again optimal.

Region 4: \(\beta \phi^2 < Z_{N-2} < (\alpha + \beta) \phi - \beta \phi^2\). In this region, the outcomes are

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \\
+ Z_{N-2} + \beta \phi - \beta (\phi + \phi^2) \\
+ Z_{N-2} - \beta \phi + \beta (\phi - \phi^2) \\
- Z_{N-2} + \beta \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2} [\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \\
- Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\
+ Z_{N-2} - \alpha \phi + \beta (\phi - \phi^2) \\
- Z_{N-2} + \alpha \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2} [\beta (\rho + \sigma) \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}],
\]

and since \(\frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}] > \frac{1}{2}[\beta (\rho + \sigma) \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}]\) implies that \(Z_{N-2} > \beta \phi^2\) which is known to hold true, then \(\theta_{N-2} = \beta\) is optimal in this region.
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Region 5: $(\alpha - \beta)\phi + \beta \phi^2 < Z_{N-2} < \beta \phi^2$.

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta (\phi + \phi^2) \\
-Z_{N-2} - \beta \phi + \beta (\phi + \phi^2) \\
-Z_{N-2} + \beta \phi - \beta (\phi - \phi^2) \\
-Z_{N-2} + \beta \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho + \sigma) \sigma S_{N-2}],
\]

and \( \frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho + \sigma) \sigma S_{N-2}] > \frac{1}{2}[\beta (\rho + \sigma) \sigma S_{N-2} + \beta (\rho - \sigma) \sigma S_{N-2}] \Rightarrow Z_{N-2} < \beta \phi^2 \) which is true in this region and hence \( \theta_{N-2} = \beta \) is optimal in this region also.

Region 6: \( -(\alpha + \beta)\phi - \beta \phi^2 < Z_{N-2} < (\alpha - \beta)\phi + \beta \phi^2 \).

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \\
-Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\
-Z_{N-2} + \alpha \phi - \beta (\phi - \phi^2) \\
-Z_{N-2} + \alpha \phi + \beta (\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[\rho^2 X_{N-2} + \beta \rho \sigma S_{N-2} + \beta (\rho + \sigma) \sigma S_{N-2}],
\]

and it this case it is clear that since \( \beta > \alpha \), it is optimal to set \( \theta_{N-2} = \beta \).
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Region 7: $-2\beta\phi - \beta\phi^2 < Z_{N-2} < -(\alpha + \beta)\phi - \beta\phi^2$.

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta\phi + \beta(\phi + \phi^2) \\
-Z_{N-2} - \beta\phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \beta\phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \beta\phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[= \frac{1}{2}[-\rho^2 X_{N-2} + \beta\rho \bar{s} S_{N-2} + \beta(\rho + \bar{s})\bar{s} S_{N-2}],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \alpha\phi - \beta(\phi + \phi^2) \\
-Z_{N-2} - \alpha\phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \alpha\phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \alpha\phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[= -\rho^2 X_{N-2},
\]

and $\frac{1}{2}[-\rho^2 X_{N-2} + \beta\rho \bar{s} S_{N-2} + \beta(\rho + \bar{s})\bar{s} S_{N-2}] > -\rho^2 X_{N-2} \Rightarrow -2\beta\phi - \beta\phi^2 < Z_{N-2}$ which is true and thus $\theta_{N-2} = \beta$ is again optimal.

Region 8: $Z_{N-2} < -2\beta\phi - \beta\phi^2$. In this case all paths are negative, so we have

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta\phi - \beta(\phi + \phi^2) \\
-Z_{N-2} - \beta\phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \beta\phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \beta\phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[= -\rho^2 X_{N-2},
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \alpha\phi - \beta(\phi + \phi^2) \\
-Z_{N-2} - \alpha\phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \alpha\phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \alpha\phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[= -\rho^2 X_{N-2},
\]

and thus we are now indifferent about the choice of $\theta_{N-2}$. Thus we can say that when $\phi \in \left(\frac{\alpha}{\beta}, \frac{\alpha + \beta}{2\beta}\right)$, the optimal strategy is to take $\theta_{N-2} = \beta$ always, just the same as it was for $\phi \in (0, \frac{\alpha}{\beta})$.

Now, to continue our derivation of the optimal strategy for the two-step model when $\alpha > 0$ and $|\alpha| < |\beta|$, we now consider how the paths are ranked when $\phi \in (\frac{\alpha + \beta}{2\beta}, 1)$. Using table 1, we have that the path
order in this scenario is given by:

\[ Z^{uu}_N(\beta) > Z^{uu}_N(\alpha) > Z^{dd}_N(\alpha) > Z^{dd}_N(\beta) = Z^{ud}_N(\beta) > Z^{dd}_N(\beta) > Z^{ud}_N(\alpha). \]

We are now able to break our analysis into regions as we have done previously so as to determine the optimal strategy within that region.

Region 1: \((-\alpha + \beta)\phi + \beta\phi^2 < Z_{N-2}\). In this case all outcomes are positive and thus we simply have

\[
(U_{N-2}|\theta_{N-2} = \beta) = \rho^2 X_{N-2} \\
(U_{N-2}|\theta_{N-2} = \alpha) = \rho^2 X_{N-2}
\]

and hence we are indifferent as to the choice of \(\theta_{N-2}\) in this region.

Region 2: \(2\beta\phi - \beta\phi^2 < Z_{N-2} < (-\alpha + \beta)\phi + \beta\phi^2\). In this region, we have

\[
(U_{N-2}|\theta_{N-2} = \beta) = \rho^2 X_{N-2}, \\
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{pmatrix} Z_{N-2} + \alpha\phi + \beta(\phi + \phi^2) \\ -Z_{N-2} - \alpha\phi + \beta(\phi + \phi^2) \\ +Z_{N-2} - \alpha\phi + \beta(\phi - \phi^2) \\ +Z_{N-2} - \alpha\phi - \beta(\phi - \phi^2) \end{pmatrix}
\]

\[= \frac{1}{2}[\rho^2 X_{N-2} - \alpha\rho\bar{\sigma}S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma}S_{N-2}],\]

and making the comparison \(\rho^2 X_{N-2} > \frac{1}{2}[\rho^2 X_{N-2} - \alpha\rho\bar{\sigma}S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma}S_{N-2}] \Rightarrow Z_{N-2} > (-\alpha + \beta)\phi + \beta\phi^2\) which is false in this region. This means that the optimal strategy in this region is \(\theta_{N-2} = \alpha\).

Region 3: \(\beta\phi^2 < Z_{N-2} < 2\beta\phi - \beta\phi^2\).

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{pmatrix} Z_{N-2} + \beta\phi + \beta(\phi + \phi^2) \\ +Z_{N-2} + \beta\phi - \beta(\phi + \phi^2) \\ +Z_{N-2} - \beta\phi + \beta(\phi - \phi^2) \\ -Z_{N-2} + \beta\phi + \beta(\phi - \phi^2) \end{pmatrix}
\]

\[= \frac{1}{2}[\rho^2 X_{N-2} + \beta\rho\bar{\sigma}S_{N-2} + \beta(\rho - \bar{\sigma})\bar{\sigma}S_{N-2}],\]
Now, comparing these function outcomes, we have that
\[ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + \alpha \phi + \beta (\phi + \phi^2) \\ -Z_{N-2} - \alpha \phi + \beta (\phi + \phi^2) \\ +Z_{N-2} - \alpha \phi + \beta (\phi - \phi^2) \\ +Z_{N-2} - \alpha \phi - \beta (\phi - \phi^2) \end{array} \right] \]

and since \( \frac{1}{2}[\rho^2 X_{N-2} + \alpha \rho S_{N-2} + \beta (\rho - \bar{\sigma}) S_{N-2}] > \frac{1}{2}[\rho^2 X_{N-2} - \alpha \rho S_{N-2} + \beta (\rho + \bar{\sigma}) S_{N-2}] \Rightarrow \frac{\alpha + \beta}{2\bar{\sigma}} > \phi \) which is false, we again have that the optimal strategy is to set \( \theta_{N-2} = \alpha \).

Region 4: \( (\alpha + \beta)\phi - \beta \phi^2 < Z_{N-2} < \beta \phi^2 \).
which is also true meaning that this position also lies above the lower bound of our region. This means that within this interval the optimal strategy changes. If \( Z_{N-2} > \frac{1}{2}(\alpha + \beta) \phi \), then \( \theta_{N-2} = \alpha \) is optimal, and if \( Z_{N-2} < \frac{1}{2}(\alpha + \beta) \phi \), then \( \theta_{N-2} = \beta \) is optimal.\(^8\)

Region 5: \((\alpha - \beta) \phi + \beta \phi^2 < Z_{N-2} < (\alpha + \beta) \phi - \beta \phi^2\).

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} - \beta \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \beta \phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \beta \phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[-\rho^2 X_{N-2} + \beta \rho \bar{S}_{N-2} + \beta(\rho + \bar{S}) \bar{S}_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} - \alpha \phi + \beta(\phi + \phi^2) \\
+Z_{N-2} - \alpha \phi + \beta(\phi - \phi^2) \\
-Z_{N-2} + \alpha \phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[\beta(\rho + \bar{S}) \bar{S}_{N-2} + \beta(\rho - \bar{S}) \bar{S}_{N-2}].
\]

Since \( \frac{1}{2}[-\rho^2 X_{N-2} + \beta \rho \bar{S}_{N-2} + \beta(\rho + \bar{S}) \bar{S}_{N-2}] > \frac{1}{2}[\beta(\rho + \bar{S}) \bar{S}_{N-2} + \beta(\rho - \bar{S}) \bar{S}_{N-2}] \Rightarrow Z_{N-2} < \beta \phi^2 \), we have that \( \theta_{N-2} = \beta \) is optimal.

Region 6: \(-(\alpha + \beta) \phi - \beta \phi^2 < Z_{N-2} < (\alpha - \beta) \phi + \beta \phi^2\).

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} - \beta \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \beta \phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \beta \phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[-\rho^2 X_{N-2} + \beta \rho \bar{S}_{N-2} + \beta(\rho + \bar{S}) \bar{S}_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} - \alpha \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \alpha \phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \alpha \phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[-\rho^2 X_{N-2} + \alpha \rho \bar{S}_{N-2} + \beta(\rho + \bar{S}) \bar{S}_{N-2}],
\]

and in this case it is clear that \( \theta_{N-2} = \beta \) is optimal.

\(^8\)If \( Z_{N-2} = \frac{1}{2}(\alpha + \beta) \phi \) then we are indifferent between \( \theta_{N-2} = \alpha \) and \( \theta_{N-2} = \beta \).
Region 7: $-2\beta \phi - \beta \phi^2 < Z_{N-2} < -(\alpha + \beta)\phi - \beta \phi^2$.

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} - \beta \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \beta \phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \beta \phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[
= \frac{1}{2}[-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \alpha \phi - \beta(\phi + \phi^2) \\
-Z_{N-2} - \alpha \phi + \beta(\phi + \phi^2) \\
-Z_{N-2} + \alpha \phi - \beta(\phi - \phi^2) \\
-Z_{N-2} + \alpha \phi + \beta(\phi - \phi^2)
\end{bmatrix}
\]

\[
= -\rho^2 X_{N-2}.
\]

Since $\frac{1}{2}[-\rho^2 X_{N-2} + \beta \rho \bar{\sigma} S_{N-2} + \beta(\rho + \bar{\sigma})\bar{\sigma} S_{N-2}] > -\rho^2 X_{N-2} \Rightarrow -2\beta \phi - \beta \phi^2 < Z_{N-2}$, we again have that $\theta_{N-2} = \beta$ is optimal.

Region 8: $Z_{N-2} < -2\beta \phi - \beta \phi^2$. In this case all paths are negative and thus

\[
(U_{N-2}|\theta_{N-2} = \beta) = -\rho^2 X_{N-2},
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = -\rho^2 X_{N-2}
\]

and thus we are indifferent about the choice of $\theta_{N-2}$ in this region.

We may summarise the findings of the above section to state that the optimal strategy, $\theta^*_{N-2}$, for the case where $\alpha > 0$ and $|\beta| > |\alpha|$ is given by the function

\[
\theta^*_{N-2} = \begin{cases} 
\alpha & \text{if } \phi > \frac{\alpha + \beta}{2\beta} \text{ and } Z_{N-2} > \frac{\alpha + \beta}{2} \phi \\
\beta & \text{otherwise}
\end{cases}
\]

(4.25)

A graphical depiction of the function describe above for the vacation call option is displayed in figure 4.8.

As has been mentioned, the analysis performed in this section to obtain the optimal strategy at two-steps from maturity was considered only for the case where $\alpha > 0$ and $|\alpha| < |\beta|$. Performing the analysis for other situations is done in exactly the same manner, namely the optimal value for $\theta_{N-1}$ is chosen, then the paths are ordered and the
function $U_{N-2}$ is computed at $\theta_{N-2} = \alpha$ and $\beta$ over all regions. We will state the result for the situations $\alpha < 0$, $|\beta| > |\alpha|$ and $\alpha < 0$, $|\alpha| > |\beta|$ and the details of these computations may be found in appendix A.

For the case where $\alpha < 0$, $|\beta| > |\alpha|$, the optimal strategy is identical to equation (4.25) derived above, and in the case where $\alpha < 0$, $|\alpha| > |\beta|$, then $\theta_{N-1}^* = \alpha$ and the optimal strategy at two steps from maturity is

$$
\theta_{N-2}^* = \begin{cases} 
\beta & \text{if } \phi > \frac{\alpha+\beta}{2\alpha} \text{ and } Z_{N-2} < \frac{\alpha+\beta}{2} \phi \\
\alpha & \text{otherwise}
\end{cases}.
$$

(4.26)

It is clear that the optimal strategy for this two step model is not simply a function of the state variables $X_{N-2}$ and $S_{N-2}$, but also of the parameters $\rho$ and $\bar{\sigma}$. This is markedly different from the optimal strategy which was derived for the continuous time versions in that they depend only on $\alpha$, $\beta$, $X_t$ and $S_t$. The inclusion of $\phi$, which is essentially a volatility parameter, greatly increases the complexity of determining the general optimal strategy.

**Figure 4.8:** An example of the optimal strategy at two steps from maturity for the vacation call option ($\alpha = 0$, $\beta = 1$).
4.4. Determination of a General Strategy

The option value function for this two step model can be easily visualised as the value of an option on a traded account may be expressed as a function of only two variables, namely $\phi$ and $Z_{N-2}$. To provide scale we must also specify $S_{N-2}$, $r$ and $\Delta t$. A plot of option values for passport and vacation call options is presented in figure 4.9.

![Figure 4.9: A plot of the value for the option on a traded account using a two-step binomial model. Model parameters are $S_{N-2} = $1, $r = 5\%$ p.a. and $\Delta t = 1$ month.](image)

Given we have derived the optimal strategy at both one and two steps from maturity, it would seem plausible, notwithstanding the tedious algebra, that this could be continued for larger models to obtain a complete analytic optimal strategy at any number of steps from maturity. However, due to the problems associated with analytically ranking the paths, this becomes an intractable task. We discuss this further in the next section.

4.4. Determination of a General Strategy

From the analysis carried out above, it is apparent that the optimal strategy at a particular node is governed by the mathematical form of the paths from that node to maturity. For example, in the case of our one and two-step models, it can be said that the optimal strategy’s dependency on $\phi$ is of order zero and one respectively. This property in
fact holds in general and is a result of the fact that paths which consist of $k$-steps are described by a $k$th order polynomial in $\phi$. The order of this polynomial may be reduced by one via a simple factorisation, though we are always left with at least a $(k-1)$ order polynomial. This is the minimum amount of information required to describe the path from a node that is $k$-steps from maturity. So, if we were to consider a three step model, then the optimal strategy at three steps from maturity will depend on $X_{N-3}$, $S_{N-3}$ and an order two polynomial in $\phi$.

To obtain this order two polynomial requires the ability to rank the paths as we did for the two step case in table 1. To obtain the rank of these paths requires comparing the polynomials pairwise to determine under what conditions one polynomial dominates the other, and this necessarily requires finding the roots of a polynomial of equivalent order. So, for the three step model we must find the roots for a quadratic equation to determine this rank. A four step model would require roots from a cubic equation and so on.

This is an enormous problem in finding optimal strategies in general. Whilst it would be extremely tedious to repeat the procedure of determining the optimal strategy over all possible variable values and working backwards via dynamic programming for a $k$-step model, it could still be done in principle if this problem of computing polynomial roots were not encountered. However, Galois theory tells us that polynomials of order greater than 4 have no analytic formula for finding their roots which are expressible in radicals, thus we cannot analytically determine the order of the paths for a model of 5 steps or greater and hence could not determine the optimal strategies for these models analytically. We point out that the continuous time version of this problem does not suffer from this dependency on $\phi$. The reason for this is due to the fact that in the limit as $\Delta t \to 0$, $\phi \to 0$. Thus this problem is not encountered in the continuous time models which makes their treatment more tractable.

While we expect that as the number of steps increases ($N \to \infty$, $\Delta t \to 0$) the optimal strategy for the binomial model would converge to the continuous time case, we will show by way of an example that
it is not an optimal strategy for our discrete time model. In other words, we will now present a specific example which will serve to show that the optimal strategy used for the continuous time models will not necessarily produce the maximum option value. We consider a 3-step model where we first compute the option value assuming that the optimal strategy is the same as that given by the continuous time framework, namely that

$$\theta^*_i = \alpha I(X_i > (\alpha + \beta)S_i) + \beta I(X_i \leq (\alpha + \beta)S_i),$$

where $i$ denotes a particular node and $S$ and $X$ are the stock price and trading account value respectively. This is then compared to the optimal option value obtained by examining all feasible optimal strategies. A complete discussion of how we obtain these optimal strategies is given later in section 4.6.2, for the time being however, we simply state that the convexity of the problem allows us to determine a finite number of strategies of which at least one must be optimal. For this example involving three steps, there are 128 possible strategies which may be optimal. The parameters involved in this example are given in table 2 below.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.08</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.35</td>
</tr>
<tr>
<td>$S_0$</td>
<td>5</td>
</tr>
<tr>
<td>$X_0$</td>
<td>1</td>
</tr>
<tr>
<td>$T$</td>
<td>90/365</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>30/365</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-1</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.5</td>
</tr>
</tbody>
</table>

**Table 2:** Parameter values for our three-step example.

Using these arguments, the value of an option on a traded account using the continuous time strategy was found to be $V_{ct} = 1.1283$ while the true option value obtained by selecting the largest of the possible values was $V^* = 1.1326$. This shows that the strategy given earlier will
not always produce the appropriate option value and hence it cannot be the optimal strategy.

![Graph showing option value vs strategy no.]

**Figure 4.10:** The option value for a given strategy. Parameter values are given in Table 2 and the red points highlight the optimal values.

We also make note of the fact that the optimal strategy is not necessarily unique. In Figure 4.10, we plot the option value against a particular strategy. What can be taken from this diagram is that there are in fact many strategies which will produce the optimal option value.

This leads us to conclude that if we wish to use a binomial approach for a practical number of steps, we will need to use numerical methods to solve the optimisation problem which governs the value of an option on a traded account, as simply applying continuous time trading strategies will not yield the correct value. Due to the structure of the problem, this is also an extremely difficult task. Before we examine these numerical methods, we first examine a special case of the option on a traded account, the passport option, for which the analytic optimal strategy may be obtained.
4.5. The Binomial Passport Option

In this section we examine a special case of an option on a traded account, the passport option. This is simply an option on a traded account where the trading position is restricted to the symmetric interval $[-1, 1]$. In determining the optimal trading strategy, which we can determine analytically, we first provide some evidence which leads us to this conclusion. The reason for this is because our proof of the optimal strategy will proceed by induction, which necessarily requires knowledge of the optimal strategy beforehand, thus we wish to provide the reader with some intuition as to where this strategy was obtained.

First, let us consider the optimal strategy at one and two-steps from maturity for a passport option. Using the strategies given by (4.23), (4.25) and (4.26) derived in the previous section, we have that one such choice for the optimal strategy for both one and two-steps from maturity is

$$\theta^*_N = -\text{sgn}(X_{N-1})$$  \hspace{1cm} (4.27)

$$\theta^*_N = -\text{sgn}(X_{N-2}).$$  \hspace{1cm} (4.28)

It is immediately clear that these strategies are identical to that obtained for the continuous time counterpart. This leads us to the conjecture that the optimal strategy for the passport option under the binomial model and continuous time model are identical.

It can however be dangerous to generalise based on just two cases, so we develop a combined visualisation-inductive argument to make our conjecture a little more concrete. We note that in the previous section on the one and two-step models that the option value can essentially be described in terms of two variables only, namely $Z_{N-k}$ which accounts for the state variables and $\phi$ which accounts for the model parameters. This means that for a $k$-step model, we can construct a surface plot of the function value for an initial choice of trading position, assuming we know what the remaining optimal strategy is. Since we know that

\footnote{Technically, the analysis would work just as well for any symmetric trading interval $[-\beta, \beta]$ as the option values scales linearly with $\beta$.}
the initial position choice must be either $\alpha$ or $\beta$, we can simply take the difference of the two value functions to determine which strategy is optimal. Since this requires knowledge of the remaining optimal strategies, we use a dynamic programming argument to make our case.

First, to define the function which we are plotting, we note that

$$U_{N-k} = \max_{\theta_{N-k}} \mathbb{E}_{N-k} \{ |X_N| \}$$

can be reduced into a function of the variables $Z_{N-k}$ and $\phi$ by factoring out $S_{N-k}\rho^k$. Thus the function, $f$ that we are considering is given by

$$f(Z_{N-k}, \phi|\theta) = \frac{(U_{N-k}|\theta_{N-k} = \theta)}{S_{N-k}\rho^k}.$$  

In order to determine the optimal strategy, we define the quantity

$$\Delta f = f(Z_{N-k}, \phi|\theta_{N-k} = 1) - f(Z_{N-k}, \phi|\theta_{N-k} = -1).$$

Essentially, all we are doing is computing the option value with the initial trading position set to $\theta_{N-k} = 1$, repeat for $\theta_{N-k} = -1$ and then take the difference between the outcomes. It is clear from this that if $\Delta f > 0$, then $\theta_{N-k} = 1$ is optimal, and if $\Delta f < 0$ the $\theta_{N-k} = -1$ is optimal. In order to determine the optimal strategy for the remaining $(k-1)$-steps, we use a dynamic programming approach where the optimal strategy for the immediate sub-problem determines the remaining trading positions.

For example, in figure 4.11a, we have a model at one step from maturity. In this case the surface for $\Delta f$ is always 0, thus both $\theta_{N-1} = 1$ and $\theta_{N-1} = -1$ are equally optimal strategies, and (4.27) represents an optimal strategy for this model. Moving to the two-step model depicted in figure 4.11b, we now use the optimal strategy obtained from the previous one-step model to compute $\Delta f$. From this plot it is clear that when $Z_{N-2} > 0$, $\Delta f < 0$ and when $Z_{N-2} < 0$, $\Delta f > 0$, which agrees with the strategy stated in (4.28). We hence continue in this manner to determine the optimal strategy for a particular number of steps from maturity. In all cases that we have examined it is found
that the optimal strategy can be given by

\[ \theta^*_N = -\text{sgn}(Z_{N-k}) = -\text{sgn}(X_{N-k}). \] (4.29)

Figure 4.11 continues this procedure for \( k = 1, 2, \ldots, 6 \) and also shows that there are many cases where we are indifferent to \( \theta = 1 \) or \( \theta = -1 \).

We thus have some strong evidence that the strategy given by (4.29) is in fact optimal for \( k = 1, 2, \ldots, N - 1 \). In the following theorem, we offer a proof of this statement.

**Theorem 4.7.** *The optimal strategy function, \( \theta^*_N \), for a passport option under the assumption of a binomial model is given by

\[ \theta^*_N = -\text{sgn}(X_{N-k}) \]

for \( k = 1, 2, \ldots, N \) where the strategy and account value occur at the same node having followed the same path.*

**Proof.** In order to make the proof as transparent as possible, we introduce some new notation. We also note that the proof will proceed forward in time, which is somewhat ironic given that we have been approaching this problem from a backward time DP perspective. First, we define the set of all binomial paths of length\(^\text{10}\) \( n \) by \( \mathcal{P}_n \). Secondly, we define the collection of all subpaths of \( \mathcal{P}_n \) as \( \pi_n \). For example,

\[
\begin{align*}
\mathcal{P}_2 &= \{uu, ud, du, dd\} \\
\pi_2 &= \{0, u, d, uu, ud, du, dd\}
\end{align*}
\]

where \( u \) and \( d \) are “symbolic” in the above presentation. Numerically, \( u \) and \( d \) are given in theorem 4.1 with \( p = q = \frac{1}{2} \), or equivalently,

\[
\begin{align*}
u &= \rho + \bar{\sigma} = \rho(1 + \phi) \\
d &= \rho - \bar{\sigma} = \rho(1 - \phi)
\end{align*}
\]

with \( \rho, \bar{\sigma} \) and \( \phi \) defined by equations (4.14), (4.15) and (4.22) respectively. We now define a stock and trading account value at some node as \( S(\mathcal{P}_n) \) and \( X(\mathcal{P}_n) \) respectively. We also define the notation that

\(^{10}\)We point out that the path \( \mathcal{P}_n \) also uniquely defines a node since the binomial tree is non-recombining.
Figure 4.11: Visual determination of the optimal strategy for a passport option. We note that the red dots simply highlight $Z_{N-k} = 0$.

$\mathcal{P}_n u$ is the set of binomial paths of length $(n + 1)$ which end in an up movement, and similarly for $\mathcal{P}_n d$. Using this notation, the recursion
which describes our stock price is
\[
\begin{align*}
S(P_{n+1}) &= \rho (1 + \phi) S(P_n) \\
S(P_{n+1}) &= \rho (1 - \phi) S(P_n)
\end{align*}
\]
and for the account value,
\[
\begin{align*}
X(P_{n+1}) &= \rho [X(P_n) + \phi S(P_n) \theta(P_n)] \\
X(P_{n+1}) &= \rho [X(P_n) - \phi S(P_n) \theta(P_n)]
\end{align*}
\]
We now begin our proof by induction on \( n \). First, set \( n = 1 \) and show that the prescribed strategy does in fact hold. Note that the objective function we are using is given by \( U \), the sum of absolute values. Thus we have,
\[
U(P_1) = \max_{\theta(P_0)} \sum_{P_1} |X(P_1)|
\]
\[
= \rho \max_{\theta(P_0)} \{|X(P_0)u| + |X(P_0)d|\}
\]
\[
= \rho \max_{\theta(P_0)} \{|X(P_0) + \phi S(P_0) \theta(P_0)| + |X(P_0) - \phi S(P_0) \theta(P_0)|\}
\]
\[
= 2\rho \max_{\theta(P_0)} \{\max \{|X(P_0)|, |\phi S(P_0) \theta(P_0)|\}\}
\]
where we have used the identity \(|a + b| + |a - b| = 2 \max\{|a|, |b|\}\). Now, since \( \theta(P_0) = \pm 1 \), \( \phi > 0 \) and \( S(P_n) > 0 \) for all \( n \geq 0 \), we have that
\[U(P_1) = 2\rho \max_{\theta(P_0)} \{\max \{|X(P_0)|, |\phi S(P_0)\theta(P_0)|\}\}\]
Clearly the function \( U(P_1) \) is independent of \( \theta_0 \), thus we may choose either \( \theta(P_0) = \pm 1 \). Thus the strategy \( \theta(P_0) = -\text{sgn}(X(P_0)) \) will satisfy \( U(P_1) \).

We now assume that the strategy \( \theta(P_j) = -\text{sgn}(X(P_j)) \) for \( j = \{1, 2, \ldots, n - 2\} \), and analyse the problem of determining the strategy which satisfies \( U(P_{n+1}) \). The reason we only need assume an optimal choice up to node \( (n - 2) \) is that the optimal strategy at node \( n \),

\[\text{We only need to consider boundary solutions due the convexity of the payoff function.}\]
within the \((n+1)\)-step model, is already known to be any "bang-bang" strategy by the principle of optimality. Therefore, we make the inductive assumption up to node \((n-2)\) then proceed to prove the optimal strategy for node \((n-1)\) within the \((n+1)\)-step model. Consider

\[
U(P_{n+1}) = \max_{\theta(P_n)} \sum_{P_{n+1}} |X(P_{n+1})|
\]

\[
= \max_{\theta(P_n)} \sum_{P_n} \{|X(P_n u)| + |X(P_n d)|\}
\]

\[
= \rho \max_{\theta(P_n)} \sum_{P_n} \left\{ \begin{array}{c}
|X(P_n) + \phi S(P_n)\theta(P_n)| \\
+ |X(P_n) - \phi S(P_n)\theta(P_n)|
\end{array} \right\}
\]

where we have used the fact that the paths emanating from node \(P_n\) are independent of each others trading strategies from future nodes to separate the value function into a sum over \(P_n u\) and \(P_n d\). Now, the problem we have here when taken from \(P_n\) is identical to a one-step problem that we have already examined, and by the principle of optimality we know that the optimal one-step strategy is still optimal in this scenario. Thus we may choose any value for \(\theta(P_n)\) as long as it is chosen to be on the boundary, that is \(\pm 1\). In this case, we will choose \(\theta(P_n)\) to be equal to \(\theta(P_{n-1})\), that is

\[
\theta(P_n) = \begin{cases}
\theta(P_{n-1} u) = \theta(P_{n-1}) \\
\theta(P_{n-1} d) = \theta(P_{n-1})
\end{cases}
\]

This is just an artificial device which is used to simplify the algebra. Thus, we have that

\[
U(P_{n+1}) = \rho \max_{\theta(P_{n-1})} \sum_{P_{n-1}} \left\{ \begin{array}{c}
|X(P_{n-1} u) + \phi S(P_{n-1} u)\theta(P_{n-1})| \\
+ |X(P_{n-1} d) + \phi S(P_{n-1} d)\theta(P_{n-1})| \\
+ |X(P_{n-1} u) - \phi S(P_{n-1} u)\theta(P_{n-1})| \\
+ |X(P_{n-1} d) - \phi S(P_{n-1} u)\theta(P_{n-1})|
\end{array} \right\}
\]
\[= \rho^2 \max_{\theta(P_{n-1})} \sum_{P_{n-1}} \left\{ \begin{array}{c} |X(P_{n-1}) + \phi(2 + \phi)S(P_{n-1})\theta(P_{n-1})| \\ + 2|X(P_{n-1}) - \phi^2 S(P_{n-1})\theta(P_{n-1})| \\ |X(P_{n-1}) - \phi(2 - \phi)S(P_{n-1})\theta(P_{n-1})| \end{array} \right\} .\]

Now, we need only show that the strategy \(\theta(P_{n-1}) = -\text{sgn}(X(P_{n-1}))\) maximises the function \(U(P_{n+1})\) to prove our assertion. To this end, define

\[
\alpha^\pm = 2\phi \pm \phi^2;
\]

\[
U^\pm = U(P_{n+1}|\theta(P_{n-1}) = \pm 1).\]

We wish to determine the value of \(\Delta U = U^+ - U^-\) over the entire domain of \(X(P_{n-1})\). Clearly if \(\Delta U > 0\), then \(\theta(P_{n-1}) = 1\), and if \(\Delta U < 0\), then \(\theta(P_{n-1}) = -1\). Similar to the previous section on the two-step option on a traded account, we must consider \(\Delta U\) over several regions. These regions are depicted in figure 4.12 below.

We now consider each region in turn. Again, to save space we omit the path argument \(P_{n-1}\) in all cases.

**Region 1:** \(\alpha^+ S < X\). In this case,

\[
U^+ = (X + \alpha^+ S) + 2(X - \phi^2 S) + (X - \alpha^- S) = 4X
\]

\[
U^- = (X - \alpha^+ S) + 2(X + \phi^2 S) + (X + \alpha^- S) = 4X
\]

\[
\Delta U = 0.
\]

We are thus indifferent about the choice of \(\theta(P_{n-1})\) in this region.

**Region 2:** \(\alpha^- S < X < \alpha^+ S\). In this case,

\[
U^+ = (X + \alpha^+ S) + 2(X - \phi^2 S) + (X - \alpha^- S) = 4X
\]
4.5. THE BINOMIAL PASSPORT OPTION

\begin{align*}
  U^- &= -(X - \alpha^+ S) + 2(X + \phi^2 S) + (X + \alpha^- S) = 2(X + \alpha^+ S) \\
  \Delta U &= 2(X - \alpha^+ S) < 0,
\end{align*}

hence \( \theta(\mathcal{P}_{n-1}) = -1 \) is optimal in this region.

Region 3: \( \phi^2 S < X < \alpha^- S \). In this case,

\begin{align*}
  U^+ &= (X + \alpha^+ S) + 2(X - \phi^2 S) - (X - \alpha^- S) = 2(X + \alpha^- S) \\
  U^- &= -(X - \alpha^+ S) + 2(X + \phi^2 S) + (X + \alpha^- S) = 2(X + \alpha^+ S) \\
  \Delta U &= -2\phi^2 S < 0,
\end{align*}

hence \( \theta(\mathcal{P}_{n-1}) = -1 \) is optimal in this region.

Region 4: \( 0 < X < \phi^2 S \). In this case,

\begin{align*}
  U^+ &= (X + \alpha^+ S) - 2(X - \phi^2 S) - (X - \alpha^- S) = 2(-X + \alpha^+ S) \\
  U^- &= -(X - \alpha^+ S) + 2(X + \phi^2 S) + (X + \alpha^- S) = 2(X + \alpha^+ S) \\
  \Delta U &= -4X < 0,
\end{align*}

hence \( \theta(\mathcal{P}_{n-1}) = -1 \) is optimal in this region.

Region 5: \( -\phi^2 S < X < 0 \). In this case,

\begin{align*}
  U^+ &= (X + \alpha^+ S) - 2(X - \phi^2 S) - (X - \alpha^- S) = 2(-X + \alpha^+ S) \\
  U^- &= -(X - \alpha^+ S) + 2(X + \phi^2 S) + (X + \alpha^- S) = 2(X + \alpha^+ S) \\
  \Delta U &= -4X > 0,
\end{align*}

hence \( \theta(\mathcal{P}_{n-1}) = 1 \) is optimal in this region.

Region 6: \( -\alpha^- S < X < -\phi^2 S \)

\begin{align*}
  U^+ &= (X + \alpha^+ S) - 2(X - \phi^2 S) - (X - \alpha^- S) = 2(-X + \alpha^+ S) \\
  U^- &= -(X - \alpha^+ S) - 2(X + \phi^2 S) + (X + \alpha^- S) = -2(X + \alpha^- S) \\
  \Delta U &= 4\phi^2 S > 0,
\end{align*}

hence \( \theta(\mathcal{P}_{n-1}) = 1 \) is optimal in this region.

Region 7: \( -\alpha^+ S < X < -\alpha^- S \)

\begin{align*}
  U^+ &= (X + \alpha^+ S) - 2(X - \phi^2 S) - (X - \alpha^- S) = 2(-X + \alpha^+ S) \\
  U^- &= -(X - \alpha^+ S) - 2(X + \phi^2 S) + (X + \alpha^- S) = -4X
\end{align*}
$$\Delta U = 2X + \alpha^+ S > 0,$$

hence $\theta(P_{n-1}) = 1$ is optimal in this region.

Region 8: $X < -\alpha^+ S$

$$U^+ = -(X + \alpha^+ S) - 2(X - \phi^2 S) - (X - \alpha^- S) = -4X$$
$$U^- = -(X - \alpha^+ S) - 2(X + \phi^2 S) - (X + \alpha^- S) = -4X$$
$$\Delta U = 0,$$

hence we are indifferent about the choice of $\theta(P_{n-1})$ in this region.

Reverting back to our original notation, the optimal strategy $\theta^*_N-k$ for $k = \{0, 1, \ldots, N-1\}$ for an arbitrary $N$ is given by

$$\theta^*_N-k = -\text{sgn}(X_{N-k}).$$

This completes the proof. $\square$

Whilst we have been able to derive the optimal strategy analytically for the passport option, if we consider a trading interval which is not symmetric, then as we showed in the previous section, such an analytic strategy does not exist at present. Thus to solve these types of problems, we must revert to numerical methods. This forms the topic of the following sections where we discuss some numerical methods appropriate to solving the optimisation problem associated with options on traded accounts.

### 4.6. Numerical Valuation

In this section we will present several numerical approaches which may be used to solve the optimisation problem associated with the valuation of options on traded accounts. In this section, we will deal with the function $U$ which is defined as

$$U_{N-k} = \max_{\{\theta\}_{N-k}} \mathbb{E}_{N-k}\{|X_N|\},$$  \hspace{1cm} (4.30)

where $\{\theta\}_{N-k}$ is the set of all $\theta$’s from node $(N-k)$ to maturity (node $N$). This means that rather than looking directly at the option value function $V^*$, we will attempt to solve $U$ from which the solution to
4.6. NUMERICAL VALUATION

$V^*$ may be obtained. In solving this problem analytically, dynamic programming was used to determine the optimal strategy functions. Unfortunately this same technique is not possible when applying it numerically. The reason for this is that the optimal strategy at some node $(N - k)$ is dependent on the state variables at that node which is in turn dependent on the strategies used at earlier nodes. Thus we cannot proceed backwards as dynamic programming would suggest as the values of those required state variables will be unknown. The alternative is then to treat the problem as a static optimisation problem and attempt to optimise over all strategies at the same time, or in other words we will treat the problem as one of mathematical programming.

Before we outline the class of algorithms that we will use to solve this problem, we will first place the problem in a more general setting by expressing it via matrix algebra.

4.6.1. Vectorisation of the Pricing Problem. We start with a simple example to show how this may be achieved. Consider the one-step problem. The function $U_{N-1}$ may be written as

$$U_{N-1} = \max_{\theta_{N-1}} \frac{\rho S_{N-1}}{2} \left[ |Z_{N-1} + \theta_{N-1} \phi| + |Z_{N-1} - \theta_{N-1} \phi| \right].$$

Now, to rewrite this problem using matrix algebra, we define the following variables:

$$\Phi_{N-1} = \begin{bmatrix} \phi \\ -\phi \end{bmatrix}; \quad \tilde{Z}^1_N = Z_{N-1} + \Phi_{N-1} \theta_{N-1}$$

where $\tilde{Z}^1_N$ is the $(2 \times 1)$ vector\textsuperscript{12} of account outcomes reduced by the stock price at one step from maturity. Using these variables, we can now write $U_{N-1}$ as

$$U_{N-1} = \frac{\rho S_{N-1}}{2} \max_{\theta_{N-1}} \sum_{i=1}^2 \{|\tilde{Z}^1_N|\}$$

\textsuperscript{12}Tilde’s will be used to denote vectors whilst bold faced variables wil denote matrices.
where $i$ refers to an element within the vector $\tilde{Z}_N^1$.

Now consider the two-step problem,

$$U_{N-2} = \max_{\{\theta\}_{N-2}} \frac{\rho^2 S_{N-2}}{4} \left[ \sum_{i=1}^4 \left| \{Z_{\theta_{N-2}} + \Phi_{N-2} \theta_{N-2}\}_i \right| \right]$$

where $\{\theta\}_{N-2} = \{\theta_{N-2}, \theta_{N-1}, \theta_{d_{N-1}}\}$. Again we may collect some of these terms into vectors and matrices to simplify the expression of the problem. Define the variables

$$\Phi_{N-2} = \begin{bmatrix} \phi & (\phi + \phi^2) & 0 \\ \phi & - (\phi + \phi^2) & 0 \\ -\phi & 0 & (\phi - \phi^2) \\ -\phi & 0 & -(\phi - \phi^2) \end{bmatrix}; \quad \tilde{\theta}_{N-2} = \begin{bmatrix} \theta_{N-2} \\ \theta_{N-1} \\ \theta_{d_{N-1}} \end{bmatrix};$$

$$\tilde{Z}_N^2 = Z_{\theta_{N-2}} + \Phi_{N-2} \tilde{\theta}_{N-2}$$

then $U_{N-2}$ may be written as

$$U_{N-2} = \frac{\rho^2 S_{N-2}}{4} \sum_{i=1}^4 |\{\tilde{Z}_N^2\}_i|$$

$$= \frac{\rho^2 S_{N-2}}{4} \sum_{i=1}^4 |\{Z_{\theta_{N-2}} + \Phi_{N-2} \theta_{N-2}\}_i|.$$
To define the matrix $\Phi_{N-k}$ we will use a recursive technique which consists of two parts. Firstly a new path is created, then the existing paths are progressed forwards in time by multiplying by up and down factors. We develop an algorithm for this purpose in proposition 4.8 below.

**Proposition 4.8.** The path matrix which is used to compute trading account value outcomes may be generated for any number of steps from maturity in the following manner. Define:

$$\tilde{A}_1 = \begin{bmatrix} \phi \\ -\phi \end{bmatrix}, \quad B_1 = [\ ].$$  

Then the path matrix $\Phi_{N-k}$ may be constructed by the following recursive algorithm

$$\tilde{A}_{i+1} = \tilde{A}_i \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_{i+1} = \begin{bmatrix} 1 + \phi & 0 \\ 0 & 1 - \phi \end{bmatrix} \otimes \Phi_{N-i},$$

$$\Phi_{N-(i+1)} = \begin{bmatrix} \tilde{A}_{i+1} & B_{i+1} \end{bmatrix},$$

where $[\ ]$ denotes the empty matrix and $\otimes$ is the Kronecker product.

Before proceeding with the intuition of the above algorithm, we briefly define the Kronecker product of two matrices. If $X$ is a $(n \times m)$ matrix and $Y$ is a $(p \times q)$ matrix, then the Kronecker product $X \otimes Y$ is the $(mp \times nq)$ block matrix

$$X \otimes Y = \begin{bmatrix} x_{1,1}Y & \cdots & x_{1,n}Y \\ \vdots & \ddots & \vdots \\ x_{m,1}Y & \cdots & x_{m,n}Y \end{bmatrix},$$

where $x_{i,j}$ is the $(i, j)$th element of $X$. A useful property of Kronecker products is that the transpose is distributive, that is $(X \otimes Y)' = X' \otimes Y'$.

Intuitively, the algorithm presented in proposition 4.8 consists of two parts. First the existing paths are advanced one-step by a multiplication with an up or down term creating the matrix $B_i$, then the new component of the path is inserted, this aspect of the algorithm being performed by the vector $\tilde{A}_i$. We make note of the fact that the
matrix $\Phi_{N-k}$ has some very interesting properties, namely that
\[ \tilde{1}_n' \Phi_{N-k} = \tilde{0}_m \]
\[ \Phi_{N-k}' \Phi_{N-k} = D_{m,m} \]
where $n = 2^k$, $m = n - 1$, $\tilde{1}_i$ is a column vector of ones of length $i$, $\tilde{0}_i$ is a column vector of zeros of length $i$ and $D_{i,i}$ is an $(i \times i)$ diagonal matrix with only positive elements on the diagonal. A proof of these properties will be provided in lemmas 4.14 and 4.15 in section 4.6.6.

Using this matrix notation, the problem of pricing an option on a traded account starting at $k$-steps from maturity may be expressed as
\[
U_{N-k} = \max_{\tilde{\theta}_{N-k}} \sum_{i=1}^{2^k} \{|Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}|_i \}
\]
subject to: $\alpha \leq \tilde{\theta}_{N-k,i} \leq \beta$

(4.31)

where $\theta_{N-k,i}$ is the $i$th element of $\tilde{\theta}_{N-k}$. In this guise, it is straightforward to show that the problem is a convex maximisation by simply proving that $U_{N-k}$ is convex. This is carried out in theorem 4.9 below.

**Theorem 4.9.** The function $U_{N-k}$ is convex for $k = 1, 2, \ldots, N$.

**Proof.** Define the constants $\lambda_1 > 0$ and $\lambda_2 > 0$ where $\lambda_1 + \lambda_2 = 1$. Consider a linear combination of $U_{N-k}$ with $\lambda_1$ and $\lambda_2$ as the weights at two distinct points $\tilde{\theta}_{N-k}^1$ and $\tilde{\theta}_{N-k}^2$. This may be reduced to
\[
\lambda_1 U(\tilde{\theta}_{N-k}^1) + \lambda_2 U(\tilde{\theta}_{N-k}^2) = \lambda_1 \sum_{i=1}^{n} \{|Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}^1|_i \} + \lambda_2 \sum_{i=1}^{n} \{|Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}^2|_i \}
\]
\[
= \sum_{i=1}^{n} \left\{ |\lambda_1 Z_{N-k} + \lambda_1 \Phi_{N-k} \tilde{\theta}_{N-k}^1|_i + |\lambda_2 Z_{N-k} + \lambda_2 \Phi_{N-k} \tilde{\theta}_{N-k}^2|_i \right\}.
\]

Now, consider the same linear combination applied to the functions arguments, namely
\[
U_{N-k}(\lambda_1 \tilde{\theta}_{N-k}^1 + \lambda_2 \tilde{\theta}_{N-k}^2) = \sum_{i=1}^{n} \{|Z_{N-k} + \Phi_{N-k}(\lambda_1 \tilde{\theta}_{N-k}^1 + \lambda_2 \tilde{\theta}_{N-k}^2)|_i \}
\]
\[
= \sum_{i=1}^{n} \left\{ |Z_{N-k} + \lambda_1 \Phi_{N-k} \tilde{\theta}_{N-k}^1|_i + |\lambda_2 \Phi_{N-k} \tilde{\theta}_{N-k}^2|_i \right\}.
\]
Since $\lambda_1 + \lambda_2 = 1$, we have that
\[
\begin{align*}
\begin{array}{c}
\lambda_1 Z_{N-k} + \lambda_1 \Phi_{N-k} & \theta_{N-k}^1 \\
+ \lambda_2 Z_{N-k} + \lambda_2 \Phi_{N-k} & \theta_{N-k}^2 
\end{array}
\end{align*}
\begin{array}{c}
i \\
i
\end{array}
\Rightarrow
\begin{align*}
\begin{array}{c}
Z_{N-k} + \lambda_1 \Phi_{N-k} & \theta_{N-k}^1 \\
+ \lambda_2 \Phi_{N-k} & \theta_{N-k}^2 
\end{array}
\end{align*}
\begin{array}{c}
i \\
i
\end{array}
\end{align*}
\]
and using the property $|a| + |b| \geq |a + b|$ we have that
\[
\lambda_1 U(\theta_{N-k}^1) + \lambda_2 U(\theta_{N-k}^2) \geq U_{N-k}(\lambda_1 \theta_{N-k}^1 + \lambda_2 \theta_{N-k}^2)
\]
which is the definition of a convex function, thus the function $U_{N-k}$ is convex.

We may further specify our problem as one of maximising an $l_1$-norm which is a subset of the more general convex maximisation problem. This connection with $l_1$-norm maximisation is important to make as norm optimisation has been a long studied area of applied mathematics. Unfortunately, minimisation of the norm is typically the process examined and this is where the vast majority of the work on optimisation of the $l_1$-norm lies. An even greater difficulty, which may in fact render our problem impossible to solve in an efficient manner, is that Bodlaender et al. (1990) showed that the $l_1$-norm maximisation is in fact NP-hard. It is for this reason that we have been unable to determine an efficient numerical algorithm which can perform the optimisation required to price options on traded accounts on binomial trees. Nonetheless, we present some algorithms which may be used to solve this type of problem, though any claims of an efficient solution using these algorithms would be grossly overstated. Whilst at present none of the algorithms are particularly efficient, and thus not practical, they do provide insight into the numerical treatment of such a problem. We thus present our work on the numerical approach in the hope that this may lead to either a new class of algorithms which may overcome efficiency problems, or that perhaps an appropriate combination of the presented techniques could yield a more efficient solution.

In the following sections, we present some of the more common methods of solving convex global maximisation problems. Much of the theory for the following sections is derived from Horst et al. (2000) and Horst and Tuy (1996), and though we may provide a brief description of
the general theory as it applies to convex maximisation, our focus is on
the application to the problem of valuing options on traded accounts.
For more detail, the reader is directed to the aforementioned texts.

4.6.2. Vertex Enumeration. Convex functions have certain prop-
erties that makes optimisation a much simpler task than for many other
non-linear functions. For example, it is well known that a local min-
umum of a convex function is also its global minimum. Another such
property that will prove useful is that the maximum of a convex func-
tion lies at the boundary of that functions arguments, or in other words,
the global maximum must lie at one of the functions vertices. For this
reason, we will call these solutions that occur at the argument bound-
aries vertex solutions. To place this property in context with respect
to our valuation problem, it is known in attempting to maximise the
expected value of the positive trading account value, that it is always
optimal to have a trading position that is equal to $\alpha$ or $\beta$. At no point
is an intermediate trading position between these two boundaries bet-
ter than a solution that is at the end points of the variable $\theta$’s range.
Another property of convex functions is that a local maximum may ex-
ist at each vertex solution. This means that we have potentially many
local maxima and we must select from these to determine the global
maximum, which is not necessarily unique. This task defines the field
of convex maximisation.$^{13}$

Since convex functions grow without bound, the act of maximising
them only makes sense if the variable over which one is maximising
are themselves bounded. These bounds are expressed by a series of
(typically linear) inequalities which define a feasible space over which
the optimisation is performed. This feasible space is often referred to
as a polytope as it takes on some geometric shape in $n$-dimensional
space. If the number of variables involved is sufficiently small, then
this feasible space can often be visualised. For example, if we consider
our two-step model, we have three variables over which to optimise,

$^{13}$Or equivalently, concave minimisation.
\( \alpha \leq \tilde{\theta}_{N-2,i} \leq \beta \) where, as stated previously, \( \tilde{\theta}_{N-2} \) is the vector of changing variables and \( i \) denotes its \( i \)th element. Given that we are considering only three changing variables, this is a scenario which can be visualised. We present such a depiction in figure 4.13. Whilst any of the infinite number of points within the polytope is a feasible point and will produce a suitable value for the function \( U_{N-k} \), what we are most interested in are those points which will yield the largest value for \( U_{N-k} \). Using the properties describe above, we know that such a point must lie at one of the polytopes vertices. So for our two-step example presented in figure 4.13, rather than having to examine the complete feasible space, we may simply consider the vertices of it, namely the 8 vertices of the 3-dimensional polytope, by computing the option value at those vertices and then choosing the largest one. Algorithms that work in this manner are called vertex enumeration methods.

\[\begin{align*}
\theta_{N-2} &\quad \theta_{N-1}^u \\
\theta_{N-1}^d &\quad \theta_{N-1}^d
\end{align*}\]

Figure 4.13: A representation of the polytope defined by the decision variables in a two-step model. Vertices are labelled \((\theta_{N-2}, \theta_{N-1}^u, \theta_{N-1}^d)\).

Unfortunately, due to the exponential scaling of the binomial process, such a task is intractable for all but a small number of steps. To
illustrate this point, we now show just how many comparisons would be required for this type of computation in general.

**Proposition 4.10.** The $k$-step binomial model for an option on a traded account has $2^k - 1$ distinct trading variables and thus will require $2^{(2^k - 1)}$ comparisons to find the global optimum using a vertex enumeration technique.

**Proof.** First, we have that at some number of steps $j$ into the tree, there will be $2^j$ distinct nodes as the tree is not generally recombining, and each of these nodes will have a trading strategy attached. Since we do not have a trading opportunity at maturity, this leaves us with a total of $(k - 1)$ steps to consider for a $k$-step model. The total number of trading variables $M(k)$ will thus be the sum of all trading strategies over all nodes starting at node 0, that is

$$M(k) = \sum_{i=0}^{k-1} 2^i = 2^k - 1$$

which is easily obtained by recognising that we have a simple geometric progression.$^{14}$ Now, since each of these variables can take two possible values ($\alpha$ or $\beta$), we need to consider all possible permutations. It is obvious that if $n$-variables each have two possible outcomes, then there are a total of $2^n$ possible states for all $n$-variables, thus for our problem, the total number of possible outcomes will be equal to $2^{(2^k - 1)}$. \(\square\)

To give an indication of just how quickly these numbers grow, table 3 lists the number of vertex enumerations required for a particular sized model. We have denoted the vertex counting function for a model consisting of $k$-steps by $V(k)$. It is clear from even the small number of evaluations of the function $V(k)$ in table 3 that this strategy will not be feasible for any reasonable number of steps that a practitioner may wish to use to value an option on a traded account. This is a direct result of the curse of dimensionality in that the non-recombining property of

$^{14}$The standard form of a finite geometric progression is given by $\sum_{i=1}^{N} ar^{N} = a \left( \frac{r^{N} - 1}{r - 1} \right)$. Setting $a = 1$, $r = 2$ and $N = k$ we obtain our result.
the trading account tree leads to the exponentially increasing number of possible outcomes and trading strategies, all of which need to be computed and compared to ascertain the global maximum.

In order to value these types of options in a more efficient manner, we wish to find methods of determining the global maximum without having to compute all the vertex solutions. Two such methods that we will examine are the cutting plane and branch and bound algorithms. These are methods which at any iteration use special rules to discard that part of the feasible space which is guaranteed to contain solutions that are less optimal than those we are currently examining. Before we do this though, we examine how we may reduce our problem, which is described over a somewhat arbitrary polytope, into one which is defined on a standard unit cube as this will prove useful in applying the algorithms just mentioned.

4.6.3. Reduction to a pseudo-Boolean Problem. As discussed above, the problem of finding the global maximum of $U_{N-k}$ is given by an optimisation on a $(2^k - 1)$-dimensional polytope. In its present form, the vertices of the polytope are given by a $(2^k - 1) \times 1$ vector populated with $\alpha$ and $\beta$ only. This polytope is unfortunately somewhat arbitrary. Since we know that the global maximum must lie at one of the polytope vertices, we may re-express the problem in terms of binary integer variables $b_i$ which may take the values 0 and 1 only. This is done by

\[
\begin{array}{|c|c|}
\hline
k & \mathcal{V}(k) = 2^{(2^k - 1)} \\
\hline
1 & 2 \\
2 & 8 \\
3 & 128 \\
4 & 32768 \\
5 & 2147483648 \\
6 & 9.2234 \times 10^{18} \\
\hline
\end{array}
\]

Table 3: The required number of vertices to be computed for a $k$-step model.
specifying the $i$th element of the vector $\tilde{\theta}_{N-k}$ in the following way,

$$\tilde{\theta}_{N-k,i} = \alpha + (\beta - \alpha)b_i. \quad (4.32)$$

Intuitively we are simply forcing the vector $\tilde{\theta}_{N-k}$ to consist of only $\alpha$ and $\beta$. Thus our function above is simply a remapping of the interval $[\alpha, \beta]$ to the unit interval $[0, 1]$. The advantage of such a transformation is that we now have a problem defined on the $(2^k - 1)$-dimensional unit hypercube which is a more standard polytope. Inserting expression (4.32) into the pricing expression (4.31) yields

$$U_{N-k} = \max_b \sum_{i=1}^{n} \left| \{ Z_{N-k} + \Phi_{N-k}(\alpha \tilde{1}_m + (\beta - \alpha)\tilde{b}) \}_i \right|$$

$$= \max_b \sum_{i=1}^{n} \left| \{ Z_{N-k} + \alpha \Phi_{N-k}\tilde{1}_m + (\beta - \alpha)\Phi_{N-k}\tilde{b} \}_i \right|$$

$$= \left\{ \begin{array}{l}
\max_b \sum_{i=1}^{n} |G\tilde{b} + H | \\
\text{subject to: } b_i = \{0, 1\}
\end{array} \right. \quad (4.33)$$

where we have defined $n = 2^k$, $m = n - 1$, $G = (\beta - \alpha)\Phi_{N-k}$ and $H = Z_{N-k} + \alpha \Phi_{N-k}\tilde{1}_m$.

There is a large amount of research that has been conducted on pseudo-Boolean optimisation problems. An example of a thorough survey is Boros and Hammer (2002). Unfortunately, we have been unable to find a method of pseudo-Boolean optimisation (as distinct from convex maximisation) which can handle the absolute value function. Whilst it appears that quadratic functions may be handled in this framework, absolute value functions appear less amenable to efficient maximisation. Later we will demonstrate how to transform our optimisation of a sum of absolute values into a quadratic programming (QP) problem, however this itself has issues which appear to be insurmountable, and in fact may be more complex than the original problem. First, we consider how we may apply the two most common global optimisation methods to our valuation problem where we apply the relaxation that
$b_i \in [0, 1]$ rather than $b_i = 0$ or $1$. We begin with a discussion of cutting planes, then examine branch and bound algorithms.

4.6.4. Optimisation by Cutting Planes. Cutting planes provide a method of computing a global optimum without having to enumerate all vertex solutions. Instead, cutting plane methods use information derived from a known local optima and its neighbouring vertices to construct a hyperplane which cuts through the feasible set of all vertices. If this plane is constructed appropriately, then it can be guaranteed that all points in the set which lie on one side of the plane cannot produce a function value which is better than that currently obtained. Intuitively, one cuts part of the feasible set from the original, and then considers the problem defined by the new, smaller feasible set. Further cuts may then be used in the same manner until a single feasible point is obtained, which must be the global maximum since all points cut from the feasible set were guaranteed to be inferior.

This method relies on being able to find a vertex of the polytope defining our feasible set, which will of course be a local maximum. For any non-empty, closed polytope defined by the intersection of a finite number of halfspaces, this is possible by employing a little computational effort. For example, as Horst et al. (2000) point out, Phase 1 of a linear programming algorithm, such as the Simplex method, can be used to achieve such a task. In the problem we are examining however, this is not necessary as we can always find a local maximum by simply specifying an arbitrary vertex. Such a vertex may be found by simply specifying a vector which is populated with 0 and 1 only, considering we have the problem set up as in equation (4.33). First we will outline the algorithm and then apply it to a simple two-step problem. This will serve to both provide some intuition, as each cut may be visualised, and also outline some drawbacks which are encountered in practice, namely the fact that a very large number of cuts are typically required to ensure that one has a global maximum. For additional detail, the reader is directed to Horst et al. (2000) and Horst and Tuy (1996) as this section draws heavily from these two references.
We will specifically be using a method known as a concavity cut which was introduced by Tuy (1964). This method gets its name as it was first employed to minimise concave functions, though convex maximisation is a mathematically identical problem. Whilst there are other types of cutting plane algorithms available, we choose this as it is applicable to any convex function. Other types of cuts usually require that the objective function satisfy additional criteria (for example the $\phi$-cut found in Horst et al. (2000) requires a quadratic objective function).

We will now provide some insight into concavity cuts. To keep this section general, we assume that the problem at hand is to maximise some convex function $f(\bar{x})$ where $\bar{x} \in \mathbb{R}^n$, subject to constraint that $\bar{x}$ must lie within some polytope $D$, or formally

$$\begin{cases}
\max f(\bar{x}) \\
\text{subject to: } \bar{x} \in D
\end{cases}$$

where $D$ satisfies the requirements for the problem to be well posed. We begin by defining an initial vertex of the polytope which represents the feasible space. As we mentioned earlier, we always have knowledge of such a vertex, for example, in our option pricing problem, one could start with $\bar{b} = \bar{1}_m$. Often a better initial vertex may be found by examining the neighbouring vertices and moving to those which produce a better function value (pivoting), though this is not entirely necessary and can be computationally expensive. Once an initial vertex has been defined and its neighbours located, then we may construct what are known as $\gamma$-extensions. Before we define $\gamma$-extensions, we point out that it is trivial to find a neighbour to some initial vertex $\bar{v}$ as simply changing one entry in the initial vertex $\bar{v}$ will produce a neighbouring vertex. For example, a vertex defined by a $(n \times 1)$ vector will have $n$ neighbours which differ to the original vertex $\bar{v}$ by only one entry.

Firstly, define the function value at the current vertex $\bar{v}$ by $\gamma$, that is $f(\bar{v}) = \gamma$. For the time being, assume that $\bar{v}$ is chosen such that it is optimal among its neighbours, thus $\gamma$ is the best current solution to
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the problem. While perhaps not technically precise, we will refer to a vertex which is known to be the best solution among its neighbours as locally optimal. Intuitively, a $\gamma$-extension, $\tilde{y}_i$, is the largest line segment which extends from the chosen vertex $\tilde{v}$ in the direction of that vertex’s neighbours, to another point (which need not necessarily lie within the feasible space) such that $f(\tilde{y}_i) \leq \gamma$. Formally, if we define the direction $\tilde{d}_i = \tilde{v}_i - \tilde{v}$ where $\tilde{v}_i$ is a neighbouring vertex of $\tilde{v}$ for $i = 1, \ldots, n$, then a $\gamma$-extension is defined by the problem

$$\begin{cases} 
\max_{\psi_i} \tilde{y}_i = \tilde{v} + \psi_i \tilde{d}_i \\
\text{subject to: } f(\tilde{y}_i) \leq \gamma.
\end{cases}$$

We point out that if we ensure that $f(\tilde{v}) = \gamma$, then by the convexity of $f$, $\psi \geq 1$. There are many ways to solve this class of problem, though perhaps the simplest is the bracket and bisect algorithm outlined in Horst et al. (2000).

Now, since the vectors $[\tilde{y}_1 - \tilde{v}, \ldots, \tilde{y}_n - \tilde{v}]$ are linearly independent for $i = 1, 2, \ldots, n$, there exists a unique hyperplane $H(y_1, \ldots, y_n)$ which contains $y_1, \ldots, y_n$. This hyperplane is defined by

$$H = \{ \tilde{x} : x = Y\lambda + v, \tilde{1}'\lambda = 1 \} = \{ \tilde{x} : \tilde{1}'Y^{-1}(\tilde{x} - \tilde{v}) = 1 \}$$

where $Y = [(\tilde{y}_1 - \tilde{v}), \ldots, (\tilde{y}_n - \tilde{v})]$ which is an $(n \times n)$ matrix, $\lambda$ is an $(n \times 1)$ vector which ensures a convex combination of the $\gamma$-extensions and $\tilde{1}_n$ is a $(n \times 1)$ vector of ones. The hyperplane $H$ generates two halfspaces defined as

$$H_- = \{ \tilde{x} : \tilde{1}'Y^{-1}(\tilde{x} - \tilde{v}) < 1 \}$$
$$H_+ = \{ \tilde{x} : \tilde{1}'Y^{-1}(\tilde{x} - \tilde{v}) > 1 \}.$$ 

Examining these two halfspaces, we have that $\tilde{v}$ is contained in the halfspace $H_-$ and more importantly that $f(\tilde{x}) \leq \gamma$ for all $\tilde{x} \in D \cap H_-$. For a proof of this statement we refer the reader to Horst et al. (2000).

\footnote{While this is not necessarily required, it does simplify exposition and a vertex that satisfies this property can always be found by pivoting techniques.}
This allows us to remove the part $D \cap H_-$ from the feasible set without eliminating any points which will yield a function value lower than the current best solution. This also provides a method of determining if we are in fact at a global maximum, for if the entire polytope $D$ is a subset of $D \cap H_-$, then there is no point in the feasible space which produces a better function value than that currently obtained at $\tilde{v}$, and thus $\tilde{v}$ must be the global maximum. In geometric terms, checking whether $D \subset D \cap H_-$ may be done by translating $H$ into a parallel hyperplane $H^*$ which supports $D \cap H_-$. If this translation occurs as a result of a movement of $H$ towards $\tilde{v}$, then $D \cap H_- \subset H_-$ and $\tilde{v}$ is the global maximum. As is shown in Horst and Tuy (1996), such a translation may be expressed numerically as a linear program. Let $c^*$ be the solution to

$$
\begin{cases}
  c^* = \max_{\tilde{x}} \tilde{1}'Y^{-1}\tilde{x} \\
  \text{subject to: } A\tilde{x} \leq \tilde{d}
\end{cases}
$$

(4.35)

where $A$ and $\tilde{d}$ represent the constraints, namely those given in the original problem which defines our feasible space, and those established by any concavity cuts previously performed. If $c^* \leq 1 + \tilde{1}'Y^{-1}\tilde{v}$, then $D \cap H_- \subset H_-$ and we have established that $\tilde{v}$ is a global maximum. If $c^* > 1 + \tilde{1}'Y^{-1}\tilde{v}$, then we cannot guarantee that we have a global maximum. Continuing the procedure is simply a matter of moving to another vertex in the remaining feasible space $D \cap H_+$ and repeating the cutting procedure. A good candidate for the next vertex is in fact that which was obtained when computing $c^*$, though we need not necessarily choose this vertex. It is important to note that we may in fact be at a global maximum, but the procedure described above cannot guarantee it without further reducing the feasible space. This is in fact one of the main drawbacks of the cutting plane methods in that whilst one may actually already have the global maximum, many other operations which involve moving from away from it are required to ascertain this fact.
We now demonstrate how such a cutting plane algorithm may be applied to a simple two-step case of our option on a traded account. In this example we will consider the problem in the form of a pseudo-Boolean function as described previously, though we consider the slightly relaxed problem where $b_i \in [0, 1]$ rather than the true pseudo-Boolean form where $b_i = 0$ or 1. The reason for this is that the concavity cut we are considering cannot easily handle such constraints as we would not be optimising over a closed polytope, but rather a set of points in $\mathbb{R}^n$. Whilst we are technically omitting useful information, this is often required to simplify the problem structure. Formally, we wish to determine the function $U_{N-2}$ given by

$$U_{N-2} = \max_b \left\{ \sum_{i=1}^{4} \{ |G_{N-2}b + H_{N-2}| \} \right\}$$

subject to: $0 \leq b_i \leq 1$

where $G_{N-2} = (\beta - \alpha) \Phi_{N-2}$ and $H_{N-2} = Z_{N-2} + \alpha \Phi_{N-2} \mathbf{1}_3$ and

$$\Phi_{N-2} = \begin{bmatrix} \phi & \phi(1 + \phi) & 0 \\ \phi & -\phi(1 + \phi) & 0 \\ -\phi & 0 & \phi(1 - \phi) \\ -\phi & 0 & -\phi(1 - \phi) \end{bmatrix}.$$  

First, we note that our problem is defined on the unit cube in $\mathbb{R}^3$, similar to that presented in figure 4.13. To make our example concrete, we will define some parameter values which we list in table 4 below. A simple way to determine the optimal strategy in this case is to simply enumerate the function value at all eight vertices. Doing so, we have that there are two optimal strategies $\tilde{b}^* = [1, 1, 0]'$ and $[1, 1, 1]'$, which gives a function value of $U_{N-2} = 0.4439$. We begin our optimisation via cutting planes by selecting an initial vertex, in this case we choose $\tilde{b}_{init} = [0, 0, 0]'$. To ensure that we are dealing with a vertex which is optimal with respect to its neighbours, we simply search among this vertex's neighbours to ascertain if a better function value can be achieved at one of those neighbours. This is repeated
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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.3</td>
</tr>
<tr>
<td>$S_{N-2}$</td>
<td>1</td>
</tr>
<tr>
<td>$X_{N-2}$</td>
<td>0</td>
</tr>
<tr>
<td>$T$</td>
<td>0.25</td>
</tr>
<tr>
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</tr>
<tr>
<td>$\alpha$</td>
<td>0</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4: Parameter values for our two-step concavity cut example.

...until a locally optimal solution is found. In this case this locally optimal solution is found to be $[1, 1, 0]'$. We set this as the initial vertex for our concavity cut to proceed from, that is $\tilde{b}_0 = [1, 1, 0]'$. Next, we construct our $\gamma$-extensions in the directions of $\tilde{b}_0$’s neighbours where we use a bracket and bisection technique to determine the appropriate values of $\psi_i$ for $i = 1, 2, 3$. We have that $\psi_1 = 2$, $\psi_2 = 2$ and $\psi_3 = 1$, thus our $\gamma$-extensions with respect to $\tilde{b}_0$ are given by

\[
\tilde{y}_1 = [1, 1, 0]' + 2 \times ([0, 1, 0]' - [1, 1, 0]') = [-1, 1, 0]'
\]
\[
\tilde{y}_2 = [1, 1, 0]' + 2 \times ([1, 0, 0]' - [1, 1, 0]') = [1, -1, 0]'
\]
\[
\tilde{y}_3 = [1, 1, 0]' + 1 \times ([1, 1, 1]' - [1, 1, 0]') = [1, 1, 1]'
\]

We may now define and compute our first concavity cut. First, we note that the matrix $Y_1$ is given by

\[
Y_1 = \begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

and thus using equation (4.34), we may now compute our cutting plane. This is presented in figure 4.14 together with the feasible space polytope and $\gamma$-extensions.

We now wish to test this point to determine if it is indeed a global maximum. Whilst we already know that the current vertex $[1, 1, 0]'$ is a global maximum via vertex enumeration, in a larger problem we...
would be unable to state this. Thus we construct our linear program to ascertain if we have a global maximum in accordance with equation (4.35). In this case we have that $\tilde{I}_3'Y_1^{-1} = [-0.5, -0.5, 1]$ and the constraints are $I_3\tilde{b} \leq \tilde{I}_3$ and $\tilde{b} \geq 0$, where $I_n$ is an $(n \times n)$ identity matrix. Performing the required linear maximisation, we have that $c^* = 1$ and $1 + \tilde{I}_3'Y^{-1}\tilde{b}_0 = 0$, and thus $c^* > 1 + \tilde{I}_3'Y^{-1}\tilde{b}_0$ so we cannot guarantee that the current vertex $\tilde{b}_0$ is a global maximum. In terms of the hyperplane $H$ translating to a supporting hyperplane $H^*$, we have actually moved away from $\tilde{b}_0$ to the vertex $[0, 0, 1]'$.

We now perform the second iteration of the algorithm. We begin at the vertex $\tilde{b}_1 = [0, 0, 1]'$ as this is the vertex at which our linear program, used to ascertain global optimality, was maximised. Horst
et al. (2000) state that the vertex generated by the linear program is a good starting point for the next iteration, so we stay true to their algorithm. Computing the function value at $\tilde{b}_1$ and its neighbouring vertices, we have that
\[
U_{N-2}([0, 0, 1]') = 0.18886 \\
U_{N-2}([0, 0, 0]') = 0 \\
U_{N-2}([1, 0, 1]') = 0.4216 \\
U_{N-2}([0, 1, 1]') = 0.4216.
\]
None of these vertices provides a better function value than the current lower bound, so we leave $\gamma = 0.4439$ and use this value to compute our $\gamma$-extensions in the directions
\[
\tilde{d}_1 = [0, 0, 0]' - \tilde{b}_1 = [0, 0, -1]' \\
\tilde{d}_2 = [1, 0, 1]' - \tilde{b}_1 = [1, 0, 0]' \\
\tilde{d}_3 = [0, 1, 1]' - \tilde{b}_1 = [0, 1, 0]'.
\]
The appropriate values for $\psi_i$ for $i = 1, 2, 3$, and thus the vectors given by the $\gamma$-extensions are
\[
\psi_1 = 3.3537 \Rightarrow \tilde{y}_1 = [0, 0, -2.3537] \\
\psi_2 = 1.0528 \Rightarrow \tilde{y}_2 = [1.0528, 0, 1] \\
\psi_3 = 1.0956 \Rightarrow \tilde{y}_3 = [0, 1.0956, 1].
\]
Using the above $\gamma$-extensions, we can compute the matrix $Y_2$ for this iteration which is given by
\[
Y_2 = \begin{bmatrix}
0 & 1.0528 & 0 \\
0 & 0 & 1.0956 \\
-3.3537 & 0 & 0
\end{bmatrix}
\]
and using this we can again construct our cutting hyperplane. The updated polytope, $\gamma$-extensions and cutting plane for this second iteration are plotted in figure 4.15. We now wish to test this vertex for global optimality. To set up our linear program for this purpose. The
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Figure 4.15: The polytope and second concavity cut hyperplane for the two-step problem. The feasible space bounds are drawn in blue while the γ-extensions from \( \tilde{b}_1 \) are drawn in red.

The constraints for this program are that \( 0 \leq \tilde{b} \leq 1 \) and

\[
\tilde{I}_3'Y_1^{-1}(\tilde{b} - \tilde{b}_0) \geq 1 \quad \Rightarrow \quad [0.5, 0.5, -1] \tilde{b} \leq 0
\]

where this additional constraint accounts for the first concavity cut. This additional constraint is easily incorporated into the constraints \( A \) and \( \tilde{d} \). Performing the required maximisation, we obtain \( c^* = 1.5644 \) and \( 1 + \tilde{I}_3'Y_2^{-1}\tilde{b}_1 = 0.7018 \), thus we cannot say that we currently have a global maximum. The new vertex at which this linear program obtains its maximum is \( \tilde{b}_2 = [1, 1, 1]' \).

At this stage we could say that the vertex \( \tilde{b}_2 \) is the only feasible vertex left after cutting all the other corner points from feasible space, and thus it must be the global optimum with function value \( U_{N-2}(\tilde{b}_2) = \)
0.4439. However, we will continue with our method of cutting and testing for global optimality via a linear program as this identification would be more difficult in a larger scale problem. Our polytope for this iteration is now defined by the constraints
\[
\begin{align*}
0 & \leq \tilde{b} \leq 0 \\
[0.5, 0.5, -1] \tilde{b} & \leq 0 \\
[-0.9498, -0.9127, 0.2979] \tilde{b} & \leq -0.7021
\end{align*}
\] 
(4.36)
where the final constraint is derived from computing \( \tilde{1}_3' Y^{-1}_2 \) and substituting into equation (4.34) to compute the cutting plane equation. From these constraints our feasible polytope is defined. What we require in this case are the neighbouring vertices of the current vertex \( \tilde{b}_2 = [1, 1, 1]' \). In this case, \( \tilde{b}_2 \) has four neighbouring vertices. These neighbours and their associated function values are
\[
\begin{align*}
U_{N-2}([1, 0, 0.5]') &= 0.4216 \\
U_{N-2}([0, 1, 0.5]') &= 0.3273 \\
U_{N-2}([1, 0.0550, 1]') &= 0.4216 \\
U_{N-2}([0.0919, 1, 1]') &= 0.4216,
\end{align*}
\]
which are all less than the current best solution, so we keep \( \gamma = 0.4439 \). Constructing our \( \gamma \)-extensions, we note that with four neighbours, if we include all of them, then \( Y_3 \) will be a \((3 \times 4)\) matrix and thus not have an inverse. This issue is easily handled by understanding that we need only include linearly independent vertices to define our cutting plane. In this case it is easy to see that the implies we need only include one of \([1, 0, 0.5]'\) or \([0, 1, 0.5]'\). In this case we choose to include \([1, 0, 0.5]'\), though this choice is arbitrary and makes no difference to our cutting plane. We now define our directions and associated \( \gamma \)-extensions
\[
\begin{align*}
\tilde{d}_1 &= [0, 1, 0.5]' - \tilde{b}_2 \quad \Rightarrow \quad \psi_1 = 2 \quad \Rightarrow \quad \tilde{y}_1 = [1, -1, 0]' \\
\tilde{d}_2 &= [1, 0.0550, 1]' - \tilde{b}_2 \quad \Rightarrow \quad \psi_2 = 2.1166 \quad \Rightarrow \quad \tilde{y}_2 = [1, -1, 1]' \\
\tilde{d}_3 &= [0.0919, 1, 1]' - \tilde{b}_2 \quad \Rightarrow \quad \psi_3 = 2.2027 \quad \Rightarrow \quad \tilde{y}_3 = [-1, 1, 1]',
\end{align*}
\]
and using the vectors \( \tilde{y}_i \) for \( i = \{1, 2, 3\} \), we can compute the matrix \( Y_3 \) and hence derive our cutting plane via equation (4.34). In this case
the cutting plane is given by \([-0.5, -0.5, 0]\). Figure 4.16 shows these features graphically. We now wish to show that the vertex \(\tilde{b}_2\)

![Figure 4.16: The polytope and third concavity cut hyperplane for the two-step problem. The feasible space bounds are drawn in blue while the \(\gamma\)-extensions from \(\tilde{b}_2\) are drawn in red.](image)

is indeed the global optimum. This requires us to perform the linear program outlined in (4.35) where the constraints are given by (4.36). This gives \(c^* = -0.4382\) and \(1 + \tilde{I}_3^{-1}b_2 = 2.1213 \times 10^{-5}\), so clearly \(c^* < 1 + \tilde{I}_3^{-1}b_2\) and we thus conclude that the current vertex \(\tilde{b}_2\) is the global maximum we desire. This completes the optimisation by cutting planes for this two-step problem.

Whilst the cutting plane method outlined above is more efficient than exhaustive approaches such as vertex enumeration, there are still some issues which prevent the general adoption of these types of algorithms for efficient valuation of options on traded accounts. Firstly, the
algorithm requires that at each iteration we can compute the vertices of the feasible polytope determined by the constraints. In general this is a very difficult and computationally expensive problem. Matheiss and Rubin (1980) present a good survey of the algorithms which may be used to determine these vertices. Efficiency may be gained by using the fact that we need actually only compute the current vertex, which is always known in this problem, and its neighbours. Whilst this is far less computationally intensive than finding all vertices, it is still a difficult task to perform, especially as constraints are added at each iteration, thus potentially changing the neighbours. Also, in the example above, even though we started the algorithm from what was a globally optimal solution, we still required several cuts in order to confirm this. Our experience leads us to believe that this problem would persist in cases where more steps are included thus requiring many cuts leading to an algorithm which is not particularly efficient. This appears to be a general problem when using cutting planes and its variants for global convex maximisation. In fact, Pardalos and Rosen (1987) make the point that cutting plane algorithms are suitable for only moderately sized problems. We next consider another class of algorithm which is commonly used in global optimisation problems, the branch and bound technique.

**4.6.5. Optimisation by Branch and Bound.** The branch and bound algorithm is a common method of solving complex optimisation problems. Again the idea behind this algorithm is to determine the global optimum without having to compute and compare the function value at all vertices. Rather than using additional inequalities to remove part of the feasible space, branch and bound algorithms use a branching procedure to partition the feasible space into smaller sets over which bounds on the objective function can be computed via a specially chosen bounding functions. If it can be determined from these bounds that a solution in one partition is always inferior to those in
another partition, then those inferior parts are eliminated from consideration. Elimination of certain parts of the feasible space in this manner is often referred to as pruning.

To place this procedure in the context of our valuation problem, branching would be performed over the trading strategies by, for example, setting the first element of $\tilde{b}$ to 0 and 1, thus splitting our feasible space into two separate partitions. We then compute upper and lower bounds of the objective function, $U$, over each of these partitions and if we can show that the upper bound in one partition is less than the lower bound in the other partition, then we can eliminate the inferior partition from further consideration. In figure 4.17, we provide a visualisation of such a procedure. In this figure, the feasible space is partitioned by considering two cases, the first case being where the first element of $\tilde{b}$, $b_1$, is equal to 1 and the second case where this element is equal to 0. Upper and lower bounds are then computed for the objective function in each of these partitions. In this case we assume that the upper bound in the partition defined by $b_1 = 0$ is less than lower bound in the partition set by $b_1 = 1$, or in the notation of the diagram, $UB_0 < LB_1$. This means that we know that it is always optimal to have $b_1 = 1$, thus we need not consider any solutions which have $b_1 = 0$. The procedure is then repeated for other elements of $\tilde{b}$ until we are either left with only one solution, which must be optimal, or alternatively a stopping criterion may be chosen. For further details on the branch and bound algorithm we refer the reader to Horst et al. (2000), Horst and Tuy (1996) and Pardalos and Rosen (1987).

While the branch and bound technique has been successfully applied to many complex optimisation problems, integer quadratic programming being a prominent example, this success relies quite strongly on the choice of the bounding functions. There is obviously no universal bounding function, with the choice of such a function being dependent on the problem structure. Choosing an appropriate bounding function is thus the crux of the algorithm. This is where the application of branch and bound to our option valuation problem becomes difficult. At present we have been unable to determine an appropriate bounding
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Figure 4.17: A graphical representation of the branch and bound procedure. In this diagram, we assume that $LB_1 > UB_0$.

function for $U$ which allows the application of the branch and bound algorithm. The difficulty is due to the fact that the present value of the trading account $X$ is a martingale. Thus the expected value of the trading account at maturity is independent of the strategy chosen during its lifetime. The strategy only becomes important when a convex payoff function, in this case the plus function which we approach via the absolute value function, is applied to the account value at maturity. This means that many functions which could have been considered appropriate will actually yield the same bound regardless of the strategy chosen. For example, one choice for the lower bound of our value function would use the property that $|x + y| \leq |x| + |y|$, or in terms of our function $U$,

$$\max_b \left| \sum_{i=1}^n \{ Z_{N-k} + \Phi_{N-k}(\alpha \hat{1}_m + (\beta - \alpha)\hat{b}) \} \right| \leq U_{N-k}.$$  

However, due to the martingale property, the term on the left in the above equation is always equal to $|Z_{N-k}|$ and is hence independent of the trading strategy. This means that employing this as a lower bounding function in a branch and bound algorithm will not eliminate any branches and thus the algorithm will essentially become a vertex enumeration. This problem has been encountered when using several other linear bounding functions.
One possible choice for the upper bound could be the log-sum-exponential function. This is described by Boyd and Vandenberghe (2009) as a smooth analogue of the max function, thus we can relate it to the absolute value function in the following way,

$$|x| = \max(x, -x) \approx \log(e^x + e^{-x}).$$

We can make this statement more precise by showing that the log-sum-exponential function is in fact an upper bound to the absolute value function.

**Theorem 4.11.** The following relation holds

$$|x| \leq \log(e^x + e^{-x})$$

**Proof.** First consider $x > 0$. We wish to show that

$$x \leq \log(e^x + e^{-x}) \iff e^x \leq e^x + e^{-x} \iff e^{-x} \geq 0$$

which is true. A similar argument holds for $x < 0$. \[\square\]

Thus, using theorem 4.11, we have that

$$U_{N-k} \leq \max_b \sum_{i=1}^n \log \left( \exp \left( \{Z_{N-k} + \Phi_{N-k}(\alpha \tilde{I}_m + (\beta - \alpha)\tilde{b})\}_i \right) + \exp \left( \{-Z_{N-k} - \Phi_{N-k}(\alpha \tilde{I}_m - (\beta - \alpha)\tilde{b})\}_i \right) \right).$$

Determining the maximum in the above function is itself a convex maximisation problem and thus difficult to solve, so whilst we have been able to determine bounds for our value function, they appear to be ill suited to the branch and bound algorithm.

Given that we have not been particularly successful in solving the $l_1$-norm maximisation problem associated with the option valuation directly, we now consider how the problem may be transformed such that it may lend itself to other techniques. To this end, we show how we may convert our $l_1$-norm maximisation into a quadratic programming problem which opens it up to other algorithms specifically designed to handle quadratic programs.

**4.6.6. Transformation to an Indefinite QP Problem.** Before we show how this transformation takes place, we first provide some
4.6. NUMERICAL VALUATION

intuition for it by considering the task of minimising the $l_1$-norm. This is a well known problem, being discussed for example in Boyd and Vandenberghe (2009), which can be transformed into a linear program. We demonstrate this with respect to our valuation problem. To keep the notation a little more concise, we express this problem in terms of the original strategy variable $\tilde{\theta}$ rather than the re-scaled variable $\tilde{b}$.

**Proposition 4.12.** The minimisation of the $l_1$-norm specified by

$$\min_{\tilde{\theta}_{N-k}} \sum_{i=1}^{n} \left| \{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i \right|$$

subject to: $\alpha \leq \tilde{\theta}_{N-k,j} \leq \beta$

may be converted into a linear program by introducing a new variable $\tilde{g}$ in the following manner,

$$\min \tilde{g}$$

$$\text{subject to: } -g_i \leq \{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i \leq g_i \cdot$$

$$\alpha \leq \tilde{\theta}_{N-k,j} \leq \beta$$

The intuition behind the above proposition is that we essentially try to shrink an interval around the objective function. The only way to make the interval as small as possible is to make the objective function as small as possible, thus achieving the desired minimisation.

We wish to use a similar idea to maximise the $l_1$-norm, however as we will show this is not as straightforward as minimisation. Again, we construct an interval $[-g_i, g_i]$ which will contain our objective function, and we will attempt to maximise the width of this interval. However, in order to do this, we require a mechanism which ensures that the only way the interval can increase is through an increase in the objective function. This could be achieved by including additional constraints (typically non-linear) or by augmenting the objective function. We choose the latter.

In order to force the growth of the sum of $\tilde{g}$ through the growth of the absolute value of the account value, we will impose a quadratic
penalty function on the linear objective function so that the new objective function will be maximised only when $|\{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i| = |g_i|$. The following proposition outlines how we may achieve this.

**Proposition 4.13.** The maximisation of the $l_1$-norm, expressed as

$$\max_{\tilde{\theta}_{N-k}} \sum_{i=1}^{n} |\{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i|$$

subject to: $\alpha \leq \tilde{\theta}_{N-k,j} \leq \beta$

may be converted into a quadratic programming problem by introducing the variable $\tilde{g}$ and the parameter $B$ in the following way

$$\max \{ \tilde{g}, \tilde{\theta}_{N-k} \} \tilde{1}'\tilde{g} - B \left[ \sum_{i=1}^{n} (g_i^2 - \{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i^2) \right]$$

subject to:

$-g_i \leq \{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i \leq g_i$,

$\alpha \leq \tilde{\theta}_{N-k,j} \leq \beta$

where $B$ is an arbitrarily large scalar. This is a quadratic penalty function approach whereby the objective function will be heavily reduced if $|\{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i| \neq |g_i|$, as desired. By replacing the sum with its vector counterpart, we may express the above problem in matrix notation

$$\max \{ \tilde{g}, \tilde{\theta}_{N-k} \} \tilde{1}'\tilde{g} - B [\tilde{g}'\tilde{g} - (Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k})'(Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k})]$$

subject to:

$-g_i \leq \{Z_{N-k} + \Phi_{N-k} \tilde{\theta}_{N-k}\}_i \leq g_i$,

$\alpha \leq \tilde{\theta}_{N-k,j} \leq \beta$

(4.37)

To provide a complete analysis of (4.37), let us express it in a more standard manner where the optimisation is performed over a single variable. This will require that we combine some of the terms in (4.37) together. We thus define

$$\tilde{x} = \begin{bmatrix} \tilde{\theta}_m \\ \tilde{g}_n \end{bmatrix}; \quad \tilde{R} = \begin{bmatrix} \tilde{0}_m \\ \tilde{1}_n \end{bmatrix}; \quad R_1 = [ \mathbf{I}_m \ 0_{m,n} ]; \quad R_2 = [ \mathbf{0}_{n,m} \ \mathbf{I}_n ]$$

where $\tilde{0}_j$ is a $(j \times 1)$ vector of zeros, $\mathbf{I}_j$ is a $(j \times j)$ identity matrix and $\mathbf{0}_{j,l}$ is a $(j \times l)$ matrix of zeros. We again use the notation that for a $k$-step model, $n = 2^k$ and $m = n - 1$. Using this notation, we have
that \( \tilde{\theta}_m = R_1 \tilde{x}, \tilde{g}_n = R_2 \tilde{x} \) and \( \tilde{R}' \tilde{x} = \tilde{1}' \tilde{g} \). We may thus rewrite the problem given by (4.37) as

\[
\min_{\tilde{x}} \tilde{x}'B(R_2^2 R - R_1' \Phi_{N-k} \Phi_{N-k} R_1)\tilde{x} - (\tilde{R} + 2B \tilde{Z}_{N-k} \Phi_{N-k} R_1)\tilde{x}
\]

subject to:

\[
M \tilde{x} \leq \tilde{\zeta}_{N-k}
\]

\[
\tilde{\lambda} \leq \tilde{x} \leq \tilde{\mu}
\]

where

\[
M = \begin{bmatrix}
\Phi_{N-k} R_1 - R_2 \\
-\Phi_{N-k} R_1 - R_2
\end{bmatrix}; \quad \tilde{\zeta}_{N-k} = \begin{bmatrix}
-\tilde{Z}_{N-k} \\
\tilde{Z}_{N-k}
\end{bmatrix};
\]

\[
\tilde{\lambda} = \begin{bmatrix}
\alpha \tilde{1}_m \\
\tilde{0}_n
\end{bmatrix}; \quad \tilde{\mu} = \begin{bmatrix}
\beta \tilde{1}_m \\
\infty \tilde{1}_n
\end{bmatrix}.
\]

This is now the standard form for a quadratic program. Solution of such a problem is highly dependent on the properties of the matrix

\[
Q = R_2^2 R - R_1' \Phi_{N-k} \Phi_{N-k} R_1.
\] (4.38)

For example, if \( Q \) is negative definite, then the local maximum is also the global maximum and standard gradient based methods may be applied which could efficiently solve the problem. We will show that \( Q \) is in fact an indefinite, diagonal matrix. In order to do this, we first require two results which we present as lemmas 4.14 and 4.15 below.

**Lemma 4.14.** The product \( \tilde{1}_n' \Phi_{N-k} \) is given by

\[
\tilde{1}_n' \Phi_{N-k} = \tilde{0}_m'
\]

where \( n = 2^k \), \( m = n - 1 \) and \( \tilde{0}_m \) is a \((m \times 1)\) vector of zeros.

**Proof.** We proceed by induction on \( k \). For \( k = 1 \), we have that \( n = 2, m = 1 \) and thus

\[
\begin{bmatrix}
1 & 1
\end{bmatrix}
\begin{bmatrix}
\phi \\
-\phi
\end{bmatrix} = 0
\]

We point out that the \( \infty \) which appears in the vector \( \tilde{\mu} \) is simply a notational device used to indicate that there is no upper bound for some elements in the vector \( \tilde{x} \).
as required. We assume the statement holds for a $k$-step model, so we now wish to show it holds for a $(k + 1)$-step model. First, using proposition 4.8 and defining $n_1 = 2^{(k+1)}$ and $m_1 = n_1 - 1$, we have that

$$\tilde{I}_{n_1}^t \Phi_{N-(k+1)} = \tilde{I}_{n_1}^t \begin{bmatrix} \tilde{A}_{k+1} & B_{k+1} \end{bmatrix}.$$ 

Now, to save space, define $P = \tilde{I}_{n_1}^t \Phi_{N-(k+1)}$. Then we have that

$$P = \tilde{I}_{n_1}^t \begin{bmatrix} 1 + \phi & 0 \\ 0 & 1 - \phi \end{bmatrix} \otimes \Phi_{N-k}$$

$$= \begin{bmatrix} \tilde{I}_{n_1}^t \tilde{A}_{k+1} \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{I}_{n_1}^t \\ 0 \end{bmatrix} \begin{bmatrix} (1 + \phi) \Phi_{N-k} & 0_{n,m} \\ 0_{n,m} & (1 - \phi) \Phi_{N-k} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (1 + \phi) \tilde{I}_{n_1}^t \Phi_{N-k} + \tilde{I}_{n_1}^t 0_{n,m} + (1 - \phi) \tilde{I}_{n_1}^t \Phi_{N-k} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \tilde{0}_m' \\ \tilde{0}_m' \end{bmatrix}$$

$$= \tilde{0}_{2m+1}'$$

where we have used the fact that $\tilde{I}_{n_1}^t \tilde{A}_{k+1} = 0$ as it is simply a vector where the top half is the constant $\phi$, and the bottom half $-\phi$. Now, $2m + 1 = 2(2^k - 1) + 1 = 2^{(k+1)} - 1 = m_1$. Thus for $k = 1, \ldots, N$, we have that $\tilde{I}_n \Phi_{N-k} = \tilde{0}_m'$ as required.

**Lemma 4.15.** The matrix given by $\Phi_{N-k}' \Phi_{N-k}$ is a $(m \times m)$ diagonal matrix with only positive elements on the diagonal.

**Proof.** We again proceed by induction on $k$. For $k = 1$, we can easily compute $\Phi_{N-1}' \Phi_{N-1}$. This is given by

$$\Phi_{N-1}' \Phi_{N-1} = \begin{bmatrix} \phi & -\phi \\ \phi & -\phi \end{bmatrix} = 2\phi^2$$

which, being a scalar, is technically a diagonal matrix with a positive entry. We now assume that this holds for a $k$-step model, that is

$$\Phi_{N-k}' \Phi_{N-k} = D_{m,m}$$

where $D_{m,m}$ is an $(m \times m)$, diagonal matrix with only positive elements on the diagonal. We now move to a $(k + 1)$-step model. We have from
4.6. NUMERICAL VALUATION

proposition 4.8 that

\[
\Phi'_{N-(k+1)} \Phi_{N-(k+1)} = \begin{bmatrix} \tilde{A}_{k+1} & B_{k+1} \end{bmatrix}' \begin{bmatrix} \tilde{A}_{k+1} & B_{k+1} \end{bmatrix}
\]

\[
= \begin{bmatrix} \tilde{A}'_{k+1} \\ B'_{k+1} \end{bmatrix} \begin{bmatrix} \tilde{A}_{k+1} & B_{k+1} \end{bmatrix}
\]

\[
= \begin{bmatrix} \tilde{A}'_{k+1} \tilde{A}_{k+1} & \tilde{A}'_{k+1} B_{k+1} \\ B'_{k+1} \tilde{A}_{k+1} & B'_{k+1} B_{k+1} \end{bmatrix}. \quad (4.39)
\]

We now compute the sub-matrices to prove our assertion. First, consider the matrix \(\tilde{A}'_{k+1} \tilde{A}_{k+1}\). Since we are taking the inner product of two vectors, this is simply the sum of the square of the elements of \(\tilde{A}_{k+1}\) which is a positive scalar. Next, consider the matrix given by \(\tilde{A}'_{k+1} B_{k+1}\). We will show that this is equal to the vector \(\tilde{0}_{2m}\). To save space, let us define \(\tilde{Z} = \tilde{A}'_{k+1} B_{k+1}\), then we may write

\[
\tilde{Z} = \phi \tilde{1}' - \phi \tilde{1} \begin{bmatrix} (1 + \phi) \Phi_{N-k} & 0_{n,m} \\ 0_{n,m} & (1 - \phi) \Phi_{N-k} \end{bmatrix}
\]

\[
= \begin{bmatrix} \phi(1 + \phi) \tilde{1}' \Phi_{N-k} - \phi \tilde{1} 0_{n,m} & \phi \tilde{1}' 0_{n,m} - \phi(1 - \phi) \tilde{1}' \Phi_{N-k} \end{bmatrix}
\]

\[
= \begin{bmatrix} \tilde{0}'_m & \tilde{0}'_m \end{bmatrix}
\]

where we have used lemma 4.14 in the third line. Finally, we turn to the matrix \(B'_{k+1} B_{k+1}\). To save space, let us define \(\hat{B} = B'_{k+1} B_{k+1}\) We have that

\[
\hat{B} = \begin{bmatrix} (1 + \phi) \Phi'_{N-k} & 0_{m,n} \\ 0_{m,n} & (1 - \phi) \Phi'_{N-k} \end{bmatrix} \begin{bmatrix} (1 + \phi) \Phi_{N-k} & 0_{n,m} \\ 0_{n,m} & (1 - \phi) \Phi_{N-k} \end{bmatrix}
\]

\[
= \begin{bmatrix} (1 + \phi)^2 \Phi'_{N-k} \Phi_{N-k} & 0_{m,m} \\ 0_{m,m} & (1 - \phi)^2 \Phi'_{N-k} \Phi_{N-k} \end{bmatrix}
\]

\[
= \begin{bmatrix} (1 + \phi)^2 D_{m,m} & 0_{m,m} \\ 0_{m,m} & (1 - \phi)^2 D_{m,m} \end{bmatrix}.
\]
Substituting these matrices into (4.39) gives
\[ \Phi'_{N-(k+1)} \Phi'_{N-(k+1)} = \begin{bmatrix} 2n\phi^2 & \tilde{\phi}_{2m}' \\ \tilde{\phi}_{2m} & D_{2m,2m} \end{bmatrix}. \]
This is a square matrix with positive entries only on the diagonal as required. To confirm that we have the appropriate dimensions, we note that the matrix \( \Phi'_{N-(k+1)} \Phi'_{N-(k+1)} \) has dimensions \( (2m+1 \times 2m+1) \). Notice that \( 2m + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1 = m_1 \) as required. □

We now proceed to show that the matrix \( Q \) is indefinite by use of lemmas 4.14 and 4.15. This is presented in the following theorem.

**Theorem 4.16.** The matrix \( Q \) given by equation (4.38) is an indefinite, diagonal matrix.

**Proof.** First, consider the matrix \( R'_2 R_2 \). It is clear that
\[ R'_2 R_2 = \begin{bmatrix} 0_{m,n} \\ I_n \end{bmatrix} \begin{bmatrix} 0_{n,m} & I_n \end{bmatrix} = \begin{bmatrix} 0_{m,m} & 0_{m,n} \\ 0_{n,m} & I_n \end{bmatrix}. \]
Second, we have that \( R'_1 \Phi'_{N-k} \Phi'_{N-k} R_1 \) is given by
\[ R'_1 \Phi'_{N-k} \Phi'_{N-k} R_1 = \begin{bmatrix} I_m \\ 0_{n,m} \end{bmatrix} D_{m,m} \begin{bmatrix} I_m & 0_{m,n} \\ 0_{m,n} & 0_{n,n} \end{bmatrix} = \begin{bmatrix} D_{m,m} & 0_{m,n} \\ 0_{n,m} & 0_{n,n} \end{bmatrix}, \]
and thus \( Q \) can be written
\[ Q = \begin{bmatrix} 0_{m,m} & 0_{m,n} \\ 0_{n,m} & I_n \end{bmatrix} - \begin{bmatrix} D_{m,m} & 0_{m,n} \\ 0_{n,m} & 0_{n,n} \end{bmatrix} = \begin{bmatrix} -D_{m,m} & 0_{m,n} \\ 0_{n,m} & I_n \end{bmatrix}. \]
Hence \( Q \) is a diagonal matrix of dimension \((m + n \times m + n)\). More importantly, it is clear that the top left \((m \times m)\) entries will always be
negative, and the bottom right \((n \times n)\) entries will be positive always. Since the eigenvalues of a diagonal matrix are simply those entries themselves, this means that \(Q\) will have both positive and negative eigenvalues, thus it is an indefinite matrix.

Global optimisation of indefinite quadratic functions is a highly non-trivial task. The reason for this is that the stationary points in such problems are guaranteed to not be a maximum or a minimum, rather they correspond to internal saddle points. This property also implies that the optimal solution must lie along an edge of the constraint boundaries, rather than at a vertex. Despite these difficulties, there exist several algorithms which may be used to handle this class of problem. These algorithms can be split into two general classes. The first class is a special application of the branch and bound method described previously. This was developed by Pardalos et al. (1987) and is also discussed in Pardalos and Rosen (1987). The intuition behind this type of algorithm is to separate the indefinite objective function into a concave part, a convex part and a linear part. Approximate solutions are then obtained by using linear bounding functions and applying branch and bound techniques. The second class of algorithm developed by Kough (1979) uses a cutting plane technique based on a generalised Benders cut developed by Geoffrion (1972). Again the separability of the objective function into a concave and convex component is exploited, though rather than proceeding with a branch and bound technique, a set of inequalities which are specific to quadratic functions are added at each iteration to reduce the feasible space in the same way as our concavity cuts did for our cutting plane algorithm discussed earlier. This method is known to converge to the optimal solution, though it is computationally expensive.

Unfortunately the algorithms we have discussed above for the indefinite quadratic program suffer similar problems as the general branch and bound or cutting plane algorithms for the maximisation of general
convex functions. Whilst some efficiency\textsuperscript{17} may be gained by the quadratic structure of the problem, in order to obtain this form we have also more than doubled its size.\textsuperscript{18} Since we already suffer from the curse of dimensionality, we have decided against following this path as the increase in the number of changing variables is likely to inhibit any efficiency gains from the more desirable problem structure.

4.7. Concluding Remarks

In this section we have examined the problem of computing the value of an option on a traded account under the assumption of a binomial model. Perhaps the most challenging aspect of this valuation is determining the optimal trading strategy which we tackle in several different ways. First, analytic optimal strategies were derived for one and two step models by an application of dynamic programming. A general analytic strategy for the special case of the passport option was derived, however obtaining the equivalent strategy for the option on a traded account was shown to be an extremely difficult, and perhaps impossible task due to the polynomial structure of the account paths.

For this reason we turned to numerical methods where it was shown that the optimisation problem associated with this valuation is a global convex maximisation problem which is known to be NP-hard. Given this result, we do not expect to have an efficient numerical solution available to us. Nonetheless, we examine several different algorithms including the cutting plane, branch and bound and indefinite QP which, given sufficient time, would solve the optimisation problem. Unfortunately we have been unable to determine an efficient algorithm which would be of practical use for pricing options on traded accounts.

Though we have been ultimately unsuccessful in valuing options on traded accounts in discrete time, we have gained significant insight into the problem at hand and have managed to value the passport option.

\textsuperscript{17}Often a specific problem structure, e.g. quadratic, may allow one to perform deeper cuts, thus requiring fewer iterations.

\textsuperscript{18}The original $k$-step option valuation problem had $2^k - 1$ changing variables while the quadratic transformation of this problem now has $2^{k+1} - 1$ changing variables.
 Whilst it was understood that the problem would grow exponentially quickly, it was not expected that determining the optimal strategy would pose as significant a challenge as it inevitably has. It was also not expected that the analytic strategy would be so complicated as to make it impossible to express in terms of simple functions. In the following sections we consider some of the implications of the results we have obtained and outline some paths for future research.
CHAPTER 5

Conclusion and Future Research

In this thesis we have thoroughly examined the problem of valuing options on traded accounts using both continuous and discrete time models. Whilst a closed form solution for the continuous time model was previously known, we have applied our own techniques to derive the same formulas. In the process, we have provided extended detail regarding these derivations and also discovered some small errors in the formulas published which appear to have gone unnoticed in the field. We have also presented our own, alternative derivation for the special case of passport options. We believe this approach to be a simpler method than those previously discussed as it doesn’t require a specific derivation of the Green’s function or the solution of the Kolmogorov forward equation to determine the transition density.

While continuous time models for options on traded accounts have been covered in some detail in the existing literature, there are still extensions left which we believe could provide a fruitful area for future research. For example, much of the present literature has only considered options on a single asset. Since this type of option is a perfect hedge for an actively managed fund, a formula for valuing options on traded accounts for several correlated assets would be a new and useful result from both a theoretical and practical standpoint. There has been some work performed on numerical methods to solve the HJB equation which governs the value function of options on traded accounts for two assets, for example Ahn et al. (1999) provides the relevant multivariate HJB equation and plots solutions for options on two assets, but little detail is provided on the actual algorithm used. Closed form solutions to such problems are however still very much an open problem. The problem is not left untouched however. Kampen (2007) has derived the
optimal strategies for such options under the assumptions of correlated geometric Brownian motions, though the derivation of the pricing formula is yet to be obtained. The approach taken by Kampen (2007) is not dissimilar to Shreve and Vecer (2000) in that a generalised version of the mean comparison theorem of Hajek (1985) is used for this purpose, though the implementation is far more involved than for the univariate case and is found to depend upon the asset correlations. A derivation of this pricing formula is by no means a trivial exercise, and thus we believe that such a challenging problem would provide fertile ground for future research.

For the binomial model of the option on a traded account, there is still much work to be done. This is because, as far as the author is aware, this thesis represents the first time that the binomial model has been thoroughly examined and applied to the problem of valuing these types of options. Whilst we have ultimately been unsuccessful in finding an efficient and practical method to value options on traded accounts using this model, we have still gained considerable insight and discovered some results which we believe add significantly to the existing literature.

Firstly, we showed that in the general $k$-step model, the analytic optimal strategy requires determining the roots of a $(k-1)$ order polynomial. It is well known that the formulas for such roots do not exist, and as such the optimal strategy for such a model may not yield an analytic solution. Luckily, for the special case of the passport option, we were able to use the additional symmetry and a clever induction argument to determine that the optimal strategy is in fact the same as that found in the continuous time case, that being

$$\theta_t^* = -\text{sgn}(X_t).$$

Whilst it may be expected that such a property would hold in general, we showed via the analytic solution to the one and two-step models that this property doesn’t hold for the more general option on a traded account. To be more precise, while the optimal trading strategy in
continuous time is given by the rather simple formula
\[ \theta_t^* = \alpha \mathbb{I}(X_t > \frac{1}{2}(\alpha + \beta)S_t) + \beta \mathbb{I}(X_t \leq \frac{1}{2}(\alpha + \beta)S_t), \]
this function doesn’t provide the the maximum option value when a binomial model is employed. Rather, the optimal strategy function for the binomial model is dependent not only on the current state variables, but also on the values of the parameters which define our binomial tree in a very complicated manner. Thus, to solve the optimisation problem associated with the option on a traded account, we must turn to numerical methods.

Unfortunately, due to the problem structure, we found that the standard approach of dynamic programming was not numerically applicable and thus we were forced to treat the problem as one of mathematical programming. Unfortunately it was found by transforming the problem into an \( l_1 \)-norm maximisation that the problem at hand is in fact NP-hard. This was an unexpected result given that the continuous time models had reasonably simple optimal trading strategy functions. We thus explored several algorithms and transformations designed to solve such large-scale global convex maximisation problems. Unfortunately, while these algorithms will determine the optimal strategy, they are not efficient enough to be considered practical for the current problem. We do however believe that this area of numerical optimisation could be a good candidate for future research. The algorithms we presented would be considered somewhat standard, however in these types of problems a specifically written algorithm is usually required. Often, better results are obtained by combining several of the discussed algorithms into a hybrid method. For example, cutting planes and branch and bound algorithms have been successfully combined to form a so called branch-and-cut method which shows promise in solving some quadratic optimisations which are similar to the problems we have encountered.

As in the continuous time case, multivariate models could also be examined. At present however, we believe that this area of research may be intractable with present algorithms and computing power. The
curse of dimensionality has proven to be a very difficult problem to overcome for our model consisting of a single asset, and this will only be exacerbated when several assets are considered. Nonetheless, our model could be extended to account for several assets without too much trouble as the dynamics we have chosen are in fact a special case of the He (1990) model which describes several correlated assets via multiple, multi-nomial trees.

Options on traded accounts, being able to describe almost any other type of option, are necessarily very difficult to value, thus any extensions to the models already presented represent a huge challenge to any potential researchers. These challenges however, also provide fertile ground for future research as they require advanced, and most likely new methods in stochastic analysis and large-scale global optimisation.
References


APPENDIX A

Derivation of the Two-Step Optimal Strategy

In this appendix we will provide detailed calculations involved in the derivation of the optimal strategy for the option on a traded account, using a binomial model, at two steps from maturity. In chapter 4 the optimal strategy at two steps from maturity was derived for the case where $\alpha > 0$ and $|\beta| > |\alpha|$. In this section, we will apply the same analysis to the remaining cases.

A.1. Derivation of $\theta^*_N - 2$ when $\alpha < 0$ and $|\beta| > |\alpha|$

The first step to obtaining the optimal strategy is to order the paths. Since this order depends on the value of $\phi$, we will break the analysis into sections. First, we need to identify where $-\frac{\alpha}{\beta}$ and $\frac{\alpha + \beta}{2\beta}$ lie relative to one another.

$$\frac{\alpha + \beta}{2\beta} > -\frac{\alpha}{\beta} \Rightarrow \beta > -3\alpha.$$  

Thus means that the relative positions of these two terms may switch depending on the values of $\alpha$ and $\beta$. First we will consider $-\frac{\alpha}{\beta} > \frac{\alpha + \beta}{2\beta}$ and later we will consider the alternative case. We also point out that since $|\beta| > |\alpha|$, this implies that $\theta^*_N = \beta$.

A.1.1. $\phi \in (0, \frac{\alpha + \beta}{2\beta})$, $\beta < -3\alpha$. We may use table 1 in chapter 4 to determine the order for this range of $\phi$. The order of the paths in this case is:

$$Z_N^{uu}(\beta) > Z_N^{du}(\alpha) > Z_N^{uu}(\alpha) > Z_N^{dd}(\beta) = Z_N^{ud}(\beta) > Z_N^{dd}(\alpha) > Z_N^{ud}(\alpha) > Z_N^{dd}(\beta)$$

**Region 1:** $Z_N > 2\beta\phi - \beta\phi^2$. In this case all outcomes are positive, and thus

$$\begin{align*}
(U_N|\theta_N = \beta) &= \frac{\rho^2 S_{N-2}}{4}[4Z_N] \\
&= \rho^2 X_N

(U_N|\theta_N = \alpha) &= \frac{\rho^2 S_{N-2}}{4}[4Z_N] \\
&= \rho^2 X_N
\end{align*}$$
thus we are indifferent about which strategy is chosen in this region.

Region 2: \(-\alpha \phi + \beta \phi + \beta \phi^2 < Z_{N-2} < 2\beta \phi - \beta \phi^2\). In this case we have that

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + 2\beta \phi + \beta \phi^2 \\
+Z_{N-2} - \beta \phi^2 \\
+Z_{N-2} - \beta \phi^2 \\
-Z_{N-2} + 2\beta \phi - \beta \phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}].
\]

Now, since \(2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4Z_{N-2} \Rightarrow Z_{N-2} < 2\beta \phi - \beta \phi^2\), we have that \(\theta_{N-2} = \beta\) is optimal in this region.

Region 3: \(\alpha \phi + \beta \phi - \beta \phi^2 < Z_{N-2} < -\alpha \phi + \beta \phi + \beta \phi^2\). The outcomes are now

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2 \\
-Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\
+Z_{N-2} - \alpha \phi + \beta \phi - \beta \phi^2 \\
+Z_{N-2} - \alpha \phi - \beta \phi + \beta \phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 2\beta \phi - 2\alpha \phi + 2\beta \phi^2].
\]

Now, since \(2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 2Z_{N-2} + 2\beta \phi - 2\alpha \phi + 2\beta \phi^2 \Rightarrow \phi < \frac{\alpha + \beta}{2\rho}\), we again have that \(\theta_{N-2} = \beta\) is optimal.

Region 4: \(\beta \phi^2 < Z_{N-2} < \alpha \phi + \beta \phi - \beta \phi^2\). In this case, we have that

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2 \\
-Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\
+Z_{N-2} - \alpha \phi + \beta \phi - \beta \phi^2 \\
-Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [4\beta \phi].
\]

Since \(2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4\beta \phi \Rightarrow Z_{N-2} > \beta \phi^2\), this implies that \(\theta_{N-2} = \beta\) is optimal.
Since \( -\theta \) and so we are again indifferent about the choice of \( -\theta \).

Region 5: \( -\alpha \phi - \beta \phi - \beta \phi^2 < Z_{N-2} < \beta \phi^2 \). In this region we have that

\[
(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + 2\beta \phi + \beta \phi^2 \\
-Z_{N-2} + \beta \phi^2 \\
-Z_{N-2} + \beta \phi^2 \\
-Z_{N-2} + 2\beta \phi - \beta \phi^2 
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2],
\]

\[
(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4\beta \phi].
\]

Since \( -2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > 4\beta \phi \Rightarrow Z_{N-2} < \beta \phi^2 \), again it is optimal to take \( \theta_{N-2} = \beta \).

Region 6: \( \alpha \phi - \beta \phi + \beta \phi^2 < Z_{N-2} < -\alpha \phi - \beta \phi - \beta \phi^2 \). We now have

\[
(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2],
\]

\[
(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \alpha \phi - \beta \phi - \beta \phi^2 \\
-Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \\
-Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \\
-Z_{N-2} + 2\alpha \phi - 2\beta \phi - 2\beta \phi^2 
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 2\alpha \phi + 2\beta \phi - 2\beta \phi^2].
\]

Since \( -2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > -2Z_{N-2} - 2\alpha \phi + 2\beta \phi - 2\beta \phi^2 \Rightarrow \phi < \frac{(\alpha + \beta)}{2\beta} \), \( \theta_{N-2} = \beta \) is again optimal.

Region 7: \( -2\beta \phi - \beta \phi^2 < Z_{N-2} < \alpha \phi - \beta \phi + \beta \phi^2 \). In this case we have

\[
(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2],
\]

\[
(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}].
\]

Since \( -2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > -4Z_{N-2} \Rightarrow -2\beta \phi - \beta \phi^2 < Z_{N-2} \), we have that \( \theta_{N-2} = \beta \) is optimal.

Region 8: \( Z_{N-2} < -2\beta \phi - \beta \phi^2 \). In this case, we have that all account outcomes are negative, thus

\[
(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],
\]

\[
(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],
\]

and so we are again indifferent about the choice of \( \theta_{N-2} \).
A.1.2. $\phi \in \left(\frac{\alpha + \beta}{2\beta}, \frac{\alpha}{\beta}\right)$, $\beta < -3\alpha$. In this case the order of the paths is:

$Z_{uu}(\beta) > Z_{du}^d(\alpha) > Z_{uu}^u(\alpha) > Z_{dd}^d(\alpha) = Z_{uu}^u(\beta) > Z_{dd}^d(\beta) > Z_{dd}(\alpha)$

Region 1: $-\alpha \phi + \beta \phi + \beta \phi^2 < Z_{N-2}$. In this region, we have

$$ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}] $$

$$ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}] $$

as all paths are positive. Thus we are indifferent about our choice of $\theta_{N-2}$ in this region.

Region 2: $2\beta \phi - \beta \phi^2 < Z_{N-2} < -\alpha \phi + \beta \phi + \beta \phi^2$. We now have that

$$ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}] $$

$$ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2] $$

Now, since $4Z_{N-2} > 2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2 \Rightarrow Z_{N-2} > -\alpha \phi + \beta \phi + \beta \phi^2$ which is false, we thus have that $\theta_{N-2} = \alpha$ is optimal in this region.

Region 3: $\beta \phi^2 < Z_{N-2} < 2\beta \phi - \beta \phi^2$. We have

$$ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2], $$

$$ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2]. $$

Now, $2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2 \Rightarrow \phi < \frac{\alpha + \beta}{2\beta}$ which is false in this region and thus $\theta_{N-2} = \alpha$ is optimal.
Region 4: $\alpha \phi + \beta \phi - \beta \phi^2 < Z_{N-2} < \beta \phi^2$. We now have

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array} {c} Z_{N-2} + 2\beta \phi + \beta \phi^2 \\ -Z_{N-2} + \beta \phi^2 \\ -Z_{N-2} + \beta \phi^2 \\ -Z_{N-2} + 2\beta \phi - \beta \phi^2 \end{array} \right]$$

$$= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2].$$

Now, since $-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > 2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2$ implies that $\frac{1}{2}(\alpha + \beta) \phi > Z_{N-2}$, we have to ascertain if $\frac{1}{2}(\alpha + \beta) \phi$ is within the bounds of the interval. It is easy to show that it is and thus if $\frac{1}{2}(\alpha + \beta) \phi > Z_{N-2}$, then it is optimal to take $\theta_{N-2} = \beta$, while if $\frac{1}{2}(\alpha + \beta) \phi < Z_{N-2}$ the $\theta_{N-2} = \alpha$ is optimal.

Region 5: $\alpha \phi - \beta \phi + \beta \phi^2 < Z_{N-2} < \alpha \phi + \beta \phi - \beta \phi^2$. We now have

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array} {c} Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2 \\ -Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\ +Z_{N-2} - \alpha \phi + \beta \phi - \beta \phi^2 \\ -Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \end{array} \right]$$

$$= \frac{\rho^2 S_{N-2}}{4} [4\beta \phi].$$

Comparing, we get $-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > 4\beta \phi \Rightarrow Z_{N-2} < \beta \phi^2$ which is true and thus $\theta_{N-2} = \beta$ is optimal.

Region 6: $-\alpha \phi - \beta \phi - \beta \phi^2 < Z_{N-2} < \alpha \phi - \beta \phi + \beta \phi^2$. In this region,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array} {c} Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2 \\ -Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\ -Z_{N-2} + \alpha \phi - \beta \phi + \beta \phi^2 \\ -Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \end{array} \right]$$

$$= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\alpha \phi + 2\beta \phi + 2\beta \phi^2].$$

Since $\beta > \alpha$, we have that $\theta_{N-2} = \beta$ is again optimal.
Region 7: \(-2\beta\phi - \beta\phi^2 < Z_{N-2} < -\alpha\phi - \beta\phi - \beta\phi^2\). We have in this region that

\[
(U_{N-2}\mid \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2],
\]

\[
(U_{N-2}\mid \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[-4Z_{N-2}],
\]

thus since \(-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > -4Z_{N-2} \Rightarrow -2\beta\phi - \beta\phi^2 < Z_{N-2}\), this means that \(\theta_{N-2} = \beta\) is optimal.

Region 8: \(Z_{N-2} < -2\beta\phi - \beta\phi^2\). In this region all paths are negative, thus

\[
(U_{N-2}\mid \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[-4Z_{N-2}],
\]

\[
(U_{N-2}\mid \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[-4Z_{N-2}],
\]

so we are indifferent about the choice of \(\theta_{N-2}\) in this region.

A.1.3. \(\phi \in (\frac{-\alpha}{\beta}, 1), \beta < -3\alpha\). In this case the order of the paths is:

\[
Z_N^{uu}(\beta) > Z_N^{uu}(\alpha) > Z_N^{dd}(\alpha) > Z_N^{ud}(\beta) = Z_N^{dd}(\beta) > Z_N^{ud}(\beta) > Z_N^{ud}(\alpha)
\]

Region 1: \(-\alpha\phi + \beta\phi + \beta\phi^2 < Z_{N-2}\). In this region all outcomes are positive, and so

\[
(U_{N-2}\mid \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[4Z_{N-2}],
\]

\[
(U_{N-2}\mid \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[4Z_{N-2}],
\]

therefore we are indifferent about the choice of \(\theta_{N-2}\).

Region 2: \(2\beta\phi - \beta\phi^2 < Z_{N-2} < -\alpha\phi + \beta\phi + \beta\phi^2\). In this case,

\[
(U_{N-2}\mid \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[4Z_{N-2}],
\]

\[
(U_{N-2}\mid \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\ -Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\
+Z_{N-2} + \alpha\phi + \beta\phi - \beta\phi^2 \\
+Z_{N-2} - \alpha\phi - \beta\phi + \beta\phi^2 \\
\end{array} \right] 
\]

\[
= \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2].
\]

Since \(4Z_{N-2} > 2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow Z_{N-2} > -\alpha\phi + \beta\phi + \beta\phi^2\) which is false, it is clear that \(\theta_{N-2} = \alpha\) in this region.
A.1. DERIVATION OF $\theta^*_N$ 

Region 3: $\beta\phi^2 < Z_{N-2} < 2\beta\phi - \beta\phi^2$. We now have that

$$(U_{N-2}[\theta_{N-2} = \beta]) = \frac{\rho^2S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + 2\beta\phi + \beta\phi^2 \\ Z_{N-2} - \beta\phi^2 \\ Z_{N-2} + \beta\phi^2 \\ -Z_{N-2} + 2\beta\phi - \beta\phi^2 \end{array} \right]$$

$$= \frac{\rho^2S_{N-2}}{4} [2Z_{N-2} + 4\beta\phi - 2\beta\phi^2],$$

$$(U_{N-2}[\theta_{N-2} = \alpha]) = \frac{\rho^2S_{N-2}}{4} [2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2].$$

Since $2Z_{N-2} + 4\beta\phi - 2\beta\phi^2 > 2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow \phi < \frac{\alpha + \beta}{2\beta}$ which is false, $\theta_{N-2} = \alpha$ is optimal.

Region 4: $\alpha\phi + \beta\phi - \beta\phi^2 < Z_{N-2} < \beta\phi^2$. In this case,

$$(U_{N-2}[\theta_{N-2} = \beta]) = \frac{\rho^2S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + 2\beta\phi + \beta\phi^2 \\ -Z_{N-2} + \beta\phi^2 \\ -Z_{N-2} + \beta\phi^2 \\ -Z_{N-2} + 2\beta\phi - \beta\phi^2 \end{array} \right]$$

$$= \frac{\rho^2S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2],$$

$$(U_{N-2}[\theta_{N-2} = \alpha]) = \frac{\rho^2S_{N-2}}{4} [2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2].$$

Since $-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > 2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow \frac{1}{2}(\alpha + \beta)\phi > Z_{N-2}$, we have that if $\frac{1}{2}(\alpha + \beta)\phi > Z_{N-2}$ then $\theta_{N-2} = \beta$ and if $\frac{1}{2}(\alpha + \beta)\phi < Z_{N-2}$, then $\theta_{N-2} = \alpha$.

Region 5: $\alpha\phi - \beta\phi - \beta\phi^2 < Z_{N-2} < \alpha\phi + \beta\phi - \beta\phi^2$. We have that

$$(U_{N-2}[\theta_{N-2} = \beta]) = \frac{\rho^2S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2],$$

$$(U_{N-2}[\theta_{N-2} = \alpha]) = \frac{\rho^2S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\ -Z_{N-2} - \alpha\phi + \beta\phi + \beta\phi^2 \\ +Z_{N-2} - \alpha\phi + \beta\phi - \beta\phi^2 \\ -Z_{N-2} + \alpha\phi + \beta\phi - \beta\phi^2 \end{array} \right]$$

$$= \frac{\rho^2S_{N-2}}{4} [4\beta\phi].$$

Since $-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > 4\beta\phi \Rightarrow Z_{N-2} < \beta\phi^2$ which is true, we have that $\theta_{N-2} = \beta$ is optimal in this region.

Region 6: $-\alpha\phi - \beta\phi - \beta\phi^2 < Z_{N-2} < \alpha\phi - \beta\phi + \beta\phi^2$. We have that

$$(U_{N-2}[\theta_{N-2} = \beta]) = \frac{\rho^2S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2],$$
A.1. DERIVATION OF $\theta^*_N - 2$

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\
-Z_{N-2} - \alpha\phi + \beta\phi + \beta\phi^2 \\
-Z_{N-2} + \alpha\phi - \beta\phi + \beta\phi^2 \\
-Z_{N-2} + \alpha\phi + \beta\phi - \beta\phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\alpha\phi + 2\beta\phi + 2\beta\phi^2].
\]

Since $-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > -2Z_{N-2} + 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow \beta > \alpha$, $\theta_{N-2} = \beta$ is optimal.

Region 7: $-2\beta\phi - \beta\phi^2 < Z_{N-2} < -\alpha\phi - \beta\phi - \beta\phi^2$. In this case, since all outcomes are negative when $\theta_{N-2} = \alpha$,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}].
\]

Since $-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > -4Z_{N-2} \Rightarrow -2\beta\phi - \beta\phi^2 < Z_{N-2}$, we again have that $\theta_{N-2} = \beta$.

Region 8: $Z_{N-2} < -2\beta\phi - \beta\phi^2$. In this case all account outcomes are negative regardless of the choice of $\theta_{N-2}$, thus

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],
\]

and we are hence indifferent as to the choice of $\theta_{N-2}$.

Having obtained the optimal strategy for $\theta_{N-2}$ in all regions when $\beta < -3\alpha$, we now repeat the procedure for the case where $\beta > -3\alpha$.

A.1.4. $\phi \in (0, \frac{-\alpha}{\beta})$, $\beta > -3\alpha$. The order of the paths in this case is:

$Z_N^{uu}(\beta) > Z_N^{du}(\alpha) > Z_N^{uu}(\alpha) > Z_N^{du}(\beta) = Z_N^{dd}(\beta) > Z_N^{dd}(\alpha) > Z_N^{uu}(\beta)$

Region 1: $2\beta\phi - \beta\phi^2 < Z_{N-2}$. In this case all outcomes are positive and thus we have that

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}]
\]

and thus we are indifferent in this region.
A.1. DERIVATION OF $\theta^*_N$

Region 2: $-\alpha \phi + \beta \phi + \beta \phi^2 < Z_{N-2} < 2\beta \phi - \beta \phi^2$. In this case we have that

$$(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + 2\beta \phi + \beta \phi^2 \\ +Z_{N-2} - \beta \phi^2 \\ +Z_{N-2} - \beta \phi^2 \\ -Z_{N-2} + 2\beta \phi - \beta \phi^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2]$$

$$(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}].$$

Since $2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4Z_{N-2} \Rightarrow Z_{N-2} < 2\beta \phi - \beta \phi^2$ which is true, we have that $\theta_{N-2} = \beta$ is optimal.

Region 3: $\alpha \phi + \beta \phi - \beta \phi^2 < Z_{N-2} < -\alpha \phi + \beta \phi + \beta \phi^2$. We now have that

$$(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2]$$

$$(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}].$$

Since $2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4Z_{N-2} \Rightarrow Z_{N-2} < 2\beta \phi - \beta \phi^2$ which is true, again $\theta_{N-2} = \beta$ is optimal.

Region 4: $\beta \phi^2 < Z_{N-2} < \alpha \phi + \beta \phi - \beta \phi^2$. We now have that

$$(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 4\beta \phi - 2\beta \phi^2]$$

$$(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4\beta \phi].$$

Since $2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4\beta \phi \Rightarrow \beta \phi^2 < Z_{N-2}$ which is true, $\theta_{N-2} = \beta$ is optimal.
Region 5: $-\alpha \phi - \beta \phi - \beta \phi^2 < Z_{N-2} < \beta \phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + 2\beta \phi + \beta \phi^2 \\ -Z_{N-2} + \beta \phi^2 \\ -Z_{N-2} + \beta \phi^2 \\ -Z_{N-2} + 2\beta \phi - \beta \phi^2 \end{bmatrix} = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4\beta \phi].$$

Since $-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > 4\beta \phi \Rightarrow Z_{N-2} < \beta \phi^2$ which is true, $\theta_{N-2} = \beta$ is optimal.

Region 6: $\alpha \phi - \beta \phi + \beta \phi^2 < Z_{N-2} < -\alpha \phi - \beta \phi - \beta \phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \alpha \phi - \beta \phi - \beta \phi^2 \\ -Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\ +Z_{N-2} - \alpha \phi + \beta \phi - \beta \phi^2 \\ -Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \end{bmatrix} = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 2\alpha \phi + 2\beta \phi - 2\beta \phi^2].$$

Since $-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > -2Z_{N-2} - 2\alpha \phi + 2\beta \phi - 2\beta \phi^2 \Rightarrow \frac{\alpha + \beta}{2\beta} < \phi$, which is true always since $\phi > 0$, thus $\theta_{N-2} = \beta$ is again optimal.

Region 7: $-2\beta \phi - \beta \phi^2 < Z_{N-2} < \alpha \phi - \beta \phi + \beta \phi^2$. Now we have that

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}].$$

Since $-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > -4Z_{N-2} \Rightarrow -2\beta \phi - \beta \phi^2 < Z_{N-2}$ which is true, $\theta_{N-2} = \beta$ is optimal.

Region 8: $Z_{N-2} < -2\beta \phi - \beta \phi^2$. In this case all outcomes are negative, so

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}].$$

and thus we are indifferent in this region.
A.1.5. $\phi \in (\frac{-\alpha}{\beta}, \frac{\alpha+\beta}{2\beta})$, $\beta > -3\alpha$. In this case the order of the paths is:

$Z_N(u \beta) > Z_N(u \alpha) > Z_N^d(u \alpha) = Z_N^d(u \beta) = Z_N^d(u \alpha) > Z_N^d(u \alpha) > Z_N^d(\beta)$

Region 1: $2\beta \phi - \beta \phi^2 < Z_{N-2}$. In this case all account outcomes are positive and thus

$$(U_{N-2} | \theta_{N-2} = \beta) = \rho^2 S_{N-2}^2 \frac{[4Z_{N-2}]}{4}$$

$$(U_{N-2} | \theta_{N-2} = \alpha) = \rho^2 S_{N-2}^2 \frac{[4Z_{N-2}]}{4},$$

thus we are indifferent about the choice of $\theta_{N-2}$ in this case.

Region 2: $-\alpha \phi + \beta \phi + \beta \phi^2 < Z_{N-2} < 2\beta \phi - \beta \phi^2$. In this case,

$$(U_{N-2} | \theta_{N-2} = \beta) = \rho^2 S_{N-2}^2 \frac{[Z_{N-2} + 2\beta \phi + \beta \phi^2] + Z_{N-2} - \beta \phi^2 + Z_{N-2} - \beta \phi^2 - Z_{N-2} + 2\beta \phi - \beta \phi^2]}{4}$$

$$= \rho^2 S_{N-2}^2 \frac{[2Z_{N-2} + 4\beta \phi - 2\beta \phi^2]}{4}$$

$$(U_{N-2} | \theta_{N-2} = \alpha) = \rho^2 S_{N-2}^2 \frac{[4Z_{N-2}]}{4}.$$

Since $2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4Z_{N-2} \Rightarrow -\alpha \phi + \beta \phi + \beta \phi^2 < Z_{N-2}$ which is true, $\theta_{N-2} = \beta$ is optimal.

Region 3: $\alpha \phi + \beta \phi - \beta \phi^2 < Z_{N-2} < -\alpha \phi + \beta \phi + \beta \phi^2$. We have that,

$$(U_{N-2} | \theta_{N-2} = \beta) = \rho^2 S_{N-2}^2 \frac{[2Z_{N-2} + 4\beta \phi - 2\beta \phi^2]}{4}$$

$$(U_{N-2} | \theta_{N-2} = \alpha) = \rho^2 S_{N-2}^2 \frac{[Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2] - Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 + Z_{N-2} - \alpha \phi - \beta \phi + \beta \phi^2]}{4}$$

$$= \rho^2 S_{N-2}^2 \frac{[2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2]}{4}.$$

Since $2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 2Z_{N-2} - 2\alpha \phi + 2\beta \phi + 2\beta \phi^2 \Rightarrow \frac{\alpha + \beta}{2\beta} > \phi$ which is true, $\theta_{N-2} = \beta$.

Region 4: $\beta \phi^2 < Z_{N-2} < \alpha \phi + \beta \phi - \beta \phi^2$. In this case,

$$(U_{N-2} | \theta_{N-2} = \beta) = \rho^2 S_{N-2}^2 \frac{[2Z_{N-2} + 4\beta \phi - 2\beta \phi^2]}{4}.$$
\( (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2 \\ -Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\ +Z_{N-2} - \alpha \phi + \beta \phi - \beta \phi^2 \\ -Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \end{bmatrix} \)

\[ = \frac{\rho^2 S_{N-2}}{4} [4\beta \phi]. \]

Since \( 2Z_{N-2} + 4\beta \phi - 2\beta \phi^2 > 4\beta \phi \Rightarrow \beta \phi^2 < Z_{N-2} \) which is true, \( \theta_{N-2} = \beta \) is optimal.

**Region 5:** \( \alpha \phi - \beta \phi + \beta \phi^2 < Z_{N-2} < \beta \phi^2 \). We have in this case,

\( (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + 2\beta \phi + \beta \phi^2 \\ -Z_{N-2} + \beta \phi^2 \\ -Z_{N-2} + \beta \phi^2 \\ -Z_{N-2} + 2\beta \phi - \beta \phi^2 \end{bmatrix} \)

\[ = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2], \]

\( (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4\beta \phi]. \)

Since \( -2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > 4\beta \phi \Rightarrow \beta \phi^2 > Z_{N-2} \) which is true, \( \theta_{N-2} = \beta \).

**Region 6:** \( -\alpha \phi - \beta \phi - \beta \phi^2 < Z_{N-2} < \alpha \phi - \beta \phi + \beta \phi^2 \). In this case,

\( (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2], \)

\( (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \alpha \phi + \beta \phi + \beta \phi^2 \\ -Z_{N-2} - \alpha \phi + \beta \phi + \beta \phi^2 \\ -Z_{N-2} + \alpha \phi - \beta \phi + \beta \phi^2 \\ -Z_{N-2} + \alpha \phi + \beta \phi - \beta \phi^2 \end{bmatrix} \)

\[ = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\alpha \phi + 2\beta \phi + 2\beta \phi^2]. \]

Since \( -2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > -2Z_{N-2} + 2\alpha \phi + 2\beta \phi + 2\beta \phi^2 \Rightarrow \beta > \alpha \) which is true always, \( \theta_{N-2} = \beta \).

**Region 7:** \( -2\beta \phi - \beta \phi^2 < Z_{N-2} < -\alpha \phi - \beta \phi - \beta \phi^2 \). Now, in this case,

\( (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta \phi + 2\beta \phi^2], \)

\( (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}] \).

Since \( -2Z_{N-2} + 4\beta \phi + 2\beta \phi^2 > -4Z_{N-2} \Rightarrow -2\beta \phi - \beta \phi^2 < Z_{N-2} \) which is true, we have that \( \theta_{N-2} = \beta \) is optimal.
Region 8: \( Z_{N-2} < -2\beta\phi - \beta\phi^2 \). In this case, all account paths are negative valued and thus

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[-4Z_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[-4Z_{N-2}],
\]

hence we are indifferent about the choice for \( \theta_{N-2} = \beta \).

A.1.6. \( \phi \in (\frac{\alpha+\beta}{2\beta}, 1) \), \( \beta > -3\alpha \). In this case the order of the paths is:

\[
Z_N^{uw}(\beta) > Z_N^{wu}(\alpha) > Z_N^{dd}(\alpha) > Z_N^{ud}(\beta) = Z_N^{du}(\beta) > Z_N^{dd}(\beta) > Z_N^{ud}(\alpha)
\]

(A.1)

Region 1: \( -\alpha\phi + \beta\phi + \beta\phi^2 < Z_{N-2} \). In this case all account outcomes are positive, thus

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[4Z_{N-2}]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[4Z_{N-2}],
\]

and thus we are indifferent about the choice of \( \theta_{N-2} \) in this region.

Region 2: \( 2\beta\phi - \beta\phi^2 < Z_{N-2} < -\alpha\phi + \beta\phi + \beta\phi^2 \). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[4Z_{N-2}]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}\begin{bmatrix}
Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\
-Z_{N-2} - \alpha\phi + \beta\phi + \beta\phi^2 \\
+Z_{N-2} - \alpha\phi + \beta\phi - \beta\phi^2 \\
+Z_{N-2} - \alpha\phi - \beta\phi + \beta\phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2].
\]

Since \( 4Z_{N-2} > 2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow Z_{N-2} > -\alpha\phi + \beta\phi + \beta\phi^2 \) which is false, we have that \( \theta_{N-2} = \alpha \) in this region.

Region 3: \( \beta\phi^2 < Z_{N-2} < 2\beta\phi - \beta\phi^2 \). In this case we have that,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}\begin{bmatrix}
Z_{N-2} + 2\beta\phi + \beta\phi^2 \\
+Z_{N-2} - \beta\phi^2 \\
+Z_{N-2} - \beta\phi^2 \\
-Z_{N-2} + 2\beta\phi - \beta\phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} + 4\beta\phi - 2\beta\phi^2]
\]
\((U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2].\)

Since \(Z_{N-2} - \alpha\phi + 2\beta\phi + 2\beta\phi^2 > 2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow \frac{\alpha + \beta}{2\beta} > \phi\) which is false, we have that \(\theta_{N-2} = \alpha\) in this region.

Region 4: \(\alpha\phi + \beta\phi - \beta\phi^2 < Z_{N-2} - \beta\phi^2\). We have that

\[(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + 2\beta\phi + \beta\phi^2 \\ -Z_{N-2} + \beta\phi^2 \\ -Z_{N-2} + \beta\phi - \beta\phi^2 \end{array} \right] \]

\[(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2].\]

Since \(-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > 2Z_{N-2} - 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow \frac{1}{2}(\alpha + \beta)\phi > Z_{N-2}\), if this condition holds then \(\theta_{N-2} = \beta\), however if \(\frac{1}{2}(\alpha + \beta)\phi < Z_{N-2}\) then \(\theta_{N-2} = \alpha\) is optimal.

Region 5: \(\alpha\phi - \beta\phi + \beta\phi^2 < Z_{N-2} - \alpha\phi + \beta\phi - \beta\phi^2\). In this case,

\[(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2] \]

\[(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\ -Z_{N-2} - \alpha\phi + \beta\phi + \beta\phi^2 \\ +Z_{N-2} - \alpha\phi + \beta\phi - \beta\phi^2 \end{array} \right] \]

\[(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [4\beta\phi].\]

Since \(-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > 4\beta\phi \Rightarrow \beta\phi^2 > Z_{N-2}\) which is true, \(\theta_{N-2} = \beta\) is optimal.

Region 6: \(-\alpha\phi + \beta\phi - \beta\phi^2 < Z_{N-2} < \alpha\phi + \beta\phi - \beta\phi^2\). In this case,

\[(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2] \]

\[(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ \begin{array}{c} Z_{N-2} + \alpha\phi + \beta\phi + \beta\phi^2 \\ -Z_{N-2} - \alpha\phi + \beta\phi + \beta\phi^2 \\ -Z_{N-2} + \alpha\phi + \beta\phi - \beta\phi^2 \end{array} \right] \]

\[(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 4\alpha\phi + 2\beta\phi + 2\beta\phi^2].\]

Since \(-2Z_{N-2} + 4\beta\phi + 2\beta\phi^2 > -2Z_{N-2} + 2\alpha\phi + 2\beta\phi + 2\beta\phi^2 \Rightarrow \beta > \alpha\) which is true always, \(\theta_{N-2} = \beta\) is optimal.
A.2. Derivation of $\theta^*_N$ when $\alpha < 0$ and $|\alpha| > |\beta|$

We now consider the optimal strategy for $\theta_{N−2}$ when $\alpha < 0$ and $|\alpha| > |\beta|$. In this scenario, it is known that the optimal choice for $\theta^N = \theta^N_{N−1} = \alpha$ and thus $U_{N−2}$ will have the following form,

$$U_{N−2} = \frac{S_{N−2}\rho^2}{4} \left[ \begin{array}{c} |Z_{N−2} + \theta_{N−2}\phi + \alpha(\phi + \phi^2)| \\
+ |Z_{N−2} + \theta_{N−2}\phi - \alpha(\phi + \phi^2)| \\
+ |Z_{N−2} - \theta_{N−2}\phi + \alpha(\phi - \phi^2)| \\
+ |Z_{N−2} - \theta_{N−2}\phi - \alpha(\phi - \phi^2)| \end{array} \right]$$

which when evaluated at $\alpha$ and $\beta$ gives,

$$(U_{N−2}|\theta_{N−2} = \beta) = \frac{S_{N−2}\rho^2}{4} \left[ \begin{array}{c} |Z_{N−2} + \beta\phi + \alpha(\phi + \phi^2)| \\
+ |Z_{N−2} + \beta\phi - \alpha(\phi + \phi^2)| \\
+ |Z_{N−2} - \beta\phi + \alpha(\phi - \phi^2)| \\
+ |Z_{N−2} - \beta\phi - \alpha(\phi - \phi^2)| \end{array} \right],$$

$$(U_{N−2}|\theta_{N−2} = \alpha) = \frac{S_{N−2}\rho^2}{4} \left[ \begin{array}{c} |Z_{N−2} + \alpha\phi + \alpha(\phi + \phi^2)| \\
+ |Z_{N−2} + \alpha\phi - \alpha(\phi + \phi^2)| \\
+ |Z_{N−2} - \alpha\phi + \alpha(\phi - \phi^2)| \\
+ |Z_{N−2} - \alpha\phi - \alpha(\phi - \phi^2)| \end{array} \right].$$
What remains is to order these paths so that the precise values for \((U_{N-2}|\theta_{N-2} = \beta)\) and \((U_{N-2}|\theta_{N-2} = \alpha)\) may be computed. To this end, we take pair-wise comparisons between all paths, the results of which are summarised in table 1 below. Given some of the results pertaining to the order of the paths, we must consider separate cases where \(\beta > 0\) and \(\beta < 0\).

<table>
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<th>&gt;</th>
<th>(Z_{uu}^N(\beta))</th>
<th>(Z_{ud}^N(\beta))</th>
<th>(Z_{du}^N(\beta))</th>
<th>(Z_{dd}^N(\beta))</th>
<th>(Z_{uu}^N(\alpha))</th>
<th>(Z_{ud}^N(\alpha))</th>
<th>(Z_{du}^N(\alpha))</th>
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<td>(\frac{-\alpha}{\beta} &lt; \phi)</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
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<tr>
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<td>(\frac{\beta}{\pi} &lt; \phi)</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>(\frac{\alpha+\beta}{2\alpha} &lt; \phi)</td>
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<tr>
<td>(Z_{du}^N(\beta))</td>
<td>=</td>
<td>F</td>
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</tr>
<tr>
<td>(Z_{dd}^N(\beta))</td>
<td>=</td>
<td>F</td>
<td>(\frac{\alpha+\beta}{2\alpha} &gt; \phi)</td>
<td>(\frac{\alpha+\beta}{2\alpha} &gt; \phi)</td>
<td>=</td>
<td>F</td>
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<tr>
<td>(Z_{uu}^N(\alpha))</td>
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<td>F</td>
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<tr>
<td>(Z_{ud}^N(\alpha))</td>
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<tr>
<td>(Z_{du}^N(\alpha))</td>
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<tr>
<td>(Z_{dd}^N(\alpha))</td>
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</tbody>
</table>

**Table 1:** Pairwise comparisons for binomial account values when \(\alpha < 0\) and \(|\alpha| > |\beta|\). As before, the table is to be read by taking an entry from the left column \((LE)\) and comparing it with an entry from the top row \((TE)\) via \(LE > TE\). T, F, = and a stated inequality indicate that the statement is true always, false always, equal or requires that inequality to be true respectively.

Using the results of table 1 above, we may determine the path order for a given scenario and hence determine which is the optimal choice for \(\theta_{N-2}\). As before there are many scenarios we must consider, each of which will be covered in the following subsections. First, we note that we need to determine which is larger between \(\frac{\alpha + \beta}{2\alpha}\) and \(\frac{\beta}{-\alpha}\) to ascertain the correct path ranking. It is straightforward to show that \(\frac{\alpha + \beta}{2\alpha} > \frac{\beta}{-\alpha}\) \(\Rightarrow \alpha < -3\beta\), thus this will be one of the conditions we use to separate our cases.

**A.2.1.** \(\phi \in (0, \frac{\beta}{-\alpha}), \beta > 0, \alpha < -3\beta\). The order for the paths in under this scenario are

\[ Z_{dd}^N(\alpha) > Z_{ud}^N(\beta) > Z_{du}^N(\beta) > Z_{dd}^N(\alpha) = Z_{ud}^N(\alpha) > Z_{du}^N(\alpha) > Z_{dd}^N(\beta) > Z_{ud}^N(\alpha) \]

**Region 1:** \(-2\alpha - \alpha\phi^2 < Z_{N-2}\). In this case we have that all account outcomes are positive and thus we are indifferent about the choice of \(\theta_{N-2}\) under this scenario.

**Region 2:** \(\beta\phi - \alpha\phi + \alpha\phi^2 < Z_{N-2} < -2\alpha - \alpha\phi^2\). In this case,

\[(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4}[4Z_{N-2}].\]
(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - 2\alpha \phi + \alpha \phi^2 \end{bmatrix}.

= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2]

Since 4Z_{N-2} > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} > -2\alpha - \alpha \phi^2 which is false, we thus have that \theta_{N-2} = \alpha is optimal in this case.

Region 3: -\beta \phi - \alpha \phi - \alpha \phi^2 < Z_{N-2} < \beta \phi - \alpha \phi + \alpha \phi^2. In this case,

(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} Z_{N-2} + \beta \phi + \alpha \phi + \alpha \phi^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ +Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2 \end{bmatrix},

= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} + 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2],

(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].

Since 2Z_{N-2} + 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \frac{\alpha + \beta}{2\alpha} > \phi

which is false always, thus \theta_{N-2} = \alpha is optimal.

Region 4: \alpha \phi^2 < Z_{N-2} < \beta \phi - \alpha \phi - \alpha \phi^2. We have that in this region,

(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ +Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2 \end{bmatrix},

= \frac{\rho^2 S_{N-2}}{4} [-4\alpha \phi],

(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].

Since -4\alpha \phi > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \alpha \phi^2 > Z_{N-2} which is false,

\theta_{N-2} = \alpha is optimal in this case.

Region 5: \beta \phi + \alpha \phi - \alpha \phi^2 < Z_{N-2} < \alpha \phi^2. In this case,

(U_{N-2} | \theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4\alpha \phi],

(U_{N-2} | \theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\ -Z_{N-2} + \alpha \phi^2 \\ -Z_{N-2} + \alpha \phi^2 \\ +Z_{N-2} - 2\alpha \phi + \alpha \phi^2 \end{bmatrix}
Since \(-4\alpha \phi > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow Z_{N-2} > \alpha \phi^2\) which is false, 
\(\theta_{N-2} = \alpha\).

Region 6: \(-\beta \phi + \alpha \phi + \alpha \phi^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha \phi^2\). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
-Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2 \\
-Z_{N-2} + \beta \phi + \alpha \phi - \alpha \phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].
\]

Since \(-2Z_{N-2} + 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2 > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha}\) which is false, so \(\theta_{N-2} = \alpha\) in this region.

Region 7: \(2\alpha \phi - \alpha \phi^2 < Z_{N-2} < -\beta \phi + \alpha \phi + \alpha \phi^2\). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].
\]

Since \(-4Z_{N-2} > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow 2\alpha \phi - \alpha \phi^2 > Z_{N-2}\) which is false, \(\theta_{N-2} = \alpha\) is optimal.

Region 8: \(Z_{N-2} < 2\alpha \phi - \alpha \phi^2\). In this case all paths are negative, thus we are indifferent about the choice of \(\theta_{N-2}\).

\[\text{A.2.2. } \phi \in (\frac{\beta}{-\alpha}, \frac{\alpha + \beta}{2\alpha}), \beta > 0, \alpha < -3\beta.\] The ranking of the paths under this scenario are:

\(Z_{N}^{\dd}(\alpha) > Z_{N}^{\ud}(\beta) > Z_{N}^{\dd}(\beta) > Z_{N}^{\ud}(\alpha) = Z_{N}^{\du}(\alpha) > Z_{N}^{\du}(\beta) > Z_{N}^{\uu}(\beta) > Z_{N}^{\uu}(\alpha)\)

Region 1: \(-2\alpha \phi - \alpha \phi^2 < Z_{N-2}\). In this case all account outcomes are positive, thus we are indifferent about the choice of \(\theta_{N-2}\).
Region 2: \(-\beta \phi + \alpha \phi + \alpha \phi^2 < Z_{N-2} < -2\alpha \phi - \alpha \phi^2\). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\
+Z_{N-2} - \alpha \phi^2 \\
+Z_{N-2} - \alpha \phi^2 \\
+Z_{N-2} - 2\alpha \phi + \alpha \phi^2
\end{bmatrix}
\]
Since $4Z_{N-2} > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow Z_{N-2} > -2\alpha\phi - \alpha\phi^2$ which is false, $\theta_{N-2} = \alpha$ is optimal.

Region 3: $\beta\phi - \alpha\phi + \alpha\phi^2 < Z_{N-2} < -\beta\phi + \alpha\phi + \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ -Z_{N-2} - \beta\phi - \alpha\phi - \alpha\phi^2 \right.
\left. + Z_{N-2} - \beta\phi + \alpha\phi - \alpha\phi^2 \right]
$$

$$= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].$$

Since $2Z_{N-2} - 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow \beta < \alpha$ which is false, we have that $\theta_{N-2} = \alpha$.

Region 4: $\alpha\phi^2 < Z_{N-2} < \beta\phi - \alpha\phi + \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ -Z_{N-2} - \beta\phi - \alpha\phi - \alpha\phi^2 \right.
\left. + Z_{N-2} - \beta\phi + \alpha\phi + \alpha\phi^2 \right]
$$

$$= \frac{\rho^2 S_{N-2}}{4} [-4\alpha\phi],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].$$

Since $-4\alpha\phi > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow Z_{N-2} < \alpha\phi^2$ which is false, $\theta_{N-2} = \alpha$.

Region 5: $\beta\phi + \alpha\phi - \alpha\phi^2 < Z_{N-2} < \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4\alpha\phi],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ -Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \right.
\left. - Z_{N-2} + \alpha\phi^2 \right]
$$

$$= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2].$$

Since $-4\alpha\phi > -2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2 \Rightarrow Z_{N-2} > \alpha\phi^2$ which is false, $\theta_{N-2} = \alpha$. 

\[ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \left[ -Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+ Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
- Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\
- Z_{N-2} + \beta \phi + \alpha \phi - \alpha \phi^2 \right] \]
\[ = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2], \]
\[ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2]. \]

Since \(-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow \frac{\alpha + \beta}{2\alpha} > \phi \) which is false, \(\theta_{N-2} = \alpha.\)

Region 7: \(2\alpha \phi - \alpha \phi^2 < Z_{N-2} < -\beta \phi + \alpha \phi + \alpha \phi^2.\) Now we have that
\[ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}], \]
\[ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2]. \]

Since \(-4Z_{N-2} > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow Z_{N-2} < 2\alpha \phi - \alpha \phi^2\) which is false, we have that \(\theta_{N-2} = \alpha.\)

Region 8: \(Z_{N-2} < 2\alpha \phi - \alpha \phi^2.\) In this case all outcomes for the account value are negative and thus we are indifferent about the choice of \(\theta_{N-2}\) in this region.

**A.2.3.** \(\phi \in (\frac{\alpha + \beta}{2\alpha}, 1), \beta > 0, \alpha < -3\beta.\) The order for the paths under this scenario is:
\[ Z_N^{ud}(\beta) > Z_N^{dd}(\alpha) > Z_N^{du}(\alpha) = Z_N^{ud}(\alpha) > Z_N^{dd}(\beta) > Z_N^{du}(\beta) > Z_N^{uu}(\beta) > Z_N^{uu}(\alpha) \]

Region 1: \(-2\alpha \phi - \alpha \phi^2 < Z_{N-2}.\) In this case all outcomes for the account are positive and thus we are indifferent about the choice of \(\theta_{N-2}.\)

Region 2: \(-\beta \phi - \alpha \phi - \alpha \phi^2 < Z_{N-2} < -2\alpha \phi - \alpha \phi^2.\) In this case,
\[ (U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4ZN - 2], \]
\[ (U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \left[ -Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\
+ Z_{N-2} - \alpha \phi^2 \\
+ Z_{N-2} - \alpha \phi^2 \\
+ Z_{N-2} - 2\alpha \phi + \alpha \phi^2 \right] \]
\[ = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2]. \]
Since $4ZN - 2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} > -2\alpha \phi - \alpha \phi^2$ which is false, $\theta_{N-2} = \alpha$.

**Region 3:** $\beta \phi - \alpha \phi + \alpha \phi^2 < Z_{N-2} < -\beta \phi - \alpha \phi - \alpha \phi^2$. In this case,

$$
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} - \beta \phi + \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2
\end{bmatrix}
$$

$$
= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2],
$$

$$
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
$$

Since $2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \beta < \alpha$ which is false always, we have that $\theta_{N-2} = \alpha$.

**Region 4:** $\beta \phi + \alpha \phi - \alpha \phi^2 < Z_{N-2} < \beta \phi - \alpha \phi + \alpha \phi^2$. In this case,

$$
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
-Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\
+Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2
\end{bmatrix}
$$

$$
= \frac{\rho^2 S_{N-2}}{4} [-4\alpha \phi],
$$

$$
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
$$

Since $-4\alpha \phi > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} < \alpha \phi^2$ which is false, so $\theta_{N-2} = \alpha$.

**Region 5:** $\alpha \phi^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha \phi^2$. In this case,

$$
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
-Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\
+Z_{N-2} - \beta \phi + \alpha \phi - \alpha \phi^2
\end{bmatrix}
$$

$$
= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2],
$$

$$
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
$$

Since $-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \frac{1}{2}(\alpha + \beta) \phi > Z_{N-2}$. If this condition holds true, then $\theta_{N-2} = \beta$ and if $\frac{1}{2}(\alpha + \beta) \phi < Z_{N-2}$ then $\theta_{N-2} = \alpha$. 
Region 6: \(2\alpha\phi - \alpha\phi^2 < Z_{N-2} < \alpha\phi^2\). In this case we have that
\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2],
\]
\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \\ -Z_{N-2} + \alpha\phi^2 \\ -Z_{N-2} + \alpha\phi^2 \\ +Z_{N-2} - 2\alpha\phi + \alpha\phi^2 \end{bmatrix}
\]
\[
= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2].
\]
Since \(-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 \geq -2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha}\) which is true, we have that \(\theta_{N-2} = \beta\) is optimal now.

Region 7: \(-\beta\phi + \alpha\phi + \alpha\phi^2 < Z_{N-2} < 2\alpha\phi - \alpha\phi^2\). In this case we have that
\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2],
\]
\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}].
\]
Since \(-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > -4Z_{N-2} \Rightarrow Z_{N-2} > -\beta\phi + \alpha\phi + \alpha\phi^2\) which is true, \(\theta_{N-2} = \beta\).

Region 8: \(Z_{N-2} < -\beta\phi + \alpha\phi + \alpha\phi^2\). In this region all account outcomes are negative and thus we are indifferent about the choice of \(\theta_{N-2}\) in this region.

**A.2.4.** \(\phi \in (0, \frac{\alpha + \beta}{2\alpha})\), \(\beta > 0\), \(\alpha > -3\beta\). The ranking of the paths in under this scenario is:

\[
Z_N^{ud}(\alpha) > Z_N^{ud}(\beta) > Z_N^{dd}(\beta) > Z_N^{dd}(\alpha) = Z_N^{ud}(\alpha) > Z_N^{uu}(\beta) > Z_N^{du}(\beta) > Z_N^{uu}(\alpha)
\]

Region 1: \(-2\alpha\phi - \alpha\phi^2 < Z_{N-2}\. In this case all outcomes are positive and thus we are indifferent about the choice of \(\theta_{N-2}\).

Region 2: \(\beta\phi - \alpha\phi + \alpha\phi^2 < Z_{N-2} < -2\alpha\phi - \alpha\phi^2\). In this case we have that
\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}],
\]
\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \\ +Z_{N-2} - \alpha\phi^2 \\ +Z_{N-2} - \alpha\phi^2 \\ +Z_{N-2} - 2\alpha\phi + \alpha\phi^2 \end{bmatrix}
\]
\[
= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].
\]
Since \(4Z_{N-2} > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2\Rightarrow Z_{N-2} > -2\alpha\phi - \alpha\phi^2\) which is false, \(\theta_{N-2} = \alpha\).

Region 3: \(-\beta\phi - \alpha\phi - \alpha\phi^2 < Z_{N-2} < \beta\phi - \alpha\phi + \alpha\phi^2\). In this case,

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2S_{N-2}}{4} \begin{bmatrix}
Z_{N-2} + \beta\phi + \alpha\phi + \alpha\phi^2 \\
+Z_{N-2} + \beta\phi - \alpha\phi - \alpha\phi^2 \\
- Z_{N-2} + \beta\phi - \alpha\phi + \alpha\phi^2 \\
+Z_{N-2} - \beta\phi - \alpha\phi + \alpha\phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2S_{N-2}}{4}[2Z_{N-2} + 2\beta\phi - 2\alpha\phi + 2\alpha\phi^2],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].
\]

Since \(2Z_{N-2} + 2\beta\phi - 2\alpha\phi + 2\alpha\phi^2 > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2\Rightarrow \phi < \frac{\alpha + \beta}{2\alpha}\) which is false, \(\theta_{N-2} = \alpha\).

Region 4: \(\alpha\phi^2 < Z_{N-2} < -\beta\phi - \alpha\phi - \alpha\phi^2\). In this case,

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta\phi - \alpha\phi - \alpha\phi^2 \\
+Z_{N-2} + \beta\phi - \alpha\phi - \alpha\phi^2 \\
- Z_{N-2} + \beta\phi - \alpha\phi + \alpha\phi^2 \\
+Z_{N-2} - \beta\phi - \alpha\phi + \alpha\phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2S_{N-2}}{4}[-4\alpha\phi],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].
\]

Since \(-4\alpha\phi > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow Z_{N-2} < \alpha\phi^2\) which is false, \(\theta_{N-2} = \alpha\).

Region 5: \(\beta\phi + \alpha\phi - \alpha\phi^2 < Z_{N-2} < \alpha\phi^2\). In this case,

\[
(U_{N-2}\theta_{N-2} = \beta) = \frac{\rho^2S_{N-2}}{4}[-4\alpha\phi],
\]

\[
(U_{N-2}\theta_{N-2} = \alpha) = \frac{\rho^2S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \\
- Z_{N-2} + \alpha\phi^2 \\
- Z_{N-2} + \alpha\phi^2 \\
+Z_{N-2} - 2\alpha\phi + \alpha\phi^2
\end{bmatrix}
\]

\[
= \frac{\rho^2S_{N-2}}{4}[-2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2].
\]

Since \(-4\alpha\phi > - 2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2 \Rightarrow Z_{N-2} > \alpha\phi^2\) which is false, we have that \(\theta_{N-2} = \alpha\).
A.2. DERIVATION OF $\theta^*_N$ 

Region 6: $-\beta \phi + \alpha \phi + \alpha \phi^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha \phi^2$. In this case we have that

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ -Z_{N-2} + \beta \phi + \alpha \phi - \alpha \phi^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].$$

Since $-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha}$ which is false, $\theta_{N-2} = \alpha$.

Region 7: $2\alpha \phi - \alpha \phi^2 < Z_{N-2} < -\beta \phi + \alpha \phi + \alpha \phi^2$. In this case we have that

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].$$

Since $-4Z_{N-2} > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow Z_{N-2} < 2\alpha \phi - \alpha \phi^2$ which is false, so $\theta_{N-2} = \alpha$.

Region 8: $Z_{N-2} < 2\alpha \phi - \alpha \phi^2$. In this case, all account outcomes are negative and thus we are indifferent about the choice of $\theta_{N-2}$ in this region.

A.2.5. $\phi \in (\frac{\alpha + \beta}{2\alpha}, \frac{\beta}{\alpha})$, $\beta > 0$, $\alpha > -3\beta$. The ranking of the paths under this scenario is:

$Z_N^{ud}(\beta) > Z_N^{dd}(\alpha) > Z_N^{du}(\alpha) = Z_N^{ud}(\alpha) > Z_N^{dd}(\beta) > Z_N^{uu}(\beta) > Z_N^{du}(\beta) > Z_N^{uu}(\alpha)$

Region 1: $-2\alpha \phi - \alpha \phi^2 < Z_{N-2}$. In this case all account paths are positive and thus we are indifferent about the choice of $\theta_{N-2}$ in this region.

Region 2: $\beta \phi - \alpha \phi + \alpha \phi^2 < Z_{N-2} < -2\alpha \phi - \alpha \phi^2$. In this case we have that

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - 2\alpha \phi + \alpha \phi^2 \end{bmatrix}$$
A.2. DERIVATION OF $\theta_{n-2}^*$

$$= \frac{\rho^2 S_{n-2}}{4} [2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2].$$

Since $4Z_{n-2} > 2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{n-2} > -2\alpha \phi - \alpha \phi^2$ which is false, $\theta_{n-2} = \alpha$.

**Region 3:** $-\beta \phi - \alpha \phi - \alpha \phi^2 < Z_{n-2} < \beta \phi - \alpha \phi + \alpha \phi^2$. In this case we have that

$$(U_{n-2} \theta_{n-2} = \beta) = \frac{\rho^2 S_{n-2}}{4} \left[ \begin{array}{c} Z_{n-2} + \beta \phi + \alpha \phi + \alpha \phi^2 \\ +Z_{n-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{n-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ +Z_{n-2} - \beta \phi - \alpha \phi + \alpha \phi^2 \end{array} \right]$$

$$= \frac{\rho^2 S_{n-2}}{4} [2Z_{n-2} + 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2],$$

$$(U_{n-2} \theta_{n-2} = \alpha) = \frac{\rho^2 S_{n-2}}{4} [2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2].$$

Since $2Z_{n-2} + 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2 > 2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \phi < \frac{\alpha \beta}{2\alpha}$ which is false, $\theta_{n-2} = \alpha$.

**Region 4:** $\beta \phi + \alpha \phi - \alpha \phi^2 < Z_{n-2} < -\beta \phi - \alpha \phi - \alpha \phi^2$. In this case

$$(U_{n-2} \theta_{n-2} = \beta) = \frac{\rho^2 S_{n-2}}{4} \left[ \begin{array}{c} -Z_{n-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\ +Z_{n-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{n-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ +Z_{n-2} - \beta \phi - \alpha \phi + \alpha \phi^2 \end{array} \right]$$

$$= \frac{\rho^2 S_{n-2}}{4} [-4\alpha \phi],$$

$$(U_{n-2} \theta_{n-2} = \alpha) = \frac{\rho^2 S_{n-2}}{4} [2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2].$$

Since $-4\alpha \phi > 2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{n-2} < \alpha \phi^2$ which is false, thus $\theta_{n-2} = \alpha$ is optimal.

**Region 5:** $\alpha \phi^2 < Z_{n-2} < \beta \phi + \alpha \phi - \alpha \phi^2$. In this case,

$$(U_{n-2} \theta_{n-2} = \beta) = \frac{\rho^2 S_{n-2}}{4} \left[ \begin{array}{c} -Z_{n-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\ +Z_{n-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{n-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ -Z_{n-2} + \beta \phi + \alpha \phi - \alpha \phi^2 \end{array} \right]$$

$$= \frac{\rho^2 S_{n-2}}{4} [-2Z_{n-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2],$$

$$(U_{n-2} \theta_{n-2} = \alpha) = \frac{\rho^2 S_{n-2}}{4} [2Z_{n-2} - 4\alpha \phi - 2\alpha \phi^2].$$
Since \(-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow \frac{1}{2}(\alpha + \beta)\phi > Z_{N-2}\). If this condition holds true, then \(\theta_{N-2} = \beta\) and if \(\frac{1}{2}(\alpha + \beta)\phi < Z_{N-2}\), then \(\theta_{N-2} = \alpha\).

Region 6: \(2\alpha\phi - \alpha\phi^2 < Z_{N-2} < \alpha\phi^2\). In this case
\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2S_{N-2}}{4}[-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2],
\]
\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2S_{N-2}}{4}\begin{bmatrix}
-Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \\
-Z_{N-2} + \alpha\phi^2 \\
+Z_{N-2} - 2\alpha\phi + \alpha\phi^2
\end{bmatrix}
\]
\[
= \frac{\rho^2S_{N-2}}{4}[-2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2].
\]

Since \(-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > -2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha}\) which is true, we have that \(\theta_{N-2} = \beta\).

Region 7: \(-\beta\phi + \alpha\phi + \alpha\phi^2 < Z_{N-2} < 2\alpha\phi - \alpha\phi^2\). In this case,
\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2S_{N-2}}{4}[-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2],
\]
\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2S_{N-2}}{4}[-4Z_{N-2}].
\]

Since \(-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > -4Z_{N-2} \Rightarrow Z_{N-2} > -\beta\phi + \alpha\phi + \alpha\phi^2\) which is true, \(\theta_{N-2} = \beta\).

Region 8: \(Z_{N-2} < -\beta\phi + \alpha\phi + \alpha\phi^2\). In this case all account outcomes are negative valued and thus we are indifferent about the choice of \(\theta_{N-2}\).

A.2.6. \(\phi \in (\frac{\beta}{\alpha}, 1), \beta > 0, \alpha > -3\beta\). The ranking for the paths is now given by:
\[
Z^u_N(\beta) > Z^d_N(\alpha) > Z^u_N(\alpha) = Z^u_N(\beta) > Z^d_N(\beta) > Z^u_N(\beta) > Z^u_N(\alpha)
\]

Region 1: \(-2\alpha\phi - \alpha\phi^2 < Z_{N-2}\). In this case all outcomes are negative and thus we are indifferent about the choice of \(\theta_{N-2}\).

Region 2: \(-\beta\phi - \alpha\phi - \alpha\phi^2 < Z_{N-2} < -2\alpha\phi - \alpha\phi^2\). In this case,
\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2S_{N-2}}{4}[4Z_{N-2}],
\]
\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2S_{N-2}}{4}\begin{bmatrix}
-Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \\
-Z_{N-2} - \alpha\phi^2 \\
+Z_{N-2} - 2\alpha\phi + \alpha\phi^2
\end{bmatrix}
\]
A.2. DERIVATION OF $\theta_{N-2}^*$

$$= \frac{\rho^2 S_{N-2}^2}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha^2].$$

Since $4Z_{N-2} > 2Z_{N-2} - 4\alpha \phi - 2\alpha^2 \Rightarrow Z_{N-2} > -2\alpha \phi - \alpha^2$ which is false, thus $\theta_{N-2} = \alpha$.

Region 3: $\beta \phi - \alpha \phi + \alpha^2 < Z_{N-2} < -\beta \phi - \alpha \phi - \alpha^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}^2}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \alpha \phi - \alpha^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha^2 \\ +Z_{N-2} - \beta \phi + \alpha \phi - \alpha^2 \\ +Z_{N-2} - \beta \phi - \alpha \phi + \alpha^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}^2}{4}[2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}^2}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha^2].$$

Since $2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha^2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha^2 \Rightarrow \beta < \alpha$ which is false always, $\theta_{N-2} = \alpha$.

Region 4: $\beta \phi + \alpha \phi - \alpha^2 < Z_{N-2} < \beta \phi - \alpha \phi + \alpha^2$. In this case we have that

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}^2}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \alpha \phi - \alpha^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha^2 \\ +Z_{N-2} - \beta \phi - \alpha \phi + \alpha^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}^2}{4}[-4\alpha \phi],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}^2}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha^2].$$

Since $-4\alpha \phi > 2Z_{N-2} - 4\alpha \phi - 2\alpha^2 \Rightarrow Z_{N-2} < \alpha^2$ which is false, $\theta_{N-2} = \alpha$.

Region 5: $\alpha^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha^2$. In this case

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}^2}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \alpha \phi - \alpha^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha^2 \\ -Z_{N-2} + \beta \phi + \alpha \phi - \alpha^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}^2}{4}[-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}^2}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha^2].$$
Since $-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > 0$, the condition holds then $\theta_{N-2} = \beta$, while if $0, \beta < 0$. In this case, the path rankings are given by:

$Z_N^d(\alpha) > Z_N^d(\beta) > Z_N^{ud}(\beta) > Z_N^{ud}(\alpha) = Z_N^{ud}(\alpha) > Z_N^{ud}(\beta) > Z_N^{un}(\beta) > Z_N^{un}(\alpha)$

**Region 1:** $-2\alpha \phi - \alpha \phi^2 < Z_{N-2}$. In this case all outcomes for the account paths are positive and thus we are indifferent about the choice of $\theta_{N-2}$.

**Region 2:** $-\beta \phi - \alpha \phi^2 < Z_{N-2}$. In this case, the path rankings are:

$(U_{N-2})_{\theta_{N-2} = \beta} = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}]$,  

$(U_{N-2})_{\theta_{N-2} = \alpha} = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2]$.  

Since $-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha}$ which is true, $\theta_{N-2} = \beta$.

**Region 7:** $-\beta \phi + \alpha \phi + \alpha \phi^2 < Z_{N-2} < 2\alpha \phi - \alpha \phi^2$. In this case we have that

$(U_{N-2})_{\theta_{N-2} = \beta} = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2]$,  

$(U_{N-2})_{\theta_{N-2} = \alpha} = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2]$.  

Since $-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > -2Z_{N-2} \Rightarrow Z_{N-2} > -\beta \phi + \alpha \phi + \alpha \phi^2$ which is true, thus $\theta_{N-2} = \beta$.

**Region 8:** $Z_{N-2} > -\beta \phi + \alpha \phi + \alpha \phi^2$. In this case all the account paths are negative and thus we are indifferent about the choice of $\theta_{N-2}$.
\[= \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].\]

Since \(4Z_{N-2} > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} > -2\alpha \phi - \alpha \phi^2\) which is false, so \(\theta_{N-2} = \alpha\).

Region 3: \(\beta \phi - \alpha \phi + \alpha \phi^2 < Z_{N-2} < -\beta \phi - \alpha \phi - \alpha \phi^2\). In this case we have

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+ Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
+ Z_{N-2} - \beta \phi + \alpha \phi - \alpha \phi^2 \\
+ Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2
\end{bmatrix}
\]

\[= \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2],\]

\[(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].\]

Since \(2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \beta < \alpha\) which is false always, \(\theta_{N-2} = \alpha\).

Region 4: \(\alpha \phi^2 < Z_{N-2} < \beta \phi - \alpha \phi + \alpha \phi^2\). In this case

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+ Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
- Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\
+ Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2
\end{bmatrix}
\]

\[= \frac{\rho^2 S_{N-2}}{4}[-4\alpha \phi],\]

\[(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4}[2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].\]

Since \(-4\alpha \phi > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} < \alpha \phi^2\) which is false, \(\theta_{N-2} = \alpha\).

Region 5: \(-\beta \phi + \alpha \phi + \alpha \phi^2 < Z_{N-2} < \alpha \phi^2\). Here we have that

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4}[-4\alpha \phi]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\
- Z_{N-2} + \alpha \phi^2 \\
- Z_{N-2} + \alpha \phi^2 \\
+ Z_{N-2} - 2\alpha \phi + \alpha \phi^2
\end{bmatrix},
\]

\[= \frac{\rho^2 S_{N-2}}{4}[-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].\]

Since \(-4\alpha \phi > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow Z_{N-2} > \alpha \phi^2\) which is false, \(\theta_{N-2} = \alpha\).
A.2. DERIVATION OF $\theta_{N-2}^*$

Region 6: $\beta \phi + \alpha \phi - \alpha \phi^2 < Z_{N-2} < -\beta \phi + \alpha \phi + \alpha \phi^2$. In this case

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} \left[\begin{array}{c}-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{N-2} - \beta \phi + \alpha \phi + \alpha \phi^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ +Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2 \end{array}\right]$$

$$= \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} - 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].$$

Since $-2Z_{N-2} - 2\beta \phi - 2\alpha \phi + 2\alpha \phi^2 > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow \beta < \alpha$ which is false always, $\theta_{N-2} = \alpha$.

Region 7: $2\alpha \phi - \alpha \phi^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha \phi^2$. In this case we have that

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} [-4Z_{N-2}],$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].$$

Since $-4Z_{N-2} > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow Z_{N-2} < 2\alpha \phi - \alpha \phi^2$ which is false, we have that $\theta_{N-2} = \alpha$.

Region 8: $Z_{N-2} < 2\alpha \phi - \alpha \phi^2$. In this region all account outcomes are negative and thus we are indifferent about the choice of $\theta_{N-2}$.

A.2.8. $\phi \in \left(\frac{\beta}{2}, \frac{\alpha + \beta}{2\alpha}\right)$, $\beta < 0$. The ranking for the paths in this region is given by:

$Z_N^{du}(\alpha) > Z_N^{dd}(\beta) > Z_N^{uu}(\alpha) = Z_N^{ud}(\alpha) > Z_N^{du}(\beta) > Z_N^{uu}(\beta) > Z_N^{uu}(\alpha)$

Region 1: $-2\alpha \phi - \alpha \phi^2 < Z_{N-2}$. In this case all outcomes are positive and thus we are indifferent about the choice of $\theta_{N-2}$.

Region 2: $-\beta \phi - \alpha \phi - \alpha \phi^2 < Z_{N-2} < -2\alpha \phi - \alpha \phi^2$. We have that in this region

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} [4Z_{N-2}]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} \left[\begin{array}{c}-Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - 2\alpha \phi + \alpha \phi^2 \end{array}\right]$$

$$= \frac{\beta^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].$$
Since $4Z_{N-2} > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow Z_{N-2} > -2\alpha\phi - \alpha\phi^2$ which is false, $\theta_{N-2} = \alpha$.

Region 3: $\beta\phi - \alpha\phi + \alpha\phi^2 < Z_{N-2} < -\beta\phi - \alpha\phi - \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \beta\phi - \alpha\phi - \alpha\phi^2 \\
+Z_{N-2} + \beta\phi - \alpha\phi - \alpha\phi^2 \\
+Z_{N-2} - \beta\phi + \alpha\phi - \alpha\phi^2 \\
+Z_{N-2} - \beta\phi - \alpha\phi + \alpha\phi^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].$$

Since $2Z_{N-2} - 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2$ $\Rightarrow$ $\beta < \alpha$ which is false always, $\theta_{N-2} = \alpha$.

Region 4: $\alpha\phi^2 < Z_{N-2} < \beta\phi - \alpha\phi + \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \beta\phi - \alpha\phi - \alpha\phi^2 \\
+Z_{N-2} + \beta\phi - \alpha\phi - \alpha\phi^2 \\
+Z_{N-2} - \beta\phi + \alpha\phi - \alpha\phi^2 \\
-Z_{N-2} - \beta\phi - \alpha\phi - \alpha\phi^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}}{4} [-4\alpha\phi]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2].$$

Since $-4\alpha\phi > 2Z_{N-2} - 4\alpha\phi - 2\alpha\phi^2 \Rightarrow Z_{N-2} < \alpha\phi^2$ which is false, $\theta_{N-2} = \alpha$.

Region 5: $\beta\phi + \alpha\phi - \alpha\phi^2 < Z_{N-2} < \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4\alpha\phi]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \\
-Z_{N-2} + \alpha\phi^2 \\
-Z_{N-2} + \alpha\phi^2 \\
+Z_{N-2} - 2\alpha\phi + \alpha\phi^2 \end{bmatrix}$$

$$= \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2].$$

Since $-4\alpha\phi > -2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2 \Rightarrow Z_{N-2} > \alpha\phi^2$ which is false, $\theta_{N-2} = \alpha$. 
A.2. DERIVATION OF \( \theta_{N-2}^* \)

Region 6: \( -\beta \phi + \alpha \phi + \alpha \phi^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha \phi^2 \). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\ +Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\ -Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\ -Z_{N-2} + \beta \phi + \alpha \phi - \alpha \phi^2 \end{bmatrix}
\]

\[
= \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].
\]

Since \(-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha} \)
which is false, \( \theta_{N-2} = \alpha \).

Region 7: \( 2\alpha \phi - \alpha \phi^2 < Z_{N-2} < -\beta \phi + \alpha \phi + \alpha \phi^2 \). We have that in
this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [-4Z_{N-2}]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2].
\]

Since \(-4Z_{N-2} > -2Z_{N-2} - 4\alpha \phi + 2\alpha \phi^2 \Rightarrow Z_{N-2} < 2\alpha \phi - \alpha \phi^2 \) which
is false, so \( \theta_{N-2} = \alpha \).

Region 8: \( Z_{N-2} < 2\alpha \phi - \alpha \phi^2 \). In this case all account outcomes
are negative and thus we are indifferent about the choice of \( \theta_{N-2} \).

A.2.9. \( \phi \in (\frac{\alpha + \beta}{2\alpha}, 1) \), \( \beta < 0 \). In this case, the path ranks are given by:

\[
Z_{N}^{ud}(\beta) > Z_{N}^{dd}(\alpha) > Z_{N}^{du}(\alpha) = Z_{N}^{ud}(\alpha) > Z_{N}^{dd}(\beta) > Z_{N}^{du}(\beta) > Z_{N}^{uu}(\beta) > Z_{N}^{uu}(\alpha)
\]

Region 1: \(-2\alpha \phi - \alpha \phi^2 < Z_{N-2} \). In this case all account paths are
negative and thus we are indifferent about the choice of \( \theta_{N-2} \).

Region 2: \(-\beta \phi + \alpha \phi - \alpha \phi^2 < Z_{N-2} < -2\alpha \phi - \alpha \phi^2 \). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\rho^2 S_{N-2}}{4} [4Z_{N-2}]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\rho^2 S_{N-2}}{4} \begin{bmatrix} -Z_{N-2} - 2\alpha \phi - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - \alpha \phi^2 \\ +Z_{N-2} - 2\alpha \phi + \alpha \phi^2 \end{bmatrix}
\]

\[
= \frac{\rho^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
\]
Since \(4Z_{N-2} > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} > -2\alpha \phi - \alpha \phi^2\) which is false, \(\theta_{N-2} = \alpha\).

Region 3: \(\beta \phi - \alpha \phi + \alpha \phi^2 < Z_{N-2} < -\beta \phi - \alpha \phi - \alpha \phi^2\). In this case we have

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} - \beta \phi + \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} - \beta \phi - \alpha \phi + \alpha \phi^2
\end{bmatrix}
\]

\[
= \frac{\beta^2 S_{N-2}}{4} [2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
\]

Since \(2Z_{N-2} - 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \beta < \alpha\) which is false, \(\theta_{N-2} = \alpha\).

Region 4: \(\beta \phi + \alpha \phi - \alpha \phi^2 < Z_{N-2} < \beta \phi - \alpha \phi + \alpha \phi^2\). We have that in this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\
-Z_{N-2} - \beta \phi + \alpha \phi + \alpha \phi^2
\end{bmatrix}
\]

\[
= \frac{\beta^2 S_{N-2}}{4} [-4\alpha \phi]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
\]

Since \(-4\alpha \phi > 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow Z_{N-2} < \alpha \phi^2\) which is false, so \(\theta_{N-2} = \alpha\).

Region 5: \(\alpha \phi^2 < Z_{N-2} < \beta \phi + \alpha \phi - \alpha \phi^2\). In this case,

\[
(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} \begin{bmatrix}
-Z_{N-2} - \beta \phi - \alpha \phi - \alpha \phi^2 \\
+Z_{N-2} + \beta \phi - \alpha \phi - \alpha \phi^2 \\
-Z_{N-2} + \beta \phi - \alpha \phi + \alpha \phi^2 \\
+Z_{N-2} + \beta \phi + \alpha \phi - \alpha \phi^2
\end{bmatrix}
\]

\[
= \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2]
\]

\[
(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2].
\]

Since \(-2Z_{N-2} + 2\beta \phi - 2\alpha \phi - 2\alpha \phi^2 < 2Z_{N-2} - 4\alpha \phi - 2\alpha \phi^2 \Rightarrow \frac{1}{2}(\alpha + \beta) \phi > Z_{N-2}\), if this holds true then \(\theta_{N-2} = \beta\) and if \(\frac{1}{2}(\alpha + \beta) \phi < Z_{N-2}\) then \(\theta_{N-2} = \alpha\).
A.2. DERIVATION OF $\theta_{N-2}^*$

Region 6: $2\alpha\phi - \alpha\phi^2 < Z_{N-2} < \alpha\phi^2$. In this case,

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} \left[ -Z_{N-2} - 2\alpha\phi - \alpha\phi^2 \atop -Z_{N-2} + \alpha\phi^2 \atop +Z_{N-2} - 2\alpha\phi + \alpha\phi^2 \right]$$

$$= \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2].$$

Since $-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > -2Z_{N-2} - 4\alpha\phi + 2\alpha\phi^2 \Rightarrow \phi > \frac{\alpha + \beta}{2\alpha}$ which is true, then $\theta_{N-2} = \beta$.

Region 7: $-\beta\phi + \alpha\phi + \alpha\phi^2 < Z_{N-2} < 2\alpha\phi - \alpha\phi^2$. In this case we have

$$(U_{N-2}|\theta_{N-2} = \beta) = \frac{\beta^2 S_{N-2}}{4} [-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2]$$

$$(U_{N-2}|\theta_{N-2} = \alpha) = \frac{\beta^2 S_{N-2}}{4} [-4Z_{N-2}].$$

Since $-2Z_{N-2} + 2\beta\phi - 2\alpha\phi - 2\alpha\phi^2 > -4Z_{N-2} \Rightarrow Z_{N-2} > -\beta\phi + \alpha\phi + \alpha\phi^2$ which is true, $\theta_{N-2} = \beta$ in this region.

Region 8: $Z_{N-2} < -\beta\phi + \alpha\phi + \alpha\phi^2$. In this case all account paths are negative and thus we are indifferent about the choice of $\theta_{N-2}$ in this region.

This completes the analysis of the two-step option on a traded account over all possible regions. Whilst we have examined a large number of regions, the optimal strategy may be summed up by two strategy functions which apply in differing scenarios.

First, assuming $|\beta| > |\alpha|$, 

$$\theta_{N-2}^* = \begin{cases} 
\alpha \text{ if } Z_{N-2} > \frac{1}{2}(\alpha + \beta)\phi \text{ and } \phi > \frac{\alpha + \beta}{2\beta} , \\
\beta \text{ otherwise}
\end{cases}$$

and if $|\beta| < |\alpha|$, 

$$\theta_{N-2}^* = \begin{cases} 
\beta \text{ if } Z_{N-2} < \frac{1}{2}(\alpha + \beta)\phi \text{ and } \phi > \frac{\alpha + \beta}{2\alpha} , \\
\alpha \text{ otherwise}
\end{cases}$$
APPENDIX B

Inverse Laplace Transforms

In this appendix we provide some inverse Laplace transforms used in this thesis. We note that these transforms are largely adapted from Abramowitz and Stegun (1965). We reproduce them here to keep the thesis self contained. Note that not all of the inverse Laplace transforms used are contained in this appendix. Those which were deemed important to the relevant section have been presented in that section. Note that we make use of the identities \( \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \) and \( \text{erfc}(z) = 2\mathcal{N}(-\sqrt{2}z) \).

\[
\mathcal{L} \left\{ \frac{1}{s-a} \right\}^{-1} = e^{at} \tag{B.1}
\]

\[
\mathcal{L} \left\{ \frac{e^{-k\sqrt{s}}}{2(\sqrt{s}+a)} \right\}^{-1} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{k^2}{4t}} - \frac{1}{2} ae^{ak+a^2t} \text{erfc} \left( a\sqrt{t} + \frac{k}{2\sqrt{t}} \right) \]
\[
= \frac{1}{\sqrt{2t}} \phi \left( \frac{k}{\sqrt{2t}} \right) - ae^{ak+a^2t} \mathcal{N} \left( -a\sqrt{2t} - \frac{k}{\sqrt{2t}} \right) \tag{B.2}
\]

\[
\mathcal{L} \left\{ \frac{e^{-k\sqrt{s}}}{2(\sqrt{s}+a)^2} \right\}^{-1} = \mathcal{L} \left\{ -\frac{\partial}{\partial a} \left( \frac{e^{-k\sqrt{s}}}{2(\sqrt{s}+a)} \right) \right\}^{-1} \]
\[
= -\frac{\partial}{\partial a} \mathcal{L} \left\{ \frac{e^{-k\sqrt{s}}}{2(\sqrt{s}+a)} \right\}^{-1} \]
\[
= e^{ak+a^2t} \left[ \frac{(1+ak+a^2t)\mathcal{N} \left( \frac{1}{2} \sqrt{2t} - \frac{k}{\sqrt{2t}} \right)}{a\sqrt{2t}} \right. \]
\[
\left. - a\sqrt{2t} \phi \left( a\sqrt{2t} + \frac{k}{\sqrt{2t}} \right) \right] \tag{B.3}
\]
APPENDIX C

Derivation of Required Problem Transformations

In this appendix we provide detailed calculations required to transform the problems outlined into simpler forms. Specifically, we will show how in chapter 3, section 3.4.2 we move from equation (3.26) to (3.27) and (3.28) to (3.29) respectively.

C.1. Factor Reduction for the HJB Equation

In this section we demonstrate how to transform the two factor problem

\[
\begin{align*}
V^*_t &= rV^* - rsV^*_s - rxV^*_x - \max_{|\theta| \leq 1} \left\{ \frac{1}{2}\sigma^2 s^2 (V^*_s + 2\theta V^*_x + \theta^2 V^*_x) \right\} \\
V^*(s, x, T) &= (X_T)^+ 
\end{align*}
\]

into the one factor equivalent

\[
\begin{align*}
-U_t &= \frac{1}{2}\sigma^2 \max_{\theta} (z - \theta)^2 U_{zz} \\
U(z, T) &= (z)^+ 
\end{align*}
\]

via the transformation \( z = \frac{x}{s} \). Given that \( V^* \) is homogeneous of degree 1 in \( s \) and \( x \), we know that Euler’s theorem for homogeneous functions will hold, that is

\[
V^* = xV^*_x + sV^*_s 
\]

(C.1)

and thus \(-rV^*_t + rxV^*_x + rsV^*_s = 0\). Thus we have that

\[
\begin{align*}
V^*_t + \max_{\theta \in [-1, 1]} \frac{1}{2}\sigma^2 s^2 (V^*_s + 2\theta V^*_x + \theta^2 V^*_x) &= 0 \\
V^*_x &= V^*_x + xV^*_x + sV^*_s \\
V^*_s &= xV^*_x + V^*_s + sV^*_s \\
V^*_t &= U + sU_t \frac{\partial}{\partial s} \\
&= U + sU_t \frac{\partial}{\partial s} \\
&= U - zU_z 
\end{align*}
\]

Now, using the relation \( V^* = sU(z, t) \), we find that \( V^*_s \) can be written as

\[
V^*_s = U + sU_s \\
= U + sU_t \frac{\partial}{\partial s} \\
= U - zU_z 
\]
since $\frac{\partial z}{\partial s} = \frac{-x}{s}$. We also have that $V^*_x$ has the form
\[
V^*_x = U_x - (zU_z)_x \\
= U_x \frac{\partial z}{\partial x} - \left( \frac{\partial z}{\partial x} U_z + z \frac{\partial z}{\partial x} \frac{\partial z}{\partial x} \right) \\
= -\frac{1}{s} z U_{zz} \tag{C.5}
\]
since $\frac{\partial z}{\partial x} = \frac{1}{s}$. So, using (C.3) and (C.4), we may rewrite (C.2) as
\[
sU_t + \max_{\theta \in [-1,1]} \frac{1}{2} \sigma^2 s^2 \left( -\frac{z}{s} V^*_x + 2 \theta s^2 V^*_x - \theta^2 \frac{s}{x} V^*_x \right) = 0,
\]
and using (C.5), we can further reduce this equation to
\[
\begin{align*}
-U_t &= \frac{1}{2} \sigma^2 \max_\theta (z - \theta)^2 U_{zz} \\
U(z, T) &= (z)^+
\end{align*}
\]
□

C.2. One Factor HJB to the Heat Equation

We wish to transform the problem
\[
\begin{align*}
-U_t &= \frac{1}{2} \sigma^2 (|z| + 1)^2 U_{zz} \\
U(z, T) &= (z)^+ \tag{C.6}
\end{align*}
\]
into the alternative problem
\[
\begin{align*}
f_\tau - f_{yy} &= \delta(y) f(y, \tau) \\
f(y, 0) &= (e^{\frac{y}{2}} - e^{-\frac{y}{2}})^+
\end{align*} \tag{C.7}
\]
via the transformed variables
\[
\tau = \frac{\sigma^2}{2} (T - t) \\
y = \text{sgn}(z) \log(|z| + 1) \\
f(y, \tau) = e^{\frac{y - |y|}{2}} U(y, \tau).
\]
where as usual the subscripts on the functions denote partial derivatives. We begin by considering the derivatives $U_t$ and $U_z$,
\[
U_t = \frac{\partial U}{\partial \tau} \frac{\partial \tau}{\partial t} = -\frac{1}{2} \sigma^2 U_\tau \tag{C.8}
\]
\[
U_z = \frac{\partial U}{\partial y} \frac{\partial y}{\partial z} \tag{C.9}
\]
Now, we note that $\frac{\partial y}{\partial z}$ is given by
\[
\frac{\partial y}{\partial z} = 2\delta(z) \log(1 + |z|) + \text{sgn}(z) \times \frac{\text{sgn}(z)}{1 + |z|} = \frac{1}{1 + |z|}. \tag{C.10}
\]
Substituting (C.10) into (C.9) gives
\[ U_z = \frac{U_y}{1 + |z|} \]
\[ U_{zz} = \frac{U_{yz}}{1 + |z|} + U_y \times \frac{-1}{(1 + |z|)^2} \text{sgn}(z) \]
\[ = \frac{U_{yy} \frac{\partial y}{\partial z}}{1 + |z|} + U_y \times \frac{-\text{sgn}(z)}{(1 + |z|)^2} \]
so substituting (C.10) into the above equation gives
\[ U_{zz} = \frac{U_{yy} - U_y \text{sgn}(z)}{(1 + |z|)^2}. \] (C.12)

Now, substituting (C.8) and (C.12) into (C.6) gives the new PDE
\[ U_\tau = U_{yy} - \text{sgn}(z) U_y. \] (C.13)

Since we may write
\[ U(y, \tau) = e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} f(y, \tau), \]
we can directly compute the derivatives in (C.13) to simplify. Doing so yields
\[ U_\tau = -\frac{1}{4} e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} f + e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} f_\tau \]
\[ = \left( -\frac{1}{4} f + f_\tau \right) e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} \] (C.14)
\[ U_y = \frac{1}{2} \text{sgn}(y) e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} f + e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} f_y \]
\[ = \left( f_y + \frac{1}{2} \text{sgn}(y) f \right) e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} \] (C.15)
\[ U_{yy} = \left[ (f_{yy} + \delta(y) f + \frac{1}{2} \text{sgn}(y) f_y) + (f_y + \frac{1}{2} \text{sgn}(y) f) \right] e^{-\frac{1}{4} \tau + \frac{1}{2} |y|} \]
\[ = \left[ f_{yy} + \text{sgn}(y) f_y + \left( \frac{1}{4} + \delta(y) \right) f \right] e^{-\frac{1}{4} \tau + \frac{1}{2} |y|}. \] (C.16)

Now, substituting these expressions into equation (C.13) gives
\[ f_\tau - f_{yy} = \delta(y) f(y, \tau) \]
which may also be written as
\[ f_\tau - f_{yy} = \delta(y) f(0, \tau) \]
where we use the property that \( \delta(x) g(x) = \delta(x) g(0) \) for any function \( g \). Whilst this is the required form for our problem, what remains is to derive the appropriate boundary condition. Our original boundary condition was \( U(z, T) = (z)^+ \), and we wish to find the equivalent form in terms of \( f \). To this end we note that when \( t = T, \tau = 0 \), thus substituting this into our expression for \( f \),
\[ f(y, 0) = U(y, 0) e^{-\frac{1}{2} |y|}. \] (C.17)
Since $y = \text{sgn}(z) \log(1 + |z|)$, it is clear that the sign of $y$ will always be the same as the sign of $z$, that is $\text{sgn}(y) = \text{sgn}(z)$. This means that

\[
\begin{align*}
  y &= \log(1 + z) \quad \text{if } z > 0 \Rightarrow z = e^y - 1 \\
  y &= -\log(1 - z) \quad \text{if } z < 0 \Rightarrow z = 1 - e^{-y}
\end{align*}
\]

(C.18)

and thus that $U(z, T) = z^+ = (e^y - 1)^+ = U(y, 0)$, so $f(y, 0)$ is given by

\[
f(y, 0) = e^{-y} U(y, 0)
\]

\[
= e^{-y} (e^y - 1)^+
\]

\[
= e^{-y} (e^y - 1)^+
\]

\[
= \left( e^{\frac{1}{2}y} - e^{\frac{1}{2}y} \right)^+, 
\]

thus our problem becomes

\[
f_\tau - f_{yy} = \delta(y) f(0, \tau)
\]

\[
f(y, 0) = \left( e^{\frac{1}{2}y} - e^{-\frac{1}{2}y} \right)^+
\]

as required. \qed
**APPENDIX D**

**Derivation of ODE Solution**

In this appendix we derive the solution to the ODE presented in section 3.5 of chapter 3. To do this, we will use a Green’s function technique. First, let us write the equation at hand. We wish to solve

\[ \ddot{f} - s \dot{f} = -h(s)\delta(y) - g^+(y). \]  

We simplify by defining

\[ H(y) = -h(s)\delta(y) - g^+(y). \]

It can be shown that the Green’s function, \( g(y, \xi; s) \), for this equation is given by

\[ g(y, \xi; s) = -\frac{e^{-|y-\xi|\sqrt{s}}}{2\sqrt{s}} \]

and thus the solution to (D.1) can be written as

\[
\begin{align*}
    f(y, s) &= \int_{-\infty}^{\infty} g(y, \xi; s)H(\xi)d\xi \\
    &= \int_{-\infty}^{\infty} \frac{e^{-|y-\xi|\sqrt{s}}}{2\sqrt{s}}[h(s)\delta(\xi) + g^+(\xi)] \\
    &= \frac{h(s)e^{-|y|\sqrt{s}}}{2\sqrt{s}} + \int_{-\infty}^{\infty} \frac{g^+(\xi)e^{-|y-\xi|\sqrt{s}}}{2\sqrt{s}}.
\end{align*}
\]

\[ \square \]