Chapter 1

Background

In this chapter we give a more comprehensive definition of a generalised Seifert space including definitions of the associated ideas: orbifolds, orbifold coverings, singular points and flat manifolds. Scott ([Sc2]) gives a good survey of classical Seifert spaces (when the general fibre is the circle: S^1) and includes good background information on 2-orbifolds and their coverings. Bonahon and Seibenmann ([BoSi]) considered classical Seifert spaces but also allowed for singularities in the total space. In their paper, they also give a good background of orbifolds. For more background information, see these papers.

1.1 Orbifolds

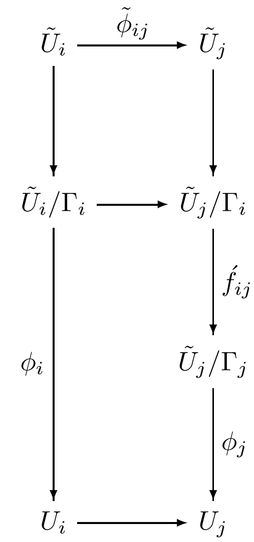
The first concept we wish to introduce is that of an orbifold. Group actions are associated to this concept. We say a group G acts on a space X *properly discontinuously* if for any compact subset C of X:

$$\{g \in G | g(C) \cap C \neq \emptyset\} \quad \text{is finite.}$$

If X is a complete Riemannian manifold and G is a group of isometries of X then G acts properly discontinuously on X if and only if it is a discrete group of isometries. In this context we will say G acts *discretely* on X. Also, we say a group G acts on a space X *freely* if the stabiliser of each point is trivial, that is no non-trivial group element has fixed points.

The quotient of a manifold by a group acting freely and properly discontinuously is again a manifold. Orbifolds provide geometric models for actions on manifolds which are discrete but not necessarily free. We will structure our definition of an orbifold to show the connection with manifolds.

Recall that a *n-manifold* is a paracompact Hausdorff space such that every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n_+ = \{x_1, \ldots, x_n | x_n \geq 0\}$. In the same way we define an *n-orbifold* to be a paracompact Hausdorff space such that every point has a neighbourhood homeomorphic to an open subset of a quotient of $\mathbb{R}^n_+$ by a finite group. More precisely an n-orbifold, B, is a paracompact Hausdorff space with an open cover $\{U_i\}$ which is closed under finite intersections. For each open set U_i, there is a finite group Γ_i and an action on $\tilde{U}_i$, an open subset of $\mathbb{R}^n_+$ such that $\phi_i : \tilde{U}_i/\Gamma_i \cong U_i$. The projections $\tilde{U}_i \to U_i$ are called the *folding charts* of the orbifold. Also if $U_i \subset U_j$ then there is an inclusion $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ equivariant with respect to f_{ij} (ie for $\gamma \in \Gamma_i$, $\tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x)$) so that the following diagram commutes:



Essentially an orbifold is a manifold with a set of marked points, so that each marked point has a finite group associated with it. These marked points determine the structure of the orbifold. Therefore the groups Γ_i or equivalently the folding charts, in the above definition are important in the structure. We call the collection of folding charts, $\{\tilde{U}_i \to U_i\}$ the *atlas of folding charts*. The atlas is not an intrinsic part of the structure of an orbifold, except if it is maximal among atlases satisfying the above conditions. An *isomorphism* of orbifolds is a homeomorphism which respects the quotient structures on open subsets of the orbifold. Any property that is based on local conditions we can give to orbifolds. For instance we can define smooth orbifolds by insisting the Γ_i act smoothly on $\tilde{U}_i$ and that the $\tilde{\phi}_{ij}$ are smooth. For the rest of this paper we will assume all orbifolds are smooth and all orbifold isomorphisms are diffeomorphisms which respect the quotient structures.

If we ignore the finite groups associated to the orbifold, then we still have a topological space, which we call the *underlying space*. The orbifold should be distinguished from its underlying space. For example $\mathbb{R}^2/\langle -I \rangle$ and $\mathbb{R}^2$ are both orbifolds with diffeomorphic underlying spaces, however they are NOT isomorphic as orbifolds.

If a point of a orbifold has a neighbourhood isomorphic to an open subset of $\mathbb{R}^n_+$ with the trivial group action, then it is called a *regular point*, otherwise it is called a *singular point* or a *critical point*. Therefore, a manifold is an orbifold whose points are all regular. By analogy with the manifold case, we call a point of an orbifold an *interior point* if it has a neighbourhood isomorphic to a open subset of a quotient of $\mathbb{R}^n$, and if a point is not interior then it is called a *boundary point*. The *boundary* of a n-orbifold is the set of boundary points and is in general a disjoint union of $(n-1)$-orbifolds (without boundary).

Lemma 1.1.1. *Let G be a finite subgroup of $GL_n\mathbb{R}$. Then G is conjugate in $GL_n\mathbb{R}$ to a subgroup of $O_n\mathbb{R}$.*

Proof. Choose an inner product on $\mathbb{R}^n$, $(-,-) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Define a new inner product, $(-,-)_G$ by averaging over G. That is define $(x,y)_G = \frac{1}{|G|}\sum_{g\in G}(gx, gy)$, which is symmetric, bilinear, positive definite and thus an inner product. The new product will also be G-invariant, ie $(gx, gy)_G = (x,y)_G$. Therefore there is an orthonormal basis of $\mathbb{R}^n$ on which G acts orthogonally. Let P be the change of basis matrix, then PGP^{-1} is a subgroup of $O_n\mathbb{R}$ and hence the lemma is proved. □

Lemma 1.1.2. *Every interior point has a neighbourhood which is diffeomorphic to the quotient of* $\mathbb{R}^n$ *by a finite subgroup of* $O_n\mathbb{R}$*, where the point maps onto the origin.*

Proof. Consider a neighbourhood, $\mathbb{R}^n/\Gamma$ (Γ a finite group) of an interior point. WLOG, suppose the interior point is the image of the origin. Let Γ' be the stabiliser of the origin. Then $\mathbb{R}^n/\Gamma'$ is diffeomorphic to a neighbourhood of the origin within this first neighbourhood. So WLOG suppose Γ fixes the origin.

Since Γ acts smoothly on $\mathbb{R}^n$, the exponential map conjugates the linear action of Γ on $T_0\mathbb{R}^n$ to the action of Γ at the origin. The previous lemma then implies the action of Γ is conjugate to an orthogonal action. So by applying an appropriate diffeomorphism to $\mathbb{R}^n$, we can modify Γ to be a subgroup of $O_n\mathbb{R}$, hence the lemma. □

In low dimensions, we can be very explicit about singular points. The only singularities of 1-orbifolds are reflector points, which have neighbourhoods diffeomorphic to the quotient of $\mathbb{R}$ by a reflection, and are isolated.

For a 2-orbifold, the lemma says the neighbourhood of a singular interior point is diffeomorphic to a quotient of $\mathbb{R}^2$ by a finite subgroup of $O_2\mathbb{R}$. The possible groups are cyclic, $\mathbb{Z}_m$, generated by a rotation of order m; order two, $\mathbb{Z}_2$, generated by a reflection; or dihedral, D_{2m}, generated by a reflection and a rotation of order $m > 1$. The quotient of $\mathbb{R}^2$ by a rotation of order m is a 2-orbifold with one singular point, called a *cone point* of order m. Cone points are always isolated. The quotient of $\mathbb{R}^2$ by a reflection is a 2-orbifold with a line of singular points, called *reflector points*. The quotient of $\mathbb{R}^2$ by a dihedral group D_{2m} has two lines of reflectors meeting at a *corner reflector* of order m. A connected component of the set of reflector points and corner reflectors is an arc or a closed loop and is called a *reflector curve*. Therefore the set of singular points, simply called the *singular set*, of a 2-orbifold is a disjoint union of cone points and reflector curves. If the orbifold is compact and has no boundary, the reflector curves are all diffeomorphic to circles. Each cone point then has a neighbourhood diffeomorphic to a disk and each reflector curve has a neighbourhood diffeomorphic to an annulus.

Note an orbifold may be homeomorphic to a manifold (as is always the case when the dimension is 1 or 2), but it is not necessarily isomorphic to a manifold. We must be careful to distinguish the orbifold from the underlying space. In particular, a reflector curve appears as part of the boundary in the underlying surface but is NOT part of the boundary of the orbifold.

We denote a 2-orbifold without boundary with underlying surface M, cone points with orders $m_1, \dots m_k$ and corner reflectors with orders $n_1, \dots n_l$ as $M(m_1, \dots m_k, \bar{n}_1, \dots \bar{n}_l)$. By the above comment, the boundaries of the underlying surface are reflector curves of the orbifold. This notation is unambiguous when there are no corner reflectors or at most one reflector curve and will only be used in those circumstances.

As an example, consider the group generated by isometries of $\mathbb{C}$: $G = \langle z \mapsto z+1, z \mapsto z+i, z \mapsto \bar{z}\rangle$ and the quotient $\mathbb{C}/G$. Since G acts discretely on $\mathbb{C}$, every point of $\mathbb{C}/G$ has a neighbourhood diffeomorphic to an open subset of a quotient of $\mathbb{R}^2$ by a finite group and therefore $\mathbb{C}/G$ is a 2-orbifold without boundary. The singular points are the image of $\{x + iy | 2y \in \mathbb{Z}\}$ (the points of $\mathbb{C}$ where G does not act freely). The singular points all have a neighbourhood diffeomorphic to an open subset of a quotient of $\mathbb{R}^2$ by a reflection. Therefore, they are all reflector points. The set of singular points is diffeomorphic to a disjoint union of two circles, ie there are two reflector curves. The underlying surface is an annulus (the boundary of which is the two reflector curves), however as an orbifold $\mathbb{C}/G$ has no boundary. Notationally (see the previous paragraph), we label $\mathbb{C}/G$ as A (for annulus).

The idea of covering maps extends to orbifolds. A continuous map of orbifolds $f : X \to Y$ is an *orbifold covering* if every y in Y has an open neighbourhood U so that its preimage $f^{-1}(U)$ is a disjoint union, ie is $\bigcup_{\lambda\in\Lambda} V_\lambda$ for some indexing set Λ, where $f|_{V_\lambda} : V_\lambda \to U$ is a natural quotient map $\mathbb{R}^n/\Gamma \to \mathbb{R}^n/\Gamma'$, where $\Gamma \le \Gamma'$ (both groups will be finite). Note the map between the underlying spaces is NOT necessarily a covering (it may have branch points).

We call an orbifold covering *regular*, if it is of the form $M/\Gamma \to M/\Gamma'$ where Γ is a subgroup of Γ' and which both act properly discontinuously on the orbifold M.

As another example, take G as above. Let G^+ be the subgroup of G of orientation preserving diffeomorphisms, then $G^+ = \langle z \mapsto z+1, z \mapsto z+i\rangle \cong \mathbb{Z}^2$. Clearly $\mathbb{C}/G^+ = T^2$, the torus, which is an orbifold since it is a manifold. Consider the quotient map $T^2 = \mathbb{C}/G^+ \to \mathbb{C}/G = A$. For every regular point of A we can choose a open neighbourhood whose preimage is a disjoint union of two open sets both diffeomorphic to the neighbourhood. Every singular point of A has a neighbourhood diffeomorphic to $D^2/\langle z \mapsto \bar{z}\rangle$ whose preimage is diffeomorphic to D^2. Therefore the map $T^2 \to A$ is an orbifold covering. However the map of the underlying surfaces cannot be a covering map since T^2 has no boundary, but A does.

If a group G acts properly discontinuously on a manifold M, the natural quotient map $M \to M/G$ is an orbifold covering. Most 2-orbifolds arise in this way. We will make this statement more precise below.

It can be proved that each orbifold X has a unique *universal orbifold cover*, and X is the quotient of its universal cover by some group G (see proposition 13.2.4 of [Th2]). We call G the *orbifold fundamental group of* X and denote it as $\pi_1^{orb}(X)$. While constructing presentations for fundamental groups of Seifert manifolds later, we implicitly construct the orbifold fundamental groups of an arbitrary 2-orbifold. For a proof of existence of universal covers and construction of an orbifold fundamental group see [Sc2] section 2.

A useful result when handling fundamental groups is van Kampen's theorem. This result generalises also to orbifold fundamental groups. See [Sc2] section 2 again, for the 2-orbifold case. The following theorem is taken from Theorem 6.8 in [Kap].

Theorem 1.1.3 (van Kampen's Theorem for orbifolds). *Let O be a connected orbifold, and let $O_1, O_2 \subset$*

O be open connected suborbifolds, such that

(i). $O = O_1 \cup O_2$,

(ii). $O_1 \cap O_2$ *is connected and*

(iii). the closures of O_1 and O_2 are suborbifolds with boundary in O such that for each i, the boundary of O_i equals the frontier of the underlying surface of O_i in the underlying surface of O.

Then $\pi_1^{orb}(O)$ is the pushout of the following diagram:

$$\begin{array}{ccc} \pi_1^{orb}(O_1 \cap O_2) & \longrightarrow & \pi_1(O_1) \\ \downarrow & & \vdots \\ \pi_1^{orb}(O_2) & \cdots\cdots\rightarrow & \pi_1^{orb}(O) \end{array}$$

An orbifold whose universal cover is a manifold is called *good*, otherwise it is called *bad*. All 1-orbifolds are good. The only bad 2-orbifolds are $S^2(p)$, $S^2(p,q)$, $D^2(\bar{p})$ and $D^2(\bar{p},\bar{q})$ where $p \neq 1$ and $p \neq q$. The following theorem regarding good orbifolds is proved in [Th2].

Theorem 1.1.4 (Thurston). *Every good 2-orbifold without boundary admits a geometric structure modelled on $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$. That is every good 2-orbifold is isomorphic to a quotient of $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ by a discrete group of isometries.*

We therefore split the good orbifolds into three categories based on their geometry: spherical, Euclidean and hyperbolic.

Other properties of good orbifolds are expressed in the next two results.

Theorem 1.1.5. *Every good compact 2-orbifold without boundary is finitely covered by a manifold.*

See [Sc2] for references to the proof.

Lemma 1.1.6. *A suborbifold of a good orbifold is also good.*

Proof. Let S be a suborbifold of a good orbifold O. Since O is good, let $\tilde{O}$ be a manifold which covers O with covering map p. Then $p^{-1}(S)$ is (in general) a union of submanifolds of $\tilde{O}$ which each cover S. Therefore S is good. □

1.2 Total holonomy of an orbifold

Later in this paper, we will consider the total holonomy of an orbifold. Using a version of the Gauss-Bonnet theorem this is connected to the euler characteristic of the orbifold. We shall now define these ideas using theorem 1.1.5. Recall that any finite simplical complex, K has an *euler characteristic* $\chi(K)$. If α_i denotes the number of i-simplices then we define $\chi(K)$ to be $\sum_{i\geq 0}(-1)^i\alpha_i$. It is well known that this is equal to $\sum_{i\geq 0}(-1)^i\beta_i$ where $\beta_i = \text{rank}(H_i(K,\mathbb{Z}))$ is the ith Betti number of K, and so $\chi(K)$ is a topological invariant. It is easily seen that if $\tilde{K}$ is a p-fold cover of K, then $p\chi(K) = \chi(\tilde{K})$.

Suppose X is a good orbifold, then by theorem 1.1.5 it is finitely covered by a manifold $\tilde{X}$. Therefore we naturally define the *(orbifold) euler characteristic of* X by the equation $p\chi^{orb}(X) = \chi(\tilde{X})$, where p is the degree of the covering $\tilde{X} \to X$. From this definition and the covering property, we can prove the following theorem for good orbifolds (see [Sc2] for a derivation or [Ta] for a more general treatment). We define the euler characteristic of bad orbifolds so that the following theorem always holds. (Note that equivalently we can use the 'volume property' of the euler characteristic: $\chi(A\cup B) = \chi(A) + \chi(B) - \chi(A\cap B)$ to determine the euler characteristic of bad orbifolds.)

Theorem 1.2.1 (Riemann - Hurwitz formula). *For a 2-orbifold X, with underlying surface Y, k_0 cone points and l reflector curves so that the ith reflector curve has k_i corner reflectors, the euler characteristic is given by the following equation:*

$$\chi^{orb}(X) = \chi(Y) - \sum_{j=1}^{k_0}\left(1 - \frac{1}{m_{0j}}\right) - \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{k_i}\left(1 - \frac{1}{m_{ij}}\right),$$

where the order of the jth cone point is m_{0j} and the order of the jth corner reflector on the ith reflector curve is m_{ij}.

Note that [Ta] shows χ^{orb} given in this theorem is the unique function which satisfies the multiplicative and volume properties and agrees with χ on manifolds.

A hyperbolic orbifold is finitely covered by a hyperbolic manifold, which has negative euler characteristic. Therefore, a hyperbolic orbifold has negative euler characteristic. Similarly, a Euclidean orbifold has zero euler characteristic and a spherical orbifold has positive euler characteristic. By direct calculation, bad orbifolds also have positive euler characteristic (which are rational numbers so that no multiple is an integer at most 2).

We now unite the euler characteristic with the total holonomy of an orbifold. Consider an oriented Riemannian 2-manifold M. Consider a geodesic path l in M. If v is a vector, tangent to l at some point x on l, then we call the vector that is tangent to y on l with the same length and orientation the *parallel translate of v by l*. Since the tangent plane (of the surface) at any point is two dimensional we can define the parallel translate of v by l for any tangent vector at x so that we get an isometry $T_xM \to T_yM$. We can also extend this definition so that l is a piecewise geodesic (the parallel translate is then a composition of parallel translates by geodesics). In particular, if l is a piecewise geodesic loop, then the parallel translate by l is a rotation of T_xM which we call the *holonomy* of l, while we call the angle of rotation the *holonomy angle*.

If l traverses anticlockwise a geodesic triangle with angles α, β and γ then it is easily seen the holonomy angle is $\alpha + \beta + \gamma - \pi$ (see [Sc2] p. 463 for a useful diagram and explanation). [When we mention the angles of a triangle or a polygon we mean the interior angles, thus we imply the interior of our region. To traverse the boundary of a region anticlockwise we mean, as usual, to traverse it in such a way as to keep the interior on the left.] Therefore when $M = \mathbb{H}^2, \mathbb{E}^2$ or $\mathbb{S}^2$ (with constant negative, zero and positive curvature respectively), the (anticlockwise) holonomy angle is the product of the curvature and the area of the triangle. By triangularisation we generalise this result to any polygon; that is the (anticlockwise) holonomy angle of a polygon equals the curvature times by the area. Note when $M = \mathbb{E}^2$, we see that the holonomy is always trivial which is clear a priori. If F is a hyperbolic or spherical orbifold, then the area of the orbifold equals the area of its fundamental region (both areas relative to the inherited metric). A good compact orbifold has a fundamental region with a finite number of sides. We define the *total (anticlockwise) holonomy angle* of a good orbifold F to be the holonomy angle of any fundamental region. Choosing a fundamental region with a finite number of sides, we see the total holonomy of a good orbifold F equals its area times by its curvature (note since the area of any fundamental region is the same, the total holonomy is also independent of the choice of fundamental region). The results here concerning area and fundamental regions are proved for the hyperbolic case in chapter 3 of Katok ([Kat]).

The classical Gauss-Bonnet theorem states that for any closed surface, F, with any Riemannian metric:

$$\int_F \kappa dA = 2\pi\chi(F),$$

where κ is the curvature. When the curvature is constant, then $\kappa A(F) = 2\pi\chi(F)$, where $A(F)$ is the area of F. If F is a good orbifold (and therefore hyperbolic, Euclidean or spherical) then area satisfies the same naturality property under finite covers that the euler characteristic does. Let $\tilde{F}$ be a manifold which covers F with degree p, then $A(\tilde{F}) = pA(F)$. We can then summarise this with the following lemma (which was proved in arbitrary dimension in [Sa]).

Lemma 1.2.2. *For a good orbifold, F, the total (anticlockwise) holonomy of F equals $\kappa A(F)$ and $2\pi\chi^{orb}(F)$.*

1.3 Orbifold Bundles

We now wish to define higher dimensional Seifert manifolds as generalisations of bundle spaces. We will now make more precise in what sense they are bundles. The definitions that appear here on orbifold bundles come from [BoSi].

Recall a *bundle map* is a continuous map $\eta : S \to B$ of spaces where there is a space F, such that for every point b of B there is an open neighbourhood, U_b and a commutative diagram:

$$\begin{array}{ccc} U_b \times F & \xrightarrow{\pi_{U_b}} & U_b \\ \downarrow{\scriptstyle \phi_b} & & \| \\ \eta^{-1}U_b & \xrightarrow[\eta]{} & U_b \end{array}$$

in which π_{U_b} is a projection and ϕ_b is a homeomorphism. The *fibre* above b is defined to be the subspace

$\eta^{-1}(b)$, which is homeomorphic to F. We call S the *total space* and say it has the structure of a *bundle*. We call B the *base* and call η the projection. (We shall always assume that F is connected.)

An *orbifold bundle map* is a continuous map $\eta : S \to B$ of orbifolds such that for every point b of B there is an open neighbourhood, U_b and a commutative diagram:

$$\begin{array}{ccc} \tilde{U}_b \times F_b & \xrightarrow{\pi_{\tilde{U}_b}} & \tilde{U}_b \\ \phi_b \downarrow & & \downarrow f_b \\ \eta^{-1} U_b & \xrightarrow[\eta]{} & U_b \end{array}$$

in which $\pi_{\tilde{U}_b}$ is a projection, ϕ_b and f_b are regular orbifold covering projections and F_b is some connected orbifold. The *fibre* above b is defined to be the suborbifold $\eta^{-1}(b)$, which will be covered by F_b. Without loss of generality, f_b can be a folding chart of B; a quotient map of $\tilde{U}_b$ under action by a finite group, ie $f_b : \tilde{U}_b \to \tilde{U}_b / \Gamma_b \cong U_b$. Similarly ϕ_b can be a folding chart of S, so long as F_b is good. We call S the *total space* and say it has the structure of an *orbifold bundle*. We call B the *base* and call η the projection.

Let G_{ϕ_b} be the group of covering translations of ϕ_b in the above commutative diagram. Then G_{ϕ_b} respects the projection $\pi_{\tilde{U}_b} : \tilde{U}_b \times F_b \to \tilde{U}_b$, however in general it will not respect the projection to F_b. We say that the bundle is *locally trivial* near b if it has a commutative diagram so that G_{ϕ_b} respects the projection to F_b and thus G_{ϕ_b} gives a product action on $\tilde{U}_b \times F_b$. If the fibre at b is compact, then it can be shown that the bundle is locally trivial near $b \in B$. (See [BoSi].)

When $\eta : S \to B$ is locally trivial, there are some basic consequences regarding the fibres. By quotienting F_b by the subgroup of G_{ϕ_b} which fixes $\tilde{U}_b$ pointwise, we see that F_b can always be chosen to be a copy of a *regular fibre*, F, that is a fibre over a nonsingular point. So the fibres above nonsingular points are isomorphic, while fibres above critical points, which are called *singular fibres* or *critical fibres*, are finitely covered by the regular fibre. The set of regular points is dense in B, so we say also the *general fibre* of a bundle is the regular fibre.

The Mobius band is often used as an example of a non-trivial fibre bundle, where the fibres are intervals and the base is a circle. The Mobius band is also the total space of an orbifold bundle with singularities in the base. The natural S^1-fibre bundle of the annulus, $p_{[-1,1]} : An = [-1,1] \times S^1 \to [-1,1]$ projects via the 2-fold cover, $\phi : An \to Mb$ to an orbifold bundle $\eta : Mb \to B$ where the general fibre is S^1 and the base, B is $[-1,1]/\mathbb{Z}_2 = [[0,1]$ (the 1-orbifold which is an interval with a reflector point at one point, 0 and a boundary point at the other). It is an orbifold bundle because we have the following commutative diagram:

$$\begin{array}{ccc} [-1,1] \times S^1 & \xrightarrow{p_{[-1,1]}} & [-1,1] \\ \phi \downarrow & & \downarrow \\ Mb & \xrightarrow[\eta]{} & [[0,1] \end{array}$$

Represent the Mobius band by $Mb = \{(s,t)|s,t \in [-1,1],\ (1,t) = (-1,-t)\}$. The map η sends (s,t) to $|t|$. So for $t \in [[0,1]$ the fibres are $\eta^{-1}(t) = F_t$, the loops $\{(s,t)\} \cup \{(s,-t)\}$. We see the fibre, F_0 above the reflector point is different. It is half the length of all other fibres, or expressed differently - it is doubly covered by all other fibres. The fibre F_0 is the only singular fibre, whereas all other fibres are regular fibres and are naturally homeomorphic to the general fibre, S^1.

If S is a manifold, then it has no singularities and all subspaces, which include the fibres, must also have no singularities and so are manifolds.

The basic properties of bundles generalise to orbifold bundles. In particular, we have the notion of an induced bundle.

Definition 1.3.1. The bundle *induced* from an orbifold bundle, $\eta : S \to B$ by a map $\bar{\phi} : \tilde{B} \to B$, is an orbifold bundle $\tilde{\eta} : \tilde{S} \to \tilde{B}$ so that the following diagram commutes:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \phi\ } & S \\ \downarrow{\scriptstyle \tilde{\eta}} & & \downarrow{\scriptstyle \eta} \\ \tilde{B} & \xrightarrow[\ \bar{\phi}\ ]{} & B. \end{array}$$

Furthermore, if $\bar{\phi}$ is a covering map then $\phi : \tilde{S} \to S$ is also a covering map.

A unique induced bundle always exists (it is the pullback and is unique up to orbifold isomorphism). We will use the notation $\bar{\phi}^*(S)$ for the bundle induced by $\bar{\phi}$.

For example the choice of a regular point $*$ as a basepoint of B, determines an inclusion $j : F = \eta^{-1}(*) \to S$.

Lemma 1.3.2. *An (orbifold) covering $\phi : \tilde{S} \to S$ of an orbifold bundle $\eta : S \to B$ has the structure of orbifold bundle.*

Proof. Each fibre of S lifts via ϕ to a disjoint union of orbifolds, each one covering the original fibre. Then the quotient map $\tilde{\eta} : \tilde{S} \to \tilde{B}$ obtained by identifying each of these orbifolds to a point, is an orbifold bundle map. The induced map $\bar{\phi}$ is a covering that completes the above commutative square, with degree that divides the degree of ϕ. □

1.4 The geometries $\mathbb{E}^n$ and flat orbifolds

The last concept we need to cover to fully define a Seifert manifold concerns the nature of the fibres. An orbifold is *flat* or geometric of type $\mathbb{E}^n$, if it is a quotient of $\mathbb{R}^n$ by a discrete cocompact group of isometries of $\mathbb{E}^n$. If α is an isometry of $\mathbb{E}^n$, then $\alpha(x) = Ax + y$, for $A \in O_n\mathbb{R}$ and $y \in \mathbb{R}^n$. We will write α as (A, y). Let $T = \{(I, y)\,|y \in \mathbb{R}^n\}$ which is isomorphic to $\mathbb{R}^n$. We call T the group of translations of $\text{Isom}(\mathbb{E}^n)$. The map

$(A, y) \mapsto A$ (which we call M), defines a surjective homomorphism with kernel T. So we get the following exact sequence:

$$0 \longrightarrow T \longrightarrow \mathrm{Isom}(\mathbb{E}^n) \xrightarrow{M} O_n\mathbb{R} \longrightarrow 1.$$

Define the group of affine maps of $\mathbb{R}^n$ to be $\mathrm{Aff}(\mathbb{R}^n) = \{\alpha \in \mathrm{Homeo}(\mathbb{R}^n) | \alpha(x) = Ax + y \ \forall x \in \mathbb{R}^n, A \in GL_n\mathbb{R}, y \in \mathbb{R}^n\}$. The group of affine maps clearly includes $\mathrm{Isom}(\mathbb{E}^n)$. Similar to above, if $\alpha(x) = Ax + y$, write α as (A, y). Also define $M : \mathrm{Aff}(\mathbb{R}^n) \to GL_n\mathbb{R}$ by $M((A, y)) = A$. Then M again has kernel T and we get the following exact sequence:

$$0 \longrightarrow T \longrightarrow \mathrm{Aff}(\mathbb{R}^n) \xrightarrow{M} GL_n\mathbb{R} \longrightarrow 1.$$

Suppose $F = \mathbb{R}^n/H$ is a compact flat n-orbifold. Then H is isomorphic to $\pi_1^{orb}(F)$. The Bieberbach theorems ([Bi]) state (i) F is finitely covered by T^n (the n-torus), or equivalently $H_T = H \cap T \cong \mathbb{Z}^n$ and $M(H) \cong H/H_T$ is a finite subgroup of $O_n\mathbb{R}$; (ii) if H and H' are isomorphic subgroups of $\mathrm{Isom}(\mathbb{E}^n)$, then they are conjugate by an element of $\mathrm{Aff}(\mathbb{R}^n)$ and so $\mathbb{R}^n/H$ and $\mathbb{R}^n/H'$ are homeomorphic. By conjugating H by an element of $\mathrm{Aff}(\mathbb{R}^n)$ if necessary, we can suppose $H_T = Z^n := \{(I, z) \,|\, z \in \mathbb{Z}^n\}$. With this condition, H is determined up to conjugation by an element (A, y) of $\mathrm{Aff}(\mathbb{R}^n)$ which normalises Z^n, ie such that A is in $GL_n\mathbb{Z}$. For example, when F is T^2 then $H = Z^2 = \left\langle \left(I, \binom{1}{0}\right), \left(I, \binom{0}{1}\right) \right\rangle$. When F is Kb we can take H to be $\left\langle \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \binom{1/2}{0} \right), \left(I, \binom{0}{1}\right) \right\rangle$ (note that other choices are possible).

The subgroup H_T is unique as the next lemma shows.

Lemma 1.4.1. *If $F = \mathbb{R}^n/H$ is a flat n-orbifold, then H_T is the unique maximal abelian normal subgroup.*

Proof. Firstly, H_T is abelian and normal. Next suppose (A, y) in H is contained in a normal abelian subgroup, N. Since N is normal, $(I, z)\,(A, y)\,(I, z)^{-1} = (I, (I - A)z)\,(A, y)$ is also in N, for all (I, z) in H_T. Multiplying by (A, y) shows $(I, (I - A)z)$ is in N. However since N is abelian, $(I, (I - A)z) = (A, y)\,(I, (I - A)z)\,(A, y)^{-1} = (I, A(I - A)z)$, so $(A - I)^2 z = 0$ for all $z \in \mathbb{Z}^n$. This forces $(A - I)^2$ to be 0. Since A has finite order, this means $A = I$. Thus N is contained in H_T, proving the result. □

Later we wish to describe presentations for groups. We shall introduce some notation concerning the presentation of H. Let $h_1, \ldots, h_q$ be some chosen generators of $\pi_1(F) = H$ and let W be the set of relations (ie for $F = T^2$, $W = \{[h_1, h_2]\}$ and for $F = K$, $W = \{h_1 h_2 h_1^{-1} h_2\}$).

1.5 Seifert manifolds

Definition 1.5.1 (Seifert bundle). A *Seifert bundle* is an orbifold bundle with base a 2-orbifold and compact flat fibres. A *Seifert orbifold* is the total space of a Seifert bundle.

Since the fibres are compact the bundle is locally trivial and therefore has a compact flat general fibre. Explicitly, if S is a Seifert orbifold, then there is an associated (locally trivial) orbifold bundle map $\eta : S \to B$

(B is the 2-orbifold base) with general (compact flat) fibre, F. In this paper we will primarily be interested in *Seifert manifolds*, that is when S has no singularities (and consequently neither has F). We are also more interested when the manifolds are compact. *Compact Seifert manifolds* will necessarily have a compact base. For the rest of this paper we will assume that all Seifert manifolds are compact, unless explicitly stated otherwise.

Lemma 1.5.2 (cf Lemma 3.2 in [Sc2]). *Let $\eta : S \to B$ be a orbifold bundle map with nonsingular total space and general fibre F. Then we have a short exact sequence:*

$$\pi_1(F) \to \pi_1(S) \to \pi_1^{orb}(B) \to 1,$$

where the maps are determined by the inclusion of fibre above the basepoint of B and η. Furthermore, if the base is good and aspherical then $\pi_1(F)$ maps injectively into $\pi_1(S)$.

Proof. Let $\tilde{S}$ be the universal cover of S. Since $\tilde{S}$ covers the orbifold bundle S it will have the structure of an orbifold bundle (by lemma 1.3.2) with, say, base $\tilde{B}$ and general fibre $\tilde{F}$. The orbifold $\tilde{B}$ will cover B, similarly $\tilde{F}$ covers F. So we get the following commutative diagram.

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\hat{\phi}} & F \\ \downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{\phi} & S \\ \tilde{\eta}\downarrow & & \downarrow\eta \\ \tilde{B} & \xrightarrow[\bar{\phi}]{} & B. \end{array}$$

Now $\tilde{B}$ is the universal cover of B (if it was not, then the bundle induced by the universal cover of $\tilde{B}$ would cover $\tilde{S}$, but $\pi_1(\tilde{S}) = 1$; contradiction). The group $\pi_1(S)$ acts on $\tilde{S}$ preserving the fibration, so there is an induced action on $\tilde{B}$. This gives a natural projection from $\pi_1(S)$ onto the covering group of $\bar{\phi}$, which is $\pi_1^{orb}(B)$ since $\bar{\phi}$ is a universal covering map. The kernel of this map consists of covering translations of $\tilde{S}$ which project to the identity map on $\tilde{B}$ and so acts freely on the fibres of $\tilde{S}$. Therefore the kernel is isomorphic to the covering group of $\hat{\phi}$. In general this is a quotient of $\pi_1(F)$ which gives the above exact sequence. If $\tilde{F}$ is simply connected then $\pi_1(F)$ is the kernel. If $\tilde{F}$ is not simply connected, then $\tilde{B}$ is not contractible and therefore must be $S^2, S^2(p)$ or $S^2(p,q)$ (p and q coprime). So in particular when the base is good and aspherical then $\pi_1(F)$ is the kernel. □

Definition 1.5.3. We shall call the sequence $\pi_1(F) \to \pi_1(S) \to \pi_1^{orb}(B) \to 1$ the *fundamental group sequence* of the orbifold bundle. An *isomorphism of fundamental group sequences* is a triple $(\phi_F, \phi, \bar{\phi})$ which make the following diagram commute:

$$\begin{array}{ccccccc}
\pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1^{orb}(B) & \longrightarrow & 1 \\
\downarrow{\scriptstyle \phi_F} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \| \\
\pi_1(F') & \longrightarrow & \pi_1(S') & \longrightarrow & \pi_1^{orb}(B') & \longrightarrow & 1
\end{array}$$

If $\phi_F = 1$ and $\bar{\phi} = 1$ (implicitly $F \cong F'$) then we call $(1, \phi, 1)$ an *equivalence of fundamental group sequences.*

Definition 1.5.4 (Monodromies). If B is aspherical, then $\pi_1(F)$ is a normal subgroup of $\pi_1(S)$ from the previous lemma. Let $A : \pi_1(S) \to \mathrm{Aut}(\pi_1(F))$ be the homomorphism which sends ξ to the automorphism $h \mapsto \xi h \xi^{-1}$. If we quotient the domain of this map by $\pi_1(F)$ and the codomain by $\mathrm{Inn}(\pi_1(F))$, then we get a map $\bar{A} : \pi_1^{orb}(B) \to \mathrm{Out}(\pi_1(F))$. We call $\bar{A}$ the *monodromy map*, and we call $\mathrm{Im}(\bar{A})$ the *group of monodromies*. Also, for $\bar{\xi} \in \pi_1^{orb}(B)$ we call $\bar{A}(\bar{\xi})$ the *monodromy* above $\bar{\xi}$.

If $F = T^n$, then $\pi_1(F)$ is abelian, and so $\mathrm{Out}(\pi_1(F)) = \mathrm{Aut}(\pi_1(F)) = GL_n\mathbb{Z}$. If $\xi \in \pi_1(S)$ projects to $\bar{\xi} \in \pi_1^{orb}(B)$, then $\bar{A}(\bar{\xi}) = A(\xi) \in GL_n\mathbb{Z}$. In this case, the group of monodromies also equals $\mathrm{Im}\, A$.