Chapter 2

Siefert fibrations determined by homotopy sequences

The main purpose of this chapter is to prove Seifert manifolds with aspherical bases are determined (up to fibre preserving homeomorphism) by their fundamental group sequences. A secondary purpose is to understand the fundamental group sequence of such manifolds to help prove results in later chapters. To begin with, we will look at affine homeomorphisms of flat manifolds (section 2.1) which help predominantly with understanding the monodromies of Seifert manifolds. The middle of the chapter is occupied with constructing the fundamental group of Seifert manifolds. When doing so we will consider presentations useful for both the current main purpose and also for later chapters. Our strategy is to first construct presentations for the restriction of a Seifert bundle to neighbourhoods of singular points (sections 2.2 and 2.3) and prove our main result for these special cases. Then we will combine them together (section 2.4) to construct the presentation. We finish the chapter with the main result and several important corollaries (section 2.5).

2.1 Affine homeomorphisms of flat manifolds

Definition 2.1.1. A self homeomorphism ϕ of a flat n-manifold $F = \mathbb{R}^n/H$ is affine if it is covered by an affine homeomorphism of $\mathbb{R}^n$. Let $[A, y]_F$ or simply $[A, y]$ denote the affine homeomorphism of F represented by (A, y).

In this section we look at representatives of affine homeomorphisms of flat manifolds, giving particular attention to those manifolds which appear as fibres for three and four dimensional Seifert manifolds (namely S^1, T^2 and K). These maps will appear later connected to how isometries of several different geometries act on $\mathbb{R}^2$ and include information about possible monodromies.

We say a homeomorphism $\hat{\phi}$ of $\mathbb{R}^n$ lifting a homeomorphism ϕ of $\mathbb{R}^n/H$ has *no fixed orbits*, if there is no $x \in \mathbb{R}^n$ such that $\hat{\phi}(x) = hx$ for some $h \in H$.

Lemma 2.1.2. *If a homeomorphism $\hat{\phi}$ of $\mathbb{R}^n$ represents a homeomorphism ϕ of $F = \mathbb{R}^n/H$ then*

(i). $\phi = 1$ if and only if $\hat{\phi} \in H$,

(ii). ϕ has finite order m if and only if $\hat{\phi}^m$ is the smallest nonzero power of $\hat{\phi}$ in H,

(iii). $\hat{\phi}$ normalises H, and

(iv). ϕ has no fixed points if and only if $\hat{\phi}$ has no fixed orbits.

Proof. The first three properties are consequences of standard covering space theory. Suppose ϕ has no fixed points. Lifting this condition we see $\hat{\phi}$ has no fixed H-orbits. Conversely, if ϕ has a fixed point, then $\hat{\phi}$ has a fixed H-orbit. If $\hat{\phi}$ has a fixed orbit, then $\hat{\phi}(Hx) = Hx$ for some $x \in \mathbb{R}^n$, or simply $\hat{\phi}(x) = hx$ for some $h \in H$. □

Lemma 2.1.3. *If $A - I$ is invertible then the affine homeomorphism (A, y) has a fixed point.*

Proof. It is easy to show $(I - A)^{-1}y$ is a fixed point. □

Lemma 2.1.4. *If $A \in GL_n\mathbb{R}$ such that $A^m = I$, then*

$$\mathrm{Im}\left(\sum_{j=0}^{m-1} A^j\right) = \ker(A - I) \tag{2.1.1}$$

$$\mathrm{Im}(A - I) = \ker\left(\sum_{j=0}^{m-1} A^j\right) \tag{2.1.2}$$

Proof. Suppose $(A - I)v = 0$, then $Av = v$ and consequently, $mv = \sum_{j=0}^{m-1} A^j v$. Therefore since m is invertible then $\ker(A - I) \subseteq \mathrm{Im}\left(\sum_{j=0}^{m-1} A^j\right)$.

Next, $(A - I)\left(\sum_{j=0}^{m-1} A^j\right) = \left(\sum_{j=0}^{m-1} A^j\right)(A - I) = A^m - I = 0$. This shows $\ker(A - I) \supseteq \mathrm{Im}\left(\sum_{j=0}^{m-1} A^j\right)$ and $\ker\left(\sum_{j=0}^{m-1} A^j\right) \supseteq \mathrm{Im}(A - I)$, which proves the first equality.

By considering the dimensions of these spaces (and the rank-nullity theorem), we get the second equality. □

Recall $Z^n = \{(I, z)\,|\,z \in \mathbb{Z}^n\} \subset \mathrm{Aff}(\mathbb{R}^n)$ and that we can consider T^n as $\mathbb{R}^n/Z^n$.

If a matrix A has finite order p, let $o(A) = p$.

Lemma 2.1.5. *If $[A, y]$ is an affine homeomorphism of T^n of finite order (m say), then firstly $(A, y)^m = (I, \tilde{e})^{-1}$ for some $\tilde{e} \in \ker(A - I) \cap \mathbb{Z}^n$ and secondly $A \in GL_n\mathbb{Z}$, $A^m = I$ and $y + \tilde{e}/m = (A - I)z$ where $z \in \mathbb{R}^n$. Furthermore, the greatest common divisor of the entries of $\tilde{e}$ and $(m/o(A))$ is 1, notationally $\gcd(m/o(A), \tilde{e}) = 1$.*

Proof. We will determine the possibilities for (A, y) by considering the properties in lemma 2.1.2. Firstly since (A, y) normalises Z^n, $A \in GL_n\mathbb{Z}$. (By conjugating $(I, x) \in Z^n$ by (A, y), we see $Ax \in \mathbb{Z}^n$ for all $x \in \mathbb{Z}^n$. Then by taking special values for x we see A has integral entries. Similarly, A^{-1} has integral entries so $A \in GL_n\mathbb{Z}$.) Since $[A, y]$ has order m, $(A, y)^m = \left(A^m, \sum_{j=0}^{m-1} A^j y\right) \in Z^n$, hence $A^m = I$ and $\left(\sum_{j=0}^{m-1} A^j\right) y \in \mathbb{Z}^n$. Let $\tilde{e} = -\left(\sum_{j=0}^{m-1} A^j\right) y$. Then $A\tilde{e} = \tilde{e}$. So,

$$\begin{aligned}\left(\sum_{j=0}^{m-1} A^j\right)(y + \tilde{e}/m) &= \left(\sum_{j=0}^{m-1} A^j\right) y + m\tilde{e}/m \\ &= -\tilde{e} + \tilde{e} \\ &= 0.\end{aligned}$$

Therefore $y + \tilde{e}/m$ is in $\ker\left(\sum_{j=0}^{m-1} A^j\right)$ which equals $\mathrm{Im}(A - I)$ by lemma 2.1.4. So $y + \tilde{e}/m = (A - I)z$ for some $z \in \mathbb{R}^n$.

Again since $[A, y]$ has order m, $(A, y)^i = \left(A^i, -i\tilde{e}/m + (A^i - I)z\right) \in Z^n$ iff m divides i. Let p be the order of A, let k be the gcd of the entries of $\tilde{e}$ and (m/p), and let $i = m/k$. Then $i\tilde{e}/m = \tilde{e}/k \in \mathbb{Z}^n$, and as p divides i, $A^i = I$. So $[A, y]$ has order i. Thus $k = 1$. □

Lemma 2.1.6 (Classical Case). *If $[A, y]$ is an affine homeomorphism of S^1 which generates a free $\mathbb{Z}_m$-action on S^1 then $A = 1$, $y = -a/m$ for an integer a coprime to m.*

Proof. By lemma 2.1.5, (A, y) is either $(1, -a/m)$ or $(-1, x)$ where a and m are coprime and $x \in \mathbb{R}$. However lemma 2.1.3 shows that the latter has a fixed point. Therefore $(A, y) = (1, -a/m)$, whose each nontrivial power clearly satisfies the no fixed orbit property. □

The following lemma shows the few values that monodromies (the automorphism $A \in GL_2\mathbb{Z}$) of loops around critical points can take.

Lemma 2.1.7 (T^2 Case). *If $[A, y]$ is an affine homeomorphism of T^2 which generates a free $\mathbb{Z}_m$-action on T^2 then either:*

(i). $A = I$, $y = \binom{-a/m}{-b/m}$, $\gcd(m, a, b) = 1$,

(ii). m *even,* $A = P\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}P^{-1}$, $y = P\binom{-a/m}{x}$, $\gcd(m, a) = 1$, *or*

(iii). m *divisible by 4,* $A = P\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}P^{-1}$, $y = P\binom{-a/m+x}{-2x}$, $\gcd(m, a) = 1$,

where $a, b \in \mathbb{Z}$; $x \in \mathbb{R}$ and $P \in GL_2\mathbb{Z}$.

Proof. The matrix A has 1 as an eigenvalue by 2.1.3 and finite order by 2.1.5. Therefore A is conjugate in $GL_2\mathbb{Z}$ to I, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The constraints on y follow from lemma 2.1.5 and the no fixed orbit property:

(i). $\left(I, \binom{-a/m}{-b/m}\right)$ ($\gcd(m,a,b)=1$):

Each nontrivial power of this satisfies the no fixed orbit property, therefore is a possible representative.

(ii). $\left(P\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)P^{-1}, P\binom{-a/m}{x}\right)$: ($P \in GL_2\mathbb{Z}$, m even, $\gcd(m/2,a)=1$ and $x \in \mathbb{R}$)

(Note $P\binom{0}{x} \in \operatorname{Im}(A-I)$, and $P\binom{-a/m}{0} \in \ker(A-I)$).

Nontrivial even powers have no fixed orbits. The odd powers are: $\left(P\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)P^{-1}, P\binom{-ia/m}{x}\right)$, where i is odd. Suppose that for some odd power i, the map has a fixed orbit, ie suppose there is an $z \in \mathbb{R}^2$ and a $(I,v) \in H$ such that $\left(P\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)P^{-1}, P\binom{-ia/m}{x}\right)(z) = (I,v)(z)$. Rearrange, premultiply by P^{-1} and let $z' = -\frac{1}{2}\binom{0}{x} + P^{-1}z$ (observe $(A-I)Pz' = P\binom{0}{x} + (A-I)z$) and $v' = P^{-1}v$ (note $v' \in \mathbb{Z}^2$) to get:

$$\begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix} z' = \begin{pmatrix}0\\ -2z_2'\end{pmatrix} = v' + \begin{pmatrix}ia/m\\ 0\end{pmatrix}$$

Therefore $ia/m = -v_1' \in \mathbb{Z}$. Since m is even and i is odd, this means a must be even. Conversely if a is even, then $m/2$ is odd (since $\gcd(m/2,a)=1$) and $(A,y)^{m/2}$ will have a fixed orbit. Therefore for $(A,y)^i$ to have no fixed orbits for all i, a is odd, which implies $\gcd(m,a)=1$.

(iii). $\left(P\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)P^{-1}, P\binom{-a/m+x}{-2x}\right)$: ($P \in GL_2\mathbb{Z}$, m even, $\gcd(m/2,a)=1$ and $x \in \mathbb{R}$)

(Note $P\binom{x}{-2x} \in \operatorname{Im}(A-I)$, and $P\binom{-a/m}{0} \in \ker(A-I)$).

Nontrivial even powers have no fixed orbits. The odd powers are: $\left(P\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)P^{-1}, P\binom{-ia/m+x}{-2x}\right)$, where i is odd. Suppose that for some odd power i, the map has a fixed orbit, ie suppose there is an $z \in \mathbb{R}^2$ and a $(I,v) \in H$ such that $\left(P\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)P^{-1}, P\binom{-ia/m+x}{-2x}\right)(z) = (I,v)(z)$. Rearrange, premultiply by P^{-1} and let $z' = \binom{0}{x} + P^{-1}z$ (observe $(A-I)Pz' = P\binom{x}{-2x} + (A-I)z$) and $v' = P^{-1}v$ (note $v' \in \mathbb{Z}^2$) to get:

$$\begin{pmatrix}0 & 1\\ 0 & -2\end{pmatrix} z' = \begin{pmatrix}z_2'\\ -2z_2'\end{pmatrix} = v' + \begin{pmatrix}ia/m\\ 0\end{pmatrix}$$

Therefore $-2z_2' \in \mathbb{Z}$ and then consequently $2ia/m \in \mathbb{Z}$. If $m/2$ is odd, we can take $i = m/2$ to achieve this, if $m/2$ is even, then since i is odd this forces a to be even which is not possible since $\gcd(m/2,a)=1$. Therefore for (A,y) to have no fixed orbits for all i, m is divisible by 4 and a is consequently odd which implies $\gcd(m,a)=1$.

$\square$

Recall that we can consider the Klein bottle, Kb as $\mathbb{R}^2/H$ where

$$H = \pi_1(Kb) = \left\langle h_1 = \left(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right), \begin{pmatrix}1/2\\0\end{pmatrix}\right), h_2 = \left(I, \begin{pmatrix}0\\1\end{pmatrix}\right)\right\rangle.$$

Notice,

$$\operatorname{Aut}(\pi_1(Kb)) = \left\{ \begin{array}{ccr} h_1 & \mapsto & h_1^{\epsilon}h_2^{c} \\ h_2 & \mapsto & h_2^{\delta} \end{array} \middle| \epsilon, \delta = \pm 1, c \in \mathbb{Z} \right\}.$$

Lemma 2.1.8. *If $[A,y]$ is an affine homeomorphism of Kb so that the automorphism of H given by conjugation by (A,y) sends h_1 to $h_1^\epsilon h_2^c$ and h_2 to h_2^δ, where $\epsilon,\delta=\pm1$ and c is an integer, then $A=\left(\begin{smallmatrix}\epsilon&0\\0&\delta\end{smallmatrix}\right)$ and $y=\left(\begin{smallmatrix}x\\-c/2\end{smallmatrix}\right)$ for some $x\in\mathbb{R}$.*

Proof. Let $M:\mathrm{Aff}_2\mathbb{R}\to GL_2\mathbb{R}$ be the map which sends (A,y) to its matrix part. Then $M(H)=\{I,\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\}$, and H is the union of two cosets, $H=\mathbb{Z}^2\cup h_1\mathbb{Z}^2$, where $M(h_1)=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. If (A,y) normalises H, then $M((A,y))=A$ normalises $M(H)$ and so (A,y) must preserve the cosets Z^2 and h_1Z^2. Since it normalises Z^2, the proof of lemma 2.1.5 shows $A\in GL_2\mathbb{Z}$. Since the conjugate of h_1 by (A,y) is in h_1Z^2, we see:

$$(A,y)\,h_1\,(A,y)^{-1}=h_1^e h_2^c$$

$$\text{ie:}\quad \left(A\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)A^{-1},A\begin{pmatrix}1/2\\0\end{pmatrix}+y-A\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)A^{-1}y\right)=\left(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\begin{pmatrix}e/2\\-c\end{pmatrix}\right),$$

for some integers c and e, where e is odd. Therefore A commutes with $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and so equals $\left(\begin{smallmatrix}\epsilon&0\\0&\delta\end{smallmatrix}\right)$, $\epsilon,\delta=\pm1$. We then reduce the equation to see $e=\epsilon$ and y has second component $-c/2$. □

Incidentally, the conjugate of h_2 by (A,y) is h_2^δ.

Lemma 2.1.9. *If $[A,y]$ is an affine homeomorphism of Kb of finite order (m say) then $(A,y)^m=(h_1^a h_2^b)^{-1}$ and one of the following holds:*

(i). $A=I$, $y=\left(\begin{smallmatrix}-a/2m\\-c/2\end{smallmatrix}\right)$, a and mc are even, $b=mc/2$ and $\gcd(m,a/2,mc/2)=1$

(ii). m odd: $A=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $y=\left(\begin{smallmatrix}-a/2m\\-c/2\end{smallmatrix}\right)$, c is even, $b=c/2$ and $\gcd(2m,a)=1$

m even: $A=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $y=\left(\begin{smallmatrix}-a/2m\\-c/2\end{smallmatrix}\right)$, a is even, $b=0$, $\gcd(m/2,a/2)=1$ and if c is even then either $m/2$ or $a/2$ is even.

(iii). $m=2$: $A=-I$, $y=\left(\begin{smallmatrix}x\\-c/2\end{smallmatrix}\right)$, $a=0$, $b=0$ and $x\in\mathbb{R}$

(iv). $m=2$: $A=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$, $y=\left(\begin{smallmatrix}x\\-c/2\end{smallmatrix}\right)$, $a=0$, $b=c$ and $x\in\mathbb{R}$

Proof. From the previous lemma, $A=\left(\begin{smallmatrix}\epsilon&0\\0&\delta\end{smallmatrix}\right)$ and y has second component $-c/2$.

Since $[A,y]$ has finite order, $(A,y)^m=(h_1^a h_2^b)^{-1}$ for some integers a,b. If a is even then $(A,y)^m=\left(I,\left(\begin{smallmatrix}-a/2\\-b\end{smallmatrix}\right)\right)$ and as in lemma 2.1.5, $y+\left(\begin{smallmatrix}a/2m\\b/m\end{smallmatrix}\right)=(A-I)\left(\begin{smallmatrix}z\\z'\end{smallmatrix}\right)=\left(\begin{smallmatrix}(\epsilon-1)z\\(\delta-1)z'\end{smallmatrix}\right)$ for some $z,z'\in\mathbb{R}$ and $\left(\begin{smallmatrix}a/2\\b\end{smallmatrix}\right)\in\ker(A-I)\cap\mathbb{Z}^2=\{\left(\begin{smallmatrix}(\epsilon+1)x\\(\delta+1)x'\end{smallmatrix}\right)|x,x'\in\mathbb{R}\}\cap\mathbb{Z}^2$, where $\gcd(m/o(A),a/2,b)=1$. Here $o(A)$ is 1 when $A=I$ or 2 otherwise. When $A=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ we additionally cannot simultaneously have $m/2$ odd, $a/2$ odd and y_2 an integer (otherwise we can halve the order). If a is odd then A is forced to be $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and m is necessarily odd. By direct calculation y is consequently $\left(\begin{smallmatrix}-a/2m\\-b\end{smallmatrix}\right)$. In this case, $y+\left(\begin{smallmatrix}a/2m\\0\end{smallmatrix}\right)=(A-I)\left(\begin{smallmatrix}z\\b/2\end{smallmatrix}\right)$ where $\left(\begin{smallmatrix}a/2m\\0\end{smallmatrix}\right)\in\ker(A-I)$ and $\gcd(2m,a,0)=1$.

In all cases $y=\left(\begin{smallmatrix}-a/2m+(\epsilon-1)z\\-c/2\end{smallmatrix}\right)$, where $z\in\mathbb{R}$, $a,c\in\mathbb{Z}$; $(\epsilon-1)a=0$ and $b=\left(\sum_{i=0}^{m-1}\delta^i\right)c/2$. These conditions lead easily to the stated results. □

Remark 2.1.10. In the previous lemma, all cases take the form $A = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \delta \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} -a/2m+(\epsilon-1)z \\ -c/2 \end{smallmatrix}\right)$ for some $z \in \mathbb{R}$, where ϵ, δ and c are determined by the automorphism $h \mapsto (A, y)\, h\, (A, y)^{-1}$.

Lemma 2.1.11 (Kb Case). *If $[A, y]$ is an affine homeomorphism of Kb which generates a free $\mathbb{Z}_m$-action on Kb then either:*

(i). m odd: $A = I$, $y = \left(\begin{smallmatrix} -a'/m \\ -c/2 \end{smallmatrix}\right)$, $\gcd(m, a') = 1$, $a = 2a'$, c even

(ii). m odd: $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} -a/2m \\ -c/2 \end{smallmatrix}\right)$, $\gcd(2m, a) = 1$, c even

(iii). $m \equiv 2 \pmod 4$: $A = I$, $y = \left(\begin{smallmatrix} -a'/m \\ -c/2 \end{smallmatrix}\right)$, $\gcd(m, a') = 2$, c odd, $a = 2a'$

(iv). $m \equiv 2 \pmod 4$: $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} -a'/m \\ -c/2 \end{smallmatrix}\right)$, $\gcd(m, a') = 1$, c odd, $a = 2a'$

where $a, a', c \in \mathbb{Z}$.

Proof. We continue from the previous lemma. Firstly, by lemma 2.1.3, A cannot be $-I$ since it has a fixed orbit. Secondly, $\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ -c/2 \end{smallmatrix}\right)\right) = h_1 \left(-I, \left(\begin{smallmatrix} x-1/2 \\ c/2 \end{smallmatrix}\right)\right)$ has a fixed H-orbit since $\left(-I, \left(\begin{smallmatrix} x-1/2 \\ c/2 \end{smallmatrix}\right)\right)$ does, so A cannot be $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ either. Therefore A is either I or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

First suppose $A = I$, then $y = \left(\begin{smallmatrix} -a/2m \\ -c/2 \end{smallmatrix}\right)$ ($a/2, c, mc/2 \in \mathbb{Z}$ and $\gcd(m, a/2, mc/2) = 1$). For simplicity, let $a' = a/2$. Now let us check the no fixed orbit property. Consider the condition that $(A, y)^i \notin H$ has a fixed orbit, ie suppose there is an $x \in \mathbb{R}^2$ and a $h \in H$, such that $\left(I, \left(\begin{smallmatrix} -ia'/m \\ -ic/2 \end{smallmatrix}\right)\right)(x) = h(x)$. Clearly h is not in Z^2. Therefore $h \in h_1 Z^2$, ie $h = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} e/2 \\ d \end{smallmatrix}\right)\right)$ where e is odd. Using this form for h, rearranging we get:

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} x = \left(I - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) x = \begin{pmatrix} e/2 + ia'/m \\ d + ic/2 \end{pmatrix}.$$

There is a solution for x if $e/2 = -ia'/m$ (e odd). We are interested in the cases when there are no solutions.

If m is odd, then there is no solution and thus each nontrivial power of (A, y) satisfies the no fixed orbit property. Since $mc/2$ is an integer and m is odd, $c/2$ is an integer. Also $1 = \gcd(m, a', mc/2) = \gcd(m, a')$. Thus we have the form labelled (i).

If m is even, then $\gcd(m/2, a') | \gcd(m, a', cm/2)$ and so $\gcd(m/2, a') = 1$. If a' is odd, take $i = -m/2$ then $-ia'/m = a'/2$ so we have a solution. So suppose a' is even. As $\gcd(m/2, a') = 1$, $m/2$ is odd (ie $m \equiv 2 \pmod 4$). Thus if $e/2 = -ia'/m$ then e will be even, so $(A, y)^i$ has no fixed orbit. Furthermore, c must be odd, as $\gcd(m, a', cm/2) = 1$. Lastly $\gcd(m/2, a') = 1$ implies $\gcd(m, a') = 2$. Thus we have the form labelled (iii).

Next consider when $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $y = \left(\begin{smallmatrix} -a/2m \\ -c/2 \end{smallmatrix}\right)$. Since $h \in H$ (in particular the generator h_1) is a covering homeomorphism, (A, y) is a possible representative iff

$$h_1\,(A, y) = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right) \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \begin{pmatrix} -a/2m \\ -c/2 \end{pmatrix}\right) = \left(I, \begin{pmatrix} (m-a)/2m \\ c/2 \end{pmatrix}\right)$$

is.

If $h_1\,(A, y)$ has the form labelled (i), then m is odd, $(m-a)/2m = -a'/m$ for some integer a' coprime to m and $c/2$ is an integer. Therefore $a = 2a' + m$, which is odd since m is. Now

$$\begin{aligned}
\gcd(2m, a) &= \gcd(m, a) \text{ since } a \text{ odd} \\
&= \gcd(m, 2a' + m) \\
&= \gcd(m, 2a') \\
&= \gcd(m, a') \text{ since } m \text{ odd} \\
&= 1.
\end{aligned}$$

So $(A, y) = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -a/2m \\ -b \end{smallmatrix}\right) \right)$ with m odd, and $\gcd(2m, a) = 1$ (which implies a is odd), which is the form labelled (ii).

If $h_1\,(A, y)$ instead has the form labelled (iii), then $m \equiv 2 \pmod 4$, $(m-a)/2m = -a'/m$ for some integer a' where $\gcd(m, a') = 2$ and c is an odd integer. Therefore $a = m + 2a'$ which is $2 \pmod 4$ since a' is even and $m \equiv 2 \pmod 4$. Now

$$\begin{aligned}
\gcd(m, a/2) &= \gcd(m/2, a/2) \text{ since } a/2 \text{ odd and } m \text{ even} \\
&= \gcd(m/2, a' + m/2) \\
&= \gcd(m/2, a') \\
&= \gcd(m/2, a'/2) \text{ since } m/2 \text{ odd} \\
&= \gcd(m, a')/2 \\
&= 1.
\end{aligned}$$

So $(A, y) = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -a"/m \\ -c/2 \end{smallmatrix}\right) \right)$ with $m \equiv 2 \pmod 4$, c odd, $a = 2a"$ and $\gcd(m, a") = 1$ (which implies $a"$ is odd), which is the form labelled (iv). □

Remark 2.1.12. In the lemma, m cannot be divisible by 4. This is because if $Kb/\mathbb{Z}_m$ is a manifold, then it must be Kb and the degree of any regular covering $Kb \to Kb$ is not divisible by 4. A consequence of this is that a Seifert manifold with Kb as the general fibre cannot have cone points in its base with order a multiple of 4.

2.2 The local picture at a cone point

We would like to know the fundamental group of the Seifert bundle restricted to a neighbourhood of various critical points. The following does so for a cone point.

Lemma 2.2.1. *Let D be a disk neighbourhood of a cone point of order m and suppose $\eta : C \to D$ is a Seifert bundle with general fibre F, then*

$$\pi_1(C) = \langle h_1, \ldots, h_q, s | W, s h_i s^{-1} = A(s) h_i, s^m \tilde{e} = 1 \rangle,$$

where the h_i correspond to the generators and W to the relations of $\pi_1(F)$, s corresponds to a section above the boundary of D, $A(s) \in \mathrm{Aut}(\pi_1(F))$ and $\tilde{e} \in \pi_1(F)$.

Proof. Since the base is good and aspherical, lemma 1.5.2 gives a short exact sequence:

$$1 \to \pi_1(F) \to \pi_1(C) \to \pi_1^{orb}(D) \to 1.$$

In particular the boundary of D lifts to a loop in C with image s in $\pi_1(C)$. Now $\pi_1^{orb}(D) = \langle \bar{s} | \bar{s}^m = 1 \rangle$, where $\bar{s}$ is the image of the boundary of D. Consequently

$$\pi_1(C) = \langle h_1, \dots, h_q, s | W, s h_i s^{-1} = A(s) h_i, s^m \tilde{e} = 1 \rangle,$$

for some $A(s) \in \mathrm{Aut}(\pi_1(F))$ and some $\tilde{e} \in \pi_1(F)$. □

Changing the section corresponds to replacing s by hs for some $h \in H = \langle h_i | W \rangle$. This will alter $A(s)$ by an inner automorphism and $\tilde{e}$ by an element of H. Note that in the above presentation, $\tilde{e}$ is an element of $\pi_1(F)$ which depends only on s, and not on the choice of generators for $\pi_1(F)$.

Our first major result is to prove that two Seifert manifolds with aspherical bases are diffeomorphic if and only if the corresponding fundamental group sequences are isomorphic. The following lemma proves this result for the special case of Seifert bundles above neighbourhoods of critical points.

Lemma 2.2.2. *Let D be a disk neighbourhood of a critical point p (ie, a cone point, a point on a reflector curve or a corner reflector). Let C and C' be affine Seifert bundles above D with general fibre F. Let $\phi : \pi_1(C) \to \pi_1(C')$ induce an isomorphism of the fundamental group sequences of C and C' which projects to the identity map of $\pi_1^{orb}(D)$.*

Then ϕ is realised by a fibre preserving homeomorphism $\theta : C \to C'$.

Proof. By lemma 1.1.2, we can take $D = D^2/\Gamma$, where Γ is a finite group of $O_2\mathbb{R}$ and p is the image of the origin. Since F is compact, the bundle is locally trivial and so for some (open) disk neighbourhood U of p in D, $\eta^{-1}(U) = (F \times \tilde{U})/\Gamma$, where $\tilde{U}$ covers U and the action of Γ on $F \times \tilde{U}$ preserves the product structure. We can extend the action of Γ to the bundle above the collar, $D - U$. So C can be taken to be $(F \times D^2)/\Gamma$, where Γ preserves the product structure. Hence we have a map $q_F : \pi_1(C) \to \mathrm{Homeo}(\tilde{F})$ and a surjection $q_D : \pi_1(C) \to \Gamma$ (where $\tilde{F}$ is the universal cover of F) such that $C \cong (\tilde{F} \times D^2)/(q_F, q_D)\pi_1(C)$. Note q_F is injective [suppose $q_F(g) = 1$, then g acts on $\tilde{F} \times D^2$ via $(q_F(g), q_D(g)) = (1, q_D(g))$ which has a fixed point. Since C is a manifold, this implies $q_D(g) = 1$ and so $g = 1$ since (q_F, q_D) is an isomorphism].

The group Γ acts freely on $F \times \{0\}$ (the fibre above the critical point), and so $F/\Gamma \cong \tilde{F}/q_F(\pi_1(C))$ is a flat manifold. Consequently there is an injective homomorphism $\alpha : \pi_1(C) \to \mathrm{Isom}(\mathbb{E}^n)$ and a corresponding homeomorphism $f : \tilde{F}/q_F(\pi_1(C)) \to \mathbb{R}^n/\alpha(\pi_1(C))$ which lifts to a homeomorphism $\tilde{f} : \tilde{F} \to \mathbb{R}^n$.

We can similarly define q'_F, α' and f' for C'.

Consider an isomorphism ϕ as in the statement of the theorem. Then $\alpha' \circ \phi \circ \alpha^{-1}$ is an isomorphism of subgroups of $\mathrm{Isom}(\mathbb{E}^n)$. The Bieberbach theorem then implies there is an affine homeomorphism $\beta : \mathbb{R}^n \to \mathbb{R}^n$ so that $\alpha'(\phi(g)) = \beta\alpha(g)\beta^{-1}$. Let β_F be the map induced by the following diagram:

$$\begin{array}{ccc} \tilde{F}/q_F(\pi_1(F)) & \xrightarrow{f} & \mathbb{R}^n/\alpha(\pi_1(F)) \\ \downarrow \beta_F & & \beta \downarrow \\ \tilde{F}/q'_F(\pi_1(F)) & \xrightarrow[f']{} & \mathbb{R}^n/\alpha'(\pi_1(F)) \end{array}$$

Then $\beta_F \times 1_{D^2}$ induces a homeomorphism $\theta : C \to C'$ (see lemma 2.1.2) which realises ϕ and is compatible with projection. □

2.3 The local picture at a reflector curve

Consider a reflector curve $\bar{\mathcal{C}}$ with k (possibly 0) corner reflectors, with orders m_j $(j = 1 \dots k)$. Then a neighbourhood, $\bar{\mathcal{A}}$, of $\bar{\mathcal{C}}$ will topologically be an annulus, with one component of the topological boundary the reflector curve. Let $\mathcal{A}$ be the canonical double cover obtained by doubling along the reflector curve and let R be the standard reflection of $\mathcal{A}$ so that $\mathcal{A}/\langle R\rangle = \bar{\mathcal{A}}$ as orbifolds. Then $\mathcal{A}$ (see the figure, for an example with $k = 3$) will be an annulus with k cone points on the middle loop, $\mathcal{C}$ with orders m_j and R will be a reflection about $\mathcal{C}$. Cut out small disks from $\mathcal{A}$, around each cone point (centred at the point) to obtain a manifold $\mathcal{A}_0$. Choose a base point on $\mathcal{C}$ between the first and kth cone points. Let $\bar{\partial}$ be a loop from the base point along the shortest path to one of the (two) boundary components of $\mathcal{A}$, around the boundary (a circle), then back along the shortest path, and let $\bar{\partial}'$ be a loop defined similarly but which goes to the other boundary component of $\mathcal{A}$ and has the same orientation (consequently $R(\bar{\partial}') = \bar{\partial}$). For $j = 1 \dots k$, let $\bar{s}_j$ be a loop from the base point which goes along a path in $\mathcal{A}_0$ to the boundary of the jth small disk, around this disk (make all $\bar{s}_j$ have the same orientation around these disks) then back along the path. Choose the $\bar{s}_j$ (in particular the paths to the boundaries of the disks), so that in $\pi_1(\mathcal{A}_0)$:

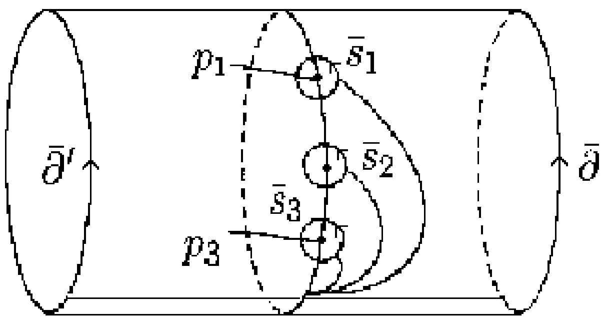


Figure 2.1: The double cover $\mathcal{A}$ of a annulus neighbourhood $\bar{\mathcal{A}}$ of a reflector curve

$$R(\bar{s}_j) = \left(\prod_{p=j+1}^{k} \bar{s}_p\right)^{-1} \bar{s}_j^{-1} \left(\prod_{p=j+1}^{k} \bar{s}_p\right), \tag{2.3.1}$$

for $j = 1, \dots, k$. See the figure.

Now $\pi_1(\mathcal{A}_0) = \langle \bar{\partial}, \bar{\partial}', \bar{s}_1, \ldots, \bar{s}_k | \bar{\partial}^{-1}\bar{s}_1\bar{s}_2\ldots\bar{s}_k\bar{\partial}' = 1\rangle$, is a free group. To obtain $\pi_1^{orb}(\mathcal{A})$ we glue on neighbourhoods of each cone point, which via van Kampen's theorem adds the relations $\bar{s}_j^{m_j} = 1$ (where m_j was the order of the cone point). Since $\mathcal{A}/\langle R\rangle = \bar{\mathcal{A}}$, $\pi_1^{orb}(\bar{\mathcal{A}})$ is the semi-direct product, $\pi_1^{orb}(\mathcal{A}) \times_R \mathbb{Z}_2$, of $\pi_1^{orb}(\mathcal{A})$ and $\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\pi_1^{orb}(\mathcal{A})$ by conjugation via the involution R. So $\pi_1^{orb}(\bar{\mathcal{A}})$ is:

$$\begin{array}{rl} \text{generators} & \bar{s}_1, \ldots, \bar{s}_k, \bar{\partial}, \bar{\partial}', \bar{r} \\ \text{relations} & \bar{s}_j^{m_j} = 1 \\ & \bar{r}^2 = 1 \\ & \bar{r}\bar{\partial}\bar{r}^{-1} = \bar{\partial}' \\ & \bar{r}\bar{s}_j\bar{r}^{-1} = \left(\prod_{p=j+1}^{k} \bar{s}_p\right)^{-1} \bar{s}_j^{-1} \left(\prod_{p=j+1}^{k} \bar{s}_p\right) \\ & \bar{\partial}^{-1}\left(\prod_{j=1}^{k} \bar{s}_j\right)\bar{\partial}' = 1. \end{array}$$

Note that $\bar{\partial}$ corresponds to the boundary of $\bar{\mathcal{A}}$. Since $\bar{\partial}' = \bar{r}\bar{\partial}\bar{r}^{-1}$ we can remove $\bar{\partial}'$ from the presentation by writing it in terms of the other generators.

We can now determine the fundamental group of a Seifert fibration above $\bar{\mathcal{A}}$.

Lemma 2.3.1. *If $\bar{\mathcal{C}}$ is a reflector curve with annulus neighbourhood, $\bar{\mathcal{A}}$, with k corner reflectors so that the jth corner reflector has order m_j and if $\bar{\eta} : \bar{\mathcal{R}} \to \bar{\mathcal{A}}$ is a Seifert bundle with general fibre F, then $\pi_1(\bar{\mathcal{R}})$ has the following presentation:*

$$\begin{array}{rl} \text{generators} & h_1, h_2, \ldots h_q, s_1, \ldots, s_k, \partial, r \\ \text{relations} & W, \\ & \xi h_p \xi^{-1} = A(\xi)(h_p) \text{ for } \xi = s_j, \partial, r \\ & s_j^{m_j}\tilde{e}_j = 1 \\ & r^2\tilde{f} = 1 \\ & r s_j r^{-1} = \left(\prod_{p=j+1}^{k} s_p\right)^{-1} s_j^{-1} \left(\prod_{p=j+1}^{k} s_p\right)\tilde{g}_j \\ & \partial^{-1}\left(\prod_{j=1}^{k} s_j\right) r\partial r^{-1} = \tilde{b}, \end{array}$$

where the h_i correspond to the generators and W to the relations of $\pi_1(F)$, ∂ corresponds to a section above the boundary of $\bar{\mathcal{A}}$, $A(\xi)$ in $\mathrm{Aut}(\pi_1(F))$ (for $\xi = s_j, \partial, r$) and $\tilde{f}$, $\tilde{e}_j$, $\tilde{g}_j$ and $\tilde{b}$ are in $\pi_1(F)$.

Proof. Since $\mathcal{A}$ is the union of cone disks along intervals it is good and aspherical by van Kampen's theorem. Hence so is $\bar{\mathcal{A}}$, and we have a short exact sequence: $1 \to \pi_1(F) \to \pi_1(\bar{\mathcal{R}}) \to \pi_1^{orb}(\bar{\mathcal{A}}) \to 1$. Therefore $\pi_1(\bar{\mathcal{R}})$ has a presentation as given in the statement of the lemma.

Note in this presentation we differentiate elements of $\pi_1^{orb}(\bar{\mathcal{A}})$ and their corresponding elements in $\pi_1(\bar{\mathcal{R}})$, by placing a bar when in $\pi_1^{orb}(\bar{\mathcal{A}})$. Note ∂ projects to $\bar{\partial}$ which corresponds to the boundary of $\bar{\mathcal{A}}$, so ∂ corresponds to a section above the boundary of $\bar{\mathcal{A}}$. $\square$

Conjugation in $\pi_1(\bar{\mathcal{R}})$ determines a homomorphism $A : \pi_1(\bar{\mathcal{R}}) \to \mathrm{Aut}(\pi_1(F))$ which induces the monodromy homomorphism from $\pi_1^{orb}(\bar{\mathcal{A}})$ to $\mathrm{Out}(\pi_1(F))$.

Later (in sections 3.3 and 3.4) it will be convenient to consider a different set of generators for this group. The following lemmas present the group with some new generators and lists some relationships between $\tilde{f}$ and $\tilde{e}_j$ etc.

Lemma 2.3.2. *Let $\sigma_j = \prod_{i=j}^{k} s_i$ and $\sigma_{k+1} = 1$. Then $s_j = \sigma_j \sigma_{j+1}^{-1}$. Then an alternate presentation for $\pi_1(\bar{\mathcal{R}})$ (as in the previous lemma) is:*

$$\begin{aligned} \text{generators} \quad & h_1, h_2, \dots h_q, \sigma_1, \dots, \sigma_k, \partial, r \\ \text{relations} \quad & W, \\ & \xi h_p \xi^{-1} = A(\xi) h_p \text{ for } \xi = \sigma_j, \partial, r \\ & \left(\sigma_j \sigma_{j+1}^{-1}\right)^{m_j} \tilde{e}_j = 1 \\ & r^2 \tilde{f} = 1 \\ & r \sigma_j r^{-1} = \sigma_j^{-1} \tilde{G}_j \\ & \partial^{-1} \sigma_1 r \partial r^{-1} = \tilde{b}, \end{aligned}$$

where $\tilde{G}_j = \prod_{i=j}^{k} A(\sigma_{i+1}) \tilde{g}_i$.

Proof. Firstly, consider what is happening topologically. In the double cover $\mathcal{A}$ of $\bar{\mathcal{A}}$ obtained by doubling over the reflector curve, the $\bar{s}_j$ generators correspond to loops around the jth cone point. The loops are not symmetric about the curve which projects to the reflector curve, thus the expression for $R(\bar{s}_j)$ is complicated. We can instead consider loops around the last $k - j + 1$ cone points (which makes sense for $j = 1, \dots, k+1$). These loops correspond to the $\bar{\sigma}_j$ introduced above. These loops are symmetric (up to homotopy that is) about $\mathcal{C}$, thus $R(\bar{\sigma}_j) = \bar{\sigma}_j^{-1}$. However since in general $\bar{\sigma}_j$ contains more than one cone point in its interior, relations arising from the cone points are now more complicated.

Secondly, consider what is happening algebraically. The first two sentences of the lemma show that $\{s_j | j = 1 \dots k\}$ and $\{\sigma_j | j = 1 \dots k\}$ are equivalent set of generators. Define $A(\sigma_j)$ to be $\prod_{i=j}^{k} A(s_i)$. Then all but the second last line of the presentation can be converted to the new set of generators in a straightforward manner. For the second last line:

$$\begin{aligned}
rs_jr^{-1} &= \sigma_j^{-1}\sigma_{j+1}\tilde{g}_j \\
r\sigma_jr^{-1} &= \prod_{i=j}^{k} rs_ir^{-1} \\
&= \prod_{i=j}^{k} \sigma_i^{-1}\sigma_{i+1}\tilde{g}_i \\
&= \left(\prod_{i=j}^{k}\left(\prod_{p=j}^{i} A(\sigma_p^{-1}\sigma_{p+1})\right)\tilde{g}_i\right)\sigma_j^{-1}\sigma_{k+1} \\
&= \left(\prod_{i=j}^{k} A(\sigma_j^{-1}\sigma_{i+1})\tilde{g}_i\right)\sigma_j^{-1} \\
&= \sigma_j^{-1}\left(\prod_{i=j}^{k} A(\sigma_{i+1})\tilde{g}_i\right)
\end{aligned}$$

Let $\tilde{G}_j = \prod_{i=j}^{k} A(\sigma_{i+1})\tilde{g}_i$, then we have the converted relation. □

Note by letting $\tilde{G}_{k+1} = 1$, we have $\tilde{g}_j = A(\sigma_{j+1})^{-1}(\tilde{G}_j)(\tilde{G}_{j+1})^{-1}$.

Lemma 2.3.3. *The following relationships exist between the various parts of the presentations of* $\pi_1(\bar{\mathcal{R}})$*:*

$$\tilde{e}_j = A(\sigma_j\sigma_{j+1}^{-1})\,(\tilde{e}_j) \tag{2.3.2}$$

$$\tilde{e}_j = A(s_j)\,(\tilde{e}_j) \tag{2.3.2'}$$

$$\left(\prod_{i=0}^{m_j-1} A(\sigma_j\sigma_{j+1}^{-1})^{-i}\left(\tilde{G}_j\left(\tilde{G}_{j+1}\right)^{-1}\right)\right)\tilde{e}_jA(\sigma_jr)\,(\tilde{e}_j) = 1 \tag{2.3.3}$$

$$\tilde{f} = A(r)\left(\tilde{f}\right) \tag{2.3.4}$$

$$\left(\tilde{f}\right)^{-1}A(\sigma_jr)\left(\tilde{f}\right) = \left(\tilde{G}_j\right)^{-1}A(\sigma_jr)\left(\tilde{G}_j\right) \tag{2.3.5}$$

$$\tilde{b}A(r)\left(\tilde{b}\right)\left(\tilde{f}\right)^{-1}A(\partial^{-1})\left(\tilde{f}\right) = A(\partial^{-1})\left(\tilde{G}_1\right) \tag{2.3.6}$$

Proof. To get the relation (2.3.2') (and hence (2.3.2) as well) conjugate $s_j^{m_j}\tilde{e}_j$ by s_j. To get the other relationships requires conjugating relations by r and then simplifying (for instance by using the other relations or inverting) to get new relations involving just the h_i. For instance, take the relation $\partial^{-1}\sigma_1r\partial r^{-1} = \tilde{b}$. Conjugate by r, simplify r^2 and $r\sigma_1r^{-1}$ via the relations to $\left(\tilde{f}\right)^{-1}$ and $\sigma_1^{-1}\tilde{G}_1$ respectively, invert then use the original relation to get (2.3.6). The relations are ordered the same way as in the relations (in lemma 2.3.2) they are based on, so that (2.3.3) corresponds to $\left(\sigma_j\sigma_{j+1}^{-1}\right)^{m_j}\tilde{e}_j = 1$ etc. □

When the general fibre is a torus, its fundamental group is abelian and the above relations may be expressed as follows.

Corollary 2.3.4. *Suppose $F = T^n$, then the relations* (2.3.2)-(2.3.6) *become*

$$\tilde{e}_j = A(\sigma_j \sigma_{j+1}^{-1})\tilde{e}_j \tag{2.3.7}$$

$$\tilde{e}_j = A(s_j)\tilde{e}_j \tag{2.3.7'}$$

$$\left(\sum_{i=0}^{m_j-1} A(\sigma_j\sigma_{j+1}^{-1})^{-i}\left(\tilde{G}_j - \tilde{G}_{j+1}\right)\right) + (A(\sigma_j r) + I)\tilde{e}_j = 0 \tag{2.3.8}$$

$$\tilde{f} = A(r)\tilde{f} \tag{2.3.9}$$

$$(A(\sigma_j r) - I)\tilde{f} = (A(\sigma_j r) - I)\tilde{G}_j \tag{2.3.10}$$

$$(A(r) + I)\tilde{b} + (A(\partial^{-1}) - I)\tilde{f} = A(\partial^{-1})\tilde{G}_1 \tag{2.3.11}$$

Proof. Since $\pi_1(T^n) \cong \mathbb{Z}^n$ is abelian, A induces a group homomorphism $A : \pi_1^{orb}(\bar{\mathcal{A}}) \to \mathrm{Out}(\pi_1(T^n)) = \mathrm{Aut}(\pi_1(T^n)) = GL_n\mathbb{Z}$. The rest follows from the lemma. ☐

There is a third way to present the group. The following lemma gives a presentation that stresses the splitting of the reflector curve into neighbourhoods of each corner reflector, and is useful in lemma 2.3.7 which follows.

Lemma 2.3.5. *Let $r_j = \sigma_j r$, so that $r_j = \left(\prod_{i=j}^k s_i\right) r$ for $j = 1, \ldots, k$ and $r_{k+1} = r$. Then $\sigma_j = r_j r_{k+1}^{-1}$ and $s_j = r_j r_{j+1}^{-1}$. Then an alternate presentation for $\pi_1(\bar{\mathcal{R}})$ (as in lemma 2.3.1) is:*

$$\begin{array}{rl} \text{generators} & h_1, h_2, \ldots h_q, r_1, \ldots, r_{k+1}, \partial \\ \text{relations} & W, \\ & \xi h_p \xi^{-1} = A(\xi)h_p \text{ for } \xi = r_j, \partial \\ & (r_j r_{j+1})^{m_j}\, \tilde{e'}_j = 1 \\ & r_j^2 \tilde{f}_j = 1 \\ & \partial^{-1} r_1 \partial r_{k+1}^{-1} = \tilde{b}, \end{array}$$

where $\tilde{e'}_j = \left(\prod_{p=0}^{m_j-1} A(s_j^{-p})\left(\tilde{f}\left(\tilde{G}_{j+1}\right)^{-1}\right)\right)\tilde{e}_j$ *and* $\tilde{f}_j = \tilde{f}\left(\tilde{G}_j\right)^{-1}$.

Let D_j be a neighbourhood in $\bar{\mathcal{A}}$ of the jth corner reflector and let C_j be the restriction of the bundle above D_j. Then $\pi_1(C_j)$ is isomorphic to the subgroup of $\pi_1(\bar{\mathcal{R}})$ generated by r_j, r_{j+1} and $\pi_1(F)$.

Proof. Algebraically, this lemma is straightforward.

Topologically, D_j is the quotient of a disk by a dihedral group. This group is generated by two reflections $\bar{r}_j$ and $\bar{r}_{j+1}$, in lines corresponding to the two parts of the reflector curve (separated by the corner reflector) in D_j. So $\pi_1(C_j)$ is isomorphic to the group generated by r_j, r_{j+1} and $\pi_1(F)$. Adjacent corner reflectors have a shared reflection. The exception to this is the first and last corner reflectors, because there is a monodromy from going around the curve. ☐

For a Seifert manifold, corner reflectors cannot occur in the base for all general fibres, as the following lemma proves. However, if a Seifert orbifold has singularities then corner reflectors can arise in the base.

Lemma 2.3.6. *A Seifert manifold with S^1 or Kb as the general fibre cannot have corner reflectors in its base.*

Proof. A singular fibre above a corner reflector is covered by the general fibre with dihedral covering group. However the only manifolds S^1 and Kb cover are themselves, and in those cases the covering group is cyclic. $\square$

Our first major result is to prove that the fundamental group sequences of Seifert manifolds with aspherical bases determine the manifold topologically. The following lemma proves this result for the special case of Seifert 3- or 4-bundles above neighbourhoods of reflector curves.

Lemma 2.3.7. *Let $\bar{\mathcal{A}}$ be an annulus neighbourhood of a reflector curve with $k \geq 0$ corner reflectors. Let $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ be Seifert bundles above $\bar{\mathcal{A}}$ with a 1 or 2 dimensional general fibre, F. Let $\phi : \pi_1(\bar{\mathcal{R}}) \to \pi_1(\bar{\mathcal{R}}')$ induce an isomorphism of the fundamental group sequences of $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ which projects to the identity map of $\pi_1^{orb}(\bar{\mathcal{A}})$.*

Then ϕ is realised by a fibre preserving homeomorphism $\theta : \bar{\mathcal{R}} \to \bar{\mathcal{R}}'$.

Proof. Let $\bar{\eta}$ and $\bar{\eta}'$ be the bundle maps for $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ respectively. Cut $\bar{\mathcal{A}}$ once along a line through the base point to open up the annulus, to obtain $\bar{\mathcal{A}}_c$. Let $\bar{\mathcal{R}}_c = \bar{\eta}^{-1}(\bar{\mathcal{A}}_c)$ and similarly define $\bar{\mathcal{R}}'_c$. Note $\pi_1^{orb}(\bar{\mathcal{A}}_c)$ is the subgroup of $\pi_1^{orb}(\bar{\mathcal{A}})$ generated by $r_1, \dots r_{k+1}$ (as in lemma 2.3.5 above), and hence $\pi_1(\bar{\mathcal{R}}_c)$ is the subgroup of $\pi_1(\bar{\mathcal{R}})$ generated by $h_1, h_2, \dots h_q, r_1, \dots, r_{k+1}$. Since ϕ is an isomorphism which projects to the identity on $\pi_1^{orb}(\bar{\mathcal{A}})$, $\phi(\pi_1(\bar{\mathcal{R}}_c)) = \phi(\pi_1(\bar{\eta}^{-1}(\bar{\mathcal{A}}_c))) = \pi_1(\bar{\eta}'^{-1}(\bar{\mathcal{A}}_c)) = \pi_1(\bar{\mathcal{R}}'_c)$. That is, ϕ induces an isomorphism $\phi_c : \pi_1(\bar{\mathcal{R}}_c) \to \pi_1(\bar{\mathcal{R}}'_c)$.

If $k = 0$ then $\bar{\mathcal{A}}_c$ is a disk neighbourhood of a portion of a reflector curve, and so ϕ_c is realised by a homeomorphism θ_c, by lemma 2.2.2.

Suppose next that $k > 0$. Then cut $\bar{\mathcal{A}}$ along lines between the corner reflectors, to get disk neighbourhoods of each of the corner reflectors. Let D_j be the neighbourhood of the jth corner reflector (numbering the points by their order in $\bar{\mathcal{A}}_c$). Let $C_j = \eta^{-1}(D_j)$ and $C'_j = \eta'^{-1}(D_j)$. By lemma 2.3.5, $\pi_1^{orb}(D_j)$ is a subgroup of $\pi_1(\bar{\mathcal{A}})$. As above, ϕ induces an isomorphism $\phi_j : \pi_1(C_j) \to \pi_1(C'_j)$. This isomorphism is realised by a fibre preserving homeomorphism θ_j by lemma 2.2.2. For $j = 1, \dots, k-1$, $\theta_j|_{C_j \cap C_{j+1}}$ and $\theta_{j+1}|_{C_j \cap C_{j+1}}$ will be isotopic since they arise from the same fundamental group data. So by altering θ_{j+1} on a neighbourhood of $C_j \cap C_{j+1}$ it can be supposed that θ_j and θ_{j+1} agree on the intersection of their domains. Thus there is a homeomorphism $\theta_c : \bar{\mathcal{R}}_c \to \bar{\mathcal{R}}'_c$ which realises ϕ_c.

To obtain $\bar{\mathcal{R}}$ from $\bar{\mathcal{R}}_c$ involves gluing the two ends together, via some fibre preserving gluing map d. WLOG, if there are corner reflectors let d map the side closest to the last corner reflector to the side closest to the first corner reflector. Then from lemma 2.3.5, $\pi_1(d) : \langle r_{k+1}, \pi_1(F)\rangle \to \langle r_1, \pi_1(F)\rangle$ is given by conjugation by ∂. So $\pi_1(\theta_c \circ d) = \phi_c \circ \pi_1(d) = c_{\phi(\partial)} \circ \phi|_{\pi_1(\mathrm{Domain}(d))}$ (c_α is conjugation by α). Similarly there is a gluing map d' which is used to construct $\bar{\mathcal{R}}'$ from $\bar{\mathcal{R}}'_c$, so that $\pi_1(d') = c_{\partial'}$. Therefore $\pi_1(d' \circ \theta_c|_{\mathrm{Domain}(d)}) = c_{\partial'} \circ \phi|_{\pi_1(\mathrm{Domain}(d))}$.

Since ϕ projects to the identity of $\pi_1^{orb}(\bar{\mathcal{A}})$, $\phi(\partial) = h_\partial \partial'$ for some h_∂ in $\pi_1(F)$. If $\pi_1(F)$ is abelian (that is when F is a circle or a torus) then $c_{\phi(\partial)} = c_{\partial'}$, so $\pi_1(\theta_c \circ d) = \pi_1(d' \circ \theta_c|_{\text{Domain}(d)})$. Thus $\theta_c \circ d$ and $d' \circ \theta_c$ restricted to the domain of d are isotopic since they arise from the same fundamental group data.

By altering θ_c on a neighbourhood of one end of $\bar{\mathcal{R}}$, it can be supposed that on the natural domains, $d' \circ \theta_c = \theta_c \circ d$. Therefore θ_c determines a homeomorphism $\theta : \bar{\mathcal{R}} \to \bar{\mathcal{R}}'$.

If $\pi_1(F)$ is not abelian then $F = Kb$ and so there are no corner reflectors and there is another way to construct θ. Let $\bar{\mathcal{C}}$ be the reflector curve in $\bar{\mathcal{A}}$. Then $\bar{\eta}^{-1}(\bar{\mathcal{C}})$ and $\bar{\eta}'^{-1}(\bar{\mathcal{C}})$ are mapping tori and deformation retracts of $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ respectively. So ϕ is an isomorphism of the fundamental groups of these two mapping tori. These mapping tori fibres are the fibres above the reflector curve and are doubly covered by F. The hypotheses on ϕ give the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle \pi_1(F), r\rangle & \longrightarrow & \pi_1(\bar{\eta}^{-1}(\bar{\mathcal{C}})) & \longrightarrow & \langle \bar{\partial}\rangle = \mathbb{Z} & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow \phi & & \| & & \| \\
1 & \longrightarrow & \langle \pi_1(F), r'\rangle & \longrightarrow & \pi_1(\bar{\eta'}^{-1}(\bar{\mathcal{C}})) & \longrightarrow & \langle \bar{\partial}\rangle = \mathbb{Z} & \longrightarrow & 1
\end{array}$$

The two mapping tori are homeomorphic (via a fibre preserving homeomorphism, $\theta_{\bar{\mathcal{C}}}$) since their homotopy exact sequences are isomorphic. By passing to double covers, $\theta_{\bar{\mathcal{C}}}$ induces a homeomorphism, $\theta_{\mathcal{C}} : \eta^{-1}(\mathcal{C}) \to \eta'^{-1}(\mathcal{C})$, which we can extend in a natural way to a homeomorphism $\mathcal{R} \to \mathcal{R}'$. We then define θ to be the projection of this homeomorphism to $\bar{\mathcal{R}}$. □

Remark 2.3.8. The above lemma can be easily generalised to fibres of higher dimension, either if $\pi_1(F)$ is abelian (ie $F = T^n$) or if the base has no corner reflectors. Is the lemma still true if F is not a torus and the base has corner reflectors?

Remark 2.3.9. If there are corner reflectors then it can be shown $\pi_1(F)$ is a characteristic subgroup of $\pi_1(\bar{\mathcal{R}})$. So all isomorphisms $\phi : \pi_1(\bar{\mathcal{R}}) \to \pi_1(\bar{\mathcal{R}}')$ induce an isomorphism of the fundamental group sequences.

2.4 Seifert manifolds over any closed 2-orbifold

Having considered cone points and seen what happens in a neighbourhood of reflector curves, we can now determine the fundamental group for a Seifert manifold over any closed 2-orbifold.

Let $\eta : S \to B$ be a Seifert bundle with B a closed 2-orbifold and general fibre F. For later reference, suppose B has k_0 cone points so that the order of the jth cone point is m_{0j}, and l reflector curves such that the ith reflector curve has k_i corner reflectors so that the order of the jth corner reflector on the ith reflector curve is m_{ij}. (Note l and k_i $(i = 0, \ldots, l)$ can be zero.) Let N be the disjoint union of neighbourhoods of critical points in B (ie disks around cone points and annuli around reflector curves). Let B_0 be the 2-manifold obtained from B by removing N. The boundary of B_0 is a (possibly empty) disjoint union of circles. Let B_a be the closed 2-manifold obtained from B_0 by gluing on a disk to each component of the boundary. We call this the closed 2-manifold associated to B. (Note if B had no reflector curves, then B_a is the underlying topological space of B.) We delete a further disk from B_0 to obtain a manifold B_e with nonempty boundary. (Thus the restriction of the fibration over B_e has a cross section.)

Let $S_0 = \eta^{-1}B_0$ and $S_e = \eta^{-1}B_e$. Then the restrictions of η to S_0 and S_e are bundle projections η_0 and η_e respectively, and η restricted to $S_0 - S_e$ is trivial (since $B_0 - B_e$ is a disk) and so $S_0 = S_e \cup (D^2 \times F)$.

To find the fundamental group of S, we shall first find the fundamental group of the Seifert manifold S_0, whose base has no singularities. Fix a base point in S_e (which by projection and inclusion fixes a base point in all other spaces mentioned).

Let G be the graph contained in B_e consisting of paths from the basepoint to loops around each of the boundary components of B_0 (each of these boundary components correspond to a singularity of B), or let G be the basepoint if B_0 has no boundary (this occurs when B has no singularities). Choose G so that the circles meet at the basepoint, that is each circle of G is a path from the basepoint to a component of the boundary, the loop of the boundary and the inverse path back. For each cone point (indexed by $j = 1, \ldots, k_0$), let $\bar{s}_{0j}$ be the loop in G around the boundary of its neighbourhood. For each reflector curve (indexed by $i = 1, \ldots, l$), let $\bar{\partial}_i$ be the loop in G around the boundary of its neighbourhood. Additionally, let $\bar{s}$ be the loop in B_e from the base point around the boundary of the disk cut out to obtain B_e. Note $\bar{s}$ is not in G.

The fundamental group of B_a is $\langle \bar{t}_1, \bar{u}_1, \ldots, \bar{t}_g, \bar{u}_g | \prod_{p=1}^{g} [\bar{t}_p, \bar{u}_p] = 1 \rangle$ when B_a is orientable of genus g and $\langle \bar{v}_1, \ldots, \bar{v}_g | \prod_{p=1}^{g} \bar{v}_p^2 = 1 \rangle$ when B_a is non-orientable with g cross caps. To avoid repetition, we will write $\pi_1(B_a) = \langle \bar{w}_1, \ldots, \bar{w}_f | \bar{R} = 1 \rangle$. Note that generally there is no direct connection between $\pi_1(B_a)$ and $\pi_1^{orb}(B)$. Then $\pi_1(B_e)$ is the free group:

$$\langle \bar{s}, \bar{s}_{01}, \ldots, \bar{s}_{0k_0}, \bar{\partial}_1, \ldots, \bar{\partial}_l, \bar{w}_1, \ldots, \bar{w}_f | \bar{R} \prod_{j=1}^{k_0} \bar{s}_{0j} \prod_{i=1}^{l} \bar{\partial}_i = \bar{s}^{-1} \rangle.$$

Furthermore we obtain $\pi_1(B_0)$ by adjoining the relation $\bar{s} = 1$.

For each of $\bar{s}_{01}, \ldots, \bar{s}_{0k_0}, \bar{\partial}_1, \ldots, \bar{\partial}_l, \bar{w}_1, \ldots, \bar{w}_f$, that is $\bar{a}$ in general, let a (ie without the bar) be a loop in S_e so that $\eta(a) = \bar{a}$, ie a is a section above $\bar{a}$. Let R be the composition of loops formed from $\bar{R}$ by replacing the $\bar{w}_i$ by w_i. Lastly let s be the loop in S_e defined by $\left(R \prod_{j=1}^{k_0} s_{0j} \prod_{i=1}^{l} \partial_i \right)^{-1}$. Then $\eta(s)$ is homotopic to $\bar{s}$, since their images in $\pi_1(B_e)$ agree.

Since $\pi_1(B_e)$ is free, $\pi_1(S_e)$ is a semi-direct product $\pi_1(F) \rtimes \pi_1(B_e)$ and so has a presentation:

$$\begin{aligned} \text{generators} \quad & h_1, \ldots, h_q, s, s_{01}, \ldots, s_{0k_0}, \partial_1, \ldots, \partial_l, w_1, \ldots, w_f \\ \text{relations} \quad & W \quad \text{(ie the relations of } \pi_1(F)), \\ & \xi h_p \xi^{-1} = A(\xi) h_p \text{ for } \xi = s_{0j}, \partial_j, w_j \\ & R \prod_{j=1}^{k} s_{0j} \prod_{i=1}^{l} \partial_i = s^{-1}; \end{aligned}$$

for some automorphisms $A(\xi)$ of $\pi_1(F)$.

From above, $S_0 = S_e \cup (D^2 \times F)$. So by Van Kampen's theorem we can determine $\pi_1(S_0)$. The loop $\partial D^2 \times \{0\}$

of $D^2 \times F$ is freely homotopic in S_e to $s\tilde{a}$ for some $\tilde{a} \in \pi_1(F)$. This element corresponds to an obstruction to extending a cross section over B_e to one over B_0. We then obtain $\pi_1(S_0)$ by adjoining the relation $s\tilde{a} = 1$ to $\pi_1(S_e)$ (ie replacing s^{-1} by $\tilde{a}$).

Now from before (lemma 2.2.1), if D_{0j} is the disk neighbourhood of the jth cone point in B and if $\eta_{0j} : C_{0j} \to D_{0j}$ is the restriction of the Seifert fibration to the neighbourhood, then

$$\pi_1(C_{0j}) = \langle h_1, \dots, h_q, s_{0j} | W, s_{0j} h_i s_{0j}^{-1} = A(s_{0j}) h_i, s_{0j}^{m_{0j}} \tilde{e}_{0j} = 1 \rangle,$$

for some $\tilde{e}_{0j}$, and where s_{0j} is a lift of the boundary of D_{0j}.

Also from before (lemma 2.3.1), if $\bar{\mathcal{A}}_i$ is the annulus neighbourhood of the ith reflector curve and if $\bar{\eta}_i : \bar{\mathcal{R}}_i \to \bar{\mathcal{A}}_i$ is the restriction of the Seifert fibration to the neighbourhood, then $\pi_1(\bar{\mathcal{R}}_i)$ has the following presentation:

$$\begin{aligned}
\text{generators} \quad & h_1, \dots h_q, s_{i1}, \dots, s_{ik_i}, \partial_i, r_i \\
\text{relations} \quad & W, \\
& \xi h_p \xi^{-1} = A(\xi) h_p \text{ for } \xi = s_{ij}, \partial_i, r_i \\
& s_{ij}^{m_{ij}} \tilde{e}_{ij} = 1 \\
& r_i^2 \tilde{f}_i = 1 \\
& r_i s_{ij} r_i^{-1} = \left(\prod_{p=j+1}^{k_i} s_{ip} \right)^{-1} s_{ij}^{-1} \left(\prod_{p=j+1}^{k_i} s_{ip} \right) \tilde{g}_{ij} \\
& \partial_i^{-1} \left(\prod_{j=1}^{k_i} s_{ij} \right) r_i \partial_i r_i^{-1} = \tilde{b}_i,
\end{aligned}$$

for some $\tilde{e}_{ij}, \tilde{f}_i, \tilde{g}_{ij}$ and $\tilde{b}_i$, and where ∂_i is a lift of the boundary of A_i.

We can then use Van Kampen's theorem to combine $\pi_1(S_0)$ with the fundamental groups of the fibration above each of the neighbourhoods. We can choose the sections above the boundaries of each neighbourhood in C_{0j}, $\bar{R}_i$ and in S_0 (which we have labelled s_{0j} and ∂_i) to be homotopic, so we can equate them.

Hence the following theorem:

Theorem 2.4.1. *Suppose S is a Seifert manifold over a closed base B, such that B has k_0 cone points, l reflector curves and k_i corner reflector on the ith reflector curve. Let m_{0j} be the order of the jth cone point, and let m_{ij} be the order of the jth corner reflector on the ith reflector curve. Let B_a be the manifold* *associated to B. Then $\pi_1(S)$ has the following presentation*

$$\begin{array}{rl}
\text{generators} & h_1, \ldots, h_q, \\
& s_{ij}\ (i = 0, \ldots, l;\ j = 1, \ldots, k_i), \\
& \partial_1, \ldots, \partial_l, \\
& r_1, \ldots, r_l, \\
& t_1, u_1, \ldots, t_g, u_g \ \textit{(if } B_a \textit{ is orientable, genus } g\textit{)} \\
& v_1, \ldots, v_g \ \textit{(if } B_a \textit{ is non-orientable, } g \textit{ cross-caps)} \\
\text{relations} & W, \\
& \xi h_p \xi^{-1} = A(\xi) h_p \text{ for } \xi = s_{ij}, \partial_j, r_j, t_j, u_j \\
& s_{ij}^{m_{ij}} \tilde{e}_{ij} = 1 \\
& r_i^2 \tilde{f}_i = 1 \\
& r_i s_{ij} r_i^{-1} = \left(\prod_{p=j+1}^{k_i} s_{ip} \right)^{-1} s_{ij}^{-1} \left(\prod_{p=j+1}^{k_i} s_{ip} \right) \tilde{g}_{ij} \quad (i \neq 0) \\
& \partial_i^{-1} \left(\prod_{j=1}^{k_i} s_{ij} \right) r_i \partial_i r_i^{-1} = \tilde{b}_i \quad (i \neq 0) \\
& \prod_{p=1}^{g} [t_p, u_p] \prod_{j=1}^{k} s_{0j} \prod_{i=1}^{l} \partial_i = \tilde{a} \ \textit{(if } B_a \textit{ is orientable)} \\
& \prod_{p=1}^{g} v_p^2 \prod_{j=1}^{k} s_{0j} \prod_{i=1}^{l} \partial_i = \tilde{a} \ \textit{(if } B_a \textit{ is non-orientable)};
\end{array}$$

2.5 Topological classification of Seifert spaces with aspherical bases

Having constructed $\pi_1(S)$ we now show its place in defining a Seifert manifold topologically. We prove if the base is aspherical then the fundamental group sequence of the manifold determines a Seifert manifold up to fibre preserving homeomorphism. Furthermore when the base is hyperbolic we prove the fundamental group determines a Seifert manifold up to fibre preserving homeomorphism.

By lemma 1.5.2, we have an exact sequence:

$$\pi_1(F) \to \pi_1(S) \to \pi_1^{orb}(B) \to 1.$$

Let H be the image of $\pi_1(F)$ in $\pi_1(S)$ via the above map. Thus $\pi_1^{orb}(B) \cong \pi_1(S)/H$. Note in the presentation in section 2.4, H is generated by $h_1, \ldots, h_q$.

Definition 2.5.1. We say S is *sufficiently complicated* if

(i). H is the unique maximal normal virtually solvable subgroup of $\pi_1(S)$ and

(ii). the inclusion of the fibre F into S induces a monomorphism, $\pi_1(F) \cong H$.

In particular, H is a characteristic subgroup of $\pi_1(S)$.

Remark 2.5.2. This notation is due to Zieschang [Zi2]. The second condition is automatic if the base is aspherical (ie hyperbolic or Euclidean).

A Seifert manifold is sufficiently complicated precisely when the base is hyperbolic, which the next two lemmas prove.

Lemma 2.5.3. *If B is a compact hyperbolic 2-orbifold, then $\pi_1^{orb}(B)$ has no nontrivial normal virtually solvable subgroups.*

Proof. First recall that a subgroup $\mathcal{G}$ of Isom($\mathbb{H}^2$) is elementary if there is a finite $\mathcal{G}$-orbit in $\tilde{H}^2$ (the closure of $\mathbb{H}^2$ in the extended complex plane). Recall also that there are three types of elementary groups. Elliptic elementary groups contain only elliptic elements (isometries with one fixed point in $\mathbb{H}^2$) and are always finite cyclic. Parabolic and hyperbolic elementary groups similarly contain parabolic and hyperbolic elements respectively (hyperbolic groups can contain elliptic elements) and the union of finite orbits contains 1 or 2 (respectively) points in the boundary of $\mathbb{H}^2$. See [Ra] for more details.

Claim: If $\mathcal{G}$ is a subgroup of Isom($\mathbb{H}^2$) and $\mathcal{G}'$ is a normal elementary subgroup of $\mathcal{G}$ then $\mathcal{G}$ is elementary or $\mathcal{G}'$ is finite.

By p 131 of [Iv] (cf Theorem 5.5.10 in [Ra]) a subgroup of Isom($\mathbb{H}^2$) is solvable if and only if it is elementary. So a virtually solvable subgroup of Isom($\mathbb{H}^2$) contains a normal elementary subgroup of finite index and so by the claim is either elementary or finite. If $\pi_1^{orb}(B)$ has a normal virtually solvable subgroup then either the subgroup is finite or $\pi_1^{orb}(B)$ is elementary. However $\pi_1^{orb}(B)$ is not elementary (see below), so the subgroup is finite. Lastly [Gr] shows a finite normal subgroup of $\pi_1^{orb}(B)$ is trivial.

Proof of claim (cf Theorem 5.5.11 in [Ra]): If $\mathcal{G}'$ is not finite, then it is elementary of hyperbolic or parabolic type. Then the union of all the finite orbits of $\mathcal{G}'$ is a one or two point set F. Let ϕ be in $\mathcal{G}$. Then

$$\phi^{-1}\mathcal{G}'\phi(F) = \mathcal{G}'F = F.$$

Hence $\mathcal{G}'\phi(F) = \phi(F)$. Therefore $\phi(F) = F$. As ϕ is an arbitrary element of $\mathcal{G}$, we deduce that $\mathcal{G}F = F$. Therefore $\mathcal{G}$ is elementary.

For completeness, we will explain why $\pi_1^{orb}(B)$ is not elementary. Aiming for a contradiction, suppose $\pi_1^{orb}(B)$ is elementary. Since B is hyperbolic it is finitely covered by an orientable manifold B' such that $\chi(B') < 0$. Therefore $\pi_1^{orb}(B)$ has a non-commutative subgroup $\pi_1(B')$. However subgroups of elementary subgroups are elementary (Exercise 2.11 in [Kat]). The group $\pi_1(B')$ is not cyclic, therefore by Corollary 2.4.3 in [Kat], $\pi_1(B')$ contains an element of order 2. This is a contradiction since $\pi_1(B')$ is torsion free. ☐

Lemma 2.5.4. *A Seifert manifold, S, is sufficiently complicated if and only if its base is hyperbolic.*

Proof. If the base B is not hyperbolic then $\pi_1^{orb}(B)$ is virtually solvable and so $\pi_1(S)$ is virtually solvable, hence H is not maximal.

Suppose the base is hyperbolic. Clearly H is virtually solvable and is normal in $\pi_1(S)$ by lemma 1.5.2. If G is a normal virtually solvable subgroup of $\pi_1(S)$, then GH/H is a normal virtually solvable subgroup of $\pi_1(S)/H \cong \pi_1^{orb}(B)$. However since the base is hyperbolic, $\pi_1^{orb}(B)$ has no nontrivial such subgroups by the previous lemma, and so GH/H is trivial. This means $G \leq H$. Hence H is the unique maximal normal virtually solvable subgroup. $\square$

Let $*$ be a basepoint for the orbifold, B (choose $*$ to be a regular point). Then a *basing* for a bundle $\eta : S \to B$ with general fibre F, is a homeomorphism:

$$j : F \xrightarrow{\cong} \eta^{-1}(*).$$

If $\eta : S \to B$ and $\eta' : S' \to B$ are two based bundles over an aspherical base B with general fibre F, then a *based bundle isomorphism* is a bundle homeomorphism $\theta : S \to S'$ such that $\theta \circ j = j'$.

Such a based isomorphism induces an equivalence, $\pi_1(\theta)$ of the fundamental group sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1^{orb}(B) & \longrightarrow & 1 \\ \| & & \| & & \downarrow{\scriptstyle \pi_1(\theta)} & & \| & & \| \\ 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(S') & \longrightarrow & \pi_1^{orb}(B) & \longrightarrow & 1. \end{array}$$

The main result in this section (theorem 2.5.10 below) that we are trying to prove is that the fundamental group sequence of a Seifert manifold with aspherical bases determines the manifold up to fibre preserving homeomorphism. This basically amounts to proving that the map $\theta \mapsto \pi_1(\theta)$ induces a surjection from the set of homotopy classes of based bundle isomorphisms over B (aspherical) with flat fibres to the set of equivalences of fundamental group sequences ($\pi_0(\mathrm{Isom}_B(\eta, \eta') \to \mathrm{Equiv}(\pi_1(\eta), \pi_1(\eta')))$. In order to help prove the main theorem, we will prove a slightly stronger result than this for the special case where B is a nonsingular 1-complex.

Lemma 2.5.5. *Let $\eta : E \to A$ and $\eta' : E' \to A$ be based F-bundles where A is a nonsingular 1-complex and F is S^1, T^2 or Kb. Then the map $\theta \mapsto \pi_1(\theta)$ induces a bijection:*

$$\pi_0(\mathrm{Isom}_A(\eta, \eta')) \to \mathrm{Equiv}(\pi_1(\eta), \pi_1(\eta')),$$

where $\pi_0(\mathrm{Isom}_A(\eta, \eta'))$ is the set of (based) homotopy classes of based bundle isomorphisms $E \to E'$, and $\mathrm{Equiv}(\pi_1(\eta), \pi_1(\eta'))$ is the set of equivalences of fundamental group sequences of $\pi_1(E)$ and $\pi_1(E')$.

Proof. We will first assume we have the special case $A = S^1$. For this special case, we will firstly show that we can reduce to the situation where $\eta = \eta'$ and hence both $\pi_0(\mathrm{Isom}_A(\eta, \eta'))$ and $\mathrm{Equiv}(\pi_1(\eta), \pi_1(\eta'))$ are groups. Secondly and thirdly we will prove the resultant group homomorphism is both surjective and injective. Lastly, we will use this special case to prove the lemma when A is a nonsingular 1-complex.

If the fundamental group sequences of η and η' are not equivalent then the above map is the trivial bijection of empty sets. Therefore suppose $\phi : \pi_1(E) \xrightarrow{\cong} \pi_1(E')$ induces an equivalence of fundamental group sequences. This means $\phi \circ \pi_1(j) = \pi_1(j')$ and $\pi_1(\eta) = \pi_1(\eta') \circ \phi$. Since $A = S^1$, E is a mapping torus $M(\alpha) = \{[f,s] | f \in F, s \in I, [f,1] = [\alpha(f), 0]\}$, for some based self-homeomorphism α of F. Similarly, E' is a mapping torus $M(\alpha')$. The basing for E, will be the map $j : F \to E$, where $j(f) = [f,0]$ (similarly $j'(f) = [f,0]$ will define the basing of E'). Let f_0 be the basepoint of F. Then let t (t' respectively) be the canonical section of E (E'), that is $t(s) = [f_0, s]$. Let $[t]$ be the image of t in $\pi_1(E)$. Then by standard covering space theory $[t]h[t]^{-1} = \pi_1(\alpha)(h)$ for all $h \in \pi_1(F)$. Since $\phi([t]) = [t']g$ for some $g \in \pi_1(F)$, $\pi_1(\alpha)$ and $\pi_1(\alpha')$ induce the same outer automorphism of $\pi_1(F)$. Since F is flat, there is a bijection $\pi_0(\text{Homeo}(F)) \to \text{Out}(\pi_1(F))$ (by the Bieberbach theorem). Therefore there is an isotopy ψ_s from 1 to $(\alpha')^{-1}\alpha$ in Homeo(F). Define $\Psi : E \to E'$ by $\Psi([f,s]) = [\psi_s(f), s]$. Then $\Psi \circ j = j'$ and so is a based bundle isomorphism.

The previous paragraph showed that if the fundamental group sequences of η and η' are equivalent then η and η' are bundle isomorphic. So we can reduce to the case where $\eta = \eta'$ after choosing such a bundle isomorphism. Hence we are now trying to prove $\pi_0(\text{Aut}_{S^1}(\eta)) \longrightarrow \text{Equiv}(\pi_1(\eta))$ is a group isomorphism. (Note Equiv$(\pi_1(\eta))$ is the group of self-equivalences of $\pi_1(\eta)$.)

For this end, we shall prove this group homomorphism is surjective. Fix a self-equivalence ϕ of the fundamental group sequence of $\pi_1(E)$. Then $\phi(h) = h$ for all $h \in \pi_1(F)$, and $\phi([t]) = [t]g$ for some $g \in \pi_1(F)$. Since $[t]$ normalises $\pi_1(F)$, $\phi([t]h[t]^{-1}) = [t]h[t]^{-1}$ and so $g \in \mathcal{Z}(\pi_1(F))$. Now the evaluation map Homeo$(F) \to F$ (that is evaluation at the basepoint of F) induces a group isomorphism $\pi_1(\text{Homeo}(F), 1_F) \to \mathcal{Z}(\pi_1(F))$ (this is clear for $F = S^1$; see [Ham] for the $F = T^2$ and $F = Kb$ cases). Let $s \mapsto \hat{g}_s$ be a loop at 1_F in Homeo(F) whose homotopy class in $\pi_1(\text{Homeo}(F), 1_F) \cong \mathcal{Z}(\pi_1(F))$ corresponds to g. Then define $\theta([f,s]) = [\hat{g}_s(f), s]$. Then θ is a bundle isomorphism which realises ϕ.

Next we shall prove the group homomorphism is injective. Suppose θ is a based bundle automorphism of E which induces the trivial automorphism on the fundamental group sequence (ie $\pi_1(\theta) = 1$). Then we will show θ is isotopic (where the isotopy is of based bundle automorphisms) to the identity bundle isomorphism.

Since θ is a bundle automorphism, $\theta([f,s]) = [\hat{\theta}_s(f), s]$ for some function $\hat{\theta} : S^1 \to \text{Homeo}(F)$. Since θ is a well defined function of $E = M(\alpha)$, $\hat{\theta}_0 \circ \alpha = \alpha \circ \hat{\theta}_1$. Since θ is based, $\hat{\theta}_0 = 1_F$ and consequently $\hat{\theta}_1 = 1_F$ as well. Therefore $\hat{\theta}$ is a loop in Homeo(F) at 1_F. Note θ is isotopic to the identity if and only if $\hat{\theta}$ is homotopic to the trivial loop, that is if and only if $[\hat{\theta}] = 1$ in $\pi_1(\text{Homeo}(F), 1_F)$.

Recall the section t is the map $s \mapsto [f_0, s]$ where f_0 is the basepoint of F. Therefore $\theta(t)$ sends s to $[\hat{\theta}_s(f_0), s]$. Since $\pi_1(\theta) = 1$, $\pi_1(\theta)([t]) = [t]$ and so $\theta(t)$ is isotopic to t. Consequently $eval_{f_0}(\hat{\theta})$ is isotopic to the trivial loop. Passing to fundamental groups, we see $\pi_1(eval_{f_0})([\hat{\theta}]) = 1$. However $eval_{f_0}$ is injective, so $[\hat{\theta}] = 1$ and so θ is isotopic to the identity. Hence the lemma for the case $A = S^1$.

Lastly, we shall prove the lemma when A is a nonsingular 1-complex. Up to homotopy equivalence, we can suppose A is a wedge of circles with the common point being the basepoint. Therefore E and E' are unions of mapping tori with a common fibre above the basepoint of A (ie for E, the basing $j : F \to \eta^{-1}(*)$ is common to all mapping tori).

A based bundle isomorphism $\theta : E \to E'$ is uniquely determined by the based bundle isomorphisms between each corresponding mapping tori. That is, for each circle L in A, we have a based bundle isomorphism $\theta_L : \eta^{-1}(L) \to (\eta')^{-1}(L)$. [Note these θ_L will all agree on the common fibre, since the isomorphisms are based.] Then this collection of θ_L uniquely determine θ.

Since A is a wedge of circles, $\pi_1(A)$ is a free group. Furthermore $\pi_1(A)$ has a set of generators $\{x\}$ which are in bijection to the set of circles in A. Suppose $\phi : \pi_1(E) \to \pi_1(E')$ induces an equivalence of the fundamental group sequences of η and η'. For each generator x of $\pi_1(A)$ there is corresponding loop L of A, and there is an unique equivalence of fundamental group sequences: $\phi_L : \pi_1(\eta^{-1}(L)) \to \pi_1((\eta')^{-1}(L))$. [The group $\pi_1(\eta^{-1}(L))$ is the preimage of the subgroup $\langle x \rangle \cong \mathbb{Z}$ by $\pi_1(\eta)$. Similarly for $\pi_1((\eta')^{-1}(L))$. Since ϕ induces the identity on $\pi_1(A)$, ϕ will send $\pi_1(\eta^{-1}(L))$ isomorphically onto $\pi_1((\eta')^{-1}(L))$.] Conversely, given a collection of these ϕ_L we can define an isomorphism ϕ. [The group $\pi_1(E)$ is a semi-direct product of $\pi_1(F)$ and $\pi_1(A)$. Therefore an isomorphism $\phi : \pi_1(E) \to \pi_1(E')$ will be defined by its image of $\pi_1(F)$ and by the image of each generator of $\pi_1(A)$ (that is an element corresponding to each generator). If ϕ is to induce an equivalence, then ϕ will act as the identity on the subgroup $\pi_1(F)$ (as all the ϕ_L do). Therefore in order to define ϕ, we need only know where each generator gets sent to, but this is determined by the ϕ_L isomorphisms.]

From above, for each loop L of A, there is a bijection $\theta_L \mapsto \pi_1(\theta_L) = \phi_L$. The map $\theta \mapsto \pi_1(\theta) = \phi$ is then a bijection, since θ and ϕ are built from the θ_L and ϕ_L respectively. ☐

Remark 2.5.6. The previous lemma is true for a larger class of fibres. In general it is true whenever the natural homomorphism $\pi_0(\text{Homeo}(F)) \to \text{Out}(\pi_1(F))$ is a bijection and evaluation at the base point induces a group isomorphism $\pi_1(\text{Homeo}(F), 1_F) \to \mathcal{Z}(\pi_1(F))$. In a series of papers, Hamstrom studied the homotopy type of the space of homeomorphisms of all possible 2-manifolds. In these papers she proves (among other things) these conditions hold for aspherical surfaces (see the last paper in the series, [Ham] for a summary of her results). [Note Scott in [Sc1] proves the corresponding results for the space $PL(F)$. Earle with Eells and Schatz in [EaEe] and [EaSc] prove the corresponding results for the space of diffeomorphisms.] These two conditions also hold for sufficiently large P^2-irreducible (Haken) 3-manifolds (this includes all flat 3-manifolds). In [Ha1], Hatcher finds the homotopy type of the space of homeomorphisms for these manifolds, hence proving these conditions. [Note in this paper Hatcher proves the result directly for $PL(F)$. Hatcher also extends this in [Ha2] to the space of diffeomorphisms.]

Thus the fibre can be S^1, any aspherical surface, or any Haken 3-manifold.

Lemma 2.5.7. *Let $\eta : S \to B$ and $\eta' : S' \to B$ be based F-bundles where B is a nonsingular 1-complex and F is flat (ie η and η' are classical fibre bundles). Let $\phi_B : \pi_1(S) \to \pi_1(S')$ induce an isomorphism of fundamental group sequences, so that the induced maps on the base and fibre are the identity. Suppose G is a subspace of B. Then any fibre preserving homeomorphism $\theta_G : \eta^{-1}(G) \to (\eta')^{-1}(G)$ which is compatible with ϕ_B (ie $\phi_B \circ \pi_1(i) = \pi_1(i') \circ \pi_1(\theta_G)$ where i and i' are the natural inclusions), extends to a fibre preserving homeomorphism $\theta_B : S \to S'$ which realises ϕ_B.*

Proof. From the previous lemma, $\theta \mapsto \pi_1(\theta)$ induces a bijection $\pi_0(\mathrm{Isom}_B(\eta, \eta')) \xrightarrow{\cong} \mathrm{Equiv}(\pi_1(\eta), \pi_1(\eta'))$. We will get the following commutative square,

$$\begin{array}{ccc} \pi_0(\mathrm{Isom}_B(\eta,\eta')) & \xrightarrow{\cong} & \mathrm{Equiv}(\pi_1(\eta),\pi_1(\eta')) \\ \downarrow & & \downarrow \\ \pi_0(\mathrm{Isom}_G(\eta|_G,\eta'|_G)) & \xrightarrow{\cong} & \mathrm{Equiv}(\pi_1(\eta|_G),\pi_1(\eta'|_G)), \end{array}$$

where the first vertical arrow is induced by restriction. The second vertical arrow is not so clear. One can define it using pullbacks and then the commutativity follows since π_1 is a functor which preserves pullbacks (see claim in lemma 2.5.8 later in this section). We shall define it here in a more ad hoc fashion. Let E and E' be the total spaces for $\eta|_G$ and $\eta'|_G$ and similarly let S and S' be the total spaces for η and η'. Let i and i' be the inclusions $i : E \to S$ and $i' : E' \to S'$. Let $\tau : \pi_1(G) \to \pi_1(E)$ and $\tau' : \pi_1(G) \to \pi_1(E')$ be sections (sections exist, since $\pi_1(G)$ is free). Choose a $\phi \in \mathrm{Equiv}(\pi_1(\eta), \pi_1(\eta'))$. For $x \in \pi_1(G)$, let $h_x = (\pi_1(i')(\tau'(x)))^{-1}\phi(\pi_1(i)(\tau(x))) \in \pi_1(S')$. By diagram chasing, we see $\pi_1(\eta')(h_x) = 1$, ie $h_x \in \pi_1(F)$. Define $\tilde{\phi} \in \mathrm{Equiv}(\pi_1(\eta|_G), \pi_1(\eta'|_G))$, by $\tilde{\phi}(h) = h$ for all $h \in \pi_1(F)$, and $\tilde{\phi}(\tau(x)) = \tau'(x)h_x$. The second vertical arrow in the above diagram is the map $\phi \mapsto \tilde{\phi}$. This map is (despite appearances) independent of the choice of sections. Furthermore, $\tilde{\phi}$ is the unique map such that $\pi_1(i') \circ \tilde{\phi} = \phi \circ \pi_1(i)$ (the proof arises from diagram chasing). The commutativity of the above square then follows.

From the data in the statement of the lemma, we have elements from three corners in the square: ϕ_B, $[\theta_G]$ and $\pi_1(\theta_G)$. We therefore have a fibre preserving homeomorphism, $\theta_B : S \to S'$ such that $[\theta_B]$ completes the square:

$$\begin{array}{ccc} [\theta_B] & \longleftrightarrow & \phi_B \\ \downarrow & & \downarrow \\ [\theta_G] & \longleftrightarrow & \pi_1(\theta_G) \end{array}$$

Therefore θ_B realises ϕ_B and the restriction of θ_B to a bundle above G is isotopic to θ_G. By modifying θ_B above a collar of G, we can suppose θ_B extends θ_G. □

The following lemma and corollary show that an isomorphism of the fundamental group sequence of a Seifert manifold induces an isomorphism of the fundamental group sequence of any submanifold which is a Seifert manifold with compatible fibration.

Lemma 2.5.8. *Let S and S' be Seifert manifolds with aspherical base B and general fibre F, and let the respective bundle maps be η and η'. Let $\phi : \pi_1(S) \to \pi_1(S')$ be an isomorphism which preserves the fundamental group sequences and projects to the identity on $\pi_1^{orb}(B)$.*

Let $f : N \to B$ be an orbifold map (where N is aspherical). Then ϕ induces an unique isomorphism $f^*(\phi) : \pi_1(f^*(S)) \to \pi_1(f^*(S'))$, such that $f^*(\phi)$ projects to the identity on $\pi_1^{orb}(N)$ and $\phi|_{\pi_1(F)} = f^*(\phi)|_{\pi_1(F)}$.

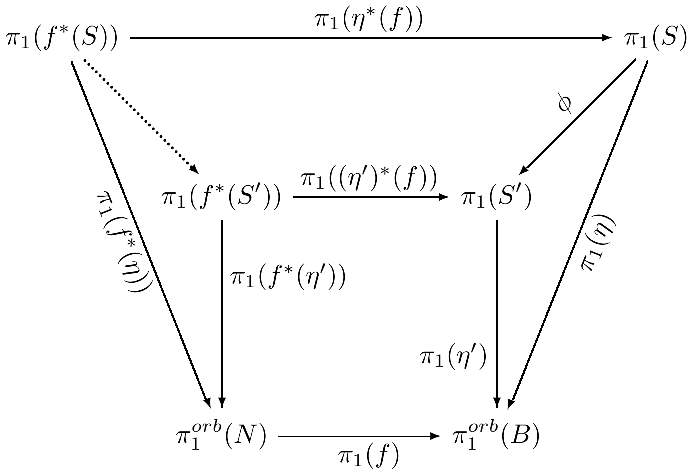
Proof. Claim: π_1 preserves pullbacks, ie $\pi_1(f^*(S)) = \pi_1(f)^*(\pi_1(S))$, the pullback of $\pi_1(\eta) : \pi_1(S) \to \pi_1^{orb}(B)$ by $\pi_1(f)$.

Let $E = \pi_1(f)^*(\pi_1(S))$. Since B is aspherical, $\pi_1(\eta)$ is a surjective homomorphism with kernel $\pi_1(F)$, and so its pullback $\pi_1(f)^*(\pi_1(\eta)) : E \to \pi_1^{orb}(N)$, is also surjective with kernel $\pi_1(F)$ by properties of pullbacks. Since $f^*(S)$ is the pullback of S, it will have the same general fibre, so $\pi_1(f^*(\eta))$ is also surjective with kernel $\pi_1(F)$. Since E is a pullback, $\pi_1(f^*(S))$ and the pair of maps $\pi_1(f^*(\eta))$ and $\pi_1(\eta^*(f))$ induce a unique map $e : \pi_1(f^*(S)) \to E$. So we get the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(f^*(S)) & \longrightarrow & \pi_1^{orb}(N) & \longrightarrow & 1 \\
\| & & \| & & \downarrow e & & \| & & \| \\
1 & \longrightarrow & \pi_1(F) & \longrightarrow & E & \longrightarrow & \pi_1^{orb}(N) & \longrightarrow & 1
\end{array}$$

By the five lemma, e is an isomorphism, hence the claim.

We can combine the pullback squares of both $\pi_1(f^*(S))$ and $\pi_1(f^*(S'))$ and the map ϕ to get the following commutative diagram.



Since $\pi_1(f^*(S'))$ is a pullback, we have a unique arrow marked by a dotted line in the diagram. We call this map $f^*(\phi)$. By reversing the role of S and S' we also get a unique map $f^*(\phi^{-1}) : \pi_1(f^*(S')) \to \pi_1(f^*(S))$. By uniqueness of maps into pullbacks, these maps are inverses and hence $f^*(\phi)$ is an isomorphism. Note $f^*(\phi)$ projects to the identity on $\pi_1^{orb}(N)$.

As remarked above, $f^*(S)$ and S will have the same general fibre, so the restriction of $\eta^*(f)$ (the pullback of f by η) to $\pi_1(F)$ will be the identity. From the above diagram, $\phi \circ \pi_1(\eta^*(f)) = \pi_1((\eta')^*(f)) \circ f^*(\phi)$. Therefore, the homomorphism $f^*(\phi)|_{\pi_1(F)}$ is $\left(\pi_1((\eta')^*(f))|_{\pi_1(F)}\right)^{-1} \circ \left(\phi|_{\pi_1(F)}\right) \circ \left(\pi_1(\eta^*(f))|_{\pi_1(F)}\right) = \phi|_{\pi_1(F)}$. ☐

Corollary 2.5.9. *Let S and S' be Seifert manifolds with aspherical base B and general fibre F. Let the respective bundle maps be η and η'. Consider an isomorphism $\phi : \pi_1(S) \to \pi_1(S')$ so that ϕ projects to the identity on $\pi_1^{orb}(B)$.*

Then for each suborbifold N of B, ϕ induces an unique isomorphism $\phi_N : \pi_1(\eta^{-1}(N)) \to \pi_1(\eta'^{-1}(N))$, so that ϕ_N projects to the identity of $\pi_1^{orb}(N)$ and $\phi_N|_{\pi_1(F)} = \phi|_{\pi_1(F)}$. ☐

We now have enough preliminaries to prove our first major result.

Theorem 2.5.10. *Suppose S and S' are Seifert spaces with 1 or 2 dimensional general fibres F and F' respectively and with aspherical bases. Then every isomorphism $\phi : \pi_1(S) \to \pi_1(S')$ which induces an isomorphism of the fundamental group sequences is realised by a fibre preserving homeomorphism $\theta : S \to S'$.*

Proof. For notation purposes, mark everything connected to S' with a prime ($'$).

Since the bases are aspherical the natural maps $\pi_1(F) \to \pi_1(S)$ and $\pi_1(F') \to \pi_1(S')$ are injective and $\pi_1(F)$ is isomorphic to $\pi_1(F')$. Hence we can suppose $F \cong F'$. Next, $\pi_1^{orb}(B) \cong \pi_1^{orb}(B')$. This means that the bases are isomorphic as orbifolds. (Since the orbifold fundamental groups of the bases are isomorphic the bases have the same geometry. The claim then follows from the Bieberbach theorems when the base is flat [as in section 1.4] and from [Zi+] Corollary 6.6.10 when the base is hyperbolic). By composing the bundle map, η, with this orbifold isomorphism, we can suppose $B = B'$ and ϕ induces the identity on $\pi_1^{orb}(B)$.

Firstly suppose, B has at least one singular point.

In order to construct the desired homeomorphism we first build it over neighbourhoods of cone points and reflector curves. We then show these homeomorphisms extend over the rest of the manifold. For each of these neighbourhoods, ϕ induces an isomorphism of fundamental groups (corollary 2.5.9). Then each of these isomorphisms can be realised by a fibre preserving homeomorphism (lemma 2.2.2 for cone points and lemma 2.3.7 for reflector curves).

Then we have an isomorphism $\phi_0 : \pi_1(S_0) \to \pi_1(S_0')$ and for each boundary component of S_0, there is a fibre preserving homeomorphism to the corresponding boundary component of S_0', which are restrictions of the homeomorphisms found in the second step. Let G be the graph defined as in section 2.4. Using isotopies we can define a fibre preserving homeomorphism $\theta_G : \eta^{-1}(G) \to \eta'^{-1}(G)$ which agrees with all the previous homeomorphisms on the boundaries. The homeomorphism θ_G is compatible with ϕ (that is, if $i : \eta^{-1}(G) \to S$ and $i' : (\eta')^{-1}(G) \to S'$ are the inclusions, then $\phi \circ \pi_1(i) = \pi_1(i') \circ \pi_1(\theta_G)$). Then θ_G extends to a fibre preserving homeomorphism $S_0 \to S_0'$ (lemma 2.5.7). By construction, this homeomorphism agrees with the homeomorphisms of the neighbourhoods of the singular points, so combining them we have a fibre preserving homeomorphism $\theta : S \to S'$ which realises ϕ.

If B is a surface (ie has no singular points), then as before cut out a disk to obtain B_e. We can choose this disk, so that the basepoint lies on its boundary. We can realise $\phi_{B_0 - B_e}$ by a homeomorphism $\eta^{-1}(B_0 - B_e) \cong D^2 \times F \to D^2 \times F \cong (\eta')^{-1}(B_0 - B_e)$ (which is essentially determined by a homeomorphism of the fibre above the basepoint). Let G be the frontier of B_e in B, then as before we have a fibre preserving homeomorphism $\theta_G : \eta^{-1}(G) \to \eta'^{-1}(G)$, which will be compatible with ϕ. Then θ_G extends to a fibre preserving homeomorphism $\theta_e : S_e \to S_e'$ (lemma 2.5.7). By combining realisation of $\phi_{B_0 - B_e}$ with θ_e, we have a fibre preserving homeomorphism $\theta : S \to S'$ which realises ϕ. □

Remark 2.5.11. The above theorem can be easily generalised to fibres of dimension 3 either if $F = T^3$ or if the base has no corner reflectors. Can this be generalised to a broader class of fibres? See remarks 2.3.8 and 2.5.6 for further comments.

Corollary 2.5.12. *Seifert spaces over aspherical bases are determined (up to fibre preserving homeomorphism) by their fundamental group sequences.* □

Corollary 2.5.13 (Zieschang). *Suppose S and S' are Seifert spaces with hyperbolic bases. Then every isomorphism $\phi : \pi_1(S) \to \pi_1(S')$ is realised by a fibre preserving homeomorphism $\theta : S \to S'$.*

Proof. Since S and S' are sufficiently complicated, by lemma 2.5.4, the subgroups H and H' are the unique maximal normal virtually solvable subgroups of $\pi_1(S)$ and $\pi_1(S')$. Since ϕ is an isomorphism, $\phi(H)$ is also a maximal normal virtually solvable subgroup of $\pi_1(S')$, and so by uniqueness is H'. Therefore ϕ maps $\pi_1(F)$ isomorphically onto $\pi_1(F')$. Then by the theorem, we have the result. □

This corollary extends Zieschang's work in [Zi2], especially his theorem 3.

Corollary 2.5.14. *Seifert spaces over hyperbolic bases are determined (up to fibre preserving homeomorphism) by their fundamental groups.* □

As we will see later in section 4.4, some but not all Seifert spaces over Euclidean bases are uniquely determined (up to fibre preserving homeomorphism) by the fundamental groups. On the other hand, there are some Seifert spaces whose total space does not have a unique Seifert fibration and so their fundamental group sequence is needed to determine the fibration.

Seifert manifolds with spherical or bad bases are generally not even determined by their fundamental group sequence or homotopy exact sequence. For example S^3 has infinitely many Seifert fibrations with the same homotopy exact sequence. Explicitly; as in p 450 of [Sc2], consider S^3 as the set $\{z_1, z_2 \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$, and $S^2 = \mathbb{C} \cup \{\infty\}$. Then the map $h_{p,q} : S^3 \to S^2$ given by $h_{p,q}(z_1, z_2) = z_1^p / z_2^q$ (p and q coprime) is a Seifert fibration $S^3 \to S^2(p, q)$ (the cone points are at 0 and ∞). The homotopy exact sequence of $h_{p,q}$ is the same for all p and q coprime. For a second example (see p 459 [Sc2] for details), $P^2 \times S^1$ has Seifert fibrations over both D^2 and P^2 (among many others) with isomorphic homotopy exact sequences.