Chapter 4

Geometric Seifert manifolds with Euclidean bases

In this chapter we will prove that all Seifert 4-manifolds with Euclidean bases are geometric. These Seifert manifolds are geometric of one of four types. The first type $\mathbb{E}^4$ was introduced in section 1.4 along with the Bieberbach theorems. After introducing the other three geometries ($\mathbb{N}il^3 \times \mathbb{E}$ in section 4.1, $\mathbb{S}ol^3 \times \mathbb{E}$ in section 4.2 and $\mathbb{N}il^4$ in section 4.3), we will prove the result in section 4.5.

4.1 The geometry $\mathbb{N}il^3 \times \mathbb{E}$

In this section we look at the geometry $\mathbb{N}il^3 \times \mathbb{E}$. We will prove that closed manifolds with this geometry are Seifert fibred.

Let Nil^3 be the nilpotent group (under multiplication) of 3×3 real upper triangular matrices of the form $\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)$ which we will write as $(\binom{x}{y}, z)$. Multiplication is given by $(\binom{x}{y}, z)(\binom{x'}{y'}, z') = (\binom{x+x'}{y+y'}, z + z' + xy')$. The centre of Nil^3 is the set of those matrices where $x = y = 0$, and so is isomorphic to $\mathbb{R}$. We therefore get the following exact sequence:

$$1 \longrightarrow \mathbb{R} \longrightarrow Nil^3 \xrightarrow{p} \mathbb{R}^2 \longrightarrow 1,$$

where $\mathbb{R}$ is the centre, and p is the projection $(\binom{x}{y}, z) \to \binom{x}{y}$. The sequence does not split.

The group Nil^3 is the model space for the geometry $\mathbb{N}il^3$. Similarly the model space for $\mathbb{N}il^3 \times \mathbb{E}$ is the group $Nil^3 \times \mathbb{R}$ which has centre $\mathbb{R}^2$ and sits in the following exact sequence:

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow Nil^3 \times \mathbb{R} \xrightarrow{p} \mathbb{R}^2 \longrightarrow 1.$$

Any isometry of $\mathbb{N}il^3$ or $\mathbb{N}il^3 \times \mathbb{E}$ preserves the coset structure modulo the centre and so will preserve the exact sequences. The isometry group of $\mathbb{N}il^3 \times \mathbb{E}$ is $\mathrm{Isom}(\mathbb{N}il^3) \times \mathrm{Isom}(\mathbb{E})$. There is an exact sequence

$1 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(\mathbb{N}il^3) \longrightarrow \mathrm{Isom}(\mathbb{E}^2) \longrightarrow 1$, and so we get the following exact sequence:

$$1 \longrightarrow \mathbb{R} \times \mathrm{Isom}(\mathbb{E}) \longrightarrow \mathrm{Isom}(\mathbb{N}il^3 \times \mathbb{E}) \xrightarrow{q} \mathrm{Isom}(\mathbb{E}^2) \longrightarrow 1.$$

These geometries fibre over $\mathbb{E}^2$ and can be considered as twisted products of the geometries of the base ($\mathbb{E}^2$) and fibre ($\mathbb{E}$ for $\mathbb{N}il^3$ and $\mathbb{E}^2$ for $\mathbb{N}il^3 \times \mathbb{E}$). Thus they are analogous to $\widetilde{SL}_2$ and $\widetilde{SL}_2 \times \mathbb{E}$ which can be considered as twisted products of the geometries of the base ($\mathbb{H}^2$) and fibre ($\mathbb{E}$ and $\mathbb{E}^2$ respectively). See [Sc2] section 4 for a detailed description of $\mathbb{N}il^3$ and see [Wa2] section 1 for a description of $\mathbb{N}il^3 \times \mathbb{E}$.

We can then prove the following proposition:

Proposition 4.1.1. *If S is a compact quotient of $\mathbb{N}il^3 \times \mathbb{E}$ by a discrete group of isometries then S is homeomorphic to a Seifert 4 manifold over a Euclidean base.*

Proof. Firstly, $S = (\mathbb{N}il^3 \times \mathbb{E})/\Gamma$ for some discrete group Γ. Let $\hat{\Gamma} = \Gamma \cap \ker(q)$ and $\bar{\Gamma} = q(\Gamma)$, where q is the projection $\mathrm{Isom}(\mathbb{N}il^3) \to \mathrm{Isom}(\mathbb{E}^2)$. The groups $\hat{\Gamma}$ and $\bar{\Gamma}$ are discrete groups of $\mathrm{Isom}(\mathbb{E}^2)$ (theorem 6.3 in [Wa1]). Then we get the Seifert fibration:

$$F = \mathbb{E}^2/\hat{\Gamma} \to S \to \mathbb{E}^2/\bar{\Gamma} = B.$$

□

Now we shall consider the isometries of $\mathbb{N}il^3 \times \mathbb{E}$ in more detail. The general element of $Nil \times \mathbb{R}$ has the form $(\binom{x}{y}, \binom{z}{w})$, where x, y, z, w are in $\mathbb{R}$. The subgroup Nil^3 consists of elements with $w = 0$. Isometries of the $\mathbb{E}$ factor act directly on w fixing everything else. Isometries of $\mathbb{N}il^3$ all preserve orientation. There are two components. The identity component consists of the action of Nil^3 on itself by left multiplication and also an action of SO_2 on Nil^3. The element R_θ in SO_2 (rotation through θ) acts via the map $(\binom{x}{y}, \binom{z}{w}) \mapsto (R_\theta\binom{x}{y}, \binom{z+\frac{1}{2}s(cy^2-cx^2-2sxy)}{w})$, where $c = \cos\theta$ and $s = \sin\theta$. The non-identity component of $\mathrm{Isom}(\mathbb{N}il^3)$ is generated by the map $\rho : (\binom{x}{y}, \binom{z}{w}) \mapsto (\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\binom{x}{y}, \binom{-z}{w}) = (\binom{x}{-y}, \binom{-z}{w})$.

A useful way to see how these isometries act is to see how they act on the centre and what they project to under the map $q : \mathrm{Isom}(\mathbb{N}il^3 \times \mathbb{E}) \to \mathrm{Isom}(\mathbb{E}^2)$. We will define a map $s : \mathrm{Isom}(\mathbb{E}^2) \to \mathrm{Isom}(\mathbb{N}il^3 \times \mathbb{E})$, so that $qs = 1$. Note the exact sequence above does not split, so s will not be a group homomorphism. Let (M, v) be an element of $\mathrm{Isom}^+(\mathbb{E}^2)$ (ie $M \in SO_2$). Then let $s(\left(M, \binom{v_1}{v_2}\right))$ be given by the composition of the action of M and the multiplication on the left by $\left(\begin{smallmatrix} 1 & v_1 & 0 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Therefore $s(\left(M, \binom{v_1}{v_2}\right))(\binom{x}{y}, \binom{z}{w}) = (M\binom{x}{y} + \binom{v_1}{v_2}, \binom{z+c}{w})$, where c is a correction term determined by $\left(M, \binom{v_1}{v_2}\right)$ and is a function of x and y (if $M = R_\theta$, $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$, then $c(x,y) = \frac{1}{2}s_\theta(c_\theta y^2 - c_\theta x^2 - 2s_\theta xy) + v_1(-s_\theta x + c_\theta y)$). Suppose $\left(M, \binom{v_1}{v_2}\right)$ has finite order, m. Since $qs = 1$, $s(\left(M, \binom{v_1}{v_2}\right))^m(\binom{x}{y}, \binom{z}{w}) = (\binom{x}{y}, \binom{z+k}{w})$ for some constant k which depends only on $\left(M, \binom{v_1}{v_2}\right)$. After multiplying $s(\left(M, \binom{v_1}{v_2}\right))$ by the central element, $\left(\begin{smallmatrix} 1 & 0 & -k/m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, we can suppose $s(\left(M, \binom{v_1}{v_2}\right))^m$ is the identity.

Suppose (M, v) is an element of $\mathrm{Isom}(\mathbb{E}^2)$ so that $\det M = -1$. Then $M\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is in SO_2. Let $s((M, v)) = s(\left(M\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), v\right)) \circ \rho$. Therefore for all $(M, v) \in \mathrm{Isom}(E^2)$, $s((M,v))(\binom{x}{y}, \binom{z}{w}) = (M\binom{x}{y} + v, \binom{(\det M)z+c}{w})$

where c is a quadratic correction term (if $M = \begin{pmatrix} c_\theta & -\delta s_\theta \\ s_\theta & \delta c_\theta \end{pmatrix}$, then $\delta = \det M$ and $c = c(x,y) = \frac{1}{2}s_\theta(c_\theta y^2 - c_\theta x^2 - \delta 2 s_\theta xy) + v_1(-s_\theta x + \delta c_\theta y)$). If $M = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $v = 0$ (and $\det M = -1$) then $s((M,v))^2$ is the identity.

Suppose $\bar{\alpha}_1, \dots \bar{\alpha}_l$ are elements of $\mathrm{Isom}(\mathbb{E}^2)$ so that their composition is the identity. Since $qs = 1$, $s(\bar{\alpha}_1)s(\bar{\alpha}_2)\dots s(\bar{\alpha}_l)(\binom{x}{y}, \binom{z}{w}) = (\binom{x}{y}, \binom{z+c}{w})$ for some constant c which depends only on the $\bar{\alpha}_i$.

If α is in $\mathrm{Isom}(\mathbb{N}il^3 \times \mathbb{E})$ then it is a composition of $s(\bar{\alpha})$ (where $q(\alpha) = \bar{\alpha}$) and an element of $\ker q$. Therefore $\alpha(\binom{x}{y}, \binom{z}{w}) = (\bar{\alpha}\binom{x}{y}, \binom{\delta(\alpha)z+z_0+c}{\epsilon(\alpha)w+w_0})$, where $\epsilon(\alpha) = \mathrm{w}(\alpha)$, $\delta(\alpha) = \mathrm{w}(\bar{\alpha})$, z_0, w_0 are real numbers and c is the correction term which is part of $s(\bar{\alpha})$. Let $T \cong \mathbb{R}^2$ be the identity component of $\ker q$, then $T = \{(\binom{x}{y}, \binom{z}{w}) \mapsto (\binom{x}{y}, \binom{z+z_0}{w+w_0})\}$. An element α acts on T via conjugation by the matrix $\mathcal{O}(\alpha) := \begin{pmatrix} \delta(\alpha & 0 \\ 0 & \epsilon(\alpha) \end{pmatrix}$.

In summary, if α is an isometry of $\mathrm{Isom}(\mathbb{N}il^3 \times \mathbb{E})$, then $\alpha(\binom{x}{y}, \binom{z}{w}) = (\bar{\alpha}\binom{x}{y}, \left(\mathcal{O}(\alpha), \binom{z_0}{w_0}\right)\binom{z}{w} + \binom{c}{0})$ where c is the quadratic correction term. Note $\mathcal{O}(\xi')\binom{c(\xi)}{0} = \binom{\mathrm{w}(\bar{\xi}')c(\xi)}{0}$.

A $\mathbb{N}il^3 \times \mathbb{E}$-manifold may have other Seifert fibrations, distinct from the one that descends by foliating $Nil^3 \times \mathbb{R}$ by cosets of the centre.

Since the commutator subgroup of $Nil^3 \times \mathbb{R}$ is $\mathbb{R} = \{\binom{0}{0}, \binom{z}{0}\}$ and its abelianisation is $\mathbb{R}^3$, there is another exact sequence

$$1 \to \mathbb{R} \to Nil^3 \times \mathbb{R} \to \mathbb{R}^3 \to 1.$$

(Moreover, the commutator subgroup is central.)

Any projection $\mathbb{R}^3 \to \mathbb{R}^2$ determines an epimorphism from $Nil^3 \times \mathbb{R}$ to $\mathbb{R}^2$ with kernel $\mathbb{R}^2$ and hence a foliation by cosets. All such foliations are preserved by left translation, although they may not be preserved by isometries which induce nontrivial rotations on the central quotient $\mathbb{R}^2$.

Lemma 4.1.2. *If $\lambda_i : Nil^3 \times \mathbb{R} \to \mathbb{R}^2$ for $i = 1, 2$ are two such projections and $\ker \lambda_i \neq \mathcal{Z}Nil^3 \times \mathbb{R}$, then there is an automorphism ϕ of $Nil^3 \times \mathbb{R}$ such that $\theta \circ \lambda_1 = \lambda_2 \circ \phi$, for some $\theta \in GL_2\mathbb{R}$.*

Proof. The automorphism group of $Nil^3 \times \mathbb{R}$ has underlying set $\left\{(A, \mu) = \left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{pmatrix}, (m_x, m_y, m_z)\right)\right\} \subset GL_3\mathbb{R} \times \mathbb{R}^3$ (note that they are not isomorphic as groups). The automorphism (A, μ) acts via

$$(A,\mu)\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\right) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} m_x x + m_y y + m_w w + (ad-bc)z + bcxy + ac\binom{x}{2} + bd\binom{y}{2} \\ ex + fy + gw \end{pmatrix}\right).$$

(This description is based on the description of the automorphism group of Nil^3 in section 1.3 of [Hi].)

To prove the result, we will prove that all such projections are equivalent to $\lambda_1 : (\binom{x}{y}, \binom{z}{w}) \mapsto \binom{y}{w}$. By assumption, $\{(\binom{0}{0}, \binom{0}{w})\}$ is not in the kernel of λ_2. Instead the kernel will be $\{(\binom{ta}{tc}, \binom{z}{te}) | t, z \in \mathbb{R}\}$ for some fixed (a, c, e). Choose another vector (b, d, f) which is not contained in the vector space generated by (a, c, e) and $(0, 0, 1)$. Let $A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{pmatrix}$ (for some nonzero g), and let $\mu = (0, 0, 0)$. Then $\lambda_2 \circ (A, \mu)$ and λ_1 will have the same kernel. However, their image will differ by an element of $GL_2\mathbb{R}$, hence the lemma. □

The lemma shows that up to automorphism, there are two types of foliations of $Nil^3 \times \mathbb{R}$ by cosets of a normal $\mathbb{R}^2$ subgroup. Does every Seifert fibration of a $\mathbb{N}il^3 \times \mathbb{E}$-manifold descend from such a foliation?

Above we considered foliations by cosets of the centre. The Seifert fibrations which arise from this we will call elliptic. We will call fibrations which arise from the other type of foliation by cosets, parabolic (the naming arises from the group of monodromies). Elliptic fibrations are the ones which are very similar to the fibrations of $\widetilde{SL}_2 \times \mathbb{E}$ manifolds. There is no analogy to parabolic fibrations for geometric Seifert fibrations with hyperbolic bases. Let us now consider the Seifert manifolds which have parabolic fibrations, ie Seifert manifolds which arise from the following foliation by cosets:

$$1 \longrightarrow G = \{(\begin{pmatrix}0\\y\end{pmatrix}, \begin{pmatrix}z\\0\end{pmatrix})\} \longrightarrow Nil^3 \times \mathbb{R} \xrightarrow{q_p} \{(x,w)\} \longrightarrow 1.$$

The isometries that preserve this foliation are the ones that normalise G considered as a subgroup of $\text{Isom}(\mathbb{N}il^3 \times \mathbb{E})$. The action of SO_2 generally does not preserve the foliation. The exceptions are the identity and R_π. Left translation preserves the foliation (since G is normal in $Nil^3 \times \mathbb{R}$), as does the maps $(\binom{x}{y}, \binom{z}{w}) \mapsto (\binom{x}{\eta y}, \binom{\eta z}{\delta w})$, $\delta, \eta = \pm 1$. Writing $Nil^3 \times \mathbb{R}$ as $\{(\binom{x}{w}, \binom{z}{y})\}$, then the set of isometries which preserve the foliation are $(\binom{x}{w}, \binom{z}{y}) \mapsto (\left(\left(\begin{smallmatrix}\epsilon\delta & 0\\ 0 & \delta\end{smallmatrix}\right), \binom{x_0}{w_0}\right)\binom{x}{w}, \left(\left(\begin{smallmatrix}\eta & \epsilon\delta\eta x_0\\ 0 & \delta\epsilon\eta\end{smallmatrix}\right), \binom{z_0}{y_0}\right)\binom{z}{y})$, where x_0, w_0, z_0 and y_0 are real numbers and $\delta, \epsilon, \eta = \pm 1$. As before if α is the isometry, $\delta = \text{w}(\alpha)$ and $\epsilon = \text{w}(\bar{\alpha})$.

These isometries satisfy the following exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \ltimes_{-I} \mathbb{R}^2 \longrightarrow Norm_{\text{Isom}(\mathbb{N}il^3 \times \mathbb{E})}(G) \longrightarrow \text{Isom}(\mathbb{E}) \times \text{Isom}(\mathbb{E}) \longrightarrow 1.$$

Definition 4.1.3. Define a *parabolic* matrix in $GL_2\mathbb{R}$ to be a matrix that is conjugate to $\pm\left(\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right)$ for some nonzero λ (a parabolic element of $PSL_2\mathbb{R}$ is defined in [Kat] section 2.1; this motivates this definition). (See the beginning of section 4.5 for details on the definition.)

Lemma 4.1.4. *Suppose $\eta : S \to B$ is a Seifert fibration of a $\mathbb{N}il^3 \times \mathbb{E}$-manifold which descends from a foliation of $Nil^3 \times \mathbb{R}$ by cosets of a normal $\mathbb{R}^2$ subgroup. Then either the group of monodromies is finite or it contains a parabolic element. In the latter case the fibre is T^2.*

Proof. From above there are two foliations (up to isomorphism) of $Nil^3 \times \mathbb{R}$ from which the Seifert fibration can arise.

In the first case, the fibre is a quotient of the subgroup $\mathcal{Z}(Nil^3) \times \mathbb{R}$. Klein bottle fibres are possible (the group of monodromies is always finite when the general fibre is Kb). Suppose the general fibre is T^2. Then $F = (\mathcal{Z}(Nil^3) \times \mathbb{R})/\hat{\Gamma}$ where $\hat{\Gamma}$ is generated by translations from the subgroup $(\mathcal{Z}(Nil^3) \times \mathbb{R}) \subset Nil^3 \times \text{Isom}(\mathbb{E}) \subset \text{Isom}(\mathbb{N}il^3 \times \mathbb{E})$. An isometry $\alpha \in \text{Isom}(\mathbb{N}il^3 \times \mathbb{E})$ acts via conjugation on these translations via the finite order matrix $\mathcal{O}(\alpha)$. Therefore the group of monodromies is conjugate to a subgroup of $\{\left(\begin{smallmatrix}\pm 1 & 0\\ 0 & \pm 1\end{smallmatrix}\right)\}$ and therefore the group of monodromies is finite.

In the second case, we can suppose the fibre is a quotient of the set $\{\binom{0}{y}, \binom{z}{0})\}$ by a subgroup $\mathbb{Z}_2 \ltimes_{-I} \mathbb{R}^2$. This implies the fibre cannot be a Klein bottle. So the general fibre is T^2 and so $F = \{\binom{0}{y}, \binom{z}{0})\}/\hat{\Gamma}$ where $\hat{\Gamma}$ is a group of translations contained in Nil^3 and hence contained in $\text{Isom}(\mathbb{N}il^3 \times \mathbb{E})$. An isometry $\alpha \in Norm_{\text{Isom}(\mathbb{N}il^3 \times \mathbb{E})}(\{\binom{0}{y}, \binom{z}{0})\})$ acts via conjugation on these translations via the matrix $\left(\begin{smallmatrix}\eta & \epsilon\delta\eta x_0\\ 0 & \delta\epsilon\eta\end{smallmatrix}\right)$, where

ϵ, δ and x_0 are determined by $q_p(\alpha) = \left(\left(\begin{smallmatrix} \epsilon\delta & 0 \\ 0 & \delta \end{smallmatrix}\right), \left(\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right) \right)$. If $S = (\mathbb{N}il^3 \times \mathbb{E})/\Gamma$ where Γ is a group of isometries, then the base $B = \mathbb{R}^2/q_p(\Gamma)$ and is therefore flat. Since B is flat and compact, there is an α such that $q_p(\alpha) = \left(I, \left(\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right) \right)$ where x_0 is nonzero. Therefore there is an element of $\pi_1^{orb}(B)$ with monodromy which is conjugate to $\left(\begin{smallmatrix} \eta & \eta x_0 \\ 0 & \eta \end{smallmatrix}\right)$ for some nonzero x_0, that is the group of monodromies contains a parabolic element. □

When a $\mathbb{N}il^3 \times \mathbb{E}$ Seifert manifold has a parabolic fibration, then it will still have an elliptic fibration. The following lemma is helpful in determining that elliptic fibration.

Lemma 4.1.5. *Suppose S is a Seifert manifold which is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ and has a parabolic fibration. Let $S = (\mathbb{N}il^3 \times \mathbb{E})/\Gamma$ where Γ is a discrete group of isometries. Then $B = \mathbb{R}^2/q_p(\Gamma)$ and $F = \mathbb{R}^2/(\Gamma \cap \ker q_p)$. Let Γ' be the subgroup of Γ: $\{\alpha \in \Gamma | \, \mathrm{w}(\alpha) = \mathrm{w}(q_p(\alpha)) = 1$ and $\mathrm{tr}(A(\alpha)) = 2\}$. Then Γ' has finite index in Γ and $\mathcal{Z}(\Gamma') \cong \mathbb{Z}^2$.*

We will see later that the elliptic fibration of S has fundamental group sequence:

$$1 \longrightarrow \mathcal{Z}(\Gamma') \longrightarrow \Gamma = \pi_1(S) \longrightarrow \Gamma/\mathcal{Z}(\Gamma') \longrightarrow 1.$$

Proof. From above, if $\alpha \in \Gamma$ then $\alpha(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\left(\begin{smallmatrix} \epsilon\delta & 0 \\ 0 & \delta \end{smallmatrix}\right), \left(\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right) \right) \left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\left(\begin{smallmatrix} \eta & \epsilon\delta\eta x_0 \\ 0 & \delta\epsilon\eta \end{smallmatrix}\right), \left(\begin{smallmatrix} z_0 \\ y_0 \end{smallmatrix}\right) \right) \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right))$. The group Γ' consists of those isometries where $\epsilon = \delta = \eta = 1$, and so Γ' has finite index in Γ. The group $q_p(\Gamma')$ is generated by translations, and so is isomorphic to $\mathbb{Z}^2$ since Γ was discrete. Also $\Gamma \cap \ker q_p = \Gamma' \cap \ker q_p$. Therefore $(\mathbb{N}il^3 \times \mathbb{E})/\Gamma'$ is a Seifert manifold with T^2 base and T^2 fibre.

If α and α' are in Γ' then $[\alpha, \alpha'](\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} x_0 y_0' - x_0' y_0 \\ 0 \end{smallmatrix}\right))$. Therefore if $\alpha \in \mathcal{Z}(\Gamma')$, $x_0 = y_0 = 0$ and so $\alpha(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 \\ w_0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} z_0 \\ 0 \end{smallmatrix}\right))$. Hence $\mathcal{Z}(\Gamma')$ is free abelian with rank at most 2.

Let $\bar{A}$ be the monodromy map associated to Γ', $\bar{A} : \mathbb{Z}^2 \cong q_p(\Gamma') \to \mathrm{Out}(\Gamma' \cap \ker q_p)$. By looking at the isometries, the image of this map is conjugate to a nontrivial subgroup of $\{\left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix}\right) | x_0 \in \mathbb{R}\}$. Therefore the group of monodromies for Γ' is infinite cyclic. The kernel of the monodromy map will therefore also be infinite cyclic. Choose a $t \in \Gamma'$ so that $q_p(t)$ generates the kernel of the monodromy map, ie $A(t) = I$ and $t(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w+w_t \end{smallmatrix}\right), \left(\begin{smallmatrix} z+z_t \\ y+y_t \end{smallmatrix}\right))$. Also, choose a $u \in \Gamma'$ so that $\bar{A}(q_p(u)) = A(u)$ generates the group of monodromies, ie $u(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x+x_u \\ w+w_u \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & x_u \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} z_u \\ y_u \end{smallmatrix}\right))$. Note $A(u) \in GL_2\mathbb{Z}$ is parabolic therefore $A(u) = P \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) P^{-1}$ for some $P \in GL_2\mathbb{Z}$ and some integer λ. So we can choose some generators of $\Gamma' \cap \ker q_p$, g and h, so that $A(u)(h) = h$ and $A(u)(g) = h^\lambda g$. This forces $h(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z+z_h \\ y \end{smallmatrix}\right))$ and $g(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z+z_g \\ y+y_g \end{smallmatrix}\right))$. Note $A(u)(g) = h^\lambda g$ implies

$$\lambda z_h = x_u y_g. \tag{4.1.1}$$

The set $\{\beta \in \Gamma' \cap \ker q_p | A(u)\beta = \beta\} \subset \{\beta \in \Gamma' | \beta(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z+z_0 \\ y \end{smallmatrix}\right))\}$ is a nonempty subgroup of $\mathcal{Z}(\Gamma')$ (and it will be generated by h). Therefore the centre of Γ' is free abelian with rank at least 1. We claim that some power of t composed with a power of g is also in the centre. Then since a power of t cannot be in $\Gamma' \cap \ker q_p$, $\mathcal{Z}(\Gamma')$ is isomorphic to $\mathbb{Z}^2$.

Consider the commutator, $[t, u]$ which acts via the isometry $(\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right)) = (\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z - x_u y_t \\ y \end{smallmatrix}\right))$. Since $q_p([t, u]) = 1$, $[t, u] \in \Gamma' \cap \ker q_p$. Additionally we can observe it equals some power of h. Therefore $-x_u y_t = a z_h$ for some integer a. Let $c = \gcd(\lambda, a)$. Then combining with equation (4.1.1), we see $\frac{\lambda}{c}y_t + \frac{a}{c}y_g = \frac{\lambda}{c}\frac{az_h}{-x_u} + \frac{a}{c}\frac{\lambda z_h}{x_u} = 0$. Therefore $t^{\frac{\lambda}{c}}g^{\frac{a}{c}}(\binom{x}{w}, \binom{z}{y}) = (\binom{x}{w+\frac{\lambda}{c}w_t}, \binom{z+\frac{\lambda}{c}z_t+\frac{a}{c}z_g}{y})$ which is in the centre of Γ', hence the claim. □

If the group of monodromies has a parabolic element, then the group must have infinite order. The following lemma then shows that the choice of base in that situation is restricted.

Lemma 4.1.6. *Suppose S is Seifert manifold with Euclidean base and general fibre T^2. If the group of monodromies has infinite order (or equivalently if there is a monodromy with infinite order) then the base cannot have cone points or corner reflectors with order greater than 2. So the base is either T^2, Kb, the annulus, the Mobius band, $S^2(2,2,2,2)$, $P^2(2,2)$, $D^2(2,2)$, $D^2(2,\bar{2},\bar{2})$ or $D^2(\bar{2},\bar{2},\bar{2},\bar{2})$.*

Proof. Suppose the base is either $S^2(3,3,3)$ or $S^2(2,4,4)$. The presentation of both is $\langle \bar{s}_1, \bar{s}_2, \bar{s}_3 | \bar{s}_i^{m_i} = 1, \bar{s}_1\bar{s}_2\bar{s}_3 = 1\rangle$, where the m_i are the orders of the cone points. By lemma 2.1.7, the monodromy for each $\bar{s}_i$ is either I or conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From the last relation we see that at least one of the $\bar{s}_i$ has a trivial monodromy, and the other two must have equal monodromies (which could also be trivial). Therefore the group of monodromies is isomorphic to $\mathbb{Z}_2$ or the trivial group. So the group of monodromies is finite.

If the base has a cone point or corner reflector with order greater than 2, then the base is finitely covered by $S^2(3,3,3)$ or $S^2(2,4,4)$ and so the group of monodromies is finite, hence the lemma. □

4.2 The geometry $\mathbb{S}ol^3 \times \mathbb{E}$

In this section we look at the geometry $\mathbb{S}ol^3 \times \mathbb{E}$. We will prove that closed manifolds with this geometry are Seifert fibred.

The group Sol^3 is a semi-direct product of $\mathbb{R}^2$ and $\mathbb{R}$ where $z \in \mathbb{R}$ acts on $\mathbb{R}^2$ via $\begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$. We will write elements of Sol^3 as $\left(\binom{x}{y}, z\right)$. Therefore we have the following exact sequence:

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow Sol^3 \xrightarrow{p} \mathbb{R} \longrightarrow 1,$$

where p is the projection $\left(\binom{x}{y}, z\right) \mapsto z$.

The group Sol^3 is the model space of the geometry $\mathbb{S}ol^3$. The group Sol^3 canNOT be expressed as a line bundle over $\mathbb{R}^2$ and so there is no Seifert 3-manifold geometric of type $\mathbb{S}ol^3$. However mapping tori with T^2 fibres and hyperbolic gluing map are geometric of type $\mathbb{S}ol^3$ ([Sc2] p. 470).

The model space for the geometry $\mathbb{S}ol^3 \times \mathbb{E}$ is $Sol^3 \times \mathbb{R}$, which has the following exact sequence:

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow Sol^3 \times \mathbb{R} \xrightarrow{p} \mathbb{R}^2 \longrightarrow 1.$$

See [Sc2] section 4 for a detailed description of $\mathbb{S}ol^3$ and see [Ue1] section 2 for description of $\mathbb{S}ol^3 \times \mathbb{E}$.

We will write elements of $Sol^3 \times \mathbb{R}$ as $\left(\binom{x}{y}, \binom{z}{t}\right)$. The identity component $\mathrm{Isom}(\mathbb{S}ol^3 \times \mathbb{E})$ is $Sol^3 \times \mathbb{R}$ which acts via left multiplication; $\left(\binom{x'}{y'}, \binom{z'}{t'}\right) : \left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\begin{pmatrix} e^{z'} & 0 \\ 0 & e^{-z'} \end{pmatrix}\binom{x}{y} + \binom{x'}{y'}, \binom{z}{t} + \binom{z'}{t'}\right)$. The other components of Isom($\mathbb{S}ol^3 \times \mathbb{E}$) are generated by the 16 isomorphisms $\left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \delta \end{smallmatrix}\right)\binom{x}{y}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & \eta \end{smallmatrix}\right)\binom{z}{t}\right)$ and $\left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\left(\begin{smallmatrix} 0 & \epsilon \\ \delta & 0 \end{smallmatrix}\right)\binom{x}{y}, \left(\begin{smallmatrix} -1 & 0 \\ 0 & \eta \end{smallmatrix}\right)\binom{z}{t}\right)$, where $\epsilon, \delta, \eta = \pm 1$. Thus an isometry has one of the two following forms; $\left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\left(\begin{smallmatrix} \epsilon e^{z'} & 0 \\ 0 & \delta e^{-z'} \end{smallmatrix}\right)\binom{x}{y} + \binom{x'}{y'}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & \eta \end{smallmatrix}\right)\binom{z}{t} + \binom{z'}{t'}\right)$ and $\left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\left(\begin{smallmatrix} 0 & \epsilon e^{z'} \\ \delta e^{-z'} & 0 \end{smallmatrix}\right)\binom{x}{y} + \binom{x'}{y'}, \left(\begin{smallmatrix} -1 & 0 \\ 0 & \eta \end{smallmatrix}\right)\binom{z}{t} + \binom{z'}{t'}\right)$. In either case, let $\mathcal{O}$ (or $\mathcal{O}(\alpha)$ if we wish to specify an isometry α) be the matrix part of the action on the $\binom{x}{y}$ part. So $\mathcal{O}$ equals $\left(\begin{smallmatrix} \epsilon e^{z'} & 0 \\ 0 & \delta e^{-z'} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & \epsilon e^{z'} \\ \delta e^{-z'} & 0 \end{smallmatrix}\right)$ respectively. Also let $\tilde{y}' = \binom{x'}{y'}$. Both of the above types project to an isometry of the base $\mathbb{R}^2$ given by the map $\left(\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \eta \end{smallmatrix}\right), \binom{z'}{t'}\right)$, which is in $\operatorname{Isom}\mathbb{E} \times \operatorname{Isom}\mathbb{E}$. For an isometry α let $\bar{\alpha}$ be its image under this projection. Then α acts via the map $(\mathcal{O}(\alpha), \tilde{y}'(\alpha)), \bar{\alpha}$.

Note that unlike the $\mathbb{H}^2 \times \mathbb{E}^2$ case we have described previously, $\mathcal{O}(\alpha)$ and $\bar{\alpha}$ are NOT independent. As we shall see later, this means we have to be careful when choosing representations for $\bar{\alpha}$ so that it agrees with the value of $\mathcal{O}(\alpha)$ [both these values will arise from the presentation].

Let K be the kernel of the projection Isom($\mathbb{S}ol^3 \times \mathbb{E}$) $\longrightarrow$ Isom($\mathbb{E}$) $\times$ Isom($\mathbb{E}$). Then K is equal to $\operatorname{Isom}\mathbb{E} \times \operatorname{Isom}\mathbb{E}$. So we have the following exact sequence:

$$1 \longrightarrow K = \operatorname{Isom}\mathbb{E} \times \operatorname{Isom}\mathbb{E} \longrightarrow \operatorname{Isom}(\mathbb{S}ol^3 \times \mathbb{E}) \xrightarrow{q} \operatorname{Isom}\mathbb{E} \times \operatorname{Isom}\mathbb{E} \longrightarrow 1.$$

Definition 4.2.1. Define a *hyperbolic* matrix in $GL_2\mathbb{R}$ to be a matrix that is conjugate to $\left(\begin{smallmatrix} \epsilon e^{\tau} & 0 \\ 0 & \delta e^{-\tau} \end{smallmatrix}\right)$ for some $\epsilon, \delta = \pm 1$ and for some nonzero $\tau \in \mathbb{R}$ (a hyperbolic element of $SL_2\mathbb{R}$ is defined in [Kat] section 2.1, this motivates this definition). (See the beginning of section 4.5 for details on the definition.)

We can now prove the following proposition.

Proposition 4.2.2. *If S is a compact quotient of $\mathbb{S}ol^3 \times \mathbb{E}$ by a discrete group of isometries then S is a Seifert 4 manifold over a Euclidean base with general fibre T^2 and with some hyperbolic monodromies.*

Proof. Firstly $S = \mathbb{S}ol^3 \times \mathbb{E}/\Gamma$ for some discrete subgroup of Isom($\mathbb{S}ol^3 \times \mathbb{E}$). Let $\hat{\Gamma} = \Gamma \cap K$ and $\bar{\Gamma} = q(\Gamma)$. The groups $\hat{\Gamma}$ and $\bar{\Gamma}$ are discrete groups of Isom($\mathbb{E}^2$) (case 3 in [Ue1] section 3). Then we get the Seifert fibration:

$$F = \mathbb{E}^2/\hat{\Gamma} \to S \to \mathbb{E}^2/\bar{\Gamma} = B.$$

Since S is compact, so is F and B. Since F is compact, $\hat{\Gamma} = \pi_1(F)$ contains a normal subgroup of translations of finite index (the index will be at most 4 since $K = \operatorname{Isom}(\mathbb{E}) \times \operatorname{Isom}(\mathbb{E})$). Since B is compact $\bar{\Gamma}$ contains a normal subgroup of translations which is abelian rank 2. This group is generated by $\left(I, \binom{t_1}{t_2}\right)$ and $\left(I, \binom{u_1}{u_2}\right)$ for some real numbers t_1, t_2, u_1, u_2 such that $\det\left(\begin{smallmatrix} t_1 & u_1 \\ t_2 & u_2 \end{smallmatrix}\right) \neq 0$. By relabelling if necessary, we can suppose $u_1 \neq 0$. Let $g \in \Gamma$ project to $\left(I, \binom{u_1}{u_2}\right)$. Then g acts on the subgroup of $\hat{\Gamma}$ consisting of translations via $\mathcal{O}(g) = \left(\begin{smallmatrix} \epsilon e^{u_1} & 0 \\ 0 & \delta e^{-u_1} \end{smallmatrix}\right)$ for some $\epsilon, \delta = \pm 1$. If $F = T^2$, then for some matrix P, $P\mathcal{O}(g)P^{-1}$ is the monodromy above $\bar{g}$. Therefore S has a hyperbolic monodromy.

Suppose $F = Kb$, aiming for a contradiction. Then WLOG $\hat{\Gamma}$ is generated by $h_1 = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \binom{a/2}{b}\right)$ and $h_2 = \left(I, \binom{0}{d}\right)$ for some real numbers a, b, d (the other possibility arises by applying the isomorphism $\left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\binom{x}{y}, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\binom{z}{t}\right)$). Then $gh_2g^{-1} = \left(I, \binom{0}{\delta e^{-u_1} d}\right)$ is not equal to $h_2^{\pm 1}$ (since $e^{-u_1} \neq \pm 1$) and so is certainly not in $\langle h_1, h_2 \rangle$. Therefore $\hat{\Gamma}$ is not normal in Γ which provides the necessary contradiction. ∎

4.3 The geometry $\mathbb{N}il^4$

In this section we look at the geometry $\mathbb{N}il^4$. We will prove that closed manifolds with this geometry are Seifert fibred.

Let Nil^4 be the group $\mathbb{R}^3 \rtimes_C \mathbb{R}$ where $C(t)$ acts on R^3 via the matrix $\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$. We will write an element of Nil^4 as $\left(\binom{x}{y}, \binom{z}{t}\right)$. Then multiplication is given by $\left(\binom{x'}{y'}, \binom{z'}{t'}\right)\left(\binom{x}{y}, \binom{z}{t}\right) = \left(\binom{x+t'y+(t')^2z/2+x'}{y+t'z+y'}, \binom{z+z'}{t+t'}\right)$.

The commutator subgroup of Nil^4 is $\{\left(\binom{x}{y}, \binom{0}{0}\right) | x, y \in \mathbb{R}\}$, which is isomorphic to $\mathbb{R}^2$. We therefore get the following exact sequence:

$$1 \longrightarrow (Nil^4)' = \mathbb{R}^2 \longrightarrow Nil^4 \xrightarrow{p} \mathbb{R}^2 \longrightarrow 1,$$

where $p\left(\binom{x}{y}, \binom{z}{t}\right) = \binom{z}{t}$.

The group Nil^4 is the model space for the geometry $\mathbb{N}il^4$. The identity component of $\mathrm{Isom}(\mathbb{N}il^4)$ is Nil^4 acting on itself by left multiplication. The other components of the isometry group are represented by the four group isomorphisms $\left(\binom{x}{y}, \binom{z}{t}\right) \mapsto \left(\binom{\epsilon x}{\epsilon\eta y}, \binom{\epsilon z}{\eta t}\right)$. By considering how the isometries act on the above exact sequence we get the following exact sequence:

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow \mathrm{Isom}(\mathbb{N}il^4) \xrightarrow{q} \mathrm{Isom}\, E \times \mathrm{Isom}\, E \longrightarrow 1.$$

See [Wa2] section 1 and [Ue1] section 2 for more details.

We shall now consider the isometries of $\mathbb{N}il^4$ in more detail. An isometry α of $\mathbb{N}il^4$ has the form $\alpha\left(\binom{x}{y}, \binom{z}{t}\right) = \left(\left(\begin{pmatrix} \epsilon & \epsilon\eta t' \\ 0 & \epsilon\eta \end{pmatrix}, \binom{x'}{y'}\right)\binom{x}{y} + \binom{\epsilon(t')^2z/2}{\epsilon t'z}, \left(\begin{pmatrix} \epsilon & 0 \\ 0 & \eta \end{pmatrix}, \binom{z'}{t'}\right)\binom{z}{t}\right)$, for some real numbers x', y', z' and t' and for some $\epsilon, \eta = \pm 1$. Let $\mathcal{O}(\alpha) = \begin{pmatrix} \epsilon & \epsilon\eta t' \\ 0 & \epsilon\eta \end{pmatrix}$. Then α acts on elements of $\ker q = \mathbb{R}^2$ via conjugation by $\mathcal{O}(\alpha)$. Let $\bar{\alpha} = q(\alpha) = \left(\begin{pmatrix} \epsilon & 0 \\ 0 & \eta \end{pmatrix}, \binom{z'}{t'}\right)$. Also let $c(\bar{\alpha})(z) = \binom{\epsilon(t')^2z/2}{\epsilon t'z}$. Then $\alpha\left(\binom{x}{y}, \binom{z}{t}\right) = \left(\left(\mathcal{O}(\alpha), \binom{x'}{y'}\right)\binom{x}{y} + c(\bar{\alpha})(z), \bar{\alpha}\binom{z}{t}\right)$.

We can now show $\mathbb{N}il^4$ manifolds are Seifert fibred.

Proposition 4.3.1. *If S is a compact quotient of $\mathbb{N}il^4$ by a discrete group of isometries then S is a Seifert 4 manifold over an Euclidean base with general fibre T^2 and with some parabolic monodromies.*

Proof. Firstly $S = \mathbb{N}il^4/\Gamma$ for some discrete group of isometries. Let $\hat{\Gamma} = \Gamma \cap \ker q$, and let $\bar{\Gamma} = q(\Gamma)$. The groups $\hat{\Gamma}$ and $\bar{\Gamma}$ are discrete groups of $\mathrm{Isom}(\mathbb{E}^2)$ (case 1 in [Ue1] section 3). Then we get the Seifert fibration:

$$F = \mathbb{E}^2/\hat{\Gamma} \longrightarrow S \longrightarrow \mathbb{E}^2/\bar{\Gamma} = B.$$

Since $\ker q = \mathbb{R}^2$, $\hat{\Gamma}$ is isomorphic to a group of translations and so $F \cong T^2$. Also the monodromy for an isometry α will be conjugate to $\mathcal{O}(\alpha)$. Since B is compact $\bar{\Gamma}$ contains a normal subgroup of translations which is abelian rank 2. This group is generated by $\left(I, \binom{z_1'}{t_1'}\right)$ and $\left(I, \binom{z_2'}{t_2'}\right)$ for some real numbers t_1', t_2', z_1', z_2' such that $\det\begin{pmatrix} z_1' & z_2' \\ t_1' & t_2' \end{pmatrix} \neq 0$. By relabelling if necessary, we can suppose $t_1' \neq 0$. Choose an α in Γ so that $q(\alpha) = \left(I, \binom{z_1'}{t_1'}\right)$. Then $\mathcal{O}(\alpha) = \begin{pmatrix} 1 & t_1' \\ 0 & 1 \end{pmatrix}$ which is parabolic. Therefore S has a parabolic monodromy. ☐

Note if α_1 and α_2 are two isometries, then

$$\alpha_1\alpha_2\left(\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\right)=\left(\left(\mathcal{O}(\alpha_1),\begin{pmatrix}x_1'\\y_1'\end{pmatrix}\right)\left(\mathcal{O}(\alpha_2),\begin{pmatrix}x_2'\\y_2'\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}+c(\bar{\alpha}_1\bar{\alpha}_2)(z)+c(\bar{\alpha}_1)(z_2'),\bar{\alpha}_1\bar{\alpha}_2\begin{pmatrix}z\\t\end{pmatrix}\right).$$

Here $c(\bar{\alpha}_1)(z_2')$ equals $\binom{\epsilon_1(t_1')^2z_2'/2}{\epsilon_1t_1'z_2'}$.

As an extension of this, if α_i $(i=1,\dots,p)$ are isometries such that $\bar{\alpha}_1\dots\bar{\alpha}_p=1$ is $\mathrm{Isom}(E^2)$. Then $\alpha_1\dots\alpha_p\left(\binom{x}{y},\binom{z}{t}\right)=\left(\left(\mathcal{O}(\alpha_1),\binom{x_1'}{y_1'}\right)\dots\left(\mathcal{O}(\alpha_p),\binom{x_p'}{y_p'}\right)\binom{x}{y}+\tilde{k},\binom{z}{t}\right)$, where $\tilde{k}\in\mathbb{R}^2$ is a constant term which depends on the $\bar{\alpha}_i$ ($\tilde{k}=\sum_{i=1}^{p-1}c(\prod_{j=1}^{i}\bar{\alpha}_i)(z_{i+1}')$).

If α is an isometry where $\bar{\alpha}$ has finite order, then firstly $\bar{\alpha}^2=1$. Secondly as an example of the above,

$$\alpha^2\left(\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\right)=\left(\left(\mathcal{O}(\alpha),\begin{pmatrix}x'\\y'\end{pmatrix}\right)^2\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\epsilon(t')^2z'/2\\\epsilon t'z'\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\right).$$

By modifying $c(\bar{\alpha})(z)$ to be $\binom{\epsilon(t')^2z/2}{\epsilon t'z}+\binom{0}{t'z'/2}$ when $\bar{\alpha}$ has finite order (and thus implicitly modifying $\binom{x'}{y'}$), then

$$\begin{aligned}\alpha^2\left(\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\right)&=\left(\left(\mathcal{O}(\alpha),\begin{pmatrix}x'\\y'\end{pmatrix}\right)^2\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\epsilon(t')^2z'/2\\\epsilon t'z'\end{pmatrix}+(\mathcal{O}(\alpha)+I)\begin{pmatrix}0\\t'z'/2\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\right)\\&=\left(\left(\mathcal{O}(\alpha),\begin{pmatrix}x'\\y'\end{pmatrix}\right)^2\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\right).\end{aligned}$$

(This is clear if $\epsilon=\eta=-1$. When ϵ or η is 1, $t'z'=0$ since $\bar{\alpha}$ has order two and so it is clear in that case too.)

This modification alters the constant $\tilde{k}$ above. However this change makes $c(\bar{\alpha})(z)$ closer to the correction term for the isometries of $\widetilde{SL}_2\times\mathbb{E}$ which will allow us to prove results concerning this geometry by appealing to results for $\widetilde{SL}_2\times\mathbb{E}$.

In summary, if α is an isometry of $\mathbb{N}il^4$, then $\alpha\left(\binom{x}{y},\binom{z}{t}\right)=\left(\left(\mathcal{O}(\alpha),\binom{x'}{y'}\right)\binom{x}{y}+c(\bar{\alpha})(z),\bar{\alpha}\binom{z}{t}\right)$, where $c(\bar{\alpha})(z)$ is a correction term.

4.4 Seifert manifolds with flat bases determined by their fundamental groups

Earlier in theorem 2.5.10 we proved that Seifert spaces over aspherical bases are determined (up to fibre preserving homeomorphism) by their fundamental group sequences. So in particular this applies to Seifert manifolds with flat bases. Are such spaces determined in this way by their fundamental group alone (as in the hyperbolic base case)? The following lemma and corollary (essentially due to Ue) show that geometric manifolds of type $\mathbb{N}il^3$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3\times\mathbb{E}$ are determined by their fundamental group.

Lemma 4.4.1. *Suppose G is a group such that for all normal subgroups G' of finite index, which contain a subgroup $A'\cong\mathbb{Z}^n$ so that $G'/A'\cong\mathbb{Z}^2$, have* $\mathrm{rank}\,H_1(G')=2$*. Suppose A_1 and A_2 are normal subgroups of G such that $A_1\cong A_2\cong\mathbb{Z}^n$ and $B_1=G/A_1$ and $B_2=G/A_2$ are flat 2-orbifold groups. Then $A_1=A_2$.*

Proof. Let η_1 and η_2 be the projections from G to B_1 and B_2.

Let B_{1T} be the maximal normal abelian subgroup of B_1. Let $G_1' = \eta_1^{-1}(B_{1T})$ and let $\eta_1' = \eta_1|_{G_1'}$. Note G_1' has finite index in G since B_{1T} has finite index in B_1.

For $i = 1, 2$, $\eta_1(A_i) \subseteq B_{1T}$ since A_i is abelian, so $A_1 = \ker \eta_1'$ and $A_2 \subset G_1'$. Let $\tilde{\eta}_2 = \eta_2|_{G_1'}$ and let $\tilde{B}_2 = \operatorname{Im} \tilde{\eta}_2 \subseteq B_2$. Note $\tilde{B}_2$ has finite index in B_2 since G_1' has finite index in G. Let $G_2' = \tilde{\eta}_2^{-1}(\tilde{B}_{2T})$ (where $\tilde{B}_{2T}$ is the maximal normal abelian subgroup of $\tilde{B}_2$) and let $\eta_2' = \tilde{\eta}_2|_{G_2'}$. Note that $\tilde{B}_{2T}$ has finite index in B_2 since $\tilde{B}_{2T}$ has finite index in $\tilde{B}_2$ which has finite index in B_2. The group A_2 is the kernel of η_2'.

The maximal normal abelian subgroup of a flat 2-orbifold group is rank 2 free abelian. Therefore B_{1T} and $\tilde{B}_{2T}$ are both isomorphic to $\mathbb{Z}^2$. Since $\operatorname{rank} H_1(G_1') = 2$ by hypothesis, the exact sequence $1 \to A_1 \to G_1' \to B_{1T} \to 1$ equals $1 \to K_1 \to G_1' \to H_1(G_1')/\text{Torsion} \to 1$, where K_1 is the natural kernel. Similarly since $\operatorname{rank} H_1(G_2') = 2$, the exact sequence $1 \to A_2 \to G_2' \to \tilde{B}_{2T} \to 1$ equals $1 \to K_2 \to G_2' \to H_1(G_2')/\text{Torsion} \to 1$, where K_2 is the natural kernel. The inclusion $G_2' \subseteq G_1'$ induces an inclusion $H_1(G_2')/\text{Torsion} \subseteq H_1(G_1')/\text{Torsion}$ and hence $A_2 = K_2 \subseteq K_1 = A_1$. By reversing the roles of A_1 and A_2 we see $A_1 = A_2$, hence the lemma. □

Remark 4.4.2. In section 3 of [Ue1], Ue essentially proved this lemma. He assumed $n = 2$ and that G is the fundamental group of a Seifert manifold but the underlying method remains in the proof above.

Corollary 4.4.3. *Suppose S and S' are geometric manifolds of type $\mathbb{N}il^3$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}$. Then S and S' are Seifert fibred and every isomorphism $\phi : \pi_1(S) \to \pi_1(S')$ is realised by a fibre preserving homeomorphism $\theta : S \to S'$.*

Proof. Geometric manifolds of type $\mathbb{N}il^4$, $\mathbb{S}ol^3 \times \mathbb{E}$ or $\mathbb{N}il^3$ are Seifert manifolds with Euclidean bases (by proposition 4.3.1, proposition 4.2.2 and [Sc2] respectively).

Claim: Suppose S and S' are Seifert manifolds with T^n fibres, Euclidean bases and for all T^n bundles over T^2, $\tilde{S}$ ($\tilde{S}'$ respectively) which cover S (S'), $\operatorname{rank} H_1(\tilde{S})$ (and $\operatorname{rank} H_1(\tilde{S}')$) are 2. Then every isomorphism $\phi : \pi_1(S) \to \pi_1(S')$ is realised by a fibre preserving homeomorphism $\theta : S \to S'$.

Take $G = \pi_1(S')$, $A_1 = \pi_1(F')$ and $A_2 = \phi(\pi_1(F))$. Then by the previous lemma $\pi_1(F') = \phi(\pi_1(F))$. Therefore ϕ induces an isomorphism of fundamental group sequences and so is realised by a fibre-preserving homeomorphism (theorem 2.5.10). Hence the claim.

To prove this corollary we will show that geometric manifolds of type $\mathbb{N}il^3$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}$ satisfy the hypotheses of the claim.

If M is a $S^1 = T^1$ bundle over T^2 then either $M = T^3$ or $\operatorname{rank}(H_1(M)) = 2$. Therefore $\mathbb{N}il^3$ manifolds (which are never covered by T^3) satisfy the hypotheses of the claim.

Let M be a T^2 bundle over T^2. Then $2 \leq \operatorname{rank}(H_1(M)) \leq 4$. In [SaFu] theorem 2(1), it is proved that $\operatorname{rank}(H_1(M)) = 4$ if and only if $M = T^4$. In List 1 in [Ue1], it is shown that if M is orientable and $\operatorname{rank}(H_1(M)) = 3$, then M is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$. If M is non-orientable and $\operatorname{rank}(H_1(M)) = 3$, then using proposition 4 of [SaFu], it can be shown that M is covered by T^4 and so M is geometric of type $\mathbb{E}^4$. By propositions 4.3.1 and 4.2.2, $\mathbb{N}il^4$ and $\mathbb{S}ol^3 \times \mathbb{E}$ manifolds cannot have Kb fibres, and so must have T^2 fibres. (The other geometries, $\mathbb{E}^4$ and $\mathbb{N}il^3 \times \mathbb{E}$ can have Kb fibres). Therefore $\mathbb{N}il^4$ and $\mathbb{S}ol^3 \times \mathbb{E}$ manifolds satisfy the hypotheses of the claim. □

Remark 4.4.4. Seifert manifolds which are geometric of type $\mathbb{E}^n$ $(n \geq 3)$ or $\mathbb{N}il^3 \times \mathbb{E}$ in general do not have unique fibreings and so the fibreing cannot be determined by the fundamental group alone (see [Ue3] for examples for the geometries $\mathbb{E}^4$ and $\mathbb{N}il^3 \times \mathbb{E}$ when there are no reflector curves).

4.5 Geometric Seifert manifolds with Euclidean bases

In [Ue1], all orientable Seifert manifolds with Euclidean base were proved to be geometric and all such manifolds were listed with their geometry. In this section, we extend this result to prove that all Seifert manifolds with Euclidean base are geometric. We also give necessary and sufficient conditions for Seifert manifolds with a Euclidean base to have one of the four possible geometries. Theorems 4.5.5 and 4.5.6 are a summary of Ue's detailed classification, using similar ideas to those used here for hyperbolic bases.

There are some similarities with theorem 4.5.5 in this section and results about three dimensional manifolds with the structure of an orbifold bundle with a flat base and flat fibres (ie S^1, T^2, Kb). Seifert 3-manifolds with flat bases are geometric of type $\mathbb{E}^3$ or $\mathbb{N}il^3$, the two cases differentiated by the euler number ([Sc2] p. 445 and 469). The group of monodromies for all Seifert 3-manifolds is finite. Thus there is some similarity to the $\mathbb{E}^4$ and $\mathbb{N}il^3 \times \mathbb{E}$ cases of the theorem. There are three cases of T^2-bundles over S^1. A T^2-bundle over S^1 is geometric of type $\mathbb{N}il^3$ if and only if it has a parabolic gluing map (see [Sc2] p. 470). A T^2-bundle over S^1 is geometric of type $\mathbb{S}ol^3$ if and only if it has a hyperbolic gluing map (see [Sc2] p. 472). A T^2-bundle over S^1 is geometric of type $\mathbb{E}^3$ if and only if it has a periodic gluing map (see [Sc2] p. 446). The three cases have similarities to the $e \in V$ cases of the theorem (which includes $\mathbb{S}ol^3 \times \mathbb{E}$ manifolds). A Kb-bundle over S^1 is geometric of type $\mathbb{E}^3$.

The key results which underpin the work in this section are theorem 2.5.10 and corollary 4.4.3, which state (respectively) that Seifert fibred $\mathbb{E}^4$ and $\mathbb{N}il^3 \times \mathbb{E}$ manifolds are determined (up to fibre preserving homeomorphism) by their fundamental group sequence and $\mathbb{N}il^4$ and $\mathbb{S}ol^3 \times \mathbb{E}$ manifolds are determined (up to fibre preserving homeomorphism) by their fundamental group. Therefore we can specify conditions for manifolds to be geometric in terms of their fundamental group sequence.

The group of monodromies of a Seifert fibration with flat base is a solvable subgroup of $GL_2\mathbb{Z}$ (since it is the image of $\pi_1^{orb}(B)$, a solvable group). A subgroup G of $GL_2\mathbb{Z}$ is solvable if and only if the subgroup $G^+ \subset SL_2\mathbb{Z}$ of matrices with determinant 1 is solvable (since G/G^+ is solvable) if and only if $G^+/\langle -I \rangle \subset PSL_2\mathbb{Z} \subset \mathrm{Isom}^+(H^2)$ is solvable (since $\langle -I \rangle$ is solvable). However the solvable subgroups of $\mathrm{Isom}^+(H^2)$ are elementary. There are three types of elementary subgroups: elliptic, hyperbolic and parabolic (see section 5.5 of [Ra]). This motivates the following definitions where a solvable subgroup G of $GL_2\mathbb{Z}$ is classified by the type of the elementary group $G^+/\langle -I \rangle$.

Definition 4.5.1. An element, A of $GL_2\mathbb{Z}$ is called:

(i). *central* if it is $\pm I$.

(ii). *elliptic* if it is conjugate in $GL_2\mathbb{R}$ to an element of $O_2\mathbb{R}$, but is not central. Thus by corollary 3.3.2 it has finite order and is conjugate in $GL_2\mathbb{Z}$ to an element of $O_2\mathbb{Z}$ or $\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$. Equivalently, A is elliptic if it has distinct eigenvalues, all with absolute value 1. Also we can state the definition simply in terms of trace and determinant; A is elliptic if $\det A = 1$ and $|\operatorname{tr} A| < 2$ or $\det A = -1$ and $\operatorname{tr} A = 0$.

(iii). *hyperbolic* if it has no eigenvalue of absolute value 1. This means it has real eigenvalues whose absolute values are distinct and so it is conjugate in $GL_2\mathbb{R}$ to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \pm\lambda^{-1} \end{smallmatrix}\right)$ where λ is real and not equal to 0 or ± 1. Equivalently, A is hyperbolic if $\det A = 1$ and $|\operatorname{tr} A| > 2$ or $\det A = -1$ and $\operatorname{tr} A \neq 0$.

(iv). *parabolic* if it is not central, elliptic or hyperbolic. Equivalently, A is parabolic if it is not central and has equal eigenvalues with absolute value 1. Consequently, it will be conjugate in $GL_2\mathbb{Z}$ to $\left(\begin{smallmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{smallmatrix}\right)$, for some nonzero λ. In terms of trace and determinant; A is parabolic if A is non-central, $\det A = 1$ and $|\operatorname{tr} A| = 2$.

Furthermore we call a matrix with finite order *periodic*. A periodic matrix is either central or elliptic.

Definition 4.5.2. A solvable subgroup, G of $GL_2\mathbb{Z}$ is called:

(i). *elliptic* if it is conjugate to a subgroup of $O_2\mathbb{R}$. Equivalently, G is elliptic if it is finite (see corollary 3.3.2). The group G is conjugate in $GL_2\mathbb{Z}$ to a subgroup of $O_2\mathbb{Z}$ or $\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$. Note all elements are central or elliptic.

(ii). *hyperbolic* if it contains a hyperbolic element. Hyperbolic elements have infinite order and thus so do hyperbolic groups. The group G is conjugate in $GL_2\mathbb{R}$ to an infinite subgroup of $\langle \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$, for some $\lambda \neq \pm 1$.

(iii). *parabolic* if it is neither elliptic nor hyperbolic. A parabolic group has infinite order and contains a parabolic element. The group G is conjugate in $GL_2\mathbb{Z}$ to an infinite subgroup of $\left\langle \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & (1) \\ 0 & -1 \end{smallmatrix}\right), -I \right\rangle$, where $\lambda \neq 0$ and $(1) = 0$ or 1.

Note that the type of a group is preserved under conjugation. Also if G' is a normal subgroup of G with finite index, then G and G' have the same type.

Lemma 4.5.3 ([SaFu]). *If $A, B \in GL_2\mathbb{Z}$ commute, then there is $C \in GL_2\mathbb{Z}$ such that $A = \pm C^p$ and $B = \pm C^q$ for some integers p and q.*

See lemma 3 in [SaFu] for a proof. □

Thus an abelian solvable subgroup of $GL_2\mathbb{Z}$ is cyclic or is an extension of a cyclic group by $\langle -I \rangle$.

Proposition 4.5.4. *A $\mathbb{E}^4$ manifold is a Seifert manifold with Euclidean base, excepting three cases.*

Proof. In [Hi], section 7.7 it is shown that the fundamental group of a flat 4-manifold has an abelian normal subgroup of rank 2, with 3 exceptions (2 orientable and 1 non-orientable). Consideration of the explicit representaion of the groups in [Br+] shows that these flat manifolds are Seifert fibred. □

The two main theorems of this section follow. They are neatly summarised by corollary 4.5.7, which is stated before the proof.

Theorem 4.5.5. *Let S be a Seifert manifold over a Euclidean base, with general fibre T^2. Then S is geometric. More precisely, let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves, such that the ith reflector curve has k_i corner reflectors so that m_{ij} is the order of the jth corner reflector on the ith reflector curve. Let A be the standard map which induces the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, $\tilde{e}_{0j}$, $\tilde{f}_i$, and $\tilde{g}_{ij}$ be the standard parts of the presentation of the fundamental group sequence of S.*

Let $e = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right)$ and let $V = \mathcal{I}_{\mathrm{w}}\mathbb{Q}^2$.

Then,

(i). S is geometric of type $\mathbb{E}^4$ if and only if the group of monodromies is elliptic (finite) and $e \in V$.

(ii). S is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ if and only if either

(i). the group of monodromies is elliptic (finite) and $e \notin V$ OR

(ii). the group of monodromies is parabolic and $e \in V$.

(iii). S is geometric of type $\mathbb{N}il^4$ if and only if the group of monodromies is parabolic and $e \notin V$.

(iv). S is geometric of type $\mathbb{S}ol^3 \times \mathbb{E}$ if and only if the group of monodromies is hyperbolic.

(Note as in previous sections, we can take V to be $\mathcal{I}_{\mathrm{w}}\mathbb{R}^2 = \sum_{\bar{\xi}\in\pi_1^{orb}(B)} \mathrm{Im}_{\mathbb{R}^2}(A(\xi) - \mathrm{w}(\bar{\xi})I)$.)

Theorem 4.5.6. *Let S be a Seifert manifold over a Euclidean base B, with general fibre Kb. Then S is geometric (of type $\mathbb{E}^4$ or $\mathbb{N}il^3 \times \mathbb{E}$). More precisely, suppose the base has l reflector curves (and consequently no corner reflectors) and k cone points, so that m_i is the order of the ith cone point. Let $A(\xi)$ be the automorphisms from the presentation of $\pi_1(S)$ which send h_1 to $h_1^{\epsilon(\xi)}h_2^{c(\xi)}$ and h_2 to $h_2^{\delta(\xi)}$ and let $\tilde{e}_i = \binom{e_{i1}}{e_{i2}}$, $\tilde{b}_i$ and $\tilde{a}$ be the standard parts of the presentation of the fundamental group sequence of S.*

Let $e = a_1 + \sum_{i=1}^{k} e_{i1}/m_i + \frac{1}{2}\sum_{i=1}^{l} b_{i1}$ and let $V = \sum_{\bar{\xi}\in\pi_1^{orb}(B)} \mathrm{Im}(\epsilon(\xi) - \mathrm{w}(\bar{\xi}))$.

Then, S is geometric of type $\mathbb{E}^4$ if and only if $e \in V$ and S is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ if and only if $e \notin V$.

Corollary 4.5.7. *If S is a Seifert manifold over a Euclidean base, then S is geometric of type $\mathbb{E}^4, \mathbb{N}il^3 \times \mathbb{E}, \mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}$. Conversely, if S is a geometric manifold of one of these types it has a Seifert fibration with Euclidean base, except for three manifolds in the $\mathbb{E}^4$ case.*

Proof. Geometric manifolds of type $\mathbb{N}il^3 \times \mathbb{E}, \mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}$ have a Seifert fibration with Euclidean base (propositions 4.1.1, 4.3.1 and 4.2.2 respectively). With three exceptions, $\mathbb{E}^4$-manifolds also have a Seifert fibration with Euclidean base (propostion 4.5.4). Then the rest of the result comes from the two theorems. □

Overview of proof of theorems 4.5.5 and 4.5.6: In [Ue1], Ue proves the first of the above theorems for the orientable case (the second theorem only arises in the non-orientable case). His method of proof is to give a detailed list of orientable Seifert 4-manifolds with Euclidean base and prove each is geometric (the lists of manifolds and their geometries: Lists I-IV in sections 4 and 6, Claims 9, 10 in section 5, and Claim 8 in section 7). Then he concludes with his Theorem B, which is essentially the same as the above corollary. The above theorems try to condense his lists (which are useful in their detail) to more generic conditions. Ue uses representatives for the isomorphism class of $\pi_1(S)$ but this is not a problem, since lemma 4.5.8 below proves $e \mod V$ is invariant under an isomorphism of fundamental group sequences.

Theorem 4.5.5 is divided into the four geometries. We will prove the result for each in term in lemmas 4.5.11, 4.5.12, 4.5.18 and 4.5.17 (where the order matches the one given in the theorem). All four cases can be proved via a similar method to theorem 3.4.1. The proofs we give highlight the differences.

Theorem 4.5.6 is proved in exactly the same way as (the second proof of) theorem 3.5.1. Namely show that the fundamental group of the orientation cover is isomorphic to a group of isometries (by using theorem 4.5.5). Then add an isometry to get a group of isometries isomorphic to $\pi_1(S)$. This amounts to proving the map $h_1 : (z, w) \mapsto (z, P \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) P^{-1} + P\binom{1/2}{0})$ is an isometry, where P is determined by the first isomorphism. For the $\mathbb{H}^2 \times \mathbb{E}^2$ case, this meant choosing the appropriate glide reflection from $\{1\} \times \mathrm{Isom}(\mathbb{E}^2) \subset \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{E}^2)$. Similarly for the $\mathbb{E}^4$ case we choose the appropriate glide reflection from $\{1\} \times \mathrm{Isom}(\mathbb{E}^2) \subset \mathrm{Isom}(\mathbb{E}^4)$. For the $\widetilde{SL}_2 \times \mathbb{E}$ case this meant showing h_1 was a composition of a reflection of the $\mathbb{E}$ factor of $\widetilde{SL}_2 \times \mathbb{E}$ and a translation from the $\mathbb{R}$ factor of $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{R} = \mathrm{Isom}(\widetilde{SL}_2)$. Similarly for the $\mathbb{N}il^3 \times \mathbb{E}$ case, we show h_1 is a composition of a reflection of the $\mathbb{E}$ factor of $\mathbb{N}il^3 \times \mathbb{E}$ and a translation determined by multiplying on the left by an element of the centre of $\mathbb{N}il^3$.

The following lemma shows that the type of the group of monodromies and the condition $e \in V$ depends only on the isomorphism class of the fundamental group sequence of S.

Lemma 4.5.8. *Let S and S' be Seifert manifolds with T^n fibres and aspherical base, and let e, V and $\mathrm{Im}(A)$ (e', V' and $\mathrm{Im}(A')$ respectively) be as in theorem 4.5.5. Suppose $\theta : S \to S'$ is a fibre preserving homeomorphism. Then $e \in V$ if and only if $e' \in V'$ and $\mathrm{Im}(A)$ is elliptic, hyperbolic or parabolic if and only if $\mathrm{Im}(A')$ is elliptic, hyperbolic or parabolic respectively.*

Proof. (Note e and V are determined from the fundamental group, so there is implicitly a basepoint.) Let θ_F be the restriction of θ to the fibre containing the basepoint. We will prove $V' = \pi_1(\theta_F)V$, $\pi_1(\theta_F)(e) = e'$ (mod V') and $\mathrm{Im}(A') = \pi_1(\theta_F)\,\mathrm{Im}(A)\,(\pi_1(\theta_F))^{-1}$. This will then imply the lemma.

Note, $\mathrm{Im}(A)$ and $\mathrm{Im}(A')$ have the same type if they are conjugate.

Any isomorphism $\pi(\theta) : \pi_1(S) \to \pi_1(S')$ (which maps $\pi_1(F)$ onto $\pi_1(F')$) can be decomposed into isomorphisms of three different types: change of fibre, change of base and change of section. [Let $\overline{\pi_1(\theta)}$ be the projection of $\pi_1(\theta)$ to an isomorphism of $\pi_1^{orb}(B)$. Then $\overline{\pi_1(\theta)}$ and the identity map $\pi_1(F) \to \pi_1(F)$ induce an isomorphism of $\pi_1(S)$ - this is called a change of base isomorphism. By composing $\pi_1(\theta)$ with the inverse of this map we can suppose $\overline{\pi_1(\theta)} = 1$. The isomorphism $\pi_1(\theta)|_{\pi_1(F)}$ and the identity map on $\pi_1^{orb}(B)$ induce an isomorphism of $\pi_1(S)$ - this is called a change of fibre isomorphism. By composing $\pi_1(\theta)$ with the inverse of this map we can suppose $\pi_1(\theta)|_{\pi_1(F)} = 1$. Let $G = \{\bar{g} \in \pi_1^{orb}(B)\}$ be a finite set of generators for $\pi_1^{orb}(B)$ and let $\pi_1(F) = \langle h_i \rangle$. Then $\pi_1(S)$ is generated by the h_i and $\{g|\bar{g} \in G\}$ where the g are lifts of the $\bar{g}$. Then by the assumptions we have put on $\pi_1(\theta)$, $\pi_1(\theta)(h_i) = h_i$ and $\pi_1(\theta)(g) = gh_g$ for some $h_g \in \pi_1(F)$. We can split this into a composition of a finite number of isomorphisms where $h_g \neq 1$ for precisely one g. An isomorphism with this property is called a change of section isomorphism. Therefore we have shown $\pi_1(\theta)$ is a composition of isomorphisms of the three types.] When the base is aspherical all these isomorphisms can be realised by fibre preserving homeomorphisms (by theorem 2.5.10). So we simply need to check each of these basic types.

Firstly, an isomorphism of type 'change of fibre' is realised by a map which uniformly applies a homeomorphism to all the fibres, this homeomorphism is given by θ_F. Thus e (which is in $\pi_1(F)$) is mapped to $\pi_1(\theta)(e)$ and $A(\xi)$ (for all $\xi \in \pi_1^{orb}(B)$) (which are automorphisms of $\pi_1(F)$) are mapped to $\pi_1(\theta_F)A(\xi)\left(\pi_1(\theta_F)\right)^{-1}$, hence the lemma for this case.

Next, an isomorphism of type 'change of base' is one that is induced by an isomorphism of the base. It will preserve the fibre above the base point, thus $\theta_F = 1_F$. It will permute the components of each of e, V and $\mathrm{Im}(A)$ and so will preserve them all, hence the lemma for this case.

Lastly, an isomorphism of type 'change of section' is one that algebraically, sends ξ to $h\xi$ for some $h \in \pi_1(F)$ and some element ξ which is a lift of an element of $\pi_1^{orb}(B)$. They fix the fibre above the base point, thus $\theta_F = 1_F$. Also they fix the base, thus $\mathrm{Im}(A)$ and V are fixed. Considering each type of generator, it can be shown that $e = e' \pmod V$. For example if $s_{0j} \mapsto hs'_{0j}$ then $a \mapsto A(d)\left(\prod_{p=1}^{j-1} A(s_{0p})\right) h + a \equiv h + a \pmod V$ and $e_{0p} \mapsto e_{0p} - \sum_{i=0}^{m-1}\left(A(s_{0j})^i h\right) \equiv m_{0p}(e_{0p}/m_{0p} - h) \pmod V$ and in the expression $e' \pmod V$ the h terms cancel, hence the lemma. ☐

The following lemma shows that the type of the group of monodromies and the condition $e \in V$ are invariant when passing to a finite cover.

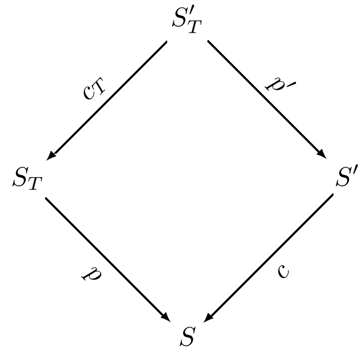
Lemma 4.5.9. *Let S and S' be Seifert manifolds with T^n fibres and flat base, and let e, V and* $\mathrm{Im}(A)$ *(e', V' and* $\mathrm{Im}(A')$ *respectively) be as in theorem 4.5.5. Suppose $c : S' \to S$ is a finite covering which preserves the bundle structure. Then $e' \in V'$ if and only if $e \in V$ and* $\mathrm{Im}(A')$ *is elliptic, hyperbolic or parabolic if and only if* $\mathrm{Im}(A)$ *is elliptic, hyperbolic or parabolic respectively.*

Note, if the Seifert fibration of S' is unique, then a finite cover $c : S' \to S$ will preserve the bundle structure.

Proof. If S' is a finite cover of S, then $\mathrm{Im}(A')$ is normal subgroup of finite index in $\mathrm{Im}(A)$ and so they have the same type, hence the second part of the lemma.

Let B (B') be the base of S (S' respectively), and let F (F') be its general fibre.

It is sufficient to prove the first part of this result for any covering where $B = B' = T^2$ and any cover which is induced by the characteristic cover of the base, $T^2 = B \to B'$ (note this amounts to 17 cases). To illustrate this, consider a covering $c : S' \to S$. Let $p : S_T \to S$ be the covering which is induced by the characteristic cover $T^2 \to B$. Similarly define $p' : S'_T \to S'$. Since $\pi_1(T^2)$ is a characteristic subgroup of $\pi_1^{orb}(B)$ and $\pi_1^{orb}(B')$, c' induces a cover $c_T : S'_T \to S_T$, which makes the following diagram commute:



Given we can prove the result for p, p' and c_T, then $e' \in V' \iff e'_T \in V'_T \iff e_T \in V_T \iff e \in V$, hence the result for c.

For each of the 17 cases above, the quotient $\pi_1^{orb}(B)/\pi_1^{orb}(B')$ is a finite solvable group. We can decompose the covering c into a chain of coverings, so that covering group of the base covering is a cyclic group with prime order. So it is sufficient to prove the result when $\pi_1^{orb}(B)/\pi_1^{orb}(B')$ is a finite cyclic group with prime order d_b. The covering c can be further decomposed into a composition of a covering c_F induced by the covering of the general fibres and a covering c_B induced by the covering of the base. The covering c_F is a map S' to a Seifert manifold S_F say. Both manifolds have B' as their base, and the general fibre of S_F is F. Furthermore c_F induces the finite cover $\hat{c} : F' \to F$. The covering $\hat{c}$ induces an injective homomorphism $\pi_1(F') = \mathbb{Z}^2 \to \mathbb{Z}^2 = \pi_1(F)$, and so induces an injective homorphism $i_f : \mathbb{Q}^2 \to \mathbb{Q}^2$. Let d_f be the degree of i_f (i_f corresponds to an invertible matrix with integral entries, whose determinant is d_f). Then $e = i_f(e')$. Furthermore, since $\pi_1(F')$ has finite index in $\pi_1(F)$ then the monodromies of S' and S_F are essentially the same and are linked by conjugation by i_f. Therefore $V = i_f(V')$ and so $e + V = i_f(e' + V')$. Therefore $e' \in V'$ if and only if $e \in V$. [Note when considering Seifert 3-manifolds, $i_f(q) = d_f q$ for $q \in \mathbb{Q}$.]

It is sufficient to prove the rest of the result by assuming $c = c_B$, ie that the general fibres of S and S' are the same. The quotient $\pi_1^{orb}(B)/\pi_1^{orb}(B')$ is generated by $\bar{\xi}_g \pi_1^{orb}(B')$ for some $\xi_g \in \pi_1(S)$. As a result $V = V' + \mathrm{Im}(A(\xi_g) - \mathrm{w}(\bar{\xi}_g))$ and $\left(\sum_{i=0}^{d_b-1} (\mathrm{w}(\bar{\xi}_g) A(\xi_g))^i\right) V = \left(\sum_{i=0}^{d_b-1} (\mathrm{w}(\bar{\xi}_g) A(\xi_g))^i\right) V' \subset V'$ (since $\mathrm{Im}(A(\xi_g) - \mathrm{w}(\bar{\xi}_g)) = \ker \left(\sum_{i=0}^{d_b-1} (\mathrm{w}(\bar{\xi}_g) A(\xi_g))^i\right) \pmod{V'}$ by lemma 2.1.4).

Claim: for each of the 17 cases, $e' = \pm\left(\sum_{i=0}^{d_b-1}(\mathrm{w}(\bar{\xi}_g)A(\xi_g))^i\right)e \pmod{V'}$. So if $e \in V$ then

$$\begin{aligned} e' + V' &= \pm\left(\sum_{i=0}^{d_b-1}(\mathrm{w}(\bar{\xi}_g)A(\xi_g))^i\right)e + V' \\ &= \left(\sum_{i=0}^{d_b-1}(\mathrm{w}(\bar{\xi}_g)A(\xi_g))^i\right)(\pm e + V) \\ &= \left(\sum_{i=0}^{d_b-1}(\mathrm{w}(\bar{\xi}_g)A(\xi_g))^i\right)V \\ &= V'. \end{aligned}$$

Thus $e' \in V'$. Also

$$\begin{aligned} e' &= \pm\left(\sum_{i=0}^{d_b-1}(\mathrm{w}(\bar{\xi}_g)A(\xi_g))^i\right)e \pmod{V} \\ &= \pm d_b e \pmod{V} \end{aligned}$$

So conversely, if $e' \in V' \subset V$, then $e \in V$, hence the lemma. [In the general case $e' = \pm i_f^{-1} d_b e \pmod{V}$ (where d_b is not necessarily prime and where i_f corresponds to a matrix with determinant d_f). Compare this with orientable Seifert 3-manifolds. In that case $A(\xi) = \mathrm{w}(\xi)$ for all ξ and $V = 0$. Then by theorem 3.6 of [Sc2] $e' = d_f^{-1} d_b e$.]

The first case we need to prove the claim for is a covering where $B = B' = T^2$ and d_b is prime. Using the standard notation for the fundamental groups, $\pi_1(S)$ is generated by t, u, h_1 and h_2. Similarly, $\pi_1(S')$ is generated by t', u', h_1 and h_2. Then up to change of section, $t' = t^{q_{11}}u^{q_{12}}$ and $u' = t^{q_{21}}u^{q_{22}}$. We can apply a bundle isomorphism to either S or S' to modify the covering. So in particular we can change $Q = \left(\begin{smallmatrix} q_{11} & q_{12} \\ q_{21} & q_{22}\end{smallmatrix}\right)$ to $Q' = MQM'$ for any $M, M' \in GL_2\mathbb{Z}$ (note M (M') corresponds to a change of base bundle isomorphism of S (S' respectively)). By taking successive differences of rows (and swapping rows if necessary) we can suppose $q'_{11} = \gcd(q_{11}, q_{21})$ which is 1 or d_b since d_b prime, and $q'_{21} = 0$ (note this is essentially half of Euclid's algorithm). By swapping t and u if necessary, and premultipling Q' by $\left(\begin{smallmatrix} 1 & -q'_{12} \\ 0 & 1\end{smallmatrix}\right)$ we can suppose $t' = t^{d_b}$ and $u' = u$.

For this covering $e = \tilde{a}$, and $e' = \tilde{a}' = \sum_{i=0}^{d_b-1} A(t^{i-d_b}u^{-1})(\tilde{a})$. Also $V' = \mathrm{Im}(A(t') - I) + \mathrm{Im}(A(u') - I)$. Therefore $e' = (\sum_{i=0}^{d_b-1} A(t)^i)\tilde{a} = (\sum_{i=0}^{d_b-1} A(t)^i)e \pmod{V'}$, hence the claim.

There are 16 more cases we need to prove the claim for. The coverings are induced from the following coverings of the bases. They are specified in terms of the standard generators.

$T^2 \to Kb$	$t' = v_2 v_1, u' = v_2^2$
$T^2 \to A$	$t' = r_1 r_2, u' = \partial_1$
$T^2 \to Mb$	$t' = r_1 v_1, u' = \partial_1$
$T^2 \to S^2(2,2,2,2)$	$t' = s_1 s_3, u' = s_1 s_2$
$T^2 \to S^2(3,3,3)$	$t' = s_1^2 s_2, u' = s_1 s_2^2$
$S^2(2,2,2,2) \to P^2(2,2)$	$s_1' = s_1, s_2' = s_2, s_3' = v s_2^{-1} v^{-1}, s_4' = v s_1^{-1} v^{-1}$
$S^2(2,2,2,2) \to D^2(2,2)$	$s_1' = s_1, s_2' = s_2, s_3' = r s_2^{-1} r^{-1}, s_4' = r s_1^{-1} r^{-1}$
$S^2(2,2,2,2) \to D^2(2,\bar{2},\bar{2})$	$s_1' = s_1, s_2' = s_2, s_3' = s_3, s_4' = r s_1^{-1} r^{-1}$
$S^2(2,2,2,2) \to D^2(\bar{2},\bar{2},\bar{2},\bar{2})$	$s_i' = s_i$
$S^2(2,2,2,2) \to S^2(4,4,2)$	$s_1' = s_2 s_1, s_2' = s_2^2, s_3' = s_2^3 s_1^3, s_4' = s_1^2$
$S^2(4,4,2) \to D^2(4,\bar{2})$	$s_1' = r s_1^{-1} r^{-1}, s_2' = s_1, s_3' = s_2$
$S^2(4,4,2) \to D^2(\bar{4},\bar{4},\bar{2})$	$s_i' = s_i$
$S^2(3,3,3) \to D^2(3,\bar{3})$	$s_1' = s_1, s_2' = s_2, s_3' = r s_1^{-1} r^{-1}$
$S^2(3,3,3) \to D^2(\bar{3},\bar{3},\bar{3})$	$s_i' = s_i$
$S^2(3,3,3) \to S^2(6,3,2)$	$s_1' = s_1^2, s_2' = s_2, s_3' = s_3 s_2 s_3$
$S^2(6,3,2) \to D^2(\bar{6},\bar{3},\bar{2})$	$s_i' = s_i$

Apart from $T^2 \to S^2(3,3,3)$ (which has degree 3) all these are 2-fold covers and all are unique (up to covering homeomorphism).

To illustrate how to prove each of these cases, we will consider two examples: $S^2(2,2,2,2) \to D^2(\bar{2},\bar{2},\bar{2},\bar{2})$ and $T^2 \to S^2(3,3,3)$. The first example is representative of the cases with reflector curves (which utilise (2.3.8) and (2.3.11)) and the second is representative of the cases with only cone points (which utilise (2.3.7')).

For the covering $S^2(2,2,2,2) \to D^2(\bar{2},\bar{2},\bar{2},\bar{2})$ defined by $s_i' = s_i$; $e = \tilde{a} + \frac{1}{2}(\tilde{b} + \tilde{e}_1/2 + \tilde{e}_2/2 + \tilde{e}_3/2 + \tilde{e}_4/2)$ and $e' = \tilde{a}' + \tilde{e}_1'/2 + \tilde{e}_2'/2 + \tilde{e}_3'/2 + \tilde{e}_4'/2$. By direct calculation e' evaluates to be $(I - A(r_1))\tilde{a} + \tilde{b} + \tilde{e}_1/2 + \tilde{e}_2/2 + \tilde{e}_3/2 + \tilde{e}_4/2$. By (2.3.8), $\left(\sum_{j=0}^{m_i-1} A(s_i^j)\right)(\tilde{G}_i - \tilde{G}_{i+1}) + (A(\sigma_i r_1) + I)\tilde{e}_i = 0$. Considering this equation modulo V', we see $(A(r_1) + I)\tilde{e}_i/m_i = \tilde{G}_{i+1} - \tilde{G}_i \pmod{V'}$ (note $\tilde{G}_5 = 0$ since there are $k = 4$ corner reflectors). By (2.3.11), $(A(r_1)+I)\tilde{b} + (A(\partial^{-1}) - I)\tilde{f} = A(\partial^{-1})\tilde{G}_1$. Considering this equation modulo V', we see $(A(r_1) + I)\tilde{b} = \tilde{G}_1 \pmod{V'}$. By combining these equations, we see $\frac{1}{2}(A(r_1) + I)\left(\tilde{b} + \sum_{i=1}^{k} \tilde{e}_i/m_i\right) = 0 \pmod{V'}$. Therefore $e' = e' - \frac{1}{2}(A(r_1)+I)\left(\tilde{b} + \sum_{i=1}^{k} \tilde{e}_i/m_i\right) \pmod{V'}$, however this reduces to $(I - A(r_1))e$, hence the claim for this case.

For the covering $T^2 \to S^2(3,3,3)$ defined by $t' = s_1^2 s_2, u' = s_1 s_2^2$; $e = \tilde{a} + \tilde{e}_1/3 + \tilde{e}_2/3 + \tilde{e}_3/3$ and $e' = \tilde{a}'$. By direct calculation, $e' = A(s_1)(I + A(s_3) + A(s_3^2))\tilde{a} + A(s_1 s_2^{-1})\tilde{e}_1 + A(s_1)\tilde{e}_2 + A(s_1)\tilde{e}_3$. Since $A(t') = A(s_1 s_3^{-1})$ and $A(u') = A(s_3^{-1} s_2)$, $A(s_1)x = A(s_2)x = A(s_3)x \pmod{V'}$ for all x. Also by (2.3.7') $A(s_i)\tilde{e}_i = \tilde{e}_i$. So for example $\frac{1}{3}(I + A(s_3) + A(s_3^2))\tilde{e}_1 = \frac{1}{3}(I + A(s_1) + A(s_1^2))\tilde{e}_1 = \tilde{e}_1 = A(s_1 s_2^{-1})\tilde{e}_1 \pmod{V'}$. Therefore e'

modulo V' is $(I + A(s_3) + A(s_3^2))e$, hence the claim for this example. $\square$

The following result is useful when looking at the torus bundles over tori which cover Seifert manifolds with flat bases.

Lemma 4.5.10. *If $T^2 = F \to S \to B = T^2$ is a torus bundle over a torus, then there are generators t and u of $\pi_1^{orb}(B)$, so that $A(t) = \pm I$. If the group of monodromies is periodic (hyperbolic or parabolic) then $A(u)$ is periodic (hyperbolic or parabolic respectively).*

Proof. Choose generators t' and u' of $\pi_1^{orb}(B)$. Since the generators commute, so will their monodromies. So by lemma 4.5.3, there is a $C \in GL_2\mathbb{Z}$ so that $A(t') = \pm C^p$ and $A(u') = \pm C^q$ for some integers p and q. Let $d = \gcd(p, q)$ and let a and b be integers such that $ap + bq = d$ (found by Euclid's algorithm). By replacing C by C^d we can suppose $d = 1$. Let $t = (t')^q(u')^{-p}$ and $u = (t')^a(u')^b$. Firstly t and u generate $\pi_1^{orb}(B)$ since the change of basis matrix $\left(\begin{smallmatrix} q & a \\ -p & b \end{smallmatrix}\right)$ has determinant 1. Secondly $A(t) = A(t')^q A(u')^{-p} = \pm I$. Note $A(u) = A(t')^a A(u')^b = \pm C^{ap+bq} = \pm C$. By changing C to $-C$ if necessary, we can suppose $A(u) = C$.

The group generated by C is at most index two inside the group of monodromies and so has the same type. Therefore $C = A(u)$ has the same type. $\square$

The following lemma proves theorem 4.5.5 for the $\mathbb{E}^4$ case.

Lemma 4.5.11. *If S is a $\mathbb{E}^4$ manifold then S is a Seifert manifold with Euclidean base, excepting three cases. Furthermore when the general fibre is T^2, the group of monodromies is finite and $e \in V$. Conversely if S is a Seifert manifold with a flat base, general fibre T^2, finite group of monodromies and $e \in V$ then S is geometric of type $\mathbb{E}^4$.*

Proof. In propostion 4.5.4 it is proved that all but three $\mathbb{E}^4$-manifolds are Seifert fibred. Consider a $\mathbb{E}^4$ manifold, S, which is Seifert fibred. Since S is geometric of type $\mathbb{E}^4$, it is finitely covered by T^4. Let e' and V' be the invariants of the fibration of T^4 which covers S. Then $e' = 0 \in \{0\} = V'$. If the general fibre of S is T^2, then by lemma 4.5.9, $e \in V$ and the group of monodromies is finite (since the group of monodromies of T^4 is trivial).

Conversely, suppose S is a Seifert manifold with flat base, general fibre T^2, finite group of monodromies and $e \in V$. Then analogously to the $\mathbb{H}^2 \times \mathbb{E}^2$ case of theorem 3.4.1, we get the following commutative diagram, where the columns are injections:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1^{orb}(B) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathrm{Isom}(\mathbb{E}^2) & \longrightarrow & \mathrm{Isom}(\mathbb{E}^2) \times \mathrm{Isom}(\mathbb{E}^2) & \longrightarrow & \mathrm{Isom}(\mathbb{E}^2) & \longrightarrow & 1.
\end{array}$$

Since $\mathrm{Isom}(\mathbb{E}^2) \times \mathrm{Isom}(\mathbb{E}^2)$ is a subgroup of $\mathrm{Isom}(\mathbb{E}^4)$, theorem 2.5.10 implies S is homeomorphic to a geometric manifold of type $\mathbb{E}^4$ (where the homeomorphism preserves the fibration). $\square$

The following lemma proves theorem 4.5.5 for the $\mathbb{N}il^3 \times \mathbb{E}$ cases.

Lemma 4.5.12. *If S is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ then S is a Seifert manifold over a Euclidean base. Furthermore when the fibre is T^2, either the group of monodromies is finite and $e \notin V$ or the group of monodromies is parabolic and $e \in V$. Conversely, if S is a Seifert manifold with flat base and T^2 fibre and either the group of monodromies is finite and $e \notin V$ or the group of monodromies is parabolic and $e \in V$ then S is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$.*

Proof. If S is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$, then lemma 4.1.4 shows that S is a Seifert manifold with flat base which is one of at least two types. Suppose the fibre is T^2.

The first type has a finite group of monodromies. Suppose (aiming for a contradiction) that $e \in V$. Then by the previous lemma S would be geometric of type $\mathbb{E}^4$ which gives the desired contradiction (since a homeomorphism of geometric manifolds implies the geometries are the same by theorem 10.1 in [Wa2]). Conversely, if S is a Seifert manifold over a flat base, general fibre T^2 and finite group of monodromies, then using the same argument as the $\widetilde{SL}_2 \times \mathbb{E}$ case of theorem 3.4.1, S is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ (see section 4.1 for details of the isometries of $\mathbb{N}il^3 \times \mathbb{E}$, and compare these with the isometries of $\widetilde{SL}_2 \times \mathbb{E}$ given in section 3.2). (The correction term, c in the $\widetilde{SL}_2 \times \mathbb{E}$ case of theorem 3.4.1 was the euler characteristic of the base, however the analogous correction term for the $\mathbb{N}il^3 \times \mathbb{E}$, does not appear to have such a nice interpretation at first glance. However it does not matter, since the geometries $\mathbb{E}^4$ and $\mathbb{N}il^3 \times \mathbb{E}$ are mutually exclusive, the correction term must be non-zero).

The second type (of lemma 4.1.4) has a parabolic group of monodromies. The following lemma proves the result for this type. □

The following lemma and corollary proves the theorem 4.5.5 for the $\mathbb{N}il^3 \times \mathbb{E}$ case when the group of monodromies is parabolic.

Lemma 4.5.13. *Let S be as in the theorem and suppose the group of monodromies is parabolic.*

Let $\mathcal{B} = \{\beta \in \pi_1(S) |\, \mathrm{w}(\beta) = \mathrm{w}(\bar{\beta}) = 1, \mathrm{tr}\, A(\beta) = 2\}$ and let $H' = \{\xi \in \pi_1(S) |\, \mathrm{w}(\xi) = 1, A(\xi) = I$ and $[\beta, \xi] = 1 \quad \forall \beta \in \mathcal{B}\}$. Then H' is a free abelian normal subgroup of $\pi_1(S)$.

The group H' is isomorphic to $\mathbb{Z}$ if and only if $e \notin V$.

The group H' is isomorphic to $\mathbb{Z}^2$ if and only if $e \in V$.

Furthermore when $e \in V$, there is a $\mathbb{N}il^3 \times \mathbb{E}$ manifold, $F' \longrightarrow S' \longrightarrow B'$, with a fundamental group sequence isomorphic to

$$1 \longrightarrow H' \longrightarrow \pi_1(S) \longrightarrow \pi_1(S)/H' \longrightarrow 1.$$

Proof. Firstly, we claim $\mathcal{B}$ is a normal subgroup of finite index in $\pi_1(S)$. Since the group of monodromies of S is parabolic, $\mathrm{Im}\, A$ is an infinite subgroup of $P\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & (1) \\ 0 & -1 \end{smallmatrix}\right), -I\rangle P^{-1}$, for some $P \in GL_2\mathbb{Z}$. Take the orientation cover of S, and then take the cover induced by the orientation cover of the base. Call this cover S'. Then $\pi_1(S') = \{\beta \in \pi_1(S) |\, \mathrm{w}(\beta) = \mathrm{w}(\bar{\beta}) = 1\}$ whose group of monodromies is an infinite subgroup of

$P\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), -I\rangle P^{-1}$. We can take a further cover so that $\operatorname{tr} A(\beta) = 2$ to get $\mathcal{B}$, thus proving the claim. Also we have shown for all $\beta \in \mathcal{B}$, $A(\beta) = P\left(\begin{smallmatrix}1&\lambda_\beta\\0&1\end{smallmatrix}\right)P^{-1}$ for some (possibly zero) integer λ_β, with not all β having trivial monodromies [otherwise $\operatorname{Im} A$ would be finite].

It is clear that H' is an abelian subgroup that lies in the centre of $\mathcal{B}$. Let us show it is normal. Suppose $\alpha \in \pi_1(S)$ and $\xi \in H'$. Then $\mathrm{w}(\alpha\xi\alpha^{-1}) = \mathrm{w}(\xi) = 1$ and $A(\alpha\xi\alpha^{-1}) = I$. Suppose $\beta \in \mathcal{B}$, then $\alpha^{-1}\beta\alpha \in \mathcal{B}$ since $\mathcal{B}$ is normal. Therefore $\beta(\alpha\xi\alpha^{-1})\beta^{-1} = \alpha((\alpha^{-1}\beta\alpha)\xi(\alpha^{-1}\beta\alpha)^{-1})\alpha^{-1} = \alpha\xi\alpha^{-1}$, and so H' is normal.

Now $\pi_1(S)$ is torsion free (since S is a manifold), therefore H' is free abelian normal subgroup of $\pi_1(S)$.

Choose a $\beta \in \mathcal{B}$ with non-trivial monodromy. Then as mentioned above, $A(\beta) = P\left(\begin{smallmatrix}1&\lambda_\beta\\0&1\end{smallmatrix}\right)P^{-1}$ for some nonzero λ_β. Since $P \in GL_2\mathbb{Z}$ we can choose generators g and h for $\pi_1(F)$ so that $A(\beta)(h) = h$ and $A(\beta)(g) = h^{\lambda_\beta}g$. Then this formula will be true for all $\beta \in \mathcal{B}$. Furthermore, the group $\langle h\rangle$ is the subgroup of $\pi_1(F)$ which commutes with all $\beta \in \mathcal{B}$. Suppose $\xi \in \pi_1(F)$. Then $\mathrm{w}(\xi) = 1$ and $A(\xi) = I$. If ξ is in H' as well, then it must commute with all elements of $\mathcal{B}$ and so lies in $\langle h\rangle$. Therefore $\pi_1(F) \cap H'$ is isomorphic to $\mathbb{Z}$.

Next, let us find all the ξ in H' which are not in $\pi_1(F)$. The base of S is Euclidean, therefore $\pi_1^{orb}(B)$ has a free abelian normal subgroup of rank 2 with finite index. According to lemma 4.5.10, this subgroup has generators t and u so that $A(t) = \pm I$ and $A(u)$ is parabolic. By taking a normal subgroup of index 2 if necessary, we can suppose $A(t)$ and $A(u)$ have positive trace. Therefore $A(t) = I$ and $A(u) = P\left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right)P^{-1}$ for some nonzero λ.

Consider $\beta \in \mathcal{B}$ and recall $A(\beta) = P\left(\begin{smallmatrix}1&\lambda_\beta\\0&1\end{smallmatrix}\right)P^{-1}$. Since $\langle\bar{t},\bar{u}\rangle$ is normal in $\pi_1^{orb}(B)$, $\beta t\beta^{-1} = t^{b_{11}}u^{b_{12}}h_t$ and $\beta u\beta^{-1} = t^{b_{21}}u^{b_{22}}h_u$ for some integers b_{ij} and some $h_t, h_u \in \pi_1(F)$. By considering the monodromies of both sides we see $b_{12} = 0$ and $b_{22} = 1$. This implies $b_{11} = 1$ since β induces an orientation preserving isomorphism of $\langle\bar{t},\bar{u}\rangle$.

The group $\langle\bar{t},\bar{u}\rangle$ has finite index in $\pi_1^{orb}(B)$. Therefore $\bar{\beta}$ acts on this group by an isomorphism with finite order. This implies $b_{21} = 0$. Consequently, $\bar{\beta}$ commutes with $\bar{t}$ and $\bar{u}$ and so is in the maximal abelian normal subgroup of $\pi_1^{orb}(B)$. Since $\operatorname{tr} A(\beta) = 2$, $\bar{\beta} \in \langle\bar{t},\bar{u}\rangle$ by definition. Therefore $\beta = t^p u^q h_\beta$, for some integers p and q and some $h_\beta \in \pi_1(F)$. Note $\lambda_\beta = q\lambda$.

The relation $[\bar{t},\bar{u}] = 1$ in $\pi_1^{orb}(B)$ lifts to $[t,u] = h_a$ for some $h_a \in \pi_1(F)$. Suppose $\xi = t^c u^d h^e g^f$ is in H'. Then all $\beta \in \mathcal{B}$ would commute with ξ, in particular t, u, h and g would commute with ξ. However h does for all values, and g does so long as $d = 0$. Since $d = 0$, t also commutes with ξ. Lastly, u commutes with ξ if and only if $h_a^c = h^{\lambda f}$.

Suppose $h_a \in \langle h\rangle$, ie $h_a = h^a$ for some integer a. Then $ac = \lambda f$. Let $\delta = \gcd(\lambda, a)$, then this has solutions $f = (a/\delta)i$ and $c = (\lambda/\delta)i$ for all integers i. Therefore in this case, $H' = \{\left(t^{\frac{\lambda}{\delta}}g^{\frac{a}{\delta}}\right)^i h^e | i,e \in \mathbb{Z}\} \cong \mathbb{Z}^2$. Instead suppose $h_a \notin \langle h\rangle$. Then c and f are both forced to be 0. Therefore $H' = H' \cap \pi_1(F) = \langle h\rangle$.

Lemma 4.5.9 implies $e \in V$ if and only if $h_a \in \operatorname{Im}(P\left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right)P^{-1} - I) = \langle h\rangle$, and hence $H' \cong \mathbb{Z}^2$ if and only if $e \in V$ and $H' \cong \mathbb{Z}$ if and only if $e \notin V$.

By definition of H', every element of $\mathcal{B}$ commutes with each element of H'. Therefore each element of $\mathcal{B}$ acts on H' by conjugation trivially. Since $\pi_1(S)$ is a finite extension of $\mathcal{B}$, the action of $\pi_1(S)$ on H' must be via a finite extension and thus $\pi_1(S)$ acts on H' by a finite group [In fact it can be shown using lemma 4.1.6, $\pi_1(S)$ acts on H' via diagonal matrices].

Suppose $e \in V$. We have so far shown $\pi_1(S)$ satisfies the following exact sequence: $H' \to \pi_1(S) \to \pi_1(S)/H'$, where $H' \cong \pi_1(T^2)$ and the group of monodromies of the sequence is finite. The maximal abelian normal subgroup of $\pi_1(S)$ is generated by g, h and t (ie it is the group generated by the union of $\pi_1(F)$ and H') and is rank 3 free abelian. The manifold S thus cannot be Euclidean, since it is not covered by a 4-torus. So for the new exact sequence involving H', $e' \notin V'$. Then analogously to the $\widetilde{SL}_2 \times \mathbb{E}$ case, $\pi_1(S)$ is isomorphic to the fundamental group of a $\mathbb{N}il^3 \times \mathbb{E}$ manifold, hence the lemma. □

Remark 4.5.14. The exact sequence:

$$1 \to H \to \mathcal{B} \to \mathcal{B}/H \to 1,$$

is isomorphic to the fundamental group sequence of a T^2-bundle over T^2 with a parabolic group of monodromies.

Also when $e \in V$, the exact sequence:

$$1 \to H' \to \mathcal{B} \to \mathcal{B}/H' \to 1,$$

is isomorphic to the fundamental group sequence of a T^2-bundle over T^2 however the group of monodromies is trivial.

Corollary 4.5.15. *Let S be a Seifert manifold as in theorem 4.5.5 and suppose the group of monodromies is parabolic and $e \in V$. Then S is diffeomorphic to a manifold S' which is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ and hence S is geometric.*

Conversely if S is a Seifert manifold which is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ and has a parabolic fibration then $e \in V$.

Proof. Suppose S is a parabolic group of monodromies and $e \in V$. Then from the lemma, there is a $\mathbb{N}il^3 \times \mathbb{E}$-manifold S' so that $\pi_1(S')$ is isomorphic to $\pi_1(S)$. This isomorphism is then realised by a diffeomorphism (theorem C of [Ue1]).

Conversely suppose S is a Seifert manifold which is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ and has a parabolic fibration (see section 4.1 for details). By lemma 4.1.5, H' is isomorphic to $\mathbb{Z}^2$. The previous lemma then implies $e \in V$. □

Remark 4.5.16. The diffeomorphism in the corollary may not preserve the bundle structure.

The following lemma proves theorem 4.5.5 for the $\mathbb{S}ol^3 \times \mathbb{E}$ case.

Lemma 4.5.17. *The manifold S is a $\mathbb{S}ol^3 \times \mathbb{E}$ manifold if and only if S is a Seifert manifold with flat base, torus general fibre and hyperbolic group of monodromies.*

Proof. Proposition 4.2.2 proves the necessity of the lemma. So we need to prove the sufficiency.

In this case $V = \mathbb{R}^2$, and so $e \in V$. In many ways it is analogous to the $\mathbb{H}^2 \times \mathbb{E}^2$ case. Indeed the proof is the same except the representation of $\pi_1^{orb}(B)$ as a subgroup of $\mathrm{Isom}(\mathbb{E}^2)$ needs to be chosen carefully (for the $\mathbb{H}^2 \times \mathbb{E}^2$ case, the representation was arbitrary).

The basic strategy is to find a representation for the maximal normal abelian subgroup of $\pi_1^{orb}(B)$. Then extend it for the rest (this involves possibly two more choices).

Since B is flat, $\pi_1^{orb}(B)$ contains a rank 2 free abelian normal subgroup of finite index. According to lemma 4.5.10 we can choose generators $\bar{t}$ and $\bar{u}$ of this group so that $A(\bar{t}) = \pm I$ and $A(\bar{u})$ is hyperbolic.

Since the group of monodromies is hyperbolic, there is a matrix $P \in GL_2\mathbb{R}$ so that $PA(\bar{u})P^{-1} = \left(\begin{smallmatrix} \epsilon e^{\tau} & 0 \\ 0 & \delta e^{-\tau} \end{smallmatrix}\right)$ for some $\epsilon, \delta = \pm 1$ and some real number τ [note $\epsilon e^{\tau} + \delta e^{-\tau} \in \mathbb{Z}$]. Consider $\bar{\alpha} \in \pi_1^{orb}(B)$. Then since $\langle \bar{t}, \bar{u} \rangle$ is normal, $\bar{\alpha}\bar{t}\bar{\alpha}^{-1} = \bar{t}^{b_{11}}\bar{u}^{b_{12}}$ and $\bar{\alpha}\bar{u}\bar{\alpha}^{-1} = \bar{t}^{b_{21}}\bar{u}^{b_{22}}$ for some integers b_{ij}. By considering monodromies $b_{12} = 0$. Since $\mathrm{Im}\, A$ is hyperbolic, either $A(\bar{\alpha}) = P^{-1}\left(\begin{smallmatrix} ? & 0 \\ 0 & ? \end{smallmatrix}\right)P$ or $A(\bar{\alpha}) = P^{-1}\left(\begin{smallmatrix} 0 & ? \\ ? & 0 \end{smallmatrix}\right)P$ (see the definition at the beginning of this section). In the former case, $A(\bar{\alpha})$ commutes with $A(\bar{u})$ and so $b_{22} = 1$. In the latter case, $A(\bar{\alpha})A(\bar{u})A(\bar{\alpha})^{-1} = \det(A(\bar{u}))A(\bar{u})^{-1}$ and so $b_{22} = -1$. If $A(\bar{t}) = -I$ then b_{21} is even in the first case and $(-1)^{b_{21}} = \det(A(\bar{u}))$ in the second. Note if $A(\bar{t}) = I$ and $\det(A(\bar{u})) = -1$, then b_{22} must equal 1. Since $\bar{\alpha}$ induces an invertible action, $b_{11} = \pm 1$. Next $\bar{\alpha}^2\bar{t}\bar{\alpha}^{-2} = \bar{t}$ and $\bar{\alpha}^2\bar{u}\bar{\alpha}^{-2} = \bar{t}^{b_{21}(b_{11}+b_{22})}\bar{u}$. Since $\langle \bar{t}, \bar{u} \rangle$ has finite index in $\pi_1^{orb}(B)$, $b_{21}(b_{11} + b_{22}) = 0$, that is either $b_{21} = 0$ or $b_{11} = -b_{22}$. Note $\mathrm{w}(\bar{\alpha}) = b_{11}b_{22}$. Therefore for orientation preserving maps $b_{21} = 0$. All orientation reversing maps differ by a map which preserves orientation, thus they all have the same value for $b_{22}b_{21}$. By replacing $\bar{u}$ by $\bar{u}\bar{t}^p$ for some integer p, we can change the value of $b_{22}b_{21}$ to $b_{22}b_{21} + b_{22}(b_{11} - b_{22})p$ which equals $b_{22}b_{21} - 2p$ for the orientation reversing maps. Therefore we can suppose $b_{22}b_{21}$ is 0 or 1.

We shall now give a representation of $\langle \bar{t}, \bar{u} \rangle$ as a subgroup of $\mathrm{Isom}(\mathbb{E}^2)$. If $b_{21} = 0$ for all $\bar{\alpha}$, let $\bar{t} = \left(I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$ and $\bar{u} = \left(I, \left(\begin{smallmatrix} \tau \\ 0 \end{smallmatrix}\right)\right)$ (where τ is defined with $A(\bar{u})$ above). If $b_{22}b_{21} = 1$ for some element, let $\bar{t} = \left(I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$ and $\bar{u} = \left(I, \left(\begin{smallmatrix} \tau \\ -1/2 \end{smallmatrix}\right)\right)$. [Note: see claim 7 in section 7 of [Ue1] of an example of when b_{21} can be nonzero. The example has the Mobius band as the base.]

Choose an element $\bar{\alpha}_=$ (if one exists) such that $b_{11} = b_{22} = -1$. Then $A(\bar{\alpha}_=) = P^{-1}\left(\begin{smallmatrix} 0 & \epsilon e^{\tau_=} \\ \delta e^{-\tau_=} & 0 \end{smallmatrix}\right)P$ for some ϵ, δ and $\tau_=$. Let $\bar{\alpha}_= = \left(-I, \left(\begin{smallmatrix} \tau_= \\ 0 \end{smallmatrix}\right)\right)$. [We claim that if $\bar{\alpha}_=$ exists, then $\bar{\alpha}_=^2 = 1$. To prove this, notice $\bar{\alpha}_=^2 = \bar{u}^a\bar{t}^b$. Conjugating by $\bar{\alpha}_=$ fixes the left hand side, but inverses the right hand side. Since $\langle \bar{u}, \bar{t} \rangle$ is torsion free, $\bar{u}^a\bar{t}^b = 1$, hence the claim.]

Choose an element $\bar{\alpha}_\mp$ (if one exists) such that $b_{11} = 1$ and $b_{22} = -1$. Then $A(\bar{\alpha}_\mp) = P^{-1}\begin{pmatrix} 0 & \epsilon e^{\tau_\mp} \\ \delta e^{-\tau_\mp} & 0 \end{pmatrix}P$ for some ϵ, δ and $\tau_\mp$. Also $\bar{\alpha}_\mp^2 = \bar{t}^q$ for some integer q. Let $\bar{\alpha}_\mp = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \tau_\mp \\ q/2 \end{smallmatrix}\right)\right)$.

If there was a $\bar{\alpha}_=$ and a $\bar{\alpha}_\mp$, let $\bar{\alpha}_\pm = \bar{\alpha}_\mp\bar{\alpha}_=$ and the action is already defined. Otherwise, choose an element $\bar{\alpha}_\pm$ (if one exists) such that $b_{11} = -1$ and $b_{22} = 1$. Then $A(\bar{\alpha}_\pm) = P^{-1}\begin{pmatrix} \epsilon e^{\tau_\pm} & 0 \\ 0 & \delta e^{-\tau_\pm} \end{pmatrix}P$ for some ϵ, δ and $\tau_\pm$. Let $\bar{\alpha}_\pm = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \tau_\pm \\ 0 \end{smallmatrix}\right)\right)$.

This is sufficient to define the representation for all of $\pi_1^{orb}(B)$. For instance, suppose $\bar{\alpha} \in \pi_1^{orb}(B)$. If

$b_{11} = b_{22} = 1$ then $\bar{\alpha} = \bar{u}^a\bar{t}^b$ for some integers a, b. Otherwise $\bar{\alpha}$ is the composition of $\bar{u}^a\bar{t}^b$ (for some integers a, b) with either $\bar{\alpha}_=$, $\bar{\alpha}_\pm$ or $\bar{\alpha}_\mp$ such that the values of b_{11} and b_{22} are the same.

Having obtained this representation, the proof for the rest of this result follows virtually the same argument as the $\mathbb{H}^2 \times \mathbb{E}^2$ case in theorem 3.4.1. □

To illustrate what is happening in the lemma, let us consider an example. Let S be a Seifert manifold over $B = D^2(2,2)$. Then $\pi_1^{orb}(B) = \langle \bar{r}, \bar{\partial}, \bar{s}_1, \bar{s}_2 | \bar{\partial}^{-1}\bar{r}\bar{\partial}\bar{r}^{-1} = 1, \bar{r}^2 = \bar{s}_i^2 = 1, \bar{s}_1\bar{s}_2\bar{\partial} = 1\rangle$. The maximal normal abelian subgroup, N is generated by $\bar{s}_1\bar{s}_2$ and $(\bar{r}\bar{s}_1)^2$. The quotient $\pi_1^{orb}(B)/N$ is isomorphic to $\mathbb{Z}_2^2$ and is generated by the images of $\bar{s}_1$ and $\bar{r}$. Suppose the monodromies of S are given by $A(\bar{s}_1) = \left(\begin{smallmatrix} 2 & -1 \\ 3 & -2 \end{smallmatrix}\right)$, $A(\bar{s}_2) = \left(\begin{smallmatrix} 2 & 1 \\ -3 & -2 \end{smallmatrix}\right)$, $A(\bar{\partial}) = A(\bar{s}_1\bar{s}_2)^{-1} = \left(\begin{smallmatrix} 7 & 4 \\ 12 & 7 \end{smallmatrix}\right)^{-1}$ and $A(\bar{r}) = I$. Then we shall use the above method to find a representation of $\pi_1^{orb}(B)$ as a subgroup of $\mathrm{Isom}(\mathbb{E}^2)$.

Firstly $\bar{t} = (\bar{r}\bar{s}_1)^2$ and $\bar{u} = \bar{s}_1\bar{s}_2$ generate N so that $A(\bar{t}) = I$ and $A(\bar{u}) = \left(\begin{smallmatrix} 7 & 4 \\ 12 & 7 \end{smallmatrix}\right)$ is hyperbolic. Next there is a $P = \left(\begin{smallmatrix} 1 & -1/\sqrt{3} \\ 1 & 1/\sqrt{3} \end{smallmatrix}\right)$ such that $PA(\bar{u})P^{-1} = \left(\begin{smallmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{smallmatrix}\right)$, where $\tau = \ln(7 - 4\sqrt{3}) \approx -2.63$. Note $PA(\bar{s}_1)P^{-1} = \left(\begin{smallmatrix} 0 & e^{\frac{\tau}{2}} \\ e^{\frac{-\tau}{2}} & 0 \end{smallmatrix}\right)$.

Next, $\bar{s}_1\bar{t}\bar{s}_1^{-1} = \bar{t}^{-1}$ and $\bar{s}_1\bar{u}\bar{s}_1^{-1} = \bar{u}^{-1}$. Also $\bar{r}\bar{t}\bar{r}^{-1} = \bar{t}^{-1}$ and $\bar{r}\bar{u}\bar{r}^{-1} = \bar{u}$. Therefore $(\bar{r}\bar{s}_1)\bar{t}(\bar{r}\bar{s}_1)^{-1} = \bar{t}$ and $(\bar{r}\bar{s}_1)\bar{u}(\bar{r}\bar{s}_1)^{-1} = \bar{u}^{-1}$. We also have the relations, $\bar{s}_1^2 = \bar{r}^2 = 1$ and $(\bar{r}\bar{s}_1)^2 = \bar{t}$.

Then we define the representation by $\bar{t} = \left(I, \binom{0}{1}\right)$ and $\bar{u} = \left(I, \binom{\tau}{0}\right)$. Let $\bar{\alpha}_= = \bar{s}_1 = \left(-I, \binom{\frac{\tau}{2}}{0}\right)$ and let $\bar{\alpha}_\mp = \bar{r}\bar{s}_1 = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \binom{\frac{\tau}{2}}{1/2}\right)$. Then $\bar{\alpha}_\pm = \bar{r} = \bar{\alpha}_\mp\bar{\alpha}_= = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \binom{0}{1/2}\right)$ and this defines the representation for all elements.

Next, we will follow the argument as the $\mathbb{H}^2 \times \mathbb{E}^2$ case in theorem 3.4.1 to find a representation of our example Seifert fibration as a $\mathbb{S}ol^3 \times \mathbb{E}$ manifold. Let $\tilde{e}_{0j}$, $\tilde{f}$, $\tilde{b}$ and $\tilde{a}$ be the standard parts of the presentation. We give a representation of $\pi_1(S)$ by $h_1\left(\binom{x}{y}, \binom{z}{t}\right) = \left(\binom{x}{y} + P\binom{1}{0}, \binom{z}{t}\right)$, $h_2\left(\binom{x}{y}, \binom{z}{t}\right) = \left(\binom{x}{y} + P\binom{0}{1}, \binom{z}{t}\right)$ and for all other generators $\xi\left(\binom{x}{y}, \binom{z}{t}\right) = \left(PA(\xi)P^{-1}\binom{x}{y} + Py(\xi), \bar{\xi}\binom{z}{t}\right)$, where $y(\xi)$ is arbitrary (for the moment) and $\bar{\xi}$ is the representation we found above. The $\bar{\xi}$ were chosen so that these maps would be isometries. For example, $s_1\left(\binom{x}{y}, \binom{z}{t}\right) = \left(\left(\begin{smallmatrix} 0 & e^{\frac{\tau}{2}} \\ e^{\frac{-\tau}{2}} & 0 \end{smallmatrix}\right)\binom{x}{y} + Py(s_1), \left(-I, \binom{\frac{\tau}{2}}{0}\right)\binom{z}{t}\right)$ which is an isometry of $\mathbb{S}ol^3 \times \mathbb{E}$.

In order to specify the representation fully, we need to find values for the $y(\xi)$. Analogously to equation (3.3.2), $y(s_j) = -\tilde{e}_{0j}/2 + (A(s_j) - I)z(s_j)$ for some $z(s_j)$. Similarly $y(r) = \tilde{f}/2 + (A(r) - I)z(r)$ for some $z(r)$. Analogously to lemma 3.4.2, $y(\partial) = -\frac{1}{2}\tilde{b} + (A(\partial) - I)z(\partial) + (A(r) + I)z'(r)$ for some $z(\partial)$ and $z'(r)$. Note in this example $(A(\partial) - I) = \left(\begin{smallmatrix} 6 & -4 \\ -12 & 6 \end{smallmatrix}\right)$ is invertible, therefore by changing the value of $z(\partial)$ we can suppose $z'(r) = 0$. [Indeed, since $A(r) = I$, $y(\partial)$ has no constraints from the relation $\partial^{-1}r\partial r^{-1}$, however to show the connection to the $\mathbb{H}^2 \times \mathbb{E}^2$ case, we will not use this explicitly.] From the relation $s_1s_2\partial = \tilde{a}$, we see $y(s_1) + A(s_1)y(s_2) + A(s_1s_2)y(\partial) = \tilde{a}$. This reduces to $e = \tilde{a} + \tilde{e}_{01}/2 + \tilde{e}_{02}/2 + \tilde{b}/2 = (A(s_1) - I)(z(s_1) - \tilde{b}/2 - \tilde{e}_{02}/2) + (A(s_1s_2s_1^{-1}) - I)A(s_1)(z(s_2) - \tilde{b}/2) + (A(s_1s_2\partial s_2^{-1}s_1^{-1}) - I)A(s_1s_2)z(\partial)$. Let $z(s_1) = \tilde{b}/2 + \tilde{e}_{02}/2$, $z(s_2) = \tilde{b}/2$ and $z(\partial) = (A(\partial) - I)^{-1}A(s_1s_2)^{-1}e$. This then defines the representation fully and the example is finished.

The following lemma proves theorem 4.5.5 for the $\mathbb{N}il^4$ case.

Lemma 4.5.18. *The manifold S is a $\mathbb{N}il^4$ manifold if and only if S is a Seifert manifold with flat base, torus general fibre and parabolic group of monodromies such that $e \notin V$.*

Proof (Sketch). Like the $\mathbb{S}ol^3 \times \mathbb{E}$ case, choose a representation of $\pi_1^{orb}(B)$ as a subgroup of $\mathrm{Isom}(\mathbb{E}^2)$ in a special way. [Basically if $A(\bar{\alpha}) = P\left(\begin{smallmatrix} \epsilon & \epsilon\eta\tau \\ 0 & \epsilon\eta \end{smallmatrix}\right)P^{-1}$, let $\bar{\alpha} = \left(\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \eta \end{smallmatrix}\right), \left(\begin{smallmatrix} ? \\ \tau \end{smallmatrix}\right)\right)$. Note $\epsilon = \mathrm{w}(\alpha)$, $\epsilon\eta = \mathrm{w}(\bar{\alpha})$ and $\eta = \det A(\alpha)$, these conditions are forced by $e \notin V$]. Under this representation, you can choose α to act via the map $(\binom{x}{y}, \binom{z}{t}) \mapsto (P^{-1}A(\alpha)P\binom{x}{y} + \binom{x'}{y'}, \bar{\alpha}\binom{z}{t})$ plus a correction term which is determined by $\bar{\alpha}$. Then the rest of the proof follows the same lines as the $\widetilde{SL}_2 \times \mathbb{E}$ case of theorem 3.4.1. In the $\widetilde{SL}_2 \times \mathbb{E}$ case, there was a correction term ($P^{-1}\binom{0}{c}$ as in equation (3.4.9)) which corresponded to the total holonomy of the base and thus was nonzero because the base was hyperbolic. For the $\mathbb{N}il^4$ case, we will again derive an equation:

$$e - P^{-1}\tilde{k} \in V.$$

Suppose, aiming for a contradiction, that $P^{-1}\tilde{k} \in V$. Then there is a $\mathbb{N}il^4$ Seifert manifold with parabolic group of monodromies such that $e \in V$. However a Seifert manifold with such properties is geometric of type $\mathbb{N}il^3 \times \mathbb{E}$ by lemma 4.5.12. This provides the contradiction, since a closed manifold cannot be geometric of more than one type. Therefore $P^{-1}\tilde{k} \notin V$ and so $e \notin V$.

The other significant deviation from the $\widetilde{SL}_2 \times \mathbb{E}$ proof is at the end of lemma 3.4.2. When proving the sufficiency of the result, we need to show that it is reversible. The following paragraph shows how we argue this is possible in this case.

Since the group of monodromies is parabolic for all $\alpha \in \pi_1(S)$, $A(\alpha) = P\left(\begin{smallmatrix} \epsilon_\alpha & \epsilon_\alpha\eta_\alpha\tau_\alpha \\ 0 & \epsilon_\alpha\eta_\alpha \end{smallmatrix}\right)P^{-1}$ for some $\epsilon_\alpha, \eta_\alpha = \pm 1$ and some $\tau_\alpha \in \mathbb{Z}$. Since $e \notin V$, $V \neq \mathbb{R}^2$ and so $\epsilon_\alpha\eta_\alpha = \mathrm{w}(\bar{\alpha})$. Since the group of monodromies is infinite, if $\bar{\alpha}$ has finite order, then it has order 2 (lemma 4.1.6), therefore $A(\alpha)$ is either I or conjugate in $GL_2\mathbb{Z}$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ by lemma 2.1.7. So $A(r) = P\left(\begin{smallmatrix} 1 & -\tau_r \\ 0 & -1 \end{smallmatrix}\right)P^{-1}$ and $A(s_j) = P\left(\begin{smallmatrix} \epsilon_{s_j} & \tau_{s_j} \\ 0 & 1 \end{smallmatrix}\right)P^{-1}$. Therefore $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \subset P\{\binom{x}{0} | x \in \mathbb{R}\} = \mathrm{Im}(A(r) + I) \subseteq \mathrm{Im}(A(r) + I) + \mathrm{Im}(A(\partial) - I)$. The result then follows as in the original lemma 3.4.2 (this is one of the cases it considers). $\square$