Chapter 3

Geometric Seifert manifolds with hyperbolic bases

In this chapter we will give necessary and sufficient conditions for Seifert 4-manifolds with hyperbolic bases to be geometric. We will introduce the two possible geometries that arise ($\mathbb{H}^2 \times \mathbb{E}^n$ in section 3.1 and $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ in section 3.2). Then we will develop our conditions over three sections. The first of these sections (section 3.3) we will consider Seifert manifolds with T^2 general fibres and no reflectors in their hyperbolic base. We then extend the result to include reflector curves (section 3.4) and Kb fibres (section 3.5). To complete the chapter we summarise our results by proving that a virtually geometric Seifert 4-manifold with hyperbolic base is geometric (section 3.6).

3.1 The geometries $\mathbb{H}^2 \times \mathbb{E}^n$

In this section we look at the geometries $\mathbb{H}^2 \times \mathbb{E}^n$ ($n \geq 1$). Our prime interest will be when $n = 2$. We will prove that closed manifolds with these geometries are Seifert fibred.

The model space for the geometry $\mathbb{H}^2 \times \mathbb{E}^n$ is $\mathbb{H}^2 \times \mathbb{R}^n$. It is also useful to consider $\mathbb{H}^2 \times \mathbb{E}^2$ as $\mathbb{H}^2 \times \mathbb{C} = \{(z, w) \in \mathbb{C}^2 | \Im z > 0\}$. The group of isometries is $\text{Isom}(\mathbb{H}^2 \times \mathbb{E}^n) = \text{Isom}(\mathbb{H}^2) \times \text{Isom}(\mathbb{E}^n)$.

There is a natural fibration of $\mathbb{H}^2 \times \mathbb{E}^n$ which is preserved by the group of isometries:

$$\mathbb{R}^n \to \mathbb{H}^2 \times \mathbb{E}^n \to \mathbb{H}^2.$$

Seifert fibrations descend from this fibration as the following proposition shows.

Proposition 3.1.1. *If S is a compact quotient of $\mathbb{H}^2 \times \mathbb{E}^n$ by a discrete group of isometries then S is a Seifert $(n+2)$-manifold over a hyperbolic base.*

Proof. Now $S = (\mathbb{H}^2 \times \mathbb{E}^n)/\Gamma$ for some discrete group of isometries of $\mathbb{H}^2 \times \mathbb{E}^n$. Let $\hat{\Gamma} = \Gamma \cap \text{Isom}(\mathbb{E}^n)$.

Then $\hat{\Gamma}$ is a discrete subgroup of Isom($\mathbb{E}^n$) and is cocompact by theorem 6.3 of [Wa1]. So $\mathbb{R}^n/\hat{\Gamma}$ will be a (flat) closed orbifold. Now $\hat{\Gamma}$ is the kernel of the projection $p : \Gamma \to \text{Isom}(\mathbb{H}^2)$. Let $\bar{\Gamma}$ be the image of this projection. Then the action of Γ gives a Seifert fibration: $\mathbb{R}^n/\hat{\Gamma} \to (\mathbb{H}^2 \times \mathbb{E}^n)/\Gamma \to \mathbb{H}^2/\bar{\Gamma}$. □

For future reference, if ξ is an isometry of $\mathbb{H}^2 \times \mathbb{E}^n$, let $\bar{\xi}$ be its image in Isom($\mathbb{H}^2$) and let $(\mathcal{O}(\xi), y(\xi))$ be its image in Isom($\mathbb{E}^n$). Thus for $(z, w) \in \mathbb{H}^2 \times \mathbb{R}^n$, $\xi(z, w) = (\bar{\xi}(z), \mathcal{O}(\xi)w + y(\xi))$.

3.2 The geometries $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$

In this section we look at the geometries $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ $(n \geq 1)$. Our prime interest will be when $n = 2$. We will prove that closed manifolds with these geometries are Seifert fibred.

The model space for the geometry $\widetilde{SL}_2$ is the universal covering space of $U(\mathbb{H}^2)$, the unit tangent bundle of $\mathbb{H}^2$. The model space is a trivial line bundle over $\mathbb{H}^2$ (however $\widetilde{SL}_2$ is not geometrically a product, rather $\widetilde{SL}_2$ can be considered as a twisted product of the geometries $\mathbb{H}^2$ and $\mathbb{E}$). The group $\widetilde{SL}_2$ is the universal cover of $PSL_2\mathbb{R}$. The latter group acts simply transitively on $U(\mathbb{H}^2)$, so a choice of base point in $\widetilde{U(\mathbb{H}^2)}$ determines diffeomorphisms $\widetilde{SL}_2 \to \widetilde{U(\mathbb{H}^2)}$ and $PSL_2\mathbb{R} \to U(\mathbb{H}^2)$. The action of $\widetilde{SL}_2$ on itself via left multiplication projects to the action of $PSL_2\mathbb{R}$ on itself. The isometry group of $\widetilde{SL}_2$ has two components, both orientable. The identity component of Isom($\widetilde{SL}_2$) is $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{R}$, the quotient of the product of $\widetilde{SL}_2$ and $\mathbb{R}$ modulo identification of the centre of $\widetilde{SL}_2$ with the group of integers $\mathbb{Z} \subset \mathbb{R}$. The isometries of $\widetilde{SL}_2$ preserve the fibration $\mathbb{R} \to \widetilde{SL}_2 \to \mathbb{H}^2$ with the $\mathbb{R}$ factor acting as translations of the fibres. Compact manifolds which are geometric of this type are precisely the Seifert 3-manifolds where the base has negative euler characteristic, and where the euler number is nonzero. See [Sc2] pp. 462-467 for the details of this description of $\widetilde{SL}_2$.

The model space for $\widetilde{SL}_2 \times \mathbb{E}$ is the universal covering space of the nonzero tangent bundle of $\mathbb{H}^2$. The hyperbolic plane $\mathbb{H}^2$ can be considered as a subset of $\mathbb{C}$ whose tangent space can be naturally associated with $\mathbb{C} \times \mathbb{C}$. Therefore, we shall identify the tangent bundle of $\mathbb{H}^2$ with the set $\mathbb{H}^2 \times \mathbb{C}$. Let v be a unit tangent vector to the hyperbolic plane at z. Then its Euclidean length is given by $|v| = \Im z$ and so we may identify the unit tangent bundle of $\mathbb{H}^2$ with the subset $\{(z, v) \in \mathbb{H}^2 \times \mathbb{C} | |v| = \Im z\}$. Hence the nonzero tangent bundle of $\mathbb{H}^2$ is the subspace $\mathbb{H}^2 \times (\mathbb{C} - \{0\})$, and so we may identify the model space of $\widetilde{SL}_2 \times \mathbb{E}$ with $\mathbb{H}^2 \times \mathbb{C}$. The universal cover $\widetilde{SL}_2 \times \mathbb{E} \longrightarrow \mathbb{H}^2 \times (\mathbb{C} - \{0\})$ is given by $(z, w) \mapsto (z, e^w)$. The model space of $\widetilde{SL}_2$ is then identified with the subset $\{(z, w) \in \mathbb{H}^2 \times \mathbb{C} | \Re w = \ln \Im z\}$.

The isometry group of $\widetilde{SL}_2 \times \mathbb{E}$ is Isom($\widetilde{SL}_2$) $\times$ Isom($\mathbb{E}$). By extension of the $\widetilde{SL}_2$ case, Isom($\widetilde{SL}_2 \times \mathbb{E}$) preserves the fibration $\mathbb{R}^2 \to \widetilde{SL}_2 \times \mathbb{E} \to \mathbb{H}^2$. We then have an associated exact sequence:

$$1 \to \mathbb{R} \times \text{Isom}(\mathbb{E}) \to \text{Isom}(\widetilde{SL}_2 \times \mathbb{E}) \xrightarrow{p} \text{Isom}(\mathbb{H}^2) \to 1$$

Note the sequence does not split. See [Wa2] section 1 for some descriptions of $\widetilde{SL}_2 \times \mathbb{E}$.

Proposition 3.2.1. *If S is a compact quotient of $\widetilde{SL}_2 \times \mathbb{E}$ by a discrete group of isometries then S is a Seifert 4-manifold over a hyperbolic base.*

Proof. Firstly, $S = (\widetilde{SL}_2 \times \mathbb{E})/\Gamma$ for Γ a discrete subgroup of $\mathrm{Isom}(\widetilde{SL}_2 \times \mathbb{E})$. Let $\hat{\Gamma} = \Gamma \cap \ker(p)$ and $\bar{\Gamma} = p(\Gamma)$. The group $\hat{\Gamma}$ is a discrete cocompact subgroup of $\mathrm{Isom}(\mathbb{E}^2)$ and $\bar{\Gamma}$ is a discrete cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ (theorem 6.3 in [Wa1]). So $\mathbb{R}^2/\hat{\Gamma}$ will be a (flat) closed orbifold. Then we have the Seifert fibration

$$F = \mathbb{R}^2/\hat{\Gamma} \to S \to \mathbb{H}^2/\bar{\Gamma} = B.$$

□

Now we shall consider how the isometries of $\widetilde{SL}_2 \times \mathbb{E}$ act in more detail. The radical of the isometry group is $\mathbb{R}^2$. The first factor (corresponding to the radical of $\widetilde{SL}_2$) acts via purely imaginary translations (where the generator of the centre: $\mathbb{Z}$, acts by $(z, w) \mapsto (z, w + 2\pi i)$), while the second factor acts via real translations. Imaginary translations of the w factor project to $\mathrm{Isom}(U(\mathbb{H}^2))$ as rotations of the tangents. Real translations of the w factor project to isometries of the nonzero tangent bundle which change the length of the tangent vectors. The quotient of the identity component of the isometry group by this radical is $PSL_2\mathbb{R} = \mathrm{Isom}^+(\mathbb{H}^2)$. Therefore to understand the rest of the isometries in the identity component of $\mathrm{Isom}(\widetilde{SL}_2 \times \mathbb{E})$ it is sufficient to construct a set-theoretic section $s : \mathrm{Isom}^+(\mathbb{H}^2) \to \mathrm{Isom}(\widetilde{SL}_2 \times \mathbb{E})$ (ie $ps = 1$). We will do this below (as well as extending this section to $\mathrm{Isom}(\mathbb{H}^2)$).

The isometry group of $\widetilde{SL}_2$ has two components, both orientation preserving, with representative for the non-identity component given by $(z, w) \mapsto (-\bar{z}, \bar{w})$. (As this corresponds to a simultaneous reflection of $\mathbb{H}^2$ and the tangent space, it is orientation preserving). The isometry group of $\mathbb{E}$ also has two components, with the non-identity component represented by reflection about 0. There is a corresponding reflection of $\mathbb{H}^2 \times \mathbb{C}$ which fixes the subspace $\widetilde{SL}_2$ and which is given by $(z, w) \mapsto (z, 2\ln(\Im z) - \bar{w})$. So $\mathrm{Isom}(\widetilde{SL}_2 \times \mathbb{E})/\mathrm{Isom}^0(\widetilde{SL}_2 \times \mathbb{E}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, with generators represented by $(z, w) \mapsto (z, 2\ln(\Im z) - \bar{w})$ and $(z, w) \mapsto (-\bar{z}, \bar{w})$. We will label each component by the corresponding element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ thought of as $\{\epsilon, \delta | \epsilon = \pm 1, \delta = \pm 1\}$ (or as the corresponding diagonal matrices, $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \delta \end{smallmatrix}\right)$).

Define an orientation function $\mathrm{w} : \mathrm{Isom}(\ ) \mapsto \{\pm 1\}$ by setting $\mathrm{w}(\alpha) = 1$ if α is orientation preserving and -1 if it reverses orientation.

We shall now define the section $s : \mathrm{Isom}^+(\mathbb{H}^2) \to \mathrm{Isom}(\widetilde{SL}_2 \times \mathbb{E})$. Suppose $\bar{\alpha}$ is in $PSL_2\mathbb{R}$ ($= \mathrm{Isom}^+(\mathbb{H}^2)$), ie $\bar{\alpha}(z) = \frac{az+b}{cz+d}$. Then $\bar{\alpha}$ acts on $PSL_2\mathbb{R} = \{z, v \in \mathbb{H}^2 \times \mathbb{C} | |v| = \Im z\}$ via the map $(z, v) \mapsto (\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2})$. Let $\log : \mathbb{C} - \{0\} \to \mathbb{R} \times (-\pi, \pi]i \subset \mathbb{C}$ be the principal value of the inverse of exp, extended so that $\log(-k) = \ln(k) + \pi i$ for $k > 0$. Let $s(\bar{\alpha})$ be the map $(z, w) \mapsto (\bar{\alpha}(z), w - \log(cz + d)^2)$. Then $s(\bar{\alpha})$ is an isometry of $\widetilde{SL}_2 \times \mathbb{E}$ (see [Wa2]) and $p(s(\bar{\alpha})) = \bar{\alpha}$, so s is a set theoretic section for p. Note $s(\bar{\alpha})$ restricts to an isometry of $\widetilde{SL}_2$ as well (since $\ln(\Im z) = \Re w$ implies $\ln(\Im \bar{\alpha}(z)) = \ln(\Im z) - \ln(|cz+d|^2) = \Re(w - \log(cz+d)^2)$). Therefore for $\alpha \in \mathrm{Isom}^0(\widetilde{SL}_2)$, $p(\alpha^{-1} sp(\alpha)) = 1$ and so $\alpha(z, w) = s(p(\alpha))(z, w) + (0, iy)$ for some $y \in \mathbb{R}$. From now on write $\bar{\alpha}$ to mean $p(\alpha)$. Let α be in $\mathrm{Isom}^0(\widetilde{SL}_2) \times \mathrm{Isom}^+(\mathbb{E})$. Then $\alpha(z, w) = s(\bar{\alpha})(z, w) + (0, y(\alpha))$, for some $y(\alpha) \in \mathbb{C}$.

We will modify s slightly to have a special form when $\bar{\alpha}$ is elliptic, which will be useful later on. From above, $s(\bar{\alpha})^m(z,w) = s(p(s(\bar{\alpha})^m))(z,w)+(0,iy) = s(\bar{\alpha}^m)(z,w)+(0,iy)$ for some $y \in \mathbb{R}$. Suppose $\bar{\alpha}$ is elliptic of order m. Then $\bar{\alpha}^m = 1$. Let $s'(\bar{\alpha})(z,w) = s(\bar{\alpha})(z,w) + (0,-iy/m)$, then $s'(\bar{\alpha})^m(z,w) = s(\bar{\alpha})^m(z,w) + (0,-iy) = s(\bar{\alpha}^m)(z,w)$ since $s(\bar{\alpha})$ commutes with imaginary translations, and so $s'(\bar{\alpha})^m = 1$. Replace $s(\bar{\alpha})$ by $s'(\bar{\alpha})$ to get the desired modification. As remarked in [Ue2], if $\bar{\alpha}$ is hyperbolic the imaginary part of the second factor is defined by the parallel translation of the unit tangent vector along the axis of $\bar{\alpha}$. We will not encounter parabolic elements, since the fundamental groups of compact hyperbolic orbifolds contain no parabolic elements ([Kat] Corollary 4.2.7), so we will not alter the action. Note s is NOT a group homomorphism; however $s(\bar{\alpha}\bar{\beta})s(\bar{\alpha})^{-1}s(\bar{\beta})^{-1}$ will be a rational translation of $2\pi i$ (ie the subgroup $\mathbb{Q}$ of $\mathbb{R}$ in $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{R}$).

We can extend our definition of s to $\mathrm{Isom}(\mathbb{H}^2)$. Take $\bar{\beta} \in \mathrm{Isom}(\mathbb{H}^2) \setminus \mathrm{Isom}^+(\mathbb{H}^2)$, then $\bar{\beta}(z) = \bar{\alpha}(-\bar{z})$ for some $\bar{\alpha} \in PSL_2\mathbb{R}$. A lift of $\bar{\beta}$ is then $s(\bar{\beta})(z,w) = s(\bar{\alpha})(-\bar{z},\bar{w})$. As before, if $\mathrm{w}(\beta) = 1$, then $\beta(z,w) = s(\bar{\beta})(z,w)+(0,y(\beta))$ for some $y(\beta) \in \mathbb{C}$. Note, $\beta^2(z,w) = s(\bar{\beta}^2)(z,w)+(0,y(\beta)+\bar{y}(\beta))$. If $\alpha \in \mathrm{Isom}(\widetilde{SL}_2 \times \mathbb{E})$ is orientation reversing, then $\alpha(z,w) = s(\bar{\alpha})(z, 2\ln(\Im z) - \bar{w}) + (0,y(\alpha))$ for some $y(\alpha) \in \mathbb{C}$.

In all this, if α is in the component labelled (ϵ,δ), then $\mathrm{w}(\alpha) = \epsilon$ and $\mathrm{w}(\bar{\alpha}) = \delta$. Furthermore, if τ_t is a translation, ie $\tau_t(z,w) = (z, w+t)$, then $\alpha\tau_t\alpha^{-1}(z,w) = (z, w + \left(\begin{smallmatrix}\epsilon & 0\\ 0 & \delta\end{smallmatrix}\right) t)$, considering $\mathbb{C} = \mathbb{R}^2$. We will label this matrix $\mathcal{O}(\alpha) := \left(\begin{smallmatrix}\epsilon & 0\\ 0 & \delta\end{smallmatrix}\right) = \begin{pmatrix}\mathrm{w}(\alpha) & 0\\ 0 & \mathrm{w}(\bar{\alpha})\end{pmatrix}$.

The lifts $s(\bar{\xi})$ are chosen so that if $\bar{\xi}$ is elliptic with order m, then $s(\bar{\xi})^m = 1$. In [Ue2], it was also observed if $\bar{l}_1\bar{l}_2\ldots\bar{l}_n$ is trivial in $\mathrm{Isom}(\mathbb{H}^2)$ and the $\bar{l}_i$ are all elliptic or hyperbolic (for example a relation of a hyperbolic orbifold), then $s(\bar{l}_1)s(\bar{l}_2)\ldots s(\bar{l}_n)$ is an imaginary translation with magnitude the holonomy corresponding to the relation, that is the composition of the parallel translations of the $\bar{l}$'s. For example the standard global relation of an orientable hyperbolic orbifold without reflectors is $\prod_{p=1}^{g}[t_p,u_p]\prod_{j=1}^{k} s_{0j}$. This relation corresponds to traversing the entire orbifold and so the holonomy corresponding to this relation is the holonomy of the orbifold. This connection can be seen in [Kat] Theorem 4.3.2 when constructing fundamental regions.

Note $s(\bar{\alpha})$ and reflection in $\widetilde{SL}_2$ commute. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_l$ are isometries of $\widetilde{SL}_2 \times \mathbb{E}$ such that $\alpha_1\alpha_2\ldots\alpha_l = 1$ and $p(\alpha_i)$ are not parabolic. Then $s(\bar{\alpha}_1)s(\bar{\alpha}_2)\ldots s(\bar{\alpha}_n)$ is also an imaginary translation, τ say, with magnitude the parallel translate corresponding to the relation. Also $(\mathcal{O}(\alpha_1), y(\alpha_1))\ldots(\mathcal{O}(\alpha_l), y(\alpha_l)) = \tau^{-1}$.

When considering the $\widetilde{SL}_2 \times \mathbb{E}$ geometry later, it is convenient to treat it as similarly to $\mathbb{H}^2 \times \mathbb{E}^2$ as possible. So we will find a new representation of $\widetilde{SL}_2 \times \mathbb{E}$ so that isometries act via an element of $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{E})^2$ plus a correction term. Let θ be the self-homeomorphism of $\mathbb{H}^2 \times \mathbb{C}$ defined by $\theta(z,w) = (z, w - \ln(\Im(z)))$. A new representation for the geometry $\widetilde{SL}_2 \times \mathbb{E}$ can then given by θ. Namely, points get sent to their image by θ and isometries get sent to their conjugate by θ. Note that $\widetilde{SL}_2$ now becomes identified with the subspace $\mathbb{H}^2 \times i\mathbb{R}$. Translations and the map $(z,w) \mapsto (-\bar{z},\bar{w})$ are preserved. Reflection in $\widetilde{SL}_2$ now becomes the map $(z,w) \mapsto (z,-\bar{w})$. For $\bar{\alpha}$ in $PSL_2\mathbb{R}$ ($\bar{\alpha}(z) = \frac{az+b}{cz+d}$), $s(\bar{\alpha})$ now becomes the map $(z,w) \mapsto (\bar{\alpha}(z), w - 2\log(\frac{cz+d}{|cz+d|}))$ (plus a purely imaginary translation if $\bar{\alpha}$ is elliptic). Note $2\log(\frac{cz+d}{|cz+d|})$ (plus possibly the translation) is purely imaginary and (as before) corresponds to the parallel translate corresponding to

$\bar{\alpha}$. Any isometry is a composition of these maps. Therefore we can say that any isometry, ξ acts (under the new representation) via an element of $\text{Isom}(\mathbb{H}^2) \times \text{Isom}(\mathbb{E})^2$ plus a purely imaginary correction term corresponding to the parallel translate of $\bar{\xi}$ ($\bar{\xi}$ is the image of ξ in $\text{Isom}(\mathbb{H}^2)$). The $\text{Isom}(\mathbb{E}) \times \text{Isom}(\mathbb{E})$ part of the isometry is given by $(\mathcal{O}(\xi), y(\xi))$ since conjugation by θ preserves translations ($\mathcal{O}(\xi)$ was defined by action on translations by conjugation and $y(\xi)$ was defined as the translation difference from $s(\bar{\xi})$).

Now let us consider $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ for $n \geq 1$. The isometry group $\text{Isom}(\widetilde{SL}_2 \times \mathbb{E}^{n-1}) = \text{Isom}(\widetilde{SL}_2) \times \text{Isom}(\mathbb{E}^{n-1})$ preserves the fibration $\mathbb{R}^n \to \widetilde{SL}_2 \times \mathbb{E}^{n-1} \to \mathbb{H}^2$. We have the associated exact sequence:

$$1 \to \mathbb{R} \times \text{Isom}(\mathbb{E}^{n-1}) \to \text{Isom}(\widetilde{SL}_2 \times \mathbb{E}^{n-1}) \xrightarrow{p} \text{Isom}(\mathbb{H}^2) \to 1$$

Similarly to proposition 3.2.1, we get the following proposition:

Proposition 3.2.2. *If S is a compact quotient of $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ by a discrete group of isometries then S is a Seifert $(n+2)$-manifold over a hyperbolic base.* □

Consider the structure of $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ in more detail. When $n = 1$ we have $\widetilde{SL}_2$, which as previously stated, is naturally contained in $\widetilde{SL}_2 \times \mathbb{E}$, and is associated to the subgroup where $\Re w = \ln(\Im z)$. For the other n, the w factor above is replaced by $\mathbb{R}^{n-2} \times \mathbb{C}$. The isometries of the $\text{Isom}(\mathbb{E}^{n-1})$ component of $\text{Isom}(\widetilde{SL}_2 \times \mathbb{E}^{n-1}) = \text{Isom}(\widetilde{SL}_2) \times \text{Isom}(\mathbb{E}^{n-1})$ $(n \neq 1)$ act on $\mathbb{R}^{n-2} \times \Re\mathbb{C}$. For the $n = 1$ case, we remove the real translations from the above exposition. We can define s in a similar way to above. Then for $n \neq 1$, $\mathcal{O}(\alpha) = \begin{pmatrix} E & 0_{1,n-1} \\ 0_{n-1,1} & \delta \end{pmatrix}$, where $0_{m,n}$ is a $m \times n$ zero matrix, E is a $n-1$ orthogonal matrix, such that $\det E = \epsilon = \text{w}(\alpha)$ and $\delta = \text{w}(\bar{\alpha})$. When $n = 1$, we have two components of the isometry group, both orientation preserving so $\text{w}(\alpha) = 1$ for all elements, and $\mathcal{O}(\alpha) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)$, where $\delta = \text{w}(\bar{\alpha})$. Note, it is convenient to consider $\mathcal{O}(\alpha) \in O_n\mathbb{R}$, therefore in the $n = 1$ case, we will tend to consider $\mathcal{O}(\alpha) = \delta$. We can change the representation of $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ as we did before, by an extension of θ, to make the isometries of $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ very similar to isometries of $\mathbb{H}^2 \times \mathbb{E}^n$.

In summary, if ξ is an isometry of $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$, then $\xi(z, w) = (\bar{\xi}(z), \mathcal{O}(\xi)w + y(\xi) + \left(\begin{smallmatrix} 0 \\ c(\xi) \end{smallmatrix}\right))$ where c is the correction term corresponding to the parallel translate of $\bar{\xi}$. Note $\mathcal{O}(\xi')\left(\begin{smallmatrix} 0 \\ c(\xi) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \text{w}(\bar{\xi}')c(\xi) \end{smallmatrix}\right)$.

3.3 Geometric Seifert manifolds with hyperbolic bases and no reflector curves

In this section we will give necessary and sufficient conditions for Seifert 4-manifolds with hyperbolic base and no reflector curves to be geometric.

Firstly, though we will give a theorem and corollary which is useful when describing the group of monodromies.

Theorem 3.3.1. *A subgroup G of $GL_n\mathbb{Z}$ is finite if and only if there is a $P \in GL_n\mathbb{R}$ such that PGP^{-1} is a subgroup of $O_n\mathbb{R}$.*

Proof. If G is finite, then G is conjugate to an orthogonal group by lemma 1.1.1. So conversely suppose G is conjugate to an orthogonal group. Since $G \subset GL_n\mathbb{Z}$, G is a discrete subgroup of $GL_n\mathbb{R}$. Since $PGP^{-1} \subset O_n\mathbb{R}$, PGP^{-1} is a discrete subgroup of a compact space and consequently is finite. Therefore G is finite. ☐

Corollary 3.3.2. *A subgroup G of $GL_2\mathbb{Z}$, is finite if and only if there is a $P \in GL_2\mathbb{R}$ such that PGP^{-1} is a subgroup of $O_2\mathbb{R}$.*

Moreover, G is conjugate (in $GL_2\mathbb{Z}$) to a subgroup of

(i). $O_2\mathbb{Z} = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \cong D_8$.

OR

(ii). $\left\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \cong D_{12}$

Proof. If G is finite, then by p 85 of [Zi3] it is conjugate in $GL_2\mathbb{Z}$ to a subgroup of $O_2\mathbb{Z}$ or $\left\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \left\langle \left(\begin{smallmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)^{-1}$ (where $a = 2 + \sqrt{3}$ and $b = 1$). The theorem gives the rest of the result. ☐

Recall that (when the base is aspherical) there is a homomorphism $A : \pi_1(S) \to \mathrm{Aut}(\pi_1(F))$, which induces the monodromy map, $\bar{A} : \pi_1^{orb}(B) \to \mathrm{Out}(\pi_1(F))$. When the fibre is T^n the group of monodromies is contained in $\mathrm{Out}(\pi_1(F)) = GL_n\mathbb{Z}$. In this case, the monodromy map gives a $\mathbb{Z}[\pi_1^{orb}(B)]$-module structure to $\mathbb{Z}^n$ and hence to $\mathbb{Q}^n$. Let $\mathcal{I}_{\mathrm{w}} = \langle \bar{\xi} - \mathrm{w}(\bar{\xi}) | \bar{\xi} \in \pi_1^{orb}(B) \rangle$ be the w-twisted augmentation ideal, and let $V = \mathcal{I}_{\mathrm{w}}\mathbb{Q}^n$ be the submodule generated by $(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)z$ for all $\bar{\xi} \in \pi_1^{orb}(B)$ and $z \in \mathbb{Q}^n$. Thus sometimes we will write V as $\sum_{\bar{\xi} \in \pi_1^{orb}(B)} \mathrm{Im}_{\mathbb{Q}^n}(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)$.

Lemma 3.3.3. *Let G be a finitely generated group with generators $\{g_1, \ldots, g_m\}$. Then the augmentation ideal $\mathcal{I} = \langle g - 1 | g \in G \rangle$ in $\mathbb{Z}[G]$ is generated as a two-sided ideal by $\{g_1 - 1, \ldots, g_m - 1\}$.*

Proof. For each g we need to prove $g - 1$ is in the ideal generated by $\{g_1 - 1, \ldots g_m - 1\}$. Since $gh - 1 = (g-1)h + (h-1) = g(h-1) + (g-1)$ and $g^{-1} - 1 = (g-1)(-g^{-1}) = -g^{-1}(g-1)$, the result follows by induction on the length of the shortest word that represents g in terms of the generators (and their inverses). ☐

Corollary 3.3.4. *Let G be a finitely generated group with generators $\{g_1, \ldots, g_m\}$. Let $R : G \to GL_n\mathbb{R}$ be a group homomorphism. Then,*

$$\sum_{g \in G} \mathrm{Im}(R(g) - I) = \sum_{i=1}^{m} \mathrm{Im}(R(g_i) - I).$$

Proof. The group homomorphism, R, gives a $Z[G]$-module structure on $\mathbb{R}^n$. Therefore $\sum_{g \in G} \mathrm{Im}(R(g) - I) = \mathcal{I}\mathbb{R}^n$. By the lemma $\mathcal{I}\mathbb{R}^n = \{g_1 - 1, \ldots, g_m - 1\}\mathbb{R}^n = \sum_{i=1}^{m} \mathrm{Im}(R(g_i) - I)$, hence the corollary. ☐

Corollary 3.3.5. *Let B be an orbifold, so that $\pi_1^{orb}(B)$ is generated by $\{g_1, \dots, g_m\}$. Let $A : \pi_1^{orb}(B) \to GL_n\mathbb{Z}$ and $\mathrm{w} : \pi_1^{orb}(B) \to \{\pm 1\}$ be group homomorphisms. Then*

$$\sum_{g \in \pi_1^{orb}(B)} \mathrm{Im}(A(g) - \mathrm{w}(g)I) = \sum_{i=1}^{m} \mathrm{Im}(A(g_i) - \mathrm{w}(g_i)I).$$

Proof. Take $R(g) = \mathrm{w}(g)A(g)$ and apply the previous result. Note $\mathrm{Im}(A(g) - \mathrm{w}(g)I) = \mathrm{Im}(\mathrm{w}(g)A(g) - I)$. □

Proposition 3.3.6. *If S is a geometric manifold of type $\mathbb{H}^2 \times \mathbb{E}^n$ or $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ ($n \geq 1$) then it is a Seifert manifold with a hyperbolic base. Furthermore if T^n is the general fibre then the group of monodromies is a finite subgroup of $GL_n\mathbb{Z}$.*

Proof. By propositions 3.1.1 and 3.2.1 respectively, $\mathbb{H}^2 \times \mathbb{E}^n$ and $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ manifolds are Seifert fibred with hyperbolic bases.

Let X be the geometry. Then $S = X/\Gamma$ for some discrete subgroup Γ of $\mathrm{Isom}(X)$. In all cases there is a projection $p : \mathrm{Isom}(X) \to \mathrm{Isom}(\mathbb{H}^2)$. Now for the fibre F, $\pi_1(F)$ is isomorphic to $\Gamma \cap \ker(p)$. Suppose the fibre is T^n, then $\Gamma \cap \ker(p)$ is isomorphic to $\mathbb{Z}^n$. However $\ker(p)$ is contained in $\mathrm{Isom}(\mathbb{E}^n)$ ($\ker(p) = \mathrm{Isom}(\mathbb{E}^{n-1}) \times \mathbb{E}$ when $X = \widetilde{SL}_2 \times \mathbb{E}^{n-1}$) and so $\Gamma \cap \ker(p)$ must consist of translations.

Now Γ acts on $\Gamma \cap \ker(p)$ by conjugation. The group $\mathrm{Isom}(X)$ acts on translations via orthogonal matrices (when $X = \widetilde{SL}_2 \times \mathbb{E}^{n-1}$ they will furthermore act via matrices of the form $\left(\begin{smallmatrix} E & 0 \\ 0 & \delta \end{smallmatrix}\right)$ where $E \in O_{n-1}\mathbb{R}$ and $\delta = \pm 1$: see section 3.2 for details), so Γ acts on $\Gamma \cap \ker(p)$ via a subgroup of $O_n\mathbb{R}$. However the group of monodromies (a subgroup of $GL_n\mathbb{Z}$) is the action of $\pi_1(S) = \Gamma$ on $\pi_1(F)$ (since $F = T^n$). Therefore the group of monodromies is a subgroup of $GL_n\mathbb{Z}$ conjugate to a subgroup of $O_n\mathbb{R}$ (the conjugation is via the isomorphism $\pi_1(F) \cong \Gamma \cap \ker(p)$), and so by corollary 3.3.1 is finite. □

We will now show that the condition that group of monodromies is finite, is not only necessary for a Seifert manifold with T^2 fibres to be geometric, but is also sufficient. The proof of the complete result is technical in places. Therefore, we do this in two stages, this section (which excludes reflector curves) and the next (which includes reflector curves). In the second stage we will try to repeat as little of this section as possible.

Theorem 3.3.7. *Let S be a Seifert manifold over a hyperbolic base, with general fibre T^2 so that the base has no reflector curves. Let the base, B, have k cone points, so that m_i is the order of the ith cone point. Let A be the standard map which induces the monodromy map, and let $\tilde{e}_i$ and $\tilde{a}$ be the standard parts of the presentation of $\pi_1(S)$.*

Then S is geometric if and only if the group of monodromies is a finite subgroup of $GL_2\mathbb{Z}$, ie it is conjugate in $GL_2\mathbb{Z}$ to a subgroup of $O_2\mathbb{Z}$ or $\left\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$.

Let $e = \tilde{a} + \sum_{i=1}^{k} \tilde{e}_i / m_i$ and let $V = \mathcal{I}_{\mathrm{w}}\mathbb{Q}^2$.

More precisely, S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if the group of monodromies is finite and

$$e \in V.$$

S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}$ if and only if the group of monodromies is finite and

$$e \notin V,$$

which implies the group $\{\mathrm{w}(\bar{\xi})A(\bar{\xi}) | \bar{\xi} \in \pi_1^{orb}(B)\}$ is conjugate in $GL_2\mathbb{Z}$ to a subgroup of $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$ or $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$.

Proof. Note for $M \in \mathbb{Z}[GL_2\mathbb{Z}]$, $\mathrm{Im}_{\mathbb{Q}^2}(M) = \mathbb{Q}^2 \cap \mathrm{Im}_{\mathbb{R}^2}(M)$, hence $V = \mathbb{Q}^2 \cap \mathcal{I}_{\mathrm{w}}\mathbb{R}^2$. So to prove a rational number e is contained in V, it is sufficient to prove $e \in \mathcal{I}_{\mathrm{w}}\mathbb{R}^2$. Therefore for the purpose of this proof, we can replace V by $\mathcal{I}_{\mathrm{w}}\mathbb{R}^2 = \sum_{\bar{\xi} \in \pi_1^{orb}(B)} \mathrm{Im}_{\mathbb{R}^2}(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)$ and write Im to mean $\mathrm{Im}_{\mathbb{R}^2}$.

Assume first S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$, so $\pi_1(S)$ is a lattice in $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{E}^2)$. Since the fibre is T^2, $\pi_1(F) = \langle h_1, h_2 | [h_1, h_2] = 1 \rangle \cong \mathbb{Z}^2$ and so the h_i act on X by translations: $(z, w) \mapsto (z, w + y(h_i))$, for some complex numbers $y(h_i)$. Let P be the matrix with $y(h_i)$ as the columns (considering $\mathbb{C} = \mathbb{R}^2$). The $y(h_i)$ are linearly independent over $\mathbb{R}$ (since $\pi_1(F)$ is a free abelian group), and P is invertible. The other generators, ξ say, project to isometries of $\mathbb{H}^2$ which are denoted as $\bar{\xi}$. They act on $\mathbb{C}$ as $(\mathcal{O}(\xi), y(\xi))$.

We shall consider the consequences of the defining relations of $\pi_1(S)$ for the actions of the generators of $\pi_1(S)$ on the "Euclidean factor", $\mathbb{C} = \mathbb{R}^2$. The monodromy relations $\xi h_i \xi^{-1} = A(\xi)h_i$ determine equations:

$$\mathcal{O}(\xi) = PA(\xi)P^{-1} \tag{3.3.1}$$

in $GL_2\mathbb{R}$, where P is the matrix with columns $y(h_i)$. This implies the subgroup $\{A(\xi) | \xi \in \pi_1(S)\}$ of $GL_2\mathbb{Z} = \mathrm{Aut}(T^2)$ is conjugate to a subgroup of $O_2\mathbb{R}$, and so we may apply corollary 3.3.2. Note this provides a more explicit proof of proposition 3.3.6 when $n = 2$.

The next relations are of the type $s_i^{m_i}\tilde{e}_i = 1$, where s_i corresponds to the ith cone point which has order m_i. Now $\bar{s}_i$ is elliptic of order m_i, so $s_i^{m_i}$ acts trivially on the $\mathbb{H}^2$ factor and acts via the map $(\mathcal{O}(s_i), y(s_i))^{m_i}$ on the $\mathbb{C}$ factor. By looking at just the $\mathbb{C}$ factor we see $(\mathcal{O}(s_i), y(s_i))^{m_i} = (I, P\tilde{e}_i)^{-1}$ or equivalently $\left(A(s_i), P^{-1}y(s_i)\right)^{m_i} = (I, \tilde{e}_i)^{-1}$. Since $\left(A(s_i), P^{-1}y(s_i)\right)$ normalises Z^2, we see $\left(A(s_i), P^{-1}y(s_i)\right)$ projects to an affine homeomorphism of $T^2 = \mathbb{R}^2/Z^2$ with order m_i. Then lemma 2.1.5 shows

$$P^{-1}y(s_i) + \tilde{e}_i/m_i = (A(s_i) - I)z(s_i), \text{ for some } z(s_i) \in \mathbb{R}^2. \tag{3.3.2}$$

The last relations to consider are

$$\prod_{p=1}^{g} [t_p, u_p] \prod_{i=1}^{k} s_i = \tilde{a}$$

when the base is orientable, and

$$\prod_{p=1}^{g} v_i^2 \prod_{i=1}^{k} s_i = \tilde{a}$$

when the base is non-orientable.

These give rise to the following equations which respectively correspond to the orientable and non-orientable base cases:

$$\sum_{i=1}^{g}\left(\prod_{1\le j<i}[\mathcal{O}(t_j),\mathcal{O}(u_j)]\right)\big(y(t_i)+\mathcal{O}(t_i)y(u_i)-\mathcal{O}(t_iu_it_i^{-1})y(t_i)-\mathcal{O}([t_i,u_i])y(u_i)\big)+ \prod_{p=1}^{g}\mathcal{O}([t_p,u_p])\sum_{i=1}^{k}\left(\prod_{1\le j<i}\mathcal{O}(s_j)\right)y(s_i)=P\tilde{a} \tag{3.3.3}$$

$$\sum_{i=1}^{g}\left(\prod_{1\le j<i}\mathcal{O}(v_j)^2\right)(\mathcal{O}(v_i)y(v_i)+y(v_i))+\prod_{p=1}^{g}\mathcal{O}(v_p)^2\sum_{i=1}^{k}\left(\prod_{1\le j<i}\mathcal{O}(s_j)\right)y(s_i)=P\tilde{a} \tag{3.3.4}$$

If now we suppose the geometry is $\widetilde{SL}_2\times\mathbb{E}$ then again the model space may be identified with $\mathbb{H}^2\times\mathbb{C}$ (see section 3.2). The generators again act via translations, while the other generators ξ act on $\mathbb{C}$ as $(\mathcal{O}(\xi),y(\xi))$ plus a correction term as discussed in section 3.2. The monodromy and cone point relations again give equations (3.3.1) and (3.3.2). The equations corresponding to the remaining relations should be modified as follows:

$$\sum_{i=1}^{g}\left(\prod_{1\le j<i}[\mathcal{O}(t_j),\mathcal{O}(u_j)]\right)\big(y(t_i)+\mathcal{O}(t_i)y(u_i)-\mathcal{O}(t_iu_it_i^{-1})y(t_i)-\mathcal{O}([t_i,u_i])y(u_i)\big)+ \prod_{p=1}^{g}\mathcal{O}([t_p,u_p])\sum_{i=1}^{k}\left(\prod_{1\le j<i}\mathcal{O}(s_j)\right)y(s_i)+\binom{0}{c}=P\tilde{a} \tag{3.3.5}$$

$$\sum_{i=1}^{g}\left(\prod_{1\le j<i}\mathcal{O}(v_j)^2\right)(\mathcal{O}(v_i)y(v_i)+y(v_i))+\prod_{p=1}^{g}\mathcal{O}(v_p)^2\sum_{i=1}^{k}\left(\prod_{1\le j<i}\mathcal{O}(s_j)\right)y(s_i)+\binom{0}{c}=P\tilde{a} \tag{3.3.6}$$

Here $c=2\pi\chi^{orb}(B)$ is the total holonomy of the base and is nonzero. This term reflects the correction factor and corresponds to parallel translation around a fundamental domain (see [Ue2] for the orientable case).

In order to simplify these expressions, we will look at these results from the perspective of a different set of generators. In light of this, we make the following definitions (in which boldface is used to highlight the original generator that we have conjugated)

$$\tau_i = \left(\prod_{1\leq j<i}[t_j,u_j]\right) t_i u_i \mathbf{t_i^{-1}} u_i^{-1} t_i^{-1} \left(\prod_{1\leq j<i}[t_j,u_j]\right)^{-1}$$

$$z(\tau_i) = -A\left(\prod_{1\leq j<i}[t_j,u_j]\right) A(t_i)P^{-1}y(u_i)$$

$$\Upsilon_i = \left(\prod_{1\leq j<i}[t_j,u_j]\right) t_i \mathbf{u_i} t_i^{-1} \left(\prod_{1\leq j<i}[t_j,u_j]\right)^{-1}$$

$$z(\Upsilon_i) = -A\left(\prod_{1\leq j<i}[t_j,u_j]\right) P^{-1}y(t_i)$$

$$\nu_i = \left(\prod_{1\leq j<i} v_j^2\right) \mathbf{v_i} \left(\prod_{1\leq j<i} v_j^2\right)^{-1}$$

$$z(\nu_i) = A\left(\prod_{1\leq j<i} v_j^2\right) P^{-1}y(v_i)$$

$$\varsigma_i = d\left(\prod_{1\leq j<i} s_j\right) \mathbf{s_i} \left(\prod_{1\leq j<i} s_j\right)^{-1} d^{-1}$$

$$z(\varsigma_i) = A(d)A\left(\prod_{1\leq j<i} s_j\right) z(s_i),$$

where $d = \prod_{p=1}^{g}[t_p,u_p]$ when the base is orientable and $d = \prod_{p=1}^{g} v_p^2$ when the base is non-orientable. Note $z(s_i)$ comes from equation (3.3.2). Notice that v_i and ν_i are orientation reversing and they are the only generators of either set that are so.

$$\text{Let } e' = \tilde{a} + A(d)\sum_{i=1}^{k}\left(\prod_{1\leq j<i} A(s_j)\right)\tilde{e}_i/m_i.$$

We can then rearrange equations (3.3.5) and (3.3.6) to both have the form:

$$e' - P^{-1}\begin{pmatrix}0\\c\end{pmatrix} = \sum (A(\xi) - \mathrm{w}(\bar{\xi})I)z(\xi) \tag{3.3.7}$$

where the sum is over these new generators, namely τ_i, Υ_i and ς_i when the base is orientable and ν_i and ς_i when non-orientable.

When $X = \mathbb{H}^2 \times \mathbb{E}^2$, $c = 0$, and so $e' \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$ (where the sum can be taken over all elements of $\pi_1^{orb}(B)$ by corollary 3.3.5 which precedes this proof). Note e' can be replaced by e (as defined in the statement of this theorem) because they are equal modulo $\sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. As stated earlier the group of monodromies is finite for both geometries.

To establish sufficiency of the conditions we will show that the groups can be realised by geometric Seifert manifolds and then invoke theorem 2.5.10. Suppose now that the conditions of the theorem for the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ hold. Then by corollary 3.3.2, $A(\xi) = P^{-1}\mathcal{O}(\xi)P$ for some $P \in GL_2\mathbb{R}$ and some group of orthogonal matrices $\{\mathcal{O}(\xi)|\xi \in \pi_1(S)\}$ and we then can choose some $z(\xi)$ to satisfy (3.3.7). Next, reverse the process to get a faithful representation of $\pi_1(S)$ as isometries of $\mathbb{H}^2 \times \mathbb{E}^2$ defined (on the generators) by $\xi \mapsto (\bar{\xi}, (\mathcal{O}(\xi), y(\xi)))$ (by construction the map will be well-defined). Therefore $\pi_1(S)$ is isomorphic to the fundamental group of a geometric Seifert manifold. Then theorem 2.5.10 implies S is homeomorphic to a Seifert geometric manifold of type $\mathbb{H}^2 \times \mathbb{E}^2$. [When $F = T^2$, let us be more explicit: If for some ξ', $(A(\xi') - \mathrm{w}(\bar{\xi}')I)$ is invertible then choose the $z(\xi)$ so that $z(\xi') = (A(\xi') - \mathrm{w}(\bar{\xi}')I)^{-1}e'$ while all the others are 0. Otherwise $\mathrm{w}(\bar{\xi})A(\xi)$ is I or conjugate in $GL_2\mathbb{R}$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. If for all generators $\mathrm{w}(\bar{\xi})A(\xi)$ is I or A for some A conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then let all $z(\xi) = 0$ except for one generator, ξ' (if there is one that is) such that $\mathrm{w}(\bar{\xi}')A(\xi') = A$, let $z(\xi') = -\mathrm{w}(\bar{\xi}')\frac{1}{2}e'$ (note in this case $e' \in \mathrm{Im}(A-I)$). If none of the previous cases arise, then there are two generators, ξ' and ξ'' say, such that $\mathrm{w}(\bar{\xi}')A(\xi')$ and $\mathrm{w}(\bar{\xi''})A(\xi'')$ are not equal but both conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ while for the rest of the generators $\mathrm{w}(\bar{\xi})A(\xi)$ is either I or conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Let $z(\xi') = -\mathrm{w}(\bar{\xi}')\frac{1}{2}p$, $z(\xi'') = -\mathrm{w}(\bar{\xi''})\frac{1}{2}(e' - p)$ where p is the intersection point of the lines $(e' + \mathrm{Im}(A(\xi'') - \mathrm{w}(\bar{\xi''})I))$ and $\mathrm{Im}(A(\xi') - \mathrm{w}(\bar{\xi}')I)$ (therefore $p \in \mathrm{Im}(A(\xi') - \mathrm{w}(\bar{\xi}')I)$ and $(e' - p) \in \mathrm{Im}(A(\xi'') - \mathrm{w}(\bar{\xi''})I)$). For the other generators let $z(\xi) = 0$.]

When $X = \widetilde{SL}_2 \times \mathbb{E}$, $c \neq 0$ and $e - P^{-1}\binom{0}{c} \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$ (where as mentioned earlier the sum can be taken over all elements of $\pi_1^{orb}(B)$ and e' can be replaced by e). Furthermore $\{\mathrm{w}(\bar{\xi})\mathcal{O}(\xi)\}$ is contained in the group $\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ (from section 3.2, $\mathcal{O}(\xi) = \left(\begin{smallmatrix} \mathrm{w}(\xi) & 0 \\ 0 & \mathrm{w}(\bar{\xi}) \end{smallmatrix}\right)$, so $\{\mathrm{w}(\bar{\xi})\mathcal{O}(\xi)\} = \left\{\left(\begin{smallmatrix} \mathrm{w}(\bar{\xi})\mathrm{w}(\xi) & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\}$). Remember $\mathcal{O}(\xi) = PA(\xi)P^{-1}$. So the group $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ is contained in a group $\langle A\rangle \subset GL_2\mathbb{Z}$ where $PAP^{-1} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $P \in GL_2\mathbb{R}$. There are two cases, either $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ is $\{I\}$ or $\langle A\rangle$. In the former case, $e = P^{-1}\binom{0}{c}$, which is nonzero, and so $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I) = 0$. In the latter case, we have $e - P^{-1}\binom{0}{c} \in \mathrm{Im}(A - I)$. Now if $P^{-1}\binom{0}{c} = (A - I)w$ for some $w \in \mathbb{R}^2$ then $\binom{0}{c} = (PAP^{-1} - I)Pw = \binom{*}{0}$, which contradicts $c \neq 0$ and so in both cases $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$.

We shall now show these conditions are not only necessary but sufficient. Suppose the group of monodromies is finite and $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. For this to happen $\dim(\sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)) < 2$. This implies $\mathrm{w}(\bar{\xi})A(\xi)$ all have 1 as an eigenvalue and so are either I or a reflection. Since $A(\xi) \in GL_2\mathbb{Z}$, all the reflections will be the same since $\mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$ will be the common -1-eigenspace. So $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ will be conjugate in $GL_2\mathbb{R}$ to a subgroup of $\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle$. Suppose $\{\mathrm{w}(\bar{\xi})A(\xi)\} = \{I\}$ and $e \neq 0$, then let $z(\xi) = 0$ for all generators and choose P so that $e - P^{-1}\binom{0}{c} = 0$ (possible since e and $c = 2\pi\chi^{orb}(B)$ are nonzero). Then we can reverse the process (as in the above paragraph) to show S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}$. Alternatively suppose $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ is conjugate to $\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ and $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. Choose a matrix Q such that $Q\{\mathrm{w}(\bar{\xi})A(\xi)\}Q^{-1} = \langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle$. Then $Qe \notin \sum \mathrm{Im}(QA(\xi)Q^{-1} - \mathrm{w}(\bar{\xi})I) = \{\binom{x}{0}|x \in \mathbb{R}\}$. Therefore $Qe = \binom{*}{a}$ for some nonzero number a. Let $P = \frac{c}{a}Q$ ($c = 2\pi\chi^{orb}(B)$). Then $P\{\mathrm{w}(\bar{\xi})A(\xi)\}P^{-1} = \langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ and $e - P^{-1}\binom{0}{c} \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. Choose a generator ξ' such that $\mathrm{w}(\bar{\xi}')A(\xi') = A$. Then let $z(\xi') = -\mathrm{w}(\bar{\xi}')\frac{1}{2}(e - P^{-1}\binom{0}{c})$ and let $z(\xi) = 0$ for all the other generators. Consequently, equation (3.3.7) is satisfied. Again we can reverse the process to show S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}$. □

Remark 3.3.8. If we allow the total space to have singularities, ie we consider a Seifert **orbifold** with hyperbolic base (without reflector curves) and general fibre the torus, then we get the same result. No singularities forces extra restrictions on what $\tilde{e}_i$ and $A(s_i)$ can be, but these are determined by the presentation and the proof makes no assumption about them.

Remark 3.3.9. The above theorem can be generalised to higher dimensional tori, so long as theorem 2.5.10 can be extended to them as well. See remark 2.5.11 for comments.

Remark 3.3.10. Let $\eta : S \to B$ be a Seifert fibration with fibre F and aspherical base. Let M be $\mathcal{Z}(\pi_1(F)) \otimes_{\mathbb{Z}} \mathbb{Q}$ considered as a $\mathbb{Z}[\pi_1^{orb}(B)]$-module with the action determined by the monodromy map. Let $e^{\mathbb{Q}}(\eta) \in H^2(\pi_1^{orb}(B), M)$ be the class corresponding to $\pi_1(S)$ as an extension of $\pi_1(S)/\mathcal{Z}(\pi_1(F))$ by $\mathcal{Z}(\pi_1(F))$ [note $\pi_1(S)/\mathcal{Z}(\pi_1(F)) \cong \pi_1^{orb}(B)$ when the general fibre is a torus]. Then $e^{\mathbb{Q}}(\eta)$ is called the *rational euler class* of the fibration. When the general fibre is the torus, it can be shown that $H^2(\pi_1^{orb}(B), M) \cong M/I_{\mathrm{w}}M (= \mathbb{Q}^2/V)$ and via this isomorphism $e^{\mathbb{Q}}(\eta)$ gets mapped to $e \mod V$.

At the end of section 13.4 in [Th2] the rational euler class for a Seifert 3-manifold is defined as the obstruction to the existence of a rational section. Can the rational euler class be similarly defined for 4-manifolds? In [NuRa], Neumann and Raymond construct the rational euler class for 3-manifolds more explicitly and prove a naturality result for finite covers (see especially their theorem 1.2). The definition of the e used in the above theorem is derived from this latter construction.

3.4 Geometric Seifert manifolds with hyperbolic bases

In this section we will give necessary and sufficient conditions for Seifert 4-manifolds with a hyperbolic base to be geometric. This extends section 3.3 which excluded the case when the base had reflector curves. This section tries to include only the parts associated to the reflector curves.

Theorem 3.4.1. *Let S be a Seifert manifold over a hyperbolic base, with general fibre T^2. Let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves, such that the ith reflector curve has k_i corner reflectors so that m_{ij} is the order of the jth corner reflector on the ith reflector curve. Let A be standard map which induces the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, $\tilde{e}_{0j}$, $\tilde{f}_i$, and $\tilde{g}_{ij}$ be the standard parts of the presentation of $\pi_1(S)$.*

Then S is geometric if and only if the group of monodromies is finite, ie it is conjugate in $GL_2\mathbb{Z}$ to a subgroup of $O_2\mathbb{Z}$ or $\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$.

Let $e = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right)$ and $V = \mathcal{I}_{\mathrm{w}}\mathbb{Q}^2$.

More precisely, S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if the group of monodromies is finite and

$$e \in V.$$

S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}$ if and only if the group of monodromies is finite and

$$e \notin V,$$

which implies the group $\{\mathrm{w}(\bar{\xi})A(\xi) | \xi \in \pi_1^{orb}(B)\}$ is conjugate in $GL_2\mathbb{Z}$ to a subgroup of $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$ or $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$.

(Note as in the previous section, we can take V to be $\mathcal{I}_{\mathrm{w}}\mathbb{R}^2 = \sum_{\bar{\xi}\in\pi_1^{orb}(B)} \mathrm{Im}_{\mathbb{R}^2}(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)$.)

Before proving the theorem, consider what is happening in the neighbourhood of a reflector curve. Consider a Seifert bundle $\eta : \bar{\mathcal{R}} \to \bar{\mathcal{A}}$ with T^2 general fibre, above an annulus neighbourhood of a reflector curve. We are trying to determine if there is, and if so what is the nature of, any injective homomorphism $\pi_1(\bar{\mathcal{R}}) \to \mathrm{Isom}(X)$ which preserves the bundle structure (where X is $\mathbb{H}^2\times\mathbb{E}^2$ or $\widetilde{SL}_2\times\mathbb{E}$). More precisely, given an injection $i : \pi_1^{orb}(\bar{\mathcal{A}}) \to \mathrm{Isom}(\mathbb{H}^2)$, we would like to determine all injections $\tilde{i} : \pi_1(\bar{\mathcal{R}}) \to \mathrm{Isom}(X)$ which makes the following diagram commute:

$$\begin{array}{ccc} \pi_1(\bar{\mathcal{R}}) & \xrightarrow{\tilde{i}} & \mathrm{Isom}(X) \\ \Big\downarrow{\scriptstyle \pi_1(\eta)} & & \Big\downarrow{\scriptstyle p} \\ \pi_1^{orb}(\bar{\mathcal{A}}) & \xrightarrow[i]{} & \mathrm{Isom}(\mathbb{H}^2), \end{array}$$

Fix a presentation of $\pi_1(\bar{\mathcal{R}})$ as given in lemma 2.3.1. Then as we saw in theorem 3.3.7, if $\tilde{i}$ exists, $\tilde{i}(h_j)(z,w) = (z, w+p_j)$ for some linearly independent $p_j \in \mathbb{R}^2$. As before let P be the matrix in $GL_2\mathbb{R}$ whose columns are these p_j. Also from before $\tilde{i}(\partial)(z,w) = (i(\bar{\partial})(z), \left(PA(\partial)P^{-1}, y(\partial)\right)(w) + \binom{0}{c(\partial)})$ for some $y(\partial)\in\mathbb{R}^2$, where $A(\partial)$ is the monodromy and where $c(\partial) = 0$ if $X = \mathbb{H}^2\times\mathbb{E}^2$ and $c(\partial)$ corresponds to the parallel translate of $i(\bar{\partial})$ if $X = \widetilde{SL}_2\times\mathbb{E}$.

The following lemma determines necessary and sufficient conditions on P and $y(\partial)$ for $\tilde{i}$ to exist.

Lemma 3.4.2. *Let $\bar{\mathcal{A}}$ be an annulus neighbourhood of a reflector curve and suppose $\eta : \bar{\mathcal{R}} \to \bar{\mathcal{A}}$ is a Seifert bundle with general fibre T^2. Suppose the reflector curve has k corner reflectors, so that the jth corner reflector has order m_j. Then an injection $i : \pi_1^{orb}(\bar{\mathcal{A}}) \to \mathrm{Isom}(\mathbb{H}^2)$, lifts to an injection $\tilde{i} : \pi_1(\bar{\mathcal{R}}) \to \mathrm{Isom}(X)$ (where X is either $\mathbb{H}^2\times\mathbb{E}^2$ or $\widetilde{SL}_2\times\mathbb{E}$) defined as above if and only if*

(i). $P\{A(\xi)|\xi\in\pi_1(\bar{\mathcal{R}})\}P^{-1} \subset O_2\mathbb{R}$ when $X = \mathbb{H}^2\times\mathbb{E}^2$ and $P\{\mathrm{w}(\bar{\xi})A(\xi)|\xi\in\pi_1(\bar{\mathcal{R}})\}P^{-1} \subset \langle\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\rangle$ when $X = \widetilde{SL}_2\times\mathbb{E}$, and

(ii).

$$P^{-1}y(\partial) + \frac{1}{2}\left(\tilde{b} + \sum_{j=1}^{k}\tilde{e}_j/m_j\right) - P^{-1}\begin{pmatrix}0\\c\end{pmatrix} \in \sum_{\bar{\xi}\in\pi_1^{orb}(\bar{\mathcal{A}})} \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I),$$

where $c = 0$ when $X = \mathbb{H}^2\times\mathbb{E}^2$ and $c = 2\pi\chi^{orb}(\bar{\mathcal{A}})$ when $X = \widetilde{SL}_2\times\mathbb{E}$.

Proof. This lemma borrows heavily from theorem 3.3.7, so instead of repeating parts of the proof of that theorem we will mention the similarities and instead concentrate on the differences. For example, in the theorem the first condition (connected to P) in the statement of the lemma was proved to be necessary.

We will first show that the above condition on $y(\partial)$ is necessary. Then we will show that all steps are reversible and hence show the condition is sufficient.

Thus firstly, suppose $i : \pi_1(\bar{\mathcal{A}}) \to \mathrm{Isom}(\mathbb{H}^2)$ does lift to a homomorphism $\tilde{i} : \pi_1(\bar{\mathcal{R}}) \to \mathrm{Isom}(X)$. By the way $\tilde{i}$ was defined above, the induced map $\pi_1(T^2) \to \ker p$ is injective (where p is the projection $\mathrm{Isom}(X) \to \mathrm{Isom}(\mathbb{H}^2)$). It follows that $\tilde{i}$ is injective.

Since $\tilde{i}(\xi)$ is an isometry of X for $\xi \in \pi_1(\bar{\mathcal{R}})$, we have $\tilde{i}(\xi)(z, w) = (i(\bar{\xi})(z), \left(PA(\xi)P^{-1}, y(\xi)\right)(w) + \binom{0}{c(\xi)})$ for some $y(\xi) \in \mathbb{R}^2$, where $\bar{\xi} = \pi_1(\eta)(\xi)$ and where $c(\xi) = 0$ if $X = \mathbb{H}^2 \times \mathbb{E}^2$ and $c(\xi)$ corresponds to the parallel translate of $i(\bar{\xi})$ if $X = \widetilde{SL}_2 \times \mathbb{E}$.

By considering the relations $s_j^{m_j}\tilde{e}_j = 1$, as we did in section 3.3, we see equation (3.3.2)) is satisfied, and so $P^{-1}y(s_j) + \tilde{e}_j/m_j = (A(s_j) - I)z(s_j)$ for some $z(s_j) \in \mathbb{R}^2$. Similarly, $P^{-1}y(r) + \tilde{f}/2 = (A(r) - I)z(r)$, for some $z(r) \in \mathbb{R}^2$.

A convenient way to look at $\pi_1(\bar{\mathcal{R}})$ for this lemma is to consider the presentation given in lemma 2.3.2, which defines a new set of generators σ_j as $\sum_{p=j}^k s_p$. Let $E_j = \sum_{p=j}^k A(\sigma_p^{-1})\tilde{e}_p/m_p$. By combining the expressions for $P^{-1}y(s_j)$ it can be shown that $P^{-1}y(\sigma_j) + A(\sigma_j)E_j = -A(\sigma_j)x(\sigma_j)$ for some $x(\sigma_j) = -\sum_{p=j}^k A(\sigma_p^{-1})(A(s_p) - I)z(s_p) = -\sum_{p=j}^k (A(\sigma_{p+1}^{-1}\sigma_p) - I)A(\sigma_p^{-1})z(s_p) \in \sum_{p=j}^k \mathrm{Im}(A(\sigma_{p+1}^{-1}\sigma_p) - I)$. However the group generated by $\sigma_{p+1}^{-1}\sigma_p$, $p = j, \ldots, k$ is also generated by σ_p, $p = j, \ldots, k$ (since $\sigma_p = \left(\prod_{q=p}^k (\sigma_{q+1}^{-1}\sigma_q)^{-1}\right)^{-1}$ and $\sigma_{k+1} = 1$), so by corollary 3.3.5, $x(\sigma_p) \in \sum \mathrm{Im}(A(\xi) - I)$ where the sum is over $\xi \in \langle \sigma_j, \ldots, \sigma_k \rangle$.

Next by considering the relation $\partial^{-1}\sigma_1 r \partial r^{-1} = \tilde{b}$ in a similar way to getting equation (3.3.5), we get:

$$A(\sigma_1 r) = A(\partial)A(r)A(\partial^{-1}) \tag{3.4.1}$$

$$\begin{aligned} &(I - A(\sigma_1 r))P^{-1}y(\partial) + (I - A(\partial))A(r)P^{-1}y(r) \\ &\qquad + A(\sigma_1 r)P^{-1}y(\sigma_1) + A(\sigma_1 r \partial)P^{-1}\begin{pmatrix} 0 \\ c \end{pmatrix} = A(\sigma_1 r \partial)\tilde{b} \end{aligned} \tag{3.4.2}$$

Here c is 0 when $X = \mathbb{H}^2 \times \mathbb{E}^2$ and c is a correction term corresponding to the holonomy of the loop projected to the base when $X = \widetilde{SL}_2 \times \mathbb{E}$. By considering the double cover of $\bar{\mathcal{R}}$ induced by doubling the base along the reflector curve, we see the above relation is preserved (except $r\partial r^{-1}$ is now a loop corresponding to a lift of the other boundary). Projecting the relation to the base, we see it corresponds to traversing the base, thus the correction is the holonomy of the base: $\mathcal{A}$, that is $c = 2\pi\chi^{orb}(\mathcal{A})$. By definition of the Euler characteristic, $c = 4\pi\chi^{orb}(\bar{\mathcal{A}})$.

We can rewrite (3.4.2) by using the expressions for $P^{-1}y(r)$, $P^{-1}y(\sigma_1)$, equations (2.3.10) and (2.3.11) and the equation $A(\xi)P^{-1}\binom{0}{c} = P^{-1}\binom{0}{\mathrm{w}(\bar{\xi})c}$. Therefore, we get:

$$\begin{aligned} &(I - A(\sigma_1 r))\left(P^{-1}y(\partial) + A(\partial)\tilde{b}/2 - A(r)E_1/2 - P^{-1}\begin{pmatrix} 0 \\ c/2 \end{pmatrix}\right. \\ &\qquad \left. - \tilde{f}/4 - A(r)x(\sigma_1)/2 - A(\partial)z(r) - (A(r) - I)z(r)/2\right) \\ &= (I + A(\sigma_1 r))\left(\tilde{G}_1/4 + A(r)E_1/2 + A(r)x(\sigma_1)/2 + (A(r) - I)z(r)/2\right) \end{aligned} \tag{3.4.3}$$

By multiplying both sides by $(I - A(\sigma_1 r))$ we see that both sides are equal to zero. Therefore

$$\Big(P^{-1}y(\partial) + A(\partial)\tilde{b}/2 - A(r)E_1/2 - P^{-1}\begin{pmatrix}0\\c/2\end{pmatrix}$$
$$-\tilde{f}/4 - A(r)x(\sigma_1)/2 - A(\partial)z(r) - (A(r)-I)z(r)/2\Big) \in \ker(I - A(\sigma_1 r)),$$

and

$$\Big(\tilde{G}_1/4 + A(r)E_1/2 + A(r)x(\sigma_1)/2 + (A(r)-I)z(r)/2\Big) \in \ker(I + A(\sigma_1 r)).$$

However, $\ker(I - A(\sigma_1 r)) = \mathrm{Im}(I + A(\sigma_1 r))$ and $\ker(I + A(\sigma_1 r)) = \mathrm{Im}(I - A(\sigma_1 r))$. So by rearranging and using equation (2.3.9), we get the following equations:

$$\begin{aligned} P^{-1}y(\partial) + A(\partial)\tilde{b}/2 - A(r)E_1/2 - P^{-1}\begin{pmatrix}0\\c/2\end{pmatrix} \\ = A(r)x(\sigma_1)/2 + (A(\partial)-I)z(r) + (A(r)+I)(z(r)/2 + \tilde{f}/8) + (A(\sigma_1 r)+I)z(\partial) \end{aligned} \tag{3.4.4}$$

$$\tilde{G}_1/4 + A(r)E_1/2 + A(r)x(\sigma_1)/2 + (A(r)-I)z(r)/2 = (I - A(\sigma_1 r))z(d), \tag{3.4.5}$$

for some $z(\partial)$ and $z(d) \in \mathbb{R}^2$.

By writing $A(\partial)\tilde{b} - A(r)E_1$ as $\tilde{b} + E_1 + (A(\partial)-I)\tilde{b} - (A(r)+I)E_1$, $(A(r)+I)$ as $(A(\sigma_1 r)+I)A(\sigma_1) + (I - A(\sigma_1))$ and for all j, $A(r)(A(\sigma_j) - I)$ as $(I - A(\sigma_j))A(\sigma_j^{-1}r)$ we see (3.4.4) becomes

$$P^{-1}y(\partial) + \frac{1}{2}\left(\tilde{b} + E_1\right) - P^{-1}\begin{pmatrix}0\\c/2\end{pmatrix} \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I), \tag{3.4.6}$$

where the sum is over a set of generators: $\{\sigma_1, \ldots, \sigma_k, \partial, \sigma_1 r\}$, and so by corollary 3.3.5 can be taken to be over $\pi_1^{orb}(\bar{\mathcal{A}})$. Observe, when $X = \widetilde{SL}_2 \times \mathbb{E}$, $\frac{1}{2}c = 2\pi\chi^{orb}(\bar{\mathcal{A}})$. Also E_1 and $\sum_{j=1}^k \tilde{e}_j/m_j$ are equal modulo $\sum_\xi (A(\xi) - \mathrm{w}(\bar{\xi})I)$.

This shows the necessity of the lemma. To show sufficiency we will show this method is reversible. The only step which does not immediately seem to be reversible is the last step. So to complete the lemma, we need to show that for any $y(\partial)$ which satisfies equation (3.4.6), there are some $x(\sigma_1) \in \sum_{j=1}^k \mathrm{Im}(A(\sigma_j) - I)$, $z(r)$, $z(\partial)$ and $z(d)$ which satisfy both equation (3.4.4) and equation (3.4.5). That is, for all $v \in \mathcal{S} = \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$, there are some $x(\sigma_1)$, $z(r)$, $z(\partial)$ and $z(d)$ such that

$$A(r)x(\sigma_1)/2 + (A(\partial)-I)z(r) + (A(r)+I)z(r)/2 + (A(\sigma_1 r)+I)z(\partial) = v \tag{3.4.7}$$

$$A(r)x(\sigma_1)/2 + \tilde{G}_1/4 + A(r)E_1/2 + (A(r)-I)z(r)/2 + (A(\sigma_1 r)-I)z(d) = 0 \tag{3.4.8}$$

In $\pi_1(\bar{\mathcal{R}})$, $\partial^{-1}\sigma_1 r \partial r^{-1} = \tilde{b}$. By considering its image by the monodromy map we see $A(\sigma_1)$ is the commutator $[A(\partial), A(r)]$ (and we also derive equation (3.4.1)). Since $\{A(\xi) | \xi \in \pi_1(S)\}$ is a dihedral group (by corollary 3.3.2), $A(\sigma_1) \neq I$ implies $(A(\sigma_1) - I)$ is invertible (lemma 3.4.3).

Suppose $\sum_{j=1}^k \mathrm{Im}(A(\sigma_j) - I) = \mathbb{R}^2$. Let $z(r) = 0$, and $z(d) = z(\partial) = v + \tilde{G}_1/4 + A(r)E_1/2$ and then choose $x(\sigma_1)$ to satisfy equation (3.4.8) (note $x(\sigma_1)$ can take any value). Equation (3.4.7) is then satisfied and so the result is proved in this case.

Instead suppose $\sum_{j=1}^k \mathrm{Im}(A(\sigma_j) - I) \neq \mathbb{R}^2$, so in particular $A(\sigma_1) = I$. Notice $z(r)$ is independent of (3.4.5) since we can absorb it into $z(d)$. If $\sum_{j=1}^k \mathrm{Im}(A(\sigma_j) - I) \subset \mathrm{Im}(A(\partial) - I) + \mathrm{Im}(A(r) + I)$, first we choose

$x(\sigma_1)$ and $z(d)$ to satisfy (3.4.8), then choose $z(r)$ and $z(\partial)$ to get (3.4.7) and the result. [Note a choice for $x(\sigma_1)$ and $z(d)$ is always possible. To see this consider equation (2.3.8) divided by $4m_j$ and taken mod $\mathrm{Im}(A(r)-I)+\sum_{p=1}^{k}\mathrm{Im}(A(\sigma_p)-I)$. By summing the set of equations over j, we see $\tilde{G}_1/4+A(r)E_1/2$ is in $\mathrm{Im}(A(r)-I)+\sum_{p=1}^{k}\mathrm{Im}(A(\sigma_p)-I)$.]

Alternatively suppose $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)\not\subset \mathrm{Im}(A(\partial)-I)+\mathrm{Im}(A(r)+I)$. Then firstly $\mathrm{Im}(A(\partial)-I)+\mathrm{Im}(A(r)+I)\neq\mathbb{R}^2$. Therefore $A(r)\neq I$ and so $A(r)=Q\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)Q^{-1}$ for some $Q\in GL_2\mathbb{R}$ (lemma 2.1.7). Also $\mathrm{Im}(A(\partial)-I)\subseteq\mathrm{Im}(A(r)+I)=Q\{\left(\begin{smallmatrix}x\\0\end{smallmatrix}\right)|x\in\mathbb{R}\}$. However since $A(\sigma_1)=I$, $A(\partial)$ commutes with $A(r)$ and so $A(\partial)=I$ or $-A(r)$. In either case, the expression $(A(\partial)-I)z(r)+(A(r)+I)z(r)/2+(A(\sigma_1 r)+I)z(\partial)$ from equation (3.4.7) is contained in $\mathrm{Im}(A(r)+I)$ so we may as well take $z(r)=0$. Also we have assumed $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)\not\subset\mathrm{Im}(A(r)+I)=Q\{\left(\begin{smallmatrix}x\\0\end{smallmatrix}\right)|x\in\mathbb{R}\}$.

Suppose $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)\supseteq\mathrm{Im}(A(r)-I)$. Hence $\mathcal{S}=\mathbb{R}^2$ and $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)=\mathrm{Im}(A(r)-I)$ because of our earlier assumption that $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)\neq\mathbb{R}^2$. We choose $x(\sigma_1)$ and $z(\partial)$ to satisfy equation (3.4.7) and then choose $z(d)$ to satisfy equation (3.4.8) [which is again possible since $\tilde{G}_1/4+A(r)E_1/2$ is in $\mathrm{Im}(A(r)-I)+\sum_{p=1}^{k}\mathrm{Im}(A(\sigma_p)-I)=\mathrm{Im}(A(r)-I)$]. Lastly suppose (aiming for a contradiction) $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)$ does not contain $\mathrm{Im}(A(r)-I)$ and is not contained in $\mathrm{Im}(A(r)+I)$. Then $\sum_{j=1}^{k}\mathrm{Im}(A(\sigma_j)-I)$ is 1-dimesional and does not equal $\mathrm{Im}(A(r)-I)$ or $\mathrm{Im}(A(r)+I)$. Therefore there is a p such that $A(\sigma_p)\neq I$. Choose p to be the largest such number. Then $A(\sigma_p)=A(s_p)A(\sigma_{p+1})=A(s_p)$ which is conjugate in $GL_2\mathbb{R}$ to $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ (lemma 2.1.7). However $A(r)A(\sigma_p)A(r)^{-1}=A(\sigma_p)^{-1}$, and therefore $A(\sigma_p)=\pm A(r)$. Both cases give the desired contradiction. ☐

Proof of Theorem 3.4.1. We will basically follow the proof of theorem 3.3.7. Suppose S is geometric of type X (where X is either $\mathbb{H}^2\times\mathbb{E}^2$ or $\widetilde{SL}_2\times\mathbb{E}$).

Since S is geometric, there is a map from $\pi_1(S)$ to $\mathrm{Isom}(X)$. By projecting to the base, we understand this map partially. By considering each of the relations we get restrictions of how the elements of $\pi_1(S)$ act, often involving the standard parts of the presentation ($\tilde{e}_{ij}$ for instance). Especially, this implies the group of monodromies is finite. By considering the relations connected with the neighbourhoods of cone points and corner reflectors ($s_{ij}^{m_{ij}}\tilde{e}_{ij}=1$), as in section 3.3, we see equation (3.3.2) is satisfied. Lemma 3.4.2 considers the relations connected with the neighbourhood of each reflector curve. Having considered the relations connected with the local information, the only relation left to consider is the relation which is connected to the global information. As before let $d=\prod_{p=1}^{g}[t_p,u_p]$ when the base is orientable and $d=\prod_{p=1}^{g}v_p^2$ when the base is non-orientable. Then the last relation to consider is

$$d\prod_{j=1}^{k_0}s_{0j}\prod_{i=1}^{l}\partial_i=\tilde{a}.$$

By considering this relation, we get an equation similar to equation (3.3.5) or (3.3.6), which is effectively the composition of the maps $(\mathcal{O}(\xi),y(\xi))$ for the relation plus a correction term. The correction term is 0 when $X=\mathbb{H}^2\times\mathbb{E}^2$. When $X=\widetilde{SL}_2\times\mathbb{E}$ the correction corresponds to the total holonomy of the base with neighbourhoods of the reflector curves taken out, which has value $2\pi\chi^{orb}\left(B\setminus\left(\bigcup_{i=1}^{l}\bar{\mathcal{A}}_i\right)\right)$. Using equation (3.3.2) ($P^{-1}y(s_{0j})+\tilde{e}_{0j}/m_{ij}=(A(s_{0j})-I)z(s_{0j})$ for some $z(s_{0j})\in\mathbb{R}^2$) and lemma 3.4.2, we can make substitutions for $y(s_{0j})$ and $y(\partial_i)$. By choosing a new set of generators and premultiplying by P^{-1}, we can get a new equation analogous to (3.3.7):

$$e' - P^{-1}\binom{0}{c} = \sum (A(\xi) - \mathrm{w}(\bar{\xi})I)z(\xi), \tag{3.4.9}$$

where e' is an expression dependent only on the presentation

$$\begin{aligned} e' &= \tilde{a} + A(d)\sum_{j=1}^{k}\left(\prod_{1\le p<j} A(s_{0p})\right)\tilde{e}_{0j}/m_{0j} \\ &\quad + A(d)\left(\prod_{j=1}^{k_0} A(s_{0j})\right)\sum_{i=1}^{l}\left(\prod_{1\le p<i} A(\partial_p)\right)\frac{1}{2}\left(\tilde{b}_i + \sum_{j=1}^{k_i}\tilde{e}_{ij}/m_{ij}\right), \end{aligned}$$

c is a correction term and the sum is over the new set of generators. The correction is 0 when $X = \mathbb{H}^2 \times \mathbb{E}^2$. When $X = \widetilde{SL}_2 \times \mathbb{E}$, $c = 2\pi\left(\chi^{orb}\left(B \setminus \left(\bigcup_{i=1}^{l} \bar{\mathcal{A}}_i\right)\right) + \sum_{i=1}^{l}\chi^{orb}(\bar{\mathcal{A}}_i)\right)$, which is equal to $2\pi\chi^{orb}(B)$.

As in theorem 3.3.7, we can use equation (3.4.9) to show that S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if the group of monodromies is finite and $e' \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$ and that S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}$ if and only if the group of monodromies is finite and $e' \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$, where the sum is over the whole group $\pi_1(S)$. It can be shown that e (as in the statement of the theorem) and e' are equal modulo $\sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$, hence the theorem. □

Lemma 3.4.3. *Suppose G is a dihedral group of order $2n$, with presentation $\langle r, s | rsr^{-1} = s^{-1}, r^2 = 1 = s^n\rangle$. The commutator subgroup G' is then $\langle s^2 | s^n = 1\rangle$ which is cyclic of order $n/2$ if n even, or cyclic of order n if n odd. Incidentally, G/G' is then $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$ when n is even or odd respectively.*

In particular, if $G = PO_2\mathbb{Z}P^{-1} \cong D_8$, then $G' = \{\pm I\}$, also if $G = P\left\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\rangle P^{-1} \cong D_{12}$, then $G' = P\left\langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)\right\rangle P^{-1}$.

Proof. Direct calculation will quickly get the result. Alternatively, the abelianisation of G can be seen to have the extra relation $s^2 = 1$, therefore G' is generated by s^2. □

Remark 3.4.4. Analogous to the remark 3.3.8, we can replace the phrase Seifert manifold with Seifert orbifold in the above theorem and have the same result.

Theorem B in [Ue2] is the above theorem restricted to the case when the total space is orientable. Ue states his result up to fiber preserving diffeomorphism and instead of checking if 'e' is in a space, checks if something he calls the 'rational euler class' is zero or not (they are equivalent). The proof of the theorem above is an extension of Ue's proof, however Ue's work is simpler since orientability forces many of the monodromies to be trivial (since $\mathrm{w}(\bar{\xi}) = \det A(\xi)$ when the total space is orientable).

The proof of the above theorem tries to avoid making assumptions about the dimension of the fibre. Indeed the necessity of the algebraic conditions is true for all dimensions. The sufficiency is not clear when the dimension is bigger than 2 (dimension 1 can be proved exactly as the dimension 2 case above). There are two problems. The first one is that theorem 2.5.10 may not extend to higher dimensions. Remark 2.5.11

shows it is not clear if the base has corner reflectors or if the dimension of the fibre is greater than 3. The second problem is that how the proof deals with corner reflectors (namely knowing the nature of $A(\sigma_{i1})$) is highly dependent on the fibre being T^2. Are these problems avoidable? By arguments similar to theorem 3.4.1, we can obtain the following partial generalisation:

Theorem 3.4.5. *Let S be a Seifert manifold over a hyperbolic base, with general fibre T^n, $1 \leq n \leq 3$. Let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves with no corner reflectors . Let A be the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, $\tilde{e}_{0j}$, $\tilde{f}_i$ be the standard parts of the presentation of $\pi_1(S)$.*

Then S is geometric if and only if the group of monodromies is finite.

Let $e = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\left(\tilde{b}_i\right)$ and let $V = \mathcal{I}_{\mathrm{w}}\mathbb{Q}^n$.

More precisely, S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^n$ if and only if the group of monodromies is finite and

$$e \in V.$$

S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ if and only if the group of monodromies is finite and

$$e \notin V,$$

which implies the group $\{\mathrm{w}(\bar{\xi})A(\xi) | \xi \in \pi_1^{orb}(B)\}$ is conjugate in $GL_n\mathbb{R}$ to a subgroup of $O_{n-1}\mathbb{R} \times \{1\}$.

The above theorem includes the classical case as given by Scott ([Sc2]).

Corollary 3.4.6. *Let S be a Seifert manifold over a hyperbolic base, with general fibre S^1. Let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves (there will be no corner reflectors). Let A be the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, and $\tilde{e}_{ij}$ be the standard parts of the presentation of $\pi_1(S)$.*

Then S is geometric.

Let $e' = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l} \tilde{b}_i$ if S is orientable or 0 if it is non-orientable.

More precisely, S is geometric of type $\mathbb{H}^2 \times \mathbb{E}$ if and only if $e' = 0$.

S is geometric of type $\widetilde{SL}_2$ if and only if $e' \neq 0$.

Proof. The group of monodromies for any Seifert 3-manifold lies in the group $GL_1\mathbb{Z} = \{\pm 1\}$. Therefore the group of monodromies is finite.

If S is orientable, $\det(A(\xi)) = \mathrm{w}(\bar{\xi})$, therefore since the fibre is one dimensional, $\sum_{\bar{\xi} \in \pi_1^{orb}(B)} \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I) = \{0\}$. Therefore $e = e' = 0$ gives an orientable manifold of type $\mathbb{H}^2 \times \mathbb{E}$ and $e = e' \neq 0$ gives an orientable manifold of type $\widetilde{SL}_2$.

If S is non-orientable, $\sum_{\bar{\xi} \in \pi_1^{orb}(B)} \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I) = \mathbb{R}$ and so $e \in V$ for all e, therefore it is geometric of type $\mathbb{H}^2 \times \mathbb{E}$. □

3.5 Geometric Klein bottled fibred 4-manifolds

In this section, we will consider the case when the fibres are Klein bottles. Recall from lemma 2.3.6, that this means the base has no corner reflectors.

In the following, we suppose $\pi_1(Kb) = \langle h_1, h_2 | h_1 h_2 h_1^{-1} h_2 = 1\rangle$.

Theorem 3.5.1. *Let S be a Seifert manifold over a hyperbolic base B, with general fibre Kb. Then S is geometric. Suppose that the base has l reflector curves and k cone points, so that m_i is the order of the ith cone point. Let $A(\xi)$ be the automorphisms from the presentation of $\pi_1(S)$ which send h_1 to $h_1^{\epsilon(\xi)} h_2^{c(\xi)}$ and h_2 to $h_2^{\delta(\xi)}$ and let $\tilde{e}_i = h_1^{e_{i1}} h_2^{e_{i2}}$, $\tilde{b}_i = h_1^{b_{i1}} h_2^{b_{i2}}$ and $\tilde{a} = h_1^{a_1} h_2^{a_2}$ be the standard parts of the presentation of $\pi_1(S)$.*

Let $e = a_1 + \sum_{i=1}^{k} e_{i1}/m_i + \frac{1}{2}\sum_{i=1}^{l} b_{i1}$ and let $V = \sum_{\bar{\xi}\in\pi_1^{orb}(B)} \mathrm{Im}(\epsilon(\xi) - \mathrm{w}(\bar{\xi}))$.

More precisely, S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if

$$e \in V.$$

S is geometric of type $\widetilde{SL}_2 \times \mathbb{E}$ if and only if

$$e \notin V,$$

ie $\epsilon(\xi) = \mathrm{w}(\bar{\xi})$ and $e \neq 0$.

We shall give two different proofs for this theorem. The first proof follows the corresponding result for T^2 fibres, theorem 3.4.1. In the second proof, we first show the orientation cover is geometric, then we add one more isometry to show the manifold is geometric.

Proof. To prove this result we will first suppose that S is geometric and then prove the necessity of the above conditions. We will then show that the process can be reversed and so the conditions are also sufficient.

Suppose a Seifert manifold with Kb fibres is geometric of type X where X is $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{SL}_2 \times \mathbb{E}$. When $X = \mathbb{H}^2 \times \mathbb{E}^2$, $h_1(w, z) = (w, P\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) P^{-1} z + P\left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right))$ and $h_2(w, z) = (w, z + P\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$, for some $P \in GL_2\mathbb{R}$. When X is $\widetilde{SL}_2 \times \mathbb{E}$, then $h_1(w, z) = (w, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) z + \left(\begin{smallmatrix} 0 \\ \mu/2 \end{smallmatrix}\right))$ and $h_2(w, z) = (w, z + \left(\begin{smallmatrix} \lambda \\ 0 \end{smallmatrix}\right))$ for some nonzero real numbers μ and λ. So effectively it is the same as the $\mathbb{H}^2 \times \mathbb{E}^2$ case with $P = \left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right)$. The other generators ξ act via $\xi(w, z) = (\bar{\xi}(w), \mathcal{O}(\xi)z + y(\xi))$ plus a correction on the $\mathbb{C}$ factor when $X = \widetilde{SL}_2 \times \mathbb{E}$. From the relation $\xi h_i \xi^{-1} = A(\xi) h_i$, we see $\mathcal{O}(\xi) = P\begin{pmatrix} \epsilon(\xi) & 0 \\ 0 & \delta(\xi) \end{pmatrix} P^{-1}$, and $y(\xi) = P\left(\begin{smallmatrix} x(\xi) \\ -c(\xi)/2 \end{smallmatrix}\right)$, where $x(\xi)$ is an arbitrary real number and $A(\xi)$ is the automorphism which sends h_1 to $h_1^{\epsilon(\xi)} h_2^{c(\xi)}$ and h_2 to $h_2^{\delta(\xi)}$.

For each cone point, the relation $s_i^{m_i} \tilde{e}_i = 1$ shows that $\left(\begin{pmatrix} \epsilon(s_i) & 0 \\ 0 & \delta(s_i) \end{pmatrix}, P^{-1} y(s_i)\right)$ represents an affine homeomorphism of $Kb = \mathbb{R}^2/H$ where $H = \left\langle \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right)\right), \left(I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)\right\rangle$, with order m_i (since it normalises H by the relation $s_i h_j s_i^{-1} = A(s_i) h_j$ and has a power in H by this relation). Lemma 2.1.9 then shows $P^{-1} y(s_i) = \left(\begin{smallmatrix} -e_{i1}/2m_i + (\epsilon(s_i) - 1) z(s_i) \\ -c(s_i)/2 \end{smallmatrix}\right)$ for some $z(s_i) \in \mathbb{R}$ (see remark 2.1.10).

For each reflector curve, we have the relations $r_i^2 \tilde{f}_i = 1$ and $\partial_i^{-1} r_i \partial_i r_i^{-1} = \tilde{b}_i$. We claim the projection of $P^{-1} y(\partial_i)$ to the first factor (which is $x(\partial_i)$) is $-b_{i1}/4 + c' + (\epsilon(\partial_i) - 1) z(\partial_i) + (\epsilon(r_i) + 1) z(r_i)$ where $z(\partial_i)$

and $z(r_i)$ are in $\mathbb{R}$, and $c' = 0$ when geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ or $c' = \frac{1}{\mu} 2\pi\chi^{orb}(\bar{\mathcal{A}}_i)$ when geometric of type $\widetilde{SL}_2 \times \mathbb{E}$ ($P = \left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right)$). We will summarise the proof of the claim which is an adaptation of lemma 3.4.2. Firstly, similar to the above paragraph, $P^{-1}y(r_i) = \left(\begin{smallmatrix} x(r_i) \\ -c(r_i)/2 \end{smallmatrix}\right)$ where $x(r_i) = -f_{i1}/4 + (\epsilon(r_i) - 1)z'(r_i)$ for some $z'(r_i) \in \mathbb{R}$. Note, if $\epsilon(r_i) = -1$ then $x(r_i)$ has no constraints. Next, we wish to form the analogue to equation (3.4.2). The element $b_i = h_1^{b_{i1}} h_2^{b_{i2}}$ acts via the map $(w, z) \mapsto \left(w, P \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{b_{i1}} P^{-1}z + P\left(\begin{smallmatrix} b_{i1}/2 \\ b_{i2}(-1)^{b_{i1}} \end{smallmatrix}\right)\right)$. The relation $\partial_i^{-1} r_i \partial_i r_i^{-1}$ acts via the map

$$(w,z) \mapsto \left(w, P \begin{pmatrix} \epsilon(\partial_i^{-1} r_i \partial_i r_i^{-1}) & 0 \\ 0 & \delta(\partial_i^{-1} r_i \partial_i r_i^{-1}) \end{pmatrix} P^{-1}z + P\begin{pmatrix} \epsilon(\partial_i)(\epsilon(r_i)-1)x(\partial_i) + (\epsilon(\partial_i)-1)x(r_i) \\ -c(\partial_i^{-1} r_i \partial_i r_i^{-1})/2 \end{pmatrix} + \begin{pmatrix} 0 \\ c'' \end{pmatrix}\right),$$

where c'' is 0 when $X = \mathbb{H}^2 \times \mathbb{E}^2$ and corresponds to the holonomy of the loop corresponding to this relation when $X = \widetilde{SL}_2 \times \mathbb{E}$. From the parallel argument in lemma 3.4.2, $c'' = 4\pi\chi^{orb}(\bar{\mathcal{A}}_i)$. Note $\left(\begin{smallmatrix} 0 \\ c'' \end{smallmatrix}\right) = P\left(\begin{smallmatrix} 2c' \\ 0 \end{smallmatrix}\right)$ where c' is defined above. By equating these two actions, and looking at the coefficient of $P\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ of the constant terms we obtain the analogue to equation (3.4.2):

$$\epsilon(\partial_i)(\epsilon(r_i)-1)x(\partial_i) + (\epsilon(\partial_i)-1)x(r_i) + 2c' = b_{i1}/2$$

When $\epsilon(r_i) = -1$, then $x(r_i)$ has no constraints and $x(\partial_i) = \epsilon(\partial_i)(-b_{i1}/4 + c' + \epsilon(\partial_i) - 1)x(r_i)/2)$. If $\epsilon(\partial_i) = 1$ then this result shows $x(\partial_i) = -b_{i1}/4 + c'$ which agrees with the claim. If $\epsilon(\partial_i) = -1$, then $x(\partial_i)$ has no constraints and again this agrees with the claim. Similarly, if $\epsilon(r_i) = 1$ then $x(\partial_i)$ has no constraints and the claim is satisfied.

From the other sort of relations, we will get expressions like equation (3.3.5) as in theorem 3.3.7. By premultiplying by P^{-1} and rewriting everything in terms of $\begin{pmatrix} \epsilon(\xi) & 0 \\ 0 & \delta(\xi) \end{pmatrix}$ and these $y(\xi)$ we get a new expression. Projection to the second factor gives a relationship between the $c(\xi)$ and $\tilde{a}$ which is forced by the presentation anyway. Projection to the first factor (noting $\mathrm{w}(\bar{s}_i) = 1$): shows that $e \in V$ when $X = \mathbb{H}^2 \times \mathbb{E}^2$ and $e \notin V$ when $X = \widetilde{SL}_2 \times \mathbb{E}$, where $V = \sum_{\bar{\xi} \in \pi_1^{orb}(B)} \mathrm{Im}(\epsilon(\xi) - \mathrm{w}(\bar{\xi}))$.

Conversely consider an arbitrary Seifert manifold with Kb fibres. Then either e is in V or it is not in V. In either case we can reverse the process to show that $\pi_1(S)$ is isomorphic to the fundamental group of a geometric manifold. Then by theorem 2.5.10, S is geometric.

Note for $h \in H$, $\epsilon(h) = 1$, so $\epsilon(\xi)$ is determined by $\bar{\xi}$. □

Alternate Proof. The basic idea is to first show that the orientation cover is geometric. We will then show that $\pi_1(S)$ is isomorphic to a group of isometries thus showing S is geometric, by 2.5.13.

Let S' be the orientation cover of S, which will also be a Seifert manifold. Since $D^2 \times Kb$ is non-orientable, S' cannot have Kb as the general fibre, and so has T^2 fibres. Let B' be the base of S'. Then the degree of the covering of $S' \to S$ equals the product of the degrees of the coverings $T^2 \to Kb$ and $B' \to B$. However S' is an orientation cover, thus the product is 2, so the first is 2 and the second is 1, which means $B' = B$.

For each $\xi \in \pi_1(S)$, let $A(\xi)_T$ be the restriction of $A(\xi)$ to the unique maximal abelian subgroup (which corresponds to the T^2 which covers Kb). Then $A(\xi)_T = \begin{pmatrix} \epsilon(\xi) & 0 \\ 0 & \delta(\xi) \end{pmatrix}$. Note $A(h_1)_T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $A(h_2)_T = I$.

Now if ξ in $\pi_1(S)$ preserves orientation, it must preserve orientation on both fibre and base, or it must reverse the orientation of both. In general, $\mathrm{w}(\xi) = \mathrm{w}(\bar{\xi})\,\mathrm{w}(A(\xi)) = \mathrm{w}(\bar{\xi})\det(A(\xi)_T) = \mathrm{w}(\bar{\xi})\epsilon(\xi)\delta(\xi)$. Therefore $\pi_1(S') = \{\xi \in \pi_1(S) | \delta(\xi) = \mathrm{w}(\bar{\xi})\epsilon(\xi)\}$ and $\pi_1(S) = \langle \pi_1(S'), h_1\rangle$. The group of monodromies of $\pi_1(S')$ equals the image of $A(-)_T$ which is contained in the finite group $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), -I\rangle$. Therefore, by theorem 3.4.1 S' is geometric and so $\pi_1(S')$ is isometric to a group of isometries. Notably $\pi_1(F)$ acts via the following isometries: $h_1^2(w,z) = (w, z + P\binom{1}{0})$ and $h_2(w,z) = (w, z + P\binom{0}{1})$ for some $P \in GL_2\mathbb{R}$ such that $P\,\mathrm{Im}\,A(-)_T P^{-1}$ is in $O_2\mathbb{R}$ (more precisely, when the geometry is $\widetilde{SL}_2 \times \mathbb{E}$, $PA(\xi)_T P^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{w}(\bar{\xi}) \end{smallmatrix}\right)$). We claim that by adjusting P as necessary, the map $(w,z) \mapsto (w, P\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)P^{-1}z + P\binom{1/2}{0})$ is an isometry whose square equals the action of h_1^2. Thus we can define the action of h_1 by this map and so $\pi_1(S)$ is isomorphic to a group of isometries. By corollary 2.5.13 this means S is geometric.

Before proving the claim, we will find conditions to separate the two geometries. The presentation of $\pi_1(S')$, in particular the standard relations, determine the geometry of S. To make these relations easier to find, we will consider a different presentation of $\pi_1(S)$. Now for each generator, ξ, of $\pi_1(S)$ which corresponds to a generator of $\pi_1^{orb}(B)$, either ξ or ξh_1 is in $\pi_1(S')$. By changing sections if necessary (and thus getting an isomorphic presentation), we can suppose $\xi \in \pi_1(S')$ for each of these generators. [Suppose $\epsilon(\xi) = \mathrm{w}(\bar{\xi})$ for all $\xi \in \pi_1(S)$. Note changing v_p to $v_p h_1$, t_p to $t_p h_1$, u_p to $u_p h_1$ or r_i to $r_i h_1$ does not change a_1, e_{i1} or b_{i1}. Changing s_i to $s_i h_1$ adds 1 to a_i and subtracts m_i from e_{i1}. Changing ∂_i to $\partial_i h_1$ adds 1 to a_i and subtracts 2 from b_i. So e and $e \mod V$ are invariant by these changes of section. If $\epsilon(\xi) \neq \mathrm{w}(\bar{\xi})$ for some ξ, then $V = \mathbb{R}$ and so $e \mod V$ is invariant by changes of section. Therefore changing the section, does not alter the conditions $e \in V$ and $e \notin V$.]

Next, consider a relation in $\pi_1^{orb}(B)$: $\bar{\xi}_1\bar{\xi}_2\ldots\bar{\xi}_p = 1$. This lifts to a relation in $\pi_1(S)$ of the form $\xi_1\xi_2\ldots\xi_p = h_1^{\alpha_1}h_2^{\alpha_2}$. Now the ξ_i were chosen to be in $\pi_1(S')$, therefore $\epsilon(\xi_1\xi_2\ldots\xi_p) = \delta(\xi_1\xi_2\ldots\xi_p)\,\mathrm{w}(\bar{\xi}_1\bar{\xi}_2\ldots\bar{\xi}_p) = \delta(\xi_1\xi_2\ldots\xi_p)$. This implies $1 = \epsilon(h_1^{\alpha_1}h_2^{\alpha_2}) = \delta(h_1^{\alpha_1}h_2^{\alpha_2}) = (-1)^{\alpha_1}$, and so α_1 is even. So relations of this type, are also relations of $\pi_1(S')$. Consequently, for each standard part of the presentation of $\pi_1(S)$: $\tilde{a}, \tilde{e}_i, \tilde{b}_i$ etc., $\tilde{\alpha} = \binom{\alpha_1}{\alpha_2}$ in general, the corresponding part of $\pi_1(S')$ is given by $\tilde{\alpha}' = \binom{\alpha_1/2}{\alpha_2}$. Let $e' = \binom{a_1/2}{a_2} + \sum_{i=1}^k \binom{e_{i1}/2m_i}{e_{i2}/m_i} + \frac{1}{2}\sum_{i=1}^l \binom{b_{i1}/2}{b_{i2}}$ and $V' = \sum_{\xi\in\pi_1(S')} \mathrm{Im}(A(\xi)_T - \mathrm{w}(\bar{\xi})I) = \sum_{\xi\in\pi_1(S')}\mathrm{Im}(\epsilon(\xi) - \mathrm{w}(\bar{\xi})) \times \sum_{\xi\in\pi_1(S')}\mathrm{Im}(\epsilon(\xi) - 1)$.

We will now show $e'_2 = a_2 + \sum_{i=1}^k e_{i2}/m_i + \frac{1}{2}\sum_{i=1}^l b_{i2} \in V'_2 = \sum_{\xi\in\pi_1(S')}\mathrm{Im}(\epsilon(\xi) - 1)$. It is sufficient to show $e'_2 = 0$ when $\epsilon(\xi) = 1$ for all ξ, so suppose $\epsilon(\xi) = 1$ and so for $\xi \in \pi_1(S')$, $\delta(\xi) = \mathrm{w}(\bar{\xi})$. Recall $A(\xi)h_1 = h_1^{\epsilon(\xi)}h_2^{c(\xi)}$. Note $c(h_1) = 0$, $c(h_2) = -2$, $c(\xi\xi') = c(\xi) + \delta(\xi)c(\xi')$ and $c(\xi^{-1}) = -\delta(\xi)c(\xi)$. By considering the image of c of the relation, $s_i^{m_i}h_1^{e_{i1}}h_2^{e_{i2}} = 1$, we see $m_i c(s_i) - 2e_{i2} = 0$, or equivalently

$$e_{i2}/m_i = c(s_i)/2.$$

By instead taking the relation $\partial_i^{-1}r_i\partial_i r_i^{-1} = h_1^{b_{i1}}h_2^{b_{i2}}$, we see $-c(\partial_i) + c(r_i) - c(\partial_i) - c(r_i) = -2b_{i2}$ (note $\delta(r_i) = \mathrm{w}(\bar{r}_i) = -1$) which reduces to

$$\frac{1}{2}b_{i2} = c(\partial_i)/2.$$

Lastly we will look at the image of c of the relation connected to the global information; $d\prod_{j=1}^k s_j \prod_{i=1}^l \partial_i = h_1^{a_1}h_2^{a_2}$, where $d = \prod_{p=1}^g [t_p, u_p]$ when the base is orientable and $d = \prod_{p=1}^g v_p^2$ when the base is non-orientable.

Note $c(d) = 0$, so we get $\sum_{j=1}^{k} c(s_j) + \sum_{i=1}^{l} c(\partial_i) = -2a_2$ which can rearrange and then use the above equations to get:

$$\begin{aligned} 0 &= a_2 + \sum_{j=1}^{k} c(s_j)/2 + \sum_{i=1}^{l} c(\partial_i)/2 \\ &= a_2 + \sum_{j=1}^{k} e_{i2}/m_i + \frac{1}{2}\sum_{i=1}^{l} b_{i2} \\ &= e'_2. \end{aligned}$$

Theorem 3.4.1 states, S' is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ ($\widetilde{SL}_2 \times \mathbb{E}$ respectively) iff $e' \in V'$ ($e' \notin V'$) which is satisfied if and only if $e = 2e'_1 \in V'_1 = V$ ($e \notin V$), since $e'_2 \in V'_2$. From above S is geometric and it will have the same geometry as S', so S is geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ ($\widetilde{SL}_2 \times \mathbb{E}$ respectively) iff $e \in V$ ($e \notin V$), hence the theorem.

We now return to prove the claim: by adjusting P as necessary, the map $(w, z) \mapsto (w, P\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) P^{-1}z + P\left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right))$ is an isometry whose square equals the action of h_1^2. To prove the claim we must consider a few cases. If the geometry is $\mathbb{H}^2 \times \mathbb{E}^2$ then we need to prove $P\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) P^{-1}$ is in $O_2\mathbb{R}$. Suppose $A(\xi)_T = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ for some ξ, then $P\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) P^{-1} = \pm PA(\xi)_T P^{-1} \in O_2\mathbb{R}$. Therefore suppose $\operatorname{Im} A(-)_T \subseteq \{\pm I\}$. Then we can conjugate the isometries by $\left(P^{-1}, 0\right)$ to get an isomorphic group of isometries. That is, we can suppose $P = I$, hence the claim. If the geometry is $\widetilde{SL}_2 \times \mathbb{E}$, then we need to prove P has the form $\left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right)$ for some nonzero real numbers λ and μ. However for this geometry, $PA(\xi)_T P^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{w}(\bar{\xi}) \end{smallmatrix}\right)$ for $\xi \in \pi_1(S')$ (since ξ is orientable), and from above $A(\xi)_T = \left(\begin{smallmatrix} \epsilon(\xi) & 0 \\ 0 & \epsilon(\xi)\,\mathrm{w}(\bar{\xi}) \end{smallmatrix}\right)$. If $\epsilon(\xi) = -1$ and $\mathrm{w}(\bar{\xi}) = 1$, then $A(\xi)_T = -I$ but $PA(\xi)_T P^{-1} = I$ which is a contradiction. Therefore, if $\epsilon(\xi) = -1$, then $\mathrm{w}(\bar{\xi}) = -1$. In this case, $A(\xi)_T = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $PA(\xi)_T P^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and so $P = \left(\begin{smallmatrix} 0 & \lambda \\ \mu & 0 \end{smallmatrix}\right)$ as desired. Instead, suppose $\epsilon(\xi) = 1$ for all $\xi \in \pi_1(S')$ (indeed since $\epsilon(h_1) = 1$, we are assuming this for all $\xi \in \pi_1(S)$). If $\mathrm{w}(\bar{\xi}) = -1$ for some $\xi \in \pi_1(S')$, then $\operatorname{Im}(A(\xi)_T - \mathrm{w}(\bar{\xi})I) = \{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) | x \in \mathbb{R}\}$ and so $V' \supseteq \{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) | x \in \mathbb{R}\}$ and $V = \mathbb{R}$. However this is impossible since from above S' being geometric of type $\widetilde{SL}_2 \times \mathbb{E}$ implies $e \notin V$ and so $V \neq \mathbb{R}$. Therefore $\mathrm{w}(\bar{\xi}) = 1$ for all ξ, which means B is orientable. As a result $A(\xi)_T = I$ for all $\xi \in \pi_1(S')$. In this situation, P may not have desired form, however by choosing an isomorphic group of isometries, we can change P to have the desired form. Let $\mu = \frac{2\pi\chi^{orb}(B)}{e'_1}$. An isomorphic action of $\pi_1(S')$ is defined by $h_1^2(w, z) = \left(w, z + \left(\begin{smallmatrix} 0 \\ \mu \end{smallmatrix}\right)\right)$, $h_2(w, z) = \left(w, z + \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$, $s_{0i}(w, z) = s(\bar{s}_{0i})\left(w, z - \left(\begin{smallmatrix} e_{i2}/m_i \\ \mu e_{i1}/2m_i \end{smallmatrix}\right)\right)$, $t_i(w, z) = s(\bar{t}_i)(w, z)$ and $u_i(w, z) = s(\bar{u}_i)(w, z)$, where $s : \operatorname{Isom}(\mathbb{H}^2) \to \operatorname{Isom}(\widetilde{SL}_2 \times \mathbb{E})$ is the section defined in section 3.2. For this group of isometries, $P = \left(\begin{smallmatrix} 0 & 1 \\ \mu & 0 \end{smallmatrix}\right)$ hence the claim. □

Remark 3.5.2. Analogous to the remark 3.3.8, we can replace the phrase Seifert manifold with Seifert orbifold in the above theorem and have a similar result (corner reflectors could then occur). The expression for e would become $a_1 + \sum_{j=1}^{k_0} (e_{0j})_1/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\left((b_i)_1 + \sum_{j=1}^{k_i} (e_{ij})_1/m_{ij}\right)$.

3.6 Virtually Geometric Seifert Manifolds

It is easily shown that if a non-orientable manifold is geometric then its orientation cover is also geometric and of the same type. However, the converse is not so clear. We will show that in our case a much stronger result is true: that Seifert manifolds with hyperbolic base which are finitely covered by a geometric manifold are themselves geometric. We call manifolds which are finitely covered by a geometric manifold, *virtually geometric.*

Theorem 3.6.1. *Let S be a Seifert 4-manifold over a hyperbolic base B, and let $\hat{S}$ be a finite cover of S.*

Then S is geometric if and only if $\hat{S}$ is.

That is, virtually geometric Seifert 4-manifolds over a hyperbolic base are geometric.

Proof. Note $\hat{S}$ is also a Seifert 4 manifold (lemma 1.3.2). If F denotes the fibre of S, and $\hat{F}$ and $\hat{B}$ denote the fibre and base of $\hat{S}$ respectively, then $\hat{F}$ (finitely) covers F and $\hat{B}$ (finitely) covers B.

Suppose first that $F = T^2$ and that $\hat{S}$ is geometric. We may assume without loss of generality that $\pi_1(\hat{S})$ is normal in $\pi_1(S)$. Hence $\pi_1(\hat{F})$ is also normal in $\pi_1(S)$, since it is characteristic in $\pi_1(\hat{S})$. If $\xi \in \pi_1(S)$ acts trivially on $\pi_1(\hat{F})$ then it acts trivially on $\pi_1(F)$, since the action of an automorphism of $\pi_1(F)$ is determined by its action on any subgroup of finite index.

Since the group of monodromies of $\hat{S}$ is finite then so is the group of monodromies of S and therefore S is geometric by theorem 3.4.1.

If $F = Kb$ there is nothing to prove, since $\mathrm{Out}(\pi_1(F))$ is finite and so the theorem follows from theorem 3.5.1.

The necessity of the condition is clear. □

Corollary 3.6.2. *Let S be a non-orientable Seifert 4-manifold over a hyperbolic base B and let its orientation cover be $\hat{S}$.*

Then S is geometric if and only if $\hat{S}$ is. □

Corollary 3.6.3. *A Seifert 4-manifold S over a hyperbolic base is geometric if and only if it has a finite cover diffeomorphic to $\tilde{B} \times T^2$ or $M^3 \times S^1$ where $\tilde{B}$ is a hyperbolic surface and M^3 is a $\widetilde{SL}_2$ 3-manifold.*

Proof. The manifolds $\tilde{B} \times T^2$ are geometric of type $\mathbb{H}^2 \times \mathbb{E}^2$ and the manifolds $M^3 \times S^1$ are geometric of type $\widetilde{SL}_2 \times \mathbb{E}$. So by the theorem, if S is finitely covered by one of these it is geometric. Conversely suppose S is geometric. Then it is finitely covered by such a manifold by theorem 9.3 in [Hi]. □