APPENDIX C: THE SINGLE ELEMENT MODEL FOR SAND

C1. GENERAL

C1.1 Elastic Behaviour

Elastic behaviour of the soil can be described by the following incremental stress-strain law:

$$d\sigma = D d\varepsilon^e \qquad \text{(C-1)}$$

where D = elastic stiffness matrix

$d\varepsilon^e$ = increments of elastic strain

$d\sigma$ = stress increments

C1.2 Elasto-plastic Behaviour

Superposition of strains is assumed so that the total strain increment may be represented as:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \qquad \text{(C-2)}$$

where $d\varepsilon^p$ = plastic strain increments

The yield criterion for the soil may be expressed as:

$$f(\sigma, \varepsilon^p) = 0 \qquad \text{(C-3)}$$

and the flow rule as:

$$d\varepsilon^p = \Lambda\left(\frac{\partial g}{\partial \sigma}\right) = \Lambda(a) \qquad \text{(C-4)}$$

where $g(\sigma, \varepsilon^p) = 0$ = plastic potential

$$a = \left(\frac{\partial g}{\partial \sigma}\right)$$

and Λ = a scalar multiplier

C2. DERIVATION - ELASTO-PLASTIC STRESS - STRAIN MATRIX, D_p

Once the soil has yielded, the incremental stress- strain law becomes:

$$d\sigma = D_P.d\varepsilon \qquad \text{(C-5)}$$

where D_P = the elasto-plastic stiffness matrix

During yielding, $$f = df = 0 \qquad \text{(C-6)}$$

But $$df = \left(\frac{\partial f}{\partial \sigma}\right)d\sigma + \left(\frac{\partial f}{\partial \varepsilon^p}\right)d\varepsilon^p$$

or $$df = b^T d\sigma + c^T d\varepsilon^p \qquad \text{(C-7)}$$

From the flow rule, as expressed by equation C-4, and the yield condition defined by equation C-6, equation C-7 becomes:

$$0 = b^T d\sigma + \Lambda(c^T)a \qquad \text{(C-8)}$$

From equations C-2 and C-4, increments of plastic strain were expressed as:

$$d\varepsilon^p = d\varepsilon - d\varepsilon^e = \Lambda(a)$$

or $$\Lambda(a) = d\varepsilon - D^{-1}(d\sigma)$$

Hence $$d\sigma = D(d\varepsilon) - \Lambda D(a) \qquad \text{(C-9)}$$

Substituting equation C-9 into C-8 gives:

$$0 = b^T(D(d\varepsilon) - \Lambda D(a)) + \Lambda(c^T)a$$

$$0 = b^T D(d\varepsilon) - \Lambda[b^T D(a) - (c^T)a]$$

Therefore $$\Lambda = [b^T D(d\varepsilon)]/[b^T D(a) - (c^T)a] \qquad \text{(C-10)}$$

or $$\Lambda = [b^T D(d\varepsilon)]/\Delta \qquad \text{(C-10a)}$$

with $$\Delta = b^T D(a) - (c^T)a \qquad \text{(C-10b)}$$

Substituting C-10 into C-9 yields:

$$d\sigma = Dd\varepsilon - Da\left[\frac{b^T Dd\varepsilon}{b^T Da - c^T a}\right] \qquad \text{(C-10c)}$$

Or, $$d\sigma = d\varepsilon\left[D - \frac{Dab^T D}{b^T Da - c^T a}\right] \qquad \text{(C-10d)}$$

Therefore it may be deduced by comparison with equation C-5 that the elasto-plastic stiffness matrix is a function of the elastic stiffness matrix, i.e.:

$$D_p = \left[I - \frac{Dab^T}{b^T Da - c^T a}\right]D \qquad \text{(C-11)}$$

Or, $$D_p = \left[I - \frac{Dab^T}{\Delta}\right]D \qquad \text{(C-11a)}$$

where I = a unit matrix

and $\Delta = b^T Da - c^T a$

C3. APPLICATION TO THE TRIAXIAL TEST

The stress state for triaxial stress conditions is conveniently represented by the deviatoric and mean stress components, q and p′, respectively, or, in terms of stress increments:

$$d\sigma = \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

where $p' = \frac{1}{3}(\sigma'_1 + 2\sigma'_3)$

and $q = (\sigma'_1 - \sigma'_3)$

Similarly, the strains can be represented by the volumetric and deviatoric strains, ε_q and ε_v, respectively, i.e. strain increments are:

$$d\varepsilon = \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{bmatrix}$$

where $\varepsilon_v = \varepsilon_1 + 2\varepsilon_3$ (C-12)

and $\varepsilon_q = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$ (C-13)

Manipulation of these triaxial stress and strain expressions leads to the following four equations for the major and minor principal components of stress and strain:

$$\varepsilon_1 = \frac{1}{3}(\varepsilon_v + 3\varepsilon_q) \quad \text{(C-14)}$$

$$\varepsilon_3 = \frac{1}{3}\left(\varepsilon_v - \frac{3}{2}\varepsilon_q\right) \quad \text{(C-15)}$$

$$\sigma'_1 = p' + \frac{2}{3}q \quad \text{(C-16)}$$

$$\sigma'_3 = p' - \frac{1}{3}q \quad \text{(C-17)}$$

C4. INCORPORATION OF STATE PARAMETER

C4.1 Elastic Behaviour - Isotropic elasticity

In the elastic phase, the volumetric strain is related to the effective mean stress by the bulk modulus, K, while the deviatoric stress is related to the deviatoric strain by the shear modulus as follows:

$$dp' = K.d\varepsilon_v \quad \text{and} \quad dq = 3G.d\varepsilon_q$$

$$\text{or} \quad \begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{bmatrix}$$

If Poisson's ratio ν is assumed to be constant and $E = E(p')$, then:

Bulk modulus $K = K(p')$

Shear modulus $G = G(p')$

C4.2 Plastic Behaviour

Plasticity is invoked when the yield function is zero.

Yield function: $f = q - Mp'$

where $M = M(\phi'(\xi))$

and $\xi = e - e_c$ = the state parameter (C-18)

with e = the current void ratio

and e_c = void ratio at critical, or steady, state for the current value of effective mean stress

Flow Rule:

The flow rule for triaxial compression was proposed by Carter, Booker and Yeung (1986), which was essentially an extension of Davis' flow rule (1969). The flow rule takes the form:

$$D = \frac{d\varepsilon_p^p}{d\varepsilon_q^p} = \frac{-6\sin\Psi}{(3-\sin\Psi)} \quad \text{(C-19)}$$

where, D = the plastic dilation of the soil

$d\varepsilon_p{}^p$ = increment of volumetric strain

$d\varepsilon_q{}^p$ = increment of deviatoric strain

Ψ = angle of dilation

Equation C-19 may be re-written as:

$$\begin{bmatrix} d\varepsilon_v^p \\ d\varepsilon_q^p \end{bmatrix} = \Lambda \begin{bmatrix} \left(\dfrac{-6\sin\Psi}{(3-\sin\Psi)} \right) \\ 1 \end{bmatrix} = \Lambda(a) = \Lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Therefore $a_1 = \dfrac{-6\sin\Psi}{(3-\sin\Psi)}$ and $a_2 = 1$ (C-20)

Bolton (1986) suggested an empirical relationship between the dilation and friction angles as follows:

$$\Psi(\xi) = (\phi' - \phi'_{cv})/0.8$$

Now, at yield, $df = dq - (Mdp' + p'dM) = 0$ (C-21)

where $dM = \left(\dfrac{\partial M}{\partial \phi'}\right)\left(\dfrac{\partial \phi'}{\partial \xi}\right) d\xi$ (C-22)

or $dM = \Omega(\phi')d\xi$

where $\Omega(\phi') = \left(\frac{\partial M}{\partial \phi'}\right)\left(\frac{\partial \phi'}{\partial \xi}\right)$

Equation C-22 recognizes that the current value of the effective friction angle is a function of the state parameter.

For triaxial compression, the gradient of the failure line is related to the effective friction angle by the expression:

$$M = \frac{6 \sin \phi'}{(3 - \sin \phi')}$$

Differentiating with respect to ϕ' yields:

$$\frac{\partial M}{\partial \phi'} = \frac{18 \cos \phi'}{(3 - \sin \phi')^2} \qquad \text{(C-23)}$$

Collins, Pender and Yan (1992) proposed that the current effective friction angle was related to the state parameter by the empirical expression:

$$\phi' = \phi'_{cv} + A(e^{-\xi} - 1) \qquad \text{(C-24)}$$

where A = a material constant

Therefore, $$\frac{\partial \phi'}{\partial \xi} = -Ae^{-\xi} \qquad \text{(C-25)}$$

Substituting equations C-23 and C-25 into equation C-22 yields the expression:

$$\Omega(\phi') = -\frac{18A(e^{-\xi}) \cos \phi'}{(3 - \sin \phi')^2}$$

Since the critical state line can be defined by a gradient, λ, and an intercept, Γ, the following expression defines the state parameter:

$$\xi = (e - e_c) = e + (\lambda \ln p' - \Gamma)$$

Therefore small changes or increments of the state parameter may be defined by:

$$d\xi = de + (\lambda dp')/p' \qquad \text{(C-26)}$$

Incremental changes in void ratio depend upon the volumetric strain, i.e.:

$$\frac{de}{(1+e_o)} = -d\varepsilon_p$$

where e_o = previous value of e

Substituting for void ratio increment, de, in equation C-26 leads to:

$$d\xi = -(1+e_o)d\varepsilon_p + \frac{\lambda dp'}{p'}$$

Since volumetric strain has plastic and elastic components:

$$d\xi = -(1+e_o)(d\varepsilon_p^e + d\varepsilon_p^p) + \frac{\lambda dp'}{p'}$$

The elastic volumetric strain may be replaced by effective mean stress divided by the bulk modulus, i.e.:

$$d\xi = -(1+e_o)(\frac{dp'}{K} + d\varepsilon_p^p) + \frac{\lambda dp'}{p'}$$

Arranging terms into effective mean stress and plastic volumetric strain components provides the expression:

$$d\xi = \left[-\frac{(1+e_o)}{K} + \frac{\lambda}{p'}\right]dp' - (1+e_o)d\varepsilon_p^p \qquad \text{(C-27)}$$

Substituting equations C-23, C-25 and C-27 into equation C-22 yields:

$$dM = \Omega(\phi')\left[\left(\frac{-(1+e_o)}{K} + \frac{\lambda}{p'}\right)dp' - (1+e_o)d\varepsilon_p^p\right] \qquad \text{(C-28)}$$

Substituting C-28 into equation C-21, i.e: df = 0 = dq - Mdp' - p'dM,

yields the expression for the flow rule:

$$0 = dq - Mdp' - p'\left[\Omega(\phi')\left[\left(\frac{-(1+e_o)}{K} + \frac{\lambda}{p'}\right)dp' - (1+e_o)d\varepsilon_p^p\right]\right]$$

Or,

$$0 = dq - Mdp' - p'\Omega(\phi')\left[\left(\frac{-(1+e_o)}{K} + \frac{\lambda}{p'}\right)dp' - (1+e_o)d\varepsilon_p^p\right]$$

Arranging stress and strain terms:

$$0 = -\left[M - \Omega(\phi')\left(\frac{(1+e_o)p'}{K} - \lambda\right)\right]dp' + (dq) + \left(\Omega(\phi')(1+e_o)p'\right)\left(d\varepsilon_p^p\right)$$

eqn. (C-29)

Comparing equation C-29 with C-7, i.e.: $df = b^T \partial\sigma + c^T \partial\varepsilon^p$,

firstly yields: $b^T = [b_1 \quad b_2]$

with $b_1 = -\left[M - \Omega(\phi')\left(\frac{(1+e_o)p'}{K} - \lambda\right)\right]$ (C-30)

and $b_2 = 1$

Secondly, the comparison of the equations reveals that:

$$c^T = [c_1 \quad 0]$$

with $c_1 = \Omega(\phi')(1+e_o)p'$ (C-31)

The zero in the matrix arises from the observation that there isn't any term in equation C-29 relating to plastic deviatoric strains.

C5. MATRIX OPERATIONS

The elastic stiffness matrix may be expressed in matrix form as:

$$D = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$$

As the elasto-plastic stiffness matrix was defined by equation C-10c, as:

$$d\sigma = Dd\varepsilon - Da\left[\frac{b^T Dd\varepsilon}{b^T Da - c^T a}\right]$$

In order to assemble the stiffness matrix, the following terms must also be expressed in matrix form, i.e.:

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \text{ and } \qquad \underline{c} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

The terms required for the elasto-plastic stiffness matrix are then deduced by simple matrix operations as follows:

$$D\underline{a} = \begin{bmatrix} D_{11}a_1 \\ D_{22}a_2 \end{bmatrix}$$

and, $$\underline{b}^T D\underline{a} = b_1 D_{11} a_1 + b_2 D_{22} a_2$$

and $$\underline{c}^T \underline{a} = c_1 a_1$$

since c_2 was shown to be zero in the previous section.

From equation C-11a, $$\Delta = \underline{b}^T D\underline{a} - c^T \underline{a}$$

Substituting the above expressions into C-10b produces the equation:

$$\Delta = b_1 D_{11} a_1 + b_2 D_{22} a_2 - c_1 a_1 \tag{C-32}$$

To complete the elasto-plastic stiffness matrix, a further term is required in matrix form, Dab^T.

Now, $$ab^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$$

Therefore, $$Dab^T = \begin{bmatrix} D_{11} a_1 b_1 & D_{11} a_1 b_2 \\ D_{22} a_2 b_1 & D_{22} a_2 b_2 \end{bmatrix}$$

From equation C-11a, $$D_p = \left[I - \frac{Dab^T}{\Delta} \right] D$$

with the unit matrix, I, defined as: $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$D_p = \begin{bmatrix} (1 - D_{11} a_1 b_1 / \Delta) & (- D_{11} a_1 b_2 / \Delta) \\ (- D_{22} a_2 b_1 / \Delta) & (1 - D_{22} a_2 b_2 / \Delta) \end{bmatrix} D$$

$$D_p = \begin{bmatrix} (1 - D_{11} a_1 b_1 / \Delta) & (- D_{11} a_1 b_2 / \Delta) \\ (- D_{22} a_2 b_1 / \Delta) & (1 - D_{22} a_2 b_2 / \Delta) \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$$

$$D_P = \begin{bmatrix} D_{11}(1 - D_{11} a_1 b_1 / \Delta) & (- D_{11} D_{22} a_1 b_2 / \Delta) \\ (- D_{11} D_{22} a_2 b_1 / \Delta) & D_{22}(1 - D_{22} a_2 b_2 / \Delta) \end{bmatrix} \tag{C-33}$$

The elasto-plastic stiffness matrix can now be generalised to:

$$D_P = \begin{bmatrix} D_{P11} & D_{P12} \\ D_{P21} & D_{P22} \end{bmatrix} \tag{C-34}$$

where $D_{P11} = D_{11}(1-D_{11}a_1b_1/\Delta)$

$D_{P21} = -D_{11}D_{22}a_2b_1/\Delta$

$D_{P12} = -D_{11}D_{22}a_1b_2/\Delta$

$D_{P22} = D_{22}(1-D_{22}a_2b_2/\Delta)$

and D_{11} and D_{22} are the elastic stiffness components of the elasto-plastic stiffness matrix

C6. APPLICATION TO TRIAXIAL TESTING

C6.1 The Conventional Triaxial Test

In this form of the triaxial test, the confining stress is held constant, i.e., $d\sigma'_3 = 0$

Therefore in terms of increments of deviatoric and mean effective stress:

$$dq = d\sigma'_1$$

and

$$dp' = d\sigma'_1/3$$

From equations C-12 and C-13, the corresponding volumetric and deviatoric strain increments were defined as:

$$d\varepsilon_p = d\varepsilon_1 + 2d\varepsilon_3$$

and

$$d\varepsilon_q = (2/3)(d\varepsilon_1 - d\varepsilon_3)$$

In the conventional triaxial test, the strain rate is controlled i.e. vertical strain increments are applied, $d\varepsilon_1$ (and are therefore known).

So the governing stress-strain law,

$$\begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} D_{P11} & D_{P12} \\ D_{P21} & D_{P22} \end{bmatrix} \begin{bmatrix} d\varepsilon_p \\ d\varepsilon_q \end{bmatrix}$$

can be re-written as:

$$\begin{bmatrix} \left(\dfrac{d\sigma'_1}{3}\right) \\ d\sigma'_1 \end{bmatrix} = \begin{bmatrix} D_{P11} & D_{P12} \\ D_{P21} & D_{P22} \end{bmatrix} \begin{bmatrix} (d\varepsilon_1 + 2d\varepsilon_3) \\ \left(\dfrac{2}{3}\right)(d\varepsilon_1 - d\varepsilon_3) \end{bmatrix} \qquad \text{(C-35)}$$

Knowing vertical strain increments, stress changes may be determined without matrix inversion, since equation C-35 represents two equations with two unknowns. The stiffness matrix changes from the elastic to the plastic stiffness matrix as yield is reached, which is dependent upon stress state.

C6.2 The Volume Controlled Triaxial Test

In this test, axial stress is incremented ($d\sigma'_1$) at a constant rate. As the sample begins to deform, the horizontal stress is changed to maintain the volume of the sample.

As before, $$dq = d\sigma'_1 - d\sigma'_3$$

and $$dp' = \frac{d\sigma'_1}{3} + \frac{2d\sigma'_3}{3}$$

From the definitions of strains (equations C-12 and C-13):

$$d\varepsilon_p = d\varepsilon_1 + 2d\varepsilon_3 = 0$$

Therefore, $$d\varepsilon_3 = -\frac{d\varepsilon_1}{2}$$

And so $$d\varepsilon_q = \frac{2(d\varepsilon_1 - d\varepsilon_3)}{3} = d\varepsilon_1$$

Therefore it can be seen that, $$d\varepsilon_3 = -\frac{d\varepsilon_q}{2}$$

Therefore the stress-strain law is simplified for the controlled volume test to the expression:

$$\begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} D_{P11} & D_{P12} \\ D_{P21} & D_{P22} \end{bmatrix} \begin{bmatrix} 0 \\ d\varepsilon_1 \end{bmatrix} \qquad \text{(C-36)}$$

or $$dp' = D_{P12}.d\varepsilon_1$$

and $$dq = D_{P22}.d\varepsilon_1$$

By incrementing $d\varepsilon_1$, the new stress state may be determined.

C6.3 The Constant Mean Stress Triaxial Test

As the title suggests, the effective mean stress is kept constant, i.e. p' = constant, and so increments of effective mean stress are zero:

$$dp' = 0$$

Therefore it follows that ,

$$d\sigma'_3 = -\frac{d\sigma'_1}{2}$$

and

$$dq = \frac{3d\sigma'_1}{2}$$

Thus, the governing stress-strain law becomes:

$$\begin{bmatrix} 0 \\ \left(\frac{3d\sigma'_1}{2}\right) \end{bmatrix} = \begin{bmatrix} D_{P11} & D_{P12} \\ D_{P21} & D_{P22} \end{bmatrix} \begin{bmatrix} d\varepsilon_p \\ d\varepsilon_q \end{bmatrix} \tag{C-37}$$

By incrementing $d\varepsilon_1$ (and therefore $d\varepsilon_q$), $d\varepsilon_p$ may be found and hence both e and $d\sigma'_1$ can be determined.