Mr Swinbourne/hb

Mr John Carter,
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25 October 1977

Dear Mr Carter,

I am pleased to advise you that you have been awarded the degree of Doctor of Philosophy. This follows consideration of the reports of the examiners of your thesis entitled "Finite Deformation Theory and its Application to Elastoplastic Soils".

The next ceremony of conferring of degrees which you could attend will be held in 1978. Alternatively, arrangements can be made for your degree to be conferred "in absentia" at a monthly meeting of the Senate. I should be glad if you would write to me indicating whether you wish your degree to be conferred "in absentia" by the Senate or at the conferring of degrees ceremony.

Yours sincerely,

Kenneth W Knight
Registrar

C.C. Professor Roderick
Professor E H Davis
Miss P Green, Rare Book Librarian

Thesis attached - undertaking re immediate availability signed.
FINITE DEFORMATION THEORY AND ITS APPLICATION

TO

ELASTOPLASTIC SOILS

by

John P. Carter, B.E.

A Thesis submitted for the Degree of Doctor
of Philosophy in the University of Sydney.

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SYNOPSIS

In this thesis an examination is made of the application of finite deformation theory to an ideal elastic, perfectly plastic soil. The work may be conveniently divided into four sections.

(a) A presentation of the theory of finite deformation for an elastoplastic material. This investigation includes the development of the governing equations using a virtual work principle that is cast in a rate form and constitutive laws that are cast in a frame indifferent manner. Plastic failure is described by a general yield condition and plastic deformation by an arbitrary flow law. The approach is quasi-static in nature.

(b) The proposal of a numerical technique to be used for the approximate solution of the governing equations of finite deformation. The numerical method is verified by means of comparisons between exact and numerical solutions.

(c) The numerical methods referred to above are then used to examine several boundary value problems of engineering interest. These include the surface loading, under conditions of plane strain, of a layer of ideal, elastic perfectly plastic, cohesive-frictional material which obeys either an associated or a non-associated flow rule. The surface loading is either applied as a line loading, a traction loading, a rigid footing (specified boundary displacements) or the build up of an embankment.

(d) Finally, the finite deformation analysis is extended to
predict the time dependent, finite consolidation behaviour of a two-phase elastoplastic soil. The latter theory is illustrated by the use of several practical examples, viz. one-dimensional consolidation and two-dimensional consolidation of a rigid strip footing resting on either an elastic or an elastoplastic soil.

It was found that for many problems in soil mechanics, e.g. line loading, uniform traction loading and rigid footing load on a homogeneous and inhomogeneous layer, the finite deformation theory predicted less settlement at a given applied load than did the usual infinitesimal theory. This result is not general, however, and counter examples are cited. Indications are given as to when a finite deformation analysis might be necessary in soil mechanics. In real engineering problems this is confined to the loading of materials of very low stiffness, particularly due to embankment building.
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PREFACE

The candidate carried out the work described in this thesis during the period 1973-1976. All of the work was conducted in the School of Civil Engineering, The University of Sydney, except for a period during 1974 when the author conducted some of this research in the Department of Civil Engineering, Kings' College, The University of London.

The candidate was supervised by Professor E.H. Davis, Professor of Civil Engineering (Soil Mechanics) except for the period spent at Kings' College when he was supervised by Professor R.E. Gibson.

The By-Laws of the University of Sydney require a candidate for the degree of Doctor of Philosophy to indicate which sections of the thesis are original. Although many references have been used during the course of this research program, any information or ideas derived from these sources has been acknowledged in the text. In accordance with the abovementioned By-Laws the Author claims originality for the following work:

(i) In chapter 2 the virtual work formulation of section 2.4 for finite deformation using a rate approach and the general rate law described by equation (2.30) are claimed as original, although analyses following a similar basic procedure have been presented by several workers.

(ii) In chapter 3 the treatment of the rate law and in particular the application to a Mohr-Coulomb type material with either an associated or a non-associated flow rule are claimed as original.

(iii) In chapter 4 the author claims as original the appli-
cation of the numerical method to the solution of the governing equations of finite deformation and all numerical results given in the text. In this chapter the author also claims as original the exact solutions for the finite expansion of an elastic cylinder, the deposition problem and the extension of the solution for the homogeneous deformation problem to a Mohr-Coulomb material and to include the effects of an initial stress state.

(iv) In chapter 5 all numerical results for the surface loading problems, in both cases of infinitesimal and finite analysis, are claimed as original. In particular, the discussion and results for a ponderable soil are claimed as original.

(v) In chapter 6 the following are claimed as original: the treatment of the finite deformation behaviour of a consolidating soil and the derivation of the governing equations, which involve an extension to three dimensions of the work of Gibson et al. (1967); the development of a numerical technique to solve the governing equations; all numerical solutions so obtained; and the exact solution for the one-dimensional finite deformation of an elasto-plastic Mohr-Coulomb material.

A number of papers were prepared and published by the author and others during the period of the author's candidature. These are submitted in support of his candidature. They are:


(2) Carter, J.P. "Discussion of Finite Elasto-Plastic Defor-


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The work described in this thesis was made possible by the award of a Commonwealth Research Scholarship and a University of Sydney Post-Graduate Research Travelling Scholarship.

I am indebted to many people for their help during the course of this work. I gratefully acknowledge the value of discussions with, and the helpful supervision of Professor R.E. Gibson. I should like to record my sincere thanks to Dr. J.R. Booker for his generous and invaluable assistance and patient understanding during the time of my candidature. The advice of my fellow workers both as members of staff and as research students in the Schools of Civil Engineering at the University of Sydney and at King's College London is gratefully acknowledged.

In particular, I wish to express my gratitude and most sincere thanks to Professor E.H. Davis for his supervision, encouragement and guidance with all aspects of this work. Finally, for their continual encouragement and help, I should like to thank my parents and my wife, Heather.

J.P.C.
All notation and symbols are defined where they first appear in the text. For convenience, the most frequently used symbols and their meanings are listed below.

$A$  
Matrix of velocity shape functions

$\mathbf{a}_i, \mathbf{a} = (a, b, c)$  
Cartesian coordinates (Lagrangian)

$\dot{\mathbf{s}}$  
Plastic strain rate vector, vector of pore pressure shape functions

$\mathbf{B}$  
Strain rate transformation matrix, footing half-width

$\mathbf{b}$  
Vector normal to yield surface

$\mathbf{C}$  
Strain rate transformation matrix

$c$  
Material cohesion

$D$  
Plasticity matrix

$D_E$  
Elasticity matrix

$D_{ijkl}$  
Constitutive law 'constants'

$d$  
Denotes an incremental quantity

$\mathbf{g}$  
Vector of velocity gradients

$\mathbf{E}$  
Young's modulus, gradient operator

$e$  
Void ratio

$\mathbf{E}_{ij}$  
Green strain tensor

$e_{ij}$  
Cauchy strain tensor

$\mathbf{F}_i, F$  
Body force vector

$f$  
Yield function

$G$  
Shear modulus

$g$  
Acceleration due to gravity

$g$  
Vector for initial strains or pore pressures

$H$  
Constitutive matrix, layer thickness
h  Piezometric head, layer thickness
h  'Load' vector
K  Stiffness matrix, permeability matrix
K_E Elastic stiffness matrix
k  Isotropic permeability
L  Coupling matrix in consolidation analysis
\ell, \ell_0 Typical length dimensions
\ell_{ij}, \ell Partial 'load' vector
m  Partial 'load' vector
N  Volume strain rate transformation matrix
n_i  Unit vector normal to surface S
\bar{n} Partial 'load' vector
0  Zero matrix
p  Constitutive matrix, magnitude of applied load
\rho  Pore pressure, magnitude of applied traction
Q  Consolidation matrix, magnitude of applied load
q  Magnitude of applied traction
q  Vector of nodal pore pressures
R  Radius, Mohr circle radius, rotation matrix
r  Radial coordinate
S  A surface, a boundary
S_g  Specific gravity
S_{ij}  Kirchoff stress tensor
T_i, T Surface traction vector
t  Time variable
u_i, \bar{u} = (u_x, u_y, u_z)^T Displacement field, vector
\bar{u}_i  Nodal displacement vector
\bar{v}  A volume, a region in space
v_i, \bar{v} = (v_x, v_y, v_z)^T Velocity field, vector
\( v_{si}, v_s \)  Velocity field, vector for solid particles
\( v_{fi}, v_f \)  Velocity field, vector for fluid particles
\( w \)  Weight
\( w \)  Width dimension
\( x_i, x = (x, y, z) \)  Cartesian coordinates (Eulerian)
\( \alpha \)  Porosity
\( \gamma \)  Density
\( \gamma_s \)  Density of solid particles
\( \gamma_f \)  Density of fluid particles
\( \Delta \)  Footing displacement, finite increment
\( \delta \)  Total nodal displacement vector
\( \theta, \xi \)  Differential operators
\( \varepsilon_{ij} \)  Almansi strain tensor
\( \eta \)  Special vector \( (1,1,1,0,0,0)^T \)
\( \theta \)  Volume strain rate, angular measure
\( \Lambda \)  Lamé parameter
\( \lambda \)  Flow rule proportionality 'constant'
\( \mu, \rho \)  Length proportionality 'constants'
\( \nu \)  Poisson's ratio
\( \xi, \eta, \zeta \)  Local cartesian coordinates
\( \sigma_{ij}, \zeta \)  Cauchy stress tensor, vector
\( \phi \)  Flow matrix
\( \phi \)  Angle of internal friction
\( \chi \)  Plasticity transformation matrix
\( \psi \)  Dilatancy angle
\( \omega \)  Loading rate, spin (rotation rate)
\( \omega_{ij}, \psi \)  Spin tensor, vector.
CHAPTER 1
INTRODUCTION AND HISTORICAL REVIEW
1.1 INTRODUCTION AND AIMS

During recent years there has been a rapid increase in the application of analytical methods to the solution of soil and soil structure problems. Design is no longer based mainly on semi-empiricism and experience. This trend is, of course, not exclusive to the field of Soil Mechanics but is common throughout the engineering world. A significant amount of current research is involved with the development of simple ideal materials to mathematically model the behaviour of real systems.

Consider the problem of the investigation of the behaviour of a particular soil. Throughout this work attention is confined to an examination of the macroscopic behaviour and, as such, the techniques of continuum mechanics are employed. The particular soil in mind must be represented by an ideal model which, as near as possible, describes the properties of the soil which are pertinent to the problem. The model must not, however, become so complex that a mathematical solution is no longer tractable.

In the formulation of theories in applied mechanics, and in particular soil mechanics, it has been a common practice to assume that strains, both elastic and plastic are infinitesimal and, that the initial geometry of a deforming body is not appreciably altered during the deformation process. These assumptions are less justified for soil than for some other common engineering materials such as steel and concrete. Theories of finite strain that relax some of these restrictive assumptions have been developed and there exists a considerable body of literature on what might be called the classical elastic large
strain theory (see section 1.2.1).

In contrast to the methods of these early investigators, many more recent studies have preferred an incremental approach to facilitate the analysis of the more general class of inelastic materials whose constitutive laws are expressible in terms of incremental or rate quantities. For such formulations the solution of a given problem is found by following a specified loading path. In most cases the governing equations cannot be solved exactly and it is necessary to adopt an approximate numerical technique. In addition these loading path methods have the inherent advantage of supplying a knowledge of stresses and displacements throughout the loading history.

In this thesis a formulation is given for the solution of problems of finite elastoplastic deformation without restriction on the magnitude of such deformations. Plastic failure is described by a general yield condition and plastic deformation by an arbitrary flow rule. The theory is developed for a general constitutive law which relates an objective stress rate to the time rate of change of a strain measure. The work will contain:

(a) an examination of previous methods of finite deformation analysis;

(b) a general formulation for the analysis of the finite deformation of many types of materials including some ideal soils;

(c) the development of a numerical method of analysis, using the finite element process of spatial discretisation, for the approximate solution of the governing equations of the theory;

(d) exact solutions for some simple problems involving finite deformation;
(e) an evaluation of the numerical technique by means of comparisons between exact and numerical solutions;

(f) a discussion of some ideal soil models and their suitability to the finite deformation formulation;

(g) some applications of the finite analysis and solution technique to an ideal elastic, perfectly plastic soil, with an examination of some relevant problems, e.g. the surface loading of soil masses by uniform load, rigid footings, and embankment building. These loadings are considered to be monotonically increasing under either purely undrained or drained conditions and attention is confined to problems of a plane strain nature;

(h) an extension of the finite deformation analysis to predict the time dependent, finite consolidation behaviour of a two phase elastoplastic soil. The latter theory is also illustrated by the use of a number of practical examples, viz. one-dimensional consolidation and two-dimensional consolidation of a rigid strip footing resting on a mass of either elastic or elastoplastic soil.

(i) a critical examination of the use of finite deformation theory in practical foundation engineering problems.

With regard to this last point, and by way of introduction, it may be expected that the theory has applications to such problems as: the penetration of embankments into soft soils; the behaviour of layers of normally consolidated clay in which both the undrained modulus and undrained shear strength increase with depth being virtually zero at the surface; the post-peak load behaviour of very sensitive clays for which large shear strains are required to attain the full softening corresponding to the remoulded strength; and bomb crater and underground cavity formation.
1.2 HISTORICAL REVIEW

The first recorded attempts to analyse the nature of the resistance of solids to rupture were those of Galileo (1638). These initial enquiries may have differed from the more recent works in their fundamental approach to the problem, but they gave the direction that was subsequently followed by many investigators. Since Galileo's time men have tried to understand the manner in which solid bodies resist loading and, in particular, have sought the means by which they could predict such behaviour. The results of their attempts have been practically significant in the fields of architecture and engineering and all other useful arts in which the material of construction was solid.

One particular field of endeavour is of some interest here. It concerns the study of the finite deformations of an elastoplastic medium, deformation that may be either instantaneous or time dependent. This particular area of applied mechanics evolved from the mathematical theories of elasticity, plasticity and consolidation, and any review of its history must necessarily cover a large range of topics. The present review is divided into five broad areas. These are:

(i) Finite elastic deformation theory,
(ii) Plasticity theory,
(iii) Infinitesimal elastoplastic deformation theory,
(iv) Finite elastoplastic deformation theory, and
(v) Consolidation theory.

These categories are by no means self contained, there exists a considerable amount of overlap between them. Where appropriate, mention is also made of the particular application of the work in the field
of soil mechanics.

1.2.1 Finite Elastic Deformation Theory

A comprehensive review of the history of the mathematical theory of elasticity from the initial enquiries of Galileo to the conclusive investigations of Saint Venant and Lord Kelvin is given in a treatise on the subject by Love (1944). Much of this early work in elasticity was applicable only in circumstances where the displacements of the deforming body were small. This presented a difficulty in the analysis of many practical problems, e.g. the deformation of the spiral spring, where the displacements are not small. Krichoff (1859, 1883) was the first to face this difficulty. Since his time many investigators have developed the general theory of elasticity which incorporates the possibility of finite displacements, strains and rotations. An historical and critical account of large elastic deformation theory and application up to about 1952 has been given by Truesdell (1952) and this has been followed by a review of progress in non-linear elasticity up to 1955 by Doyle and Erickson (1956).

Murnaghan (1937) discussed the finite deformation of an elastic solid and later (1951) produced a text on the subject which included discussion of both isotropic and non-isotropic media. Novozhilov (1953) also presents the general theory in a text that includes the deformation of plate and shell structures.

Green and Zerna (1954) gave a treatment of the general theory of elasticity in their text and this was followed by another
book by Green and Adkins (1960). The latter gives a systematic
exposition of some of the main topics of the non-linear theory of
large elastic deformations including an examination of the strain
energy function of the crystal classes and various forms of the
stress-strain relation. The authors also include chapters on
exact solutions and approximate methods. Both of these texts pro-
vide a large number of source references.

Several workers have been concerned with the practical
application of the large elastic deformation theory to substances
like rubber and so called "Mooney" materials. Amongst them were
Mooney (1940), Treloar (1949, 1956) and Hill (1973a, 1973b, 1973c).
Others have been interested in the problem of uniqueness and sta-
bility of finite deformation solutions (Hill, 1957a, 1961; Truesdell,
1952, 1956).

An incremental approach, to facilitate the more general
class of inelastic materials whose constitutive laws are expressible
in terms of incremental or rate quantities, has been adopted by
several authors (e.g. Biot, 1934, 1938, 1939a, 1939b, 1939c, 1940a, 1940b;
Green, Rivlin and Shield, 1952; Hill, 1959), although the basic ideas
were due to Cauchy (1829) and Saint Venant (1868). Biot's work rep-
resented a fundamental departure from classical trends. He applied
Cartesian concepts and elementary mathematical methods to the non-
linear elasticity theory. "The explicit introduction of a local ro-
tation field into the three dimensional equations leads to a theory
which separates the physics from the geometry" (Biot 1965, pV).
This work was later incorporated in a text (Biot 1965).
With the advent of the digital computer, incremental forms of analysis became increasingly popular. Solutions to erstwhile complex finite elasticity problems became possible by the use of loading path techniques and incremental formulations (e.g. Argyris, 1965a). In particular, the finite element method was employed as a numerical device in this area (Oden, 1967; Hartz and Nathar, 1970). Applications of this technique to plane stress problems were considered by Oden (1966), Becker (1966), Oden and Slato (1967), Oden and Kubitza (1967) and Brebbia and Connor (1969). Problems of finite plane strain were solved using finite elements by Oden (1968) and applications involving three-dimensional bodies and bodies of revolution were discussed by Oden and Key (1970, 1971). Kavanagh (1969) used the method in connection with his experimental work on finite elasticity. A survey of this recent work has been given in a paper by Oden (1969) and the theory and some examples are presented in a text by the same author (Oden, 1972).

1.2.2 Plasticity Theory

The scientific study of the plasticity of metals began in the latter half of the nineteenth century with the work of Tresca (1864), Saint Venant (1870) and Lévy (1870). Interest in the field lapsed for about fifty years but some important contributions were made during this period by Haar and von Kármán (1909) and von Mises (1913). Then after 1920 the subject was advanced by several German writers (e.g. Prandtl, 1924; Hencky, 1924). A more complete history of the development of metal plasticity is provided in a text by Hill (1950).
During this early period there seemed to be little appreciation of the necessity of considering the strain rate behaviour of plastic materials, although Saint Venant (1870) and Lévy (1870) had proposed relations between the stress ratio and the strains. In 1930 Geiringer discovered the equations relating the stress and velocity fields for a rigid plastic, purely cohesive material. A growing awareness in the period 1930 to 1950 of the necessity of considering both stress and velocity fields for plastic materials culminated with the publication of Hill's book (1950).

After 1950 there was a period of rapid development. Hill had proven extremum and variational principles and Drucker, Greenberg and Prager (1951) were able to cast these in the particularly useful form of limit theorems for a purely cohesive material. The same authors (1952) later extended these limit theorems to a more general class of materials with an associated flow rule. Bishop (1953) clarified the relation between the limit theorems and uniqueness for a purely cohesive, rigid plastic material and outlined steps necessary to establish uniqueness.

However, it was not in the area of metal plasticity that the first attempts were made to analyse the behaviour of plastic materials. The first attempts are to be seen in the earth pressure calculations of Coulomb (1773) and Rankine (1857). As it is one of the aims of the present work to examine the application of finite deformation theory to elastoplastic soil, it is appropriate to outline some of the major advances in the use of plasticity theory in this area. This summary can be conveniently be divided into two sections: (a) the use of plasticity theory in the determination of the stability of soil masses
loaded in various manners; and (b) the investigation of various yield
criteria and plastic flow laws in an effort to obtain a more accurate
description of elemental soil behaviour.

(a) In determining solutions to problems of stability of soil
masses most researchers focussed attention on the bearing capacity
of strip or circle footings or the limiting heights of soil slopes.
The soil mass was, in general idealised as a cohesive or cohesive
frictional, rigid plastic. In order to obtain a complete solution
to any problem one must be able to show that certain requirements have
been met. These are:

(i) a statically admissible stress field must be found for any
regions of the loaded body that are plastic;

(ii) a kinematically admissible velocity field must be found;

(iii) the stress field must be capable of extension into the
rigid region (outside the plastic region) without violation of the
yield criterion;

Kötter (1888, 1903) was the first to combine equilibrium
equations with the Mohr-Coulomb yield criterion to produce the fami-
liar characteristic equations for plane deformations. In 1920 Prandtl
showed that the two-dimensional problem, for a cohesive frictional
weightless material, is hyperbolic and in subsequent years found, what
are now realised to be incomplete solutions to several problems (Shield,
1954b). These results were subsequently applied by Reissner (1924) and
Novotorsev (1938) to certain particular problems on the bearing capaci-
ty of footings on weightless soil. Cox (1962) has also found complete
solutions for the smooth circular and smooth strip footing.
However, the important inclusion of soil weight considerably complicates the mathematical solution. Consequently, many approximate methods have been developed for calculating the bearing capacity of a ponderable soil. The work of Lundren and Mortensen (1953), de Jong (1957), Spencer (1962), Dembicki, Dravtchenko and Sibille (1964), Booker (1970) and Davis and Booker (1971) falls into this category.

Traditionally though, the so-called limit equilibrium method was used to obtain approximate solutions for the stability problems in soil mechanics. In these analyses it is usual to make an assumption about the shape of the failure surface in the soil mass and then to analyse the statics of the surface. (To do so usually requires further arbitrary assumptions regarding the stresses or forces involved.) An example of this analysis may be seen in the bearing capacity solutions of Terzaghi (1943) and the slope stability studies of Taylor (1937) and Bishop (1955). These traditional solutions do not, strictly speaking, meet the requirements for a true lower bound answer. Davis and Booker (1971), using a numerical technique for the integration of the equations of plasticity, examined rigorously the solutions of Terzaghi and found that generally his bearing capacity factors were optimistic, particularly for ideal soils with large (>30°) friction angles.

There are instances in the literature where the authors have met the precise requirements for a true upper and/or lower bound answer. The solutions of Meyerhof (1951), Shield and Drucker (1953), Finn (1967) and Davis (1969) are some examples. In such cases the limit theorems of Drucker, Greenberg and Prager (1951, 1952) are very useful. They provide an approximate means of evaluating the solution by placing bounds on the collapse load. Statically admissible and kinematically admissible
solutions form lower and upper bounds respectively, to the true answer. In many instances it is possible to evaluate bounds that are very close so that the solution is of considerable engineering value. These limit theorems have only been proved for, and are thus only applicable to, materials with an associated flow rule.

Recently an interesting paper appeared which presented a technique of direct limit analysis using rigid-plastic finite elements (Hung, 1976). Collapse loads may be found by conversion of the theoretical nonlinear problem into an equivalent linear elastic problem and performing a reasonable number of analysis iterations. Lysmer (1970) suggested a similar method of limit analysis based on a division of the continuum into a distinct number of regions and the use of linear programming to optimise the statically admissible lower bound solution.

(b) The second area of application of plasticity theory in soil mechanics has been considered by Drucker (1953b) and Shield (1955). They found a valid generalisation of the Tresca yield criterion suitable for soil and Cox, Eason and Hopkins (1961) made a detailed examination of the equations which resulted from this generalisation. Drucker, Gibson and Henkel (1957) have examined the consequences of introducing a strain hardening model and obtained good qualitative agreement with observed behaviour of soils. Other work hardening models have been introduced by Jenicke and Shield (1959). More recently, intensive research has been carried out at Cambridge University into the stress-strain behaviour of clays (e.g. Roscoe, Schofield and Wroth; Roscoe, Schofield and Thurairajah, 1963a, 1963b; Burland, 1965; Roscoe and Burland, 1968) and some of the proposed models involve strain
hardening.

Independently Hansen (1958), Mroz (1963) and Davis (1969) postulated a class of materials which obey a non-associated flow rule and a similar class of materials has been suggested by Strogenov (1967). The adoption of such rules results in a more accurate description of real soil behaviour but it does have some surprising consequences. In the case of a non-associated flow rule material the limit theorems no longer hold. This have been shown by Davis (1969). Also, uniqueness of solutions cannot be established for such a material.

Further discussion of constitutive models for elastoplastic soils is contained in chapter III. To avoid repetition the relevant literature is not presented here.

Finally, a review of the field of soil plasticity has recently been published in a text by Chen (1975), which contains a very large source list.

1.2.3 Infinitesimal Elastoplastic Deformation Theory

The extreme difficulty in obtaining an exact elastoplastic solution to any other than a very simple problem, is due basically to the change in the form of governing equations, from elliptic partial differential equations for elasticity to hyperbolic equations for plasticity. Further difficulties arise due to the fact that the elastic-plastic boundary is changing with continued loading, from the fact that the stress-strain relationships for loading and unloading
are different and also because the geometry of the body may change appreciably during its loading history. Even without these complications, there are few solutions available for problems that consider only non-linear elasticity. One important point needs emphasis. While in linear problems the solution is always unique this is no longer the case in many non-linear situations. Thus, if a solution is achieved it may not necessarily be the solution sought. Some insight into the nature of the problem may be necessary and, on occasion, small-step, incremental approaches are essential to obtain meaningful answers.

Since the initial elastic analysis of Turner et al. (1956) and later Clough (1960) many investigators have adapted the technique of finite elements (Zienkiewicz, 1971a) to the analysis of continuum problems for both linear and non-linear systems. Initial attempts at analysing non-linear elastic continua were performed by Argyris (1965b).

A brief review is now given of the literature concerned with the analysis of infinitesimal deformations of elastoplastic materials. We leave aside, for the moment, the added complication of finite changes in geometry. The first such analyses were those of Pope (1965), Swedlow, Williams and Yang (1965), Reyes and Deere (1966), Marcal and King (1967), Akyuz and Merwin (1968) and others, who employed the familiar variable stiffness method. Subsequently attempts were made to achieve more efficient solutions. The so-called "initial strain" method was proposed by Argyris (1969b) and later Zienkiewicz, Valliapan and King (1969) suggested a technique which they termed the "initial stress" approach. All of the above three solution procedures usually required iteration within any load step to approximate the stress dependent quantities.
In all elasto-plastic analyses the size of the load increments affects the solution because of the latter's dependence on stress history. Yamada, Yoshimura and Saburi (1968) contend that the true sequence of yielding can only be followed if the size of each load increment is adjusted so that a maximum of only one element yields within any load step. This method has the advantage that few iterations are needed as load steps are generally quite small. This advantage must, however, be weighed against the obvious disadvantage that, if a large number of elements are employed in an effort to achieve greater accuracy, then a large number of load steps will also be necessary.

The use of infinitesimal elasto-plastic and non-linear elastic analyses in the realm of soil mechanics has become popular in recent years. Ang and Harper (1964) formulated equations relating stress and strain increments for a Prandtl-Ruess material which obeys a Tresca yield criterion. Marcal (1965) applied the theory together with a von Mises yield criterion in the formulation of a stiffness method while Yamada et al. (1968) extended the analysis to a form suitable for finite element formulation. These analyses were appropriate for purely cohesive materials, but none had application to a frictional or cohesive-frictional material. Reyes and Deere (1966) showed that if the normality principle is accepted in conjunction with a modified von Mises criterion, then cohesive frictional materials can be analysed numerically. Later a form of flow rule suggested by Davis (1969) was employed by Zienkiewicz and Best (1969) in an attempt to analyse materials with non-associated flow rules by the finite element method. Further investigation of the use of such flow rules in numerical techniques has been reported by Ring (1975), Davis, Ring and Booker (1974) and Davis and Booker (1973a, 1975). Some attempts have also been made at incorporating the 'critical state' soil models, proposed by Roscoe and his co-workers at Cambridge, into
finite element calculations. Amongst these are the works of Smith and Kay (1971), Zienkiewicz (1971b), Chung and Lee (1972), Simpson and Wroth (1972), Wroth and Simpson (1972) and Simpson (1973).

Considerable attention has been given to the problem of a horizontal clay layer subjected to vertical loads such as those transmitted by a footing. Girijavallabhan and Reese (1968) considered a circular footing bearing on an undrained clay using an isotropic non-linear elastic model for the clay. They compared finite element predictions with the results of footing tests and obtained good agreement. Others to compare numerical predictions with model tests or field data include Desai and Reese (1970), Smith (1970), Duncan and Chang (1970), D'Appolonia, Poulos and Ladd (1971), Penman, Burland and Charles (1971) and Desai (1971).

The problem of a strip footing on a clay layer has been treated numerically by Zienkiewicz, Valliapan and King (1969), Fernandez and Christian (1971) and Davidson and Chen (1974). The latter two groups both performed finite strain as well as small strain analyses of this problem. In addition, the behaviour of a circular footing on a shallow layer of undrained, strain softening clay has been analysed using a finite element technique by Höeg (1972).

Many other examples of the use of incremental elasto-plastic and non-linear elastic analyses in geomechanics can be found in the volumes edited by Desai (1972, 1976).
1.2.4 Finite Elastoplastic Deformation Theory

The problem of the finite deformation of an elastoplastic continuum is one that has received much attention in the last two decades. Initially, attempts were made to examine the conditions necessary for a rigorous solution, some works of Hill (1958, 1959, 1961) serve as examples. He established a sufficient condition for the uniqueness of the boundary-value problem set by given velocities and nominal traction rates on its boundary. No restriction was placed on changes in geometry. The solution, when unique, is shown to be characterised by an extremum principle. Hill compares these theorems with those for rigid-plastic solids (Hill, 1957a, 1957b).

Several workers considered various ways of decomposing the strain rate into elastic and plastic parts. Among them were Green and Naghdi (1965), Sedov (1966), Lee (1969), Freund (1970) and Hahn (1974). Naghdi and Trapp (1975) were concerned with the restrictions on the constitutive equations of elastic-plastic materials in the presence of large deformations. Their theory is purely mechanical and the non-linear equations developed are of a rate type. Maier and Drucker (1973) have considered the presence of geometric effects on the equilibrium equations. They show that even in the presence of finite geometry changes normality of constituent elements implies normality for the system.

Some exact and numerical solutions for simple isothermal loading and unloading processes, produced by normal and shear stress combinations, have been presented by Lehman (1973).
The invention of the digital computer has given rise to a great amount of activity in this area. Just as in the case of finite elastic deformations, and indeed many other areas of engineering science, the solutions to many problems are now within reach, when previously their attempt would have been unthinkable. A great many incremental formulations have been put forward to make use of the numerical capabilities of these computers. Most have utilised finite element schemes and these include diverse formulations of both Lagrangian (material) and Eulerian (spatial) type. (Further details of the Lagrangian and Eulerian approaches are given in the next chapter). Considerable attention has likewise been given to the selection of a suitable constitutive law for use in these finite deformation analyses.

Much of this recent work has been devoted to formulating analyses for plate and shell problems including large displacement but small strains (a survey is given by Marcal, 1967). These formulations are, however, inappropriate for applications to bulky geometries such as occur in many problems in soil and continuum mechanics. Some of the more general formulations are now reviewed.

An incremental finite element theory for large strain and large displacement is given by Kitigawa, Seguchi and Tomita (1972). They use a convected coordinate system and consider an infinitesimal increment of deformation superposed on the current configuration resulting from the known finite deformation. No numerical examples are provided but an explicit formulation for a tetrahedral element is given.

Hibbit, Marcal and Rice (1970) used a Lagrangian scheme to
derive finite element rate equilibrium equations from the principle of virtual work for large deformations. Their work is a direct extension of the Prandtl-Reuss theory, taking into account a frame indifferent stress rate, and has relevance to the behaviour of metals. They suggest a linear relation between the incremental Kirchoff (1852) stress tensor and the incremental Green (1839) strain tensor. Needleman (1972) has a similar Lagrangian scheme to Hibbit et al. but he derives his equations from a variational principle due to Hill (1959).

Hofmeister, Greenbaum and Evensen (1971) also use a Lagrangian approach in an attempt to obtain a large strain elasto-plastic analysis of two-dimensional structures. They use an incremental variational principle to develop finite element equations for a piecewise linear solution which includes an equilibrium check. The constitutive equation adopted relates the incremental Cauchy stress tensor to the incremental Green strain tensor. Numerical examples presented include the problems of a notched tensile specimen and plate bending.

Felippa and Sharifi (1973) adopted a Lagrangian formulation. They intended to place no limitation on the size of an increment of deformation. However, the increment size is limited in any case by the need to ensure that the constitutive rate moduli do not change significantly from one increment to the next. In these circumstances their approach corresponds to that of Hibbit et al.

Zienkiewicz and Nayak (1971) presented a unified formulation for large deformation, large strain and plasticity problems. Lagrangian forms were preferred but an alternative Eulerian system was given. Iso-
parametric finite elements were used to solve several example problems including those of a thick cantilever, shallow arch, bellows and axisymmetric extrusion. This work is closely related to that of Nayak (1971).

Another formulation which employs a finite element mesh representing the current state of deformation has been used by Yaghmi and Popov (1971). They used the same variational principle as Felippa and Sharifi to derive governing equations and solve an elastic and an elastoplastic example problem. Sharifi and Popov (1973) extended this method to the elasto-plastic analysis of infinitesimal strain but finite rotation problems.

An Eulerian presentation is also given by Osias (1972, 1973) and subsequently by Osias and Swedlow (1974). This work admits non-symmetric constitutive laws through a Galerkin method. A rate view point is adopted and objectivity of formulation is preserved by the introduction of the Jaumann (1905) stress rate. Some solutions are presented for simple homogeneous deformation problems.

Meeking and Rice (1975) adopt an Eulerian formulation and derive governing equations based on Hill's (1959) variational principle for incremental deformations. This formulation is suited to the study of isotropically hardening Prandtl-Reuss materials. The necking bifurcation analysis of a bar in plane strain tension is presented as a numerical example.

In the following chapter it is pointed out that in describing finite deformations one must be precise when using the continuum mecha-
nics terminology. Gunasekera and Alexander (1973) have used an Eulerian formulation with equations for a Prandtl-Reuss material. There seems to be some confusion in their work as to what interpretation should be given to the symbol $\sigma$, used to represent some stress. This matter is discussed further by Meeking and Rice (1974). Some ambiguity in the interpretation of stress is also apparent in the work of Argyris and Chan (1973), who preferred the Eulerian approach.

In the field of soil mechanics very few attempts have been made at the application of a finite deformation analysis to soil behaviour. It is well known that large deformations can occur when very soft soils are loaded with plastic flow being an important mechanism. The most obvious example in engineering practice concerns the deformations that occur when road or railway embankments are built across marsh or swamp lands. The swamp material is usually highly organic and provides little resistance to loading. A finite deformation theory might thus be useful for the prediction of the penetration of such embankments into the soft soil. A good description of an example of this type of problem is contained in a paper by Casagrande (1960).

One of the first attempts at analysis in this area was by Thoms and Arman (1970). The objective of their study was to determine the displacements and stresses beneath embankments constructed over soft organic soils using experimental models and a numerical technique employing the finite element method. In the experimental phase of the study grid and photoelastic methods were used to determine deformations and stresses in the model soil, which consisted of a soft gelatin mix with an embedded ink grid. In the numerical study of the finite deformation behaviour an approach due to Argyris (1965a) was used to analyse
the elastic material. No account was taken of any plastic behaviour. The authors concluded from this study that the Bousinesq (1885) elastic solutions were conservative for such a problem.

This work was followed by two more papers on the same problem. In the first Thoms, Arman and Pequet (1972) recognised the time dependence of the behaviour of embankments on soft soil. In this numerical study a diffusion analysis was crudely coupled with the finite deformation analysis proposed by the first two authors. In this paper a description is also given of the test apparatus used to measure the material parameters used in their theory. Later Thoms, Pequet and Arman (1976) used the theory to predict the behaviour of a highway embankment constructed with clam shell over South Louisiana (U.S.A.) marshland in 1974. They reported good agreement between predicted and measured values of settlement over a time period of 120 days following construction. In this problem it is not clear whether finite deformation analysis was important. The authors reported that a settlement of approximately one foot was measured beneath a five foot high embankment on a layer of unspecified depth. The density of the clam shell material was only 65 lb/ft\(^3\). No predictions using small strain theory were given.

Fernandez and Christian (1971) performed an analytical study of a strip footing on undrained clay with both material and geometric nonlinearities included in the formulation. Hyperbolic nonlinear elastic and elasto-plastic Tresca models were used to describe the clay. The finite element technique was employed and the analysis was based on the formulation of Biot (1965). The results of this early attempt at a finite deformation analysis are very interesting but inconclusive.
The load displacement curves were irregular and a lack of equilibrium between stresses and nodal forces was reported. For the material parameters used (Young's modulus/cohesion ~ $10^3$) changing geometry should have little effect on the results at load levels below the small strain limit values.

Another analytical study of the static response of a homogeneous clay stratum to footing loads was conducted by Davidson and Chen (1974). They performed both small and large strain finite element analyses of drained and undrained behaviour with the clay being modelled as a linear elastic-perfectly plastic material with a Drucker-Prager (extended von Mises) yield criterion and an associated flow rule. The finite strain analysis was based on an incremental virtual work principle with no allowance being made for body force. The adopted constitutive law relates the Cauchy infinitesimal strain tensor to an incremental stress tensor formed from the difference between the Cauchy stresses in the deformed and undeformed states, but with both measured relative to their respective local coordinate systems which differ by the increment of rotation. For values of total and effective stress parameters, of the order given by Young's modulus/cohesion ~ $1000$, the results of small and large deformation analyses were found to differ only near the small strain limit load. This is a similar conclusion to that found by Fernandez and Christian. It is not certain from the work of Davidson and Chen whether or not they assume that a limit load must exist for the finite strain analysis. In fact no such limit need exist. They do not pursue their solutions much beyond the theoretical small strain limit values. It is quite conceivable that after considerable change in geometry has occurred, the loads may continue to increase (or perhaps decrease in some cases).
The results of Davidson and Chen are also presented by Chen (1975) and Davidson and Chen (1976) and the analyses have been extended by Snitbahn and Chen (1976) to the case of soil slopes. In this latter work load displacement results are presented for typical soil slopes all the way up to nominal failure, defined by bulging or loss of ground. The loading is defined by incrementing the soil density. They show that considerable differences occur between the load displacement relations predicted by small and large strain analyses and in fact that the finite strain analysis produces the more critical results in this problem. It should be noted that such a finding is problem dependent and indeed it will be shown in this work that the opposite may well be true in some cases.

1.2.5 Consolidation Theory

An important problem in foundation engineering is that of predicting the behaviour of a foundation resting on saturated clay. When subjected to loading this material generally undergoes a process of time dependent deformation including that termed consolidation. Terzaghi (1925, 1943) first investigated the phenomenon of consolidation for one dimensional conditions. Biot (1941a, 1941b) subsequently extend this theory to three dimensional conditions, and since then many investigators have been concerned with obtaining both closed form and numerical solutions to consolidation problems with various boundary loadings (for references, see Introduction, Chapter 6).

In the formulations of Terzaghi and Biot the authors restricted their attentions to conditions of infinitesimal strain and thus the
theory they developed is only strictly applicable to situations in which the geometry varies only slightly during loading. Gibson, England and Hussey (1967) recognised this limitation and developed a one dimensional theory which accounted for such finite deformation. Using a Lagrangian formulation the equations governing the one-dimensional consolidation are derived with no imposition on the magnitude of strain and an allowance for the variation in soil compressibility and permeability during consolidation. Furthermore, although Darcy's law is assumed to be valid, it is recast in a form in which it is the relative velocity of the soil skeleton and the pore fluid that is related to the excess pore fluid pressure gradient. The problem of the consolidation of a thin clay layer, whose self weight stresses are negligible compared with those applied, is examined in detail.

Mesri and Rokhsar (1974) have also presented a theory and some numerical results for one dimensional consolidation that considers finite strain, the variations of compressibility and permeability during the consolidation process and the effects of a critical pressure and secondary compression. The governing equations of this theory were solved using an explicit finite difference approximation.

Some similar work involving geometrical non-linearity in one dimensional consolidation has been presented by Smiles and Poulos (1969), Monte and Krizek (1976) and De Simone and Viggiani (1976). In the latter case the authors show how the theory of Gibson, England and Hussey may form the basis for a numerical treatment of the problem. Some numerical results are presented along with a case history of field behaviour.
All of the abovementioned theories of consolidation assume that the skeleton of the soil is elastic, although in some cases this elasticity is nonlinear. Recently a theory and numerical solution method was presented that examines the consolidation of an ideal permeable soil whose skeleton is composed of an elasto-plastic material (Small, 1977; Small, Booker and Davis, 1976). This theory unites the usually separately considered aspects of analytical soil mechanics: time dependent consolidation; and plastic yielding leading to the collapse of the soil mass. The resulting equations are obtained from the principle of virtual work and apply for infinitesimal deformations. The importance of the rate of loading in determining the collapse load of structures founded on soil is demonstrated by the use of an example.

The work of Small, Booker and Davis was extended by Carter, Small and Booker (1976) and Carter, Booker and Small (1977) to include the possibility of finite deformations during the consolidation of ideal soils whose skeletons are elastic or elastoplastic.

Having now reviewed much of the relevant literature from the fields of elasticity, plasticity, consolidation and soil mechanics generally, the following chapters will endeavour to present a rational method of analysis for finite deformations, and then to apply the theory to some problems of interest. Much of the material reviewed in this chapter will again be referred to in the following pages.
CHAPTER 2

FINITE DEFORMATION FORMULATION
2.1 INTRODUCTION

A mathematical treatment of the problem of the finite deformation of a continuum body is the objective of this chapter. The word "deformation" is used, in general, to encompass all the changes in the location in space and in the geometrical configuration of the body in some specified period of time, due to the action of all forces acting on it during that time. Later it will be used to refer specifically to a measure of the change that takes place in any given period.

In any attempt to describe finite changes a choice must be made between a number of alternative descriptions, a problem which does not arise in the infinitesimal theory. The choice must be made between coordinate systems, the measure of deformation (strain) or deformation rate, the measures of stress or stress rate and even the form of the constitutive law governing the physical behaviour of the material. Some of the alternatives that are in common use in continuum mechanics theory are presented together with a brief description of their respective advantages. Finally, a consistent rate description to be employed throughout the remainder of this work, is proposed.

Throughout this chapter the shorthand technique of tensor notation is used. In addition, Appendix 2A contains explicit expressions of the basic definitions and governing equations, in three dimensions, using matrix notation. This is more convenient when discussing the finite element solution method.

The material in this chapter is organised so that the basic definitions of stress and strain are presented first in section 2.2,
together with some discussion on the various ways of describing
constitutive behaviour. For the greater part, the concept of time
variation is not a real consideration. The treatment of section
2.2 loosely follows a similar presentation of Fernandez and

A rate description of finite changes is introduced in
section 2.3. This uses a quasi-static approach employing a dependence
of field quantities upon a pseudo time variable. The non-uniqueness
of stress rate and strain rate description is discussed, closely
following a similar presentation of Prager (1961a, 1961b).

In section 2.4 a virtual work formulation is used to derive
exact equations governing an arbitrary increment of finite deformation
of the continuum body. It is then shown in section 2.5 that in the
limit as the magnitude of deformation is reduced to zero, the governing
equations of finite deformation reduce to the more familiar equations
of the infinitesimal theory.

Some recent work has been devoted to formulating analyses for
plate and shell problems involving large displacements but small strains
(a survey is given by Marcal, 1967). These formulations are inappropriate
for applications to bulky geometries such as occur in many problems in
soil mechanics. Some attempts to develop a more complete solution method
for the finite deformation problem have been reported by Felippa (1966),
Davidson and Chen (1974), Osias and Swedlow (1974), Osias (1973), Hibbitt,
Marcal and Rice (1970), Hofmeister, Greenbaum and Evensen (1971) and
Carter, Booker and Davis (1976).
2.2 SOME FUNDAMENTAL DEFINITIONS

Because of the non-uniqueness of the means of describing finite changes in a body subject to loading, it is important to be precise when using the commonly occuring terminology. A discussion of some alternative systems of description is appropriate.

2.2.1 Coordinate Systems

The description of the motion of a continuum will be restricted to the use of a Cartesian coordinate system. The question is purely geometrical and neither the causes which gave rise to the deformation nor the law according to which the body resists them are of any importance at present. These matters are discussed later.

For any body undergoing some displacement in space, that may involve any or all of translation, straining and rotation, there are two means of description of the movement.

Consider that at some time \( t_o \) a body of matter is located within the region in space defined by the symbol \( V_o \). This region is bounded by the surface \( S_o \). At some time \( t \) later (for our present argument it matters not, how much later) the same body occupies a position in space designated by the symbol \( V \) and is bounded by the surface \( S \).

Let a typical material particle of the body be located at time \( t_o \) at a point in space \( P_o \) described by the Cartesian coordinate \( a_i \), \( i = 1, 2, 3 \). At time \( t \) later this same material particle occupies a position \( P \) described by the new coordinate \( x_i \), \( i = 1, 2, 3 \). (see Fig. 2.1)
The relationship between the two coordinates $a_i$ and $x_i$ is given by

$$x_i = a_i + u_i$$ \hspace{1cm} i = 1, 2, 3 \hspace{1cm} (2.1)$$

where $u_i$ represents the displacement field for this movement and may be considered as either a function of $a_i$, i.e. $u_i = u_i(a_k)$ or as a function of $x_i$, i.e. $u_i = u_i(x_k)$. The former case is called the Lagrangian description while the later is the Eulerian description. This well established nomenclature, that is familiar in the corresponding situation in hydrodynamics, will be retained in the following, even though it cannot be justified on historical grounds (Prager, 1961a; Truesdell, 1952).

When motions of unrestricted magnitude are to be described careful distinction must be made between these two systems. When the movements are considered to be infinitesimal the two descriptions are assumed to coincide, i.e.

$$x_i = a_i$$ \hspace{1cm} (2.2)$$

In any formulation used to analyse finite movements certain different advantages ensue from adopting either description. From the standpoint of physics, Lagrangian variables are particularly suited to the description of the motion of a continuum for which the initial state is not an arbitrarily chosen reference state but a natural state of the continuum, such as the homogeneous stress-free state of an elastic body. The use of Lagrangian variables then enables us to treat all kinematical questions in a particularly simple manner. For example this system is useful in boundary loading problems as one can always locate its position. In exchange, however, for this advantage certain
complications must be accepted in the description of the statics of deformation, as will be seen in section 2.2.3. In fact it will be seen that the chief advantage of the Eulerian variables is the simplicity of form of the governing equations of finite deformation that results from their use. This renders Eulerian variables attractive from the mathematical point of view.

2.2.2 Descriptions of Strain

In particular, the word "deformation" is used to describe a change in distance between points of a body and the word "strain" is used to describe a measure of unitary deformation. Potentially, any tensor formed from the displacement gradients qualifies as a strain measure if it vanishes identically for all rigid body motions. Ambiguity may arise since there are in fact an infinite number of possible strain measures; a general expression is given by Karni and Reiner (1962). Various forms of the strain - displacement relations are clearly explained by Parks and Durelli (1964).

Some of the more common strain tensors will be given here. It is, however, convenient to first examine a simple homogeneous deformation problem to gain some insight into their basic differences. (This example has been discussed by Fernandez and Christian, 1971).

Consider the extension of a bar under a uniform tension (Fig. 2.2). The axial strain in this situation can be defined in the following ways.

\[
\frac{\ell - \ell_0}{\ell_0} \quad \text{(Cauchy, 1827)}
\]  

(2.3)
(b) \[ \frac{L - L^0}{L^0} \] (Swainger, 1947) \hspace{1cm} (2.4)

(c) \[ \frac{1}{2} \left( \frac{L}{L^0} \frac{L^0}{L} - 1 \right) \] (Green, 1839) \hspace{1cm} (2.5)

(d) \[ \frac{1}{2} \left( 1 - \frac{L}{L^0} \frac{L^0}{L} \right) \] (Almanso, 1911) \hspace{1cm} (2.6)

(e) \[ \int \frac{dL}{L} = \ln \left( \frac{L}{L^0} \right) \] (Hencky, 1931; also Ludwik, 1909) \hspace{1cm} (2.7)

where \( L \) is the instantaneous length. We note that if the rod length doubles, \( \frac{L}{L^0} = 2 \), the values of strain are: (a) 100%; (b) 50%;
(c) 150%; (d) 37.5% and; (e) 66%. Notice the large variation in magnitude of the different measures of the same phenomenon. Of course as the magnitude becomes smaller, the measures of strain more closely agree e.g. if \( \frac{L}{L^0} = 1.1 \) then the values of strain are: (a) 10%;
(b) 9.1%; (c) 10.5%; (d) 8.7% and; (e) 9.5%. From this simple example it is obvious that any discussion of finite strain should also include its definition.

For the general three-dimensional situation only the Green (1839), the Almanso (1911) and the Cauchy (1827) strain tensors are of interest here, and the first two are given by,*

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i} \cdot \frac{\partial u_k}{\partial a_j} \right) \] (Green Strain Tensor) \hspace{1cm} (2.8)

* Unless otherwise stated repeated indices imply summation.
\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial u_k}{\partial x_j} \right) \]  

(Almansi Strain Tensor)

Furthermore, if the components of displacement \( u_i \) are such that their derivatives are so small that the squares and products of the partial derivatives of \( u_i \) are negligible, then equations (2.8, 2.9) become identical in form, since derivatives of the displacement relative to a point \( a_i \) or a point \( x_i \) are equivalent (to within higher degree terms in the displacement). In other words \( \frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial x_j} \). The strain tensor then reduces to the Cauchy Infinitesimal Strain Tensor

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} \right) \]  

(2.10)

Both Green and Almansi tensors are obtained by examining the change in the square of the length of a segment. The Green tensor is obtained when it is expressed as a function of the undeformed geometry, and the Almansi tensor when it is expressed as a function of the deformed geometry (Prager, 1961a; Fung 1965; Green and Zerna, 1954).

Preference for one definition of strain over another may depend upon its relation to stress. For example, a particular strain definition may give a desirable linear relation between the stress and strain tensors within a certain range of behaviour. This situation, however, is dependent upon the material and is considered in chapter 3.

2.2.3 Descriptions of Stress

The concept of stress is unique as opposed to the concept of
strain. The Cauchy (or Eulerian) stress tensor $\sigma_{ij}$ referred to the deformed configuration is the $i$th component of traction on a surface normal to the $j$th coordinate direction. The symmetric Cauchy stress tensor provides a complete description of the loading state at a point in a deformed body. More detailed descriptions of what is meant by the Cauchy stress tensor can be found in many general texts (e.g. Prager, 1961a; Fung, 1965; Green and Zerna, 1954; Love, 1944; Sokolnikov, 1956) but the definition above should not, however, obscure the simple well known interpretation of stress. The usual continuum mechanics convention of tension positive will be followed in this thesis.

There are several other tensors defined to give some required forces on some given differential areas. They are called tensors by extension and are defined for convenience when formulating a theory. The forces that they give on an area are not the real ones acting on those areas. Two such stress tensors are commonly used in the classical theory of finite elasticity. They are the Lagrangian, $L_{ij}$, and the Kirchoff, $K_{ij}$, stress tensors (Kirchoff, 1852).

Several advantages of the Lagrangian variables were pointed out in section 2.2.1. As is to be expected, however, the use of these variables also entails some disadvantages. These stem from the fact that the stresses transmitted in the instantaneous state must now be referred to the initial state in a way that is physically artificial though mathematically consistent.

If $\sigma_{ij}$ is the stress tensor referred to the instantaneous state, the infinitesimal force $dF_j$ transmitted in the differential
surface element $\text{d}S$ with unit normal vector $n_i$ is

$$\text{d}P_j = \sigma_{ij} n_i \text{d}S$$  \hfill (2.11)

Referred to the initial state this can be expressed as

$$\text{d}p_j = L_{ij} n_{oi} \text{d}S_0$$  \hfill (2.12)

where $n_{oi}$ is the normal to the differential surface element $\text{d}S_0$ in the undeformed condition. $L_{ij}$ is the Lagrangian stress tensor and is related to $\sigma_{ij}$ by (Prager, 1961a)

$$L_{ij} = \frac{\rho_o}{\rho} \frac{\partial a_i}{\partial x_m} \sigma_{mj}$$  \hfill (2.13)

where $\rho_o$ and $\rho$ are the densities of the material before and after deformation.

As relation (2.13) shows, the Lagrangian stress tensor is not symmetric as a rule, and this is very inconvenient in formulating constitutive relations. This difficulty can be avoided by using the following transformation

$$\text{d}P_{oj} = \frac{\partial a_i}{\partial x_j} \cdot \text{d}P_i$$  \hfill (2.14)

The new tensor will give the force $\text{d}P_{oj}$ on the differential area $\text{d}S_o$ before deformation and is related to $\sigma_{ij}$ by

$$S_{ij} = \frac{\rho_o}{\rho} \cdot \frac{\partial a_i}{\partial x_m} \cdot \frac{\partial a_i}{\partial x_k} \cdot \sigma_{km}$$  \hfill (2.15)

$S_{ij}$ is called the Kirchoff stress tensor.
It should be noted that as the tensors $L_{ij}$ and $S_{ij}$ have no obvious physical meaning, then any idea of what they represent can only be obtained from their respective definitions.

2.2.4 Equilibrium

It is convenient here to express the equilibrium equations in terms of each of the three stress tensors mentioned above.

Eulerian
\[
\frac{\partial U_{ij}}{\partial x_j} + F_i = 0 \quad , \quad T_i = n_{ij} S_{ij} \tag{2.16}
\]
(or Cauchy)

Lagrangian
\[
\frac{\partial L_{ij}}{\partial a_j} + F_{oi} = 0 \quad , \quad T_{oi} = n_{oj} L_{ji} \tag{2.17}
\]

Kirchoff
\[
\frac{\partial}{\partial a_j} \left( S_{ik} \frac{\partial x_i}{\partial a_k} \right) + F_{oi} = 0 \quad , \quad T_{oi} = S_{ji} \frac{\partial x_i}{\partial a_j} n_{oj} \tag{2.18}
\]

where $F_{oi}$, $F_i$ are the body forces per unit volume of the undeformed and deformed volumes respectively, $T_i$ are the surface tractions referred to the deformed area and $T_{oi}$ are the surface tractions referred to the undeformed area but keeping the direction they had in the deformed position. Equivalent virtual work expressions can also be derived from equations (2.16, 2.17, 2.18) and may be found in many of the standard texts (e.g. Prager, 1961a; Fung, 1965). Notice the relative mathematical simplicity of the Euler treatment compared with the others.

2.2.5 Constitutive Relations

The constitutive equation is a tensor equation which estab-
lishes a relation between statical and kinematical tensors, for instance the stress or stress rate and the strain or strain rate (the rate quantities are defined in section 2.3).

The problem that poses itself is which tensors should be used in a general constitutive law? In particular, which form of the constitutive equation and which type of formulation (i.e. Lagrangian or Eulerian) would be suited to the description of soil behaviour? In deciding the answer to this question a number of factors should be considered, though the final solution may indeed only be a question of personal preference. In any given problem the solution should be unique* no matter which formulation is adopted, the difference is of course provided for in the constitutive relation (e.g. Carter, 1975).

The ease with which the required material parameters may be measured should be one of the considerations when choosing an approach to problem solution. Establishing a relation between $S_{ij}$ and $E_{ij}$ or $e_{ij}$, say, may be difficult in practice as $S_{ij}$ has no physical meaning. $L_{ij}$ is inconvenient to use in material laws for the same reason and because it is not symmetric. The tensor $\sigma_{ij}$ may also prove an inconvenient quantity to use because of its frame dependence.

As soil is in most instances non-linear an incremental approach to problem solution suggests itself. In such circumstances it seems reasonable to postulate that there exists a relation, which may of course depend on the previous history of the body, between the

* Leaving aside the questions of instability and non-uniqueness of some elasto-plastic solutions.
increment in stress and the increment in strain and that as the time interval becomes infinitesimal this will reduce to a relation between stress rate and strain rate. An investigation of the nature of this relation is the subject of the next section.

2.3 A RATE APPROACH

In this section a rate approach is used to describe the deformations of a body. As before we focus attention on a particular increment of motion in the time interval $t_0$ to $t$. The displacement field $u_i$ for this increment is measured relative to the configuration at $t_0$ and thus vanishes when $t = t_0$, while the Cauchy stress field $\sigma_{ij}$ is measured at the instantaneous configuration at time $t$. The total motion of the body from time 0 to $t$ can be regarded as a sequence of such incremental motions. The typical increment is depicted in Fig. 2.1. The approach is thus quasi-static in nature.

2.3.1 Deformation Rate

In using a rate approach we must not lose sight of the fact, mentioned earlier, that there exists more than one way of describing finite motions. Strain and strain rate, like displacement, can be defined only with respect to some reference state. This matter and its relevance to a rate approach is described lucidly by Prager and Hodge (1951, p. 119). They are now quoted:

"For some continuous media, e.g. the Newtonian viscous fluid, the choice of reference state is arbitrary; for other media, e.g. the Hookean elastic solid, there exists a natural reference state in which the medium is free from stress. A medium of this
second kind may be said to 'remember' its natural state, and the stress-strain law of such a medium will involve the strain with respect to the natural state; i.e. the mechanical behaviour of the medium will depend on the difference between instantaneous and natural states. On the other hand, the arbitrary reference state used in defining the strain in a medium of the first kind cannot be expected to influence the mechanical behaviour. Thus, the stress-strain law for a medium of the first kind will not involve strain itself, but only the rate of straining. Moreover, in defining the instantaneous rate of straining in such a medium, we need not use a fixed reference state, but may, at any instant, use the instantaneous state for reference purposes. In this manner we obtain the rate of straining defined with respect to the deformed medium (Eulerian strain rates).

In a medium of the second kind, on the other hand, it is reasonable to define the rate of straining as the time derivative of strain defined with respect to the natural state. We shall term this concept the rate of straining with respect to the undeformed medium (Lagrangian strain rates). Obviously, these two manners of defining the rate of straining will lead to different results only in the case of finite deformations".

We now proceed to formulate the mathematical expression for these concepts and in doing so a choice must be made from the alternative descriptions. In this present work the Eulerian point of view is regarded as fundamentally more significant than the Lagrangian, a
point of view that is shared by Murnaghan (1937), Seth (1935), Cocker and Filon (1931) and more recently Osias and Swedlow (1974).

Using the Eulerian description, the instantaneous rate of deformation may be described by the velocity gradient

\[
\frac{\partial v_i}{\partial x_j} = \lambda_{ij} + \omega_{ij} \tag{2.21}
\]

where

\[
v_i = \frac{\partial}{\partial t} u_i(x_k, t)
\]

\[
= \frac{d}{dt} u_i(x_k, t) \quad \text{for} \ k = 1, 2, 3 \tag{2.22}
\]

The symmetric deformation rate tensor \( \lambda_{ij} \) and the skew-symmetric spin tensor \( \omega_{ij} \) are given as

\[
\lambda_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{2.23}
\]

\[
\omega_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \tag{2.24}
\]

Of course the Lagrangian deformation rates may be retrieved through the relation

\[
\frac{\partial v_i}{\partial x_j} = \frac{\partial v_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial x_j} \tag{2.25}
\]

2.3.2 Stress Rate

For materials that are either elastic or elasto-plastic it is usual (Prager, 1961b) to express the rate of deformation as a func-
tion of an appropriately defined rate of stress (or vice versa).
An important problem in continuum mechanics has been that of selecting a suitable definition of stress rate. It has commanded the attention of, amongst others, Jaumann (1905), Truesdell (1953), Green (1956) and Cotter and Rivlin (1955), each adopting a slightly different definition.

As Prager notes (1961b),
"the stress rate must obviously satisfy the following condition: if a stressed continuum performs a rigid body motion and the stress field is independent of time when referred to a coordinate system that participates in this motion, the stress rate vanishes identically".

Its definition must, therefore, contain a rotary term to compensate for the fact that the stress components with respect to a fixed coordinate system change, even when there is no change in the stress components with respect to a coordinate system that participated in the instantaneous rotation of the neighbourhood of a considered particle. The example of a bar in simple tension that is rigidly rotated is often quoted (Truesdell, 1955).

It can readily be seen that the restriction of objectivity is not severe enough to lead to a unique definition. As Prager (1961a, 1961b, 1962) and Oldroyd (1950) have observed, the difference between acceptable definitions of stress rate must necessarily consist of a linear combination of deformation rates. Thus in any constitutive law which expresses the stress rate as a linear combination of the deformation rates, such terms can always be absorbed into this linear relationship, so that the difference between the various definitions is
only illusory. As a consequence of this freedom of choice, many
definitions of stress rate are found in the literature. This situ-
ation is directly analogous to that which arises when one attempts
to define finite strain (see previous discussion, section 2.2.2).

Some of the principal definitions of stress rate found in
the literature are given here. Recalling that \( \sigma_{ij}(x_k, t) \), \( k = 1, 2, 3 \)
denotes the time dependent Cauchy stress field in the continuum then:

(a) Jaumann's (1905) definition is given by

\[
\dot{\sigma}_{ij} = \sigma_{ij} - \sigma_{ik} \omega_{jk} - \sigma_{jk} \omega_{ik} \tag{2.26}
\]

where \( \omega_{ij} \) is the skew symmetric spin tensor of section 2.3.1 and the
superior dot is used to represent the material derivative.

(b) the definition of Cotter and Rivlin (1955) is given as

\[
\sigma'_{ij} = \sigma_{ij} + \frac{\partial \nu_k}{\partial x_j} \cdot \sigma_{ki} + \frac{\partial \nu_i}{\partial x_j} \cdot \sigma_{kj} \tag{2.27}
\]

(c) Oldroyd's (1950) definition (sometimes known as Green's
(1956) definition) is

\[
\sigma''_{ij} = \sigma_{ij} - \sigma_{ik} \cdot \frac{\partial \nu_i}{\partial x_k} - \sigma_{jk} \cdot \frac{\partial \nu_i}{\partial x_k} \tag{2.28}
\]

(d) Truesdell (1953) proposed the following definition of stress
rate

\[
\sigma'''_{ij} = \sigma_{ij} + \sigma_{ij} \cdot \frac{\partial \nu_k}{\partial x_k} - \sigma_{ik} \cdot \frac{\partial \nu_j}{\partial x_k} - \sigma_{jk} \cdot \frac{\partial \nu_i}{\partial x_k} \tag{2.29}
\]
Prager (1961b) notes that the vanishing of the Jaumann stress rate implies stationary behaviour of the stress invariants. This condition does not hold, however, for definitions (b), (c) and (d). Furthermore, when describing deformations of an elastic, perfectly plastic solid it is common practice to assume that the total deformation rate is formed from the superposition of elastic and plastic components. For such solids two criteria are therefore used to judge a given variation of stress in time: the stress rate in the elastic constituent, and the rate of change of the yield function in the plastic constituent. Prager suggests that the yield function should be stationary when the stress rate vanishes. (This point is discussed further with regard to anisotropic strength in chapter 3). Only Jaumann's definition of stress rate satisfies these conditions and as such will be adopted in the remainder of this dissertation.

2.3.3 A General Rate Law

In choosing an appropriate elasto-plastic constitutive equation for use in this work it is the intention here to preserve the character of the infinitesimal theory while generalising to allow for the finite deformation. The basic equation must also be readily applicable to the description of soil behaviour. The generalisation requires precise interpretation of the time rates which must be objective.

The general linear relationship between the objective stress rate and the deformation rate which is proposed can be expressed in the form

\[ \dot{\sigma}_{ij} = D_{ijkl} \dot{e}_{kl} + g_{ij} \]  

(2.30a)
where $\dot{\sigma}_{ij}$ is the Jaumann stress rate, 
$\dot{e}_{kl}$ is the deformation rate defined by equation (2.23), 
$D_{ijkl}$ are constants that may depend upon the current state 
and perhaps all previous states, in some specified 
way, and where the symmetric tensor $g_{ij}$ is included to account for the presence of any features such as pore pressures (i.e. an effective stress law) or thermal and shrinkage effects. If these are absent then $g_{ij} = 0$. By the use of a piecewise linearity of the values of the constants $D_{ijkl}$ any combination of experimentally observed or intuitively assumed material behaviour may be analysed by this theory. Such multi-linear behaviour is readily incorporated into the numerical technique described in chapter 4. The explicit determination of $D_{ijkl}$ for a material with a general yield criterion and an arbitrary flow rule is given in the next chapter.

Equation (2.30a) may also be conveniently be expressed as

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{e}_{kl} + \sigma_{ik} \omega_{jk} + \sigma_{jk} \omega_{ik} + g_{ij} \quad (2.30b)$$

2.4 VIRTUAL WORK FORMULATION

An exact set of equations governing the behaviour of the body in deforming from $V_0$ to $V$ in the time interval $t_0$ to $t$ is now sought. No limit is imposed, at this stage, on the magnitude of the displacements $u_i$. For the body of Fig. 2.1 the stress field $\sigma_{ij} = \sigma_{ij}(a_k, t_0)$, $k = 1, 2, 3$ is in equilibrium at time $t_0$ with a traction set $T_{oi} =$
\( \mathbf{n}_{i} \mathbf{t}_{i} \mathbf{n}_{i} \) acting over the surface \( S_{i} \) (\( \mathbf{n}_{i} = \) unit normal to \( S_{i} \)), and with bodyforces \( \mathbf{F}_{i} \) acting within \( V_{o} \). Similarly, for the body at time \( t \) the stress field \( \mathbf{S}_{i} = \mathbf{S}_{i}(x, k, t) \), \( k = 1, 2, 3 \) is in equilibrium with a traction set \( \mathbf{T}_{i} = \mathbf{n}_{i} \mathbf{t}_{i} \) acting over \( S_{i} \) (\( \mathbf{n}_{i} = \) unit normal to \( S_{i} \)) and with bodyforces \( \mathbf{F}_{i} \) acting within \( V \). At times \( t_{o} \) and \( t \) velocities are specified on \( S_{o} \) and \( S_{i} \) respectively*.

The velocity field \( \mathbf{v}_{i} \) of particles within \( V \) satisfies the velocity boundary conditions along \( S_{i} \). As well, the incremental velocity field \( \mathbf{d} \mathbf{v}_{i} \) is compatible with the symmetric incremental deformation rate tensor \( \mathbf{d} \mathbf{l}_{i j} \) and satisfies the boundary conditions on \( S_{i} \). Use may thus be made of the principle of virtual work and hence at time \( t \)

\[
\int_{V} \mathbf{d} \mathbf{l}_{i j} \mathbf{S}_{i j} \mathbf{d} \mathbf{V} = \int_{V} \mathbf{d} \mathbf{v}_{i} \mathbf{F}_{i} \mathbf{d} \mathbf{V} + \int_{S_{i}} \mathbf{d} \mathbf{v}_{i} \mathbf{T}_{i} \mathbf{d} \mathbf{S} \tag{2.31}
\]

Equation (2.30b) may be integrated in time to yield

\[
\mathbf{S}_{i j} - \mathbf{S}_{i j} = \int_{t_{o}}^{t} \left\{ \mathbf{D}_{i j k l} \mathbf{S}_{k l} + \mathbf{S}_{i k j} \mathbf{w}_{k} + \mathbf{S}_{j k i} \mathbf{w}_{k} + \mathbf{q}_{i j} \right\} \mathbf{d} \mathbf{t} \tag{2.32}
\]

Substituting equation (2.32) into equation (2.31) gives

\[
\int_{V} \mathbf{d} \mathbf{l}_{i j} \left[ \mathbf{S}_{i j} + \int_{t_{o}}^{t} \left\{ \mathbf{D}_{i j k l} \mathbf{S}_{k l} + \mathbf{S}_{i k j} \mathbf{w}_{k} + \mathbf{S}_{j k i} \mathbf{w}_{k} + \mathbf{q}_{i j} \right\} \mathbf{d} \mathbf{t} \right] \mathbf{d} \mathbf{V} = \int_{V} \mathbf{d} \mathbf{v}_{i} \mathbf{F}_{i} \mathbf{d} \mathbf{V} + \int_{S_{i}} \mathbf{d} \mathbf{v}_{i} \mathbf{T}_{i} \mathbf{d} \mathbf{S} \tag{2.33}
\]

This is an exact equation governing the behaviour of the body in deforming from \( V_{o} \) to \( V \). The solution of this integral equation will, for all but the simplest problems, require an approximate technique.

A discussion of a suitable solution method forms the subject matter

*More complicated boundary conditions are easily incorporated into the theory.
of chapter 4.

2.5 THE LIMIT OF INFINITESIMAL DEFORMATION

The virtual work equations (2.33) are valid irrespective of deformation magnitude. Reduction of these equations to more familiar forms of the infinitesimal theory follows from the assumption that the change in the configuration of the body due to loading is negligible. As a result, rotation terms may be neglected and the spatial integrations of equation (2.33) may be performed over the original body. This then becomes

\[
\int \int \int_{V_o} \left\{ \frac{D_{ijkl}}{t_o} + g_{ij} \right\} dt \, dV = \int \int \int_{V_o} \sigma_{ij} \, dV + \int \int_{S_{OT}} \sigma_{ij} \, dS \quad (2.34)
\]

In the absence of plastic flow the \( D_{ijkl} \) of equation (2.30a) remain constant for all \( t \). If none of the effects represented by \( g_{ij} \) are present and if we evaluate the time integral over the range \( 0, t \), then

\[
\int \int \int_{V_o} \sigma_{ij} \, dV = \int \int \int_{V_o} \sigma_{ij} \, dV + \int \int_{S_{OT}} \sigma_{ij} \, dS \quad (2.35a)
\]

where in this case \( \sigma_{ij} \) is considered as the change in the stress tensor in the time interval \( 0, t \). From equation (2.35a) we may also imply that

\[
\int \int \int_{V_o} \delta_{ij} \, dV = \int \int \int_{V_o} \delta_{ij} \, dV + \int \int_{S_{OT}} \delta_{ij} \, dS \quad (2.35b)
\]

which is the familiar virtual work expression of the infinitesimal theory.
Fig 2.1 DEFORMATION MAPPING
APPENDIX 2A

MATRIX REPRESENTATION OF THE GOVERNING EQUATIONS

The aim here is the conversion from tensor notation to matrix notation of all essential definitions and governing equations of this chapter. The matrix approach takes account of the symmetry and skew symmetry of certain tensors. Again only a Cartesian description is given for the general three dimensional case.

Let the vectors \( \mathbf{a} = (a, b, c) \) and \( \mathbf{x} = (x, y, z) \) contain the coordinates \( a_1 \) and \( x_1 \) respectively. For convenience we have replaced \( a_1, a_2, a_3, x_1, x_2, x_3 \) by \( a, b, c, x, y, z \) respectively. Similarly let the vector \( \mathbf{u}^T = (u_x, u_y, u_z) \) contain the displacement components \( u_i \). We now define some additional vectors.

(a) For the deformation gradients \( \lambda_{ij} \) and \( \omega_{ij} \) define

\[
\mathbf{d}^T = (\mathbf{\lambda}^T, \mathbf{\omega}^T)
\]

where

\[
\mathbf{\lambda} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \text{ is the vector of strain rates}
\]

(2A.2)

\[
\mathbf{\omega} = \frac{\partial \mathbf{u}}{\partial t} \text{, is the vector of spin or rotation rates}
\]

(2A.3)

and

\[
\mathbf{v} = (v_x, v_y, v_z)^T = \frac{\partial}{\partial t} \mathbf{u}(a, t)
\]

(2A.4)

\[
= \frac{\partial}{\partial t} \mathbf{u}(x, t)
\]

is the vector of velocity components,

with \( \partial^T = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \)
\[ \zeta = \begin{bmatrix} \partial/\partial y & -\partial/\partial x & 0 \\ 0 & \partial/\partial z & -\partial/\partial y \\ -\partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \]

(b) The Cauchy stress \( \sigma_{ij} \) is represented by the vector of stress components,

\[ \sigma^T = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}) \]  \( \text{(2A.5)} \)

(c) The Jaumann stress rate \( \dot{\sigma}_{ij} \) is represented by,

\[ \dot{\sigma} = \sigma + M \dot{d} \]  \( \text{(2A.6)} \)

where \( M = \begin{bmatrix} -\sigma_{xy} & 0 & \sigma_{zx} \\ \sigma_{xy} & -\sigma_{yz} & 0 \\ 0 & 0 & -\sigma_{zz} \end{bmatrix} \)

\( (6 \times 6) \)

(d) The general rate law can be written,

\[ \dot{\sigma} = D \dot{\varepsilon} + g \]  \( \text{(2A.8a)} \)

where \( g \) corresponds to the symmetric tensor \( g_{ij} \) and \( D \) is a matrix whose components depend on the current state and perhaps all previous states in some specified way (see chapter 3). Equation (2A.8a) may also be expressed as

\[ \dot{\sigma} = H \dot{d} + g \]  \( \text{(2A.8b)} \)
where $H = (D, O)$.

Alternatively we may write

\[
\dot{\sigma} = P \dot{\alpha} + q \tag{2A.8c}
\]

where $P = H - M$.

The virtual work equation (2.33) which governs exactly the behaviour of the body during an increment of deformation in the period $t_o$ to $t$ has the matrix equivalent

\[
\int_{V}^{t} \left\{ \sigma_o + \int_{t_o}^{t} (P \dot{\alpha} + q) \, dt \right\} \, dv = \int_{V}^{t} \dot{v}^{T} \tau dv + \int_{V}^{t} \dot{v}^{T} T ds \tag{2A.9}
\]

where $\sigma_o = \sigma_o(a, t_o)$ corresponds to $\sigma_{oij}$ and $T$ and $F$ correspond to $T_i$ and $F_i$ respectively.
CHAPTER 3

CONSTITUTIVE RELATIONS FOR SOIL
3.1 INTRODUCTION

In any analytical study of the behaviour of a soil mass subject to loading, indeed of any body subject to loading, it is necessary to select a suitable mathematical model to describe the elemental response of the soil to applied stress. Such models are termed "stress-strain laws", though in fact they may involve relations between quantities other than stress and strain (e.g. stress rate and strain rate).

A soil can be considered to be an aggregate of both organic and inorganic particles or grains. These particles do not fill the entire space occupied by the soil but are bound together by geometrical constraint, by intergranular adhesion and by friction to form a skeleton. The space between adjacent particles may be occupied by liquid and gas.

Because of the complicated particulate nature of the mineral skeleton the stress-strain behaviour of soil is exceedingly complex. It is this complex nature that accounts for the existence of more than one model to describe soil behaviour. Some of these models will be described in this chapter and one that is particularly suited to the finite strain formulation is also presented. The latter will then be adopted throughout the remainder of this thesis.

The validity of any conclusions that may be drawn from analytical studies of soil behaviour are, of course, dependent upon the degree to which the mathematical model approximates real soil behaviour. However, it should be noted that in all models the real behaviour must be
drastically idealised in order to make the mathematical analysis tractable. In addition, the solution to any problem, as posed for the ideal soil, must be within the means of the investigator. Thus a conflict of interest arises when choosing a model between the accuracy of description and the amenability to mathematical treatment. It is for this reason that one may find a soil described as a linear elastic material in one context (say) and a viscoelastic material in another.

All models are oversimplifications of reality and are usually the major source of discrepancy in the comparison of theoretical prediction and experimental fact.

For the moment we shall ignore any real time dependence of the soil to applied stress. Phenomena such as consolidation and viscous effects such as creep will not be considered in this chapter. Consolidation behaviour is discussed in Chapter 6. For the present we consider the soil to be a single phase material. We begin with a discussion of one of the simplest soil models.

3.1.1 Elastic Model

Soil is often idealized as an elastic material. When loads are applied to an elastic body the deformation is instantaneously dependent upon only the load and independent of how the load was applied. The behaviour of such a soil is characterised by a number of elastic constants \( D_{ijkl} \) and the stress strain law is usually written in the form

\[
\sigma_{ij} = D_{ijkl} \varepsilon_{kl} \tag{3.1}
\]
where $\sigma_{ij}$ and $\varepsilon_{kl}$ are the Cauchy stress and infinitesimal strain tensors of the previous chapter.

The elastic model was employed by Davis and Poulos (1963, 1968) and Poulos (1967) to perform settlement analyses due to various loadings on soil layers. Similar work has been presented by Kerisel and Quatre (1968) and Egorov et al. (1957).

Real soil is, in general, not an ideal elastic material in that stress and strain are not linearly related, strains are not fully recoverable on removal of the applied stress, and strains are not independent of time. Nevertheless, over a limited stress range, the stress-strain response of the soil can reasonably be assumed to be linear and provided that the soil moduli $D_{ijkl}$ can be determined over the correct ranges of initial stress and stress increments, the use of elastic theory should lead to satisfactory predictions of behaviour. Methods of determining the soil parameters are described by Davis and Poulos (1963, 1968) and Ladd and Foote (1974) and have been conveniently summarised by Poulos (1975).

The linear elastic model has been extended by several investigators (for references see Hagmann, 1971) to take account of local yielding of the soil. The new model is termed the bi-linear elastic model and it involves the use of a different set of elastic moduli in the stress range after local yield has occurred as well as some allowance for irrecoverable strain. Multi-linear and piecewise linear models are developments of the bi-linear model.

There are several shortcomings associated with an adoption of the bi-linear model as opposed to a true elasto-plastic model. The most
important of these is that in the bi-linear model an increment of strain maintains the same principal directions as the corresponding increment of stress, whereas for the perfectly plastic model the plastic strain increments have the same principal directions as the current stress at any given instant. This difference in stress-strain behaviour may result in a different deformation behaviour for the loaded soil.

3.1.2 Elasto-plastic Model

The possibility of obtaining closed form solutions to many problems may explain the widespread use of a linear elastic model when treating problems in soil mechanics. However, soil is fundamentally a plastic material. Residual deformations generally result after cycles of loading and unloading even at low stress levels.

The application of the mathematical theory of plasticity to soils has involved the efforts of many researchers (e.g. Drucker and Prager, 1952; Shield, 1954a; and later Davis, 1969). Much of the early work was concerned with the application of limit analysis to determine the collapse of an ideal plastic material.

The availability of high speed computers has made possible the determination of the response of an ideal elasto-plastic soil throughout the full loading history including a determination of the collapse state. Techniques such as those of finite differences (Ang and Harper, 1964) and finite elements (e.g. Zienkiewicz Valliapan and King, 1969) have been used together with the loading path methods to solve such problems. (A discussion of both finite difference and finite element techniques is given by Ring, 1975).
For the ideal elasto-plastic material several features determine the overall behaviour. These are:

(i) A set of elastic parameters which determine the behaviour of an element of ideal soil at stress levels below that of the yield condition.

(ii) The yield function. A suitable failure criterion is required to determine at what stress level plastic behaviour begins.

(iii) A flow law, which determines the plastic volume behaviour of an element of ideal soil once it has reached the yield state.

Most theories assume that the infinitesimal strain can be considered as the superposition of a plastic component of strain upon an elastic component, i.e.

\[ e_{ij} = e_{ij}^E + e_{ij}^P \]  

(3.2)

where \( e_{ij}^E \) and \( e_{ij}^P \) are the elastic and plastic components of the Cauchy strain tensor, respectively. This is not a necessary requirement of plastic behaviour and, indeed, some authors have assumed otherwise (e.g. Hahn, 1974 assumes a multiplicative decomposition of the measure of deformation).

Some of the early work in this field was concerned with a material that is linearly elastic up to yield, deforms plastically at constant volume and which, in soil mechanics terms, is purely cohesive. Either the von Mises (1913) or Tresca (1864) yield criterion was employed. Such theory is directly applicable to investigation of the
immediate or undrained response of saturated clay, since such soil behaves in a manner close to that of the ideal material of the theory. The well known Mohr-Coulomb yield criterion (Mohr, 1900; Coulomb, 1773) has been applied to cohesive frictional materials together with both associated and non-associated flow rules (see sections 3.2 and 3.3). This has been particularly useful when analysing the drained or long term behaviour of many soils. More recently the Mohr-Coulomb model has been used when treating the consolidation behaviour of an elasto-plastic soil (Small, Booker and Davis, 1976).

Drucker and Prager (1952) discussed an extension of the well known von Mises yield criterion and subsequently Drucker (1953a) presented the so-called extended Tresca yield criterion. Bishop (1966) has attempted to correlate these two criteria and that of Mohr-Coulomb with experimental data and has concluded that the Mohr-Coulomb criterion best predicts soil failure. Kirkpatrick (1957) also reached this conclusion for sands. Roscoe, Schoefield and Thurairajah (1963a) contend that the available experimental data (particularly triaxial extension tests) are not sufficiently reliable to allow one of the criteria to be favoured over the others. At this time they suggested "it would therefore seem reasonable to assume, until reasonable evidence is produced to the contrary, that the simplest failure criterion, namely the extended von Mises, is relevant to soils" (op. cit. p. 127). Later Roscoe and Burland (1968) showed that the Mohr-Coulomb criterion of rupture could in fact be used in conjunction with the Cambridge theory for "wet" clay to successfully predict the behaviour of such a material under triaxial and plane strain conditions.
Some recent work has been aimed at extending the elastic perfectly plastic models to account for the phenomenon of strain softening exhibited by many real soils (e.g. Höeg, 1972, 1973; Lo and Lee, 1972; Prévost and Höeg, 1975).

3.1.3 The Cambridge Models

The suggestion that soil might be modelled as a plastic work hardening material was first proposed by Drucker, Gibson and Henkel (1957). Subsequently, extensive research has been carried out at Cambridge University into the stress-strain behaviour of clays. Roscoe, Schofield and Wroth (1958) published work that contained the basis for a number of strain hardening models for soil. Later, Roscoe, Schofield and Thurairajah (1963b) utilised the strain hardening theory of plasticity to formulate a complete stress-strain model for normally consolidated clay in the triaxial test. This model has since become known as the Cam-clay model (Schofield and Wroth, 1968). Burland (1965) suggested a modified version of the Cam-clay model that will strain soften rather than strain harden.

In recent years the results of the Cambridge research has been put into a form suitable for practical application (Burland, 1969, 1971; Wroth and Simpson, 1972).

3.1.4 Hyperbolic Models

Based on the results of compression triaxial tests Konder (1963) and Konder and Zelasko (1963) have suggested that the relationship between stress and strain for many sands and clay can be approxi-
mated by a hyperbola. Shear failure is the main phenomenon involved in the triaxial test so the functional relationship can be written

\[ \tau = \frac{\gamma}{a + b\gamma} \]  

(3.3)

where \( \tau \) is the maximum shear stress, \( \gamma \) is twice the Cauchy shear strain (i.e. the engineering shear strain) and \( a \) and \( b \) are constants. This model and its application in large strain theory are discussed further by Fernandez and Christian (1971). A particular shortcoming of the hyperbolic models is their inability to represent the strain softening behaviour exhibited by many soils.

3.2 A GENERAL RATE LAW

The analysis of an ideal elasto-plastic material undergoing infinitesimal deformation has been examined on several occasions (Zienkiewicz, 1971a; Marcal and King, 1967; Zienkiewicz Vallaipan and King, 1969). More recently there has been an attempt to extend this analysis to the problem of finite deformation. However, these investigations have been restricted to materials with an associated flow rule (Osias and Swedlow, 1974; Davidson and Chen, 1974; Fernandez and Christian, 1971). In this section the constitutive equations governing a material with an arbitrary yield condition and a general flow rule will be developed. The discussion will proceed using the matrix notation introduced in Appendix 2A.

The deformation rate \( \dot{\lambda} \) is considered as the superposition of an elastic \( \dot{\lambda}_E \) and a plastic component \( \dot{\lambda}_P \), thus

\[ \dot{\lambda} = \dot{\lambda}_E + \dot{\lambda}_P \]  

(3.4)
This is analogous to the small strain theory - see equation (3.2).

The elastic rate component is related to the Jaumann stress rate through the elastic rate law

\[ \dot{s} = D_{E} \dot{E} + q_{E} \]  \hspace{1cm} (3.5)

where \( D_{E} \) is a matrix of elastic constants and \( q_{E} \) is the elastic vector corresponding to the tensor \( q_{ij} \) (equation 2.30).

The plastic rate component may be written in the form

\[ \dot{\lambda}_{p} = \lambda_{a} \] \hspace{1cm} (3.6)

where the vector \( \lambda_{a} \) depends upon the current state of the material and may in fact depend on all previous states*, and \( \lambda \) is a multiplier signifying that there are no viscous effects present. \( \lambda \) has to be positive to fulfil the necessary requirements that the rate of plastic work must always be positive (Davis and Booker, 1973a; Davis, Ring and Booker, 1974). The importance of this feature in relation to numerical computation is discussed in the next chapter.

It will be assumed that the general yield criterion can be written in the form

\[ f(\dot{\sigma}) = 0 \] \hspace{1cm} (3.7)

where \( f \) is termed the yield function. The nature of equation (3.7) indicates that the material behaves as a perfect plastic, i.e.

* A correct determination of the properties \( a, D_{E} \) occurring in equations (3.5, 3.6) would involve the testing of an element whose history (load path) was identical to the given element.
\[ df = \frac{\partial f}{\partial \sigma} \cdot d\sigma = b^T \cdot d\sigma = 0 \]  
\( (3.8a) \)

where \( b^T = \left( \frac{\partial f}{\partial \sigma}, \frac{\partial f}{\partial \sigma}, \ldots, \frac{\partial f}{\partial \sigma} \right) \)  
\( (3.8b) \)

Many investigators have assumed that the material has an associated flow rule, i.e.

\[ a \propto b \]  
\( (3.9) \)

but this will not be a necessary assumption in this work. An alternative expression for \( a \), that applies to a Mohr-Coulomb material with a non-associated flow rule is given in section 3.3.

Consider now an element in a plastic state, which deforms from an initial position A to an adjacent position B in the time interval \( t_0 \) to \( t \), as shown schematically in Fig. 3.1. Thus initially at \( t_0 \)

\[ f(\sigma(t_0)) = f(\sigma_{xx}(t_0), \sigma_{yy}(t_0), \ldots, \sigma_{zx}(t_0)) = 0 \]  
\( (3.10a) \)

The question now arises as to the form of the yield criterion when the element is in position B. One reasonable assumption is that any form of strength anisotropy is intrinsic to the element so that

\[ f(\sigma_{\xi\xi}(t), \sigma_{\eta\eta}(t), \ldots, \sigma_{\zeta\zeta}(t)) = 0 \]  
\( (3.10b) \)

Combining equations (3.10a, b and 3.8) leads to

\[ b^T \cdot d\sigma = 0 \]  
\( (3.11) \)
where \( \mathbf{b} \) is the vector normal to the yield surface, defined above, and \( \mathbf{\dot{\xi}} \) is the Jaumann stress rate.

It is interesting to note that according to this assumption as an element rotates the form of the yield criterion (referred to the initial set of axes \( X, Y, Z \)) changes. Thus a material which is anisotropic, but whose anisotropy is homogeneous, gradually develops an inhomogeneity of anisotropy as different elements rotate by different amounts. Of course this does not occur (according to this formulation) if the material is initially isotropic and in such a case

\[
\mathbf{b}^T \mathbf{\dot{\xi}} = \mathbf{b}^T \mathbf{\dot{\eta}} = 0 \quad (3.12)
\]

It is now possible to derive a relationship between the deformation and stress rates. Equations (3.4, 3.5, 3.6) may be combined to show that

\[
\mathbf{D}_E \mathbf{\dot{\xi}} = \mathbf{\dot{\xi}} - \mathbf{g}_E + \lambda \mathbf{\dot{\eta}} \quad (3.13)
\]

where \( \mathbf{g} = \mathbf{D}_E \mathbf{a} \)

Now on premultiplying equation (3.13) by \( \chi = \left(1 - \frac{\mathbf{ab}^T}{\mathbf{g}^T \mathbf{b}}\right) \) it is found that

\[
\mathbf{\dot{\xi}} - \chi \mathbf{g}_E = \chi \mathbf{D}_E \mathbf{\dot{\xi}} \quad (3.14)
\]

This is precisely the form of equation (2A.8a) considered in the previous chapter with

\[
\mathbf{D} = \chi \mathbf{D}_E = \left(1 - \frac{\mathbf{ab}^T}{\mathbf{g}^T \mathbf{b}}\right) \mathbf{D}_E \quad (3.15)
\]
\[ q = \chi_{q_E} = \left( 1 - \frac{a_T^T}{a_T^T} \right) q_E \] (3.16)

and thus the formulation for the general rate law applies to the elastoplastic material.

The ideal model may be used to analyse the deformations of materials other than soil as long as the components of the matrix D can be determined. In the next section this matrix is defined for a material with a Mohr-Coulomb failure criterion and deforming under conditions of plane strain.

Before proceeding, however, it is convenient to point out, that according to this formulation, if a material possesses infinite strength and if the elastic parameters of \( D_E \) are constant (i.e. they are independent of stress level), then the deformations will be independent of load path.* This is the definition of an elastic material as used in this thesis.

3.3 RATE LAW FOR A 'MOHR-COULOMB' MATERIAL

The general equations of the preceding section are now particularised for an isotropic soil obeying the Mohr-Coulomb failure criterion and deforming under conditions of plane strain. In such circumstances the matrix of elastic constants, \( D_E \) is given by

\[
D_E = \begin{bmatrix}
\Lambda + 2G & \Lambda & 0 \\
\Lambda & \Lambda + 2G & 0 \\
0 & 0 & C
\end{bmatrix}
\] (3.17)

* When obtaining an approximate solution to any problem involving an 'elastic' material the magnitude of load steps may, however, be governed by the need for numerical accuracy and stability (see Chapter 4).
where \( \Lambda \) and \( G \) are analogous to the Lame parameters of the classical small strain theory of elasticity. They are related to quantities \( E \) and \( \nu \) through

\[
\Lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}
\]

\[
G = \frac{E}{2(1+\nu)}
\]

The quantities \( E \), \( G \) and \( \nu \) will be termed the Young's modulus, Shear modulus and Poisson's ratio respectively. In the limit of small deformation they are the same as the corresponding parameters of the classical theory.

The failure criterion can be written

\[
f(\sigma) = (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 - \sin^2\phi(\sigma_{xx} + \sigma_{yy} - 2c \cot\phi)^2 = 0 \quad (3.18)
\]

where \( c, \phi \) are the cohesion and angle of friction respectively, and tensile stresses are reckoned positive. The vector \( \vec{b} \) which is normal to the yield surface is given by,

\[
\vec{b} = 4R(\sin\phi + \cos2\theta, \sin\phi - \cos2\theta, 2\sin2\theta)^T \quad (3.19)
\]

where \( R = \left\{ \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2 \right\}^{1/2} \) is the radius of the Mohr circle of stress and \( \theta \) is the angle between the major principal stress direction and the \( x \) axis.

For the material with an associated flow rule the plastic component of the deformation rate adopts the form,
\[ \mathbf{\dot{\lambda}} = \lambda_a = 4\lambda R (\sin\phi + \cos2\theta, \sin\phi - \cos2\theta, 2\sin2\theta)^T \]  

(3.20)

As is well known a flow rule described by equation (3.20) predicts a rate of dilatency far in excess of that observed in real soils.* Davis (1969) has postulated a class of ideal soils which have a flow rule that may be expressed for conditions of plane strain as,

\[ \mathbf{\dot{\lambda}} = \lambda_a = 4\lambda R (\sin\psi + \cos2\theta, \sin\psi - \cos2\theta, 2\sin2\theta)^T \]  

(3.21)

where \( \psi \) is a measure of the rate of dilatency of the soil so that \( \psi = \phi \) corresponds to a soil with an associated flow rule and \( \psi = 0 \) corresponds to a material which deforms plastically at constant volume.

The expressions for \( \dot{a} \) (equation 3.21), \( \dot{b} \) (3.19) and \( \dot{D}_E \) (3.17) may now be combined in the manner specified by equation (3.15) to obtain the plasticity matrix \( D \) for the Mohr-Coulomb material.

* An important exception is the undrained deformation of saturated clays.
FIG. 3.1 ELEMENT ROTATION
CHAPTER 4

NUMERICAL SOLUTION METHOD
4.1 INTRODUCTION

In solving problems that involve finite deformations of a body, the ultimate aim is to obtain knowledge of its geometrical configuration and the associated stress field at any given time, i.e. at any load level. Specifically we require knowledge of the full history of the deformation mapping \( x_i(a_k, t), \) \( k = 1, 2, 3 \) and the stress tensor \( \sigma_{ij}(x_k, t), \) \( k = 1, 2, 3. \) Evaluation of this solution for an elasto-plastic material will require the solution of the governing virtual work equation (2.33), an expression that is, in fact, an integral equation in time. To do this involves a number of operations:

(i) the solution of equation (2.33) to obtain the velocity field \( v_i \) in the time varying domain \( V, \)

(ii) integration of the velocity field with respect to time to determine \( V, \) and

(iii) integration of the constitutive equations (2.30c) with respect to time to determine the stress field \( \sigma_{ij} \) in \( V. \)

An approximate solution of the governing equations of chapter 2 becomes necessary because of the difficulty of obtaining exact solutions to all but the simplest problems (e.g. homogeneous deformation under proportional loading: Osias and Swedlow, 1974; Carter, Booker and Davis, 1976). Rather than seeking a continuous solution for the time varying configuration and stress field we restrict our attention to the behaviour of these quantities at a finite number of times during the deformation. The total deformation is approximated as a sequence of incremental deformations. Adoption of the incremental approach makes use of the load path method of solu-
tion that is familiar from previous infinitesimal elastoplastic ana-
yses (e.g. Zienkiewicz, Valliapan and King, 1969; Christian and Marr,
1972; Ring, 1975). While the time integration is effected by the
loading path method, the spatial integration of the field quantities
will be performed using the method of finite elements (Zienkiewicz

The incremental approach to problem solving allows spatial
and time integration to proceed either (a) sequentially or (b) simul-
taneously.

(a) Sequential Integration: In this procedure spatial
integration of the instantaneously linear virtual work equations pro-
vides the velocity field in $V$ at time $t_0$. Subsequent integration of
the velocity field and constitutive equations over a time increment $\Delta t$
provides the configuration and stress field at a new time $t_0 + \Delta t$.
A new spatial problem for the velocity at $t_0 + \Delta t$ may then be defined.
In this procedure no iteration is required* and solution accuracy is de-
pendent upon the step size used. Increased time step size provides a
less accurate solution at reduced expense and vice versa. This is the
method preferred by Osias and Swedlow (1974; see also Osias, 1973).

(b) Simultaneous Integration: In this method both time and
spatial integration within any increment are effected together. Solution
for the dependent variables as a function of time requires an iterative
procedure as time dependent coefficients in the governing equations are
approximated by some 'mean' values. The chief advantage of this method
is that generally time steps larger than those of method (a) may be
employed to obtain the same order of solution accuracy. In any case

* Unless elastic unloading occurs.
method (a) is a particular application of (b) with the use of only one iteration.

In both methods the incremental approach to problem solution, superficially at least, imposes no restriction on loading type, geometry or deformation magnitude.* The usefulness of the rate analysis is limited only by the availability of the material data required to fully define the constitutive behaviour.

It is emphasised that in this analysis the current loading history begins at time \( t = 0 \). The body may, however, be in a stressed condition at this time, the initial stress state having arisen due to the action of all forces on the body prior to time \( t = 0 \) (e.g. the presence of geostatic stresses in soil layers prior to foundation loading). The importance of correctly accounting for the presence of such an initial stress state is demonstrated in a later section. This feature is, of course, not a consideration in the small strain theory where the presence of an initial stress state affects only the pattern and sequence of yielding of an elasto-plastic material.

The material of this chapter is organised as follows. In section 4.2 the numerical solution technique is described. Its capability, accuracy, applicability and limitations are then examined, in section 4.3, with the use of a number of simple examples for which exact solutions can also be found. Finally, some general conclusions are presented in section 4.4.

* Some special numerical problems may arise - see sections 4.3 and 4.4.
4.2 FINITE ELEMENT EQUATIONS

In the following presentation it is convenient to adopt
the matrix notation of Appendix 2A for the governing equations of
the general, three dimensional, finite deformation analysis.
Equations governing numerical solution of plane strain problems
are given in Appendix 4A.

For the body of Fig. 2.1 we assume that at time $t$ the
stress field $\mathbf{\sigma}(x, t)$ is in equilibrium with a total traction set $\mathbf{T}$
acting over the surface $S_T$ and with bodyforces $\mathbf{F}$ within $V$. We
allow the vector $\mathbf{v}$ to denote the velocity field of all particles
within $V$, which satisfies the velocity boundary conditions. The
incremental velocities $\mathbf{\dot{v}}$ are compatible with the incremental strain
rates $\mathbf{\ddot{v}}$. Hence the equation of virtual work may be expressed as,

$$
\int_{V}^{T} \mathbf{\sigma} \mathbf{v}^{\dot{\Omega}} \mathbf{dt} + \int_{S_T}^{T} \mathbf{F} \mathbf{v}^{\dot{\Omega}} \mathbf{dv} + \int_{V}^{T} \mathbf{v}^{\dot{\Omega}} \mathbf{F}^{\dot{\Omega}} \mathbf{dv} = \int_{S_T}^{T} \mathbf{v}^{\dot{\Omega}} \mathbf{T}^{\dot{\Omega}} \mathbf{ds}
$$

(2A.9)

where all symbols are defined in Appendix 2A

An approximate solution of equation (2A.9) can be obtained
using the Finite Element method. If the continuous deforming body
is divided into a discrete number of conforming elements then suppose
that the field of displacements $\mathbf{u}$ that occurs in time interval $t_0$ to
t can adequately be represented by values at the connecting nodes 1,
2, ... N and let,

$$
\mathbf{\Delta u}^T = (u_1^{T}, u_2^{T}, \ldots u_N^{T}) \mathbf{= u}^T(t) - \mathbf{u}^T(t_0)
$$

(4.1)
Note that $\ddot{\mathbf{z}} = \ddot{\mathbf{z}}(t)$ represents the total nodal displacement in the time interval $(0, t)$. Then suppose that the continuous velocity field $\mathbf{v}$ can be approximated by,

$$\mathbf{v} = A\dot{\ddot{\mathbf{z}}} = A\dot{\dot{\ddot{\mathbf{z}}}}$$

(4.2)

where the form of $\lambda$ will depend upon the particular element used and will, in general, depend upon its current position. It follows then that the vector $\dot{\dot{\mathbf{z}}}$ of velocity gradients and $\leq$ the strain rates may be related to $\dot{\dot{\ddot{\mathbf{z}}}}$ by the approximations,

$$\dot{\mathbf{d}} = B\dot{\dot{\mathbf{z}}}$$

(4.3a)

and

$$\leq = C\dot{\dot{\mathbf{z}}}$$

(4.3b)

where $B = (\partial \xi)A$, $C = 3A$

and $\partial$ and $\xi$ are as defined in Appendix 2A.

Substituting into equation (2A.9) it is found that for arbitrary variations $\dot{\dot{\mathbf{z}}}^T$ consistent with the velocity boundary conditions,

$$\dot{\dot{\mathbf{z}}}^T \left[ \int_C \left\{ \int_{t_0}^t B\dot{\mathbf{d}} \, dt \right\} \, d\mathbf{v} - h \right] = 0$$

(4.4)

and thus that

$$\int_C \left\{ \int_{t_0}^t B\dot{\mathbf{d}} \, dt \right\} \, d\mathbf{v} = h$$

(4.5a)

where

$$h = -\int_C \left\{ \dot{\mathbf{g}} + \int_{t_0}^t g \, dt \right\} \, d\mathbf{v} + \int_A^T P \, \dot{\mathbf{d}} + \int_A^T T \, d\mathbf{S}$$

(4.5b)
In many cases the conservation of mass can be used to simplify the integral containing the bodyforce, however, in some circumstances it is convenient to consider such forces as coming into being over a period of time. This may be used as a convenient numerical device when material is added to the body or may correspond to an actual physical loading such as in a centrifuge apparatus.

The basic finite element equation (4.5) will, in general, require a numerical solution. Loading path methods have been discussed in section 4.1 and the solution according to method (b), i.e. an incremental iterative procedure, is as follows. For any finite but small step in time $\Delta t$, we approximate all time dependent quantities by their average values for this particular step. Equation (4.5) then reduces to the approximate set of equations

$$\overline{K}\Delta \overline{u} = \overline{h} \quad (4.6)$$

where

$$\overline{K} = \int \frac{\overline{C}^{T}\overline{P}BdV}{V}$$

$$\overline{h} = -\int \frac{\overline{C}^{T}(\overline{Q} + \frac{\partial}{\partial \Delta t})dV + \int \frac{\overline{A}^{T}\overline{F}dV + \int \frac{\overline{A}^{T}\overline{T}ds}}{S_T}}$$

The superior bar denotes some average or representative value of the quantity for the current time step and the spatial integrations are carried out over the 'mean' configuration. In general, the set of incremental displacement equations, (4.6), will be non-symmetric. This lack of symmetry arises from two causes: the inclusion in the analysis of the effect of rotations; and because a non-associated flow rule is used to model soil behaviour. If the material had an associated flow rule and if the effect of rotations in any problem
were considered insignificant and neglected, then the matrix $\bar{K}$ would reduce to a symmetric form. Once convergence has been achieved in any one load step the cumulative quantities must be updated. These include the nodal co-ordinates which are adjusted to incorporate the increments of nodal displacement. Thus each new step begins with a new 'initial' geometry. The solution method described above may be termed a "variable stiffness" technique. If it proves more convenient, an initial stress technique may be employed (Zienkiewicz, Valliapan and King, 1969). Details of these methods are given in section 4.2.2.

4.2.1 Numerical Approximations

For any typical iteration in the overall solution procedure an equation set (4.6) must be solved. In establishing these equations a number of numerical approximations have to be made. For the simultaneous integration procedure described in section 4.1 these approximations concern: (a) the method of estimating the 'mean' quantities when performing the spatial integrations; (b) the method of determining the load vector terms $\bar{h}$. Both of these matters are now discussed.

(a) Spatial Integration Approximation

Quantities that have to be represented by some average value include all variables that are dependent upon the geometry e.g. the nodal co-ordinates, the element stresses and the domains, $\bar{v}$ and $\bar{s}_T$ of spatial integration. Thus when calculating the stiffness matrix $\bar{K}$ and the load vector $\bar{h}$ a number of possibilities were investigated. These are:
Type (I)

All average quantities may be approximated by their values at the beginning of the current load step. This in fact corresponds to a solution technique that only employs only one iteration per load (time) step, i.e. the sequential integration scheme described previously.

Type (II)

In this alternative integration scheme the average quantities are evaluated at a configuration that is the numerical mean or mid-point of the initial and final configuration within the current load step. This 'half-way' condition will of course change with each iteration until convergence is achieved.

Type (III)

In this scheme the average quantities are given values corresponding to the final configuration of the body at any load step. The final configuration must of course also be estimated and iteration will be necessary within each load step.

(b) Load Vector Approximation

When determining the load vector it matters not when the examination begins. Only a knowledge of the current configuration of the body and its current stress state are required. To be precise we require at any time a knowledge of \( g_0 \), \( T \), \( F \), \( S_T \) and \( V \). In the physical situation these quantities are determined as a result of testing. If we are part-way through a numerical computation then: the stress state \( g_0 \) has been calculated previously from the constitutive laws; the boundary tractions \( T \), representing the applied loading,
are always known; and the bodyforces $\tau$ can be obtained from conservation of mass* or from a knowledge of material added to the body. However, in such a numerical computation the use of a finite time step $\Delta t$ must be accompanied by the use of some representative geometry as mentioned above, i.e. $\bar{V}$ and $\bar{S}_T$. Two possible approximations for the 'load vector' $\dot{\tau}$ are suggested. These are:

**Type (1)**

This approximation is as given by equation (4.6),

$$\dot{\tau} \approx \dot{\bar{\tau}} = - \int_{\bar{V}} C^T (\tau_0 + g\Delta t) \, dV + \int_{\bar{V}} \bar{A}_T \bar{F} \, dV + \int_{\bar{S}_T} \bar{A}_T \bar{T} \, dS \tag{4.7}$$

Implicit in this method of calculation is the inclusion of rotation effects (via matrix $\bar{C}$) and hence the stiffness matrix $\bar{K}$ also involves rotation terms.

**Type (2)**

An approximation which is computationally simpler but in most cases less accurate is,

$$\dot{\tau} \approx \dot{\bar{\tau}} = \int_{\bar{V}} \bar{A}_T \Delta \bar{F} \, dV + \int_{\bar{S}_T} \bar{A}_T \Delta \bar{T} \, dS \tag{4.8}$$

where $\Delta \bar{T}$ and $\Delta \bar{F}$ given by,

$$\Delta \bar{T} = T(t) - T(t_0)$$

$$\Delta \bar{F} = F(t) - F(t_0) \tag{4.9}$$

which represents the change in traction and bodyforce that occurs during the time interval $\Delta t$. Consistent with this approximation is

*Except when analysing situations such as centrifuge loading where it may be obtained from the angular velocity.
the neglect of rotation terms and hence matrix \( \overline{K} \) is then given as,

\[
\overline{K} = \int_B \frac{\mathbf{B}^T \mathbf{D} \mathbf{B} dV}{V} \quad (4.10)
\]

It will be shown later that the approximation (2) is incapable of giving the correct solution for the analysis of a body which is highly stressed at time \( t = 0 \), when the current loading history commences, but may be adequate in other situations. This may partially account for the fact that in most of the recently published work on finite deformation problems, the authors conveniently examine only the finite deformation of an initially unstressed body.

Both load vector approximations (1) and (2) were used in combination with integration approximations (I), (II) & (III) and their various combinations are displayed in Table 4.1. The calculation schemes A - F were applied to some simple problems which are described in section 4.3.

<table>
<thead>
<tr>
<th>Load Vector Approx.</th>
<th>Spatial Integration Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(I)</td>
</tr>
<tr>
<td>(1)</td>
<td>A</td>
</tr>
<tr>
<td>(2)</td>
<td>D</td>
</tr>
</tbody>
</table>

Table 4.1: Code for Calculation Schemes

4.2.2 Equation Solution

We now examine the manner in which the system of equations, that governs the finite deformation behaviour, may be solved. There
are several alternative techniques.

4.2.2 (i) Variable Stiffness Method

For any iteration of the solution method, the increment of displacement $\Delta \delta$ is obtained from solution of equation (4.6). As noted above matrix $\bar{K}$ may be non-symmetric and as such may be expressed as the sum of symmetric and anti-symmetric parts, i.e.

$$\bar{K} = \bar{K}_S + \bar{K}_A$$

where

$$\bar{K}_S = \frac{1}{2}(\bar{K} + \bar{K}^T)$$

$$\bar{K}_A = \frac{1}{2}(\bar{K} - \bar{K}^T)$$

Hence, in iterative form (4.6) may be expressed as

$$\bar{K}_S \Delta \delta^{(i)} = \bar{n}^{(i)} - \bar{K}_A \Delta \delta^{(i-1)}$$

(4.11)

where the superscript $(i)$ corresponds to values at the $i$th iteration. Equation (4.11) applies not only to the total stiffness matrix but also at an elemental level. Thus when establishing the total stiffness matrix, $\bar{K}$, it proves more convenient to set up $\bar{K}_S$ and to form the contribution to the load vector $-\bar{K}_A \Delta \delta^{(i-1)}$ for each element. It is more economical to solve the equation set (4.11) than to solve the non-symmetrical set (4.6). The stiffness matrices are also highly banded if the nodes are carefully numbered, hence use may be made of an efficient equation solution algorithm for symmetric, banded equations. For the results presented in this thesis the Crout-Cholesky method was employed.
4.2.2 (ii) Initial Stress Method

For many problems it proves uneconomical to set up the stiffness matrix at every iteration in the solution procedure. In such cases use may be made of the initial stress technique (Zienkiewicz, Valliapan and King, 1969).

The stiffness matrix \( \overline{K} \) may be expressed as \( \overline{K} = K_E + \overline{K}_C \) where \( K_E \) is the stiffness matrix of the body at time \( t = 0 \) before it is deformed by the current loading, and \( \overline{K}_C \) represents the difference between \( \overline{K} \) and \( K_E \). Hence the governing equation set (4.6) may be written in iterative form as

\[
K_E \Delta \delta(i) = \bar{\gamma}(i) - \overline{K}_C \Delta \delta(i-1)
\]  

(4.12)

As in the variable stiffness method the vector \( \overline{K}_C \Delta \delta(i-1) \) may be formed at element level. The chief advantage of this technique is that the stiffness matrix \( K_E \) need only be established once and the Crout-Cholesky factorisation need only be performed once.

4.2.3 Negative Plastic Work Rate

In approximate solutions convergence is not the only criterion of satisfactory computation if parts of the body have become plastic. As mentioned in chapter 3, it is necessary to check that \( \lambda \), the multiplier which signifies that viscous effects are absent, is everywhere positive in order to ensure that the rate of plastic work is positive.
The occurrence of negative values for some elements indicates either numerical inaccuracy or that, in the exact solution, the stress state at the points represented by these elements returns from a plastic to an elastic state. In either case straightforward continuation of the solution is unjustified, since it is clear that the basic constitutive laws of the ideal material are being contravened. This limit to the numerical solution may be avoided by ensuring that no negative rate of plastic work occur in the converged solution. This can be done by incorporating in the iteration procedure of the numerical algorithm the possibility that any plastic element may become elastic as well as the usual possibility that an elastic element may become plastic (Davis and Booker, 1973; Davis, Ring and Booker, 1974).

4.2.4 Yield Surface Corrections

The incremental solution technique (b) described earlier allows for the solution of stress increments which must be added to the existing state of stress at time $t_0$ to obtain a current distribution of total stress at time $t = t_0 + \Delta t$. There exists the possibility in any element that the new total stress may violate the yield criterion, i.e. the new stress state may lie outside the yield surface, particularly as an element first passes from an elastic to a plastic condition. Such stress states are not permitted in the theory of perfect plasticity and it is usual in elastoplastic finite element analyses to correct the state of stress in such elements so that it will always lie on the yield surface.

There exists no unique method of correcting these offending
stress states back onto the yield surface. Several proposals have been suggested and some of these have been discussed by Christian and Marr (1972). At the present time there appears no real justification for preferring one method of correction to another (Ring, 1975).

The correction technique used in this work involves bringing the offending stress state back to the yield surface along a path normal to that surface, while at the same time holding constant the directions of principal stresses. Equations to determine this correction for the case of plane strain deformation are set out in Appendix 4B.

Of course, once the stress state has been adjusted to satisfy the yield criterion it may no longer be in equilibrium. Thus further iterations may be required in the solution procedure, so that both equilibrium and the yield criterion are adequately satisfied.

4.3 VERIFICATION OF THE SOLUTION METHOD

The verification of the numerical solution technique is the aim of this section. This checking procedure involves three important features.

(i) Numerical solutions are checked against exact solutions. Because of the non-linear nature of finite deformation problems it is no surprise that exact solutions are possible only in the very simplest of cases, e.g. problems involving homogeneous deformation.

(ii) In problems that involve more complex stress distributions,
closed form solutions are often obtainable only when one assumes that
gometry changes are not significant. Numerical finite deformation
olutions may adequately agree with these closed form solutions for
aterials which are very stiff, i.e. materials which have high deforma-
tion moduli. Such agreement offers another useful check on the validi-
ty of the numerical solution procedure.

(iii) Both numerical and rigorous finite strain solutions should
be examined to ascertain whether they predict any physically absurd
results. It should be noted that the possibility of such absurdities
is not an exclusive feature of the finite deformation theory. The
small strain theory is also capable of such predictions as, for example,
in the case of the surface loading of an elastic layer of infinite
lateral extent, where the upper boundary may pass through the lower
base at sufficiently large but finite values of the applied load.

The abovementioned features are now examined.

4.3.1 Homogeneous Extension or Compression - A Test of Numerical
Accuracy

The simple problem of the extension and compression of an
initially stress free, square section under conditions of plane strain
has been solved on several occasions. Fernandez and Christian (1971)
have presented a solution for an elastic material and Osias and
Swedlow (1974) have presented solutions for a bi-linear strain har-
dening material. Neither of these solutions allowed for the presence
of an initial, non-zero stress state in the square section. The exact
solution to such a problem for a material with a Mohr-Coulomb yield
criterion with cohesion $c$, friction angle $\phi$, dilatancy angle $\psi$ and obeying the rate law of equations (2A.8a, 3.15, 3.16) is presented in Appendix 4C. This problem is defined in Fig. 4.1.

It is the principal aim of this subsection to test the numerical algorithm described previously, by comparing its results with those of the exact solution method. It should be noted at this juncture, that it would not be necessary to employ a finite element discretisation to obtain a numerical solution to this simple problem. However, in view of our immediate objective the block is modelled in this case by a single, constant strain rate, quadrilateral element.

4.3.1 (i) Body Initially Unstressed

For the moment consider the case where the body is initially unstressed. Results for load as a function of the stretch ratio are given in Fig. 4.2 for associated ($\psi = \phi$) and non-associated ($\psi \neq \phi$) flow rules, as well as the results for a material with infinite strength ($E/c = 0$). The dilatancy rate is seen to have a significant effect on the load stretch response for those materials considered. The volume behaviours are depicted in Fig. 4.3. Note the elastic dilatation in the case $E/c = 0$, and the different volume responses for $\psi = \phi$ and $\psi \neq \phi$. As expected the response for $\psi = 0$ is one of constant volume once the block yields.

The values of $E/c$ used here are not intended to be representative of any real material but were chosen as a severe test of the numerical capabilities of the method. Note that even over a large range of distortions, e.g. stretch ratios in the range 0.1 to 30, the
numerical solution showed excellent agreement with the exact solution. (This may not necessarily be the case when non-homogeneous deformations are analysed using more than one element). The computation scheme used to obtain the above finite element results was that of method E (see Table 4.1).

This problem was also used to make an evaluation of the calculation schemes summarised by Table 4.1. Finite element computations were performed for the homogeneous extension and compression of an elastic material for which \( \nu = 0.3 \). The results for load stretch response are presented in Figs. 4.4a and b. A number of points are worth noting:

(a) for the block that is unstressed there is no difference between the finite element results obtained using load vector approximations (1) and (2), i.e. results are the same for schemes A and D, for B and E and for C and F.

(b) the spatial integration approximations (I), (II) and (III) each produce different results. Method (II) provides the best agreement with the exact solutions (i.e. schemes B and E).

4.3.1 (ii) Body Initially Stressed

Consider now the homogeneous deformation problem for the case when the body is initially in a state of non-zero stress (see Fig. 4.1). The general solution to this problem is presented in Appendix 4C. Some particular solutions for the exact load stretch response of an elastic material with \( \nu = 0.3 \) when \( P_o = 0 \) are plotted in Figs. 4.5a and b. Note that in both extension and compression the value of the initial stress \( Q_o/EI_o \) has a significant effect on the load deformation behaviour.
Some further explanation is necessary.

It must be remembered that for this problem, the quantity held constant is the total load \( Q_o \) in the \( y \) direction. This is not the same as the problem in which the normal traction \( \sigma_{yy} \) is held constant (in the latter case it would follow that the load-stretch response is not at all a function of the initial stress in the \( y \) direction). While \( Q_o \) is maintained constant \( \sigma_{yy} \) in fact varies as the deformation progresses.

Consider, for example, the extension of a material that was initially in a state of compression in the \( y \) direction and unstressed in the \( x \) direction. As the block is extended in the \( x \) direction the stress component \( \sigma_{yy} \) must decrease in a compressive sense in order that \( Q_o \) remains constant. This change in \( \sigma_{yy} \) is in fact a change in the tensile direction. The block thus undergoes a change in \( \sigma_{yy} \) and \( \sigma_{xx} \), both tensile. It follows from this argument that the load \( P \) required to extend the body will be greater for greater initial compressive forces \( Q_o \), a feature that is evident from Fig. 4.5a. This peculiar result may not have been intuitively obvious at first. It may be worth noting here then, that this phenomenon may not appear when real materials are deformed in this way. Most engineering materials would yield at values of initial stress well below that required to produce this separation phenomenon.

Numerical agreement with the exact solution is shown in Figs. 4.6a and b for the case \( Q_o/E \theta_o = -1 \), \( P_o = 0 \). Satisfactory numerical results were obtained using calculation scheme (B), i.e.
load vector approximation (1). Note, however, that these numerical results could not have been obtained using the load vector approximation (2) which would produce only one response, that for \( Q_o = 0 \), independent of the magnitude of \( Q_o \).

4.3.1 (iii) Homogeneous Extension with Rotation - the Jaumann Stress Rate

Until now all the test problems presented have involved no rotation. In all these cases the Jaumann stress rate \( \dot{\sigma} \) reduces to the more simple material derivative \( \dot{\sigma} \). In problems that involve rotations of elements of the deforming body the quantity \( \dot{\sigma} \) does, theoretically, give an objective measure of the stress rate. It is, however, important to test if this is so for the numerical procedure presented earlier.

A problem designed to test the numerical reliability of \( \dot{\sigma} \) is defined in the inset to Fig. 4.7a. An initially square, and unstressed block of elastic material, for which \( \nu = 0 \), is simultaneously stretched along one of its axes (that which at \( t = 0 \) was parallel to the x coordinate direction) to a stretch ratio of 8.5 and rigidly rotated through an angle of \( \pi/2 \) radians. Numerically this was modelled in thirty equal steps of extension \( \Delta \rho = 0.25 \) and simultaneous rotation \( \Delta \theta = \pi/60 \). The load stretch relationship for such a deformation is plotted in Fig. 4.7a. It can be seen that the computation schemes (A), (B), (C) all compare favourably with the exact solution. For the specified step size used here the best comparison was that of scheme (C). Fig. 4.7b shows a comparison of the numerical and exact stress components \( \sigma_{xx}, \sigma_{yy} \) and \( \sigma_{xy} \) during this deformation. No doubt greater accu-
racy could be achieved using a large number of smaller deformation increments. It should also be noted that the use of \( \dot{\sigma} \) as a stress rate instead of \( \hat{\sigma} \) will lead to gross errors in the numerical solution of this problem.

Similar results to these are reported by Osias and Swedlow (1974) which together with the present investigation support the use of the Jaumann stress rate in numerical computation.

4.3.1 (iv) Importance of the Constitutive Law

In section 2.2.3 the matter of the definition of a suitable objective stress rate for use in constitutive laws was discussed in some detail. The importance of paying close attention to the accepted definition of stress rate is demonstrated here by the use of a simple example. This example has previously been reported by the author (Carter, 1975).

Described in the inset to Fig. 4.8 is an initially square, initially unstressed section of elastic material which is extended under plane strain conditions. Each of the two curves of Fig. 4.8 corresponds to the adoption of a different constitutive law, each relating different stress rates to the same strain rate through identical material constants. These two laws are:

(i) The general rate law given in equation (2A.8). In this example involving no rotation equation (2A.8) reduces to

\[
\dot{\sigma} = D \dot{\varepsilon}
\]

(4.13)

where \( \dot{\varepsilon} \) is defined in Appendix 2A.
(ii) The second constitutive law is expressible as

\[ \dot{\sigma} = H \dot{d} \]  

(4.14)

where both $H$ and $d$ are defined in Appendix 2A. The stress rate $\dot{\sigma}$ was suggested by Biot (1965, pp. 63-67) and is related to $\dot{\sigma}$ by

\[ \dot{\sigma} = \dot{\sigma} + S d \]  

(4.15)

where

\[
S = \begin{bmatrix}
0 & \sigma_{xx} & -\sigma_{xy} & 0 \\
\sigma_{yy} & 0 & -\sigma_{xy} & 0 \\
\lambda \sigma_{xy} & \lambda \sigma_{xy} & -(\sigma_{xx} + \sigma_{yy}) & 0 \\
\end{bmatrix}
\]

Equation (4.15) may be expressed in the form

\[ \dot{\sigma} = (H - S) \dot{d} \]  

(4.16a)

and for this case involving zero rotation we may write

\[ \dot{\sigma} = (D - R) \dot{\ell} \]  

(4.16b)

where $R$ is matrix $S$ with the last column deleted.

Equations (4.15, 4.16) exemplify the point made in section 2.2.3 that the difference between acceptable definitions of stress rate consist merely of a linear combination of strain rates.

The two curves of Fig. 4.8 were computed (both exactly and by the numerical technique) on the assumption that the matrix $D$ (i.e. Young's modulus $E$, and Poisson's ratio $\nu$) was constant for all states of stress. The two curves are markedly different at larger values of stretch ratio. Of course curve (i) could have been obtained using the
stress rate \( \dot{\sigma} \) but then \( D \) would have to depend on the stress state in the correct way and similarly curve (ii) could have been obtained using \( \dot{\sigma} \).

For most engineering materials, including soil, yielding occurs at sufficiently small strain for distinction between alternative stress rates to be of little consequence in the elastic range. Finite deformation behaviour of massive configurations of such materials will be dominated by finite plastic strains.

To avoid confusion it will be restated that, except for this current example, the general rate law of equation (2A.8) is used to obtain all the numerical results presented in this thesis.

4.3.1 (v) Rigid Plastic Behaviour

It is true that, in the previous test examples, material properties have been chosen so that finite deformation behaviour has been governed largely by finite elastic strain rather than finite plastic strain. In any validation of the solution technique it is important to check as many possibilities as one is able.

Thus we come now to consider the homogeneous finite deformation of an initially square block of material which is rigid until the onset of plastic yielding. In this simple problem of homogeneous stress state, yielding will occur in a purely cohesive material when the applied traction has a value of \( 2c \) (both tensile or compressive). For definiteness we confine our attention to the compression problem, whence first yield occurs at \( \sigma_{xx} = -2c \). All other stress components are of course zero.
If we further assume that the material is incompressible then it is a simple matter to show that the load-stretch relationship is given by the equation,

$$\frac{P}{2\sigma_o} = -\frac{1}{\rho}$$  \hspace{1cm} (4.17)

where the symbols have their usual meaning.

This relation is plotted as the broken line in Fig. 4.9. Also shown are the load-stretch relations for a number of incompressible materials which are deformable in the elastic range. Each of these curves eventually joins, and from then on is coincident with, that curve which corresponds to the rigid plastic. The intersection point in each case represents first yield within the elastoplastic material. The position of the intersection is of course a function of the stiffness of the material. The more stiff is the material, i.e. the greater is the E/c value, then the sooner (in terms of stretch ratio) the elastoplastic behaviour tends to that of a rigid plastic. Also shown in Fig. 4.9 are finite element solutions corresponding to several values of E/c. These were obtained numerically using $\nu = 0.48$ to approximate elastic incompressibility, and show good agreement with the exact solutions for $\nu = 0.5$.

4.3.2 The Cylinder Problem - Approaching the Limits Of Infinitesimal Theory

In the examples presented previously in this chapter attention has been focused mainly on the numerical accuracy obtainable with the approximate solution technique. Numerical results have compared
favourably with exact solutions. Little attention was paid to the relation between deformation and strength parameters adopted and those of real materials. Although the previous example demonstrated that rigid plastic solutions may sometimes be approximated using large $E/c$ values, the notion that small strain theory is the special limit of large strain theory was not pursued in depth. Some attention is now given to this important feature.

We consider the expansion of an initially thick-walled cylinder due to an internal pressure under conditions of plane strain. The cylinder is composed of an elastoplastic Tresca (1864) material with an associated flow rule. Before proceeding with the solution a brief historical summary is relevant.

Much attention has been paid to the problem of the expansion of a cylinder due to internal pressure for both rigid plastic and elastoplastic materials. For a survey of this work reference may be made to a text by Hill (1950) and to papers by Hodge and White (1950), Allen and Sopwith (1951), Beliaev and Sinitzky (1938), Steele (1952) and Palmer (1972). Koiter (1953) gives a lucid description of the basic assumptions and results of most of these works as well as presenting an infinitesimal strain solution for a Tresca material with an associated flow rule.

For a cylinder expanding under conditions of plane strain, the problem of unrestricted plastic flow has been dealt with by Prager and Hodge (1951), using the Von Mises theory (1913), and by Hill, Lee and Tupper (1947) using a treatment based on the Prandtl-Reuss theory (Prandtl, 1924; Reuss, 1930). The former approach assumes that the
material is incompressible and neglects elastic strains in the plastic region while the latter includes an account of the elastic component of strain but assumes that changes in geometry are small.

In this section the restrictive assumptions of the previous two works are abandoned. The present analysis applies to materials possessing deformation moduli of any magnitude, no restriction is placed on the elastic component of strain or the magnitude of the overall deformation. This simple analysis is presented in Appendix 4D and is the same as that previously presented by the author (Carter, 1976). The problem is defined in Fig. 4.10.

The numerical technique already outlined in general was applied to the expansion of a cylinder whose finite element idealisation is shown in Fig. 4.11. The mesh is shown at the stress free configuration at time $t = 0$ when $b_I/a_I = 2$.

In addition there are two closed form solutions to this problem that are, strictly speaking, applicable only under certain circumstances. These are:

(a) A solution reported by Hill (1950) and Koiter (1953) and strictly applicable only when $G/c = \infty$. This is the conventional small strain solution that does have practical application for materials of high deformation modulus. According to this theory the maximum internal pressure, $p_H$, which is determined by complete yield within the cylinder is given by,

$$p_H/c = 2\ln(b_I/a_I) \quad (4.18a)$$
(b) A solution for materials of infinite strength (i.e. \( G/c = 0 \)) is presented in the Appendix 4E and the current value of the outer radius \( b \) is plotted against the internal pressure \( p \) in Fig. 4.12 for the case when Poisson's ratio \( \nu = 0.3 \). For \( G/c = 0 \) the maximum pressure \( p_c \) which corresponds to elastic instability is given by,

\[
p_c/G = 2\ln(b/I/a_I)
\]

(4.18b)

We thus note the interesting result that,

\[
p_c/p_H = G/c
\]

(4.19)

A solution for the material with \( G/c = 0 \) was obtained using the numerical technique described above and numerical displacement results are compared with the rigorous solution in Fig. 4.12. The elastic dilatation that is a consequence of adopting the material rate law of equation (4D.4) is evident from Fig. 4.13 where the volume is plotted as a function of the outer radius \( b \). \( (V_I, V \) are the cross-sectional areas at times \( 0, t \) respectively).

Neither of the closed form solutions (a) and (b) are strictly applicable to real materials as these possess finite values of \( G/c \). However, either may provide adequate predictions of the behaviour of cylinders made from many materials in engineering use.

An examination of the behaviour of materials with a variety of \( G/c \) values has been made using the numerical technique described above. The relationship between the outer radius and the internal pressure for cylinders with \( \nu = 0.3 \) is plotted in Fig. 4.14. Note that for stiffer materials, e.g. \( G/c = 38.5 \), the closed form solution (a)
gives a very good approximation to the expansion behaviour. The stresses obtained for the case \( G/c = 38.5 \) (i.e. \( E/c = 100 \)) are shown in Fig. 4.15 to be in close agreement with those of the closed form solution. For less stiff materials the elastoplastic behaviour becomes more like that predicted by the analytical method (b). The curve for \( G/c = 0 \) has no unique position on Fig. 4.14 because the ordinate is plotted as \( p/c \), but it is noted that for values of \( G/c \) less than about 0.385 the elastoplastic behaviour is almost identical with that predicted by method (b); at least for the range of displacements plotted. This may not continue to be the case when the cylinder expands further to become one with thin walls.

The relationship between the maximum pressure \( p_{\text{max}} \) and the material properties is given in Fig. 4.16. As \( G/c \) is increased \( p_{\text{max}} \) approaches that predicted by method (a), while for lower values of \( G/c \) the limiting value is that of \( p_c \), the elastic instability pressure.

The numerical results presented in this section were obtained using linear finite elements. The results were in fact duplicated using plane strain quadrilateral elements as shown in Fig. 4.17 thus providing further verification of the quadrilateral finite element computer program. The mesh geometries at various internal pressures are presented in Fig. 4.18 for the case when \( G/c = 0 \). Even when the elements become highly elongated, e.g. at \( p/G = 1.3 \), the accuracy is still maintained.

In summary then, this problem has served as a useful check on the limiting solutions from finite strain theory. For materials with high deformation moduli, the predictions from small strain theory
and the numerical finite deformation analysis adequately agree.

4.3.3 The Deposition Problem - Some Absurd Extensions

We now examine a curious problem. A very interesting result is obtained when finite deformation theory is carried to its logical conclusion. The problem is defined in Fig. 4.19 and it concerns the deposition of a material under one-dimensional conditions. One could imagine it physically as the filling of a hole with smooth sides, or an infinitely long trench with smooth walls. The material to be deposited may be either elastic or an elastoplastic material which has a Young's modulus $E$ and Poisson's ratio $\nu$ which remain constant, a cohesion $c$, a friction angle $\phi$ and dilation determined by the angle $\psi$. Before deposition the material has a uniform density $\gamma$.

A solution for the height of fill material $h$ as a function of the weight of fill $W$, is given in Appendix 4F. In both the elastic and elastoplastic case note that there exists a critical height $h_c$ given by equations (4F.7, 21) above which the fill material cannot rise. A situation is reached at which the addition of further material causes a settlement in the underlying fill equivalent to the volume occupied by the additional material. Finite deformation theory thus predicts the creation of an infinitely dense material in this case. This is of course a physical absurdity. Hence if real materials do deform according to the rate law of equation (2A.8) then some modification is necessary. For example Young's modulus, rather than remaining constant throughout the entire deformation history, could perhaps be stress dependent.

To return to the problem at hand we shall demonstrate again
the accuracy obtainable with the numerical technique, albeit with physically absurd consequences. Consider the deposition of a frictional material. Fig. 4.20 shows the relationship between the height of fill and the weight \( W \) (per unit plan area). Note that material with a positive dilation rate reaches a greater height than that which flows plastically at constant volume. This result is not unexpected. Note also the close agreement between exact and numerical solutions.

Fig. 4.21 shows the growth of the fill with the addition of further material. At each stage in computation a pair of pictures is shown. The first shows the additional quadrilateral finite element (shaded) before the gravity field is applied to it. The second shows the equilibrium position of the entire fill in the presence of the gravity field. At larger fill weights the vertical increments in height of each element reflect inversely the variation in material density throughout the fill. This variation in density is shown again in Fig. 4.22 where the vertical stress distribution \( \sigma_{yy} \) is shown at various stages in the deposition process. Note again the good numerical agreement.

This problem was not only itself of curious interest, it served as a preliminary check on the algorithm used in modelling the building of granular embankments on soft clay layers. The latter problem is discussed in the next chapter.

4.3.4 Squashing Between Rough Rigid Plates — A More General Test

All of the examples presented so far in this chapter have had
at least one common feature. Exact solutions have been easy to discover because either (a) the problem was one-dimensional or (b) the deformation was homogeneous.

The real use of the numerical technique described in the preceding pages is, of course, in solving more difficult problems where exact solutions are not possible. Eventually the question must be posed, "how accurate are the numerical solutions to these problems?"

Although no exact solutions to situations of a more general nature have been found for elastoplastic materials, there are some solutions available involving finite changes in geometry of a rigid plastic material (for examples see Hill, 1950). It may be possible then to test the numerical solutions, in the limit as E/c becomes large, for something other than a simple well defined case.

Problems such as the pseudo-steady flow examples described by Hill (1950) are sufficiently general, however, they pose major computational difficulties. These difficulties concern the imposition of boundary conditions. It was thus necessary to choose a problem of a different kind. One such problem is that concerning the squashing between rough rigid plates of a rigid perfectly plastic material that obeys a Tresca yield criterion - see inset to Fig. 4.23a. The exact solution for this problem was first presented by Hill, Lee and Tupper (circa 1945) and is described in Hill's text (1950, p. 226).

Since the idealised material is rigid when stressed below the yield limit, the plates cannot move together so long as a remaining non-plastic strip of finite width spans the block. Hill notes
that for compression to be possible, not only is it necessary for the plastic zones to be fused but it is also necessary that "the two slip lines through the centre should be entirely contained within the plastic zone up to their junction with the plates" (Hill, 1950, p. 227). The moment when the two plates first approach each other is referred to as the "yield-point" of the block as a whole, and the corresponding load on the plates is known as the yield point load. Hill has presented a solution for the actual pressure distribution q as well as the average pressure $q_{AV}$ over the plates at the yield point. The latter is given very nearly by

$$\frac{q_{AV}}{2c} = \frac{3}{4} + \frac{w}{4h} \left( \frac{w}{h} \geq 1 \right)$$  \hspace{1cm} (4.20)

where $2w$ is the width of the block, and $2h$ is the height. The actual contact pressure distribution q is plotted in Fig. 4.23a for a block of initial dimension, $w/h = 4.5$.

The numerical technique was used to analyse this problem for the case of a block composed of elastoplastic Tresca material. Before any load is applied the block is assumed to overhang the plates at either end by an amount equal to 11% of the initial half width - see finite element mesh Fig. 4.25. The calculated contact pressure distribution at "yield-point" (achieved in this numerical case when plastic yielding has spread from the edge of the plate to the centre) is also shown in Fig. 4.23a for the case of a material with $E/c = 100$. Notice that due to elastic compression the value of $w/h$ has increased from 4.5 at zero applied load to a value of 4.52 at the calculated "yield-point". At $w/h = 4.52$ the numerical solution, $q_{AV} \approx 3.8c$ and that of Hill, $q_{AV} = 3.76c$ are in good agreement. A sharp increase in contact
stress near the edge of the plate is obvious and this reflects the presence there of a stress singularity. Nevertheless, this stress distribution is in general agreement with that of Hill.

Hill (1950, p. 230) contends that this solution "is valid no matter what the amount of compression, since (for a rigid plastic) the overhang remains rigid. As the compression continues we have to deal, in effect, with a series of blocks of increasing width/height ratio". The contact stress distribution obtained numerically for a material with $E/c = 100$ has been compared with those of Hill after a finite compression, i.e. at $w/h = 5.102$, see Fig. 4.23b. It can be seen that in contrast to the "first-yield" stresses the agreement is no longer as good. The stress distribution near the edge again reflects the stress singularity but near the centre of the plate it is no longer satisfactory, considerable oscillations are obvious. Generally these stresses are too large and this is reflected in a plot of average contact pressure versus the $w/h$ ratio, see Fig. 4.24.

Hill's solution for the rigid plastic ($E/c = \infty$), i.e. equation (4.20), is plotted as a straight line. The numerical solutions for a variety of $E/c$ values from 1 to 100 are also plotted. Observe that the curve for the case we were considering above, $E/c = 100$, eventually lies above the Hill solution. It seems that once the "yield-point" has been reached the numerical solutions for higher $E/c$ may be in error. It seems reasonable to assume that as in the smooth plate problem (section 4.3.1(v)) all the curves for finite $E/c$ should be bounded above or else eventually merge with the solution of Hill for $E/c = \infty$. If this is so then the implication is that numerical solutions become inaccurate once the compression is dominated by plastic strain. There are several factors that may at least partially account for this post
"yield-point" discrepancy.

(a) Mesh Effect. Apart from the perhaps obvious point that more elements may be necessary, (particularly near the centre of the block) for greater accuracy, there are several problems that arise in respect of this or any finite element network used to discretise the block. Probably most important of these is the inability of a regular finite element mesh to model the stress singularity which occurs at the edge of the rough plate. In addition Hill's solution for the rigid plastic assumes that slip will occur along the rough interface between the block material and the plates. This is possible only where the shear stress is equal to the shear strength. In numerical calculations this slip had to be modelled by intense shearing of actual elements in contact with the plate. Better results may have been obtained if some measure such as a joint element was employed. (Of course in a finite deformation analysis there would be the problem of how to account for the case when a node originally in contact with the plate slipped and became part of the overhang). The intense shearing that was a feature of the present method of analysis is demonstrated vividly in Fig. 4.25 for the case when \( E/c = 10 \). Note the extreme distortion of elements in contact with the plate. It is difficult to know just what effect this very large change in shape of the elements has on the solution. In fact this feature highlights one of the very real difficulties involved in obtaining finite deformation solutions to most problems in general. If a Lagrangian mesh is employed, i.e. one that is embedded in the material and hence follows the deformation, its suitability as a model for the deforming material decreases as the deformation magnitude increases. We are thus faced with a dilemma. In order to achieve "finite strains" we often must
allow the elements themselves to become severely deformed whence their suitability for complicated problems is in question. This problem and some possible solutions are discussed in the conclusions to this chapter.

(b) The Rigid Overhang. Hill's solution is valid for the rigid plastic no matter what reduction in thickness occurs. It is implicit in his solution that the overhang material outside the plates is unstressed and therefore remains rigid. For the more realistic situation of an elastoplastic material the overhang could clearly not remain completely unstressed and rigid and the cases examined by the numerical method show that this is so, see the distortion of the overhang material in Fig. 4.25.

It may well be pointed out here that the numerical difficulty described above is a function of the complexity of the problem. Recall that good agreement was obtained for the problem of squashing between smooth rigid plates, see Fig. 4.9, indeed only one element was required for a numerical solution. This good agreement was obvious even when the deformation was dominated by large plastic strains. However, once we introduce the extra, restrictive boundary condition such as plate roughness a numerically accurate solution becomes far more difficult to achieve. This finding is in agreement with those of Nagtegaal, Parks and Rice (1974).

4.4 CONCLUSIONS

Several conclusions may be drawn from the material presented in this chapter.
Efficient numerical solution capability has been developed for problems of two-dimensional deformation under conditions of plane strain. The validity of the numerical analysis has been evaluated by considering a variety of elastic and elastoplastic finite deformation problems for which exact solutions are available. It is concluded generally that the best methods of numerical computation are those described as schemes B and E of Table 4.1. (Scheme C was in fact slightly superior in one problem, with B and E also proving adequate). Schemes B and E were thus used to obtain the results presented in the remainder of this work.

Unlike E, the scheme B is capable of making more than just a simple account for the presence of initial stresses at \( t = 0 \) (when the current loading commences) in the finite deformation analysis. Instead of the coupled nature of the initial stress effect the best that can be managed with scheme E is a simple superposition. The scheme B is also capable of including the effects due to finite rotations. Notwithstanding the above, scheme E may be practically useful (it will involve less computer time) when initial stress effects and rotations are insignificant.

It is further concluded that generally the finite deformation formulation leads to results that are in agreement with classical solutions of the small strain theory for sufficiently stiff materials. There are some difficulties, however, and these are usually associated with some restrictive type boundary condition (see section 4.3.4 and Nagtegaal et al. 1974).

The numerical formulation is also capable of predicting some
physically absurd results when carried to its logical extension; a feature that is not exclusive to finite deformation theory. Many such instances exist in the classical theory.

Further, it is emphasised that care should be exercised when interpreting solutions, particularly those to complicated problems involving many elements which have become highly distorted. There are limits to the complexities, particularly those of stress distribution, which a finite element mesh may model. With regard to the dilemma mentioned previously, where elements must themselves become highly distorted in order to reach "large strains" and thus reduce numerical accuracy, a suggestion for further work may be in order. If a Lagrangian mesh is employed it may perhaps be advisable to 'update' the mesh at convenient intervals during the loading. When the deformation becomes sufficiently large and elements become severely distorted, the stress distribution should be noted, and the 'distorted' mesh replaced by a new mesh which fits the boundary of the body and whose elements possess aspect ratios that are likely to provide more accurate numerical results. The analysis would then continue for the body in the given stress state. Such a procedure would of course involve considerably more computer time than the usual Lagrangian mesh procedure adopted throughout this thesis.
Fig 4.1 HOMOGENEOUS DEFORMATION
Fig 4.2a HOMOGENEOUS EXTENSION
Fig 4.2b  HOMOGENEOUS COMPRESSION
Fig 4.3a HOMOGENEOUS EXTENSION - VOLUME BEHAVIOUR
Fig 4.3b HOMOGENEOUS COMPRESSION
VOLUME BEHAVIOUR

Key same as Fig 4.2b
Fig 4.4a HOMOGENEOUS EXTENSION
COMPARISON OF CALCULATION SCHEMES.
$P_0 = Q_0 = 0$
$\gamma = 0.3$
$E/c = 0$

Fig 4.4b HOMOGENEOUS COMPRESSION
COMPARISON OF CALCULATION SCHEMES
Fig 4.5a HOMOGENEOUS EXTENSION
INITIAL STRESS EFFECT.
Fig 4.5b HOMOGENEOUS COMPRESSION
INITIAL STRESS EFFECT.
Fig 4.6a HOMOGENEOUS EXTENSION

COMPARISON BETWEEN NUMERICAL & ANALYTICAL RESULTS
Fig 4.6b HOMOGENEOUS COMPRESSION PROBLEM
COMPARISON BETWEEN ANALYTICAL & NUMERICAL RESULTS.

Analytic

- Scheme B (Every fifth point shown)

\[ \frac{Q_o}{E \varepsilon_o} = -1 \]

\[ P_o = 0 \]

\[ V = 0.3 \]
\( \gamma = 0 \quad E/c = 0 \)

- **Analytic**
- **Scheme A**
- **Scheme B**
- **Scheme C**

**Fig. 4.7a** HOMOGENEOUS EXTENSION WITH SIMULTANEOUS ROTATION
Fig 4.7b HOMOGENEOUS EXTENSION WITH ROTATION
Fig 4.8 HOMOGENEOUS EXTENSION - ELASTIC MATERIAL
Fig 4.9 SQUASHING BETWEEN SMOOTH RIGID PLATES.
FIGURE 4.10 CYLINDER EXPANSION

(a) time = 0

(b) time = t₀

(c) time = t
FIG. 4.11 FINITE ELEMENT MESH AT $t = 0$
\[ \frac{p_c}{G} = 1.386 \]

- **Small Strain**
- **Finite Deformation - Exact**
- **Finite Deformation - Numerical**

\[ \frac{G}{c} = 0 \]
\[ v = 0.3 \]

**FIG. 4.12 ELASTIC CYLINDER EXPANSION**
\( G/c = 0 \)
\( v = 0.3 \)

\[ \frac{v}{v_i} \]

\[ \frac{b}{b_i} \]

**FIG. 4.13** ELASTIC CYLINDER - VOLUME BEHAVIOUR
Fig. 4.14 Elasto-Plastic Cylinder Expansion
Fig. 4.15 Comparison of Stress Distributions

- Small Strain - Exact
- Numerical, G/c = 38.5

ρ = Radius of Elasto - Plastic Interface
FIG. 4.16 MAXIMUM PRESSURE V MATERIAL PROPERTIES
Fig 4.17  CYLINDER EXPANSION
PROBLEM DEFINITION
Fig 4.18 ELASTIC CYLINDER EXPANSION

MESH GEOMETRY AS A FUNCTION OF INTERNAL PRESSURE
Fig 4.19 ONE DIMENSIONAL DEPOSITION
Fig 4.20 RELATIONSHIP BETWEEN DEPTH OF FILL AND WEIGHT OF DEPOSITED MATERIAL.
\[
\frac{\xi}{E} = 0.25 \quad 0.5 \quad 0.75 \quad 1.0 \quad 1.25 \quad 1.5 \quad 2.0 \quad 2.5 \quad 3.0
\]

\(c = 0\) \quad \phi = \psi = 30^\circ \quad \nu = 0

Fig 4.21 GROWTH OF FILL - FINITE ELEMENT PICTURE
$c = 0 \quad \psi = \phi = 30^\circ \quad v = 0$

- **Exact Finite Strain**
- **Finite Element – Finite Strain**
- **Small Strain – Linear Solution**

\[ \sigma_{yy} = -\gamma(h - y) \]

\[ \gamma_y/E \]

\[ \sigma_y/E \]

$W/E = 3.0$

$W/E = 1.5$

$W/E = 0.5$

Fig 4.22 VARIATION OF VERTICAL STRESS WITH DEPTH
Fig 4.23a SQUASHING BETWEEN ROUGH RIGID PLATES.

CONTACT STRESSES AT 'YIELD POINT.'
Fig 4.23b SQUASHING BETWEEN ROUGH RIGID PLATES.
CONTACT STRESSES AT $w/h = 5.1$. 

Hill

Prandtl

Numerical for $w/h = 5.102$
($E/c = 100$)
Fig 4.24 SQUASHING BETWEEN ROUGH RIGID PLATES.
AVERAGE PRESSURE v PLATE SEPARATION.
Fig 4.25 SQUASHING BETWEEN ROUGH RIGID PLATES.

\[ \frac{w}{h} = 6.43 \quad \frac{q_{av}}{2c} = 2.20 \]

\[-E/c = 10.\]
APPENDIX 4A

PLANE STRAIN PROBLEMS

One of the most common finite elements used in infinitesimal analyses is the constant strain triangle. For the rate analysis described earlier we adopt a constant strain rate triangle as a basic element - see Fig. 4A.1. The nodes i, j, k are numbered in an anti-clockwise direction with each having two components of velocity, e.g.

\[ \dot{\delta}_i = (v_x, v_y)^T \]  \hspace{1cm} (4A.1)

and the six components of the element velocities are listed as

\[ \dot{\delta}_e = (\dot{\delta}_i, \dot{\delta}_j, \dot{\delta}_k)^T \]  \hspace{1cm} (4A.2)

Assuming that the velocity gradients are constant within this triangle allows us to write,

\[ v_x = \alpha_1 + \alpha_2 x + \alpha_3 y \]  \hspace{1cm} (4A.3)

\[ v_y = \beta_1 + \beta_2 x + \beta_3 y \]

The instantaneous rate of deformation may be described by the vector of elemental velocity gradients

\[ \dot{d}_e = \begin{pmatrix} \eta_e^T \\ \omega_e^T \end{pmatrix} \]  \hspace{1cm} (4A.4)

where

\[ \eta_e = \begin{pmatrix} \frac{\partial v_x}{\partial x} \\ \frac{\partial v_y}{\partial y} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \end{pmatrix} \]
\[ \omega^e = \left( \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \right) \]

It follows that the vectors \( \dot{\omega}^e \) and \( \dot{\lambda}^e \) may be related to \( \dot{\gamma}^e \) by

\[ d^e = B^e \dot{\gamma}^e \]  \hspace{1cm} (4A.5a)

and

\[ \lambda^e = C^e \dot{\gamma}^e \]  \hspace{1cm} (4A.5b)

The assumption of equation (4A.3) leads to expressions for \( B^e \) and \( C^e \) as follows,

\[ C^e = \frac{1}{2\Delta} \begin{bmatrix}
  b_1 & 0 & b_j & 0 & b_k & 0 \\
  0 & c_i & 0 & c_j & 0 & c_k \\
  c_i & b_1 & c_j & b_j & c_k & b_k \\
\end{bmatrix} \]  \hspace{1cm} (4A.6a)

\[ B^e = \frac{1}{2\Delta} \begin{bmatrix}
  2\Delta C^e (3x6) \\
  c_i & -b_i & c_j & -b_j & c_k & -b_k \\
\end{bmatrix} \]  \hspace{1cm} (4A.6b)

with

\[ b_i = y_j - y_k \] etc.

\[ c_i = x_k - x_j \] etc.

\[ 2\Delta = \det \begin{vmatrix}
  1 & x_i & y_i \\
  1 & x_j & y_j \\
  1 & x_k & y_k \\
\end{vmatrix} \]

We note here that the assumption of equations (4A.3) is not identical to assuming a linear distribution of displacement within an element, i.e. if we write
\[ \eta^e = B^e \Delta \delta^e \]
\[ \xi^e = C^e \Delta \delta^e \]

where
\[ \eta^e = (\xi^e, \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x})^T \]
\[ \xi^e = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right)^T \]

are the displacement gradients (strains), then matrix \( B^e \) is not identical to \( B^e \) nor \( C^e \) to \( C^e \). In practice, if we adopt small load steps, then to sufficient accuracy we may use the following approximations

\[ B^e \approx B^e \]
\[ C^e \approx C^e \]

(4A.8)

For the plane strain case the matrix \( P \) used in the rate law, equation (2A.8), is given by

\[ P = \begin{bmatrix} -\sigma_{xy} \\ D(3x3) \sigma_{xy} \\ \frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \end{bmatrix} \]

(4A.9)

where \( \sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})^T \) are the Cartesian stress components, tension positive.

For an elastic material or an elastoplastic material obeying a Mohr-Coulomb yield criterion and a non-associated flow rule the matrix \( D \) is given in section 3.3.
In much of the numerical work presented in this thesis plane strain quadrilateral elements were used in computation. These were formed from four plane strain triangles by the nodal condensation of the common node located at the centroid of the quadrilateral (Zienkiewicz, 1971, p. 113) - see Fig. 4A.2. In such cases the representative stress was formed from the area weighted average of the stresses of the four component triangles. The nodal displacement of the common (interior) node was assumed to be the arithmetic average of the displacements of the four exterior nodes.
Fig 4A.1 CONSTANT STRAIN RATE TRIANGLE.
\[ x_5 = \frac{1}{4} (x_1 + x_2 + x_3 + x_4) \]
\[ y_5 = \frac{1}{4} (y_1 + y_2 + y_3 + y_4) \]

Fig 4A-2 CONSTANT STRAIN RATE QUADRILATERAL.
APPENDIX 4B

YIELD SURFACE CORRECTION

This appendix presents the equations which govern the correction of invalid stress states to the yield surface. The yield criterion is that for a Mohr-Coulomb material, which for plane strain conditions may be expressed as

\[ \sigma_3 = N_{\phi} \sigma_1 - 2c\sqrt{N_{\phi}} \]  \hspace{1cm} (4B.1)

where

\[ N_{\phi} = \frac{1 + \sin \phi}{1 - \sin \phi} \]

\( c, \phi \) are the cohesion and angle of friction of the material, and

\( \sigma_1, \sigma_3 \) are the major and minor principal stresses respectively within any element (tension positive).

This yield surface is shown pictorially in Fig. 4B.1. All stress states within the shaded region are elastic. All stress states on the failure surface are of course plastic. Stress states outside the region, such as \( P(\sigma_1^*, \sigma_3^*) \) are invalid and are to be corrected normally to the yield surface \( \sigma_3 = N_{\phi} \sigma_1 - 2c\sqrt{N_{\phi}} \). We therefore correct normal to this line, i.e. reduce \( P(\sigma_1^*, \sigma_3^*) \) to \( P'(\sigma_1', \sigma_3') \). The outward normal to this yield line is given by the vector \( (1, -N_{\phi}) \). Hence

\[ \sigma_1' = \sigma_1^* - N_{\phi}k \]  \hspace{1cm} (4B.2a)

\[ \sigma_3' = \sigma_3^* + k \]  \hspace{1cm} (4B.2b)

\( (\sigma_1', \sigma_3') \) must obey equation (4B.1), i.e.
\[ (\sigma_3^* + k) = N_\phi (\sigma_1^* - N_\phi k) - 2c'N_\phi \]  

(4B.3)

and hence

\[ k = \frac{N_\phi \sigma_1^* - 2c'N_\phi - \sigma_3^*}{1 + N_\phi^2} \]  

(4B.4)

Thus the corrected principal stresses are

\[ \sigma_1' = \frac{\sigma_1^* + 2c'N_\phi - \sigma_3^*}{1 + N_\phi^2} \]  

(4B.5a)

\[ \sigma_3' = \frac{N_\phi \sigma_1^* - 2c'N_\phi + N_\phi^2 \sigma_3^*}{1 + N_\phi^2} \]  

(4B.5b)

To determine the Cartesian components of the corrected stress, we assume that the direction of the principal stresses remains the same both before and after correction. Hence the corrected stress components \((\sigma_{xx}', \sigma_{yy}', \sigma_{xy}')\) are

\[ \sigma_{xx}' = \sigma_{xx}' + R' \cos 2\theta \]  

(4B.6a)

\[ \sigma_{yy}' = \sigma_{yy}' - R' \cos 2\theta \]  

(4B.6b)

\[ \sigma_{xy}' = R' \sin 2\theta \]  

(4B.6c)

where

\[ p' = \frac{1}{2}(\sigma_1' + \sigma_3') \]  

(4B.6d)

\[ R' = \frac{1}{2}(\sigma_1' - \sigma_3') \]  

(4B.6e)

\[ \theta = \frac{1}{2} \tan^{-1} \frac{2|\sigma_{xy}|}{\sigma_{xx}' - \sigma_{yy}'} \]  

(4B.6f)

\((\sigma_{xx}', \sigma_{yy}', \sigma_{xy}')\) are the Cartesian components of the uncorrected stress state which has principal values \((\sigma_1^*, \sigma_3^*)\).
Fig 4B.1 MOHR-COULOMB FAILURE SURFACE.

PLANE STRAIN.
An exact solution for the load-stretch response of an initially square block of material deforming under conditions of plane strain is presented in this appendix. The problem is defined in Fig. 4.1. The extension or compression is considered to begin at time 0 when the body is subject to an initial stress state. The initial loading is shown in Fig. 4.1a and it is assumed that the force $P$ in the $x$ direction is varied while the force $Q_0$ in the $y$ direction is maintained at its initial value (Fig. 4.1b).

The material of the block obeys a Mohr-Coulomb criterion and deforms according to the constitutive law of equations (2A.8, 3.15 and 3.16). We first consider the purely elastic response.

(i) Elastic Response

In this simple case of homogeneous deformation the non-zero deformation rates are given by

$$\varepsilon_{xx} = \frac{3u}{\delta x} \quad (4C.1)$$
$$\varepsilon_{yy} = \frac{3v}{\delta y} \quad (4C.2)$$

where $u(x, t)$ and $v(y, t)$ are the displacement components in the $x$ and $y$ directions respectively.

If $Q_0$ is maintained constant then for all time $t$ we may write
\[ \sigma_{yy} = \frac{\sigma_o / h_o}{\rho} \] \hfill (4C.3)

so that
\[ \dot{\sigma}_{yy} = -\frac{(\sigma_o / h_o) \rho}{\rho^2} \] \hfill (4C.4)

If, for convenience, we allow \( l_{xx} = k \) a constant independent of \((x, y, t)\), then together with the boundary condition \( \rho = 1 \) at \( t = 0 \), the solution of equation (4C.1) in terms of overall deformation is
\[ \rho = e^{kt} \] \hfill (4C.5)

Using the constitutive relation, together with the equation (4C.4) gives
\[ \dot{\xi}_{yy} = -Bk e^{-kt} - Ck \] \hfill (4C.6)

where
\[ B = \frac{(\sigma_o / h_o)}{(\Lambda + 2G)} \]
\[ C = \left( \frac{\Lambda}{\Lambda + 2G} \right) \]

Hence a particular solution of equation (4C.2) is
\[ \mu = \rho^{-C} \exp\{-B(1-1/\rho)\} \] \hfill (4C.7)

The total applied load \( P \) per unit length is given by
\[ P = \sigma_{xx} \mu l_o \] \hfill (4C.8)

In the case of zero rotation
\[ \sigma_{xx} = \sigma_{xx} \approx (\Lambda + 2G) \xi_{xx} + \Lambda \dot{\xi}_{yy} \] \hfill (4C.9)
\[ = Ak - Bke^{-kt} \]
where \[ A = \frac{4G(\Lambda + \sigma)}{(\Lambda + 2G)} \]

which together with the boundary condition \( t = 0, \sigma_{xx} = \left( \frac{P}{E} \right) \) gives

\[ \sigma_{xx} = \left( \frac{P}{E} \right) + A\ln\theta + B(1-1/\rho) \]  \hspace{1cm} (4C.10)

Thus the elastic solution follows as

\[
\frac{P}{E\theta} = \left[ \frac{1}{(1-\nu)^2} \ln\theta - \left( \frac{1}{1-\nu} \right) \left( \frac{\sigma_{xx} - \sigma}{E} \right)(1 - 1/\rho) \right] \]

\[
\times \left[ \frac{\rho - \nu/(1-\nu)}{(1-\nu)} \exp \left\{ - \frac{\sigma_{xx} - \sigma}{E} \left( \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \right)(1 - 1/\rho) \right\} \right] \hspace{1cm} (4C.11)
\]

(ii) Elastoplastic Response

We present now the elastoplastic solution for the case when \( \sigma = P = 0, \) i.e. the body is unstressed at \( t = 0. \) The extension to the more general case is not difficult.

For plastic failure within the material, the value of \( \sigma_{xx} \) is limited according to the type of loading, i.e. extension or compression. In both cases \( \sigma_{yy} = 0, \) so that for a Mohr-Coulomb type of material \( \sigma_{xx} \) is limited by

(a) extension: \[ \sigma_{xx} = 2\nu/1/N_\phi \]  \hspace{1cm} (4C.12)

(b) compression: \[ \sigma_{xx} = -2\nu/N_\phi \]  \hspace{1cm} (4C.13)

where \[ N_\phi = \frac{1 + \sin\phi}{1 - \sin\phi} \]

and tension is reckoned positive.
The constitutive law in this special case takes the form

\[
\begin{pmatrix}
\dot{\sigma}_{xx} \\
\dot{\sigma}_{yy}
\end{pmatrix} = \left[ 1 - \frac{a b^T}{a^T b} \right] \begin{bmatrix}
\Lambda + 2G & \Lambda \\
\Lambda & \Lambda + 2G
\end{bmatrix} \begin{pmatrix}
\ell_{xx} \\
\ell_{yy}
\end{pmatrix}
\]

(4C.14)

where \( a = \begin{bmatrix}
\Lambda + 2G & \Lambda \\
\Lambda & \Lambda + 2G
\end{bmatrix} \)

\[
a = \begin{pmatrix}
2\sigma_{xx}(1-\sin\phi\sin\psi) + 4c\cos\phi\sin\psi \\
2\sigma_{xx}(-1-\sin\phi\sin\psi) + 4c\cos\phi\sin\psi
\end{pmatrix}
\]

\[
b = \begin{pmatrix}
2\sigma_{xx}(1-\sin^2\phi) + 2c\sin2\phi \\
2\sigma_{xx}(-1-\sin^2\phi) + 2c\sin2\phi
\end{pmatrix}
\]

Substituting into equation (4C.14) the material properties \( c, \phi, \psi \) and the appropriate limiting stress of (4C.12, 13), and utilising the fact that \( \sigma_{yy} = 0 \), allows a relationship to be obtained between \( \ell_{yy} \) and \( \ell_{xx} \). The solution for the elastoplastic material is then obtained in the same manner as that for the elastic material after replacing equation (4C.6) by its elastoplastic equivalent.
APPENDIX 4D

SIMPLIFIED EQUATIONS FOR PLANE STRAIN - RADIAL SYMMETRY

The plane strain cylinder expansion problem is described in Fig. 4.10. In this problem the use of symmetry allows simplifications to be made to the general equations of finite deformation given in chapter 2.

We need only consider the radial movement $u$ of a typical material particle of the cylinder in the time interval $(t_0, t)$. In this interval we assume that the particle moves from a position $r_0$ to $r$ so that $u = r - r_0$ as a result of a change in pressure $p_0$ to $p$. The velocity of the particle at time $t$ is given by,

$$ v = \frac{\partial}{\partial t} u(r_0, t) = \frac{\partial}{\partial t} u(r, t) \quad (4D.1) $$

Since there are no rotations and no radial or tangential shear deformations associated with the expansion, then a complete description of the deformation is given, in Eulerian terms, by the vector of rate quantities

$$ \dot{\mathbf{x}}_R = \left( \frac{\partial v}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial r} \right) \quad (4D.2) $$

A complete description of the stress field at $r$ at time $t$ is also given by

$$ \sigma_R = (\sigma_{rr}, \sigma_{\theta \theta}) \quad (4D.3) $$

where $\sigma_{rr}$ and $\sigma_{\theta \theta}$ are the radial and tangential normal stress components,
respectively, tensile stresses are reckoned positive.

The rate law equation (2A.8) then simplifies to

\[ \dot{\sigma}_R = D_{R} \dot{\epsilon} \]

(4D.4)

where for the elastic material

\[ D_R = \begin{bmatrix} \Lambda + 2G & \Lambda \\ -\Lambda & \Lambda + 2G \end{bmatrix} \]

(4D.5)

This material yields according to the Tresca criterion which may be written

\[ \sigma_1 - \sigma_3 = 2c \]

(4D.6)

where \( \sigma_1 > \sigma_2 > \sigma_3 \) are the principal stresses and \( c \) is the yield limit in pure shear, or the cohesion. We assume without proof that the intermediate principal stress \( \sigma_2 \) is \( \sigma_{xx} \), the longitudinal component of normal stress (Koiter, 1953 has shown that this assumption is valid in the case of plane strain and infinitesimal deformation when \( b_1/a_1 \leq 5.75 \)). Once it has yielded the material flows plastically according to an associated flow law and the matrix \( D_R \) is given by

\[ D_R = (\Lambda + G) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

(4D.7)

Following the analysis of chapter 2, the virtual work expression governing the finite deformation behaviour reduces to,

\[ \int_V \dot{\sigma}^T \dot{\epsilon} \ dv = \int_V \dot{\sigma}^T \dot{\epsilon} \ dv = \int_{S} \dot{\epsilon}^T \dot{\sigma} \ dv \]

(4D.8)
where $\Delta \delta_{r0}$ is the vector of stress components at $r_0$ at time $t_0$.

Numerical solution of equation (4D.8) proceeds as in section 4.2, and if $\Delta \delta_{r}$ is the vector of incremental radial displacements of the nodes of linear elements then the final equation set to be solved at each time step is

$$\mathbf{K}_{r} \Delta \delta_{r} = \bar{\mathbf{f}}_{r}$$

(4D.9)

where

$$\mathbf{K} = \int \frac{\mathbf{B}^T \mathbf{B}}{V} \, dV$$

$$\bar{\mathbf{f}}_{r} = \int \frac{\mathbf{p} \mathbf{n} \mathbf{d}s - \int \frac{\mathbf{B}^T \Delta \mathbf{A}}{V} \, dV}{S}$$

$$\mathbf{n} = (1, 0, \ldots, 0) \quad (N \text{ terms where } N=\text{No. of nodes})$$

$$\mathbf{B} = \begin{bmatrix} \partial/\partial r \\ 1/r \end{bmatrix} \Delta \mathbf{A}$$

and

$$\mathbf{v} = \frac{\mathbf{A}}{\Delta \delta_{r}}$$

The superior bar takes the same meaning as before. Equation set (4D.9) may require an iterative solution procedure.
APPENDIX 4E

FINITE EXPANSION OF AN ELASTIC CYLINDER

It is convenient to record here the exact solution for the expansion of a cylinder of material possessing infinite strength \( G/c = 0 \). Consider the cylinder of Fig. 4.10c which had the initial stress free configuration shown in Fig. 4.10a. For an infinitesimal change in pressure \( dp \), the outer and inner radii change by amounts \( db \) and \( da \) respectively, where from the elastic small strain theory we have,

\[
\begin{align*}
    db &= \left[ \frac{2(1 - \nu^2) b}{(b/a)^2 - 1} \right] \frac{dp}{E} \quad (4E.1) \\
    da &= \left[ \frac{1 + \nu}{(b/a)^2 - 1} \right] \cdot \left\{ \frac{1}{(1 - 2\nu) a + b^2/a} \right\} \frac{dp}{E} \quad (4E.2)
\end{align*}
\]

Let the parameter \( R = b/a \), then on combination of equations (4C.1 and 2) we obtain

\[
\frac{dR}{R} = -2\frac{dp}{G} \quad (4E.3)
\]

whence

\[
R = R_I e^{-2p/G} \quad (4E.4)
\]

where \( R_I = b_I/a_I \).

The maximum pressure \( p_c \) will occur when \( R = 1 \), i.e.

\[
\frac{p_c}{G} = 2\ln R_I \quad (4E.5)
\]

Current values of radii \( a \) and \( b \) at pressure \( p \) may be recovered from
\[ \ln \left( \frac{a}{a_I} \right) = \frac{1}{2} \left\{ (1-2\nu) \ln \left( \frac{(R/R_I)^2}{(1-R^2)/(1-R_I^2)} \right) - \ln \left( \frac{1-R^2}{1-R_I^2} \right) \right\} \quad (4E.6) \]

\[ \ln \left( \frac{b}{b_I} \right) = \ln \left( \frac{R}{R_I} \right) + \ln \left( \frac{a}{a_I} \right) \quad (4E.7) \]
APPENDIX 4F

THE DEPOSITION PROBLEM

In this appendix is presented the rigorous solution for the problem of one dimensional deposition of a material in the presence of a gravity field. The problem is defined in Fig. 4.19.

Assume that deposition begins at time \( t = 0 \) when \( h = 0 \) and that at some time \( t \) later the height of fill contains a mass \( M \) of weight \( W \) (per unit plan area) of deposited material, where \( W = Mg \) and \( g \) is the acceleration due to gravity.

We consider the deposition of two different types of material (i) an elastic material and (ii) an elastoplastic material.

(i) Elastic Material

For convenience assume that elastic material is being deposited at a constant rate and that the mass density and unit weight of material before deposition are \( m \) and \( \gamma = mg \) respectively. \( \gamma \) and \( m \) are assumed constant.

Consider a typical increment in time \( dt \) during which the overall height of fill increases by an amount \( dh \). The increase in height is calculated from the sum of the volume occupied by the additional material and the settlement of the previously deposited, underlying layer (downwards settlements are negative). Since mass is added at a unit rate then

\[
\Delta h = dt + ds
\]  
(4F.1)
where we calculate the incremental settlement $ds$ from

$$ds = \int_0^h (\varepsilon_{yy} \, dt) \, dy$$  \hspace{1cm} (4F.2)$$

The quantity $\varepsilon_{yy}$ is the vertical normal component of the strain rate.

Noting that $\varepsilon_{xx} = \varepsilon_{zz} = 0$, then from equation (2A.8) we find that,

$$\dot{\varepsilon}_{yy} = \frac{\sigma_{yy}}{(\lambda+2\mu)}$$  \hspace{1cm} (4F.3)$$

The stress rate $\dot{\sigma}_{yy}$ is given simply by

$$\dot{\sigma}_{yy} = -mg = -\gamma$$  \hspace{1cm} (4F.4)$$

and tensile stresses are reckoned positive. Hence on substitution

into equation (4F.2) $ds$ is given by

$$ds = -\left(\frac{\gamma}{\lambda+2\mu}\right) \, h \, dt$$  \hspace{1cm} (4F.5)$$

Equations (4F.1, 5) and the boundary condition $h = 0$ at $t = 0$ leads to

a solution for $h$ as a function of the weight of fill material, i.e.

$$h = \left(\frac{\lambda+2\mu}{\gamma}\right) \left(1 - \exp\left\{-\frac{W}{\lambda+2\mu}\right\}\right)$$  \hspace{1cm} (4F.6)$$

where $W = mg t$.

Notice that as $W \to \infty$ a critical height, $h_c$, is reached.

$$h_c = h(W \to \infty) = \left(\frac{\lambda+2\mu}{\gamma}\right)$$  \hspace{1cm} (4F.7)$$

To determine the stress distribution as a function of both $\gamma$
and t consider a typical infinitesimal element of the elastic material.

Notice that: (1) The element is deposited at time \( t = t_o \), when
\[
h = h_o = \left( \frac{\alpha + 2G}{\gamma} \right) \left( 1 - \exp \left\{ \frac{-\gamma t_o}{\alpha + 2G} \right\} \right).
\]
Its coordinate location is then \( y = h_o \).

For all \( t < t_o \) the element does not exist (at least as far as the deposited layer is concerned) and

\[
\sigma_{yy} = 0 \text{ for } t < t_o \tag{4F.8}
\]

(2) For all \( t > t_o \) the change in vertical stress for this particular element is defined by \( \sigma_{yy} = -\gamma \) so that for \( t > t_o \)

\[
\sigma_{yy} = -\gamma (t - t_o) = -(W - W_o) \tag{4F.9}
\]

where \( W_o = W(t = t_o) \).

(3) The stress state described by equations (4F.8, 9) is valid only for this particular infinitesimal element. During the time interval \( t_o \) to \( t \) this element changes its location in space relative to this fixed coordinate \( y \). The element that was deposited at \( y = h_o \) at time \( t = t_o \) is then located at \( y = y' \) at time \( t \). To determine \( y' \) let this element change its location by an amount \( dy \) in the time interval \( dt \), then

\[
dy = \int_{0}^{y} \left( \frac{\gamma}{\alpha + 2G} dy \right) dt
\]

\[
= -\left( \frac{\gamma}{\alpha + 2G} \right) y \ dt
\]

On noting the boundary conditions we find that
\[ y' = h_o \exp \left\{ -\frac{\gamma(t-t_o)}{(\Lambda + 2G)} \right\} \quad (4F.10) \]

(5) In summary, the stress distribution at time \( t \) may be determined by tracing the deformation history of each particle as it is deposited, i.e.

\[ \sigma_{yy}(y') = -\gamma(t-t_o) = -(W-W_o) \quad \text{for } t \geq t_o \]

where \[ y' = \left(\frac{\Lambda + 2G}{\gamma}\right) \exp\left\{ -\frac{W-W_o}{\Lambda + 2G} \right\} \exp\left\{ \frac{-W}{\Lambda + 2G} \right\} \quad (4F.11) \]

(ii) Elastoplastic Material

Assume that the material to be deposited is an ideal Mohr-Coulomb material whose shear strength is determined by the usual parameters \( c \) and \( \phi \).

When the material is elastic the stress components are given by equation (4F.11, 12). If tension is positive then since \( \sigma_{xx} \geq \sigma_{yy} \) and \( \sigma_{xx} = \sigma_{zz} \) we may write the yield criterion as (for example)

\[ \sigma_{xx} = \frac{1}{N_\phi} \sigma_{yy} + 2c \sqrt{\frac{1}{N_\phi}} \quad (4F.13) \]

where \[ N_\phi = \frac{1 + \sin \phi}{1 - \sin \phi} \]

The stress components at first yield are thus defined by equating (4F.12 and 13), and thus

\[ \sigma_{yy} = \frac{-2c}{N_\phi} \left[ \frac{1}{N_\phi} - \frac{\nu}{1-\nu} \right] \quad (4F.14) \]
at first yield.

A complete solution to the deposition problem for a material which has $c \neq 0$ is difficult to trace. The spread of plastic yielding throughout the deposited material must be known as a function of time. However, in the special case of a purely frictional material, i.e. $c = 0$, with $\frac{1}{N_\phi} > \frac{\nu}{1-\nu}$ this presents no real problem. The material yields as soon as it is deposited, e.g. for $c = 0$, $\phi = 30^\circ$ then $\nu < 1/4$. It is these special materials that we consider here.

The solution for the depth as a function of time is obtained as follows:

Each deformation rate may be written as the sum of an elastic and a plastic component, e.g.

$$\dot{\ell}_{yy} = \dot{\ell}_{yy}^E + \dot{\ell}_{yy}^P$$

(Superscript E = Elastic, P = Plastic)

Adopt the plastic flow law as in section 3.3, i.e.

$$\frac{\dot{\ell}_{yy}^P}{\dot{\ell}_{xx}^P} = -\frac{1}{N_\psi}$$

where $N_\psi = \frac{1 + \sin \psi}{1 - \sin \psi}$

The elastic deformation rates $\dot{\ell}_{yy}^E$ and $\dot{\ell}_{xx}^E$ may be determined from the rate law of equation (2A.8) as,

$$\dot{\ell}_{yy}^E = \dot{A}_\psi \dot{\ell}_{xx} + \dot{B}_\psi \dot{\ell}_{yy}$$

(4F.17)
\[ \ell_{xx}^E = B \sigma_{xx} + A \sigma_{yy} \]  

(4F.18)

where the constants \( A \) and \( B \) are given by

\[ A = \frac{\Lambda + 2G}{4G(h + G)} \]
\[ B = \frac{-\Lambda}{4G(h + G)} \]

On substitution of equations (4F.13, 16, 17, 18) into equation (4F.15) and noting that \( \ell_{xx} = 0 \), then we find

\[ \ell_{yy} = C_{yy} \left[ A \left( 1 + \frac{1}{N \psi \phi} \right) + B \left( \frac{1}{N \phi} + \frac{1}{N \psi} \right) \right] \]  

(4F.19)

The analysis then follows that for the elastic material after replacing equation (4F.3) by (4F.19), and it can be shown that

\[ h = \frac{1}{\gamma F} (1 - \exp \{-WF\}) \]  

(4F.20)

where

\[ F = A \left( 1 + \frac{1}{N \phi \psi} \right) + B \left( \frac{1}{N \phi} + \frac{1}{N \psi} \right) \]

In this case the critical height is given by

\[ h_c = \frac{1}{\gamma F} \]  

(4F.21)
CHAPTER 5
APPLICATIONS TO SOIL
MECHANICS PROBLEMS
5.1 INTRODUCTION

In this chapter an examination is made of some typical boundary value problems in theoretical soil mechanics. Many of these problems have been examined in the past using both exact and numerical techniques and in most cases the analyses assumed infinitesimal deformation. The works of Fernandez and Christian (1971) and Davidson and Chen (1974) are two exceptions. The present examination contains not only the results of conventional infinitesimal analyses but also the results obtained using the previously described approaches to finite deformation analysis.

The type of problem studied involves the surface loading of both a homogeneous and an inhomogeneous clay layer. In the latter case the inhomogeneity is defined by a linear increase of modulus and/or strength with depth below the surface of the layer. In all problems the loading is of a plane strain nature with the load applied as either a line load; a uniform normal traction; a rigid footing; or the building up of an embankment made of frictional material.

A number of different finite element networks were used to obtain the numerical results presented in this chapter and these are depicted in Figs. 5.1 to 5.5. Each figure shows the mesh configuration at \( t = 0 \) before the application of any load. The mesh is assumed to be embedded in the layer and will follow the deformation. As such each network may be described as a Lagrangian or material mesh.
5.2 SURFACE LOADINGS ON HOMOGENEOUS MATERIALS

Before the discovery of numerical discretisation processes such as the finite element method, mathematical solutions to problems of the boundary loading of a continuum were usually much simpler for the case of a homogeneous, isotropic material than for an inhomogeneous material (an important exception is discussed later - see section 5.3.1). Since soil deposits can effectively be considered as continua, and since many important problems in soil mechanics and foundation engineering are of a boundary loading type, it has been common practice to idealise the soil as a homogeneous mass. Although in the real situation a soil deposit may not in fact be uniform, representative properties are often determined and deemed to apply for the entire material. Hence the solution to problems for the ideal homogeneous mass are seen to have some application to real situations. It is in this vein that we now look at some special types of boundary loading problems for such homogeneous, isotropic materials. Solutions are presented for both elastic and elastoplastic materials as well as for both undrained and drained types of behaviour. Unlike previous efforts at finite analysis (e.g. Fernandez and Christian, 1971 and Davidson and Chen, 1974) the present work also considers the importance of material self-weight and the use of a non associated flow rule to describe plastic volume behaviour.

5.2.1 Line Load on an Elastic Layer

As a matter of interest as well as a matter of some practical importance consider the problem of a layer of soil subject to a line load under conditions of plane strain. The soil is underlain by a rough rigid base (Fig. 5.6). For the moment ignore any plastic yielding
within the soil which is assumed to be an isotropic, homogeneous, ideal elastic material.

Calculating stresses and displacements in a layered elastic system is a problem which often arises in engineering analysis and design, especially in the field of soil mechanics. Problems of this type have received attention from several investigators, e.g. Marguerre (1931); Biot (1935); Pickett (1938); and Burmister (1943, 1945, 1956). More recently Lemcoe (1960) obtained expressions for the distribution of line load stresses in a layer and independently Poulos (1966, 1967) obtained solutions for the stresses and displacements due to line loading. All of these authors restricted their attention to the case of infinitesimal deformation and most of their results have been conveniently summarised by Poulos and Davis (1974). One particular feature of the line load solutions is worth noting here. There exists a discontinuity in vertical displacement and all stress components at the point of application of the line load on the surface.

The finite deformation theory and numerical procedure of the previous chapters were applied to this problem. It is assumed that throughout the loading the soil remains elastic with constant Young's modulus $E$ and Poisson's ratio $\nu = 0.4$. As mentioned above, displacements immediately below the loading are meaningless. However, the finite element model is unable to predict the presence of such discontinuities and will yield a finite value of displacement at the loaded node. For these reasons a comparison is made between the finite element surface displacement and those of Poulos (1966) at a distance $x = 0.1H$ from the point of load application. - See Fig. 5.7 (the finite deformation solutions were obtained using scheme E of Table 4.1). As the
load is increased the displacement becomes less like that predicted by small strain theory. In an engineering sense the small strain results seem adequate for values of load less than about 0.1\(W_EH\) where the difference in settlement predictions is of the order of 20%. At such a load the settlement is about 20% of the layer depth. In real soils it is likely that displacements of this magnitude (\(>0.2H\)) would be accompanied by very severe plastic yielding. Thus for this problem it appears that, within the range of loading for which a soil can be considered elastic, the classical theory adequately predicts the behaviour, at least for most engineering purposes.

The accuracy of the finite element infinitesimal solutions is verified in Figs. 5.8 and 5.9 where components of surface displacement \(\rho_x\) and \(\rho_z\) and the normal stress component \(\sigma_{zz}\) are compared with the solutions of Poulos. Note the accuracy of the numerical solutions even quite close to the load itself.

As a matter of further interest the mesh configurations for the finite deformation analysis are shown in Fig. 5.10 corresponding to various values of applied load. This method of giving a pictorial description of the deformation history will repeatedly be used in this and the following chapters.

5.2.2 Traction Loading on an Elastic Layer

Consider the plane strain problem of a normal traction loading, of intensity \(q\), applied downwards over a cross-sectional width 2\(B\) of the surface of a layer of homogeneous, isotropic, elastic material. The initial depth of the layer, before any load is applied,
is $H = 10B$ (see Fig. 5.11). The deformation of the layer is determined by the material properties $E$, the Young's modulus and $v$, the Poisson's ratio. Initially we consider only the small strain solution.

5.2.2 (i) Infinitesimal Solution

Solutions, assuming infinitesimal deformation, for the problem of Fig. 5.11 have been presented by Poulos (1967) and Giroud (1973). Some results for the central displacement, as a function of the Poisson's ratio $v$ are given in Table 5.1.

$$\text{Central Vertical Displacement} = \frac{2B}{E} \cdot q \cdot I_c$$

<table>
<thead>
<tr>
<th>$v$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_c$</td>
<td>1.866</td>
<td>1.856</td>
<td>1.750</td>
<td>1.609</td>
<td>1.402</td>
<td>1.112</td>
</tr>
</tbody>
</table>

Table 5.1: Displacement Influence Factors, $I_c$ (after Giroud, 1973)

When considering a finite element solution to the problem of Fig. 5.11 it is not possible to place a lateral boundary at infinity in the finite element mesh. In lieu of the infinite lateral extent of the layer, it is convenient to place a perfectly smooth, rigid boundary at a distance $L$ from the centreline - see Fig. 5.12. It is then of some interest to know what effect the placement of this smooth boundary has on the solution of the problem. In order to investigate this point an analytical solution using Fourier series was carried out. Some results for vertical displacement, $w$, under the centre of the traction loading are plotted in Fig. 5.13 for a number of values of
Poisson's ratio. Notice, that as expected, when \( L \) becomes large the solution approaches that for the problem of Fig. 5.11. At the other extreme, i.e. \( L/B=1 \), we obtain the one-dimensional solution. At a value of \( L/B=10 \) there is little difference between this solution and the solution for \( L/B=\infty \). The worst discrepancy is about 4% when \( \nu = 0.5 \). The value \( L/B=10 \) was thus considered to give an adequate representation of the layer of infinite lateral extent.

A finite element solution for this infinitesimal elastic problem was obtained using the mesh shown in Fig. 5.1. A comparison of central displacements obtained by finite element and Fourier analyses is given in Table 5.2.

\[
\text{Central Vertical Displacement} = \frac{2B}{E} \cdot q \cdot I_c
\]

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>Finite Element</th>
<th>Fourier ( L/B=10 )</th>
<th>Giroud ( L/B=\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.48 )</td>
<td>1.044</td>
<td>1.072</td>
<td>1.17*</td>
</tr>
<tr>
<td>( 0.5 )</td>
<td>-</td>
<td>1.10†</td>
<td>1.112</td>
</tr>
</tbody>
</table>

* Extrapolated value
† Interpolated value

Table 5.2: Displacement Influence Factors, \( I_c \)

Note that not only does the finite element solution using \( \nu = 0.48 \) give a good approximation to the analytic solutions for \( \nu = 0.48 \) and \( L/B = 10, \infty \) but it also shows good agreement with the solution for an incompressible material, \( \nu = 0.5 \) and \( L/B = \infty \).
The stress components beneath the centreline of the strip footing have been calculated by the finite element method for $\nu = 0.48$ and these have been plotted in Fig. 5.14. They compare favourably with the results of Poulos (1967) who used $\nu = 0.5$.

The effect on displacement of either a smooth or rough base can be seen in Fig. 5.15 where the vertical deflections of points along the surface have been plotted. These numerical solutions show little difference whereas Ueshita and Meyerhof (1968) suggest that in the smooth case when $\nu = 0.5$ the central displacement should be some 25% larger. This indicates the restraining effect on horizontal movement of the lateral boundary when the layer has a smooth base.

The conclusion that $\nu = 0.48$ gives a good approximation for incompressible behaviour and that for a layer with a rough base the use of $L/B = 10$ is adequate for a layer of infinite lateral extent are valid in the case of infinitesimal analysis. They may not, however, retain their validity when the analysis is extended to include the possibility of finite deformation. We consider now a finite deformation problem.

5.2.2 (ii) Finite Deformation Solution

For finite strain problems, boundary conditions have to be specified more fully than for infinitesimal theory. In the traction loading problem it is assumed that $q$ remains uniform in intensity and always acting normal to the originally horizontal surface, even when that surface becomes significantly distorted. It is also assumed that this traction is always applied to the same physical part of the surface,
even when this part has a length different from its original length of 2B.

The problem of just such a boundary loading applied to an elastic layer underlain by a smooth rigid base has been solved using the mesh of Fig. 5.1 and the numerical technique described in chapter 4. The relationship between the magnitude of the applied traction and the central vertical displacement is plotted in Fig. 5.16 and a pictorial description of the deformation is given in Fig. 5.17(a) to (d). Note that even at a central displacement equal to approximately 80% of the layer depth the small strain solution differs from that of finite deformation by only about 10%. This difference is of course a function of applied traction. The finite strain theory predicts that the elastic layer becomes stiffer as the settlement increases. It is not possible to demonstrate numerically, but the finite deformation theory should predict that an infinite pressure is required to produce a settlement equal to the layer depth, i.e. when the traction load would bear directly on the rigid base. The small strain theory does not of course do this, and it predicts physically impossible settlements (i.e. settlements that are greater than the layer depth) at finite load levels.

Due to extreme distortion of some elements of the Lagrangian mesh the solution becomes numerically unstable and it is not possible to proceed with the finite analysis much beyond the situation depicted in Fig. 5.17(d). In addition, the validity of applying these finite deformation results to a layer of infinite lateral extent, particularly at the higher load levels, is in some doubt. As can be seen in Fig. 5.17(d), after extreme distortion a large amount of heave of the layer surface is obvious at the lateral boundary \( x = L = 10B \). In other words, to rep-
resent adequately the gross deformations of a layer of infinite lateral extent, one may have to choose a value of \( L \) far greater than that of 10B. Such a value will of course be a function of \( H/B \), \( q/E \) and \( v \).

5.2.3 Traction Loading on an Elastoplastic Layer

If the layer, which is subjected to traction loading as described in the preceding section, is now idealised as an elastoplastic material we have an added complication. The finite deformation analysis of such a problem must consider the finite strength of the material. In particular we consider the clay to be a purely cohesive material. As previously mentioned such an idealisation is valid for the undrained behaviour of most clays. To approximate incompressible behaviour in the elastic range we again assign a value of 0.48 to Poisson's ratio. We assume also that the clay is weightless.

5.2.3 (i) Infinitesimal Deformation Solution

A small strain analysis of this problem was performed using the finite element meshes described as number 1 (Fig. 5.1) and number 2 (Fig. 5.2). No slip was permitted along the line \( MM' \) of mesh 2. The relationship between the applied traction \( q \) and the central displacement is shown in Fig. 5.18. Note that the curve which corresponds to Mesh 2 approaches asymptotically the limit load of \( q = (2 + \pi)c \) while that for mesh 1 has no well defined collapse load. We thus conclude that the numerical predictions of elastoplastic behaviour are mesh dependent, and further, that a mesh which may be used to provide satisfactory elastic results may not be adequate for some infinitesimal elastoplastic predictions (e.g. Mesh 1).
5.2.3 (ii) Finite Deformation Solution

A finite deformation solution for this problem has also been found and the relationship between the traction and the central displacement is plotted in Fig. 5.19 for the case when \( E/c = 100 \). The calculation for this problem was arbitrarily terminated at a value of \( q \) near 5.6c. This is in contrast to the infinitesimal theory, where unconfined plastic flow signifies general failure at a traction of \((2 + \pi)c\). No definite value can be assigned to the ultimate bearing capacity when finite strain theory is employed for this problem, yet such a value is predicted by the infinitesimal theory - see Fig. 5.19.

The difference between the two predicted load deformation behaviours only becomes significant at values of traction greater than \((2 + \pi)c\). This point of departure is a function of the stiffness to strength ratio \( E/c \). The relationship between the central vertical displacement and the traction magnitude is plotted in Fig. 5.20 for a wide range of ratios from \( E/c = 0 \) (infinitely strong material, finite elastic strain effects only) to \( E/c = 200 \). On this figure the ordinate is plotted as \( q/E \) and the abscissa as displacement/B. An alternative representation of these results is obtained if we use \( q/c \) as the ordinate and \( E(\text{displacement})/cB \) as the abscissa - see Fig. 5.21. In addition Fig. 5.22a to d shows the progressive distortion of the surface and growth of the plastic regions for one of the less stiff cases, i.e. \( E/c = 10 \). We note also that at lower values of \( E/c \), the more coarse mesh, No. 1, provided a finite deformation solution not substantially different from that of mesh 2. This reflects the increased importance of large elastic strains over large plastic strains at lower \( E/c \).
5.2.4 Rigid Footings on an Elastoplastic Layer

Consider now a boundary loading problem different to that described in previous sections. The behaviour of a rigid footing resting on a homogeneous clay layer is of some practical interest. Some solutions are given for the behaviour in plane strain of a rigid strip footing on the ideal elastoplastic materials that correspond to the two extremes of clay behaviour, viz. a purely cohesive material for undrained behaviour and a cohesive frictional material for the drained behaviour. This section commences with some small strain solutions.

5.2.4 (i) Infinitesimal Solutions - Undrained Behaviour

The problem of the indentation of a clay layer by a rigid strip footing is described in Fig. 5.23. The properties of the ideal clay for both undrained and drained behaviours are also shown. Several Lagrangian finite element meshes were used to discretise the layer and their results compared. The meshes used were

(a) Mesh No. 1 of Fig. 5.1
(b) Mesh No. 2 of Fig. 5.2
(c) Mesh No. 3 of Fig. 5.3

Note that mesh 3 has a similar but more coarse layout than mesh 2. In all the above cases the loading is specified by a given boundary motion. The magnitude of the applied force is then backfigured from the nodal values and checked for equilibrium with the element stresses.

There exists a very important difference between this type of loading and the traction loading described earlier. Because of the rigi-
dity of the footing and the nature of the prescribed boundary movement there exists a stress concentration at the corner of the footing (point M of Figs. 5.2 and 3). Because of the extremely high stress gradients in the vicinity of M it is not possible to adequately model such a singularity using conventional finite element arrays.

In an endeavour to more accurately represent such a situation the following measures have been adopted. On the figures 5.2 and 5.3 the line MM' may be a potential rupture surface. This means that nodes situated on MM' are in fact dual nodes, with one member of each pair initially coincident and considered to belong to the elements on either side of MM'. There are two possible types of behaviour that may be exhibited along MM'. These are

(a) perfect bonding between elements on either side and adjacent to MM' is assumed with each member of the dual node pair always being coincident with its partner. This will reproduce the behaviour of regular finite element analyses.

(b) once the shear stress at any point along MM' is reached then rupture occurs. Provision for this slipping of nodes past each other is accomplished by the application of forces to the dual nodes along the rupture surface to ensure compatibility up until the point when the material shear strength is reached. This technique differs from the use of joint elements (Goodman et al., 1968) in that dual nodes are not connected by joint stiffness terms inserted into the global stiffness matrix. Details of the mathematical formulation for such behaviour have been fully described by Rowe, Booker, Balaam (1976).

Consider first the infinitesimal analysis of a smooth rigid
footing resting on the ideal weightless, undrained clay layer for which
\( E/c = 100 \) and \( v = 0.48 \). For the moment consider also that no slip is
allowed to occur along \( MM' \). The results of the infinitesimal analysis
are given in Fig. 5.24 where footing settlement \( \Delta \) is plotted against
footing load \( P \). The applied load was displacement defined with incre-
ments of \( \Delta \) equal to 0.05B. The three curves show the effect, on load-
deformation behaviour, of using the different finite element represen-
tations. The average pressure \( P/2B \) at collapse for the infinitesimal
analysis is given by the Prandtl value of \( (2 + \pi)c \), as shown in the
figure. Quite obviously, the finer the discretisation then the more
accurate the result, e.g. compare the results of meshes 1 and 2. A
feature worth noting also, is that in the case of a rigid footing and
for the meshes considered, the use of triangular elements in the
neighbourhood of the loaded footing, produced more accurate results
than the use of only quadrilaterals, e.g. compare results of mesh 1
with those of meshes 2 and 3.

We consider now the possibility of allowing slip to occur
along \( MM' \) for the smooth footing problem. Load displacement relations
for several analyses are shown in Fig. 5.25. There is little difference
between the small strain solutions, with either slip or no slip, for
the finer mesh, No. 2. The point to note, however, is the considerable
improvement in the predicted collapse load in the case of slip in the
coarse mesh, No. 3. Thus for the case of homogeneous materials, the
main benefit to be gained from incorporating a potential rupture surface
into the finite element mesh may merely be one of economy; i.e. a saving
on elements.

The difference in computed results, for the case of a rough
rigid footing, as opposed to a smooth one, can be seen in Fig. 5.26. In both cases the computation procedures were identical, except for the additional displacement boundary condition for the rough footing. Both used mesh No. 2 with no slip along MM'. Material properties were the same in both cases. It will be seen that the use of a rough footing results generally in less displacement for any given load, particularly near the Prandtl collapse value. Both smooth and rough footings should, theoretically, have the same collapse load. The difference in the finite element solutions shows how the extra, restrictive boundary condition effects the computed result. The numerically obtained value of the collapse load, (i.e. the point of zero slope on the P,Δ curve) is not as well defined for the rough as for the smooth footing. This observation is consistent with the findings of others (Nagtegaal, Parks and Rice, 1974).

5.2.4 (ii) Finite Strain Solutions - Undrained Behaviour

The problem of a rough, fully rigid footing, bearing on a weightless, undrained clay has also been analysed using the finite deformation theory. Unlike the infinitesimal case, certain computational difficulties arise in the finite analysis if the footing base is considered to be completely smooth, incapable of developing any shear stress between itself and the clay material. If this were the case then finite element mesh nodes in the soil, initially located beneath the footing and in contact with the loaded area, may be free to move horizontally as well as vertically. Eventually some of these nodes may move far enough so as to emerge from beneath the footing into the stress free surface region. The computational difficulty associated with this phenomenon is essentially one of book-keeping, i.e. knowing exactly when a
given node changes from being in contact with the footing, and hence the load, to being not in contact.

To avoid such a difficulty we confine the analysis to that of a rough footing capable of developing any shear stress along the interface with the soil. The finite element mesh 2 of Fig. 5.2 shown at the zero load configuration was used for all calculations. As with the previous finite deformation problems, the same material mesh is used throughout the loading to trace the deformation history.

For the case of a layer of weightless clay material, having the same undrained properties as before, i.e. $E/c = 100$, $\nu = 0.48$, and loaded by a rough rigid footing, results of the finite deformation analysis allowing no slip along $MM'$ are given in Fig. 5.27(a) to (d). The progressive distortion of the layer and the growth of the plastic region are shown on this figure. Extreme distortion of elements close to the footing is obvious in Fig. 5.27(d). Note also the manner in which the footing deforms the clay layer. Immediately beneath the footing an almost triangular wedge of material, which itself suffers relatively little distortion, is seen to penetrate the rest of the layer, leading the footing downwards as it were. Beneath this wedge a narrow band of intensely sheared material is seen to move upwards diagonally towards the edge of the footing, along the leading edge of the wedge. This gives a vivid picture of an aspect of the well known Prandtl velocity field.

As with the traction loading problems, the end to computation is arbitrarily selected. Even though the plastic zone extends completely from beneath the footing to an appreciable part of the free boundary
(e.g. Fig. 5.27(d)), the addition of further load, and the consequent further deformation, could presumably continue indefinitely. Again this is in contrast to the infinitesimal theory, where unconfined plastic flow signifies final general failure. This point is brought out by the load \((P)\), settlement \((\Delta)\) curves given in Fig. 5.28. As before, the curve for the infinitesimal case tends to a finite collapse load in the region of the Prandtl value. (The finite element approximation for the average pressure in this case of a rough footing is in fact about \(5.6c\) to \(5.7c\)). However, in the case of the finite deformation analysis no such collapse load is obvious. The curve continues well above the small strain limiting value. The difference in the two behaviours is due almost entirely to the fact that the geometry of the clay layer is allowed to change with increase in load in the finite deformation analysis. This variation in geometry will also effect the resultant stress distribution and hence the pattern of yielding within the layer. The apparent increase in load carrying capacity is evident only at larger deformations and in fact both the infinitesimal and finite deformation analyses produce nearly identical curves for much of the load (average pressure) range between zero and \((2 + \pi)c\). This trend is in agreement with the results of analyses by previous authors (Fernandez and Christian, 1971; Davidson and Chen, 1974). The increased load carrying capacity predicted by the finite deformation theory may be due partially to a membrane type action of the surface elements of the finite element mesh. At large loads these elements are generally plastic and are stressed such that parallel to the surface they are in tension \((\sim 2c)\) and perpendicular to the surface the stress component is approximately zero. As the indentation continues, so the membrane effect increases, providing additional support for the footing. Figs. 27(a) to (d) provide a description of this behaviour where the elements in question
exhibit increasing tensile strain, in the appropriate direction, as the load is increased. It should be noted, however, that there is some doubt whether the membrane effect is a real phenomenon or an artifact produced by the finite element solution method employed.

The present results were obtained using a step size of \( \Delta = 0.05\) as before, and in the range plotted they were little affected by the choice of computation schemes B or E (Table 4.1) for this case of a weightless clay. No finite deformation solution was obtained for the case when slip was allowed to occur along MM'.

The finite deformation behaviour of the rough rigid footing on a weightless clay layer is a function of the deformation to strength parameter ratio E/c. To demonstrate this effect a suite of curves is presented in Fig. 5.29 for a range of E/c values from 0 (infinitely strong material, finite elastic strain effects only) to E/c = 100. These results are also presented in the alternative form - see Fig. 5.30. Note that as E/c is increased the difference between finite and infinitesimal deformation predictions is reduced. In fact Davidson and Chen (1974) found that for a material with E/c = 1000 there was no plottable difference in load deflection curves at loads up to the small strain collapse value. Figs. 5.31(a) to (c) show the progressive distortion of the Lagrangian mesh and the growth of the plastic regions for the case when E/c = 10. Comments made in relation to the deformation pictures for E/c = 100 (Figs. 5.27(a) to (d)) also apply here.

It may be of interest to note that in order to obtain a converged solution to these problems generally less computational effort is required for materials with lower E/c values, when large strains
are predominantly elastic, than it does for materials with higher E/c values, when large plastic strains dominate. In addition, the need to use a fine mesh is more important for materials with higher values of the parameter E/c. In fact for materials with E/c less than about 25, the use of mesh 1 of Fig. 5.1 yielded results that were almost identical to those from the use of mesh 2. This feature was also observed in the case of the traction loading problem.

5.2.4 (iii) Drained Behaviour

An effective stress analysis is now presented for the problem of a rough rigid footing on a clay layer (see Fig. 5.23) loaded slowly, i.e. under drained conditions. The clay material is assumed to be a weightless, elastoplastic, cohesive-frictional material with $E'/c' = 100$, $\nu' = 0.3$ and $\phi' = \psi' = 30^\circ$. Load-displacement results are plotted in Fig. 5.32 for both infinitesimal and finite deformation analyses. These were obtained using mesh 2 with no slip allowed on MM' and a step size of 0.02B. The solution from the infinitesimal analysis approaches a limit value which is above the correct collapse value given by Prandt, but which is fortuitously close to the Terzaghi approximation. As in the case of undrained behaviour no such limit load is obtained from the finite deformation analysis, only an arbitrary end to computation is selected. Again the load to produce a given settlement is greater by the finite deformation prediction.

5.2.4 (iv) Finite Deformation of a Ponderable Soil

We come now to consider the case in which the layer of elastoplastic material is in a stressed condition prior to the application of
the rigid footing load. It is assumed that this stress state has arisen
due to the deposition of the clay layer in a gravity field. If the
density of clay material is \( \gamma \) we assume that the state of stress at
time, \( t = 0 \) (when \( P = 0 \)) is given by

\[
\begin{align*}
\sigma_{zz} &= -\gamma(H-z) \\
\sigma_{xx} &= k\sigma_{zz} \\
\sigma_{xz} &= 0
\end{align*}
\]

where \((H-z)\) is the depth below the surface, measured relative to the
location of that surface at time, \( t = 0 \). For definiteness it is
assumed that \( k = 1 \). The deformation parameters for the clay are \( E/c = 100, \phi = \psi = 0 \) and \( v = 0.48 \) (i.e. undrained). The effect of the pre-
sence of this initial stress state on the load deformation behaviour
can be seen in Fig. 5.33. The curves indicate that as the parameter
\( \gamma B/E \) is increased, then the load-deformation behaviour departs further
from the solution for \( \gamma B/E = 0 \). This phenomenon is partially explained
by the fact that heave of the clay surface occurs beyond the footing
contact, see mesh pictures of Figs. 5.27 and 5.31. The heave of this
clay occurs generally in the direction opposite to that of gravity and
so one would expect that greater energy and thus load will be required
to heave a ponderable clay than a weightless one. An idea of the
magnitude of quantities involved can be obtained if we consider an
example. For the given geometry of this problem and typical values of
\( E = 100,000 \text{ lb/ft}^2, \ c = 1,000 \text{ lb/ft}^2 \) and \( \gamma = 100 \text{ lb/ft}^3 \), curve 3 of
Fig. 5.32 would be appropriate for a footing of total width 200 ft on
a layer whose initial depth was 1000 ft. Density would thus have little
effect for a footing of total width less than about 20 ft on a layer of
the same material, initial depth 100 ft.

It can also be seen from the chain-dotted line on Fig. 5.33
that the load for a given penetration increases with increase in density almost exactly by an amount equal to the density times the penetration; in soil mechanics terms, \( N_\gamma = 1 \).

Computationally, these results were obtained using mesh 2 of Fig. 5.2, allowing no slip along \( MM' \), a step size of 0.05B and the calculation scheme (B) of Table 4.1. They could not have been obtained using the scheme (E). The reasons have been discussed previously in Chapter 4.

The assumption of a ponderable clay with a constant undrained cohesion is, from a fundamental point of view, inconsistent and can only be justified as an approximation when the footing is small so that the natural increase in cohesion with depth is insignificant within the relevant depth of say one footing breadth. An examination is now made of the behaviour of clays which are inhomogeneous with respect to both strength and modulus.

5.3. SURFACE LOADINGS ON INHOMOGENEOUS MATERIALS

In many circumstances the idealisation of a soil layer as a homogeneous material is unrealistic. Although for small footings, it is sufficiently accurate to use some average value of the strength and deformation parameters, \( G \) (or \( E \)) and \( c \), to obtain predictions of the deformation and bearing capacity behaviour, for wide footings and embankments the variation of \( c \) and \( G \) with depth may have a significant effect. It is commonly recognised that the undrained shear strength \( c_u \) and the shear modulus \( G \), increase with depth, the ratios \( \frac{c_u}{p} \) and \( \frac{G}{p} \), where \( p \) is the effective overburden stress, being substantially a constant for
a particular deposit.

If we consider that the soil skeleton obeys a Mohr-Coulomb yield criterion with constant drained cohesion $c'$ friction angle $\phi'$ and dilation angle $\psi' = 0$, then Small, Booker and Davis (1976) have noted that, under plane strain conditions, the undrained cohesion of any such soil element is given by

$$c_u = \left\{ \frac{N \left( \sigma'_{30} + c' \cot \phi' \right) - \left( c'_{10} + c' \cot \phi \right)}{1 + N_{\phi}} \right\}$$

$$+ \left\{ \frac{\sigma'_{30} - \sigma'_{10}}{2} \right\}$$

(5.1)

where $N_{\phi} = \frac{1 + \sin \phi'}{1 - \sin \phi'}$

and $\sigma'_{10}$, $\sigma'_{30}$ are the major, minor principal stress components associated with the element due to self weight alone. For soil layers which have the water table at the surface, these stress components are given by

$$\sigma'_{10} = \gamma_{sub} (H-z)$$

$$\sigma'_{30} = K_o \sigma'_{10}$$

(5.2)

where $(H-z)$ is the depth below the surface and $\gamma_{sub}$ is the submerged density of the clay material. Substituting equation (5.2) into (5.1) gives

$$\frac{c_u}{c'} = \left\{ \frac{2N_{\phi}}{1 + N_{\phi}} \right\} + \left\{ \frac{\gamma_{sub} (H-z)}{c'} \right\} \left\{ \frac{1 + K_o}{2} \left( \frac{N_{\phi} - 1}{N_{\phi} + 1} \right) \right\}$$

(5.3)

This may alternatively be written in the form

$$c_u = c_o + \rho (H-z)$$

(5.4)

Thus $c_u$ is a linear function of the depth $(H-z)$ below the original surface.
Most of the previous attempts at analysing the behaviour of inhomogeneous clays fall into two categories:

(a) an examination of the settlement of footings founded on such material due to loads in the elastic or working range (see Gibson, 1967; Gibson, Brown and Andrews, 1971; Gibson and Sills, 1971; Brown and Gibson, 1972, 1973; Awojobi and Gibson, 1973; Gibson and Kalsi, 1974; Gibson, 1974; Brown, 1974; Awojobi, 1974, 1975; and Simons and Rodrigues, 1975), and

(b) an examination of the ultimate load required for collapse of such footings (see Raymond, 1967; Gibson and Morgenstern, 1962; Davis and Booker, 1973b; Booker and Davis, 1972).

The author is unaware of any published attempts at a more complete analysis of the behaviour of foundations resting on such material. The purpose of this section is to investigate this type of behaviour including that in the range of loading between the extremes (a) and (b), described above. Some consideration will also be given to the finite deformation analysis of footings on this material. At first sight such analyses would seem to be of the utmost importance, especially in the case when G and c approach a value of zero at the surface of the layer. Intuitively, the strains immediately below the footing should be indefinite in this limiting case. Thus a finite deformation analysis might be expected to give a significantly different and more realistic answer than Gibson's infinitesimal theory.

Throughout what follows it is assumed that the strength and elastic moduli are intrinsic properties of an element of soil, i.e. they are locked in. As each element is deformed the properties are considered to be transported along with it. Thus a soil layer which has a
well defined inhomogeneity, such as the Gibson type soil, undergoes a finite deformation and this inhomogeneity becomes more random, as different soil elements deform by different amounts. To commence the present investigation consider the simpler case of uniform strip loading on an elastic layer.

5.3.1 Traction Loading on an Elastic Layer

5.3.1 (i) Infinitesimal Solution

Consider the traction loading problem described in Fig. 5.34. The infinitesimal elastic solution for the surface settlements due to this loading have been found by Gibson, Brown and Andrews (1971) for the case of a layer of inhomogeneous, incompressible, elastic material, of infinite lateral extent, underlain by a rough rigid base and whose shear modulus increases linearly with depth from a value of zero at the surface, i.e. \( G_0 = 0 \).

When attempting to duplicate numerically, using the finite element method, the solution of Gibson et al. several factors affect the results. Amongst these are some that have already been investigated for a homogeneous material, i.e. (a) In lieu of the infinite lateral extent of the layer, a smooth rigid boundary to the finite element network was placed at a distance \( L \) from the centreline, see Fig. 5.35. What effect does this have on the solution?

(b) How accurately does a value of \( V = 0.48 \) approximate incompressible behaviour for such an inhomogeneous material?
The answer to the first of these questions can be determined rigorously. The method of Gibson et al. (1971) has been extended by the author to enable a solution to be obtained for the problem described in Fig. 5.35. In the Gibson solution the displacement field is expressed as a Fourier integral expansion. In the present solution it has been expressed as a Fourier series expansion. Otherwise the present solution follows the procedure of Gibson et al. The results of this analysis are presented in Fig. 5.36(a) to (e). The general trend to observe is the rather obvious one that, as the lateral boundary is placed further from the load, then the more like Gibson's original solution the present answers become. However, in all cases there exists a value of L/B for which the Gibson answers are adequately reproduced e.g. see the curve for L/B = 5 on Fig. 5.36(b).

The answer to question (b), posed above, has only been determined numerically for one given geometry, i.e. H/B = 2.5 and L/B = 5. The technique described by Booker and Small (1975b) and later extended by Booker, Carter and Small (1976), used to obtain finite element solutions for incompressible materials, has been applied to this problem. The results of this analysis are presented in Fig. 5.37 where the vertical displacement of the point of intersection of the centreline and the surface is plotted as a function of the Poisson's ratio, ν, of the soil material. Also shown in the figure is the exact answer for ν = 0.5. It is obvious from this figure that the solution changes rapidly with changes in ν in the neighbourhood of ν = 0.5, e.g. the solutions for ν = 0.48 and ν = 0.5 differ by about 33%.
5.3.1 (ii) Finite Deformation Solution

The problem described in Fig. 5.35 was analysed using the numerical method for finite deformations, presented earlier in chapter 4, for the particular case of L/B = 5 and H/B = 2.5. Finite deformation solutions were obtained for both (a) the approximation to incompressibility, i.e. \( \nu = 0.48 \) and (b) the value of \( \nu = 0.5 \). The latter was obtained by a simple extension of the method of Booker and Small (1975b) for infinitesimal elastic problems. For finite deformation problems the extension is quite straightforward, involving the use of the Booker, Small technique in each iteration of the nonlinear solution. Further details are not given here.

The results of these finite deformation analyses are presented in Fig. 5.38. Of immediate note is the fact that at any given finite load there corresponds a finite settlement. These results are by no means conclusive but it would seem that the tendency for infinite 'strains' at the surface where \( G = 0 \) has been offset by the restraining action of the rest of the material which has finite \( G \). Of note also is the difference between the curves for \( \nu = 0.48 \) and \( \nu = 0.5 \). The discrepancy becomes more serious as the traction magnitude is increased. As expected a finite deformation analysis is more important in this particular case of material inhomogeneity than it is for a layer of homogeneous elastic material, c.f. Fig. 5.38 and Fig. 5.16. However, even when the displacement is about 25% of the footing width, the load by finite strain theory is still only about 30% greater than that by Gibson's infinitesimal theory. For smaller displacements, within the usual range of practical interest, the infinitesimal theory is clearly accurate enough, provided the assumption of elasticity is valid.
The Lagrangian finite element mesh used to obtain these results is shown in Fig. 5.4 at its initial configuration when \( q = 0 \). In Fig. 5.39 the surface profiles at selected load levels are presented for the layer with \( v = 0.5 \). We note that at low levels of load the position of the boundary at \( L = 5B \) has little effect on the solution which is almost the same as that for a layer of infinite lateral extent. However, this may not be the case as the magnitude of traction is increased.

5.3.2 Traction Loading on an Elastoplastic Layer

So far in section 5.3 we have considered the inhomogeneous material to be entirely elastic, throughout the entire range of loading. Consider now the traction loading of a material of finite strength. The linear variation of cohesion with depth and other aspects of the problem are defined in Fig. 5.40. The finite element mesh used to represent the layer is shown in Fig. 5.5. Notice that this mesh differs from those used to represent a homogeneous material because of the inclusion of more elements at finer vertical spacings near the surface of the layer. This is in order to more accurately represent the change in material properties at shallow depths.

5.3.2 (i) Infinitesimal Analysis

Several authors have presented solutions for the bearing capacity of footings on inhomogeneous layers of this type. The works of Davis and Booker (1973b) and later Salençon, Florentin and Gabriel (1976) are notable examples. Both of these papers considered a rigid footing. Nevertheless, their solutions may have relevance to this
problem. We consider first the load deformation relationship obtained using infinitesimal theory which is plotted in Fig. 5.41 for the case when $E_o/c_o = 100$, $E_o/3mB = 100$, $c_o/pB = 1$, and $v = 0.48$. These parameters indicate only a moderate increase in strength with depth. The numerical solution asymptotes to a value of applied traction equal to about $8c_o$. Also shown in Fig. 5.41 are the analytic collapse loads, determined by Davis and Booker for a smooth rigid footing $(6.6c_o)$ and a rough rigid footing $(7.7c_o)$.

5.3.2 (ii) Finite Analysis

Also plotted in Fig. 5.41 is the solution from finite deformation theory. Note, as before, that no definite collapse load may be assigned to the layer. The traction load merely continues to indent the layer as its magnitude increases. Plastic deformation will be dominant over elastic deformation when $q/c_o > 8$.

5.3.3 Rigid Footings on an Elastoplastic Layer

Consider the problem of a rigid footing resting on a layer of inhomogeneous material of this type – as described in the inset to Fig. 5.42. The results that follow were all computed using Mesh 5 of Fig. 5.5.

5.3.3 (i) Infinitesimal Analysis

Firstly, using an infinitesimal analysis, an examination is made of the effect on load settlement behaviour of the degree of inhomogeneity and of the predictive ability of the numerical technique for
such materials.

Two classes of inhomogeneity are investigated. These are the cases in which the surface modulus and strength are both either zero or non-zero at the surface. In addition we consider cases of rough and smooth rigid footings.

(a) Non-zero surface modulus and strength:

The case in which \( E_o/c_o = 100 \) was examined. Fig. 5.42 shows the results of infinitesimal analyses for a smooth footing on a layer with \( E_o/3\mu B = 100 \) and \( c_o/\mu B = 1 \). These parameters represent only a moderate increase in modulus and cohesion with depth and the average pressure at collapse is given analytically by Davis and Booker as \( 6.6c_o \). This represents an increase of about 30% over \( (2 + \pi)c_o \), the bearing capacity for a homogeneous clay with strength \( c_o \). The two curves shown on Fig. 5.42 correspond to the two cases of slip and no slip along the line MM' in mesh 5. For the smooth footing it can be seen that an allowance for rupture lowers the curve considerably. It should further be noted that even with this difference in response both curves show no definite tendency to asymptote horizontally to a collapse value. The numerical calculations have proceeded well beyond the Davis-Booker collapse value.

This situation is repeated in the case of a rough footing on the same material – see Fig. 5.43. The plasticity collapse value in the rough case is \( 7.7c_o \) but again this is exceeded by the numerical results. The difference between slip and no slip is not as significant in this case as it was for the smooth footing.
Similar results were found when the rate of increase in strength and modulus was made greater, i.e. $E_o/3mB = 10$ and $c_o/\rho B = 0.1$. From Davis and Booker (1973b) the smooth and rough bearing capacities for the new material are then $14.7c_o$ and $17.8c_o$ respectively. Numerical curves for these footings are shown on Figs. 5.44 and 5.45 respectively. Note, that in this case of greater inhomogeneity, the discrepancy between the numerical results and the analytical collapse values is greater than before.

(b) Zero surface modulus and strength:

The problem of a rigid footing resting on a layer for which $E_o = c_o = 0$ is now studied. The problem is completely defined by specifying the strength and modulus at a depth below the surface equal to the footing half width $B$. These are $E/3mB = 1$ and $c/\rho B = 1$.

Consider first the collapse loads that apply for such a material. Davis and Booker quote the analytic value for average pressure at $0.5\rho B$ for both the rough and smooth case. It must be remembered, however, that in a finite element analysis a value of zero may not be assigned to the modulus nor, in this particular case when $\phi = 0$, to the cohesion $c$. Hence for the finite element calculations the strength and modulus variation is modelled as shown in Fig. 5.46. The presence of a finite value of cohesion in the top layer of elements may contribute to the numerically calculated bearing capacity. There is some argument then in favour of suggesting that such a model for the strength inhomogeneity does not truly represent a layer with zero $c_o$ at all, but perhaps a layer with a crust value as described by Davis and Booker (1973b). For the present crust problem these authors have determined the following average footing pressures at collapse: rough =
0.91μB; smooth = 0.71μB.

Load-displacement curves for this problem were obtained allowing slip along the line MM' and these are plotted in Fig. 5.47. In the case of the smooth footing the initial linear-elastic portion of the curve differs from the prediction due to Gibson et al. (1971) by about 15%. This discrepancy is thought to be due mainly to the use of ν = 0.48 as an approximation for incompressibility. (The effect of ν in problems of this type was discussed earlier - see also Fig. 5.37) Note that solutions for both rough and smooth footings indicate a collapse load far in excess of the no crust value. In the smooth case the numerical value is in reasonable agreement with the crust answer, being about 10% above. The numerical value for the rough case is about 30% too large. Again this may reflect the presence of the extra restrictive boundary condition for a rough footing (Nagtegaal et al., 1974).

In conclusion it may be said that an inability to numerically produce satisfactory small strain solutions to the problem of a footing on an inhomogeneous layer has been demonstrated. The greater is the degree of inhomogeneity then the worse, it seems, are the numerical solutions. The type of mesh used may have been a factor contributing to this inability but the further expense of using a finer mesh and computation scheme was not possible and perhaps not even justified. There is some evidence to suggest that the inability of present techniques to model the stress singularity at the edge of the strip footing may be a contributing factor. In all cases studied the solution that allowed for slip gave a better result than that without but obviously did not give an entirely satisfactory improvement.
5.3.3 (ii) Finite Analysis

In the light of the above mentioned inadequacies of the finite element method for such infinitesimal analyses it was considered that a detailed examination of the finite deformation behaviour of these materials was not warranted. Instead we present only one such solution for a rough, rigid footing resting on the same material as in the previous example. The results obtained allowing slip along MM' are presented in Fig. 5.48. Again the finite deformation curve lies above the infinitesimal solution and it indicates no collapse load. Note also the coincidence of the two curves in their linear portion at lower loads. Because of the numerical inability to assign a value of zero exactly to $E$ in a finite element analysis these results are by no means conclusive but together with the results of Fig. 5.38 these large strain solutions indicate what might be the real behaviour of a Gibson material. As expected for such a material, yielding begins soon after the load is given a finite value, because $c_o$ also has a zero (or very small crust) value - see Fig. 5.49. Despite the occurrence of the yielded zones, the overall load-deformation relationship is linear for a considerable portion of its history. This is no doubt due to the restraining effect of the surrounding, elastic elements (material). This linearity, even in the presence of such yielded zones, indicates that Gibson's analysis of this type of problem (e.g. Gibson, 1967; Gibson, Brown and Andrews, 1971), which considers the material as entirely elastic, is still relevant. Even though the numerical technique was incapable of giving an accurate picture of the behaviour near the Davis and Booker collapse values there is some numerical evidence in favour of accepting Gibson's analysis in the range of working loads.
5.4 EMBANKMENTS ON SOFT CLAY

Thusfar, all of the examples presented defined the applied loading by either specifying a boundary traction or a type of boundary movement. We consider now a loading of a different type, i.e. the building up of a very long embankment in stages over a soft clay layer. This problem is of some practical interest and in fact there are reported instances where traditional methods of stability and settlement analysis have not given satisfactory predictions of behaviour of these type of structures. For example, Casagrande (1960) has described what he terms "an unsolved problem of embankment stability on soft ground". Some of the more important conclusions of his study were

(i) "using only conventional soils tests and stability analyses it is not possible to carry out a satisfactory design of granular embankments of this type on soft clay"

(ii) "there exists at present no reliable theoretical or experimental method which would permit the determination of the magnitude of the earth pressure within the fill".

It is not suggested that the present method of finite deformation analysis will completely answer all of the unknowns of Casagrande's problem. However, it does allow an examination of the ideal situation, i.e. the coupled behaviour of an embankment and clay layer, which may be of use in some practical situations. Presented now is an example which indicates the general trend in behaviour and highlights the importance of performing a finite deformation analysis when self weight of the bank and clay material is considered. No account is taken of any time dependent strength or deformation be-
haviour, although this may be accomplished, as seen in the next chapter.
The successive buildup of a granular embankment on soft clay is analysed.*

The clay is considered to be an undrained, purely cohesive
material with $E_c/c_c = 50$ and $v_c = 0.48$ (to approximate incompressible
behaviour) and of initial depth $D$. The embankment material is considered
to be purely frictional with $v_s = 0.3$, $\phi_s = \psi_s = 35^\circ$ modulus $E_s$ and unit
weight $\gamma_s$, where $E_s/\gamma_s D = 125$. The ratio of the moduli of embankment
material to that of the clay layer is $E_s/E_c = 10$. (Subscripts $c$, $s$
refer to the clay and the frictional material respectively).

The initial half-base width of the embankment was assumed to
be $0.6D$ and construction of the embankment, having a side slope of $30^\circ$,
was assumed to take place in a number of finite lifts. This was modelled
numerically, at any stage, by adding a row of elements above the current
crest level, then applying the gravity load to these elements. The new
row of elements were added so that before gravity was applied they produced
a horizontal crest. Numerical computation was achieved using the ap-
proximation scheme B.

Fig. 5.50 shows the elevation of the crest centre (above the
rigid base of the clay layer) at various stages during construction and
for various ratios of density of embankment material to initial density
of clay. In all cases the clay was in a state of hydrostatic total
stress prior to embankment loading. The separation of the curves at
higher bank weights occurs for similar reasons to those that explained
the difference in load-deflection behaviour of rigid footings when clay
self weight was included. i.e. when heave occurs in the clay layer it
requires more load (energy) to heave a ponderable clay than a weightless

* The computer program used for problems such as this, in which material
is added to the boundary, has been verified previously with the prob-
lem of one-dimensional deposition - see section 4.3.3.
For low values of $\gamma_c/\gamma_s$ there is a critical load (weight of added fill material) beyond which any addition of fill to the crest of the embankment causes a settlement greater than the height of material added. For larger values of $\gamma_c/\gamma_s$, crest elevation is limited only by the production of a triangular apex in the embankment cross-section. It should be noted that this feature is not only a function of $\gamma_c/\gamma_s$ but also of the strength and deformation properties of the clay material as well as the geometry of the bank section. For example, in the case of a wider bank built on similar material the 'critical load' feature will occur at more realistic values of the ratio $\gamma_c/\gamma_s$, i.e. $\gamma_c/\gamma_s > 0.5$ or different values of $E_s/\gamma_s D$.

For the current problem, some idea of the deformation history of the embankment and clay layer may be obtained from Fig. 5.51(a) to (d), for the case when $\gamma_c/\gamma_s = 1$. In this figure the finite element mesh is shown at various stages of construction.

It may be of some interest to note that, if an infinitesimal analysis or a finite analysis which neglected the importance of the initial stress state (e.g. scheme E) was performed for this problem, then the four curves of Fig. 5.50 would all coincide. (The coincidence would be with each other and not necessarily with any of the present solutions). The density of the clay layer would have no influence at all on these results. Moreover, there would be no effect on the pattern of yielding in an infinitesimal analysis because initially $K_o = 1$. Indeed, for the solutions of Fig. 5.50 the four cases nearly coincide in the early stages of construction with each tending to a
common solution at zero embankment load.

Figures 5.52(a) and (b) illustrate the 'critical load' feature mentioned above. For the case when \( \gamma_c / \gamma_s \rightarrow 0 \) finite element mesh pictures are shown at two stages during the construction. The first is at a load of \( W/(\gamma_s D^2) = 0.144 \) when the crest elevation is at about 1.1D. The second is after more load has been added to bring \( W/(\gamma_s D^2) \) up to 0.183 but sufficient deformation has occurred so that the crest has settled back to about the same elevation of 1.1D, having been slightly higher for intermediate loads.

5.5 DISCUSSION

For typical soil mechanics problems no general statement can be made about whether a finite deformation analysis gives predictions of behaviour which are more or less optimistic than those of the conventional infinitesimal analysis. With regard to load-deformation behaviour it can be said that less optimistic predictions are obtained using the infinitesimal theory for problems such as line loading, traction loading, and rigid footing loading on both homogeneous and inhomogeneous layers. In such problems a collapse load is indicated by infinitesimal theory while no such limiting load is indicated by finite deformation theory. In other problems, e.g. the slope stability studies of Snitbahn and Chen (1976) and the problem, discussed in chapter 6, of a footing near a vertical cut, the finite deformation solutions may be more critical. For example, while a true collapse load may not be predicted by finite deformation theory, the displacements may go beyond the working engineering range at loads below that predicted by the infinitesimal theory.
It has also been shown that in the problem of embankment building over soft clay a finite deformation analysis was essential in order to obtain a true picture of the behaviour.
Fig 5.1 MESH 1 AT ZERO TRACTION LOAD.
Footing \{ 
  \text{Perfectly flexible-traction } q \\
  \text{Perfectly rigid - total load } P \\
\} \\

\text{MM}' may be a potential rupture surface \\
\text{(i.e. dual nodes)}

Fig 5.2  \text{FINITE ELEMENT MESH 2}
Perfectly flexible - traction $q$

Perfectly rigid - total load $P$

Footing

$\text{MM}'$ may be a potential rupture surface
(i.e. dual nodes)

Fig 5.3 FINITE ELEMENT MESH 3.
Fig 5.4 FINITE ELEMENT MESH 4.
Perfectly flexible - traction $q$
Perfectly rigid - total load $P$

MM' may be potential rupture surface (i.e., dual nodes)

Fig 5.5 FINITE ELEMENT MESH 5.
Fig 5.6 LINE LOAD—PROBLEM DEFINITION
Fig 5.7 VERTICAL SURFACE DISPLACEMENT AT X = 0.1H
Fig 5.8 COMPARISON OF SURFACE DISPLACEMENT.

INFINITESIMAL ANALYSIS.
Fig 5.9 COMPARISON OF VERTICAL STRESS NEAR CENTRE-LINE INFINITESIMAL ANALYSIS.
Fig 5.10 LINE LOAD PROBLEM - MESH GEOMETRY.
Fig 5.11 TRACTION LOADING PROBLEM.
Fig 5.12 MODIFIED TRACTION LOADING PROBLEM.
\[ \frac{H}{B} = 10 \]

\[ w = \frac{2Bq}{E} I_c \]

Fig 5.13 DISPLACEMENT INFLUENCE FACTORS FOR STRIP LOADING.
Fig 5.14 STRESS COMPONENTS - TRACTION LOADING
INFINITESIMAL ANALYSIS.
Fig 5.15 TRACTION LOADING - VERTICAL DISPLACEMENTS.
Fig 5.16 TRACTION LOADING ON ELASTIC MATERIAL.
Fig 5.17 TRACTION LOADING ON ELASTIC MATERIAL.
$E/c = 100$

$\sqrt{ } = 0.48$

$\phi = 0$

Fig 5.18 TRACTION LOADING - MESH EFFECT

INFINITESIMAL ANALYSIS.
Finite Analysis

Infinitesimal Analysis

E/c = 100
ν = 0.48
ϕ = 0

Fig 5.19 TRACTION LOADING ON HOMOGENEOUS MATERIAL.
$v = 0.48$
\[\phi = 0\]

Fig 5.20 TRACTION LOADING - EFFECT OF $E/c$. 
Fig 5.21 TRACTION LOADING - EFFECT OF $E/c$. 

$\sqrt{V} = 0.48$

$\phi = 0$

$E/c = 100$

$(2 + \pi)$

Infinitiesinal Analysis for all finite $E/c$

Finite Analysis
FIG. 5.22 DEVELOPMENT OF PLASTIC REGIONS, HOMOGENEOUS CASE.
Material Properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>General Symbol</th>
<th>Undrained</th>
<th>Drained</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Modulus</td>
<td>$E$</td>
<td>$E$</td>
<td>$E'$</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td>$\nu$</td>
<td>$\nu_u = 0.5$</td>
<td>$\nu' = 0.3$</td>
</tr>
<tr>
<td>Cohesion</td>
<td>$c$</td>
<td>$c_u$</td>
<td>$c'$</td>
</tr>
<tr>
<td>Friction Angle</td>
<td>$\phi$</td>
<td>$\phi_u = 0$</td>
<td>$\phi = 30^\circ$</td>
</tr>
<tr>
<td>Dilatancy Angle</td>
<td>$\psi$</td>
<td>$\psi_u = 0$</td>
<td>$\psi = 0^\circ, 30^\circ$</td>
</tr>
<tr>
<td>Bulk Density</td>
<td>$\gamma$</td>
<td>$\gamma$</td>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

Fig 5.23 RIGID FOOTING ON HOMOGENEOUS MATERIAL.
Fig. 5.24 SMOOTH RIGID FOOTING - MESH EFFECT
INFINITESIMAL ANALYSIS.
Fig 5.25 SMOOTH RIGID FOOTING ON HOMOGENEOUS MATERIAL.
Fig 5.26 SMOOTH AND ROUGH RIGID FOOTING ON HOMOGENEOUS MATERIAL—INFINITESIMAL ANALYSIS.
Fig. 5.27 DEFORMATION AND GROWTH OF PLASTIC REGION.

\[ \frac{P}{2Bc} = 5.1 \quad \frac{A}{B} = 0.19 \]

\[ \frac{P}{2Bc} = 7.2 \quad \frac{A}{B} = 0.58 \]

E/c = 100
E/c = 100
μ = 0.48
φ = ψ = 0
γ = 0

---

Fig 5.28 ROUGH RIGID FOOTING ON HOMOGENEOUS MATERIAL.
Fig 5.29 ROUGH RIGID FOOTING ON HOMOGENEOUS MATERIAL — EFFECT OF $E/c$
Fig 5.30 ROUGH RIGID FOOTING ON HOMOGENEOUS MATERIAL — EFFECT OF $E/c$. 

\[ \frac{P}{2Bc} \]

\[ \sqrt{1 + 0.48 \phi} = 100 \]

---

Infinitesimal Analysis (for all $E/c$)

Finite Analysis
Fig. 5.31 DEFORMATION AND GROWTH OF PLASTIC REGION

$\frac{P}{2Bc} = 4.8 \quad \frac{\Delta}{B} = 1 \quad \frac{P}{2Bc} = 7.3 \quad \frac{\Delta}{B} = 2$

$E/c = 10$
Fig 5.32 ROUGH RIGID FOOTING ON HOMOGENEOUS MATERIAL - DRAINED ANALYSIS.
Fig 5.33 ROUGH RIGID FOOTING - DENSITY EFFECT.
Fig 5.34 Traction loading on an inhomogeneous elastic layer.
Fig 5.35 INHOMOGENEOUS LAYER WITH LATERAL RESTRAINT.
Fig 5.36(a) UNIFORM LOAD ON AN INHOMOGENEOUS LAYER WITH LATERAL RESTRAINT.

<table>
<thead>
<tr>
<th>$H/B$</th>
<th>$L/B$</th>
<th>$w2m/q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.25</td>
<td>10.0</td>
</tr>
<tr>
<td>0.5</td>
<td>1.25</td>
<td>5.0</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>2.5</td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig 5.36(b)
\[
\frac{H}{B} = 0.2
\]
\[
\sqrt{\lambda} = 0.5
\]

Fig 5.36 (e)
Fig 5.37 CENTRAL VERTICAL DISPLACEMENT AS A FUNCTION OF POISSON'S RATIO. INHOMOGENEOUS LAYER.
Fig 5.38 TRACTION LOADING ON AN INHOMOGENEOUS ELASTIC MATERIAL.
Fig 5.39 INHOMOGENEOUS ELASTIC LAYER.

SURFACE PROFILES.
Fig 5.40 TRACTION LOADING ON AN INHOMOGENEOUS ELASTO-PLASTIC LAYER.
Fig 5.41 TRACTION LOADING ON INHOMOGEOUS LAYER.
Fig 5.42 SMOOTH RIGID FOOTING ON AN INHOMOGENEOUS LAYER - SMALL STRAIN ANALYSIS.
Fig 5.43 ROUGH RIGID FOOTING ON INHOMOGENEOUS LAYER — SMALL STRAIN ANALYSIS.
\[ \frac{E_o}{c_o} = 100 \]
\[ \frac{E_o}{3mB} = 10 \]
\[ \frac{c_o}{\rho B} = 0.1 \]
\[ \nu = 0.48 \]

**Fig 5.44 SMOOTH RIGID FOOTING ON INHOMOGENEOUS LAYER - SMALL STRAIN ANALYSIS.**
Fig 5.45 ROUGH RIGID FOOTING ON AN INHOMOGENEOUS LAYER - SMALL STRAIN ANALYSIS.
Fig 5.46 **FINITE ELEMENT MODEL OF ACTUAL STRENGTH OR MODULUS VARIATION.**
\[
\frac{E(z'=B)}{3mB} = 1
\]
\[
\frac{c(z'=B)}{\varphi B} = 1
\]
\[E_o = c_o = 0\]
\[z' = H - z\]
\[\sqrt{\varphi} = 0.48\]
\[E/c = 100\]

Rough with slip

Davis, Booker solutions

0.91
Rough, Crust

0.71
Smooth, Crust

0.5
Rough and Smooth—no crust

\[
\frac{P}{2\varphi B^2}
\]

\[z' \]

2B

\[z\]

Inhomogeneous

2OB

\[\Delta \]

OB

\[B\]

\[\frac{\Delta}{B}\]

Fig 5.47 RIGID FOOTING ON AN INHOMOGENEOUS LAYER—SMALL STRAIN ANALYSIS.
\[ z' = H - z \]
\[ \frac{E(z' = B)}{3mB} = 1 \]
\[ \frac{c(z' = B)}{\tau B} = 1 \]
\[ v = 0.48 \]
\[ E/c = 100 \]
\[ E_o = c_o = 0 \]

\[ \frac{P}{2\rho B^2} \]

\[ \text{Crust} \]
\[ 0.91 \]

\[ \text{Inhomogeneous} \]

\[ O.5 \]

\[ \text{No crust} \]
\[ (\text{Davis, Booker}) \]

Fig 5.48 ROUGH RIGID FOOTING ON AN INHOMOGENEOUS LAYER.
Fig 5.49 PLASTIC REGIONS AT VARIOUS LOADS.

<table>
<thead>
<tr>
<th>Region</th>
<th>$\frac{P}{2\phi B^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>0.45</td>
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<tr>
<td></td>
<td>0.66</td>
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<td></td>
<td>0.85</td>
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<tr>
<td></td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>1.17</td>
</tr>
<tr>
<td></td>
<td>1.30</td>
</tr>
</tbody>
</table>

$E = 3mz'$
$c = \phi z'$

$\frac{E}{c} = 100$
$\sqrt{\phi} = 0.48$

$\frac{E(z'B)}{3mB} = 1$
$\frac{c(z'B)}{\phi B} = 1$
$H = \text{Elevation of crest centre above rigid base of clay layer}$

$D = \text{Original depth of clay layer}$

$W = \text{Weight per unit length of embankment}$

$E_s / \gamma_s D = 12.5$

$\gamma_c / \gamma_s = 1$

$0.5$

$0.1$

$0$

FIG. 50 ELEVATION OF CREST CENTRE V EMBANKMENT WEIGHT
(a) Clay Layer - No Embankment

(b) \( \frac{W}{(Y_s D^2)} = 0.106 \)

(c) \( \frac{W}{(Y_s D^2)} = 0.179 \)

(d) \( \frac{W}{(Y_s D^2)} = 0.211 \)

FIG. 5.51 GROWTH OF PLASTIC REGION FOR EMBANKMENT PROBLEM \( \left( \frac{Y_c}{Y_s} = 1 \right) \)
\( W/(\gamma_s D^2) = 0.144 \)

\( W/(\gamma_s D^2) = 0.183 \)

FIG5: ILLUSTRATION OF THE 'CRITICAL LOAD' FEATURE FOR \( \gamma_c/\gamma_s = 0 \)
CHAPTER 6
FINITE CONSOLIDATION OF AN ELASTOPLASTIC SOIL
6.1 INTRODUCTION

The analysis described in the previous chapter has dealt with the finite deformation of an elastoplastic material with special reference to soils. The ideal soil was treated as a single phase material and interaction between solid, fluid and possibly gaseous phases within the soil was ignored. As such the previous analysis can only be used to predict the behaviour of soils under either totally drained or totally undrained conditions.

However, an important problem in foundation engineering is that of predicting the behaviour of a foundation resting on saturated clay and such materials consist of two phases, a compressible solid phase (the soil skeleton) and a liquid phase (the water filling the pores). When the foundation is first loaded the skeleton tends to compress and so excess pore pressures develop and the foundation undergoes an initial settlement. The pore water then tends to flow from regions of higher excess pore pressure to regions of lower excess pore pressure. As this dissipation of excess pore pressure occurs the foundation settles and ultimately reaches a final settlement.

This process of consolidation described above was first investigated by Terzaghi (1925, 1943) for one dimensional conditions. Subsequently Biot (1941a, 1941b) extended Terzaghi's theory to three dimensional situations. However, exact solutions to problems involving the consolidation of a soil are not easy to obtain. This is not surprising when it is considered that the equations of consolidation combine the complexities of an elastic or even an elastoplastic problem with those of a diffusion process. For this reason exact solutions
have been found only to problems in which the body under consideration has an elastic skeleton, a particularly simple geometry and is subject to simple boundary conditions (see for example Mandel, 1953; McNamee and Gibson, 1960a, 1960b; Gibson and McNamee, 1957; and Gibson Schiffman and Pu, 1970).

In most practical problems it is necessary to employ numerical techniques to integrate the equations of Biot's theory. Some approximate methods for the case when the soil skeleton is elastic have been developed by various authors (e.g. Sandhu and Wilson, 1969; Christian and Boehmer, 1970; Hwang, Morgenstern and Murray, 1971; Booker, 1973; and Booker and Small, 1975a). More recently, a theory of consolidation for an ideal two phase soil with an elastoplastic skeleton has been proposed, together with an approximate finite element solution technique by Small, Booker and Davis, (1976).

In the formulations of Terzaghi and Biot the authors restricted their attention to conditions of infinitesimal strain and thus the theory they developed is only strictly applicable to situations in which the geometry varies only slightly during loading. Gibson, England and Hussey (1967) recognised this limitation and developed a one dimensional theory which accounted for such finite deformation. Mesri and Rokhsar (1974) and DeSimone and Viggiani (1976) have also included some account for finite strain in their numerical treatment of one-dimensional consolidation. Smiles and Poulos (1969) examined the one-dimensional problem with no restriction on the magnitude of strain and an allowance for the variation in flow parameters with variation in void ratio, in an endeavour to explain the phenomenon of secondary consolidation.
Recently a paper appeared in which a mathematical model was developed to represent the one-dimensional large strain consolidation of a fully saturated clay (Monte and Krizek, 1976). The model, which closely follows that of Gibson et al., was applied to a material in the stress range in which a 'slurry' is transformed to a 'soil'. Experimental results from a series of permeability tests further suggest that the relationship between the logarithm of the coefficient of permeability and the void ratio is not a straight line for the entire range of void ratio considered. The authors also indicate, for the particular clay investigated, the effective stress range in which it was found that the classical small strain theory can adequately describe the deformation-time response.

In this chapter the finite deformation analysis for single phase materials, presented in previous chapters, is extended to obtain an incremental analysis for both the soil skeleton and the pore fluid while taking into account the coupling of the two processes. As before a quasi-static approach is adopted. The theory of finite consolidation is generalised from one to three dimensional conditions. Governing equations are cast in a rate form and the laws which determine deformation and pore fluid flow are presented in a frame indifferent manner. The analysis presented is applicable to an ideal elastic, perfectly plastic soil with a general yield criterion and an arbitrary flow law. A numerical technique is described that provides an approximate solution to the governing equations. The theory and solution technique are illustrated by several examples of practical interest.

It is also shown that in the limit of small deformation the governing equations reduce to those of the familiar Biot theory for a
soil with an elastic skeleton, and to those of the theory of Small et al. for a soil with an elastoplastic skeleton.

6.2 GOVERNING EQUATIONS

The basic equations governing the coupled behaviour of an elastoplastic soil skeleton and its pore fluid are now presented.

6.2.1 Effective Stress-Strain Behaviour

As before, we consider a typical increment in the motion of the consolidating body as depicted by the deformation mapping of Fig. 6.1. Suppose that at some time $t_0$ a consolidating soil occupies a region in space $V_0$ bounded by the surface $S_0$. Part of this surface $S_{OT}$ is subject to specified traction $T_{oi}$ while on the remainder $S_{OD}$ the velocities are assumed to be zero. Portion of the surface $S_{OP}$ is free to drain, the remainder is impermeable.* At some later time $t$ the body will have moved to a region $V$ bounded by a surface $S$. The traction specified, velocity specified, permeable and impermeable portions of $S$ will be denoted by $S_T$, $S_D$, $S_P$, $S_I$ respectively.

Again, adopting a Cartesian reference frame, consider the deformation mapping

$$x_i = a_i + u_i \quad i = 1, 2, 3$$

(6.1)

where $a_i$ and $x_i$ are the coordinates of a specified material point of the body.

*The extension to more complicated boundary conditions, both hydraulic and with regard to the body's motion, is straightforward and will not be given here.
soil skeleton at times $t_0$ and $t$ respectively and $u_i$ represents the displacement of this solid particle in the time interval $t_0$ to $t$, measured relative to the position of the body at time $t_0$.

In an Eulerian description, the instantaneous rate of deformation may be described by the velocity gradient,

$$\frac{\partial v_{si}}{\partial x_j} = l_{ij} + \omega_{ij} \tag{6.2}$$

where

$$v_{si} = \frac{\partial u_i(a_k, t)}{\partial t}$$

$$= \frac{\partial u_i(x_k, t)}{\partial t} \quad k = 1, 2, 3 \tag{6.3}$$

is the velocity of the soil skeleton.

The symmetric deformation rate tensor $l_{ij}$ and the skew symmetric spin tensor $\omega_{ij}$ are defined in chapter 2 and, for convenience, are repeated here

$$l_{ij} = \frac{1}{2} \left( \frac{\partial v_{si}}{\partial x_j} + \frac{\partial v_{sj}}{\partial x_i} \right) \tag{2.23}$$

$$\omega_{ij} = \frac{1}{2} \left( \frac{\partial v_{si}}{\partial x_j} - \frac{\partial v_{sj}}{\partial x_i} \right) \tag{2.24}$$

In earlier chapters the necessity to employ a frame indifferent stress rate in the constitutive laws has been discussed. In the consolidation analysis presented here the stress rate due to Jaumann will be used (as before). We repeat its definition,

$$\dot{\sigma}_{ij} = \sigma_{ij} - \sigma_{ik} \omega_{jk} - \sigma_{jk} \omega_{ik} \tag{2.26}$$
where $\sigma_{ij}$ denotes the Cauchy total stress field at time $t$ with tensile stresses reckoned positive.

A general linear relationship between the objective stress rate and the deformation rate (i.e. the effective stress-strain law) can be written in the form,

$$\dot{\sigma}_{ij} + p\delta_{ij} = D_{ijkl}\dot{\varepsilon}^{kl} \tag{6.4}$$

where $p$ is the pore pressure at time $t$ (taken positive when compressive),

$$\delta_{ij}$$ is the Kronecker delta, and

$D_{ijkl}$ are the material constants for the drained behaviour of the soil.

These 'constants' $D_{ijkl}$ may of course be a function of both position and time. By the use of a piecewise linearity of their values any combination of experimentally observed and intuitively assumed skeleton behavior may be analysed by this theory. Such multi-linear behaviour is readily incorporated into the numerical technique described later. In particular, the expression for $D_{ijkl}$ for a skeleton material which obeys a Mohr-Coulomb yield criterion and a non-associated flow rule are given in chapter 3.

An alternative form of equation (6.4) useful in subsequent developments is

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}^{kl} + \sigma_{ik}\dot{\varepsilon}^{jk} + \sigma_{jk}\dot{\varepsilon}^{ik} - p\delta_{ij} \tag{6.5}$$
6.2.2 Fluid Flow Behaviour

It will be assumed that the movement of the fluid through the soil is governed by Darcy's law (1856) but, as observed by Gibson et al. (1967) some care is necessary in formulating this in a consistent form. Thus if the fluid has an actual velocity \( v_{fi} \), then the superficial velocity of the fluid relative to the skeleton is \( \alpha(v_{fi} - v_{si}) \), where \( \alpha \) is the soil porosity in the neighbourhood of \( x_i \) at time \( t \). This superficial velocity is proportional to the hydraulic gradient, i.e.

\[
\alpha(v_{fi} - v_{si}) = -k_{ij} \frac{\partial h}{\partial x_j}
\]  
(6.6)

where \( h = \frac{P}{\gamma_f} + x_k b_k \),

\( \gamma_f \) is the unit weight of the pore fluid,

\( k_{ij} \) are the permeability coefficients which depend upon position and time, and

\( b_k \) are the components of a unit vector parallel to the direction of gravity.

Some care is also necessary in the definition of the permeability coefficients and this matter is discussed in Appendix 6A.

6.2.3 Mass Flow for Solid and Fluid Phases

Consider a physical element of the soil skeleton which has unit weight and porosity (\( \gamma_s, \alpha \)) at time \( t \). Conservation of mass leads to the equation
\[
\frac{d}{dt} \left\{ \frac{\gamma_s}{g} (1-\alpha) \right\} + \theta \frac{\gamma_s}{g} (1-\alpha) = 0 \tag{6.7}
\]

where \( \theta = \dot{\varepsilon}_{ii} \) is the rate of volume strain.

If it is supposed that the material of the soil skeleton is much less compressible than the soil consisting of both the solid and fluid phases, then \( \gamma_s \) is constant, so that

\[
\dot{\theta} = \frac{\alpha}{(1-\alpha)} \tag{6.8}
\]

Similarly, considering the mass flow of fluid into and out of a specified physical element, with velocity \( v_{fi} \) and unit weight \( \gamma_f \) at time \( t \), then it is found that

\[
\frac{d}{dt} \left\{ \frac{\gamma_f}{g} \alpha \right\} + \theta \frac{\gamma_f}{g} \alpha = -\frac{\partial}{\partial x_i} \left\{ \frac{\gamma_f}{g} \alpha (v_{fi} - v_{si}) \right\} \tag{6.9}
\]

Again assuming that the fluid is much less compressible than the two phase soil, then

\[
\dot{\alpha} + \alpha \dot{\theta} = -\frac{\partial}{\partial x_i} \left\{ \alpha (v_{fi} - v_{si}) \right\} \tag{6.10}
\]

Equations (6.8, 6.10) may be combined to obtain the overall volume behaviour of the soil

\[
\theta = -\frac{\partial}{\partial x_i} \left\{ \alpha (v_{fi} - v_{si}) \right\} \tag{6.11}
\]
### 6.2.4 Virtual Work Expressions

The total stress distribution within the soil must always satisfy the conditions of equilibrium, so that at time $t$

$$
\frac{\partial}{\partial x_j} \{\sigma_{ij}\} + F_i = 0 \quad (6.12)
$$

where $F_i = \{\gamma_s (1-\alpha) + \gamma_f \alpha\} b_i$ is the body force vector.

For our purposes a more convenient form of equation (6.12) incorporating the stress boundary conditions is the expression of virtual work, equation (2.31). When the rate law, equation (6.5), for the two phase soil is introduced into (2.31) it becomes

$$
\int_{V_{ij}} \int_{t_0}^{t} \left[ f_{ijkl} \delta_{kl} + \sigma_{ik} \omega_{jk} + \sigma_{jk} \omega_{ik} \right] dt - (p-p_0) \delta_{ij} \\text{d}V = R_i \quad (6.13)
$$

where $R_i = \int_{V} \sigma_{ij} \text{d}V + \int_{S_T} \text{d}S \text{d}S - \int_{S_i} \sigma_{ij} \text{d}V$

$\sigma_{ij}$, $p_0$ are the total stress and pore pressure distributions within $V$, at time $t$, and

$\text{d}v_{si}$ are the virtual velocities of the soil skeleton, which are compatible with the virtual deformation rates $\delta_{ij}$ and the velocity boundary conditions on $S_D$.

The remaining symbols of equation (6.13) have their usual meaning.

Similarly the volume behaviour, equation (6.11), can be replaced by the integral formulation,
\[ f \left[ \alpha (v_{fi} - v_{si}) \frac{\partial}{\partial x_i} \{dp\} - \theta dp \right] dv = 0 \quad (6.14a) \]

which on introduction of Darcy's law becomes,

\[ f \left\{ \frac{\partial}{\partial x_j} \{h\} k_{ji} \frac{\partial}{\partial x_i} \{dp\} + \theta dp \right\} dv = 0 \quad (6.14b) \]

where virtual pore pressure changes are consistent with the boundary conditions on \( S_p \).

### 6.2.5 The Limit of Infinitesimal Deformation

It has been shown in section 2.5 that in the limit, as the magnitude of the deformation is reduced to zero, then the governing equations of the finite theory for a single phase material reduce to the more familiar forms of the infinitesimal theory. The aim here is to demonstrate this feature in the case of a two phase material.

The assumption that negligible change in the original geometry occurs during the consolidation process results in the reduction of (6.13) to,

\[ \int_{v_o}^{t} \left\{ \int_{i,j} \frac{F_i dV}{v_o} \right\} dt - (p - p_o) \delta_{ij} \int_{v_o}^{t} dv_o \]

\[ = \int_{v_o}^{t} dv_i F_i dv_o + \int_{v_o}^{t} dv_i S_{OT} \]

\[ \quad \text{bis} \]

The same assumption implies that Darcy's law takes the form

\[ v_{fi} = -\frac{k_{ij} \frac{\partial p}{\partial x_j}}{\gamma_f} \]
where now $p$ represents the excess pore pressure over values at $t = 0$.

We may thus imply that equation (6.14) reduces to

$$\int V_0 \left\{ \frac{\partial}{\partial x_j} [p] k_{oij} \frac{\partial}{\partial x_i} [dp] + \theta dp \right\} \, dv = 0 \quad (6.14)_{\text{bis}}$$

where $k_{oij}$ is the tensor of permeabilities at time $t = 0$.

Equations (6.13) bis and (6.14) bis are simply the governing equations of the infinitesimal theory due to Small et al. (1976).

If the consolidating soil has an elastic skeleton then the constants $D_{ijkl}$ are independent of time. The time integration may now be performed over the range $(0, t)$ to obtain,

$$\int V_0 \left( \sum_{ijkl} D_{ijkl} e_{ki} e_{lj} - p \delta_{ij} \right) \, dv = \int V_0 \left( \sum_{ijkl} \delta_{ij} u_{ik} u_{lj} + u_{ij} T_{ij} T_{ij} \right) \, dv$$

where $e_{ij}$ is the Cauchy strain tensor and $u_{ij}$ are the incremental displacements compatible with $e_{ij}$ and the displacement boundary conditions.

For this material the governing equation thus reduce to those of the Biot theory.

6.3 APPROXIMATE SOLUTION

Equations (6.13, 6.14) are exact expressions governing the finite consolidation behaviour of an elastoplastic soil. It is not, in general, possible to find closed form solutions to these equations. However, numerical solutions may be found, using the finite element technique to perform the spatial integrations and a marching process to perform the time integration. The numerical solution method is a
direct extension of that used for single phase soils, described in chapter 4.

In developing the numerical formulation it is convenient to use the notation introduced in Appendix 2A. The essential equations are now summarised.

Adopting the cartesian reference frame \((X, Y, Z)\), let the field quantities \(u_i, v_{si}, v_{fi}\) be the components of vectors \(u_s, v_s, v_f\) respectively where

\[
\begin{align*}
\bar{u}^T &= (u_x, u_y, u_z) \\
\bar{v}_s^T &= (v_{sx}, v_{sy}, v_{sz}) \\
\bar{v}_f^T &= (v_{fx}, v_{fy}, v_{fz})
\end{align*}
\]

For convenience we have replaced subscripts \(i = 1, 2, 3\) by \(x, y, z\) respectively. Since \(\sigma_{ij}\) is symmetric we define \(\bar{\sigma}\) a vector of stress components as

\[
\bar{\sigma}^T = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})
\]

Noting the symmetry and skew symmetry of \(\bar{\ell}_{ij}\) and \(\bar{\omega}_{ij}\), we define vectors of deformation rate \(\bar{\ell}\) and \(\bar{\omega}\) as

\[
\begin{align*}
\bar{\ell} &= \bar{\partial} \bar{v}_s \\
\bar{\omega} &= \bar{\xi} \bar{v}_s
\end{align*}
\]

where the operators \(\bar{\partial}\) and \(\bar{\xi}\) are defined in Appendix 2A. In this notation the rate law, equation (6.5), is expressible as,

\[
\bar{\sigma} = \bar{Pd} - \bar{p} \bar{\eta}
\]  
(6.15)
where \( \mathbf{\dot{d}}^T = (\mathbf{\dot{l}}^T, \mathbf{\omega}^T) \)
\[ \mathbf{n}^T = (1, 1, 1, 0, 0, 0) \]

The matrix \( \mathbf{P} \) is also previously defined (Chapters 2 and 3) for an elastoplastic material. We note here that the quantity \( q \) of the rate law, equation (2A.8c), which was included in the general analysis of chapter 2 to allow for such features as an effective stress law, is thus given by

\[ q = -p \mathbf{n} \]  \( \quad (6.16) \)

In this notation Darcy's law takes the form

\[ \alpha (\mathbf{v}_f - \mathbf{v}_s) = -K \mathbf{\nabla} h \]  \( \quad (6.17) \)

where \( K \) is the matrix of permeabilities \( k_{ij} \).

The governing equations of the previous section thus become,

\[ \int_{V} \frac{d}{dt} \left\{ \mathbf{l}^T \left[ \int_{0}^{t} \mathbf{P} d\mathbf{t} - (\mathbf{P} - \mathbf{P}_0) \mathbf{n} \right] d\mathbf{V} = \mathbf{R} \]  \( \quad (6.18) \)

and

\[ \int_{V} \left\{ (\mathbf{\nabla} h)^T \mathbf{T} \left[ (\mathbf{\nabla} \mathbf{p}) + \mathbf{0} \mathbf{d} \mathbf{p} \right] \right\} d\mathbf{V} = 0 \]  \( \quad (6.19) \)

where \( \mathbf{R} = \int_{V} \mathbf{d} \mathbf{v}^T \mathbf{F} d\mathbf{V} + \int_{V} \mathbf{d} \mathbf{v}^T \mathbf{T} d\mathbf{v} - \int_{S} \mathbf{d} \mathbf{l}^T \mathbf{G} d\mathbf{V} \)

and \( \mathbf{T}, \mathbf{F}, \mathbf{G} \) correspond to \( \mathbf{T}_i, \mathbf{F}_i \) and \( \mathbf{G}_{oij} \) respectively.

The problem described by equations (6.18, 6.19) can be solved
approximately as follows:

(i) Suppose that the deforming body is represented by a number of finite elements and that the continuous displacement and pore pressure fields can adequately be described by their values at the connecting nodes 1, 2, ..., N and let,

\[ \Delta \dot{\delta}^T = \delta^T(t) - \delta^T(t_0) = (u^T_1, u^T_2, ..., u^T_N) \]

\[ \dot{q}^T = q^T(t) = (p_1, p_2, ..., p_N) \]

The subscripts in the above definitions refer to values at a particular node and note that \( q = q(t) \) represents the nodal pore pressures at time \( t \) while \( \dot{\delta} = \dot{\delta}(t) \) represents the total nodal displacements in the time interval \( (0, t) \).

(ii) Suppose that the continuous fields \( v_s \) and \( p \) can be adequately approximated in terms of nodal values so that

\[ v_s = A \dot{\delta} = \dot{A} \delta \] \hspace{1cm} (6.20a)

\[ p = a^T q \] \hspace{1cm} (6.20b)

where the form of \( A \) and \( a \) depend upon the particular type of element used and will in general depend upon its current position.

(iii) In terms of nodal quantities, the velocities and pore pressure gradients may be written,

\[ \dot{\delta} = B \delta \] \hspace{1cm} (6.21a)
\[ \dot{q} = c\delta \]  
\[ \theta = N^T\delta \]  
\[ \varphi \mu = E\varphi \]  
\[ \varphi h = \left( \frac{1}{\gamma_f} \right) E\varphi + i_q \]  

where \[ B = \begin{pmatrix} 2 \end{pmatrix} A, \quad C = \partial A, \quad N^T = \eta^TC \]  
\[ E^T = \begin{pmatrix} \partial_a \partial_a \partial_a \\ \partial_x \partial_y \partial_z \end{pmatrix} \]  

and \( i_q \) is the vector containing the terms \( b_i \).

(iv) Equations (6.18, 6.19) can now be approximated by

\[ \dot{\delta}^T \int_{\theta}^{t} C^T \{ \int_{\varphi}^{t} \rho B \dot{\delta} dt \} dV - \dot{\delta}^T L \Delta \varphi = \dot{\delta}^T \mu \]  
(6.23)

- \[ \dot{\varphi}^T L \delta \]  
- \[ \dot{\varphi}^T \phi \varphi = \dot{\varphi}^T \nu \]  
(6.24)

where \( \Delta \varphi = \varphi(t) - \varphi(t_0) \)

\[ L^T = \int_{N} a^T dV \]  
\[ \phi = \frac{1}{\gamma_f} \int_{E^T \kappa E} dV \]  

\[ m = \int (A^T - C^T \sigma) dV + \int \Lambda^T \sigma dS \]  
\[ n = \int E^T \kappa I^T i_q dV \]  

Equations (6.23, 6.24) apply for arbitrary \( \dot{\delta}^T \) and \( \dot{\varphi}^T \), hence the set of approximating equations becomes
\[ \int_{C^T}^{t} \{ \int_{V_t} P \delta \psi \, dt \} \, dv - L^T \Delta q = m \]  
\[ - \dot{\bar{\psi}} - q \dot{\bar{\Phi}} = \bar{n} \]  

6.3.1 Numerical Method

Equations (6.25, 6.26) are a set of differential-integral equations and they may be integrated from \( t_0 \) to \( t \) to obtain the following approximation

\[ \begin{bmatrix} \bar{\varphi} \\ -\bar{L}^T \\ -\bar{L} \end{bmatrix} \begin{bmatrix} \Delta \bar{\psi} \\ \Delta \bar{\delta} \\ \Delta \bar{q} \end{bmatrix} = \begin{bmatrix} m \\ \bar{n} \Delta t + \bar{\Phi} q_0 \Delta t \end{bmatrix} \]  

(6.27)

where \( \bar{\varphi} = \int_{V} (C^T P \bar{\psi}) \, dv \), \( \Delta t = t-t_0 \), \( q_0 = q(t_0) \)

The superior bar denotes that the quantity is evaluated for some average or representative value, integrations being performed over some representative configuration. It can be seen from equation (6.27) that if the solution is known at a time \( t_0 \), it can be marched forward to obtain the solution at \( t_0 + \Delta t \), however, it should be noted that since \( \bar{\varphi}, \bar{L}, \bar{\Phi}, \bar{n}, \bar{\bar{n}} \) may all contain average quantities it may be necessary to solve equations (6.27) iteratively for each time step.

The parameter \( \beta \) corresponds to the approximation

\[ \int_{t_0}^{t} \Phi \delta \psi \, dt = \bar{\Phi} (q_0 + \beta \Delta q) \Delta t \]

In order to ensure stability of the marching process it is necessary to choose \( \beta > \frac{1}{2} \) (Booker and Small, 1975a).
6.4 EXAMPLES

A number of examples are now presented to illustrate the foregoing theory. The finite consolidation of two separate classes of soil are considered. In the first instance the soil is considered as an elastic two phase continuum and in the latter it is assumed that the skeleton is an ideal elastoplastic material obeying a Mohr-Coulomb yield criterion and a non-associated flow rule. In both cases the soil is assumed to be isotropic and homogeneous with regard to both deformation and flow properties.

Both one and two dimensional examples are presented. For the plane strain case the matrix P of the constitutive equation (6.15) is defined in chapter 3 for both materials.

6.4.1 Finite Elastic Consolidation

6.4.1 (i) One-dimensional Finite Elastic Consolidation

The problem of the one-dimensional consolidation of an elastic soil is analysed for the situation in which the traction \( q \) is applied instantaneously to the soil surface at time \( t = 0 \) and thereafter held constant and drainage occurs only at the surface, see Fig. 6.2a. Under these conditions a solution for the settlement as a function of time \( t \) can be seen to depend on the following parameters: \( q/E' \); \( \gamma_f H/E' \); \( \nu' \); \( e_o \) and \( S_g \), where \( E' \) and \( \nu' \) are the drained Young's modulus and Poisson's ratio respectively for the soil; \( H \) is the initial depth of the layer; \( \gamma_f \) is the unit weight of the pore fluid; \( e_o \) is the initial void ratio of the soil, assumed uniform; \( k \) is the soil permeability; and \( S_g \) is the specific gravity of the solid particles.
For the results presented the following material properties were chosen: $e_0 = 10$; $v' = 0.3$; $S = 2.65$. Figs. 6.3 and 6.4 show some solutions for the degree of settlement $U$ as a function of the dimensionless parameters $q/E'$ and $\gamma_f H/E'$ and the dimensionless time $T = c_v t / H^2$ where $c_v$ is the usual one-dimensional consolidation coefficient. The curves indicate that shallow stiff layers exhibit a consolidation behaviour more like the Terzaghi prediction than do deeper, less stiff layers. For any given soil a further departure from the classical behaviour is observed as the magnitude of the consolidating pressure is increased. Note that to obtain the Terzaghi solution it is necessary for both of the parameters $\gamma_f H/E'$ and $q/E'$ to approach zero. Numerical agreement with this solution is shown in Fig. 6.4.

Not only is the degree of settlement with time of interest in this problem, but so is the magnitude of the final settlement. In Fig. 6.5 is shown the relationship between the applied traction and the final settlement $S_F$. For the range of traction magnitudes considered, i.e. $0 \leq q/E' \leq 1.0$, it was found that the parameter $\gamma_f H/E'$ had no appreciable effect on the magnitude of $S_F$, even for the large initial value of void ratio $e_0 = 10$. This is consistent with the fact that the saturated density of the soil, given by

$$\gamma_{sat} = \left( \frac{S_q + e}{1 + e} \right) \gamma_f$$  \hspace{1cm} (6.28)

changes by only about 5% (from $1.15 \gamma_f$ to $1.21 \gamma_f$) when the void ratio changes from 10 to 5 as it does in the extreme case examined here, i.e. $q/E' = 1.0$. (Note that $S_F$ is proportional to the change in $e$). The density parameter will of course have a greater effect when $q/E' \gg 1$.
but such a range of values is considered unrealistic.

An exact solution for the ultimate settlement can be found for the case $\gamma_f H/E' = 0$, see Appendix 6B. Excellent agreement was found between this exact solution and the numerical solution of Fig. 6.5 in the range $0 < q/E' < 1$. The finite element mesh used to obtain these one-dimensional results is shown in Fig. 6.2b at the configuration for zero load, $t = 0$.

6.4.1 (ii) Two-dimensional Finite Elastic Consolidation - A Rigid Footing

The plane strain problem of the finite consolidation of a rigid, permeable strip footing resting on the surface of a saturated clay layer is described in the inset to Fig. 6.6. As with the one-dimensional problem the solution for the settlement, $\Delta$, with time can be shown to depend upon amongst others, the parameters $Q/2BG$ and $\gamma_f B/G$. $G$ is the shear modulus for the soil, $B$ is the footing half width and $Q$ is the total applied load per unit length of footing.

To approximate an instantaneous loading the vertical force on the footing was increased linearly with time from zero at $T = 0$ to its ultimate value at $T = 0.0001$ over a number of steps. The dimensionless time $T$ is given in this problem as $T = c \sqrt{t/D^2}$ where $D$ is the layer depth at $t = 0$. To model a rigid footing the load was applied as a series of nodal forces to several very stiff (compared to the soil) footing elements.

Some results for the footing settlement $\Delta$ as a function of the dimensionless time $T$ are given in Fig. 6.6 for the case $\gamma_f B/G = 0.1$,.
\( \varepsilon_0 = 10, \nu' = 0.3 \) and \( \rho \). The curves show that the settlement behaviour is more unlike the small strain prediction for larger values of the parameter \( Q/2BG \) (i.e. as either the load is increased and/or less stiff soils are loaded). According to the finite theory a settlement equal to the layer depth would be approached as \( Q/2BG \) approaches an infinite value. This is not the case for infinitesimal theory where physically impossible settlements are predicted at finite load levels, see for example the curve for \( Q/2BG = 10 \) of Fig. 6.6.

Fig. 6.7 shows the configuration of the finite element mesh at various times for the case \( Q/2BG = 5 \). For large values of the parameter \( Q/2BG \) severe distortion of elements occurs, particularly near the edge of the footing. This may render the calculation unrealistic, especially if numerically the void ratio becomes zero in any element.

6.4.2 Finite Elastoplastic Consolidation

In the previous examples of this chapter attention has been restricted to the case of a soil with an elastic skeleton. The following examples examine soils in which plastic yielding of the skeleton may occur. The skeleton behaviour is governed by drained elastic parameters \( E', \nu' \) and drained cohesion \( c' \), friction angle \( \phi \) and dilation angle \( \psi \) (see chapter 3). The shear modulus for such a soil is given by

\[
G = \frac{E'}{2(1+\nu')}
\]  

(6.30)

Small et al. (1976) note that the undrained behaviour of such a material will be governed by the undrained parameters,
\[ E_u = \frac{3E^*}{2(1+\nu')}, \quad (6.31) \]

\[ \nu_u = 0.5, \quad (6.32) \]

and when \( \psi = 0 \),
\[ c_u = \frac{N_\phi (\sigma'_{10} + c' \cot\phi) - (\sigma'_{10} + c' \cot\phi)}{1+N_\phi} + \left(\frac{\sigma'_{10} - \sigma'_{30}}{2}\right), \quad (6.33) \]

where \( E_u, \nu_u, c_u \) are the undrained Young's modulus, Poisson's ratio and cohesion respectively, and \( \sigma'_{10} \) and \( \sigma'_{30} \) are the major and minor principal stresses in the soil at \( t = 0 \). The constant \( N_\phi \) is given by

\[ N_\phi = \frac{1 + \sin\phi}{1 - \sin\phi}. \quad (6.34) \]

The time varying response of a porous elastoplastic soil is dependent upon the rate of loading \( \omega \). Here we define \( \omega \) to be

\[ \omega = \frac{d(g/c')}{dT}. \quad (6.35) \]

where \( T = c_v t/H^2 \) as before. Small et al. point out that at very slow loading rates the soil behaves in a drained fashion. As the load rate is increased the behaviour becomes more like that of the undrained response.

6.4.2 (i) One-dimensional Finite Elastoplastic Consolidation

Consider first the one-dimensional problem described in Fig. 6.8. A layer of initial height \( H \) is subject to a proportional loading which commences at \( t = 0 \). At time \( t \) later, when the consolidating
pressure is \( q \), the layer depth is \( \rho H \).

In the limit for an elastoplastic material with \( \gamma_f H/G > 0 \) exact solutions may be found for the long term (drained, \( \omega = 0 \)) response of the layer. These exact solutions are presented in Appendix 6B and are plotted in Fig. 6.9 for a material with \( c' = 0 \), \( \phi = 30^\circ \), \( \psi = 0^\circ \). In this case the material remains elastic throughout the loading as long as \( \nu' > 0.25 \).

The numerical technique described above was used to determine the effect of the loading rate on the finite deformation response of this material. These results are presented in Fig. 6.10. Notice that like the infinitesimal strain results of Small et al., an increase in loading rate \( \omega \) produces behaviour more like the undrained response. Note also that in the one-dimensional problem the undrained settlement is zero. Similar results are shown in Fig. 6.11 for a cohesive frictional material with \( G/c' = 10 \), \( \nu' = 0.2 \), \( \phi = 30^\circ \), \( \psi = 0^\circ \) and again \( \gamma_f H/G > 0 \).

The importance of density in the response of such a soil is shown in Figs. 6.12 and 6.13. Solutions for \( \gamma_f H/G = 0 \) and \( 10^3 \) show no significant difference over the range \( 0 \leq q/G \leq 5 \) examined at the loading rates \( \omega = 0.1, 10 \). This is a similar result to that obtained previously for an elastic soil. It is important to note, however, that generally smaller computational steps are required to give satisfactory results as \( \gamma_f H/G \) is increased. For example a step size in \( q/G \) equal to 0.1 gave satisfactory results when \( \gamma_f H/G = 10^3 \) but a step size of 0.5 was found unsatisfactory.
6.4.2 (ii) Two-dimensional Finite Elastoplastic Consolidation -

A Rigid Footing

The application of the numerical technique to a problem of two-dimensional consolidation is now presented. We consider the problem of a perfectly rough and rigid strip footing resting on a layer of soil which overlies a smooth rigid base. In order to completely define the problem it is assumed that there is no resultant horizontal force on any vertical section, see the inset to Fig. 6.14. (This problem is similar to that solved by Small et al., 1976, who obtained infinitesimal strain solutions for a perfectly flexible footing.)

In the present problem only the case when $\gamma_f B/G + 0$ is examined, i.e. the density effect is ignored. In addition, the material of the layer was assumed to have the following drained properties $\nu' = 0.3$, $\phi' = 30^\circ$, $\psi' = 0^\circ$. Initially the problem for a material with $E'/c' = 200$ is examined and a number of constant load rates $\omega$ were used in the analysis. Fig. 6.14 shows the results of the numerical analyses. Consider first the results of infinitesimal deformation theory. Notice that in one extreme case when $\omega = 143.0$, the load settlement curve is asymptotic to the analytic collapse load for $\phi = 0$ (Booker, 1972), while in the other extreme when $\omega = 0.143$, the solution asymptotes to the analytic collapse load for a material with $\phi = \phi'$ and $\psi = \phi'$ (Booker, 1972). For the intermediate loading rate $\omega = 1.43$, the solution asymptote is between the drained and undrained collapse loads. This is a further example in infinitesimal theory of the phenomenon described by Small et al. (1976); that as the loading rate is increased there is a transition from drained failure to undrained failure. For this problem the soil skeleton behaves in a drained manner when the loading rate is
less than about 0.143 and in an undrained manner when the loading rate is greater than about 143.0.

The results of a finite deformation analysis for this problem are also shown on Fig. 6.14. It can be seen that both small and large strain predictions are almost identical until very near the small strain collapse loads where for all loading rates examined the finite deformation curve then lies slightly below the infinitesimal result. This discrepancy is small and we cannot be certain if it is numerical or if it represents a true difference.

The above problem has also been analysed for a less stiff material, one with \( E' / c' = 20 \), and the results are shown in Fig. 6.15. The above discussion is also applicable to these results. The finite element representation of the deformation history for this material when the loading rate \( \omega = 0.143 \) is given in Fig. 6.16 as are plastic regions.

It may be conjectured that the shape of the load deformation curves at very large settlements is as shown schematically in Fig. 6.17. The initial portion will be as given by the present analysis. As the finite settlement continues the load will decrease and the middle portion will be given approximately by the solutions of Booker (1972) for layers of ever decreasing thickness, (although it must be remembered that the surface is no longer plane). At some stage during the loading, when the ratio of the footing width to the vertical separation of the footing and the base of the layer becomes large enough, the curve will be given approximately by Hill's (1950) solution for the squashing between rough rigid plates. Ultimately the load must approach an infinite value as the footing bears on the base of the layer.
6.5 DISCUSSION AND CONCLUSIONS

A consistent formulation for the consolidation of an elastoplastic soil which incorporates the effects due to significant changes in geometry has been proposed. Only some very simple example problems have been solved in an effort to ascertain the importance of certain parameters. In all cases the ideal material has been of a fairly simple nature. This was in the absence of adequate experimental data, but additional material non-linearities may easily be incorporated. Nevertheless, some evaluation of the present solutions will show their merit as a model for real clay behaviour.

In Fig. 6.4 the relationship between the total final settlement of an elastic soil under one-dimensional conditions, and the magnitude of the applied traction (effective stress) was plotted. It will be remembered that the magnitude of this settlement was little affected by the density parameter and so can be considered to be independent of the void ratio \( e \). We may thus present these results differently, as a plot of a void ratio parameter against the logarithm to base 10 of the effective stress (i.e. the applied traction). Because the layer height is proportional to \( 1 + e \) and the load settlement response is independent of \( e \), we use \( \frac{1+e}{1+e_0} \) as the ordinate, where \( e_0 \) is the void ratio when \( q = 0 \). Plotted in Fig. 6.18 is such a relationship for a number of values of Poisson's ratio. Both finite deformation and infinitesimal deformation results are shown. The shape of these curves is typical for the consolidation of a clay before and beyond its preconsolidation pressure. It may be tempting at this stage to explain the phenomenon of oedometer consolidation on the basis of this theory, but it should be remembered that the responses plotted on Fig. 6.18 are fully recoverable...
since the soil skeleton is elastic. The results of Fig. 6.18 were calculated assuming that $E'$ was independent of the effective stress $\sigma'$ (or $q$) but it would be possible to discover some functional relationship between $E'$ and $\sigma'$ that may fit this theory to given experimental results. There are fundamental objections to such a procedure, however, as it is merely equivalent to a multilinear elastic analysis. We should look to the true elastoplastic theory to account for features such as irrecoverability.

Some typical predictions for the loading and subsequent unloading of a one-dimensional specimen are given in Fig. 6.19 for a material with an elastoplastic skeleton. The unloading began once the effective stress reached a value of $q=100c'$ throughout the layer. Note that the settlement recovery was almost complete despite the fact that plastic yielding commenced early during the loading sequence. This yielding was of a deviatoric nature only. Thus these results suggest that the plastic yielding that occurs during consolidation should be determined by a hydrostatic stress component as well as perhaps a deviatoric component, at least in the case of one-dimensional behaviour. This conclusion is consistent with the observations of others (e.g. Drucker, Gibson and Henkel, 1957). The present analysis is easily modified to allow for the inclusion of such plastic behaviour and this is a suggestion for further work.
(a) Problem Definition

(b) Mesh at $t = 0$

Fig. 6.2 1-D ELASTIC CONSOLIDATION.
\[ T = \frac{c_v t}{H^2} \]

- \[ \frac{q}{E'} = 1 \]
- \[ \varepsilon_o = 10 \]
- \[ v' = 0.3 \]
- \[ S_g = 2.65 \]
- \[ \frac{\gamma_f H}{E'} = 20 \]
- \[ \frac{\gamma_f H}{E'} = 10 \]
- \[ \frac{\gamma_f H}{E'} = 0 \]

**Fig. 6.3 1-D Finite Consolidation - \( \gamma_f H/E' \) Effect**
$T = \frac{C_v t}{H^2}$

![Graph showing consolidation process with various curves]

- **Terzaghi**
- **Finite Element**

As $\frac{q_p}{E'}$, $\frac{\gamma_f H}{E'} \rightarrow 0$

- $q_p/E' = 1$
- $\gamma_f H/E' = 10$
- $\varepsilon_0 = 10$
- $v' = 0.3$
- $S_g = 2.65$

**Figure 6.4** 1-D Finite Consolidation - $q_p/E'$ Effect
FIG. 6.5 RELATIONSHIP BETWEEN FINAL SETTLEMENT AND APPLIED TRACTION FOR 1D - FINITE ELASTIC CONSOLIDATION
FIG. 6.6 TWO DIMENSIONAL FINITE CONSOLIDATION
Free to drain

Homogeneous Elastoplastic

Impermeable

(a) time $t = 0$

(b) time $t$

FIG. 6.8 ONE DIMENSIONAL COMPRESSION
$c = 0 \quad \gamma_f = 0$
$
\phi = 30^\circ \quad \psi = 0^\circ$

FIG. 6.91-D COMPRESSION - DRAINED SOLUTIONS
Fig. 6.10 1-D COMPRESSION - EFFECT OF LOADING RATE

- $c = 0$, $\gamma_f = 0$
- $\phi = 30^\circ$, $\psi = 0^\circ$
- $v' = 0.2$

-q/G vs $\rho$:
- Exact drained solution
- Numerical finite elastoplastic consolidation

Graph showing the relationship between $q/G$ and $\rho$. The graph includes curves for different loading rates ($\omega = 100$, $10$, $1$, $0.1$).
FIG. 6.11 1-D COMPRESSION - EFFECT OF LOADING RATE
$G/C = 10 \quad \nu' = 0.2$

$S_g = 2.65 \quad e_0 = 10 \quad K_0 = 1$

$\phi = 30^\circ \quad \psi = 0^\circ$

$\omega = 0.1$

$\gamma_f H/G = 0$

$\square \gamma_f H/G = 10^3$

(every fifth step plotted)

FIG. 6.12 1-D COMPRESSION - EFFECT OF $\gamma_f H / G$
FIG. 6.131-D COMPRESSION - EFFECT OF $\gamma_f H/G$

$G/c = 10, \gamma' = 0.2$

$S_g = 2.65, e_o = 10, K_o = 1$

$\phi = 30^\circ, \psi = 0^\circ$

$w = 10$

$\gamma_f H/G = 0$

$\gamma_f H/G = 10^3$

(every fifth step plotted)
FIG 6.14 EFFECT OF LOADING RATE ON FOOTING BEHAVIOUR
FIG. 6.15 EFFECT OF LOADING RATE ON FOOTING BEHAVIOUR
FIG. 6.16 2-D CONSOLIDATION - MESH GEOMETRIES

FOR \( \omega = 0.143 \), \( \frac{E'}{c'} = 20 \)
FIG. 6.17 POSSIBLE SHAPE OF LOAD-SETTLEMENT CURVE
FIG. 6.18 1-D FINITE ELASTIC CONSOLIDATION
FIG. 6.19 1-D FINITE ELASTOPLASTIC CONSOLIDATION

LOADING AND UNLOADING
APPENDIX 6A

THE PERMEABILITY MATRIX

We present here the derivation of matrix $K$ of equation (6.17) for the general case of three dimensional consolidation.

Consider an element of soil with centre $P_0(a, b, c)$ which deforms from an initial position $A$ at time $t_0$ to an adjacent position $B$ with centre $P(x, y, z)$ in a time interval $dt$, as shown schematically in Fig. 6A.1. Initially the flow properties are characterised by

$$K_0 = \begin{bmatrix}
k_{xx} & k_{xy} & k_{xz} \\
- & k_{yy} & k_{yz} \\
- & - & k_{zz}
\end{bmatrix}$$

so that Darcy's law at $A$ may be written as

$$\alpha_0 (v_{fo} - v_{so}) = -K_0 \nabla h_0$$

(6A.1)

where $h_0$, $\alpha_0$ are the head and porosity in the vicinity of $P_0$ respectively, and $(v_{fo} - v_{so})$ is the vector of velocity components (measured with respect to $x$, $y$, $z$ axes) of the fluid phase relative to the solids at $P_0$. The quantity $\nabla h_0$ is given by

$$\nabla h_0 = \left(\frac{\partial \alpha_0}{\partial a}, \frac{\partial \alpha_0}{\partial b}, \frac{\partial \alpha_0}{\partial c}\right)^T$$

(6A.2)

The question now arises as to the form of Darcy’s law when the element of soil is in position $B$. One reasonable assumption is that the form of any flow anisotropy is intrinsic to the element so that
\[ \alpha (v'_f - v'_s) = -Kv'h \]  \hspace{1cm} (6A.3)

Where \( h, \alpha \) represent the same quantities as before only measured at \( P \) at time \( t_0 + dt \), and
\[ \nabla'h = \begin{pmatrix} \frac{\partial h}{\partial \xi} \\ \frac{\partial h}{\partial \eta} \\ \frac{\partial h}{\partial \zeta} \end{pmatrix}^T \]  \hspace{1cm} (6A.4)

The vector \( (v'_f - v'_s) \) contains the components of the relative velocity at \( P \) but measured with respect to rotated axes \( (\xi, \eta, \zeta) \). The relationship between \( (v'_f - v'_s) \) and \( (v_f - v_s) \), the relative velocity vector at \( P \) measured with respect to \( (x, y, z) \), is given by
\[ (v'_f - v'_s) = R(v_f - v_s) \]  \hspace{1cm} (6A.5)

Where \( R \) is the matrix which corresponds to the appropriate rotation of coordinate axes. (e.g. for plane deformations
\[ R = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \]

Where \( \gamma \) is the angle of rotation between \( (x, y) \) and \( (\xi, \eta) \).

Considering \( (x, y, z) \) as the independent variables then
\[ \nabla'h = R\nabla'h \]  \hspace{1cm} (6A.6)

Where
\[ \nabla'h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{pmatrix}^T \]

Equation (6A.3) may thus be transformed to give Darcy's law at \( P \) in the form
\[ \alpha (v_f - v_s) = -Kv'h \]  \hspace{1cm} (6A.7)

Where
\[ K = R^T K_0 R \]

\( K \) thus remains symmetric during the rotation. It is interesting to note that, according to the above assumption, as an element rotates the form
of Darcy's law (referred to the initial set of axes) changes. Thus a material which is anisotropic, but whose initial anisotropy is homogeneous develops an inhomogeneity of anisotropy as different elements rotate by different amounts. Of course this does not occur (according to this formulation) if the material is initially isotropic and in such a case

\[
\alpha (v_f - v_s) = k \bar{V}h
\]  

(6A.8)

where \( k \) is the isotropic permeability.
FIG. 6A1 ELEMENT ROTATION
APPENDIX 6B

ONE-DIMENSIONAL FINITE DEFORMATION

The aim of this appendix is to present the solutions for the finite deformation of an elastic and an elasto-plastic material under one-dimensional conditions. The problem has been described in Figs. 6.2a and 6.8a. We consider that the material is weightless and that its constitutive law is as given in Chapter 3 for a Mohr-Coulomb material obeying a non-associated flow rule.

(i) Elastic Solution

Consider first the elastic behaviour of such a mass. The constitutive law, equation (2A.8) reduces to

\[ \dot{\sigma}_{yy} = (\Lambda + 2\mu) \dot{\varepsilon}_{yy} \]  

(6B.1)

for this problem as \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \dot{\varepsilon}_{xx}, \dot{\varepsilon}_{yy}, \dot{\varepsilon}_{zz} \) are all identically zero. The other non-zero stress rates are given by

\[ \dot{\sigma}_{xx} = \dot{\sigma}_{zz} = \left( \frac{\Lambda}{\Lambda + 2\mu} \right) \dot{\sigma}_{yy} \]  

(6B.2)

We arbitrarily assign the rate quantity \( \dot{\varepsilon}_{yy} \) equal to a constant \( k \), so that

\[ \dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y} = k \]  

(6B.3)

where \( v(y, t) \) is the displacement in the y direction. Using the condition that \( \rho = 1 \) at \( t = 0 \), the solution of equation (6B.3) in terms of overall deformation is,

\[ \rho = e^{kt} \]  

(6B.4)

Since \( \dot{\varepsilon}_{yy} = q \) then equation (6B.1) may be written,
\[ q = (\Lambda + 2G) k_{yy} \]
\[ = (\Lambda + 2G) k \]  
\[ (6B.5) \]

Hence \[ q = (\Lambda + 2G) kt \]  
\[ (6B.6) \]

since \( q = 0 \) at \( t = 0 \). The solution for \( \rho \) as a function of \( q \) is thus

\[ \rho = \exp \left\{ \frac{q}{\Lambda + 2G} \right\} = \exp \left\{ \frac{q}{(1-\nu)/2G} \right\} \]  
\[ (6B.7) \]

(ii) Elastoplastic Solution

In the elastic range we know from equation (6B.2) that

\[ \sigma_{xx} = \sigma_{zz} = \frac{\Lambda}{\Lambda + 2G} \sigma_{yy} = \frac{\nu}{1-\nu} \sigma_{yy} \]  
\[ (6B.8) \]

However, the Mohr-Coulomb yield criterion states that

\[ \sigma_{xx} = \frac{1}{N_{\phi}} \sigma_{yy} + 2c \frac{\sigma_{yy}}{N_{\phi}} \]  
\[ (6B.9) \]

where in this case \( \sigma_{xx} \) and \( \sigma_{yy} \) are the major and minor principal stresses respectively (tension positive) and as usual \( c \) is the material cohesion and

\[ N_{\phi} = \frac{1 + \sin \phi}{1 - \sin \phi} \]

As in Appendix 4F we note that the material will first yield when \( \sigma_{yy} \) and \( \sigma_{xx} \) satisfy both (6B.8) and (6B.9), i.e. when

\[ \sigma_{yy} = q_{F,Y.} = \frac{2c}{\sqrt{N_{\phi}} \frac{\nu}{1-\nu} - \frac{1}{N_{\phi}}} \]  
\[ (6B.10) \]

and

\[ q_{F,Y.} = \exp \left\{ \frac{q_{F,Y.}}{\Lambda + 2G} \right\} \]  
\[ (6B.11) \]
Note that for purely frictional materials \( c = 0 \) and yield will begin at \( q = 0 \) if

\[
\frac{\nu}{1-\nu} \leq \frac{1}{N_\phi}
\]

As in Appendix 4F we may express the elastoplastic constitutive relation for a purely frictional material as

\[
\sigma_{yy} = \kappa \frac{\dot{\varepsilon}_{yy}}{1 - \nu}
\]  

(68.12)

where

\[
\frac{1}{\kappa} = A \left( 1 + \frac{1}{N_\psi N_\phi} \right) + B \left( \frac{1}{N_\psi N_\phi} \right)
\]

\[
A = \frac{\Lambda + 2G}{4G(\Lambda + G)}
\]

\[
B = \frac{-\Lambda}{4G(\Lambda + G)}
\]

\[
N_\psi = \frac{1 + \sin \psi}{1 - \sin \psi}
\]

The solution for \( \rho \) as a function of \( q \) thus proceeds as before, only with the boundary condition that \( \rho = \rho_{F,Y} \) when \( q = q_{F,Y} \), so that

\[
\rho = \rho_{F,Y} \exp \left\{ \frac{q - q_{F,Y}}{\kappa} \right\}
\]  

(68.13)

Note that in the above solutions a downward traction must be given a negative sign.
CHAPTER 7
CONCLUDING REMARKS
A formulation has been presented which allows an analysis of the deformation of elastoplastic continua. The treatment is more general than conventional infinitesimal theories which adopt, as a basic premise, the notion that the imposed loading causes deformations that do not significantly change the geometric configuration of the body. The theory here has disregarded such an assumption. The results obtained from the present approach and the infinitesimal approach have been compared and certain conclusions may be drawn from the material of the previous chapters.

The basic equations which were given in chapter 2 provide an exact description of the deformation behaviour of an ideal material. They are self-consistent and applicable for both large and small deformations. The validity of the solutions to these equations, obtained for the various boundary loadings considered in this thesis, is dependent upon the accuracy of the approximate numerical solutions. The latter in turn depends upon the numerical procedure (algorithm) and the spatial and time discretisation process.

Because of the non-linear nature of finite deformation problems only a limited checking on the validity of the present approach has been possible. For simple well defined problems, such as homogeneous deformation or essentially one-dimensional problems, the numerical technique has proved extremely reliable. For problems of a more complicated nature there are certain reservations. The stiffness formulation used here was shown to be, at times, inadequate when used to obtain a measure of small strain collapse. (Similar conclusions have been reached by Nagtegaal et al., 1974). In many cases the values so obtained are an overestimate of the true collapse load. There are
many examples where this occurs; in particular see the work on rough rigid footings resting on both homogeneous and inhomogeneous materials. The greater the degree of inhomogeneity then the worse the estimate of collapse. Some improvement was found when potential rupture planes were inserted into the finite element mesh, as discussed in chapter 5. When the stiffness formulation is used for finite deformation solutions there may be some carry over, or even accentuation, of this type of error.

The stiffness formulation of the finite element technique has been commonly used in the last decade. It is a technique well tried in the solution of elasticity problems but when extended to problems involving plastic yielding it must, of its very nature, become difficult to use with reliability once unrestrained plastic flow occurs. Nevertheless, given the use of such a technique, solution improvement may be obtained by reducing discretisation error, i.e. increasing the number of elements and reducing the time step.

Because the stiffness formulation was used to obtain approximate solutions to finite deformation problems, perhaps such results are also open to some question. It is clear from examples studied in this work that finite deformation results for low values of E/c are likely to be reliable since the finite deformation behaviour of the body is then dominated by finite elastic strains. The use of the numerical results to extrapolate to rigid plastic behaviour (E/c = ∞) presents great difficulty using this approach. In fact the case of a rigid plastic may need to be treated as a special limiting case and a quite different solution procedure need to be formulated.
A major matter for conjecture is the relationship between finite deformation solutions for elastoplastic materials and the solution for a rigid plastic. Must the solutions for all materials with finite E/c behave in the manner suggested by Fig. 7.1? That is, must all the load-displacement curves be bounded above by the rigid plastic solution? There are certain problems where this has been shown to be true, e.g. the squashing between smooth rigid plates as presented in chapter 4. Another case where this is so is the special problem shown in Fig. 7.2. Consider three blocks of purely cohesive material arranged initially as shown by the unbroken lines in Fig. 7.2. These blocks deform according to the broken line picture with zero vertical load on the outer blocks and a uniform load, total magnitude P, over the central block. The arrangement is of course a special case of squashing between two different sets of smooth plates specially aligned. This provides a very crude but simple approximation to the surface loading problems of chapter 5. It is easy to show that exact load-displacement relations for such a simple model, using finite deformation theory, are as shown in Fig. 7.3. All curves for materials with finite E/c are bounded above and in fact eventually merge with the rigid plastic solution, in the manner of the smooth plate problem.

In problems of a more complicated nature, where the stress and displacement fields are no longer essentially uniform, finite deformation solutions are difficult or impossible to obtain. Hill (1950) presented a solution for one such problem, that of squashing between rough rigid plates, which he claims was valid for all finite plate separations. It was shown in chapter 4 that numerical load-displacement solutions for materials with a finite E/c lay above Hill's
curve (see Fig. 4.24), contrary to the trend demonstrated with smooth plate squashing and the preceding example.

It is important to also examine the behaviour of numerical solutions for the surface loading problems of chapter 5 in the limit as $E/c \to \infty$. For a problem as complicated as the indentation of a layer of purely cohesive material due to a rough rigid footing, an exact solution for the finite deformation behaviour of a rigid plastic has not been found. However, Davis (1961), using an infinitesimal limit analysis, has obtained upper and lower bounds to the bearing capacity of a rigid footing seated at the bottom of a vertical walled trench in a purely cohesive rigid plastic. The relationship between the depth of burial and the upper and lower limits of the collapse load are plotted in Fig. 7.4. Also shown on the figure are the numerical solutions for the indentation problem for materials with various $E/c$ values. It can be seen that these curves cross and eventually lie above the solution range of Davis. It is not suggested that the trench problem and the indentation problem are identical. Should they be comparable, then it is possible that the discrepancy shown on Fig. 7.4 (and also that shown on Fig. 4.24 for squashing between rough rigid plates) may be due entirely to numerical discretisation error, but again it is emphasised that there is no general proof that the solution for a rigid plastic should bound above all solutions for elastoplastics.

In general it may be stated that the numerical technique described in this thesis may confidently be used to tell when a finite deformation analysis for an elastoplastic material gives results that are significantly different from the infinitesimal analysis. It is the amount of such a difference that must be treated with some caution.
When is a finite deformation analysis important in soil mechanics problems? For most of the cases presented in this work, i.e. line loading, uniform traction loading and rigid footing loading on both homogeneous and inhomogeneous layers, it can be said that, from an engineering point of view, the infinitesimal theory provides a less optimistic prediction of the behaviour of the body than the finite deformation theory. In such cases the finite deformation analysis predicts less displacement due to any given load; see Fig. 7.5 for a general representation of this behaviour. However, for these problems it was found that if the deforming material possessed a stiffness to strength ratio E/c greater than about 100, then the finite deformation predictions were very close to the predictions of the infinitesimal theory at loads up to about a value given by the latter for collapse. Such conclusions are not, necessarily, perfectly general. It must be remembered that in all problems of downward loading on a layer of finite depth, the applied load must approach an infinite value as the magnitude of settlement beneath the load approaches the layer depth. Nevertheless, the loading predicted by the finite deformation analysis need not necessarily be monotonically increasing. Behaviour of the type depicted schematically in Fig. 7.6 may be possible (an example was cited in chapter 6). In such cases the finite deformation predictions are, in an engineering sense, more critical. The possibility of geometric softening cannot be discounted. It has also been shown that in the case of an embankment loading on soft clay layers a finite deformation analysis might be essential.

Throughout this work the actual calculations have been performed for elastic, perfectly plastic materials. It is expected that the behaviour of real soils particularly clays, would in many cases be
more accurately modelled by a work softening material (and in some cases a work hardening material). Further, it is expected that allowance for change in geometry will be particularly important in the analysis of such a material, where the full range of softening behaviour is achieved at strains of high magnitude but only possibly within narrow zones. The present formulation might easily be extended to the finite deformation analysis of such materials.
Fig. 7.1
Fig 7.2 BLOCK MODEL.
Fig 7.3 BLOCK MODEL RESULTS.
Fig 7.4 COMPARISON BETWEEN BURIED FOOTING AND INDENTATION PROBLEM.
Fig. 7.5
Fig. 7.6
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FINITE DEFORMATION THEORY AND ITS
APPLICATION TO ELASTOPLASTIC SOILS

by

John P. Carter, B.E.

Summary

In this thesis an examination is made of the application of finite deformation theory to an ideal elastic, perfectly plastic soil. The work may be conveniently divided into four sections.

(a) A presentation of the theory of finite deformation for an elastoplastic material. This investigation includes the development of the governing equations using a virtual work principle that is cast in a rate form and constitutive laws that are cast in a frame indifferent manner. Plastic failure is described by a general yield condition and plastic deformation by an arbitrary flow law. The approach is quasi-static in nature.

(b) The proposal of a numerical technique to be used for the approximate solution of the governing equations of finite deformation. The numerical method is verified by means of comparisons between exact and numerical solutions.

(c) The numerical methods referred to above are then used to examine several boundary value problems of engineering interest. These include the surface loading, under conditions of plane strain, of a layer of ideal, elastic perfectly plastic, cohesive-frictional material which obeys either an associated or a non-associated flow rule. The surface loading is either applied as a line loading, a traction loading, a rigid footing (specified boundary displacements) or the build up of an embankment.

(d) Finally, the finite deformation analysis is extended to predict the time dependent, finite consolidation behaviour of a two-phase elastoplastic soil. The latter theory is illustrated by the use of several practical examples, viz. one-dimensional consolidation
and two-dimensional consolidation of a rigid strip footing resting on either an elastic or an elastoplastic soil.

It was found that for many problems in soil mechanics, e.g. line loading, uniform traction loading and rigid footing load on a homogeneous and inhomogeneous layer, the finite deformation theory predicted less settlement at a given applied load than did the usual infinitesimal theory. This result is not general, however, and counter examples are cited. Indications are given as to when a finite deformation analysis might be necessary in soil mechanics. In real engineering problems this is confined to the loading of materials of very low stiffness, particularly due to embankment building.
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