Topics in Multi-dimensional Signal Demodulation

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DEDICATION

I dedicate this work to my mother Kathleen Larkin,

and to the memory of my father Joseph Larkin.
DECLARATION

The work presented in this thesis is my own, except where otherwise acknowledged. I am sole author of the work presented in chapters 1-2, and 4-6. The original idea for unravelling the autocorrelation appearing in chapter 3 was developed in conjunction with my supervisor, Colin Sheppard. Michael Oldfield at CISRA wrote the software framework for the Fourier analysis of complex images. An essential precursor to my own work in chapter 4 is the experimental development of a new transform by myself, Donald Bone, and Michael Oldfield. The precursor is necessary for a clearer understanding of chapter 4 and is lodged as a document in support of this thesis.

Kieran G. Larkin
PUBLISHED ARTICLES

The work presented in various chapters has appeared in journals and conference proceedings during the last few years or has been submitted for publication.

Chapter 2


Chapter 3


Chapter 4


Chapter 5

RELATED PUBLICATIONS


P. Hariharan, K. G. Larkin, and M. Roy, “The geometric phase: interferometric

C. J. R. Sheppard, and K. G. Larkin, “Effect of numerical aperture on interference

C. J. R. Sheppard, and K. G. Larkin, “Vectorial pupil functions and vectorial transfer

C. J. R. Sheppard, and K. G. Larkin, “Similarity theorems for fractional Fourier

C. J. R. Sheppard, and K. G. Larkin, “Focal shift, optical transfer function, and phase

Hilbert transform for 3D visualization of differential interference contrast microscope

JOSA,A, (accepted for publication, 2001).

C. J. R. Sheppard, and K. G. Larkin, The Wigner function for highly convergent
three-dimensional wavefields, Optics Letters (accepted for publication, 2001).

C. J. R. Sheppard, and K. G. Larkin, “The optical transfer function and phase space
mappings,” Optik, (accepted for publication, 2001).


nonlinear and spatially nonuniform phase shifts”, J. Opt. Soc. Am., A 14, (4), 918-

K. Hibino, K. G. Larkin, B. F. Oreb, and D. I. Farrant, “Phase-shifting algorithms for
Problems in the demodulation of one, two, and three-dimensional signals are investigated. In one-dimensional linear systems the analytic signal and the Hilbert transform are central to the understanding of both modulation and demodulation. However, it is shown that an efficient nonlinear algorithm exists which is not explicable purely in terms of an approximation to the Hilbert transform. The algorithm is applied to the problem of finding the envelope peak of a white light interferogram. The accuracy of peak location is then shown to compare favourably with conventional, but less efficient, techniques.

In two dimensions (2-D) the intensity of a wavefield yields to a phase demodulation technique equivalent to direct phase retrieval. The special symmetry of a Helmholtz wavefield allows a unique inversion of an autocorrelation. More generally, a 2-D (non-Helmholtz) fringe pattern can be demodulated by an isotropic 2-D extension of the Hilbert transform that uses a spiral phase signum function. The range of validity of the new transform is established using the asymptotic method of stationary phase. Simulations of the algorithm confirm that deviations from the ideal occur where the fringe pattern curvature is larger than the fringe frequency.

A new self-calibrating algorithm for arbitrary sequences of phase-shifted interferograms is developed using the aforementioned spiral phase transform. The algorithm is shown to work even with discontinuous fringe patterns, which are known to seriously hamper other methods. Initial simulations of the algorithm indicate an accuracy of 5 milliradians is achievable.

Previously undocumented connections between the demodulation techniques are uncovered and discussed.
# TABLE OF CONTENTS

**Topics in Multi-dimensional Signal Demodulation**

1. **Introduction**.............................................................................................................. 1
   - 1.1 Overview.............................................................................................................. 1
   - 1.2 Selection of Multi-dimensional Demodulation Problems....................... 3
   - 1.3 Scope of the thesis............................................................................................ 6
   - 1.4 References......................................................................................................... 7

2. **An Efficient Digital Algorithm for Envelope Detection**.......................... 9
   - 2.1 Background to White Light Interferometry.............................................. 9
   - 2.2 Structure of a White Light Interferogram............................................... 11
   - 2.3 Ideal Envelope and Phase Detection......................................................... 12
   - 2.4 Approximations to Hilbert Transform Envelope Detection............... 17
   - 2.5 Calculation of Phase and Modulation using Phase-Shifting Algorithms .................................................. 24
   - 2.6 Application of Spatial Phase-Shift Algorithms to Envelope Detection .................................................. 28
   - 2.7 Interpretation of Calculated Correlogram Envelopes............................ 33
   - 2.8 Numerical Simulation of Algorithms......................................................... 43
   - 2.9 Conclusion........................................................................................................ 48
2.10 Connections ........................................................................................................ 49
2.11 Acknowledgements .......................................................................................... 55
2.12 References and Notes .................................................................................... 55

3. Demodulation of Two-Dimensional Electromagnetic Intensity ........ 69

3.1 Introduction ........................................................................................................... 69
3.2 Fourier Representation of Problem .................................................................. 72
3.3 Coordinate Transform ....................................................................................... 75
3.4 Computational Procedure .................................................................................. 78
3.5 Experimental Verification .................................................................................. 82
3.6 Conclusion ........................................................................................................... 87
3.7 Connections ........................................................................................................ 88
3.8 Acknowledgement ............................................................................................. 91
3.9 References and notes ........................................................................................ 91

4. Natural Demodulation of Two-Dimensional Fringe Patterns:

Stationary Phase Analysis of the Spiral Phase Quadrature Transform 97

4.1 Introduction ........................................................................................................ 97
4.2 A quite remarkable transform ........................................................................... 98
4.3 Two Dimensional Fringe Patterns ................................................................... 99
4.4 Stationary Phase Expansion of a Complex Pattern \( p(x,y) \) .................. 100
4.5 Stationary Phase Expansion of a Real Fringe Pattern \( f(x,y) \) ............... 111
4.6 Heuristic development of the Spiral Phase Quadrature Transform .. 116
4.7 Deviations From the Ideal Quadrature Transform ........................... 117
4.8 Conclusion ........................................................................................ 124
4.9 Connections ...................................................................................... 124
4.10 Appendix: Practical Aspects of the Fringe Orientational Factor . 127
4.11 Acknowledgments .......................................................................... 132
4.12 References and notes ...................................................................... 132

5. The Ultimate Phase-Shifting Algorithm? ........................................ 137
5.1 A cornucopia of phase-shifting algorithms ...................................... 137
5.2 Spatio-temporal algorithms .............................................................. 140
5.3 Can all of the spatial information be used to calibrate the temporal
phase shift? .............................................................................................. 145
5.4 Inter-frame phase difference estimation using the vortex transform 147
5.5 Some Accuracy Estimates ................................................................ 160
5.6 Discussion .......................................................................................... 163
5.7 References and Notes ....................................................................... 163

6. Conclusion .......................................................................................... 173
6.1 Final Comments ............................................................................... 173
6.2 Bibliography ..................................................................................... 177

7. Appendix: Document lodged in support of thesis ......................... 183
LIST OF TABLES

Table 2.1 RMS peak location error for various algorithms for optimal sampling.

Table 2.2 RMS peak location error for various algorithms for 3 times undersampling.

Table 5.1 Performance of phase-shift calibrating algorithm applied to fringe pattern of figure 5.4

LIST OF FIGURES

Figure 1.1 Comparison table for spherical aberration.

Figure 1.2 Manual fringe pattern analysis.

Figure 2.1 Typical white light interferogram, $g(z)$.

Figure 2.2 Modulus of the Fourier transformed correlogram $|G(w)|$.

Figure 2.3 The spectral responses of the numerator and denominator filters of the five-sample phase-shifting algorithm.

Figure 2.4 Envelope demodulation for all three algorithms using a 90° step size.

Fig 2.5 Envelope demodulation for all three algorithms using a 45° step size.
Figure 2.6  Envelope demodulation for all three algorithms using a 135° step size.

Figure 2.7  Error in the predicted peak position (in units of sample step size) as a function of initial selected peak position.

Figure 3.1(a)  Region of support for the pupil function $F(m,s)$.

Figure 3.1(b)  Regions of support for the autocorrelation of the pupil function.

Figure 3.2  The spatial frequency coordinate system and angular coordinate system.

Figure 3.3  The rhombic overlap region in the autocorrelation of $F(m,s)$.

Figure 3.4  Effect of coordinate transformation for a typical grid pattern.

Figure 3.5  Greyscale plot of a typical intensity pattern $g(x,z)$.

Figure 3.6  Greyscale plot of the modulus of the Fourier transformed intensity, $|G(m,s)|$.

Figure 3.7  Greyscale plot of the initial angular spectrum $F(m,s)$ as a magnitude (left) and a phase (right) component.

Figure 3.8  Greyscale plot of $\tilde{F}(\theta_1), \tilde{F}(\theta_2)$ from the direct phase retrieval algorithm of equation (3.12).
Figure 3.9  The phase profile recovered using equation 3.14.

Figure 3.10  Greyscale plot of an intensity pattern with a uniform random noise.

Figure 3.11  Greyscale plot of $\hat{f}(\theta_j)\hat{f}^*(\theta_j)$ from the direct phase retrieval algorithm applied to the noisy intensity map.

Figure 3.12  The recovered phase profile from the noisy intensity map.

Figure 3.13(a) Pupil function magnitude and phase.

Figure 3.13(b) Wavefield magnitude and phase.

Figure 3.13(c) Spectral correlation function, magnitude and phase.

Figure 3.13(d) Spectral correlation function; statistically enhanced pseudocolour representation of magnitude and phase.

Figure 4.1(a) A typical fringe pattern with smooth and differentiable amplitude and phase.

Figure 4.1(b) Underlying phase function of the fringe pattern in figure 1(a).

Figure 4.2(a) The definition of orientation angle from the phase gradient.

Figure 4.2(b) The definition of orientation angle from fringe angle.

Figure 4.3(a) Simple square root of orientation phase map (modulo $\pi$).
Figure 4.3(b)  Unwrapped orientation phase map (modulo $2\pi$). Greyscale encoding means black represents $-\pi$ and white represents $+\pi$.

Figure 4.4  The magnitude of the Hessian of the phase.

Figure 4.5(a)  Sixth root of the magnitude of the relative curvature of the phase.

Figure 4.5 (b)  Sixth root of the magnitude of actual error in the phase derived using the vortex operator on the fringe pattern of figure 4.1(a).

Figure 4.6  Relative magnitude of the vortex operator derived magnitude

Figure 5.1  Upper plot shows sequential inter-frame differences (dotted connecting lines) on a unit circle representing phase angle.

Figure 5.2(a)  Flowchart for automatic phase-step calibration method.

Figure 5.2(b)  Continuation of flowchart for automatic calibration method.

Figure 5.3  Simple weight function calculated from the estimated errors in the contingent analytic function (relative to the output from the PSA).

Figure 5.4  Fringe pattern used for testing the phase-shift calibration algorithm.

Figure 5.5  Phase error showing the classic second harmonic structure, and the disappearance of the vertical half-period discontinuity.