CHAPTER 1

Introduction

Braid groups were introduced in 1925 by Emil Artin [5], primarily as a tool for the study of knots and links. The importance of braids in this area is highlighted by Alexander’s Theorem [1] which states that every knot (or link) may be obtained by closing a braid; see Figure 1.1 for an example. The braid group $B_n$ is the set of (homotopy classes of)

\[ \text{Figure 1.1. Representing the trefoil knot (left) as a closed braid (right).} \]

braids with $n$ strings, under the group operation of concatenation. Markov’s Theorem [47] (proved by Joan Birman in [9]) gives necessary and sufficient conditions for two braids to represent the same knot or link. Several authors have used braids, and generalisations such as singular braids, to define recursive formulae for link invariants; see for example the survey article by Birman [8] and references therein. Singular braids, which may be thought of as braids in which certain pairs of strings have been glued together at points called double points or singular points, arise when studying the way in which a knot simplifies by changing over-crossings to under-crossings (or vice versa). For example, the trefoil knot in Figure 1.1 may be un-knotted by changing, say, the left-most crossing to a crossing of the opposite kind; see Figure 1.2. At the moment the strings touch, as one passes through the other, a

\[ \text{Figure 1.2. Un-knotting the trefoil.} \]

singular knot is produced. A singular version of Alexander’s Theorem, which was proved
1. INTRODUCTION

in [8], states that any singular knot (or link) may be obtained by closing a singular braid; see Figure 1.3 for an example. (See also [35] for a singular version of Markov’s Theorem.)

Figure 1.3. Representing the singular trefoil knot (left) as a closed singular braid (right).

The singular braid monoid $SB_n$, introduced in [7, 8], is the monoid of (rigid-vertex-isotopy classes of) singular braids with $n$ strings, under the monoid operation of concatenation. Any singular braid may be resolved into a (non-singular) braid by changing a double point to either an over-crossing or an under-crossing. In general there will be $2^k$ ways to resolve a singular braid with $k$ double points, some of which will give rise to equivalent braids. For example, the singular braid $\beta \in SB_2$ pictured in the right hand side of Figure 1.3 may be resolved to give the two braids $\beta_1, \beta_2 \in B_2$ pictured in Figure 1.4. We saw in

Figure 1.4. Two resolutions $\beta_1 \in B_2$ (left) and $\beta_2 \in B_2$ (right) of the singular braid $\beta \in SB_2$ (centre).

Figure 1.1 that $\beta_1$ closes to give the trefoil and, with the aid of Figure 1.5, we see that $\beta_2$ closes to give the unknot. It is then natural to ask how much information about a singular braid may be recovered when presented with all of its possible resolutions. One of the most popular topics in the theory of singular braids is Birman’s Conjecture [8] which

Figure 1.5. The braid $\beta_2 \in B_2$ closes to give the unknot.
1. INTRODUCTION

effectively states that a singular braid is completely determined by its resolutions (counted according to multiplicities). The conjecture (which has recently been solved positively by Luis Paris [53]; see also [2, 3, 26, 30, 36, 41]) formally states that the map $SB_n \to \mathbb{C}[B_n]$, which maps a singular braid to the formal sum of its resolutions, is injective.

Thus singular braids provide a useful tool for studying the way in which a knot or link may be simplified by allowing strings to pass through each other. One of the motivations of the present work is to look at this problem from a slightly different angle. Rather than focusing on the moment at which the strings of a knot pass through each other, and considering singular braids, what if one focuses on the actual process of strings passing through each other? Turning our attention again to the braids $\beta_1$ and $\beta_2$ pictured in Figure 1.4, we see that if we make the first string of $\beta_1$ pass through the second string in a neighbourhood of the upper-most crossing, then we obtain the braid $\beta_2$. This idea will lead us to the definition of the permeable braid monoid $P_B$. Another possibility is to allow the strings to touch at a point and then, at the moment of touching, forget where the upper and lower portions of the strings came from, and then part as newly reconstituted strings. This leads to the definition of the factorisable braid monoid $F_B$. Yet another way to simplify a knot or link is to snip one or more strings. This idea leads to the definition of the inverse braid monoid $IB_n$ which has been studied by Easdown and Lavers [21]. The inverse braid monoid is the monoid of (homotopy classes of) partial braids. A partial braid may be thought of as a braid in which some strings have been removed; see Figure 1.6 for an example. The inverse braid monoid is a braid analogue of $I_n$, the symmetric inverse

![Figure 1.6. A partial braid in $IB_3$.](image)

semigroup on the finite set $\{1, \ldots, n\}$, in the sense that there is a natural map $IB_n \to I_n$ which extends the map $B_n \to S_n$ that sends a braid from $B_n$ to its associated permutation in the symmetric group $S_n$. For example, the partial permutation $f \in I_3$ associated to the partial braid pictured in Figure 1.6 is defined by $1f = 2$, $2f = 3$, and $3 \notin \text{dom}(f)$.

Using duality in category theory, FitzGerald and Leech [33] introduced a new kind of transformation semigroup known as the dual symmetric inverse semigroup. The dual symmetric inverse semigroup on a set $X$ is the inverse monoid $I_X^*$ of all bijections between quotients of $X$. At present it is not known whether there is a natural braid analogue of $I_n^*$,
the dual symmetric inverse semigroup on \( \{1, \ldots, n\} \). However, two of the monoids we study in this thesis, the factorisable braid monoid \( \mathfrak{F}B_n \) and the permeable braid monoid \( \mathfrak{P}B_n \) alluded to above, are both braid analogues of \( \mathfrak{F}_n^* \), the largest factorisable inverse submonoid of \( \mathcal{I}_n^* \). Both of these monoids are derived from the braid group by modifying the notion of equivalence of braids, and allowing certain “compatible” pairs of strings to come together during a homotopy, momentarily creating a double point, and then parting as reconstituted strings.

As an illustration of the ideas involved, consider the braid \( \beta \in \mathcal{B}_4 \) pictured in Figure 1.7.

![Figure 1.7. The braid \( \beta \in \mathcal{B}_4 \).](image.png)

We examine the manner in which \( \beta \) may be unravelled to produce the trivial braid \( 1 \in \mathcal{B}_4 \) pictured in Figure 1.8.

![Figure 1.8. The trivial braid \( 1 \in \mathcal{B}_4 \).](image.png)

We first claim that \( \beta \) may not be continuously deformed into 1. This fact actually follows rather quickly using the general theory of pure braids (braids with trivial permutation; see Section 2.4) but we include the following direct argument for the sake of completeness. Suppose there is a continuous deformation (a homotopy) from \( \beta \) to 1. Then by considering the effect this homotopy has on the first and fourth strings, we see that there must be a homotopy between the braids \( \beta' \) and \( 1' \) pictured in Figure 1.9.
By rescaling the homotopy (if necessary), we obtain a homotopy from $\beta'$ to $1'$ which fixes the right-most string of $\beta'$. Projecting $\beta'$ onto a horizontal plane yields a loop at the first point which contains the fourth point, while the projection of $1'$ yields a degenerate loop at the first point (which does not contain the fourth point); see Figure 1.10.

However, the fourth point remaining inside a loop at the first point is an invariant of any continuous deformation of $\beta'$ which fixes the right-most string. The fact that this invariant fails in the horizontal projection of $1'$ gives the required contradiction.

Suppose now that we modify the notion of braid equivalence. Imagine that the first and fourth strings of $\beta$ are made of some suitably compatible material which allows them to pass through each other. (This is an instance of working within the permeable braid monoid.) Then $\beta$ becomes equivalent to the braid $\gamma \in B_4$ pictured in Figure 1.11.
The braid $\gamma$ may then be represented as a commutator $\gamma = xyx^{-1}y^{-1}$ as in Figure 1.12.

![Figure 1.12. Representing $\gamma$ as a commutator.](image)

Now $x$ and $y$ are canonical generators of a free subgroup of the braid group on the first three strings. (This well-known fact is indirectly implied by Theorem 2.14 below.) Thus there is no hope of finding a homotopy from $\beta$ to 1, even if we allow the first and fourth strings to pass through each other.

Suppose now that we allow the first and fourth strings to touch and, at the moment they touch, forget where the respective parts of the strings came from, and then part as newly reconstituted strings. (We are now working in the factorisable braid monoid.) Figure 1.13 displays a homotopy of this type which completely unravels $\gamma$!

![Figure 1.13. A merge-and-part homotopy from $\gamma$ to 1.](image)

These calculations show that the equation $xyx^{-1}y^{-1} = 1$ holds in the factorisable braid monoid. With a suitable renaming of generators, this verifies one of the relations in a presentation (see for example Theorem 6.22 below).
Overview of Thesis

We begin in Chapter 2 by reviewing basic material on semigroups and presentations. We also devote a section to some classical results from Artin’s seminal papers on the theory of braids [5, 6].

In Chapter 3 we investigate the class of factorisable inverse monoids (FIMs). We review the definition and basic properties of FIMs before exploring several key examples including coset monoids. We then review a structure theorem which describes how any FIM may be built up from a group and a semilattice, and use this to describe presentations of an arbitrary FIM.

We then begin our study of three (factorisable inverse) braid monoids derived from Artin’s braid group $B_n$. These monoids are the inverse braid monoid $IB_n$, the factorisable braid monoid $FB_n$, and the permeable braid monoid $PB_n$. We study all three monoids using much the same techniques; the key property we exploit is factorisability. In Chapter 4 we define $IB_n$, $FB_n$, and $PB_n$ geometrically as monoids of equivalence classes of braids (or partial braids in the case of $IB_n$). We show that all three monoids are FIMs, and then investigate the way in which they may be constructed algebraically from the braid group and certain semilattices. In Chapter 5 we obtain presentations of the three monoids, using the general theory of FIMs developed in Chapter 3, and the algebraic constructions of Chapter 4. We then derive several different presentations of each monoid, including the presentation of $IB_n$ discovered in [21]. These presentations all extend Artin’s presentation of the braid group. We will also see that every presentation of $IB_n$ and $FB_n$ yields a presentation of $I_n$ and $F_n$ respectively. In particular, we will be able to deduce Popova’s presentation of $I_n$ [55] and FitzGerald’s presentation of $F_n$ [32].

In Chapter 6, motivated by questions such as “how can we tell if two partial braids are equivalent?” or “how can we tell if two braids are merge-and-part (or permeable) equivalent?”, we define “pure submonoids” $IP_n \subseteq IB_n$, $FP_n \subseteq FB_n$, and $PP_n \subseteq PB_n$. These submonoids are designed to be analogues of the pure braid group $P_n \subseteq B_n$. We will see that $IP_n$, $FP_n$, and $PP_n$ each decompose as the union of their maximal subgroups, each of these subgroups being isomorphic to either a pure braid group $P_k$ for some $0 \leq k \leq n$ (in the case of $IP_n$), or a quotient of the pure braid group $P_n$ (in the case of $FP_n$ and $PP_n$). We obtain presentations, and semidirect product decompositions, of these maximal subgroups analogous to Artin’s presentation and decomposition of $P_n$ [6]. We use these results to give answers to the questions posed at the beginning of this paragraph. Then in Chapter 7 we use results obtained in Chapters 4, 5, and 6 to give presentations of $IP_n$, $FP_n$, and $PP_n$ which all extend Artin’s presentation of $P_n$. 
In Chapter 8 we focus on the inverse braid monoid $\mathcal{IB}_n$. We identify a submonoid of $\mathcal{IB}_n$ which is naturally isomorphic to $\mathcal{POI}_n$, the “order-preserving submonoid” of $\mathcal{I}_n$ studied by Fernandes [31] and others. We then use a presentation of $\mathcal{POI}_n$ (which differs from Fernandes’) and Artin’s presentation of $\mathcal{B}_n$ to derive another presentation of $\mathcal{IB}_n$. We also define $\mathcal{POIB}_n$, a braid analogue of $\mathcal{POI}_n$ consisting of “order-preserving partial braids”, and give a presentation of $\mathcal{POIB}_n$ using our presentation of $\mathcal{POI}_n$ and Artin’s presentation of the pure braid group $\mathcal{P}_n$. We then discuss further applications of these techniques to the study of $\mathcal{IB}_n \setminus \mathcal{B}_n$, the semigroup of all strictly-partial braids, and the dual symmetric inverse semigroup $\mathcal{I}_n^*$. We conclude with an explanation of “why” the braid monoids $\mathcal{SB}_n$ and $\mathcal{PB}_n$ do not feature in this chapter.

Appendix A is a catalogue of several key presentations derived throughout the thesis.

**General notation**

We now take the opportunity to fix some of the “standard” notation we will be using throughout. We denote by

- $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$ the set of integers;
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots \}$ the set of positive integers;
- $\mathbb{N} = \{0, 1, 2, \ldots \}$ the set of natural numbers;
- $n = \{1, \ldots, n\}$ the set of the first $n$ positive integers (for any $n \in \mathbb{N}$);
- $\mathbb{R}$ the set of real numbers.

All numbers used throughout this thesis are assumed to be integers unless specified otherwise. Thus a statement such as “suppose that $2 \leq i \leq 5$” should be read as “suppose that $i$ is an integer such that $2 \leq i \leq 5$”.

If $i_m, \ldots, i_n$ is a sequence of integers and either

(i) $n = m - 1$ and the subscripts are understood to be increasing, or

(ii) $n = m + 1$ and the subscripts are understood to be decreasing,

then the sequence is understood to be empty. Thus, for example, we interpret the set $n$ described above to be empty if $n = 0$. Likewise, if $x_m, \ldots, x_n$ are elements of a monoid, then the product $x_m \cdots x_n$ is understood to be the empty product (the identity of the monoid) if either (i) or (ii) above holds.
A *groupoid* is a pair \((S, \cdot)\) where \(S\) is a set and \(\cdot\) is a binary operation on \(S\). That is, for each \(s, t \in S\) there is an element of \(S\), denoted \(s \cdot t\), called the *product* of \(s\) by \(t\). In practice, we will often write \(st\) for the product \(s \cdot t\). Further, we will sometimes say that \(S\) is a groupoid under \(\cdot\) or, if we are using juxtaposition to denote multiplication, we will simply say that \(S\) is a groupoid. Two elements \(s, t \in S\) are said to *commute* if \(st = ts\), and \(S\) is said to be *commutative* if each pair of elements of \(S\) commute. An *idempotent* of \(S\) is an element \(e \in S\) such that \(e^2 = e\).

A groupoid \(S\) is a *semigroup* if its operation is associative; that is, for all \(s, t, u \in S\) we have \((st)u = s(tu)\). The set of idempotents of a semigroup \(S\) is denoted \(E(S)\).

A semigroup \(S\) is a *monoid* if there exists an element \(1 \in S\) such that \(1s = sl = s\) for all \(s \in S\); this element is called the *identity element* of \(S\) and is sometimes denoted \(1_S\). If \(S\) is a semigroup, then we write \(S^1\) for either \(S\) if \(S\) has an identity, or the semigroup obtained from \(S\) by artificially adjoining an identity \(1\) to \(S\).

A monoid \(S\) is a *semilattice* if \(S\) is commutative and \(S = E(S)\). The reason for this terminology is that if \(S\) is a monoid of commuting idempotents, then a partial order on \(S\) may be defined, for \(s, t \in S\), by

\[ s \leq t \quad \text{if and only if} \quad st = t, \]

with respect to which every pair of elements \(s, t \in S\) have a unique least upper bound, namely \(st\). (Note that it is usual for authors to define the opposite partial order so that products coincide with greatest lower bounds.)

A monoid \(S\) is a *group* if for each \(s \in S\) there exists \(t \in S\) such that \(st = ts = 1\). The element \(t\) is necessarily unique, and is called the *group* inverse of \(s\) which we denote by \(s^{-1}\). The *group of units* of a monoid \(M\) is the submonoid of all invertible elements of \(M\). This submonoid is a group and is denoted by \(G(M)\).
2.1. SEMIGROUPS AND GROUPS

A semigroup \( S \) is an inverse semigroup if for each \( s \in S \) there exists a unique \( t \in S \) such that \( sts = s \) and \( tst = t \). This element \( t \) is called a (semigroup) inverse of \( s \), and is also denoted by \( s^{-1} \). A semigroup \( S \) is inverse if and only if

(i) for each \( s \in S \) there exists \( t \in S \) such that \( sts = s \); and

(ii) idempotents of \( S \) commute.

See for example [54] for a proof of this non-trivial equivalence. In practice, to show that a given semigroup is inverse, this equivalent criterion is more commonly used than the original definition. If \( S \) is an inverse monoid, then the set \( E(S) \) forms a submonoid of \( S \) called the semilattice of idempotents of \( S \). Antipodal examples of inverse semigroups include

(i) groups, in which the notion of group inverse and semigroup inverse coincide; and

(ii) semilattices, in which every element is its own inverse.

An equivalence relation \( \sim \) on a groupoid \( S \) is a congruence if \( s_1 s_2 \sim t_1 t_2 \) whenever \( s_1, s_2, t_1, t_2 \in S \) with \( s_1 \sim t_1 \) and \( s_2 \sim t_2 \). We may then form the quotient groupoid

\[
S/\sim = \{ [s] \mid s \in S \}
\]

where we have denoted the \( \sim \)-class of \( s \in S \) by \([s]\). The multiplication on \( S/\sim \) is defined, for \( s, t \in S \), by \([s][t] = [st]\). If \( S \) is a semigroup, a monoid, a semilattice, a group, or an inverse semigroup, then \( S/\sim \) is a semigroup, a monoid, a semilattice, a group, or an inverse semigroup respectively. All of these facts are obvious except the last, which follows by Lallement’s Lemma; see for example [54].

Congruences on groups correspond to normal subgroups in a natural way. Suppose that \( G \) is a group. If \( \sim \) is a congruence on \( G \), then it is routine to check that \( G/\sim = G/N_\sim \) where \( N_\sim \) is the normal subgroup \( \{ g \in G \mid g \sim 1 \} \) of \( G \). Conversely, if \( N \) is a normal subgroup of \( G \), then \( G/N = G/\sim_N \) where \( \sim_N \) is the congruence \( \{(g, h) \in G \times G \mid gh^{-1} \in N\} \) on \( G \).

If \( S \) and \( T \) are groupoids (resp. semigroups, monoids, semilattices, groups, inverse semigroups), then a function \( \varphi : S \to T \) is called a groupoid (resp. semigroup, monoid, semilattice, group, inverse semigroup) homomorphism if \((st)\varphi = (s\varphi)(t\varphi)\) for all \( s, t \in S \), with the additional requirement that \( 1_S\varphi = 1_T \) in the case that \( S \) and \( T \) are monoids. An injective (resp. surjective, bijective) homomorphism is called an embedding (resp. epimorphism, isomorphism). The kernel of a homomorphism \( \varphi : S \to T \) is the congruence \( \ker \varphi = \ker(\varphi) = \{(s, t) \in S \times S \mid s\varphi = t\varphi\} \). The Fundamental Homomorphism Theorem states that \( S/\ker(\varphi) \) is isomorphic to \( \text{im}(\varphi) \), the image of \( \varphi \). If \( G \) is a group and \( \varphi : G \to S \) a homomorphism, then we will call \( \varphi \) a group homomorphism (regardless of whether \( S \) is a group); in this case, the normal subgroup \( N_{\ker(\varphi)} \) as described in the previous paragraph
2.2. TRANSFORMATION SEMIGROUPS

coincides with the usual definition \( \{ g \in G \mid g \varphi = 1 \varphi \} \) of the kernel of \( \varphi \). Without causing confusion, we will sometimes denote this normal subgroup by \( \ker(\varphi) \).

2.2. Transformation Semigroups

Let \( X \) be a set, which is fixed throughout this section. The transformation semigroup on \( X \), denoted \( T_X \), is the semigroup of all transformations (functions) \( X \to X \). The semigroup operation on \( T_X \) is composition. In practice, we will usually write \( \alpha \beta \) for the composite \( \alpha \circ \beta \) of two transformations \( \alpha, \beta \in T_X \). The transformation semigroup plays a fundamental role in the theory of semigroups because of Cayley's Theorem.

**Theorem 2.1** (Cayley's Theorem). Let \( S \) be a semigroup. Then \( S \) embeds in \( T_X \) for some set \( X \).

The proof of this theorem involves showing that the map \( \varphi : S \to T_{S^1} \) defined, for \( s \in S \), by \( s \varphi = \varphi_s \) where

\[
\varphi_s : S^1 \to S^1 : t \mapsto ts
\]

for each \( t \in S^1 \)

is an embedding. For more details see for example [39].

The partial transformation semigroup on \( X \), denoted \( \mathcal{PT}_X \), is the semigroup of all partial transformations on \( X \). A partial transformation on \( X \) is a function \( : A \to X \) for some subset \( A \subseteq X \); the domain and image (range) of \( \alpha \) are denoted by \( \text{dom}(\alpha) \) and \( \text{im}(\alpha) \) respectively. The product of two partial transformations \( \alpha, \beta \in \mathcal{PT}_X \) is \( \alpha \beta : C \to X \) where \( C = (\text{im}(\alpha) \cap \text{dom}(\beta)) \alpha^{-1} \) and, for \( x \in C \), we have \( x(\alpha \beta) = (x \alpha) \beta \). Since \( T_X \subseteq \mathcal{PT}_X \) we immediately have the following.

**Theorem 2.2.** Let \( S \) be a semigroup. Then \( S \) embeds in \( \mathcal{PT}_X \) for some set \( X \).

The symmetric inverse semigroup on \( X \), denoted \( \mathcal{I}_X \), is the inverse subsemigroup of \( \mathcal{PT}_X \) consisting of all partial permutations (injective partial transformations) on \( X \). The symmetric inverse semigroup plays a fundamental role in the theory of inverse semigroups because of the Wagner-Preston Theorem. For a proof see for example [54].

**Theorem 2.3** (The Wagner-Preston Theorem). Let \( S \) be an inverse semigroup. Then \( S \) embeds in \( \mathcal{I}_X \) for some set \( X \).
The symmetric group on $X$, denoted $S_X$, is the subgroup of $I_X$ consisting of all permutations (bijective) on $X$. In the proof of Theorem 2.1, the transformations that arise are permutations if $S$ is a group. Thus we have the following.

**Theorem 2.4.** Let $G$ be a group. Then $G$ embeds in $S_X$ for some set $X$.  

The semigroups $T_X$, $PT_X$, $I_X$, and $S_X$ are all monoids with identity $1 = id_X$. Further, $T_X$, $I_X$, and $S_X$ are submonoids of $PT_X$. All of these monoids have the same group of units, namely $S_X$. The various inclusions are pictured in the following diagram.

See [14] for a more comprehensive diagram which includes several additional submonoids of $PT_X$. If $n \in \mathbb{N}$, we write $T_n$ (resp. $PT_n$, $I_n$, $S_n$) for $T_n$ (resp. $PT_n$, $I_n$, $S_n$), recalling that $n = \{1, \ldots, n\}$.

When performing calculations involving (partial) transformations on $n$, we will be greatly aided by the pictures which we now describe. Let $\alpha \in PT_n$. We draw an upper and lower row of $n$ dots representing the elements of $n$ in increasing order (from left to right). A line is drawn from the $i$th dot in the top row to the $j$th dot in the bottom row if and only if $i \in \text{dom}(\alpha)$ and $i\alpha = j$. This diagram is called the *picture* of $\alpha$. In Figure 2.1 we have drawn the picture of the map $\alpha \in PT_8$ for which $\text{dom}(\alpha) = \{1, 3, 6, 7\}$ and $1\alpha = 3$, $3\alpha = 4$, $6\alpha = 8$, $7\alpha = 3$.

![Figure 2.1](image.png)

**Figure 2.1.** The element $\alpha \in PT_8$ defined by $1\alpha = 3$, $3\alpha = 4$, $6\alpha = 8$, $7\alpha = 3$.

Pictures are very useful when performing calculations involving composition. For any $\alpha, \beta \in PT_n$, we obtain the picture of $\alpha \beta$ by

(i) placing the picture of $\alpha$ above the picture of $\beta$ in such a way that lower dot $i$ of $\alpha$ is superimposed on upper dot $i$ of $\beta$;
2.3. Presentations

(ii) erasing the middle row of dots and any lines which do not now join an upper dot of $\alpha$ to a lower dot of $\beta$; and

(iii) straightening out any crooked lines.

See Figure 2.2 for an example.

\[ \alpha = \quad \beta = \quad \rightarrow \quad \rightarrow \quad = \alpha \beta \]

Figure 2.2. The product (composite) of two elements $\alpha, \beta \in \mathcal{PT}_1$.

2.3. Presentations

Presentations encode all of the information about a semigroup and its multiplicative structure in a set of generators and relations. To say that a semigroup $S$ has presentation $\langle X \mid R \rangle$ means that

(i) $S$ is generated by $X$;

(ii) $R$ is a set of equations, or “relations”, satisfied between the elements of $X$; and

(iii) any equation which holds among the elements of $X$ may be derived from those in $R$.

In practice, we will use a much more formal definition involving alphabets, words, and congruences. This section is devoted to providing such a definition. Further, since shortcuts are available when working with monoids and groups, we will treat these separately.

But first, as a simple example of the idea, let $S$ be the subsemigroup of $\mathcal{I}_2$ consisting of the maps $a, b, c, d, e$ pictured in Figure 2.3.

\[ \begin{array}{cccccc}
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\

  a & b & c & d & e
\end{array} \]

Figure 2.3. The elements of the semigroup $S = \{a, b, c, d, e\} \subseteq \mathcal{I}_2$.

We now exhibit a presentation of $S$ which is, in fact, derivable from a special case of Theorem 8.17.
Proposition 2.5. Let $X = \{a, b\}$ and let $R$ be the set of equations
\begin{align*}
a^3 &= a^2 = b^2 = b^3 & \text{(R1)} \\
aba &= a, \ bab &= b. & \text{(R2)}
\end{align*}
Then $S$ has presentation $\langle X \mid R \rangle$.

Proof. Now $S$ is generated by $X$ since we have $c = ba, d = ab,$ and $e = a^2$. This may easily be verified by drawing pictures as in Figure 2.2. It is also easy to check, using pictures, that the stated equations do indeed hold in $S$. To complete the proof we must show that any equation among the elements of $X$ is a consequence of those in $R$. Notice first that any product of the elements $X$ with a repeated letter may be transformed into $a^2$ using equation (R1). Any other product is of the form $(ab)^i, (ba)^j, (ab)^i a,$ or $(ba)^j b$ for some $i \geq 1$ or $j \geq 0$. It is easy to see that these words may be transformed into $ab, ba, a,$ and $b$ (respectively) using only equations (R2). So now suppose that $u$ and $v$ are products of the elements of $X$ such that the equation $u = v$ holds in $S$. Then using equations (R1—R2) we may transform each of $u$ and $v$ into one of $a, b, ab, ba, a^2$. Since each of these five products is a different element of $S$, and the equation $u = v$ holds in $S$, the transformed products must be identical. Working backwards through the previous use of equations (R1—R2), we see that the equation $u = v$ is a consequence of the equations in $R$. This completes the proof that $S$ has presentation $\langle X \mid R \rangle$. \hfill \Box

The calculations in this example may seem ad hoc, but in fact we made use of a set of unique normal forms. See Chapter 8, and also [21, 29, 31], for more proofs of this type. We will spend the rest of this section developing a more formal and systematic framework for dealing with presentations.

2.3.1. Semigroup Presentations.

Let $X$ be a set. A word over $X$ is a string $x_1 \cdots x_k$ where $k \geq 0$ and $x_1, \ldots, x_k \in X$. If $k, l \geq 0$ and $x_1, \ldots, x_k, y_1, \ldots, y_l \in X$, we define the product of the words $u = x_1 \cdots x_k$ and $v = y_1 \cdots y_l$ to be the word $uv = x_1 \cdots x_k y_1 \cdots y_l$. It is easy to check that this product is associative and that the empty word, denoted 1, is an identity. Thus the set $X^*$ of all words over $X$ forms a monoid known as the free monoid on $X$. The set $X^+ = X^* \setminus \{1\}$ of all nonempty words over $X$ forms a subsemigroup of $X^*$ known as the free semigroup on $X$.

Suppose that $R \subseteq X^+ \times X^+$ and let $R^\sharp$ denote the smallest congruence on $X^+$ which contains $R$ (that is, $R^\sharp$ is the intersection of all congruences on $X$ containing $R$). We say that $R^\sharp$ is the congruence generated by $R$. It follows easily that for any $(w, w') \in R^\sharp$ there
exist words \( w = w_1, w_2, \ldots, w_r = w' \in X^+ \) such that for each \( i \) we have \( w_i = u_i z_i v_i \) and \( w_{i+1} = u_i z_{i+1} v_i \) where \( u_i, v_i \in X^* \) and either \((z_i, z_{i+1}) \in R\) or \((z_{i+1}, z_i) \in R\). We say that a semigroup \( S \) has semigroup presentation \( \langle X \mid R \rangle \) if \( S \cong X^+/R^* \) or, equivalently, if there is an epimorphism \( X^+ \to S \) with kernel \( R^* \). If \( f : X^+ \to S \) is such an epimorphism, we will say that \( S \) has semigroup presentation \( \langle X \mid R \rangle \) via \( f \). An element of \( X \) is called a generator.

An element \((w_1, w_2) \in R \) is called a relation, and is often written as an equation: \( w_1 = w_2 \).

To reconcile this notion of presentation with that given at the beginning of Section 2.3, we examine the semigroup \( T = X^+/R^* \). Now elements of \( T \) are \( R^*\)-classes of nonempty words over \( X \). We will denote the \( R^*\)-class of a word \( w \in X^+ \) by \( \tilde{w} \). Put \( \tilde{X} = \{ \tilde{x} \mid x \in X \} \).

Then \( T \) is generated by \( \tilde{X} \) since if \( w = x_1 \cdots x_k \in X^+ \), then we have \( \tilde{w} = \tilde{x}_1 \cdots \tilde{x}_k \). This shows that (i) holds. Let \( \tilde{R} \) be the set of equations \( \{ \tilde{w} = \tilde{w}' \mid (w, w') \in R \} \). Then these equations hold in \( T \) since \( R \subseteq R^* \). Thus (ii) holds. Finally, to show that (iii) holds, we must show that any equation which holds in \( T \) is a consequence of equations from \( \tilde{R} \). So suppose that \( \tilde{w} = \tilde{w}' \) holds in \( T \) where \( w, w' \in X^+ \). Then \( (w, w') \in R \) so there exist words \( w = w_1, w_2, \ldots, w_r = w' \in X^+ \) such that for each \( i \) we have \( w_i = u_i z_i v_i \) and \( w_{i+1} = u_i z_{i+1} v_i \) where \( u_i, v_i \in X^* \) and either \((z_i, z_{i+1}) \in R \) or \((z_{i+1}, z_i) \in R \). Then for each \( i \), \( \tilde{z}_i = \tilde{z}_{i+1} \) or \( \tilde{z}_{i+1} = \tilde{z}_i \) is an equation in \( \tilde{R} \), and it follows by a simple induction that \( \tilde{w} = \tilde{w}' \) is a consequence of equations from \( \tilde{R} \).

### 2.3.2. Monoid Presentations.

Let \( X \) be a set. Suppose that \( R \subseteq X^* \times X^* \) and let \( R^* \) denote the congruence on \( X^* \) generated by \( R \). We say that a monoid \( M \) has monoid presentation \( \langle X \mid R \rangle \) if \( M \cong X^*/R^* \) or, equivalently, if there is an epimorphism \( X^* \to M \) with kernel \( R^* \). If \( f : X^* \to M \) is such an epimorphism, we will say that \( M \) has monoid presentation \( \langle X \mid R \rangle \) via \( f \). Again we call the elements of \( X \) and \( R \) generators and relations respectively, and we typically write relations as equations.

Monoid presentations may be described in terms of semigroup presentations. Suppose that \( X \) is a set, and \( R \subseteq X^* \times X^* \). Then a monoid \( M \) has monoid presentation \( \langle X \mid R \rangle \) if and only if \( M \) has semigroup presentation \( \langle X' \mid R' \rangle \) where \( X' = X \cup \{ e \} \), for some new letter \( e \), and \( R' = \tilde{R} \cup \{ (xe, x), (ex, x) \mid x \in X' \} \). Here we have written \( \tilde{R} \) for the set of relations obtained from \( R \) by replacing each relation \((w_1, w_2) \in R \) with \((\tilde{w}_1, \tilde{w}_2) \) where, for \( w \in X^* \), we have written

\[
\tilde{w} = \begin{cases}  w & \text{if } w \neq 1 \\ e & \text{if } w = 1 \end{cases}
\]
2.3.3. Group Presentations.

Let $X$ be a set and let $X^{-1} = \{x^{-1} \mid x \in X\}$ be a set, disjoint from $X$, consisting of formal inverses for the elements of $X$. If $x \in X$, then we write $(x^{-1})^{-1} = x$. Let

$$R_F = \{(xx^{-1}, 1) \mid x \in X \cup X^{-1}\}.$$ 

The free group on $X$ is defined to be the quotient $F(X) = (X \cup X^{-1})^*/R_F$. It is easily checked that $F(X)$ is a group; the inverse of the $R_F$-class of a word $w = x_1^{\pm 1} \cdots x_k^{\pm 1}$ is the $R_F$-class of the word $x_k^{\mp 1} \cdots x_1^{\mp 1}$ which we denote by $w^{-1}$. To avoid cumbersome notation, we will generally denote elements of $F(X)$ simply as words over $X \cup X^{-1}$, identifying two words $w_1$ and $w_2$ if $(w_1, w_2) \in R_F$.

Suppose that $R \subseteq F(X) \times F(X)$, and let $R^2$ denote the congruence on $F(X)$ generated by $R$. We say that a group $G$ has group presentation $\langle X \mid R \rangle$ if $G \cong F(X)/R^2$ or, equivalently, if there is an epimorphism $F(X) \to G$ with kernel $R^2$. If $f : F(X) \to G$ is such an epimorphism, we will say that $G$ has group presentation $\langle X \mid R \rangle$ via $f$. Again, elements of $X$ and $R$ are called generators and relations respectively.

It is also possible to describe group presentations in terms of monoid or semigroup presentations.

An equivalent way to define group presentations makes use of the relationship between congruences and normal subgroups of a group. Again, suppose that $R \subseteq F(X) \times F(X)$ and let $N_R$ be the normal closure in $F(X)$ of the set $\{w_1w_2^{-1} \mid (w_1, w_2) \in R\}$. That is, $N_R$ is the subgroup of $F(X)$ generated by

$$\{w(w_1w_2^{-1})w^{-1} \mid w \in F(X), (w_1, w_2) \in R\}.$$ 

Then for any $w_1, w_2 \in F(X)$ we have $(w_1, w_2) \in R^2$ if and only if $w_1w_2^{-1} \in N_R$ so that $F(X)/R^2 = F(X)/N_R$. This allows us to say that a group $G$ has group presentation $\langle X \mid R \rangle$ if $G \cong F(X)/N_R$.

2.3.4. Tietze Transformations.

Tietze transformations are elementary operations which may be applied to a presentation in order to produce an “equivalent” presentation. For simplicity, we state the following theorem in the context of semigroup presentations, although it holds for monoid (resp. group) presentations by suitably changing statements concerning free semigroups to the corresponding statements concerning free monoids (resp. free groups). A proof in the context of group presentations may be found in [46] for example.
2.3. PRESENTATIONS

Theorem 2.6. Suppose that \( S \) is a semigroup with semigroup presentation \( \langle X \mid R \rangle \) via \( f \). Then \( S \) has semigroup presentation

(i) \( \langle X \mid R \cup \{(w, w')\} \rangle \) via \( f \), for any \((w, w') \in R^2\);  
(ii) \( \langle X \mid R \setminus \{(w, w')\} \rangle \) via \( f \), for any \((w, w') \in R \) such that \( \langle R \setminus \{(w, w')\} \rangle \rangle = R^2 \);
(iii) \( \langle X \cup \{y\} \mid R \cup \{(y, w)\} \rangle \) for a new letter \( y \notin X \) and any word \( w \in X^+ \), via \( f' : (X \cup \{y\})^+ \rightarrow S : \begin{cases} x \mapsto xf & \text{for each } x \in X \\ y \mapsto wf \end{cases} \)
(iv) \( \langle X \setminus \{z\} \mid R\phi \rangle \) via \( f|_{(X \setminus \{z\})^+} \), for any \((z, w) \in R^2 \) with \( z \in X \) and \( w \in (X \setminus \{z\})^+ \), where \( \phi : X^+ \rightarrow (X \setminus \{z\})^+ \) is defined by \( x\phi = x \) for each \( x \in X \setminus \{z\} \) and \( z\phi = w \).

An application of the above theorem is called a Tietze transformation. The respective parts of the theorem may be thought of as having the following actions on the presentation:

(i) adding a relation which is a consequence of the original relations;
(ii) removing a redundant relation;
(iii) adding a generator which is defined in terms of the original generators; and
(iv) removing a superfluous generator, replacing any occurrence of it in a relation by an equivalent word.

2.3.5. Using Presentations to Define Homomorphisms.

We now describe how presentations may be used to define homomorphisms. The following discussion, with appropriate modifications, also applies to monoid and group presentations.

Suppose that \( S \) is a semigroup with semigroup presentation \( \langle X \mid R \rangle \) via \( f \), and denote by \([w]\) the \( R^2 \)-class of \( w \in X^+ \). Let \( T \) be another semigroup and suppose that we have a map \( \varphi : X \rightarrow T \). Then \( \varphi \) extends to a homomorphism \( X^+ \rightarrow T \), also denoted \( \varphi \), by defining \( (x_1 \cdots x_k)\varphi = (x_1\varphi) \cdots (x_k\varphi) \) for \( x_1, \ldots, x_k \in X^+ \). If \( R \subseteq \ker \varphi \), then \( \varphi \) induces a homomorphism \( \Phi : X^+ / R^2 \rightarrow T \) defined, for \( w \in X^+ \), by \([w]\varphi = w\varphi \). This then allows us to define a homomorphism \( \Phi : S \rightarrow T \), for \( w \in X^+ \), by \((wf)\Phi = [w]\Phi \).

2.3.6. The Power Set of a Finite Set.

Let \( n \) be an integer and put \( 2^n = \{A \mid A \subseteq n\} \), the power set of \( n = \{1, \ldots, n\} \), which we consider as a semilattice under \( \cap \). Let \( X_T = \{\varepsilon_1, \ldots, \varepsilon_n\} \) and let \( R_T \) be the set of relations

\[
\begin{align*}
\varepsilon_i^2 &= \varepsilon_i & \text{for all } i \\
\varepsilon_i \varepsilon_j &= \varepsilon_j \varepsilon_i & \text{for all } i, j. \quad \text{(T1)}
\end{align*}
\]
The following result is well-known and \((2^n \cap \cdot)\), or its dual \((2^n \cup \cdot)\), is called the free semilattice on \(n\). We include a proof for completeness.

**Theorem 2.7.** The power set \(2^n\) has monoid presentation \(\langle X_T \mid R_T \rangle\) via 
\[
\varepsilon_i \mapsto n \setminus \{i\} \quad \text{for each } i.
\]

**Proof.** Let \(\alpha : X_T^* \to 2^n\) denote the map in the statement of the theorem. It is clear that \(\alpha\) is an epimorphism, so it suffices to show that \(\ker \alpha = R_T^\varepsilon\). For the remainder of this proof, we write \(\varepsilon = R_T^\varepsilon\). It is clear that \(\varepsilon \subseteq \ker \alpha\), so suppose that \((w_1, w_2) \in \ker \alpha\). Now by (T1) and (T2), we have 
\[
w_1 \sim \varepsilon_{i_1} \cdots \varepsilon_{i_k} \quad \text{and} \quad w_2 \sim \varepsilon_{j_1} \cdots \varepsilon_{j_{\ell}}
\]
for some \(i_1, \ldots, i_k, j_1, \ldots, j_{\ell} \in n\) with \(i_1 < \cdots < i_k\) and \(j_1 < \cdots < j_{\ell}\). Thus 
\[
n \setminus \{i_1, \ldots, i_k\} = w_1 \alpha = w_2 \alpha = n \setminus \{j_1, \ldots, j_{\ell}\},
\]
from which it follows that \(k = \ell\) and \(i_t = j_t\) for all \(t \in k\), so that 
\[
w_1 \sim \varepsilon_{i_1} \cdots \varepsilon_{i_k} = \varepsilon_{j_1} \cdots \varepsilon_{j_{\ell}} \sim w_2,
\]
completing the proof. \(\square\)

### 2.3.7. The Join Semilattice of Equivalence Relations on a Finite Set.

Suppose that \(X\) is a set. We denote by \(\mathfrak{Eq}_X\) the set of all equivalence relations on \(X\). The *join* \(\mathcal{E}_1 \vee \mathcal{E}_2\) of two equivalence relations \(\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{Eq}_X\) is defined to be the smallest equivalence relation on \(X\) which contains \(\mathcal{E}_1 \cup \mathcal{E}_2\). It is easily checked that \(\mathfrak{Eq}_X\) forms a semilattice under \(\leq\). For any integer \(n\), we denote \(\mathfrak{Eq}_n\) by \(\mathfrak{Eq}_n\). For \(1 \leq i < j \leq n\), let \(\mathcal{E}_{ij}\) denote the equivalence 
\[
\{(r, s) \mid r = s \text{ or } \{r, s\} = \{i, j\}\} \in \mathfrak{Eq}_n.
\]

Let \(X_E = \{\mathcal{E}_{ij} \mid 1 \leq i < j \leq n\}\), and let \(R_E\) denote the set of relations 
\[
\begin{align*}
\varepsilon_{ij}^2 &= \varepsilon_{ij} & \text{for all } i, j & \quad (E1) \\
\varepsilon_{ij} \varepsilon_{kl} &= \varepsilon_{kl} \varepsilon_{ij} & \text{for all } i, j, k, l & \quad (E2) \\
\varepsilon_{ij} \varepsilon_{jk} &= \varepsilon_{jk} \varepsilon_{ij} = \varepsilon_{ik} \varepsilon_{ij} & \text{for all } i, j, k. & \quad (E3)
\end{align*}
\]

**Theorem 2.8** (FitzGerald [32]). The equivalence relation semilattice \(\mathfrak{Eq}_n\) has monoid presentation \(\langle X_E \mid R_E \rangle\) via 
\[
\varepsilon_{ij} \mapsto \mathcal{E}_{ij} \quad \text{for each } i, j.
\]
2.4. THE BRAID GROUP

An element \( E \in \mathfrak{S}_n \) may be described pictorially by

(i) drawing a row of \( n \) points labelled \( 1, \ldots, n \) from left to right;

(ii) joining the vertices by edges in such a way that for each \( i, j \in n \) there is a path from vertex \( i \) to vertex \( j \) if and only if \( (i, j) \in E \).

Such a graphical representation is called a picture of \( E \) and is, of course, not unique. However, pictures prove extremely useful when performing calculations involving joins since a picture of \( E_1 \vee E_2 \), where \( E_1, E_2 \in \mathfrak{S}_n \), may be obtained by superimposing pictures of \( E_1 \) and \( E_2 \). A picture of the generator \( E_{ij} \in \mathfrak{S}_n \) is shown in Figure 2.4.

![Figure 2.4. The generator \( E_{ij} \in \mathfrak{S}_n \).](image)

In Figure 2.5 we use pictures to show that relation (E3) holds in \( \mathfrak{S}_n \).

\[
E_{ij} \vee E_{jk} = \begin{array}{c}
\ldots \\
i \\
\ldots \\
\ldots \\
j \\
\ldots \\
k \\
\ldots \\
n
\end{array} = E_{jk} \vee E_{ik} = E_{ik} \vee E_{ij}
\]

![Figure 2.5. Relation (E3): \( E_{ij} \vee E_{jk} = E_{jk} \vee E_{ik} = E_{ik} \vee E_{ij} \) if \( i < j < k \).](image)

2.4. The Braid Group

Since its introduction in 1925 by Emil Artin [5], the braid group has been studied by many mathematicians for many different reasons. There are numerous ways to define braids; braids can be thought of as geometric objects involving strings, homeomorphisms of a punctured plane, automorphisms of a free group, and many other diverse objects. We follow the geometric approach initiated by Artin. The definitions given here are largely based on those from [38] and [50]; see also [8].

Let \( n \) be a positive integer which is fixed throughout this section. We define sets of points \( \{P_1, \ldots, P_n\} \) and \( \{P'_1, \ldots, P'_n\} \) in \( \mathbb{R}^3 \) by

\[
P_i = (i, 0, 1) \quad \text{and} \quad P'_i = (i, 0, 0) \quad \text{for each} \ i \in n.
\]
2.4. THE BRAID GROUP

We define a string to be a (continuous) embedding \( s \) from the unit interval \([0, 1]\) into \( \mathbb{R}^3 \) such that

(i) \( s(0) = P_i \) and \( s(1) = P'_j \) for some \( i, j \in n \); and

(ii) the \( z \)-coordinate of \( s(t) \) is \( 1 - t \) for each \( t \in [0, 1] \).

The image \( \text{im}(s) \) of a string \( s : [0, 1] \to \mathbb{R}^3 \) is completely determined by \( s \), and completely determines \( s \). As such, we will generally blur the distinction between a string and its image in \( \mathbb{R}^3 \).

A braid on \( n \) (or an \( n \)-string braid, or simply a braid) is a collection \( \beta = (s_1, \ldots, s_n) \) of strings such that \( s_i(0) = P_i \) for each \( i \in n \), and \( s_i(t) \neq s_j(t) \) for each \( t \in [0, 1] \) if \( i, j \in n \) with \( i \neq j \). See Figure 2.6 for an example.

![Figure 2.6. A braid on \{1, 2, 3, 4\}.](image)

We denote by \( B_n \) the set of all braids on \( n \). Notice that any braid \( \beta = (s_1, \ldots, s_n) \in B_n \) induces a permutation \( \beta \in S_n \) defined, for \( i \in n \), by \( s_i(1) = P'_i \). Thus the permutation \( \beta \in S_4 \) associated to the braid \( \beta \in B_4 \) pictured in Figure 2.6 is the 3-cycle \((1, 3, 4)\).

We now describe how the set \( B_n \) may be turned into a groupoid. If \( s \) and \( t \) are two strings such that \( s(1) = t(0) \), then we define a string \( st : [0, 1] \to \mathbb{R}^3 \) by

\[
(st)(t) = \begin{cases} 
T_1(s(2t)) & \text{for } t \in [0, \frac{1}{2}] \\
T_2(t(2t - 1)) & \text{for } t \in [\frac{1}{2}, 1]
\end{cases}
\]

where \( T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^3 \) are the affine transformations defined, for \((x, y, z) \in \mathbb{R}^3\), by

\[
T_1(x, y, z) = (x, y, \frac{z + 1}{2}) \quad \text{and} \quad T_2(x, y, z) = (x, y, \frac{z}{2}).
\]

Now suppose that \( \beta = (s_1, \ldots, s_n) \) and \( \gamma = (t_1, \ldots, t_n) \) are two braids. We then define the product of \( \beta \) and \( \gamma \) to be the braid \( \beta\gamma = (s_1t_{i_\beta}, \ldots, s_nt_{i_\gamma}) \). See Figure 2.7 for an example.
Informally, to calculate the product $\beta \gamma$ of two braids $\beta, \gamma \in B_n$, one joins the bottom of $\beta$ to the top of $\gamma$, and then uniformly shrinks the resulting object until it lies between the $z = 0$ and $z = 1$ planes.

Two braids $\beta = (s_1, \ldots, s_n)$ and $\gamma = (t_1, \ldots, t_n)$ are said to be equivalent if there is a homotopy through braids from $\beta$ to $\gamma$ or, more formally, if there are continuous maps

$$F_j : [0, 1] \times [0, 1] \to \mathbb{R}^3$$

for each $j \in n$ such that

(i) $F_j(0, t) = s_j(t)$ and $F_j(1, t) = t_j(t)$ for each $t \in [0, 1]$;

(ii) for each $u \in [0, 1]$, the maps $s^u_j : [0, 1] \to \mathbb{R}^3$ defined, for $t \in [0, 1]$, by

$$s^u_j(t) = F_j(u, t)$$

are strings; and

(iii) for each $u \in [0, 1]$, the collection $(s^u_1, \ldots, s^u_n)$ is a braid.

In this case, we write $\beta \sim \gamma$. Informally, $\beta \sim \gamma$ if $\beta$ can be continuously deformed, through braids, to $\gamma$.

We denote the $\sim$-class of a braid $\beta \in B_n$ by $[\beta]$. Pictured in Figure 2.8 is a braid which is equivalent to the product $\alpha \beta$ calculated in Figure 2.7.

In general, determining equivalence of braids is nontrivial.

**Problem 2.9.** *Given two braids $\beta, \gamma \in B_n$, determine whether or not $\beta \sim \gamma$.***
A solution to this problem, of which there are many, will be useful throughout this thesis. We will outline one solution later in this section once we have developed more of the theory of braids.

Now it is clear that \( \sim \) is a groupoid congruence on \( B_n \). Thus we may form the quotient groupoid

\[
\mathcal{B}_n = B_n / \sim = \{ [\beta] \mid \beta \in B_n \}
\]

with multiplication defined, for \( \beta, \gamma \in B_n \), by \([\beta][\gamma] = [\beta \gamma]\). One may easily check that this product on \( \sim \)-classes is associative so that \( \mathcal{B}_n \) forms a semigroup. In fact, \( \mathcal{B}_n \) is a group with identity \([1]\) where the identity braid, sometimes called the trivial braid, \( 1 \in B_n \) is pictured in Figure 2.9 (for \( n = 5 \)).

![Figure 2.9. The identity braid 1 ∈ B₅.](image)

The inverse of \([\beta]\) for \( \beta = (s_1, \ldots, s_n) \in B_n \) is \([\beta]^{-1} = [\beta^{-1}]\) where \( \beta^{-1} = (s_1^{-1}, \ldots, s_n^{-1}) \) with each string \( s_j^{-1} \) defined, for \( t \in [0, 1] \), by \( s_j^{-1}(t) = T(s_j(T^{-1}(t))) \), where \( T: \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by \( T(x, y, z) = (x, y, 1 - z) \) for all \((x, y, z) \in \mathbb{R}^3\). See Figure 2.10 for an example.

![Figure 2.10. The inverse of a braid: β⁻¹ β⁻¹ ∼ β⁻¹ β ∼ 1.](image)

The group \( \mathcal{B}_n \) is known as the braid group on \( n \). Now if \( \beta, \gamma \in \mathcal{B}_n \) and \( \beta \sim \gamma \), then we clearly have \( \overline{\beta} = \overline{\gamma} \) so that we may define a map \( \overline{\cdot} : \mathcal{B}_n \to S_n \) by \( \overline{[\beta]} = \overline{\beta} \) for each \( \beta \in \mathcal{B}_n \) which is easily seen to be an epimorphism. The kernel of this map, denoted \( \mathcal{P}_n \), is known as the pure braid group on \( n \) which consists of all equivalence classes of braids with trivial permutation; such a braid is called a pure braid.
To avoid cumbersome notation, we will usually blur the distinction between a braid $\beta \in B_n$ and its equivalence class $[\beta] \in B_n$. Thus, we will view elements of $B_n$ as braids, identifying two braids if they are $\sim$-equivalent. At certain times however it will be convenient to make a clear distinction between a braid and its equivalence class; the reader will be alerted when this is the case.

2.4.1. Presentations of the Braid Group.

Artin gave a group presentation of $B_n$ in his seminal paper on braids [5]; see also [6]. Since then, there have been numerous proofs of Artin’s presentation.

For $1 \leq i \leq n-1$ we denote by $\xi_i$ and $\xi_i^{-1}$ the ($\sim$-class of the) braids pictured in Figure 2.11.

$$\xi_i$$

Figure 2.11. The braids $\xi_i$ (left) and $\xi_i^{-1}$ (right) in $B_n$.

Put $X_B = \{\sigma_1, \ldots, \sigma_{n-1}\}$, and let $R_B$ denote the set of relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1 \quad (B1)
\]

\[
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1. \quad (B2)
\]

**Theorem 2.10** (Artin [5]). *The braid group $B_n$ has group presentation $(X_B \mid R_B)$ via

\[
\sigma_i \mapsto \xi_i \quad \text{for each } i. \quad \square
\]

We now describe an alternative presentation of $B_n$, known as the band presentation, given in [10]. For $1 \leq i < j \leq n$ we denote by $\xi_{ij}$ and $\xi_{ij}^{-1}$ the braids pictured in Figure 2.12.

$$\xi_{ij}$$

Figure 2.12. The braids $\xi_{ij}$ (left) and $\xi_{ij}^{-1}$ (right) in $B_n$.

It is easy to check that for each $1 \leq i < j \leq n$ we have

\[
\xi_{ij} = (\xi_{j-1}^{-1} \cdots \xi_{i+1}^{-1}) \xi_i (\xi_{i+1} \cdots \xi_{j-1}) = (\xi_i \cdots \xi_{j-2}) \xi_{j-1} (\xi_{j-2}^{-1} \cdots \xi_{i-1}^{-1}).
\]
Put $X'_B = \{\sigma_{ij} \mid 1 \leq i < j \leq n\}$ and let $R'_B$ denote the set of relations
\[
\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \quad \text{if } i < j < k < l \text{ or } i < k < l < j \quad (B1)'
\]
\[
\sigma_{ij}\sigma_{jk} = \sigma_{jk}\sigma_{ik} = \sigma_{ik}\sigma_{ij} \quad \text{if } i < j < k. \quad (B2)'
\]

**Theorem 2.11** (Birman, Ko, and Lee [10]). The braid group $B_n$ has group presentation $\langle X'_B \mid R'_B \rangle$ via
\[ \sigma_{ij} \mapsto \varsigma_{ij} \quad \text{for each } i, j. \]

**Remark 2.12.** The reader may have noticed that relation (B2)' is essentially the same as relation (E3) in FitzGerald’s presentation of $E_qn$ in Theorem 2.8. In fact, interpreting the braid presentation as a semilattice presentation (that is, by implicitly assuming the presence of relations which declare the generators to be commuting idempotents) we obtain FitzGerald’s presentation of $E_qn$. This inspires the definition, for an arbitrary Coxeter group $W$ (see [40]), of a semilattice $E_{qW}$, which is related to $W$ in the same way that $E_qn$ is related to the symmetric group $S_n$; see [22]. The elements of $E_{qW}$ may be realised in several different ways: as reflection subgroups of $W$, as equivalence relations on the Coxeter complex of $W$ or, when $W$ is of type $A, B, D,$ or $I$, as equivalence relations on a much simpler object (such as $n$ in the type $A$ case).

### 2.4.2. A Presentation of the Pure Braid Group.

In [6], Artin discovered a presentation of the pure braid group. For $1 \leq i < j \leq n$ we denote by $\alpha_{ij}$ and $\alpha_{ij}^{-1}$ the braids pictured in Figure 2.13.

![Figure 2.13](image-url)

**Figure 2.13.** The braids $\alpha_{ij}$ (left) and $\alpha_{ij}^{-1}$ (right) in $B_n$.

Notice that each $\alpha_{ij}$ is a pure braid. Again, it is easy to check that for each $1 \leq i < j \leq n$ we have
\[ \alpha_{ij} = (\varsigma_{j-1} \cdots \varsigma_{i+1})\varsigma_i^2(\varsigma_{i+1} \cdots \varsigma_{j-1}) = (\varsigma_{i}^{-1} \cdots \varsigma_{j-2}^{-1})\varsigma_{j-1}(\varsigma_{j-2} \cdots \varsigma_{i}). \]
Let \( X_P = \{a_{ij} \mid 1 \leq i < j \leq n\} \), and let \( R_P \) denote the set of relations
\[
\begin{align*}
  a_{rs}a_{ij}a_{rs}^{-1} &= a_{ij} & \text{if } i < r \text{ or } i > s & \quad (P1) \\
  a_{rs}a_{sj}a_{rs}^{-1} &= (a_{sj}^{-1}a_{rj}^{-1})a_{sj}(a_{rj}a_{sj}) & \quad (P2) \\
  a_{rs}a_{rj}a_{rs}^{-1} &= a_{sj}^{-1}a_{rj}a_{sj} & \quad (P3) \\
  a_{rs}a_{ij}a_{rs}^{-1} &= (a_{sj}^{-1}a_{rj}^{-1})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) & \text{if } r < i < s, & \quad (P4)
\end{align*}
\]
where in each relation we have \( s < j \).

**Theorem 2.13** (Artin [6]). The pure braid group \( \mathcal{P}_n \) has group presentation \( \langle X_P \mid R_P \rangle \) via
\[
  a_{ij} \mapsto \alpha_{ij} \quad \text{for each } i, j.
\]

In particular, \( \mathcal{P}_n \) is generated by \( \{\alpha_{ij} \mid 1 \leq i < j \leq n\} \). We will frequently make use of this fact without reference to Theorem 2.13.

Let \((w, w') \in R_P\), and let \( 1 \leq i < j \leq n \). The reader may easily check that the sum of the exponents of \( a_{ij} \) in \( w \) is the same as the corresponding exponent sum for \( w' \). This allows us to define a map \( \exp_{ij} : \mathcal{P}_n \to (\mathbb{Z}, +) \) by
\[
  \alpha_{rs} \mapsto \begin{cases} 
    1 & \text{if } r = i \text{ and } s = j \\
    0 & \text{otherwise.}
  \end{cases}
\]
In [6], Artin also described a normal form (called the *combed form*) of a pure braid. For each \( 2 \leq j \leq n \) let \( \mathcal{U}_j \) be the subgroup of \( \mathcal{P}_n \) generated by the set \( \{\alpha_{ij} \mid 1 \leq i < j\} \).

**Theorem 2.14** (Artin [6]). The pure braid group has a semidirect product decomposition
\[
  \mathcal{P}_n = \mathcal{U}_n \rtimes \left( \mathcal{U}_{n-1} \rtimes \cdots \rtimes (\mathcal{U}_3 \rtimes \mathcal{U}_2) \cdots \right).
\]
Further, each \( \mathcal{U}_j \) is a free group of rank \( j - 1 \).

For \( 2 \leq j \leq n \) put \( X^j_P = \{a_{ij} \mid 1 \leq i < j\} \), and write \( \sim_P \) for the congruence \( R^j_P \). As a result of Theorems 2.13 and 2.14, any word \( w \in F(X_P) \) is \( \sim_P \)-equivalent to a word of the form \( w_n \cdots w_2 \) where each \( w_j \in F(X^j_P) \) is uniquely determined. If \( w, w' \in F(X_P) \) with \( w \sim_P w_n \cdots w_2 \) and \( w' \sim_P w'_n \cdots w'_2 \) where each \( w_j, w'_j \in F(X^j_P) \), then \( w \sim_P w' \) if and only if \( w_j = w'_j \) for each \( 2 \leq j \leq n \).

### 2.4.3. A Solution to Problem 2.9.

The results concerning \( \mathcal{P}_n \) above allow us to outline a solution to Problem 2.9, namely the problem of determining whether two given braids \( \beta \) and \( \gamma \) are \( \sim \)-equivalent. Now \( \beta \sim \gamma \) if
and only if $\beta \gamma^{-1} \sim 1$, and it is clearly impossible to have $\beta \gamma^{-1} \sim 1$ if the permutation $\beta \gamma^{-1}$ is not the identity map. Thus, Problem 2.9 reduces to the following problem.

**Problem 2.15.** Given a pure braid $\beta$, determine whether or not $\beta \sim 1$.

There exist many varied solutions to this problem; see for example [5, 6, 8, 10, 34, 38, 50] and references therein. In [50], an algorithm is given which expresses a pure braid $\beta \in P_n$ as a product $\beta = u_n \cdots u_2$ where each $u_j$ is a product of the generators of $U_j$. For each $j$, let $\hat{u}_j \in F(X^j)$ be the word obtained from $u_j$ by changing each $\alpha_{ij}^{\pm 1}$ to $a_{ij}^{\pm 1}$. Then by Theorem 2.14 and the comments following it, $\beta \sim 1$ if and only if each word $\hat{u}_j$ is (freely equivalent to) the trivial word.
CHAPTER 3

Factorisable Inverse Monoids

Many of the monoids we study in this thesis belong to the class of factorisable inverse monoids. For this reason we devote a chapter to the general theory of the monoids in this class. We begin by reviewing the definition and listing several central examples, including coset monoids, before proving a well-known structure theorem. We then use the structure theorem to prove results concerning presentations which will be useful in subsequent chapters.

3.1. Factorisable Inverse Monoids

3.1.1. Definitions and Basic Properties.

Let $M$ be an inverse monoid. Recall that we denote by

\[ E(M) = \{ e \in M \mid e^2 = e \} \quad \text{and} \quad G(M) = \{ g \in M \mid gg^{-1} = g^{-1}g = 1 \} \]

the semilattice of idempotents and group of units of $M$ respectively. We say that $M$ is factorisable if $M = E(M)G(M)$ in which case we also have $M = G(M)E(M)$. That is, $M$ is factorisable if and only if every element of $M$ is a product of an idempotent and a unit.

In general, $E(M)G(M)$ is the largest factorisable inverse submonoid of $M$. We call this submonoid the factorisable part of $M$ and denote it by $F(M)$. See the seminal paper by Chen and Hsieh [11] for further details.

Suppose that $M$ is a factorisable inverse monoid and put $E = E(M)$ and $G = G(M)$. For $e \in E$ let

\[ G_e = \{ g \in G \mid eg = e \}. \]

It is easy to check that $G_e$ is a subgroup of $G$ for each $e \in E$, and that $G_e \subseteq G_{ef}$ for each $e, f \in E$. It is also immediate that $gg_e g^{-1} = g_{geg^{-1}}$ for each $e \in E$ and $g \in G$. The subgroups $G_e$ will play a prominent role in our study of factorisable inverse monoids.

The rest of this section is devoted to providing examples of factorisable inverse monoids (henceforth FIMs). Antipodal examples include groups and semilattices. We will now investigate the (factorisable parts of the) symmetric, and dual symmetric, inverse semigroups.
3.1. FACTORISABLE INVERSE MONOIDS

3.1.2. The Symmetric Inverse Semigroup.

Let $X$ be a set. Recall that we denote by $\mathcal{I}_X$ the symmetric inverse semigroup on $X$. We now collect some results concerning $\mathcal{I}_X$. Proofs of the various parts may be found in [11] and [54].

**Proposition 3.1.** The symmetric inverse semigroup $\mathcal{I}_X$ is an inverse monoid. Further,

(i) $E(\mathcal{I}_X) = \{ \text{id}_A | A \subseteq X \}$ is isomorphic to $2^X = \{ A | A \subseteq X \}$ considered as a semilattice under $\cap$;

(ii) $G(\mathcal{I}_X) = \{ \alpha \in \mathcal{I}_X | \text{dom}(\alpha) = \text{im}(\alpha) = X \} = \mathcal{S}_X$ is the symmetric group on $X$;

(iii) $F(\mathcal{I}_X) = \{ \alpha \in \mathcal{I}_X | |X \setminus \text{dom}(\alpha)| = |X \setminus \text{im}(\alpha)| \}$; and

(iv) $\mathcal{I}_X$ is factorisable if and only if $X$ is finite.

A partial permutation $\alpha \in \mathcal{I}_X$ such that $|X \setminus \text{dom}(\alpha)| = |X \setminus \text{im}(\alpha)|$ is said to be uniform. Thus $F(\mathcal{I}_X)$, the factorisable part of $\mathcal{I}_X$, is the monoid of all uniform partial permutations. This monoid is denoted by $\mathfrak{S}_X$. In particular, $\mathcal{I}_n = \mathfrak{S}_n$ is factorisable for any integer $n$. A partial permutation $\alpha \in \mathcal{I}_X$ on the infinite set $X = \{1, 2, 3, \ldots \}$ which is not uniform is pictured in Figure 3.1. In this example we have $|X \setminus \text{dom}(\alpha)| = 0$ while $|X \setminus \text{im}(\alpha)| = 1$.

![Figure 3.1. An element of $\mathcal{I}_X \setminus \mathfrak{S}_X$ where $X = \{1, 2, 3, \ldots \}$](image)

3.1.3. The Dual Symmetric Inverse Semigroup.

Let $X$ be a set. The set of all equivalence relations on $X$ is denoted $\mathcal{E}_qX$. Recall that the join $E_1 \vee E_2$ of two equivalences $E_1, E_2 \in \mathcal{E}_qX$ is the smallest equivalence on $X$ which contains $E_1 \cup E_2$, and that $\mathcal{E}_qX$ forms a semilattice under $\vee$. It is clear that $E_1 \subseteq E_2$ if and only if $E_1 \vee E_2 = E_2$.

For any $E \in \mathcal{E}_qX$, we may form the **quotient** $X/E$ which consists of all $E$-classes of $X$. If $E_1, E_2 \in \mathcal{E}_qX$, and if $A = X/E_1$ and $B = X/E_2$, then we write $A \prec B$ if $E_1 \subseteq E_2$. We also define $A \vee B = X/(E_1 \vee E_2)$. Thus $A \prec B$ if and only if each member of $A$ is a subset of a member of $B$.

A **block bijection** on $X$ is a bijection $\theta : A \to B$ between two quotients of $X$. The quotients $A$ and $B$ are called the **domain** and **image** of $\theta$ respectively, and we write $A = \text{dom}(\theta)$ and $B = \text{im}(\theta)$. 
Suppose that \( \theta : A \to B \) is a block bijection, and choose an indexing set \( I \) such that \( A = \{A_i \mid i \in I\} \) and \( B = \{B_i \mid i \in I\} \). If \( J \subseteq I \) and if
\[
A = \bigcup_{j \in J} A_j \quad \text{and} \quad B = \bigcup_{j \in J} B_j,
\]
then we put
\[
A \theta = \bigcup_{j \in J} (A_j \theta) \quad \text{and} \quad B \theta^{-1} = \bigcup_{j \in J} (B_j \theta^{-1}).
\]
If \( A \preceq A' \) and \( B \preceq B' \), then we put
\[
A' \theta = \{A \mid A \in A'\} \quad \text{and} \quad B' \theta^{-1} = \{B \theta^{-1} \mid B \in B'\}.
\]
The product of two block bijections \( \theta_1 : A \to B \) and \( \theta_2 : C \to D \) is \( \theta_1 \theta_2 : E \to F \) where
\[
E = (B \lor C) \theta_1^{-1} \quad \text{and} \quad F = (B \lor C) \theta_2
\]
and, for \( E \in E \), we have \( E(\theta_1 \theta_2) = (E \theta_1) \theta_2 \).

This multiplication turns the set of all block bijections on \( X \) into an inverse monoid known as the dual symmetric inverse semigroup on \( X \), and denoted by \( \mathcal{I}_X^{\ast} \). This monoid was introduced and studied in [33]; see also [18, 43]. As the name suggests, there is a natural duality between \( \mathcal{I}_X \) and \( \mathcal{I}_X^{\ast} \); see [33] for more details. If \( n \) is an integer, we will write \( \mathcal{I}_n^{\ast} \) for \( \mathcal{I}_n \).

Denoting by \( 1 = \{(x, x) \mid x \in X\} \in \mathcal{E}q_X \) the identity equivalence, we may identify the quotient \( X/1 \) with \( X \) in a natural way. We may then identify a block bijection \( X/1 \to X/1 \) with a permutation \( X \to X \). Also, for \( E \in \mathcal{E}q_X \) we will write \( id_E \) for the identity map on \( X/E \). We now collect some results concerning \( \mathcal{I}_X^{\ast} \); see [33] for proofs.

**Proposition 3.2.** The dual symmetric inverse semigroup \( \mathcal{I}_X^{\ast} \) is an inverse monoid. Further,

(i) \( E(\mathcal{I}_X^{\ast}) = \{id_E \mid E \in \mathcal{E}q_X\} \) is isomorphic to the semilattice \( \mathcal{E}q_X \) under \( \lor \);

(ii) \( G(\mathcal{I}_X^{\ast}) = \{\theta \in \mathcal{I}_X^{\ast} \mid \text{dom}(\theta) = \text{im}(\theta) = X/1\} \) is (isomorphic to) the symmetric group \( S_X \) on \( X \);

(iii) \( F(\mathcal{I}_X^{\ast}) = \{\theta \in \mathcal{I}_X^{\ast} \mid |A \theta| = |A| (\forall A \in \text{dom}(\theta))\} \); and

(iv) \( \mathcal{I}_X^{\ast} \) is factorisable if and only if \( |X| \leq 2 \). \( \square \)

A block bijection \( \theta \in \mathcal{I}_X^{\ast} \) such that \( |A \theta| = |A| \) for each \( A \in \text{dom}(\theta) \) is said to be uniform. Thus \( F(\mathcal{I}_X^{\ast}) \), the factorisable part of \( \mathcal{I}_X^{\ast} \), is the monoid of all uniform block bijections on \( X \).
This monoid is denoted by $\mathcal{I}_n^*$, and if $n$ is a positive integer we will write $\mathcal{I}_n^*$ for $\mathcal{I}_n^*$. An example of a block bijection in $\mathcal{I}_3^*$ which is not uniform is $\theta : A \to B$ where

$$A = \{\{1\}, \{2, 3\}\} \quad \text{and} \quad B = \{\{1, 2\}, \{3\}\}$$

and $\{1\} \mapsto \{1, 2\}$, $\{2, 3\} \mapsto \{3\}$.

A picture of a block bijection $\theta \in \mathcal{I}_n^*$, with $\text{dom}(\theta) = n/\mathcal{E}_1$ and $\text{im}(\theta) = n/\mathcal{E}_2$ where $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}_n$, may be constructed by

(i) drawing an upper and lower row of $n$ dots representing the elements of $n$ in increasing order (from left to right);

(ii) drawing a picture of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) on the upper (resp. lower) row of dots;

(iii) joining (the connected component representing) each equivalence class $A \in n/\mathcal{E}_1$ in the top row to the equivalence class $A\theta \in n/\mathcal{E}_2$ in the bottom row.

For example, let $A = \{A_1, A_2, A_3, A_4\}$ and $B = \{B_1, B_2, B_3, B_4\}$ where

$$A_1 = \{1, 2\} \quad A_2 = \{3\} \quad A_3 = \{4, 6, 7\} \quad A_4 = \{5, 8\}$$

$$B_1 = \{2, 4\} \quad B_2 = \{5, 6, 7, 8\} \quad B_3 = \{3\} \quad B_4 = \{1\}$$

and define $\theta : A \to B$ in $\mathcal{I}_8^*$ by $A_i\theta = B_i$ for each $i$. A picture of $\theta$ is illustrated in Figure 3.2.

![Figure 3.2. A picture of a block bijection $\theta \in \mathcal{I}_8^*$](image)

To draw a picture of the product $\theta_1\theta_2$ of two block bijections $\theta_1, \theta_2 \in \mathcal{I}_n^*$,

(i) place a picture of $\theta_1$ above a picture of $\theta_2$ in such a way that lower dot $i$ of $\theta_1$ is superimposed on upper dot $i$ of $\theta_2$; and

(ii) erase the middle row of dots and use the resulting diagram to determine which vertices must be joined in the picture of $\theta_1\theta_2$.

An example has been carried out in Figure 3.3.

![Figure 3.3. The product of two block bijections $\theta_1, \theta_2 \in \mathcal{I}_4^*$](image)
An alternative picture in the special case of a \textit{uniform} block bijection $\theta = \text{id}_E \pi \in \mathfrak{S}_n^*$ may be obtained by superimposing a picture of $E \in \mathfrak{C}_n$ on the upper row of dots of a picture of $\pi \in \mathcal{S}_n$. See Figure 3.4 for an example.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure3.4.png}
\caption{Two pictures of a uniform block bijection in $\mathfrak{S}_5^*$.}
\end{figure}

A picture of the product of two uniform block bijections $\theta_1 = \text{id}_E \pi_1, \theta_2 = \text{id}_E \pi_2 \in \mathfrak{S}_n^*$ may be obtained by

(i) placing a picture of $\theta_1$ above a picture of $\theta_2$, and erasing the middle row of dots as usual;

(ii) “sliding” the picture of $E_2$ up the lines of $\pi_1$;

(iii) simplifying the picture of the equivalence which is now on the top row of dots, and straightening out any crooked lines.

An example has been carried out in Figure 3.5.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure3.5.png}
\caption{The product of two uniform block bijections $\theta_1, \theta_2 \in \mathfrak{S}_4^*$.}
\end{figure}

\section*{3.2. Coset Monoids and Embeddings}

Coset monoids were introduced by Schein in \cite{57, 58} and have been studied by many others; see for example \cite{19, 25, 28, 44, 48, 49, 51}. Coset monoids are especially important examples of (factorisable) inverse monoids since every inverse semigroup embeds in the coset monoid of some group \cite{48}; see also \cite{19} for a description of a universal property possessed by coset monoids. In this section we review the definition of coset monoids, and some results concerning embeddings of FIMs in the coset monoid of their group of units. We also consider in detail the symmetric, and dual symmetric, inverse semigroups.
3.2. COSET MONOIDS AND EMBEDDINGS

Let $G$ be a group, and let $\mathcal{S}(G) = \{H \mid H \leq G\}$ be the set of subgroups of $G$. The *join* $H \vee K$ of two subgroups $H, K \in \mathcal{S}(G)$ is defined to be $(H \cup K)$, the smallest subgroup of $G$ containing $H \cup K$. Let

$$\mathcal{C}(G) = \{Hg \mid H \leq G, \ g \in G\}$$

denote the set of all cosets of all subgroups of $G$. A product $*$ is defined on $\mathcal{C}(G)$, for $H, K \leq G$ and $g, l \in G$, by

$$(Hg) * (Kl) = (H \vee gKg^{-1})gl,$$

which may be shown to be the smallest coset of a subgroup of $G$ which contains the set product $HgKl$. It is easy to check that $*$ is associative and that $\mathcal{C}(G)$ is a monoid with identity $\{1\}$. The monoid $\mathcal{C}(G)$ is called the *coset monoid of $G$*. The proof of the following is routine.

**Proposition 3.3.** The monoid $\mathcal{C}(G)$ is a factorisable inverse monoid under $*$ with

$$\mathcal{E}(\mathcal{C}(G)) = \mathcal{S}(G) \quad \text{and} \quad \mathcal{G}(\mathcal{C}(G)) = \{\{g\} \mid g \in G\} \cong G. \quad \square$$

Let $\mathcal{C}$ denote the class of FIMs $M$ for which, writing $E = \mathcal{E}(M)$ and $G = \mathcal{G}(M)$, the map

$$E \to \mathcal{S}(G) : e \mapsto G_e \quad \text{for each} \ e \in E$$

is a semilattice embedding of $E$ in $\mathcal{S}(G)$. The following was shown in [19].

**Theorem 3.4.** A monoid $M$ embeds in $\mathcal{C}(\mathcal{G}(M))$ if $M \in \mathcal{C}$. \quad \square

This result was proved by showing that if $M \in \mathcal{C}$ with $E = \mathcal{E}(M)$ and $G = \mathcal{G}(M)$, then the semilattice embedding $E \to \mathcal{S}(G) : e \mapsto G_e$ ($e \in E$) may be extended to a monoid embedding $M \to \mathcal{C}(G) : eg \mapsto G_egg$ ($e \in E, g \in G$).

It was shown in [25] that the condition $M \in \mathcal{C}$ is also *necessary* for a finite FIM $M$ to embed in $\mathcal{C}(\mathcal{G}(M))$. Examples were given in [25] of (necessarily infinite) FIMs $M \not\in \mathcal{C}$ which embed in $\mathcal{C}(\mathcal{G}(M))$. See also [28] for a way to generalise the notion of a coset monoid which allows the elements of any FIM, for which the map $e \mapsto G_e$ is injective, to be represented by cosets of their group of units.

Now let $X$ be a set. We conclude this section by again considering the FIMs $\mathfrak{F}_X$ and $\mathfrak{F}^*_X$. In particular, we will show that $\mathfrak{F}_X \in \mathcal{C}$ if and only if $X$ is empty (Proposition 3.5) while, in contrast, $\mathfrak{F}^*_X \in \mathcal{C}$ if and only if $X$ is finite (Theorem 3.10).
3.2. COSET MONOIDS AND EMBEDDINGS

3.2.1. The Monoid of Uniform Partial Permutations on $X$.

Recall that $\mathcal{F}_X = F(I_X)$ is the monoid of all uniform partial permutations on $X$. Put $G = G(\mathcal{F}_X) = S_X$, and

$$E = E(\mathcal{F}_X) = \{ \text{id}_A \mid A \subseteq X \}.$$ 

One may easily check that for any $A \subseteq X$,

$$G_{\text{id}_A} = \{ \pi \in G \mid a\pi = a \ (\forall a \in A) \}$$

is the pointwise stabiliser of $A$. We denote this subgroup by $\text{Stab}(A)$.

**Proposition 3.5.** The monoid $\mathcal{F}_X$ belongs to the class $\mathcal{C}$ if and only if $X = \emptyset$.

**Proof.** Now $\mathcal{F}_\emptyset = E(\mathcal{F}_\emptyset)$ consists solely of the empty map and as such is a member of $\mathcal{C}$. Suppose now that $X$ is nonempty. Then for any $x \in X$, $\text{Stab}(X) = \text{Stab}(X \setminus \{x\}) = \{1\}$ so that $\mathcal{F}_X \notin \mathcal{C}$. \qed

**Remark 3.6.** If $|X| \geq 2$, then the map

$$A \to G_{\text{id}_A} = \text{Stab}(A) \text{ for each } A \subseteq X$$

is not even a semilattice homomorphism. To see this, suppose that $x, y \in X$ and $x \neq y$. Then, writing $A = X \setminus \{x\}$ and $B = X \setminus \{y\}$, we have

$$G_{\text{id}_A} \lor G_{\text{id}_B} = \text{Stab}(A) \lor \text{Stab}(B) = \{1\} \lor \{1\} = \{1\},$$

while the transposition $t_{xy}$, which interchanges $x$ and $y$, is in

$$\text{Stab}(X \setminus \{x, y\}) = \text{Stab}(A \cap B) = G_{\text{id}_{A \cap B}} = G_{\text{id}_{A \cap B} \text{id}_B}.$$ 

3.2.2. The Monoid of Uniform Block Bijections on $X$.

Recall that $\mathcal{F}_X^* = F(I_X^*)$ is the monoid of all uniform block bijections on $X$. Put $G = S_X$, which we identify with $G(\mathcal{F}_X^*)$, and

$$E = E(\mathcal{F}_X^*) = \{ \text{id}_\mathcal{E} \mid \mathcal{E} \in \mathcal{Eq}_X \}.$$ 

Again it is easy to check that for any $\mathcal{E} \in \mathcal{Eq}_X$,

$$G_{\text{id}_\mathcal{E}} = \{ \pi \in G \mid (x, x\pi) \in \mathcal{E} \ (\forall x \in X) \}$$

is the subgroup of $G$ consisting of all permutations on $X$ which fix the $\mathcal{E}$-classes of $X$ setwise. We call this subgroup the stabiliser of $\mathcal{E}$, and denote it by $\text{Stab}(\mathcal{E})$. The next result also follows from Proposition 4.11 below which was first proved in [19]. We include a direct proof here for completeness.
Proposition 3.7. Suppose that $X$ is finite. Then for each $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}_X$ we have

$$\text{Stab}(\mathcal{E}_1) \lor \text{Stab}(\mathcal{E}_2) = \text{Stab}(\mathcal{E}_1 \lor \mathcal{E}_2).$$

Proof. Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}_X$. We then have

$$\text{Stab}(\mathcal{E}_1) = \left\langle t_{xy} \mid (x, y) \in \mathcal{E}_1^\circ \right\rangle \quad \text{and} \quad \text{Stab}(\mathcal{E}_2) = \left\langle t_{xy} \mid (x, y) \in \mathcal{E}_2^\circ \right\rangle,$$

where, for $\mathcal{E} \in \mathcal{C}_X$, we have written $\mathcal{E}^\circ = \mathcal{E} \setminus \{(x, x) \mid x \in X\}$. Now

$$\text{Stab}(\mathcal{E}_1 \lor \mathcal{E}_2) = \text{id}_{\mathcal{E}_1 \lor \mathcal{E}_2} = \left\langle t_{xy} \mid (x, y) \in (\mathcal{E}_1 \lor \mathcal{E}_2)^\circ \right\rangle.$$ 

It suffices to show that the generators of $\text{Stab}(\mathcal{E}_1 \lor \mathcal{E}_2)$ are in $\text{Stab}(\mathcal{E}_1) \lor \text{Stab}(\mathcal{E}_2)$. So suppose that $(x, y) \in (\mathcal{E}_1 \lor \mathcal{E}_2)^\circ$. Then there exist distinct $x = x_1, x_2, \ldots, x_k = y \in X$ such that, without loss of generality,

$$(x_1, x_2) \in \mathcal{E}_1^\circ, \quad (x_2, x_3) \in \mathcal{E}_2^\circ, \quad \ldots, \quad (x_{k-1}, x_k) \in \mathcal{E}_i^\circ$$

where $i \in \{1, 2\}$. But then

$$t_{xy} = t_{x_1 x_k} = (t_{x_1 x_2} \cdots t_{x_k x_1}) t_{x_2 x_3} (t_{x_3 x_4} \cdots t_{x_k x_1}) \in \text{Stab}(\mathcal{E}_1) \lor \text{Stab}(\mathcal{E}_2),$$

completing the proof. \hfill $\square$

In particular, it follows that $\mathfrak{F}_X^* \in \mathcal{C}$ if $X$ is finite. Our next goal is to show that $\mathfrak{F}_X^* \not\in \mathcal{C}$ if $X$ is infinite. It suffices to show this when $X$ is countable. Thus, for the remainder of this section, we assume that $X = \{1, 2, 3, \ldots\}$. We define equivalences $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}_X$ diagrammatically in Figure 3.6, where we have also pictured the equivalence $\mathcal{E}_1 \lor \mathcal{E}_2$.

\begin{figure}[h]
\centering
\begin{array}{c|cccccccccccc|c}
\hline
\mathcal{E}_1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\
\hline
\mathcal{E}_2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\
\hline
\mathcal{E}_1 \lor \mathcal{E}_2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\
\end{array}
\caption{The equivalences $\mathcal{E}_1$ (top), $\mathcal{E}_2$ (middle), and $\mathcal{E}_1 \lor \mathcal{E}_2$ (bottom) in $\mathcal{C}_X$.}
\end{figure}

We will show that $\text{Stab}(\mathcal{E}_1) \lor \text{Stab}(\mathcal{E}_2) \neq \text{Stab}(\mathcal{E}_1 \lor \mathcal{E}_2)$.

Denote the equivalence classes of $\mathcal{E}_1 \lor \mathcal{E}_2$ by $C_1, C_2, \ldots$ from left to right. For $n \geq 1$ let $\gamma_n \in \text{Stab}(\mathcal{E}_1 \lor \mathcal{E}_2)$ be the permutation which reverses the order of the elements of $C_n$ and leaves all other elements of $X$ fixed. That is, if $C_n = \{c_1, \ldots, c_{2n+1}\}$ then

$$\gamma_n = t_{c_1 c_{2n+1}} t_{c_2 c_{2n}} \cdots t_{c_n c_{n+2}}.$$
3.3. THE STRUCTURE OF FACTORISABLE INVERSE MONOIDS

Lemma 3.8. Suppose that $n \geq 1$ and that $\gamma_n = \delta_1 \delta_2 \delta_3 \cdots \delta_k$ where $\delta_1, \delta_3, \ldots \in \text{Stab}(\mathcal{E}_1)$ and $\delta_2, \delta_4, \ldots \in \text{Stab}(\mathcal{E}_2)$. Then $k > n$.

Proof. Put $\gamma'_n = \gamma_n|_{C_n}$ and $\delta'_i = \delta_i|_{C_n}$ for each $i$, so that $\gamma'_n = \delta'_1 \delta'_2 \delta'_3 \cdots \delta'_k$. It is well-known (see for example [40]) that any reduced factorisation of $\gamma'_n$ into a product of simple transpositions (of the form $t_{c_i c_{i+1}}$ for some $i$) must involve $(2n+1) = (2n+1)n$ simple transpositions. But from the form of $\mathcal{E}_1$ and $\mathcal{E}_2$, each $\delta'_i$ is a product of at most $n$ simple transpositions. Thus a reduced expression of $\gamma'_n = \delta'_1 \delta'_2 \delta'_3 \cdots \delta'_k$ must involve $(2n+1)n \leq kn$ simple transpositions, whence $k \geq 2n+1 > n$.

Corollary 3.9. Let $\gamma \in \text{Stab}(\mathcal{E}_1 \cup \mathcal{E}_2)$ be the permutation which reverses the order of each $C_n$. Then no factorisation $\gamma = \delta_1 \delta_2 \delta_3 \cdots \delta_k$ exists with $\delta_1, \delta_3, \ldots \in \text{Stab}(\mathcal{E}_1)$ and $\delta_2, \delta_4, \ldots \in \text{Stab}(\mathcal{E}_2)$.

Proof. Suppose that $\gamma = \delta_1 \delta_2 \delta_3 \cdots \delta_k$ is such a factorisation. For $i \in \{1, \ldots, k\}$, define $\hat{\delta}_i \in G$, for $x \in X$, by

$$x \hat{\delta}_i = \begin{cases} x \delta_i & \text{if } x \in C_k \\ x & \text{otherwise.} \end{cases}$$

Then $\gamma_k = \hat{\delta}_1 \hat{\delta}_2 \hat{\delta}_3 \cdots \hat{\delta}_k$ with $\hat{\delta}_1, \hat{\delta}_3, \ldots \in \text{Stab}(\mathcal{E}_1)$ and $\hat{\delta}_2, \hat{\delta}_4, \ldots \in \text{Stab}(\mathcal{E}_2)$, contradicting Lemma 3.8.

In particular, we have $\text{Stab}(\mathcal{E}_1) \cup \text{Stab}(\mathcal{E}_2) \neq \text{Stab}(\mathcal{E}_1 \cup \mathcal{E}_2)$ which, together with Proposition 3.7, proves the following.

Theorem 3.10. The monoid $\mathfrak{G}_X$ belongs to the class $\mathcal{C}$ if and only if $X$ is finite.

3.3. The Structure of Factorisable Inverse Monoids

We now work towards a structure theorem for FIMs. We first give a method for constructing a FIM from a group and semilattice, and then show that, up to isomorphism, every FIM arises in this manner. The results of this section are part of a well-known and more general theory [43] but we provide direct proofs for completeness.

Suppose that $G$ is a group and that $E$ is a semilattice. Suppose also that for each $g \in G$ we have an automorphism

$$\varphi_g : E \rightarrow E : e \mapsto e^g \quad \text{for each } e \in E$$


such that the map
\[ \varphi : G \to \text{Aut}(E) : g \mapsto \varphi_g \] for each \( g \in G \)

is an antihomomorphism. We may then form the semidirect product
\[ E \rtimes G = E \rtimes \varphi G = \{(e, g) \mid e \in E, \ g \in G\} \]
with multiplication defined, for \( e_1, e_2 \in E \) and \( g_1, g_2 \in G \), by
\[ (e_1, g_1)(e_2, g_2) = (e_1e_2^{g_1}, g_1g_2). \]

Put \((1, G) = \{(1, g) \mid g \in G\}\) and \((E, 1) = \{(e, 1) \mid e \in E\}\). It is easy to verify the following.

**Lemma 3.11.** The monoid \( E \rtimes G \) is a factorisable inverse monoid with group of units \((1, G) \cong G\) and semilattice of idempotents \((E, 1) \cong E\).

Suppose that for each \( e \in E \), there is a subgroup \( G_e \leq G \) such that \( G_1 = \{1\} \) and
\[
\begin{align*}
gG_eg^{-1} &= G_{eg} \quad \forall e \in E, \ g \in G \quad (G_e1) \\
g_e \lor g_f &\subseteq g_{ef} \quad \forall e, f \in E \quad (G_e2) \\
e^g &= e \quad \forall e \in E, \ g \in G_e. \quad (G_e3)
\end{align*}
\]
Define an equivalence \( \sim \) on \( E \rtimes G \) by
\[
(e_1, g_1) \sim (e_2, g_2) \quad \text{if and only if} \quad e_1 = e_2 \quad \text{and} \quad g_1g_2^{-1} \in G_{e_1}.
\]

**Lemma 3.12.** The equivalence \( \sim \) is a congruence.

**Proof.** Suppose that \((e_1, g_1) \sim (f_1, h_1)\) and \((e_2, g_2) \sim (f_2, h_2)\) for some \( e_1, e_2, f_1, f_2 \in E \) and \( g_1, g_2, h_1, h_2 \in G \). Then \( e_1 = f_1, e_2 = f_2 \), and \( g_1h_1^{-1} \in G_{e_1}, g_2h_2^{-1} \in G_{e_2} \). But then
\[

g_1g_2(h_1h_2)^{-1} = (g_1h_1^{-1})h_1(g_2h_2^{-1})h_1^{-1} \\
\in G_{e_1}h_1G_{e_2}h_1^{-1} \\
= G_{e_1}G_{e_2}^{h_1} \\
\subseteq G_{e_1} \lor G_{e_2}^{h_1} \\
\subseteq G_{e_1e_2}^{h_1} \quad \text{by (G}_e1\text{)}
\]
Also, since \( g_1h_1^{-1} \in G_{e_1} \subseteq G_{e_1} \vee G_{e_2} \subseteq G_{e_1e_2} \), we have
\[
e_1e_2^{h_1} = (e_1e_2^{h_1})^{g_1h_1^{-1}} \quad \text{by } (G_e3)
\]
\[
e_1^{g_1h_1^{-1}}(e_2^{h_1})^{g_1h_1^{-1}} \quad \text{since } \varphi_{g_1h_1^{-1}} \text{ is an automorphism}
\]
\[
e_1e_2^{(g_1h_1^{-1})h_1} \quad \text{by } (G_e3) \text{ and the fact that } \varphi \text{ is an antihomomorphism}
\]
\[
e_1e_2^{g_1},
\]
so that \((e_1, g_1)(e_2, g_2) \sim (f_1, h_1)(f_2, h_2). \square\)

Thus we may form the quotient \((E \times G)/\sim\). For \(e \in E, g \in G\), let \([e, g]\) denote the \(\sim\)-class of \((e, g) \in E \times G\). Also write \([1, G] = \{[1, g] \mid g \in G\}\) and \([E, 1] = \{[e, 1] \mid e \in E\}\). The proof of the following is straightforward.

**Proposition 3.13.** The natural map \((e, g) \mapsto [e, g]\) is injective on \((1, G)\) and \((E, 1)\). Thus \((E \times G)/\sim\) is a factorisable inverse monoid with group of units \([1, G] \cong G\) and semilattice of idempotents \([E, 1] \cong E\). \(\square\)

**Proposition 3.14.** Let \(M\) be a factorisable inverse monoid with group of units \(G\) and semilattice of idempotents \(E\). Then \(M \cong (E \times G)/\sim\) arises from the construction above.

**Proof.** For \(e \in E\) and \(g \in G\) we define \(e^g = geg^{-1}\). The maps \(\varphi_g : e \mapsto e^g\) are automorphisms of \(E\), and \(\varphi : G \to \text{Aut}(E) : g \mapsto \varphi_g\) is clearly an antihomomorphism. Thus we may form \(E \times G\) as above. For \(e \in E\) let \(G_e = \{g \in G \mid eg = e\}\). Then the \(G_e\) are subgroups of \(G\). Since \(G_1 = \{1\}\) and conditions \((G_e1 - G_e3)\) are satisfied, we may form \((E \times G)/\sim\). It finally remains to observe that the map
\[
(E \times G)/\sim \to M : [e, g] \mapsto eg \quad \text{for each } e \in E \text{ and } g \in G
\]
is a well defined isomorphism. \(\square\)

### 3.4. Presentations of Factorisable Inverse Monoids

We now make use of Propositions 3.13 and 3.14 to describe a presentation of an arbitrary FIM \(M\). The ingredients are presentations of \(E = E(M)\) and \(G = G(M)\), information about the (anti-)action of \(G\) on \(E\), and generating sets for the subgroups \(G_e\). We will then examine the ways in which these presentations may be simplified if the subgroups \(G_e\) satisfy various other properties. The results of this section will appear in [20].
Suppose now that $M$ is an arbitrary FIM with $E = E(M)$ and $G = G(M)$. Then by Proposition 3.14, we may identify $M$ with $(E \times G)/\sim$ using the notation of Section 3.3. Suppose that $E$ and $G$ have monoid presentations $\langle X_E \mid R_E \rangle$ and $\langle X_G \mid R_G \rangle$ via $\eta : X_E^* \to E$ and $\zeta : X_G^* \to G$ respectively. We also assume that $X_E$ and $X_G$ are disjoint, and that $\eta$ and $\zeta$ are injective when restricted to $X_E$ and $X_G$ respectively. For each $e \in E$ and $g \in G$, choose words $\hat{e} \in X_E^*$ and $\hat{g} \in X_G^*$ such that $\hat{e}\eta = e$ and $\hat{g}\zeta = g$. We always make these choices in such a way that $\widehat{xe} = x$ and $\widehat{yg} = y$ for each $x \in X_E$ and $y \in X_G$. Put

$$R_\times = \{(yx, x\eta \zeta y) \mid x \in X_E, y \in X_G\}.$$ 

It is well known (see for example [42]) that $E \times G$ has presentation

$$\langle X_G \cup X_E \mid R_G \cup R_E \cup R_\times \rangle$$

via

$$x \mapsto \begin{cases} (x\eta, 1) & \text{if } x \in X_E \\ (1, x\zeta) & \text{if } x \in X_G. \end{cases}$$

Suppose now that for each $e \in E$ we have a subset $S_e \subseteq G$ such that $G_e$ is generated as a subgroup by $S_e$. (We could take $S_e = G_e$, but in applications we would choose $S_e$ minimally to avoid superfluous relations.) Put

$$R_\sim = \{ (\hat{e}g, \hat{e}) \mid e \in E, g \in S_e \}.$$ 

**Theorem 3.15.** The factorisable inverse monoid $M \cong (E \times G)/\sim$ has presentation

$$\langle X_G \cup X_E \mid R_G \cup R_E \cup R_\times \cup R_\sim \rangle$$

via

$$\phi : x \mapsto \begin{cases} [x\eta, 1] & \text{if } x \in X_E \\ [1, x\zeta] & \text{if } x \in X_G. \end{cases}$$

**Proof.** Put $\approx = (R_G \cup R_E \cup R_\times \cup R_\sim)^\sharp$. Now $\phi$ is surjective since $\eta$ and $\zeta$ are surjective and $(E \times G)/\sim$ is factorisable, so it remains to show that $\ker \phi = \approx$. Now $\approx \subseteq \ker \phi$ since, as may be easily checked, the relations hold as equations in $(E \times G)/\sim$ after substituting the images of the generators. Suppose now that $(w_1, w_2) \in \ker \phi$. Using $R_\times \cup R_G \cup R_E$, we have

$$w_1 \approx \hat{e}_1\hat{g}_1 \quad \text{and} \quad w_2 \approx \hat{e}_2\hat{g}_2$$

for some $e_1, e_2 \in E$ and $g_1, g_2 \in G$. But then

$$[e_1, g_1] = [e_1, 1][1, g_1] = w_1\phi = w_2\phi = [e_2, 1][1, g_2] = [e_2, g_2],$$

as desired.
so that \( e_1 = e_2 \) and \( g_1g_2^{-1} \in G_{e_1} \). Thus \( g_1g_2^{-1} = h_1 \cdots h_k \) for some \( h_1^{\pm 1}, \ldots, h_k^{\pm 1} \in S_{e_1} \) and

\[
\begin{align*}
  w_1 &\approx \hat{e}_1\hat{g}_1 \approx \hat{e}_1\hat{g}_1g_2^{-1}\hat{g}_2 & \text{by } R_G \\
  &\approx \hat{e}_1\hat{h}_1 \cdots \hat{h}_k\hat{g}_2 & \text{by } R_G \\
  &\approx \hat{e}_1\hat{g}_2 & \text{by } R_{\sim} \text{ and } R_G \\
  &= \hat{e}_2\hat{g}_2 \approx w_2.
\end{align*}
\]

This completes the proof. \( \square \)

We complete this section by proving that if the subgroups \( G_e \) satisfy the stronger condition

\[
G_e \vee G_f = G_{ef} \quad \forall e, f \in E \quad (G_e, 2)'
\]

then the set \( R_{\sim} \) defined above may be replaced by

\[
R'_{\sim} = \{(x\hat{g}, x) \mid x \in X_E, \; g \in S_{x\eta}\}.
\]

An example of a FIM which satisfies condition \( (G_e, 2)' \) is \( \mathbb{F}^*_X \) for a finite set \( X \); see Proposition 3.7. In contrast, \( \mathcal{I}_X \) does not satisfy \( (G_e, 2)' \) if \( |X| \geq 2 \); see Remark 3.6.

**Theorem 3.16.** Suppose that the factorisable inverse monoid \( M \cong (E \rtimes G)/\sim \) satisfies condition \( (G_e, 2)' \). Then \( M \) has presentation

\[
\langle X_G \cup X_E \mid R_G \cup R_E \cup R_{\times} \cup R'_{\sim} \rangle
\]

via \( \phi \).

**Proof.** Put \( \cong = (R_G \cup R_E \cup R_{\times} \cup R'_{\sim})' \). Since \( R'_{\sim} \subseteq R_{\sim} \) it suffices, by the previous theorem, to show that \( R_{\sim} \subseteq \cong \). Let \( e \in E \) and \( g \in G_e \). We prove that \( \hat{e}\hat{g} \cong \hat{e} \). Now \( e = (x_1\eta) \cdots (x_k\eta) \) for some \( x_1, \ldots, x_k \in X_E \). By \( (G_e, 2)' \), we have \( G_e = G_{x_1\eta} \vee \cdots \vee G_{x_k\eta} \), and so \( g = g_1 \cdots g_{\ell} \) for some \( g_1, \ldots, g_{\ell} \in G_{x_1\eta} \cup \cdots \cup G_{x_k\eta} \). For each \( i \in \{1, \ldots, \ell\} \), there exists \( m_i \in \{1, \ldots, k\} \) such that \( g_i \in G_{x_{m_i}\eta} \), and so \( g_i = h_{i_1} \cdots h_{i_{\ell_i}} \) for some \( h_{i_1}^{\pm 1}, \ldots, h_{i_{\ell_i}}^{\pm 1} \in S_{x_{m_i}\eta} \). But then

\[
\begin{align*}
  \hat{e}\hat{g} &\cong x_1 \cdots x_{k}(\hat{h}_{i_1} \cdots \hat{h}_{i_{\ell_1}}) \cdots (\hat{h}_{i_{\ell_i}} \cdots \hat{h}_{i_{m_i}}) & \text{by } R_E \text{ and } R_G \\
  &\cong x_1 \cdots x_kx_{m_1}(\hat{h}_{i_1} \cdots \hat{h}_{i_{\ell_1}}) \cdots (\hat{h}_{i_{\ell_i}} \cdots \hat{h}_{i_{m_i}}) & \text{by } R_E \\
  &\cong x_1 \cdots x_kx_{m_1}(\hat{h}_{i_2} \cdots \hat{h}_{i_{\ell_2}}) \cdots (\hat{h}_{i_{\ell_i}} \cdots \hat{h}_{i_{m_i}}) & \text{by } R'_{\sim} \text{ and } R_G \\
  &\cong x_1 \cdots x_k(\hat{h}_{i_2} \cdots \hat{h}_{i_{\ell_2}}) \cdots (\hat{h}_{i_{\ell_i}} \cdots \hat{h}_{i_{m_i}}) & \text{by } R_E \\
  &\cong x_1 \cdots x_{k} & \text{by a simple induction} \\
  &\cong \hat{e} & \text{by } R_E,
\end{align*}
\]

completing the proof. \( \square \)
We remark that the proof of Theorem 3.16 works almost unchanged if condition $(G_e2)'$ is replaced by

$$(\forall e \in E) \ (\exists x_1, \ldots, x_k \in X_E) \ e = (x_1 \eta) \cdots (x_k \eta) \text{ and } G_e = G_{x_1 \eta} \vee \cdots \vee G_{x_k \eta}, \ (G_e2)''$$

or the even weaker condition

$$G_e = \bigvee_{\substack{x \in X_E \ e(x \eta) = e}} G_{x \eta} \quad (\forall e \in E). \quad (G_e2)'''$$

We state the following result for convenience.

**Theorem 3.17.** Suppose that the factorisable inverse monoid $M \cong (E \rtimes G)/\sim$ satisfies condition $(G_e2)''$ or $(G_e2)'''$. Then $M$ has presentation

$$\langle X_G \cup X_E \mid R_G \cup R_E \cup R_\times \cup R_\circ \rangle$$

via $\phi$. \qed
CHAPTER 4

Factorisable Inverse Braid Monoids

In this chapter we introduce and study several monoids which are natural geometric extensions of the braid group \( B_n \). The first monoid we study is the inverse braid monoid \( IB_n \). This monoid was introduced by Easdown and Lavers in [21] where, among other things, the authors showed that \( IB_n \) was inverse, and gave a monoid presentation of \( IB_n \). This monoid is also a braid analogue of the symmetric inverse semigroup \( I_n \) in the sense that there is a natural epimorphism \( \sim : IB_n \to I_n \) which extends the map \( \sim : B_n \to S_n \) defined in Section 2.4. In Section 4.1 we show that \( IB_n \) is factorisable, a fact which is implicit in various calculations in [21]. We then provide an algebraic construction, in the style of Section 3.3, and show that this construction yields a monoid which is naturally isomorphic to \( IB_n \).

We then introduce the factorisable braid monoid \( FB_n \), a natural braid analogue of \( F_n \), the monoid of uniform block bijections on \( n \). We define \( FB_n \) geometrically as a monoid whose elements are equivalence classes of braids, where “equivalence” involves homotopies of braids in which certain pairs of strings are allowed to “merge and part”. By altering the notion of equivalence and allowing strings to “permeate” (a weaker definition) we obtain a new geometric object which we call the permeable braid monoid \( PB_n \). We adapt our treatment of \( IB_n \) to study \( FB_n \) and \( PB_n \) showing, in particular, that both monoids are factorisable and inverse. We also provide algebraic constructions which yield monoids isomorphic to \( FB_n \) and \( PB_n \). The algebraic constructions of the three monoids will play a central role when we study presentations in Chapters 5 and 7, and solve decision problems in Chapter 6. We fix a positive integer \( n \) for the remainder of the chapter.

Before we come to these definitions, we will prove a result concerning braids which will prove extremely useful on many occasions. Recall that for \( 1 \leq i < j \leq n \) we defined braids

\[
\varsigma_{ij} = (s_{j-1}^{-1} \cdots s_{i+1}^{-1})s_i(s_{i+1} \cdots s_{j-1}) = (s_i \cdots s_{j-2})s_{j-1}(s_{j-2}^{-1} \cdots s_i^{-1})
\]

\[
\alpha_{ij} = (s_{j-1} \cdots s_{i+1})s_i^2(s_{i+1}^{-1} \cdots s_{j-1}^{-1}) = (s_i^{-1} \cdots s_{j-2}^{-1})s_{j-1}^2(s_{j-2} \cdots s_i).
\]

See also Figures 2.12 and 2.13. For the proofs of Lemmas 4.1 and 4.2 we define braids \( \delta_{ij} \) and \( \gamma_{ij} \), for \( 1 \leq i < j \leq n \), by

\[
\delta_{ij} = (s_2 \cdots s_{j-1})(s_i \cdots s_{i-1}) \quad \text{and} \quad \gamma_{ij} = (s_2^{-1} \cdots s_{j-1}^{-1})(s_i^{-1} \cdots s_{i-1}^{-1}).
\]
Notice in particular that
\[ 1 \delta_{ij} = 1 \gamma_{ij} = i \quad \text{and} \quad 2 \delta_{ij} = 2 \gamma_{ij} = j \]
for all \( 1 \leq i < j \leq n \). See Figure 4.1 for an illustration.

Using Figures 2.12, 2.13, and 4.1, it is easy to check that for any \( 1 \leq i < j \leq n \) we have
\[ \varsigma_{ij} = \delta_{ij}^{-1} \varsigma_{i} \delta_{ij} \quad \text{and} \quad \alpha_{ij} = \gamma_{ij}^{-1} \varsigma_{i} \gamma_{ij}. \]

To make the statement of the following results simpler, we will write \( \varsigma_{ji} = \varsigma_{ij} \) and \( \alpha_{ji} = \alpha_{ij} \) for all \( 1 \leq i < j \leq n \).

**Lemma 4.1.** Suppose that \( 1 \leq i < j \leq n \) and \( \beta \in \mathcal{B}_n \). Then there exists \( \gamma \in \mathcal{P}_n \) such that
\[ \beta^{-1} \varsigma_{ij} \beta = \gamma^{-1} \varsigma_{i} \beta \beta \gamma. \]

**Proof.** Write \( x = \beta^{-1} \varsigma_{ij} \beta \) and \( k \tilde{\beta}, l = j \tilde{\beta} \). Replacing \( \beta \) by \( \varsigma_{ij} \beta \) if necessary, we may assume that \( k < l \). We then have
\[ \varsigma_{ij} = \delta_{ij}^{-1} \varsigma_{i} \delta_{ij} = \delta_{ij}^{-1} \delta_{kl} \varsigma_{kl} \delta_{kl}^{-1} \delta_{ij}. \]

Let \( \beta_1 = \delta_{kl}^{-1} \delta_{ij} \beta \). Then
\[ x = \beta^{-1} \delta_{ij}^{-1} \delta_{kl} \varsigma_{kl} \delta_{kl}^{-1} \delta_{ij} \beta = \beta_1^{-1} \varsigma_{kl} \beta_1. \]

We also have
\[ k \tilde{\beta}_1 = k \delta_{kl}^{-1} \delta_{ij} \beta = 1 \delta_{ij} \beta = i \beta = k \quad \text{and} \quad l \tilde{\beta}_1 = l \delta_{kl}^{-1} \delta_{ij} \beta = 2 \delta_{ij} \beta = j \beta = l. \]

Now let \( \beta_2 \) be the braid obtained by removing the \( k \)th and \( l \)th strings from \( \beta_1 \) and replacing them by straight strings which pass in front of all the other strings. By construction, it is clear that \( \beta_2 \) commutes with \( \varsigma_{kl} \), and that \( \tilde{\beta}_2 = \tilde{\beta}_1 \). Now put \( \gamma = \beta_2^{-1} \beta_1 \in \mathcal{P}_n \). We then have
\[ x = \beta_1^{-1} \varsigma_{kl} \beta_1 = \beta_1^{-1} \beta_2 \varsigma_{kl} \beta_2^{-1} \beta_1 = \gamma^{-1} \varsigma_{kl} \gamma, \]
and the proof is complete. \( \square \)

**Lemma 4.2.** Suppose that \( 1 \leq i < j \leq n \) and \( \beta \in \mathcal{B}_n \). Then there exists \( \gamma \in \mathcal{P}_n \) such that
\[ \beta^{-1} \alpha_{ij} \beta = \gamma^{-1} \alpha_{i} \beta \beta \gamma. \]
4.1. The Inverse Braid Monoid

In this section we review the definition of the inverse braid monoid $IB_n$ introduced by Easdown and Lavers in [21]. As in [21], we define $IB_n$ as a monoid whose elements are equivalence classes of braids with strings missing. We then show that $IB_n$ is a factorisable inverse monoid, and show how $IB_n$ may be constructed algebraically from the braid group $B_n$ and the power set $(2^n, \cap)$.

4.1.1. The Geometric Definition of $IB_n$.

Recall from Section 2.4 that a string is a (continuous) embedding $s$ from the unit interval $[0, 1]$ into $\mathbb{R}^3$ such that

(i) $s(0) = P_i = (i, 0, 1)$ and $s(1) = P'_j = (j, 0, 0)$ for some $i, j \in n$; and

(ii) the $z$-coordinate of $s(t)$ is $1 - t$ for each $t \in [0, 1]$.

A partial braid on $n$ is a collection of $0 \leq m \leq n$ strings $\beta = (s_{i_1}, \ldots, s_{i_m})$ such that

(i) $1 \leq i_1 < \cdots < i_m \leq n$;

(ii) $s_{i_l}(0) = P_{i_l}$ for each $l \in m$; and

(iii) $s_{i_l}(t) \neq s_{i_{l'}}(t)$ for each $t \in [0, 1]$ if $l, l' \in m$ with $l \neq l'$.

Thus a partial braid is essentially a braid in which some strings are missing; a (full) braid may be considered as a partial braid with $m = n$ strings. Notice also that the definition allows an "empty" braid with no strings. See Figure 4.2 for some examples including a full braid (left) and the empty braid (right).

![Figure 4.2. Some partial braids on \{1, 2, 3, 4\}.
}
We denote by $IB_n$ the set of all partial braids on $n$, noting in particular that $B_n \subseteq IB_n$. Notice that any partial braid $\beta = (s_{i_1}, \ldots, s_{i_m}) \in IB_n$ induces a partial permutation $\tilde{\beta}$ of $n$ defined, for $l \in m$, by $s_{i_l}(1) = P_{i_l}\beta$.

We now define a multiplication of partial braids. Suppose that $\beta = (s_{i_1}, \ldots, s_{i_m})$ and $\gamma = (t_{j_1}, \ldots, t_{j_{m'}})$ are two partial braids on $n$. Suppose also that $\text{dom}(\tilde{\beta}\tilde{\gamma}) = \{k_1, \ldots, k_l\}$ with $k_1 < \cdots < k_l$. We define $\beta\gamma = (s_{k_1}t_{k_1}\beta, \ldots, s_{k_l}t_{k_l}\beta)$. See Figure 4.3 for an example.

![Figure 4.3. The product of two partial braids.](image)

Informally, to calculate the product $\beta\gamma$ of two partial braids $\beta, \gamma \in IB_n$, one joins the bottom of $\beta$ to the top of $\gamma$, removes any string fragments which do not connect an upper point to a lower point, and then uniformly shrinks the resulting object until it lies between the $z = 0$ and $z = 1$ planes. Notice that the empty braid acts as a (left and right) zero with respect to this product.

Two partial braids $\beta = (s_{i_1}, \ldots, s_{i_m})$ and $\gamma = (t_{j_1}, \ldots, t_{j_{m'}})$ are said to be equivalent if there is a homotopy through partial braids from $\beta$ to $\gamma$ or, more formally, if $m = m'$ and there are continuous maps $F_r : [0,1] \times [0,1] \to \mathbb{R}^3$ for each $r \in m$ such that

(i) $F_r(0,t) = s_r(t)$ and $F_r(1,t) = t_r(t)$ for each $t \in [0,1]$;

(ii) for each $u \in [0,1]$, the maps $s^u_r : [0,1] \to \mathbb{R}^3$ defined, for $t \in [0,1]$, by $s^u_r(t) = F_r(u, t)$ are strings; and

(iii) for each $u \in [0,1]$, the collection $(s^u_{i_1}, \ldots, s^u_{i_m})$ is a partial braid.

In this case, we write $\beta \sim \gamma$. We denote the $\sim$-class of a partial braid $\beta \in IB_n$ by $[\beta]$. It is clear that $\sim$ is a groupoid congruence on $IB_n$. Thus we may form the quotient groupoid

$$\mathcal{IB}_n = IB_n/\sim = \{[\beta] \mid \beta \in IB_n\}$$

with multiplication defined, for $\beta, \gamma \in \mathcal{IB}_n$, by $[\beta][\gamma] = [\beta\gamma]$. One may easily check that this product on $\sim$-classes is associative so that $\mathcal{IB}_n$ forms a semigroup. It was shown in [21] that $\mathcal{IB}_n$ is an inverse monoid with identity $[1]$. The inverse of $[\beta]$ for $\beta = (s_{i_1}, \ldots, s_{i_m}) \in IB_n$ is $[\beta]^{-1} = [\beta^{-1}]$ where $\beta^{-1} = (s_{i_1}^{-1}, \ldots, s_{i_m}^{-1}) \in IB_n$, writing
im(\tilde{\beta}) = \{j_1, \ldots, j_m\}$ with $j_1 < \cdots < j_m$, and each string $s_{j_{\gamma}}^{-1}$ is defined as earlier. See Figure 4.4 for an example.

$$\beta = \quad \beta^{-1} = \quad \beta\beta^{-1} = \quad \beta^{-1}\beta =$$

**Figure 4.4.** The inverse of a partial braid: $\beta\beta^{-1}\beta \sim \beta$ and $\beta^{-1}\beta\beta^{-1} \sim \beta^{-1}$.

Now we clearly have $\tilde{\beta} = \tilde{\gamma}$ for any $\beta, \gamma \in \mathbf{IB}_n$ such that $\beta \sim \gamma$. As a result we may define a map $^- : \mathbf{IB}_n \to \mathbf{I}_n$ by $[\beta] = \beta$ for each $\beta \in \mathbf{IB}_n$. Notice that the restriction of $^-$ to $\mathbf{B}_n$ yields the map $^- : \mathbf{B}_n \to \mathbf{S}_n$ defined in Section 2.4.

For $A \subseteq n$ let $1_A \in \mathbf{IB}_n$ denote the partial braid obtained from $1 \in \mathbf{B}_n$ by removing all strings whose intersection with the $z = 1$ plane is $P_i$ for some $i \notin A$. In particular, $1_n$ (resp. $1_\emptyset$) is the identity (resp. empty) braid. See Figure 4.5 for an example.

$$1 = \quad 1_A =$$

**Figure 4.5.** The partial braids $1, 1_A \in \mathbf{IB}_6$ where $A = \{2, 3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$.

**Theorem 4.3.** The inverse braid monoid $\mathbf{IB}_n$ is a factorisable inverse monoid with

$$\mathbf{E}(\mathbf{IB}_n) = \{[1_A] \mid A \subseteq n\} \cong (2^n, \cap) \quad \text{and} \quad \mathbf{G}(\mathbf{IB}_n) = \mathbf{B}_n.$$  

**Proof.** The fact that $\mathbf{IB}_n$ is an inverse monoid, and the statements concerning $\mathbf{E}(\mathbf{IB}_n)$ and $\mathbf{G}(\mathbf{IB}_n)$ were proved in [21]. Suppose now that $\beta \in \mathbf{IB}_n$ and put $A = \text{dom}(\tilde{\beta})$. Choose a braid $\gamma \in \mathbf{B}_n$ such that every string of $\beta$ is a string of $\gamma$. Then we clearly have $1_A\gamma \sim \beta$ so that $[\beta] = [1_A][\gamma]$. \hfill \Box

As with (full) braids, we will generally blur the distinction between a partial braid $\beta \in \mathbf{IB}_n$ and its equivalence class $[\beta] \in \mathbf{IB}_n$. Thus we will view elements of $\mathbf{IB}_n$ as partial braids, identifying two partial braids if they are $\sim$-equivalent.
4.1. THE INVERSE BRAID MONOID

4.1.2. An Algebraic Construction of $\mathcal{IB}_n$.

We now provide a second way to construct $\mathcal{IB}_n$, using the methods of Section 3.3. For the remainder of this section we denote by $B$ the braid group $B_n = G(\mathcal{IB}_n)$, and we also write $T = 2^n$ for the power set of $n$ considered as a semilattice under $\cap$. If $A \in T$ we write $A^c$ for the complement $n \setminus A$.

For $A \in T$ and $\beta \in B$ we define

$$A^\beta = \{a \in n \mid a\beta \in A\} = \{a\beta^{-1} \mid a \in A\}.$$  

One may easily check that for each $\beta \in B$ the map

$$\varphi_\beta : T \to T : A \mapsto A^\beta \quad \text{for each } A \in T$$

is an automorphism of $T$, and that

$$\varphi : B \to \text{Aut}(T) : \beta \mapsto \varphi_\beta \quad \text{for each } \beta \in B$$

is an antihomomorphism. Thus we may form the semidirect product

$$T \rtimes B = T \rtimes_\varphi B = \{(A, \beta) \mid A \in T, \beta \in B\}$$

with multiplication defined, for $A_1, A_2 \in T$ and $\beta, \gamma \in B$, by

$$(A_1, \beta)(A_2, \gamma) = (A_1 \cap A_2^\beta, \beta \gamma).$$

For $A \in T$ we define $B_A$ to be the subgroup of $B$ generated by the set

$$\{\beta^{-1}\varsigma_i \beta \mid i\beta \in A^c\} \cup \{\beta^{-1}\varsigma_i^2 \beta \mid i\beta \in A^c \text{ or } (i+1)\beta \in A^c\}.$$  

Generators from each of these subsets are illustrated in Figure 4.6 below.

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure4.6.png}
\caption{The braids $\beta^{-1}\varsigma_i \beta$ (left) and $\beta^{-1}\varsigma_i^2 \beta$ (right) for the case in which $k = i\beta < l = (i + 1)\beta$.}
\end{figure}
One may readily check that these subgroups satisfy the conditions
\[
\begin{align*}
\beta B A \beta^{-1} &= B A^\beta \quad \text{for all } A \in T \text{ and } \beta \in B \quad (B_A 1) \\
B A_1 \vee B A_2 &\subseteq B A_1 \cap A_2 \quad \text{for all } A_1, A_2 \in T \quad (B_A 2) \\
A^2 &= A \quad \text{for all } A \in T \text{ and } \beta \in B_A. \quad (B_A 3)
\end{align*}
\]
Thus, by Lemma 3.12, we may define a congruence \( \sim \) on \( T \times B \) by
\[
(A_1, \beta) \sim (A_2, \gamma) \quad \text{if and only if} \quad A_1 = A_2 \text{ and } \beta \gamma^{-1} \in B A_1,
\]
and form the quotient \((T \times B)/\sim\). We denote the \( \sim \)-class of \((A, \beta) \in T \times B\) by \([A, \beta]\).

Now suppose that \( \beta \in B \) and \( A \in T \). If \( i \in n \) we say that the \( i \)th string of \( \beta \) is in \( A \) (resp. not in \( A \)) if \( i \in A \) (resp. \( i \notin A \)). We denote by \( \beta_A \) the (equivalence class of the) partial braid obtained from \( \beta \) by removing all strings of \( \beta \) which are not in \( A \). If \( A \in T \) and \( \beta, \gamma \in B \), we will write \( \beta \sim_A \gamma \) if \( \beta_A = \gamma_A \). For \( A \in T \) and \( \beta \in B \), we denote by \([\beta]_A = \{\gamma \in B \mid \gamma \sim_A \beta\} \) the \( \sim_A \)-class of \( \beta \).

**Theorem 4.4.** Let \( A \in T \). Then \( B_A = [1]_A = \{\beta \in B \mid \beta \sim_A 1\} \).

**Proof.** We first show that \( x_A = 1_A \) for any generator \( x \) of \( B_A \).

Suppose first that \( x = \beta^{-1} \xi_i \beta \) where \( 1 \leq i \leq n - 1 \) and \( \beta \in B \) such that \( i \beta, (i + 1) \beta \in A^c \). Referring to the left diagram in Figure 4.6, it is clear that removing strings \( k = i \beta \) and \( l = (i + 1) \beta \) from \( x \) leaves (a partial braid which is \( \sim \)-equivalent to) \( 1_{\{k, l\}}^c \). It then follows that \( x_A = 1_A \).

Secondly, suppose that \( x = \beta^{-1} \xi_i^2 \beta \) where either \( i \beta \in A^c \) or \( (i + 1) \beta \in A^c \). Let \( h \) denote \( i \beta \) if \( i \beta \in A^c \), or \( (i + 1) \beta \) otherwise. Referring to the right diagram in Figure 4.6, it is clear that removing string \( h \) from \( x \) leaves (a partial braid which is \( \sim \)-equivalent to) \( 1_{\{h\}}^c \). It follows that \( x_A = 1_A \).

This shows that \( B_A \subseteq [1]_A \). To show the reverse inclusion, suppose that \( \beta \in [1]_A \).

We first treat the case in which \( A = r = \{1, \ldots, r\} \) for some \( 0 \leq r \leq n \). Since \( \beta_A = 1_A \) we see that \( i \beta = i \) for all \( i \in r \). Thus we may choose a braid \( \delta = \xi_i \cdots \xi_k \) such that each \( r + 1 \leq i_s \leq n - 1 \) and \( \delta = \beta \). Then we have \( \beta = (\beta \delta^{-1}) \delta \) and, in particular, \( \beta \delta^{-1} \in P_n \).

Thus, by Theorem 2.14, we have \( \beta \delta^{-1} = u_n \cdots u_2 \) for unique braids \( u_2 \in U_2, \ldots, u_n \in U_n \).

Since \( \beta_A = 1_A \) we see that \( u_r = \cdots = u_2 = 1 \). Thus \( \beta = u_n \cdots u_{r+1} \delta \) is a product of terms of the form
\[
a_{ij}^{\pm 1} \text{ with } j \in A^c \quad \text{and} \quad \xi_i \text{ with } i, i+1 \in A^c.
\]
These elements are certainly in $B_A$, showing that $\beta \in B_A$.

To complete the proof, let $A \in T$ be arbitrary, and choose $\gamma \in B$ such that $A\gamma = r$ for some $0 \leq r \leq n$. Now since $\beta_A = 1_A$ we must have $(\gamma \beta \gamma^{-1})_{A\gamma} = 1_{A\gamma}$. By the previous paragraph, $\gamma \beta \gamma^{-1}$ is a product of terms of the form
\[ \alpha_{ij}^{\pm 1} \text{ with } j \in (A\gamma)^c \quad \text{and} \quad \varsigma_i \text{ with } i, i + 1 \in (A\gamma)^c \]
from which it follows that $\beta$ is a product of terms of the form
\[ \gamma^{-1} \alpha_{ij}^{\pm 1} \gamma \text{ with } j \in (A\gamma)^c \quad \text{and} \quad \gamma^{-1} \varsigma_i \gamma \text{ with } i, i + 1 \in (A\gamma)^c \]
We show now that each of these terms is in $B_A$. Now if $j \in (A\gamma)^c = (A\gamma)^c \gamma$ then $j \gamma \in A^c$ so that $\gamma^{-1} \alpha_{ij}^{\pm 1} \gamma \in B_A$. Similarly, if $i, i + 1 \in (A\gamma)^c$, then $\gamma^{-1} \varsigma_i \gamma \in B_A$. Thus $\beta \in B_A$. $\square$

We may now show that the construction of $(T \times B)/\sim$ yields a monoid isomorphic to $IB_n$.

**Theorem 4.5.** Suppose that $A_1, A_2 \in T$ and $\beta, \gamma \in B$. Then $(A_1, \beta) \sim (A_2, \gamma)$ if and only if $\beta_{A_1} = \gamma_{A_2}$. Thus $(T \times B)/\sim$ and $IB_n$ are isomorphic via the map
\[ [A, \beta] \mapsto \beta_A \quad \text{for all } A \in T \text{ and } \beta \in B. \]

**Proof.** Suppose first that $(A_1, \beta) \sim (A_2, \gamma)$. Then we have $A_1 = A_2$ and $\beta \gamma^{-1} \in B_{A_1}$.

From Theorem 4.4 we then have $(\beta \gamma^{-1})_{A_1} = 1_{A_1}$ from which it follows that
\[ \beta_{A_1} = (\beta \gamma^{-1})_{A_1} = (\beta \gamma^{-1})_{A_1} \gamma = 1_{A_1} \gamma = \gamma_{A_1} = \gamma_{A_2}. \]

Next suppose that $\beta_{A_1} = \gamma_{A_2}$. Then we must have $A_1 = A_2$ and
\[ (\beta \gamma^{-1})_{A_1} = \beta_{A_1} \gamma^{-1} = \gamma_{A_2} \gamma^{-1} = (\gamma \gamma^{-1})_{A_2} = 1_{A_2} = 1_{A_1} \]
so that $\beta \gamma^{-1} \in B_{A_1}$ by Theorem 4.4, showing that $(A_1, \beta) \sim (A_2, \gamma)$.

The isomorphism statement follows from the easily checked fact that for all $A_1, A_2 \in T$ and $\beta, \gamma \in B$ we have $\beta_{A_1 \cap A_2} \gamma_{A_1 \cap A_2} \sim (\beta \gamma)_{A_1 \cap A_2}$. $\square$

Theorems 4.4 and 4.5 will be useful when we give presentations of $IB_n$ in Chapter 5, and submonoids of $IB_n$ in Chapters 6 and 7. Also of use will be the following result which follows directly from the definition of the subgroups $B_A$, together with Lemmas 4.1 and 4.2, and the observation that
\[ \varsigma_i = \varsigma_{i,i+1} \quad \text{and} \quad \varsigma_i^2 = \alpha_{i,i+1} \text{ for each } i \in \{1, \ldots, n-1\}. \]

**Lemma 4.6.** If $A \in T$, then $B_A$ is generated by the set
\[ \{ \beta^{-1} \varsigma_{ij} \beta \mid \beta \in \mathcal{P}_n, \ i, j \in A^c \} \cup \{ \beta^{-1} \alpha_{ij} \beta \mid \beta \in \mathcal{P}_n, \ i \in A^c \text{ or } j \in A^c \}. \] $\square$
4.2. The Factorisable Braid Monoid

In this section we introduce the factorisable braid monoid $\mathcal{F}B_n$. Following the same pattern as the previous section, we define $\mathcal{F}B_n$ geometrically as a monoid whose elements are equivalence classes of braids. We then show that $\mathcal{F}B_n$ is a factorisable inverse monoid, and we also provide an algebraic construction, involving the braid group and the join semilattice of equivalence relations on $n$, which yields (a monoid isomorphic to) $\mathcal{F}B_n$. We also show that $\mathcal{F}B_n$ embeds in $C(B_n)$, the coset monoid of the braid group $B_n$.

4.2.1. The Geometric Definition of $\mathcal{F}B_n$.

Suppose that $s$ and $t$ are disjoint strings (see Figure 4.7).

\[
P \bigg\langle \bigg\rangle \quad Q \bigg\rangle
\]

**Figure 4.7.** The strings $s$ (left) and $t$ (right).

We say that a homotopy causes $s$ and $t$ to merge and part if

(i) the homotopy first causes $s$ and $t$ to come together just once at say $P$ and $Q$ (see Figure 4.8),

\[
P = Q
\]

**Figure 4.8.** The strings merging.

(ii) and then part, reconstituting as two strings made up of respective upper and lower strands. See Figure 4.9 for a catalogue of the possible configurations in the neighbourhood of the crossing a moment before and after merge-and-part.

\[
(1) \quad (2) \quad (3) \quad (4) \quad (5) \quad (6) \quad (7)
\]

**Figure 4.9.** Possible configurations before and after merge-and-part.

Note that configurations (1), (2), and (3) can be interchanged using a “normal” homotopy.

Let $\mathcal{E} \in \mathcal{E}_n$. We say that the $i$th and $j$th strings of a braid are $\mathcal{E}$-related (resp. $\mathcal{E}$-unrelated) if $(i, j) \in \mathcal{E}$ (resp. $(i, j) \not\in \mathcal{E}$). If $\beta, \gamma \in B_n$, then we write $\beta \sim_{\mathcal{E}} \gamma$ if there is a
homotopy from \( \beta \) to \( \gamma \) during which \( \mathcal{E} \)-unrelated strings never touch, and \( \mathcal{E} \)-related strings may merge and part (one at a time, a finite number of times). We call such a homotopy an \( \mathcal{E} \)-homotopy, and we denote the \( \sim_{\mathcal{E}} \)-class of a braid \( \beta \in \mathcal{B}_n \) by \([\beta]_{\mathcal{E}}\).

If \( \mathcal{E} \in \mathcal{E}_n \) and \( \beta \in \mathcal{B}_n \), we define

\[
\mathcal{E}^\beta = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid (i,j)\bar{\beta} \in \mathcal{E}\} = \{(i,j)\bar{\beta}^{-1} \mid (i,j) \in \mathcal{E}\}.
\]

Here, for \( i_1, \ldots, i_k \in \mathbb{N} \) and \( \pi \in S_n \), we have written \((i_1, \ldots, i_k)\pi = (i_1\pi, \ldots, i_k\pi)\).

We now define the factorisable braid monoid to be the set

\[
\mathfrak{F}_{\mathcal{B}_n} = \{[\beta]_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{E}_n, \beta \in \mathcal{B}_n\}
\]

equipped with the operation defined, for \( \mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}_n \) and \( \beta, \gamma \in \mathcal{B}_n \), by

\[
[\beta]_{\mathcal{E}_1} [\gamma]_{\mathcal{E}_2} = [\beta\gamma]_{\mathcal{E}_1 \vee \mathcal{E}_2}.
\]

One may check that \( \mathfrak{F}_{\mathcal{B}_n} \) is a monoid under this operation, with identity \( [1]_1 \), where we have written \( 1 \in \mathcal{E}_n \) for the identity equivalence relation; that is \( 1 = \{(x,x) \mid x \in \mathbb{N}\} \). It is also clear that for any \( \beta, \gamma \in \mathcal{B}_n \) we have \( \beta \sim_1 \gamma \) if and only if \( \beta = \gamma \). Thus we will identify \( [\beta]_1 = \{\beta\} \) with \( \beta \) for any \( \beta \in \mathcal{B}_n \). Under this identification, \( \mathcal{B}_n \) is a subgroup of \( \mathfrak{F}_{\mathcal{B}_n} \), indeed the group of units as we will shortly see.

Recall that \( \mathfrak{F}_n^* \) is the monoid of uniform block bijections on \( \mathbb{N} \). We define a map

\[
\tilde{\psi} : \mathfrak{F}_{\mathcal{B}_n} \rightarrow \mathfrak{F}_n^* : [\beta]_{\mathcal{E}} \mapsto [\overline{\beta}]_{\mathcal{E}} = \text{id}_{\mathcal{E}} \bar{\beta} \quad \text{for each } \mathcal{E} \in \mathcal{E}_n \text{ and } \beta \in \mathcal{B}_n.
\]

One may easily check that \( \tilde{\psi} \) is a well-defined epimorphism, and that the restriction of \( \tilde{\psi} \) to \( \mathcal{B}_n \) is the permutation map defined in Section 2.4.

**Theorem 4.7.** The factorisable braid monoid \( \mathfrak{F}_{\mathcal{B}_n} \) is a factorisable inverse monoid with

\[
\mathcal{E}(\mathfrak{F}_{\mathcal{B}_n}) = \{[1]_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{E}_n\} \cong (\mathcal{E}_n, \vee) \quad \text{and} \quad \mathcal{G}(\mathfrak{F}_{\mathcal{B}_n}) = \{[1]_{\mathcal{E}} \mid \beta \in \mathcal{B}_n\} = \mathcal{B}_n.
\]

**Proof.** Suppose first that \( \mathcal{E} \in \mathcal{E}_n \) and \( \beta \in \mathcal{B}_n \) such that

\[
[\beta]_{\mathcal{E}} = [\beta]_{\mathcal{E}} [\beta]_{\mathcal{E}} = [\beta\beta]_{\mathcal{E} \vee \mathcal{E}}.
\]

Then \( \mathbb{N}/(\mathcal{E} \vee \mathcal{E}^\beta) = \text{dom}(\overline{[\beta]_{\mathcal{E}} \mathcal{E} \vee \mathcal{E}}) \) = \text{dom}(\overline{[\beta]_{\mathcal{E}}}) = \mathbb{N}/\mathcal{E} \) whence \( \mathcal{E} \vee \mathcal{E}^\beta = \mathcal{E} \), and

\[
[1]_{\mathcal{E}} = [\beta\beta]_{\mathcal{E} \vee \mathcal{E}} = [\beta]_{\mathcal{E}} [\beta\beta]_{\mathcal{E}} = [\beta]_{\mathcal{E}} [\beta\beta]_{\mathcal{E}} = [\beta]_{\mathcal{E}} [1]_{\mathcal{E}} \mathcal{E} \vee \mathcal{E} = [\beta]_{\mathcal{E}} [1]_{\mathcal{E}} \mathcal{E} = [\beta]_{\mathcal{E}} [1]_{\mathcal{E}} \mathcal{E} = [\beta]_{\mathcal{E}}.
\]

This proves that \( \mathcal{E}(\mathfrak{F}_{\mathcal{B}_n}) = \{[1]_{\mathcal{E}} \mid \mathcal{E} \in \mathcal{E}_n\} \). Now if \( \mathcal{E}_1, \mathcal{E}_2 \in \mathcal{E}_n \), then we clearly have \( [1]_{\mathcal{E}_1} [1]_{\mathcal{E}_2} = [1]_{\mathcal{E}_1 \vee \mathcal{E}_2} \), showing that idempotents of \( \mathfrak{F}_{\mathcal{B}_n} \) commute, and that the map

\[
\phi : \mathcal{E}_n \rightarrow \mathcal{E}(\mathfrak{F}_{\mathcal{B}_n}) : \mathcal{E} \mapsto [1]_{\mathcal{E}} \quad \text{for each } \mathcal{E} \in \mathcal{E}_n
\]
is an epimorphism. To show that $\phi$ is injective, suppose that $E_1, E_2 \in E_q_n$ with $[1]_{E_1} = [1]_{E_2}$. Then we have $id_{E_1} = [1]_{E_1} = [1]_{E_2} = id_{E_2}$, and it follows that $E_1 = E_2$, completing the proof that $E(B_n) \cong (E_q_n, \lor)$.

Let $E \in E_q_n$ and $\beta \in B_n$. Then it is easy to check that

$$[\beta]_E [\beta^{-1}]_{E\beta^{-1}} [\beta]_E = [\beta]_E$$

and

$$[\beta^{-1}]_{E\beta^{-1}} [\beta]_E [\beta^{-1}]_{E\beta^{-1}} = [\beta^{-1}]_{E\beta^{-1}}$$

which, together with the fact that idempotents of $\mathfrak{B}B_n$ commute, shows that $\mathfrak{B}B_n$ is an inverse monoid. It also follows that the (unique) inverse of $[\beta]_E$ is $[\beta^{-1}]_{E\beta^{-1}}$ for each $E \in E_q_n$ and $\beta \in B_n$.

To prove the statement concerning $G(\mathfrak{B}B_n)$, notice that if $E \in E_q_n$ and $\beta \in B_n$ are such that $[1]_{E} = [\beta]_E [\beta^{-1}]_{E\beta^{-1}} = [1]_{E}$, then we must have $E = 1$ by the injectivity of $\phi$ defined above. This shows that $G(\mathfrak{B}B_n) \subseteq B_n$. The reverse inclusion is clear.

Finally, to show that $\mathfrak{B}B_n$ is factorisable, it is easy to check that for any $E \in E_q_n$ and $\beta \in B_n$ we have $[\beta]_E = [1]_{E}[\beta]_1$.

$$\square$$

4.2.2. An Algebraic Construction of $\mathfrak{B}B_n$.

We now provide a second way to construct $\mathfrak{B}B_n$, using the methods of Section 3.3. For the remainder of this section we denote by $B$ the braid group $B_n = G(\mathfrak{B}B_n)$. We also write $E = E_q_n$ for the join semilattice of equivalence relations on $n$.

Now one may easily check that for each $\beta \in B_n$ the map

$$\varphi_\beta : E \rightarrow E : E \mapsto E^\beta$$

is an automorphism of $E$, and that

$$\varphi : B \rightarrow \text{Aut}(E) : \beta \mapsto \varphi_\beta$$

is an antihomomorphism. Thus we may form the semidirect product

$$E \rtimes B = E \rtimes_\varphi B = \{(E, \beta) \mid E \in E, \beta \in B\}$$

with multiplication defined, for $E_1, E_2 \in E$ and $\beta, \gamma \in B$, by

$$(E_1, \beta)(E_2, \gamma) = (E_1 \lor E_2^\beta, \beta \gamma).$$

For $E \in E$ we define $B_E$ to be the subgroup of $B$ generated by the set

$$\{\beta^{-1} i_\beta (i, i + 1) \bar{\beta} \in E\}.$$
One may readily check that these subgroups satisfy the conditions
\[
\begin{align*}
\beta B_\mathcal{E} \beta^{-1} &= B_\mathcal{E} & \text{for all } \mathcal{E} \in E \text{ and } \beta \in B & \quad (B_\mathcal{E}1) \\
B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2} &\subseteq B_{\mathcal{E}_1 \vee \mathcal{E}_2} & \text{for all } \mathcal{E}_1, \mathcal{E}_2 \in E & \quad (B_\mathcal{E}2) \\
\mathcal{E}^\beta &= \mathcal{E} & \text{for all } \mathcal{E} \in E \text{ and } \beta \in B_\mathcal{E}. & \quad (B_\mathcal{E}3)
\end{align*}
\]
Thus, by Lemma 3.12, we may define a congruence \( \sim \) on \( E \times B \) by
\[
(\mathcal{E}_1, \beta) \sim (\mathcal{E}_2, \gamma) \quad \text{if and only if} \quad \mathcal{E}_1 = \mathcal{E}_2 \text{ and } \beta \gamma^{-1} \in B_{\mathcal{E}_1},
\]
and form the quotient \( (E \times B)/\sim \). We denote the \( \sim \)-class of \( (\mathcal{E}, \beta) \in E \times B \) by \([\mathcal{E}, \beta]\).

**Theorem 4.8.** Suppose that \( \mathcal{E} \in E \). Then \( B_\mathcal{E} = [1]_\mathcal{E} = \{ \beta \in B \mid [\beta \sim_\mathcal{E} 1] \} \).

**Proof.** Suppose first that \( 1 \leq i \leq n - 1 \) and \( (i, i + 1) \beta \in \mathcal{E} \) so that \( \beta^{-1} \mathcal{E}_i \beta \) is a generator of \( B_\mathcal{E} \). Referring again to the left diagram in Figure 4.6, we see that \( \beta^{-1} \mathcal{E}_i \beta \sim_\mathcal{E} 1 \) using a homotopy which causes the black strings in the middle of the diagram to merge and part between configurations (4) and (1) in Figure 4.9. This shows that that \( B_\mathcal{E} \subseteq [1]_\mathcal{E} \).

To show the reverse inclusion, suppose that \( \beta \in [1]_\mathcal{E} \), and that \( H \) is an \( \mathcal{E} \)-homotopy from \( \beta \) to 1. If no strings touch during \( H \) then \( \beta = 1 \in B_\mathcal{E} \). Otherwise, suppose that \( H = H_1 H_2 \) is the composite of two homotopies \( H_1 \) and \( H_2 \) such that exactly one pair of strings merge and part during \( H_2 \). Let \( \gamma \) be the braid obtained by applying \( H_1 \) to \( \beta \). By an induction hypothesis we have \( \beta \gamma^{-1} \in B_\mathcal{E} \). Now \( H_2 \) may be replaced (if necessary) by a composite of homotopies \( H_3 H_4 H_5 \), pictured in Figure 4.10, where no strings touch during \( H_3 \) and \( H_5 \), and \( \tau \in \{1, \mathcal{E}_i^{\pm1}, \mathcal{E}_i^{\pm2}\} \) for some \( i \), depending on the configurations catalogued in Figure 4.9.

**Figure 4.10.** The homotopies \( H_3, H_4, \) and \( H_5 \).

Since no strings touch during \( H_3 \) and \( H_5 \) we have \( \gamma = \gamma_1 \gamma_2 \) and \( \gamma_1 \tau \gamma_2 = 1 \). Since \( H_4 \) causes the \( i \)th and \( (i + 1) \)th strings (in the middle portion of the diagram) to merge and
part, we must have \((i, i + 1) \overline{\gamma} = 1\). But then \(1 = \gamma_1 \gamma_2 (\overline{\gamma}_2 \tau \gamma_2) = \gamma (\overline{\gamma}_2 \tau \gamma_2)\) so that \(\gamma = \overline{\gamma}_2 \tau \gamma_2 \in B_\mathcal{E}\), since \(\overline{\gamma}_2 = \gamma_1^{-1}\). Thus \(\beta = (\overline{\gamma}^{-1}) \gamma \in B_\mathcal{E}\) completing the proof. 

We may now show that the construction of \((E \times B)/\sim\) yields a monoid isomorphic to \(\mathfrak{F}\mathcal{B}_n\).

**Theorem 4.9.** Suppose that \(\mathcal{E}_1, \mathcal{E}_2 \in E\) and \(\beta, \gamma \in B\). Then \((\mathcal{E}_1, \beta) \sim (\mathcal{E}_2, \gamma)\) if and only if \([\beta]_{\mathcal{E}_1} = [\gamma]_{\mathcal{E}_2}\). Thus \((E \times B)/\sim\) and \(\mathfrak{F}\mathcal{B}_n\) are isomorphic via the map

\[
[\mathcal{E}, \beta] \mapsto [\beta]_{\mathcal{E}} \quad \text{for all } \mathcal{E} \in E \text{ and } \beta \in B.
\]

**Proof.** By Theorem 4.8 we have

\[
(\mathcal{E}_1, \beta) \sim (\mathcal{E}_2, \gamma) \iff E_1 = E_2 \text{ and } \beta \gamma^{-1} \in B_{\mathcal{E}_1}
\]

\[
\iff E_1 = E_2 \text{ and } \beta \gamma^{-1} \sim_{\mathcal{E}_1} 1
\]

\[
\iff E_1 = E_2 \text{ and } \beta = (\beta \gamma^{-1}) \gamma \sim_{\mathcal{E}_1} 1 \gamma = \gamma
\]

\[
\iff [\beta]_{\mathcal{E}_1} = [\gamma]_{\mathcal{E}_2}.
\]

The isomorphism statement is now clear in light of the respective rules for multiplication in both monoids. 

Theorems 4.8 and 4.9 will be useful when we give presentations of \(\mathfrak{F}\mathcal{B}_n\) in Chapter 5, and submonoids of \(\mathfrak{F}\mathcal{B}_n\) in Chapters 6 and 7. Also of use will be the following result which follows immediately from the definition of the subgroups \(B_{\mathcal{E}}\) together with Lemma 4.1.

**Lemma 4.10.** If \(\mathcal{E} \in E\), then \(B_{\mathcal{E}}\) is generated by the set \(\{\beta^{-1} \varsigma_{ij} \beta \mid \beta \in \mathcal{P}_n, (i, j) \in \mathcal{E}\}\).

### 4.2.3. Embedding \(\mathfrak{F}\mathcal{B}_n\) in the Coset Monoid of the Braid Group.

We now show that \(\mathfrak{F}\mathcal{B}_n\) embeds in \(\mathcal{C}(B)\), the coset monoid of the braid group \(B = \mathcal{B}_n\).

**Proposition 4.11.** The map \(\psi : \mathcal{E} \mapsto B_{\mathcal{E}} (\mathcal{E} \in E)\) is a semilattice embedding \(E \to \mathcal{C}(B)\).

**Proof.** By Theorem 4.8 and the proof of Theorem 4.7 we see that \(\psi\) is injective. By \((B_{\mathcal{E}})\) we have \(B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2} \subseteq B_{\mathcal{E}_1 \vee \mathcal{E}_2}\), so it remains to prove the reverse inclusion. Since we obviously have equality if either \(\mathcal{E}_1 = 1\) or \(\mathcal{E}_2 = 1\), we may assume that \(\mathcal{E}_1 \neq 1\) and \(\mathcal{E}_2 \neq 1\). Let \(\beta \in \mathcal{P}_n\) and \(1 \leq i < j \leq n\) such that \((i, j) \in \mathcal{E}_1 \vee \mathcal{E}_2\). By Lemma 4.10, it suffices to show that \(\tau = \beta^{-1} \varsigma_{ij} \beta \in B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2}\). Now \(\tau \in B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2}\) if and only if

\[
\varsigma_{ij} = \beta \tau^{-1} \beta^{-1} \in \beta (B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2}) \beta^{-1} = (\beta B_{\mathcal{E}_1} \beta^{-1}) \vee (\beta B_{\mathcal{E}_2} \beta^{-1}) = B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2} = B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2}.
\]

Thus the proof will be complete if we can show that \(\varsigma_{ij} \in B_{\mathcal{E}_1} \vee B_{\mathcal{E}_2}\). Since \((i, j) \in \mathcal{E}_1 \vee \mathcal{E}_2\), there exist distinct \(i_1, \ldots, i_k \in n\) such that \(i_1 = i, i_k = j\), and for each \(r\) we have either
(i_r, i_{r+1}) \in \mathcal{E}_1 \text{ or } (i_r, i_{r+1}) \in \mathcal{E}_2. \text{ Renaming } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \text{ if necessary, we may also assume for convenience that } (i_1, i_2) \in \mathcal{E}_1. \text{ Now if } k = 2 \text{ then we are done, so suppose that } k \geq 3. \text{ By relation (B2)' (see Theorem 2.11), we have}

\[ \varsigma_{ij} = \begin{cases} 
\varsigma_{i_2j_2}i_{i_2j_2}^{-1} & \text{if } i_2 < i < j \\
\varsigma_{i_2j_2}^{-1}i_{i_2j_2} & \text{if } i < i_2 < j \\
i_{i_2j_2}^{-1}\varsigma_{i_2j_2} & \text{if } i < j < i_2.
\end{cases} \]

Now, writing \( \varsigma_{pq} = \varsigma_{qp} \) if \( 1 \leq q < p \leq n \), we see that in any case we have \( \varsigma_{ij} \in B_{\mathcal{E}_1} \) by Lemma 4.10. By an induction hypothesis, \( \varsigma_{ij} \in B_{\mathcal{E}_1} \lor B_{\mathcal{E}_2} \) showing that \( \varsigma_{ij} \in B_{\mathcal{E}_1} \lor B_{\mathcal{E}_2} \) and completing the proof.

By Proposition 4.11 and Theorem 3.4 we have the following.

**Theorem 4.12.** The map \([\beta]_{\mathcal{E}} \mapsto B_{\mathcal{E}}\beta \ (\mathcal{E} \in E, \beta \in B)\) is an embedding \( \mathcal{E}B_n \rightarrow C(B_n) \). \( \square \)

**Remark 4.13.** For each \( \mathcal{E} \in E \) and \( \beta \in B \) we see, by Theorem 4.8, that

\[ [\beta]_{\mathcal{E}} = [1]_{\mathcal{E}}\beta = B_{\mathcal{E}}\beta \]

so that \( \mathcal{E}B_n \) is in fact a submonoid of \( C(B_n) \).

### 4.3. The Permeable Braid Monoid

In this section we introduce the permeable braid monoid \( \mathcal{P}B_n \). This monoid is closely related to the factorisable braid monoid \( \mathcal{E}B_n \) introduced in the previous section. As is the case with \( \mathcal{E}B_n \), the elements of \( \mathcal{P}B_n \) are equivalence classes of braids, one equivalence class for each equivalence relation on \( n \). We simply modify the notion of \( \approx_{\mathcal{E}} \)-equivalent braids by allowing \( \mathcal{E} \)-related strings to “permeate” rather than merge and part, thus defining the notion of \( \approx_{\mathcal{E}} \)-equivalent braids.

#### 4.3.1. The Geometric Definition of \( \mathcal{P}B_n \).

Suppose that \( s \) and \( t \) are disjoint strings (as in Figure 4.7). We say that a homotopy causes \( s \) and \( t \) to permeate if

(i) the homotopy first causes \( s \) and \( t \) to come together just once at say \( P \) and \( Q \) (as in Figure 4.8),

(ii) and then part, reconstituting as two strings made up of respective upper and lower strands. See Figure 4.11 for a catalogue of the possible configurations in the neighbourhood of the crossing a moment before and after permeating.
Figure 4.11. Possible configurations before and after permeating.

Note that again configurations (1), (2), and (3) can be interchanged using a normal homotopy.

Let $E \in \mathcal{E}q_n$. Recall that the $i$th and $j$th strings of a braid are $E$-related (resp. $E$-unrelated) if $(i, j) \in E$ (resp. $(i, j) \notin E$). If $\beta, \gamma \in \mathcal{B}_n$, then we write $\beta \approx_E \gamma$ if there is a homotopy from $\beta$ to $\gamma$ during which $E$-unrelated strings never touch, and $E$-related strings may permeate (one at a time, a finite number of times). We call such a homotopy an $(E, 2)$-homotopy, and we denote the $\approx_E$-class of $\beta \in \mathcal{B}_n$ by $[\beta]_E$. We now define the permeable braid monoid to be the set

$$\mathcal{P}B_n = \{ [\beta]_E | E \in \mathcal{E}q_n, \beta \in \mathcal{B}_n \}$$

equipped with the operation defined, for $E_1, E_2 \in \mathcal{E}q_n$ and $\beta, \gamma \in \mathcal{B}_n$, by

$$[\beta]_{E_1} [\gamma]_{E_2} = [\beta \gamma]_{E_1 \lor E_2}.$$

One may easily check that $\mathcal{P}B_n$ is a monoid with identity $[1]_1$. Again, for $\beta, \gamma \in \mathcal{B}_n$ we have $\beta \approx_1 \gamma$ if and only if $\beta = \gamma$, so we identify $[\beta]_1 = \{ \beta \}$ with $\beta$ for each $\beta \in \mathcal{B}_n$.

**Theorem 4.14.** The permeable braid monoid $\mathcal{P}B_n$ is a factorisable inverse monoid with

$$E(\mathcal{P}B_n) = \{ [1]_E | E \in \mathcal{E}q_n \} \cong \mathcal{E}q_n, \lor \} \quad \text{and} \quad G(\mathcal{P}B_n) = \{ [\beta]_1 | \beta \in \mathcal{B}_n \} = \mathcal{B}_n.$$

**Proof.** As in the proof of Theorem 4.7, the key step is to show that $[\beta]_{E_1} = [\gamma]_{E_2}$ implies $E_1 = E_2$ for any $\beta, \gamma \in \mathcal{B}_n$ and $E_1, E_2 \in \mathcal{E}q_n$. To show that this is the case, note first that since $\approx_E \subseteq \approx_\mathcal{E}$ for any $E \in \mathcal{E}q_n$, we may define a map

$$\nu : \mathcal{P}B_n \to \mathfrak{B} \mathcal{B}_n : [\beta]_E \mapsto [\beta]_E$$

for each $E \in \mathcal{E}q_n$ and $\beta \in \mathcal{B}_n$, which is clearly an epimorphism. Now if $\beta, \gamma \in \mathcal{B}_n, E_1, E_2 \in \mathcal{E}q_n$, and $[\beta]_{E_1} = [\gamma]_{E_2}$, then we have

$$[\beta]_{E_1} = [\gamma]_{E_2} \quad \text{whence} \quad (E_1, \beta) \sim (E_2, \gamma) \quad \text{by Theorem 4.9, and so} \quad E_1 = E_2.\quad \text{The proof of the theorem may now be completed by suitably modifying the proof of Theorem 4.7.} \quad \Box$$
Now if $\beta, \gamma \in B_n$ and $E \in \mathcal{E}q_n$ are such that $\beta \approx_E \gamma$, then we clearly have $\tilde{\beta} = \tilde{\gamma}$. Thus we may define a map

$$- : \mathcal{PB}_n \to S_n : [\beta]_E \mapsto [\tilde{\beta}]_E = \tilde{\beta}$$

for each $E \in \mathcal{E}q_n$ and $\beta \in B_n$.

which, again, is easily seen to be an epimorphism whose restriction to $B_n$ is the permutation map $- : B_n \to S_n$ of Section 2.4.

**Remark 4.15.** It is also possible to extend $- : B_n \to S_n$ to an epimorphism

$$\mathcal{PB}_n \to \mathfrak{S}^*_n : [\beta]_E \mapsto \text{id}_E \tilde{\beta}$$

for each $E \in \mathcal{E}q_n$ and $\beta \in B_n$.

This map is simply the composite of the epimorphism $\nu : \mathcal{PB}_n \to \mathfrak{S}B_n$ defined in the proof of Theorem 4.14 with the map $- : \mathfrak{S}B_n \to \mathfrak{S}^*_n$ of Section 4.2.1. However, it will be more convenient to use the map $- : \mathcal{PB}_n \to S_n$ defined above when we define the pure permeable braid monoid in Chapter 6.

### 4.3.2. An Algebraic Construction of $\mathcal{PB}_n$. 

For the remainder of this section we write $B = B_n$ and $E = \mathcal{E}q_n$. Again we start with the semidirect product

$$E \ltimes B = \{ (E, \beta) \mid E \in E, \beta \in B \}$$

with multiplication defined, for $E_1, E_2 \in E$ and $\beta, \gamma \in B$, by

$$(E_1, \beta)(E_2, \gamma) = (E_1 \vee E_2, \beta \gamma).$$

For $E \in E$ we define $B_E^{(2)}$ to be the subgroup of $B$ generated by the set

$$\{ \beta^{-1} E^{i+1} \beta \mid (i, i+1) \tilde{\beta} \in E \}.$$

One may readily check that these subgroups satisfy the conditions

$$\beta B_E^{(2)} \beta^{-1} = B_{E^\beta}^{(2)}$$

for all $E \in E$ and $\beta \in B$ \quad (B_E^{(2)}1)

$$B_{E_1}^{(2)} \vee B_{E_2}^{(2)} \subseteq B_{E_1 \vee E_2}^{(2)}$$

for all $E_1, E_2 \in E$ \quad (B_E^{(2)}2)

$$E^\beta = E$$

for all $E \in E$ and $\beta \in B_{E}^{(2)}$. \quad (B_E^{(2)}3)

Thus, by Lemma 3.12, we may define a congruence $\approx$ on $E \ltimes B$ by

$$(E_1, \beta) \approx (E_2, \gamma) \quad \text{if and only if} \quad E_1 = E_2 \quad \text{and} \quad \beta \gamma^{-1} \in B_{E_1}^{(2)},$$

and form the quotient $(E \ltimes B)/\approx$. We denote the $\approx$-class of $(E, \beta) \in E \ltimes B$ by $[E, \beta]$. The proofs of the following two theorems are almost identical to those of Theorems 4.8 and 4.9.

**Theorem 4.16.** Suppose that $E \in E$. Then $B_E^{(2)} = [1]_E = \{ \beta \in B \mid \beta \approx_E 1 \}$. \hfill $\Box$
Theorem 4.17. Suppose that $E_1, E_2 \in E$ and $\beta, \gamma \in B$. Then $(E_1, \beta) \approx (E_2, \gamma)$ if and only if $[\beta]_{E_1} = [\gamma]_{E_2}$. Thus $(E \times B)/\approx$ and $\mathfrak{P}B_n$ are isomorphic via the map

$$[E, \beta] \mapsto [\beta]_E \quad \text{for all } E \in E \text{ and } \beta \in B.$$ 

The next two results follow immediately from the definition of the subgroups $B^{(2)}_E$ and Lemmas 4.1 and 4.2 respectively. They will both prove useful at different times.

Lemma 4.18. If $E \in E$, then $B^{(2)}_E$ is generated by the set $\{\beta^{-1}\alpha_{ij} \beta \mid \beta \in \mathcal{P}_n, (i, j) \in E\}$. \hfill $\square$

Lemma 4.19. If $E \in E$, then $B^{(2)}_E$ is generated by the set $\{\beta^{-1}\alpha_{ij} \beta \mid \beta \in \mathcal{P}_n, (i, j) \in E\}$. \hfill $\square$

We conclude this chapter by showing that, unlike the factorisable braid monoid $\mathfrak{F}B_n$, the permeable braid monoid $\mathfrak{P}B_n$ does not always belong to the class $\mathcal{C}$ defined in Section 3.2. Recall that for $1 \leq i < j \leq n$, we defined

$$E_{ij} = \{(r, s) \mid r = s \text{ or } \{r, s\} = \{i, j\}\} \subset E.$$ 

Proposition 4.20. Suppose that $n \geq 3$. Then $B^{(2)}_{E_{12}} \vee B^{(2)}_{E_{23}} \neq B^{(2)}_{E_{12} \vee E_{23}}$.

Proof. By Lemma 4.19, we see that

$$B^{(2)}_{E_{12}} = \langle \beta^{-1}\alpha_{12}\beta \mid \beta \in \mathcal{P}_n \rangle \quad \text{and} \quad B^{(2)}_{E_{23}} = \langle \beta^{-1}\alpha_{23}\beta \mid \beta \in \mathcal{P}_n \rangle.$$ 

Now for any $\beta \in \mathcal{P}_n$ we clearly have

$$\exp_{13}(\beta^{-1}\alpha_{12}\beta) = \exp_{13}(\beta^{-1}\alpha_{23}\beta) = 0,$$

where the map $\exp_{13} : \mathcal{P}_n \to (\mathbb{Z}, +)$ was defined after Theorem 2.13. Thus $\exp_{13}(\gamma) = 0$ for any $\gamma \in B^{(2)}_{E_{12}} \vee B^{(2)}_{E_{23}}$. On the other hand we have $(1, 3) \in E_{12} \vee E_{23}$ so that $\alpha_{13} \in B^{(2)}_{E_{12} \vee E_{23}}$. However, $\exp_{13}(\alpha_{13}) = 1$ so that $\alpha_{13} \not\in B^{(2)}_{E_{12}} \vee B^{(2)}_{E_{23}}$. \hfill $\square$

This shows that $\mathfrak{P}B_n$ does not belong to $\mathcal{C}$ if $n \geq 3$. However, since $\mathfrak{P}B_1 = B_1 = \{1\}$ we see that $\mathfrak{P}B_1 \in \mathcal{C}$. Also, since $E(\mathfrak{P}B_2) = \{[1]_1, [1]_{E_{12}}\}$ and $B_1 \neq B_{E_{12}}$ we have $\mathfrak{P}B_2 \in \mathcal{C}$. So we have proved the following.

Theorem 4.21. The permeable braid monoid $\mathfrak{P}B_n$ belongs to $\mathcal{C}$ if and only if $n \leq 2$. \hfill $\square$

Remark 4.22. Although it follows from Theorem 3.4 that $\mathfrak{P}B_n$ embeds in $C(B_n)$ for $n \in \{1, 2\}$, we are unable to use Theorem 3.4 to conclude that $\mathfrak{P}B_n$ does not embed in $C(B_n)$ for $n \geq 3$. By results of [25], if $n \geq 3$ there is no embedding $\mathfrak{P}B_n \to C(B_n)$ which maps $B_n \subseteq \mathfrak{P}B_n$ onto $G(C(B_n)) \cong B_n$. However, if $n \geq 2$, it is possible to construct strict
(that is, non-surjective) embeddings $\mathcal{B}_n \rightarrow \mathcal{B}_n$. For example, let $z \in \mathcal{B}_n$ be a nontrivial central element (see for example [12]). Then, due to the homogeneity of the relations in Artin’s presentation, we may define a homomorphism $\psi : \mathcal{B}_n \rightarrow \mathcal{B}_n$ by $\zeta_i \psi = \zeta_i z$ for each $i$. It is a pleasant exercise, using homogeneity and the fact that $z$ has a non-zero exponent sum, to show that $\psi$ is injective but not surjective. It is unknown to the author whether a strict embedding $\mathcal{B}_n \rightarrow \mathcal{B}_n$ (such as $\psi$) may be used to construct an embedding $\mathfrak{P}\mathcal{B}_n \rightarrow \mathcal{C}(\mathcal{B}_n)$.

4.4. Visualising the Elements of $\mathfrak{F}\mathcal{B}_n$ and $\mathfrak{P}\mathcal{B}_n$

It will prove extremely useful at times to picture the elements of the braid monoids $\mathfrak{F}\mathcal{B}_n$ and $\mathfrak{P}\mathcal{B}_n$. Now elements of both monoids are equivalence classes of braids and there will generally be infinitely many braids in any one of these equivalence classes. In what follows, we describe a way to picture the elements of the semidirect product $E \rtimes B$ where again we have written $E = \mathcal{E}q_n$ and $B = \mathcal{B}_n$. These pictures are braid analogues of the pictures described at the end of Section 3.1 for uniform block bijections. They will allow us to obtain geometric interpretations of concepts such as the action of $B$ on $E$ and the rule for multiplication in $E \rtimes B$, and will also prove useful in subsequent chapters when performing calculations in $\mathfrak{F}\mathcal{B}_n$ and $\mathfrak{P}\mathcal{B}_n$.

Suppose that $\mathcal{E} \in E$ and $\beta \in B$. We obtain a picture of $(\mathcal{E}, \beta) \in E \rtimes B$ by superimposing a picture of $\mathcal{E}$ (see Section 2.3.7) on the upper row of dots of a picture of $\beta$. For example, the picture of $(\mathcal{E}, \beta) \in \mathcal{E}q_6 \rtimes \mathcal{B}_6$ is illustrated in Figure 4.12 where

$$\beta = \zeta_3 \zeta_4 \zeta_1 \zeta_5^{-1} \zeta_2 \zeta_3 \in \mathcal{B}_6 \quad \text{and} \quad \mathcal{E} = \mathcal{E}_{12} \lor \mathcal{E}_{21} \lor \mathcal{E}_{56} \in \mathcal{E}q_6.$$

Figure 4.12. A picture of an element of $\mathcal{E}q_6 \rtimes \mathcal{B}_6$.

If $\mathcal{E} \in E$ and $\beta \in B$, then the equivalence $\mathcal{E}^\beta$ may be calculated pictorially by superimposing a picture of $\mathcal{E}$ on the lower row of dots of a picture of $\beta$, and then “sliding” the picture of $\mathcal{E}$ up the strings of $\beta$. A picture of $\mathcal{E}^\beta$ will now be be superimposed on the upper row of
4.4. VISUALISING THE ELEMENTS OF $\mathfrak{B}_n$ AND $\mathfrak{P}_n$

dots of $\beta$. In Figure 4.13 we have calculated the equivalence $\mathcal{E}^\beta$ where $\mathcal{E} \in \mathcal{E}_q_6$ and $\beta \in \mathcal{B}_6$ are defined as above.

![Picture of $\mathcal{E}$](image1)

![Picture of $\mathcal{E}^\beta$](image2)

**Figure 4.13.** The action of a braid $\beta \in \mathcal{B}_6$ on an equivalence $\mathcal{E} \in \mathcal{E}_q_6$.

If $\mathcal{E}_1, \mathcal{E}_2 \in E$ and $\beta, \gamma \in B$, then a picture of the product $(\mathcal{E}_1, \beta)(\mathcal{E}_2, \gamma) = (\mathcal{E}_1 \vee \mathcal{E}_2^\beta, \beta\gamma)$ may be calculated by

(i) placing a picture of $(\mathcal{E}_1, \beta)$ above a picture of $(\mathcal{E}_2, \gamma)$;
(ii) erasing the middle row of dots and sliding the picture of $\mathcal{E}_2$ up the strings of $\beta$
until it becomes a picture of $\mathcal{E}_2^\beta$, superimposed on the picture of $\mathcal{E}_1$;
(iii) contracting the resulting object so that it lies between the $z = 0$ and $z = 1$ planes,
and simplifying the picture of $\mathcal{E}_1 \vee \mathcal{E}_2^\beta$.

An example of this procedure has been carried out in Figure 4.14.

![Picture of $\mathcal{E}_1$](image3)

![Picture of $\mathcal{E}_2$](image4)

![Picture of $\mathcal{E}_1 \vee \mathcal{E}_2^\beta$](image5)

**Figure 4.14.** The product of two elements $(\mathcal{E}_1, \beta), (\mathcal{E}_2, \gamma) \in \mathcal{E}_q_4 \rtimes \mathcal{B}_4$.

The reader is invited to draw pictures of the products $(1, \varsigma_1\varsigma_2)(\mathcal{E}_{12}, 1)$ and $(\mathcal{E}_{23}, 1)(1, \varsigma_1\varsigma_2)$ in $\mathcal{E}_q_3 \rtimes \mathcal{B}_3$ in order to show that $(1, \varsigma_1\varsigma_2)(\mathcal{E}_{12}, 1) = (\mathcal{E}_{23}, 1)(1, \varsigma_1\varsigma_2)$. This verifies a special case of one of the relations in a presentation (see Theorems 5.48 and 5.69).
CHAPTER 5

Presentations of Factorisable Inverse Braid Monoids

We now turn our attention to the task of finding presentations of the braid monoids \( \mathcal{I}B_n \), \( \mathfrak{S}B_n \), and \( \mathfrak{P}B_n \) introduced in Chapter 4. Since these are all factorisable inverse monoids, we will be able to obtain presentations for free by making use of presentations of the group of units and the semilattice of idempotents together with one of Theorems 3.15, 3.16, and 3.17. After obtaining these initial presentations, we will then derive a number of different presentations of each monoid. We will also use the presentations of \( \mathcal{I}B_n \) and \( \mathfrak{F}B_n \) to deduce presentations of \( \mathcal{I}_n \) and \( \mathfrak{F}_n \) (respectively) including well-known presentations such as Popova’s presentation of \( \mathcal{I}_n \) [55], and FitzGerald’s presentation of \( \mathfrak{S}^*_n \) [32]. When studying \( \mathfrak{F}B_n \) in Section 5.2 we will also uncover an interesting connection with the singular braid monoid \( \mathfrak{S}B_n \) (see [7, 9]). Various results of this chapter appear in [23, 24, 27].

We fix a positive integer \( n \) for the remainder of this chapter, and denote by \( B = \mathcal{B}_n \) the braid group on \( n \). Recall that \( X_B = \{ \sigma_1, \ldots, \sigma_{n-1} \} \), and that \( R_B \) is the set of relations

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{if } |i - j| > 1 \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{if } |i - j| = 1.
\end{align*}
\]

(B1) (B2)

Let \( X_B^{-1} = \{ \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1} \} \) be a set of formal inverses for the elements of \( X_B \), and denote by \( R_F \) the set of free group relations

\[
\sigma_i^{\pm 1} \sigma_i^{\mp 1} = 1
\]

for all \( i \).

(F)

Thus, by Theorem 2.10, \( B \) has monoid presentation \( \langle X_B \cup X_B^{-1} \mid R_B \cup R_F \rangle \) via \( \zeta : (X_B \cup X_B^{-1})^* \to B : \sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} \) for each \( i \).

All presentations considered in this chapter are assumed to be monoid presentations. We denote the congruence \( (R_B \cup R_F)^\sim \) on \( (X_B \cup X_B^{-1})^* \) by \( \sim_B \), and we choose a set of words \( \{ \hat{\beta} \mid \beta \in B \} \subseteq (X_B \cup X_B^{-1})^* \) such that \( \hat{\beta} \zeta = \beta \) for all \( \beta \in B \). For convenience we will assume that

\[
\begin{align*}
\hat{\zeta}_i &= \sigma_i & \text{for each } i \\
\hat{\alpha}_{ij} &= (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{i+1}^{-1} \cdots \sigma_j^{-1}) & \text{for each } i, j \\
\hat{\zeta}_{ij} &= (\sigma_j^{-1} \cdots \sigma_{i+1}^{-1}) \sigma_i (\sigma_{i+1} \cdots \sigma_{j-1}) & \text{for each } i, j.
\end{align*}
\]
Now if \( w = \sigma^{\varepsilon_1}_{i_1} \cdots \sigma^{\varepsilon_k}_{i_k} \in (X_B \cup X_B^{-1})^* \), we will write \( w^{-1} = \sigma^{-\varepsilon_k}_{i_k} \cdots \sigma^{-\varepsilon_1}_{i_1} \in (X_B \cup X_B^{-1})^* \), so that \( ww^{-1} \sim_B w^{-1}w \sim_B 1 \) for all \( w \in (X_B \cup X_B^{-1})^* \). We may further suppose that for each \( \beta \in B \) we have \( \overline{\beta}^{-1} = \overline{\beta}^{-1} \). We also define a homomorphism

\[
\overline{\cdot} : (X_B \cup X_B^{-1})^* \rightarrow S_n : w \mapsto \overline{w} = \overline{w} \xi
\]

for each \( w \in (X_B \cup X_B^{-1})^* \).

It will also be convenient to record an elementary result concerning the words \( \xi_{ij} \) which follows by Theorem 2.10 and the comment after Figure 4.1.

**Lemma 5.1.** Suppose that \( 1 \leq i < j \leq n \). Then

\[
\xi_{ij} \sim_B \overline{w_{ij}^{-1}} \sigma_1 w_{ij}
\]

where \( w_{ij} = (\sigma_2 \cdots \sigma_{j-1})(\sigma_1 \cdots \sigma_{i-1}) \).

\[\square\]

### 5.1. The Inverse Braid Monoid

In this section we study presentations of the inverse braid monoid \( \mathcal{IB}_n \). In Theorem 5.13 we derive the presentation of \( \mathcal{IB}_n \) found by Easdown and Lavers in [21]. Our method (applying Theorem 3.15) differs from the method used in [21] where the authors show that every word in the generators is equivalent, using the relations, to a unique normal form; see also Chapter 8 for more proofs of this kind. Starting from the presentation in Theorem 5.13, we derive several new presentations of \( \mathcal{IB}_n \) (see Theorems 5.15 and 5.18 and Remarks 5.16 and 5.20). As in [21], we then show that each presentation of \( \mathcal{IB}_n \) yields a presentation of the symmetric inverse semigroup \( \mathcal{I}_n \) (see Proposition 5.21 and Theorems 5.22, 5.24, 5.26, and 5.27).

#### 5.1.1. A Presentation of \( \mathcal{IB}_n \)

We use the notation of Section 4.1. In particular we write \( T = 2^n \) for the power set of \( \mathbf{n} \) considered as a semilattice under \( \cap \), and for \( A \subseteq \mathbf{n} \) we will denote the complement \( \mathbf{n} \setminus A \) by \( A^c \). Recall that \( X_T = \{ \varepsilon_1, \ldots, \varepsilon_n \} \) and that \( R_T \) is the set of relations

\[
\varepsilon_i^2 = \varepsilon_i \quad \text{for all } i \quad \text{(T1)}
\]

\[
\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad \text{for all } i, j \quad \text{(T2)}
\]

so that, by Theorem 2.7, \( T \) has presentation \( \langle X_T \mid R_T \rangle \) via

\[
\eta : \varepsilon_i \mapsto \{i\}^c \quad \text{for each } i \in \mathbf{n}.
\]
We will now gather further information in order to apply Theorem 3.15. It is clear that for each \( i \in \mathbb{N} \) and \( 1 \leq r \leq n - 1 \) we have

\[
\{i\}^c \mapsto \begin{cases} 
  \{i + 1\}^c & \text{if } r = i \\
  \{i - 1\}^c & \text{if } r = i - 1 \\
  \{i\}^c & \text{otherwise.}
\end{cases}
\]

That is, \( \{i\}^c \mapsto \{is_r\}^c \) where \( s_r = t_{r,r+1} = \bar{c}_r^\pm \) denotes the simple transposition which interchanges \( r \) and \( r + 1 \). Thus we may take \( R_\alpha \) to be the set of relations

\[
\sigma_r^\pm \varepsilon_i = \varepsilon_{is_r} \sigma_r^\pm 
\]

for all \( i, r \).

Now for \( A \in T \) with \( A^c = \{i_1, \ldots, i_k\} \) and \( i_1 < \cdots < i_k \), let \( \varepsilon_A = \varepsilon_{i_1} \cdots \varepsilon_{i_k} \in X_T^* \), noting that \( \varepsilon_A\varepsilon_A = A \). Thus, by Theorem 4.4 and the definition of the subgroups \( B_A \), we may take \( R_\wedge \) to be the set of relations

\[
\varepsilon_A \beta^{-1} \sigma_i \beta = \varepsilon_A & \quad \text{if } A \in T, \beta \in B, \text{ and } i, (i + 1) \beta \in A^c \\
\varepsilon_A \beta^{-1} \sigma_i^2 \beta = \varepsilon_A & \quad \text{if } A \in T, \beta \in B, \text{ and } i \beta \in A^c \text{ or } (i + 1) \beta \in A^c.
\]

Recall that for \( A \subseteq \mathbb{N} \) and \( \beta \in B \) we denote by \( \beta_A \in \mathcal{IB}_n \) the partial braid obtained from \( \beta \) by removing all strings which are not in \( A \). By Theorems 3.15 and 4.5 we have the following.

**Corollary 5.2.** The inverse braid monoid \( \mathcal{IB}_n \) has presentation

\[
\langle X_B \cup X_B^{-1} \cup X_T \mid R_B \cup R_F \cup R_T \cup R_\alpha \cup R_\wedge \rangle
\]

via

\[
\sigma_i^\pm \mapsto \zeta_i^\pm \quad \text{for each } i \\
\varepsilon_i \mapsto 1_{\{i\}^c} \quad \text{for each } i.
\]

The partial braid \( 1_{\{i\}^c} \) is the identity braid with the \( i \)th string missing; see Figure 5.1.

**Figure 5.1.** The partial braid \( 1_{\{i\}^c} \in \mathcal{IB}_n \).

We will now work towards simplifying the presentation in Corollary 5.2. With this in mind, let \( e = \varepsilon_1 \). Then by \( R_F \cup R_\alpha \) we see that the relations

\[
\varepsilon_i = \sigma_{i-1} \cdots \sigma_1 e \sigma_1^{-1} \cdots \sigma_{i-1}^{-1} \quad \text{for all } i
\]

(\text{*})
are in \((R_B \cup R_F \cup R_T \cup R_\times \cup R_-)^2\). Thus we may remove all generators \( \varepsilon_i \) with \( i \neq 1 \), replacing their every occurrence in the relations by the word on the right hand side of \((1)\), which we denote by \( e_i \). (Note in particular that \( e_1 = e \).) We denote the resulting relations by \((T1)'\), \((T2)'\), \((\times)'\), \((\sim 1)'\), and \((\sim 2)'\). The entire sets of relations which have been modified in this way will be denoted by \( R_T', R_\times', R_- \). Let \( X_{IB} = X_B \cup X_B^{-1} \cup \{ e \} \).

**Corollary 5.3.** The inverse braid monoid \( IB_n \) has presentation

\[
\langle X_{IB} \mid R_B \cup R_F \cup R_T' \cup R_\times' \cup R_-' \rangle
\]

via

\[
\begin{align*}
\sigma_i^{\pm 1} &\mapsto \zeta_i^{\pm 1} & \text{for each } i \\
e &\mapsto 1_{(1)e}. \\
\end{align*}
\]

For the rest of this section, we denote the congruence \((R_B \cup R_F \cup R_T' \cup R_\times' \cup R_-')^2\) on \( X_{IB}^* \) by \( \approx \).

**Lemma 5.4.** The following relations are in \((R_B \cup R_F \cup R_T' \cup R_\times' \cup R_-')^2\):

\[
\begin{align*}
\sigma_1^2 &= e \quad \text{(IB1)} \\
\sigma_i \sigma_i &= \sigma_i e \quad \text{if } i \neq 1 \quad \text{(IB2)} \\
\sigma_i e \sigma_i &= \sigma_i e \sigma_i e = e \sigma_i e \quad \text{(IB3)} \\
\sigma_1^2 &= \sigma_1^2 e = e. \quad \text{(IB4)} \\
\end{align*}
\]

**Proof.** Now relation (IB1) is part of \((T1)'\), and (IB2) is part of \((\times)'\). For (IB3) and (IB4) it suffices, by Corollary 5.3, to show that \( \tau \zeta_1 \tau \zeta_1 = \zeta_1 \tau \zeta_1 \tau = \tau \zeta_1 \tau \) and \( \tau \zeta_1^2 = \zeta_1^2 \tau = \tau \) where we have written \( \tau = 1_{(1)}e \). Figure 5.2 below shows that the first equation holds.

![Figure 5.2](relation.png)

**Figure 5.2.** Relation (IB3): \( \tau \zeta_1 \tau \zeta_1 = \zeta_1 \tau \zeta_1 \tau = \tau \zeta_1 \tau \), where \( \tau = 1_{(1)}e \).

The reader is invited to draw a picture to show that the second equation also holds. \( \square \)
Let $R_{IB}$ denote the set of relations (F), (B1—B2), and (IB1—IB4), and denote by $\sim_{IB}$ the congruence $R_{IB}^\sim$. By Lemma 5.4 we see that $\cong = (R_{IB} \cup R_{T} \cup R_{x} \cup R_{s})^\sim$. Our aim is to show that $R_{IB}^\sim \subseteq R_{IB}^\sim \subseteq R_{IB}^\sim$, so that $\sim_{IB} = \cong$ and $\mathcal{IB}_n$ has presentation $(X_{IB} \mid R_{IB})$ via the map described in Corollary 5.3.

**Lemma 5.5.** If $w \in (X_B \cup X_B^{-1})^*$ and $w \zeta \in \mathcal{P}_n$ then we $\sim_{IB} ew$.

**Proof.** By (IB2), (IB4), and (F) we see that $\hat{e}_{ij}w \sim_{IB} e\hat{e}_{ij}w$ for each $1 \leq i < j \leq n$. Now if $w \in (X_B \cup X_B^{-1})^*$ and $w \zeta \in \mathcal{P}_n$, then we have $w \sim_{IB} \hat{e}_{p_1q_1} \ldots \hat{e}_{p_{\ell}q_{\ell}}$ for some $p_1, \ldots, p_\ell, q_1, \ldots, q_\ell \in \mathcal{P}$ with $p_j < q_j$ for each $j \in \mathcal{P}$. The result now follows by induction on $\mathcal{P}$. 

For $i \in \mathcal{P}$ put $w_i = \sigma_{i-1} \ldots \sigma_1$, so that $e_i = w_i ew_i^{-1}$ by definition.

**Corollary 5.6.** If $w \in (X_B \cup X_B^{-1})^*$, $w \zeta \in \mathcal{P}_n$, and $i \in \mathcal{P}$, then $we_i \sim_{IB} e_iw$.

**Proof.** Now $(w_i^{-1}ww_i)\zeta = (w_i\zeta)^{-1}(w\zeta)(w_i\zeta) \in \mathcal{P}_n$ and so, by (F) and Lemma 5.5, we have

$$we_i \sim_{IB} w_i(w_i^{-1}ww_i)ew_i^{-1} \sim_{IB} w_i(e_i^{-1}ww_i)w_i^{-1} \sim_{IB} e_iw,$$

completing the proof. 

**Lemma 5.7.** Suppose that $1 \leq r \leq n-1$, $i \in \mathcal{P}$, and $\mu \in \{\pm 1\}$. Then $\sigma_r \mu e_i \sigma_r^{-\mu} \sim_{IB} e_{ir}$.

**Proof.** First note that we have $\sigma_r \mu e_i \sigma_r^{-\mu} \sim_{IB} \sigma_r^{-1} e_i \sigma_r \sim_{IB} e_i \sigma_r$ by (F) and Corollary 5.6. Thus it suffices to prove the lemma for any choice of $\mu$.

We consider the cases $r > i$, $r = i$, $r = i - 1$, and $r < i - 1$ separately. If $r > i$ then by (B1) we have $\sigma_r w_i \sim_{IB} w_i \sigma_r$ from which it follows, by (IB2) and (F), that $\sigma_r e_i \sigma_r^{-1} \sim_{IB} e_i$. If $r = i$ then $\sigma_r w_i = w_{i+1}$ and so $\sigma_r e_i \sigma_r^{-1} = e_{i+1}$. If $r = i - 1$ then $\sigma_r^{-1} w_i \sim_{IB} w_{i-1}$ from which it follows that $\sigma_r^{-1} e_i \sigma_r^{-1} \sim_{IB} e_{i-1}$. If $r < i - 1$ then we have

$$\sigma_r w_i = \sigma_r \sigma_{i-1} \ldots \sigma_{r+2} \sigma_{r+1} \sigma_r \sigma_{r-1} \ldots \sigma_1 \\
\sim_{IB} \sigma_{i-1} \ldots \sigma_{r+2} \sigma_{r+1} \sigma_r \sigma_{r-1} \ldots \sigma_1 \quad \text{by (B1)} \\
\sim_{IB} \sigma_{i-1} \ldots \sigma_{r+2} \sigma_{r+1} \sigma_r \sigma_{r-1} \ldots \sigma_1 \quad \text{by (B2)} \\
\sim_{IB} \sigma_{i-1} \ldots \sigma_{r+2} \sigma_{r+1} \sigma_r \sigma_{r-1} \ldots \sigma_1 \sigma_{r+1} \quad \text{by (B1)} \\
= w_i \sigma_{r+1}$$

so that, again by (IB2) and (F), $\sigma_r e_i \sigma_r^{-1} \sim_{IB} e_i$. 

\qed
This shows that $R'_s \subseteq R^2_{IB}$. The next lemma follows from Lemma 5.7 and a simple induction.

**Lemma 5.8.** Suppose that $i \in n$ and $w \in (X_B \cup X_B^{-1})^*$. Then $w^{-1}e_iw \sim_{IB} e_iw$.

**Proof.** This shows that $R'_s \subseteq R^2_{IB}$. The next lemma follows from Lemma 5.7 and a simple induction.

**Lemma 5.9.** Suppose that $i \in n$. Then $e_i^2 \sim_{IB} e_i$.

**Proof.** Now we have $e_i^2 = w_i\epsilon w_i^{-1}w_i\epsilon w_i^{-1} \sim_{IB} w_i\epsilon w_i^{-1} \sim_{IB} w_i\epsilon w_i^{-1} = e_i$ using relations (F) and (IB1).

**Lemma 5.10.** Suppose that $i, j \in n$. Then $e_i e_j \sim_{IB} e_j e_i$.

**Proof.** If $i = j$ then there is nothing to show so suppose that $i \neq j$. First notice that by (F), (IB3), and (IB4) we have

$$e_1 e_2 = e \sigma_1 e \sigma_1^{-1} \sim_{IB} e \sigma_1 e \sigma_1^{-2} \sim_{IB} \sigma_1 e \sigma_1^{-2} \sim_{IB} e \sigma_1 e \sigma_1^{-2} e \sim_{IB} \sigma_1 e \sigma_1^{-2} e = e_2 e_1.$$ 

Now choose $w \in (X_B \cup X_B^{-1})^*$ such that $(i, j) = (1, 2)\overline{w}$. Then by Lemma 5.8, (F), and the previous calculation, we have

$$e_i e_j = e_\overline{w} e_2 \overline{w} \sim_{IB} w^{-1}e_i w w^{-1} e_2 w \sim_{IB} w^{-1}e_1 w e_2 w \sim_{IB} w^{-1}e_1 w e_2 \sim_{IB} e_2 \overline{w} \overline{w} = e_j e_i,$$

and the proof is complete.

Lemmas 5.9 and 5.10 show that $R'_s \subseteq R^2_{IB}$. For $A \in T$ with $A^c = \{i_1, \ldots, i_k\}$ and $i_1 < \cdots < i_k$, put $e_A = e_{i_1} \cdots e_{i_k} \in X^*_{IB}$.

**Lemma 5.11.** Suppose that $A \in T$, $1 \leq i \leq n - 1$, and $w \in (X_B \cup X_B^{-1})^*$ with $i \overline{w}, (i + 1)\overline{w} \in A^c$. Then $e_A w^{-1} \sigma_i w \sim_{IB} e_A$.

**Proof.** Notice first that by (F) and (IB3) we have

$$e_1 e_2 \sigma_1 = e \sigma_1 e \sigma_1^{-1} \sigma_1 \sim_{IB} e \sigma_1 e \sigma_1^{-1} \sigma_1 \sim_{IB} e \sigma_1 e \sigma_1^{-1} = e_1 e_2.$$ 

Now put $w_{i,i+1} = (\sigma_2 \cdots \sigma_i)(\sigma_1 \cdots \sigma_{i-1})$. Then, noting that $\hat{\zeta}_{i+1} = \hat{\zeta}_i = \sigma_i$, and that $1 \overline{w}_{i,i+1} = i$ and $2 \overline{w}_{i,i+1} = i + 1$, we see that

$$e_i e_{i+1} \sigma_i \sim_{IB} (w_{i,i+1}^{-1} e_i w_{i,i+1})(w_{i,i+1}^{-1} e_2 w_{i,i+1})(w_{i,i+1}^{-1} \sigma_1 w_{i,i+1})$$

by Lemmas 5.1 and 5.8.

$$\sim_{IB} w_{i,i+1}^{-1} e_1 e_2 \sigma_1 w_{i,i+1}$$

by (F)

$$\sim_{IB} w_{i,i+1}^{-1} e_1 e_2 w_{i,i+1}$$

by the previous calculation

$$\sim_{IB} e_i e_{i+1}$$

by (F) and Lemma 5.8.
Thus

\[ e_A w^{-1} \sigma_i w \sim_{IB} e_A e_i e_{(i+1)w} w^{-1} \sigma_i w \]
\[ \sim_{IB} e_A (w^{-1} e_i w) (w^{-1} e_{i+1} w) w^{-1} \sigma_i w \]
\[ \sim_{IB} e_A w^{-1} e_i e_{i+1} \sigma_i w \]
\[ \sim_{IB} e_A w^{-1} e_i e_{i+1} w \]
\[ \sim_{IB} e_A e_i e_{(i+1)w} \]
\[ \sim_{IB} e_A \]

by Lemmas 5.9 and 5.10

by Lemma 5.8

by (F)

by the previous calculation

by (F) and Lemma 5.8

by Lemmas 5.9 and 5.10,

completing the proof.

\[ \square \]

**Lemma 5.12.** Suppose that \( A \in T \), \( 1 \leq i \leq n - 1 \), and \( w \in (X_B \cup X_B^{-1})^* \) with either \( iw \in A^c \) or \((i + 1)w \in A^c \). Then \( e_A w^{-1} \sigma_i^2 w \sim_{IB} e_A \).

**Proof.** Using the word \( w_{i,i+1} \) from the proof of the previous lemma, notice first that

\[ e_i \sigma_i^2 \sim_{IB} (w_{i,i+1}^{-1} e_1 w_{i,i+1}) (w_{i,i+1}^{-1} \sigma_i^2 w_{i,i+1}) \]
\[ \sim_{IB} w_{i,i+1}^{-1} e_1 \sigma_i^2 w_{i,i+1} \]
\[ \sim_{IB} w_{i,i+1}^{-1} e_1 w_{i,i+1} \]
\[ \sim_{IB} e_i \]

by Lemmas 5.1 and 5.8

by (F)

by (IB4)

by Lemma 5.8,

and

\[ e_{i+1} \sigma_i^2 \sim_{IB} (w_{i,i+1}^{-1} e_2 w_{i,i+1}) (w_{i,i+1}^{-1} \sigma_i^2 w_{i,i+1}) \]
\[ \sim_{IB} w_{i,i+1}^{-1} e_2 \sigma_i^2 w_{i,i+1} \]
\[ = w_{i,i+1}^{-1} \sigma_1 e_1 \sigma_1^{-1} \sigma_i^2 w_{i,i+1} \]
\[ \sim_{IB} w_{i,i+1}^{-1} \sigma_1 e_1 \sigma_1^{-1} w_{i,i+1} \]
\[ = w_{i,i+1}^{-1} e_2 w_{i,i+1} \]
\[ \sim_{IB} e_{i+1} \]

by Lemmas 5.1 and 5.8

by (F)

by (F) and (IB4)

by Lemma 5.8.

Now put

\[ k = \begin{cases} 
  i & \text{if } iw \in A^c \\
  i + 1 & \text{if } iw \not\in A^c \text{ but } (i + 1)w \in A^c 
\end{cases} \]
We then have
\begin{align*}
e_{A}w^{-1} \sigma_i^2 w &\sim_{IB} e_{A}e_{k\pi} w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{B}^i w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w \\
&\sim_{IB} e_{A}\pi \\
&\sim_{IB} e_{A}
\end{align*}
by Lemmas 5.9 and 5.10

\begin{align*}
e_{A}w^{-1} \sigma_i^2 w &\sim_{IB} e_{A}e_{k\pi} w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w \\
&\sim_{IB} e_{A}\pi \\
&\sim_{IB} e_{A}
\end{align*}
by Lemma 5.8

\begin{align*}
e_{A}w^{-1} \sigma_i^2 w &\sim_{IB} e_{A}e_{k\pi} w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w \\
&\sim_{IB} e_{A}\pi \\
&\sim_{IB} e_{A}
\end{align*}
by (F)

\begin{align*}
e_{A}w^{-1} \sigma_i^2 w &\sim_{IB} e_{A}e_{k\pi} w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w \\
&\sim_{IB} e_{A}\pi \\
&\sim_{IB} e_{A}
\end{align*}
by the previous calculations

\begin{align*}
e_{A}w^{-1} \sigma_i^2 w &\sim_{IB} e_{A}e_{k\pi} w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w \\
&\sim_{IB} e_{A}\pi \\
&\sim_{IB} e_{A}
\end{align*}
by Lemma 5.8

\begin{align*}
e_{A}w^{-1} \sigma_i^2 w &\sim_{IB} e_{A}e_{k\pi} w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w^{-1} \sigma_i^2 w \\
&\sim_{IB} e_{A}w^{-1}e_{k}w \\
&\sim_{IB} e_{A}\pi \\
&\sim_{IB} e_{A}
\end{align*}
by Lemmas 5.9 and 5.10.

This completes the proof. \hfill \Box

Lemmas 5.11 and 5.12 show that \( R'_\sim \subseteq R^4_{IB} \). Thus \( \sim_{IB} = \approx \) and we have proved the following.

**Theorem 5.13** (Easdown and Lavers [21]). The inverse braid monoid \( IB_n \) has presentation \( \langle X_{IB} | R_{IB} \rangle \) via

\[
\sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} \quad \text{for each } i
\]
\[
e \mapsto 1_{\{1\}^c}.
\]

\hfill \Box

**Remark 5.14.** The presentation of \( IB_n \) given by Easdown and Lavers in [21] differs from ours slightly. There they map the generator \( e \) (called \( \varepsilon \) in [21]) to the partial braid \( 1_{\{n\}^c} \). Their relations are obtained from ours by changing each generator \( \sigma_i \) in (IB2—I4) to \( \sigma_{n-i} \).

### 5.1.2. Alternative Presentations of \( IB_n \)

In this section we give further presentations of \( IB_n \). Our first alternative presentation also uses the braid generators \( X_B \cup X_B^{-1} \) along with a single extra generator, although this generator is mapped to a different partial braid (see Theorem 5.15, and also Remark 5.16 for another similar presentation). We then enlarge the set of generators and relations, thereby obtaining further presentations which display more of the symmetry possessed by \( IB_n \) (see Theorem 5.18 and Remark 5.20).

We begin with the presentation \( \langle X_{IB} | R_{IB} \rangle \) of \( IB_n \) given in Theorem 5.13, and introduce a new generator \( f \) along with the relation

\[ f = e\sigma_1. \]

Now by (IB4) we have \((e\sigma_1)\sigma_1 \sim_{IB} e\sigma_1\) and it follows that we may remove \( e \) as a generator, replacing its every occurrence in the relations by \( f\sigma_1 \). The modified relations are thus (F)
and (B1—B2), together with

\[ f\sigma_1 f\sigma_1 = f\sigma_1 \]
\[ f\sigma_1 \sigma_i = \sigma_i f\sigma_1 \text{ if } i \neq 1 \]
\[ f\sigma_1 \sigma_1 f\sigma_1 = \sigma_1 f\sigma_1 \sigma_1 f\sigma_1 = f\sigma_1 \sigma_1 f\sigma_1 \]
\[ f\sigma_1 \sigma_1^2 = \sigma_1^2 f\sigma_1 = f\sigma_1. \]

Using (F) we see that the first of these relations is equivalent to

\[ f\sigma_1 f = f, \quad (\text{IB1})' \]

and, using (B1) and (F), that the second line of relations is equivalent to

\[ f\sigma_1 \sigma_2 = \sigma_2 f\sigma_1 \]
\[ f\sigma_i = \sigma_i f \quad \text{if } i \geq 3. \quad (\text{IB3})' \]

Then, using (F), we see that the fourth line of relations is equivalent to

\[ f\sigma_1^2 = \sigma_1^2 f = f \quad (\text{IB5})' \]

and, using (F) and (IB5)', that the third line of relations is equivalent to

\[ f^2 \sigma_1 = \sigma_1 f^2 = f^2. \quad (\text{IB4})' \]

Let \( X'_{IB} = X_B \cup X_B^{-1} \cup \{f\} \), and let \( R'_{IB} \) be the set of relations (F), (B1—B2), together with \((\text{IB1—IB5})'\). We have shown the following.

**Theorem 5.15.** The inverse braid monoid \( IB_n \) has presentation \( \langle X'_{IB} \mid R'_{IB} \rangle \) via

\[ \sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} \quad \text{for each } i \]
\[ f \mapsto (\zeta_1)_1^e. \]

\[ \square \]

**Remark 5.16.** If we made the substitution \( g = \sigma_1 e \) instead of \( f = e\sigma_1 \) then we obtain a presentation of \( IB_n \) with generators \( X_B \cup X_B^{-1} \cup \{g\} \). The relations in this presentation are precisely those obtained by changing \( f \) to \( g \) in relations \((\text{IB1})'\) and \((\text{IB3—IB5})'\), and replacing relation \((\text{IB2})'\) by

\[ \sigma_2 \sigma_1 g = \sigma_1 g \sigma_2. \]

In this presentation, the generator \( g \) is sent to the partial braid \((\zeta_1)_1^e\). The partial braids \((\zeta_1)_1^e\) and \((\zeta_1)_2^e\) are pictured in Figure 5.3.
We now return to the presentation of $\mathcal{IB}_n$ given in Theorem 5.15. We will introduce a larger set of generators and relations in order to find a presentation which displays more of the symmetry possessed by $\mathcal{IB}_n$. With this in mind, we rename $f = f_1$ and introduce new generators $f_2, \ldots, f_n$ along with the relations

\[ f_i = w_{i,i+1}^{-1} f_1 w_{i,i+1} = (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}) (\sigma_i^{-1} \cdots \sigma_2^{-1}) f_1 (\sigma_2 \cdots \sigma_i) (\sigma_i \cdots \sigma_{i-1}) \]  

which define them in terms of the original generators. For $1 \leq i \leq n - 1$, let $\theta_i = (\varsigma_i f_i)$ so that $\theta_i$ is the image of $w_{i,i+1}^{-1} f_1 w_{i,i+1}$ under the map in the statement of Theorem 5.15. See Figure 5.4 for an illustration.

Let $X_{\mathcal{IB}}'' = X_B \cup X_B^{-1} \cup \{f_1, \ldots, f_{n-1}\}$, and denote by $\sim_{\mathcal{IB}''}$ the congruence on $(X_{\mathcal{IB}}'')^*$ generated by $R_{IB}$ and (D).

**Lemma 5.17.** The following relations are in $\sim_{\mathcal{IB}''}$:

\[
\begin{align*}
    f_i \varsigma_i f_i &= f_i & \text{for all } i & \quad \text{(IB1)''} \\
    \varsigma_i f_i &= f_{i+1} \varsigma_{i+1} & \text{for all } i \leq n - 2 & \quad \text{(IB2)''} \\
    \varsigma_i f_j &= f_j \varsigma_i & \text{if } |i - j| > 1 & \quad \text{(IB3)''} \\
    \varsigma_i \varsigma_j f_i &= f_j \varsigma_i \varsigma_j & \text{if } |i - j| = 1 & \quad \text{(IB4)''} \\
    \varsigma_i f_i^2 &= f_i^2 = f_i^2 \varsigma_i & \text{for all } i & \quad \text{(IB5)''} \\
    \varsigma_i^2 f_i &= f_i = f_i \varsigma_i^2 & \text{for all } i. & \quad \text{(IB6)''}
\end{align*}
\]

**Proof.** By Theorem 5.15 it suffices to show that the relations hold as equations in $\mathcal{IB}_n$ when the generators $\varsigma_i$ and $f_i$ are replaced by $\varsigma_i$ and $\theta_i$ respectively. We do this for relations (IB1)'' and (IB2)'' in Figures 5.5 and 5.6. The reader is invited to draw pictures for the remaining relations. \qed

By Lemma 5.17 we may add relations (IB1—IB6)'' to the presentation. We may clearly remove relations (IB1)' and (IB3—IB5)' from the presentation since these relations are
contained in various parts of relations (IB1—IB6). Let $R''_{IB}$ denote the set of relations (F), (B1—B2), and (IB1—IB6)$''$.

**Theorem 5.18.** The inverse braid monoid $IB_n$ has presentation $(X''_{IB} | R''_{IB})$ via

$$\sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1} \quad \text{for each } i$$

$$f_i \mapsto \theta_i \quad \text{for each } i.$$

**Proof.** For the duration of this proof we will denote by $\equiv$ the congruence $(R''_{IB})$. It remains only to show that relations (IB2)$'$ and (D) are in $\equiv$. For (IB2)$'$ we have

$$f_1 \sigma_i \sigma_2 \equiv f_1 \sigma_1 \sigma_2 \sigma_1 \sigma_1^{-1} \quad \text{by (F)}$$

$$\equiv f_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1^{-1} \quad \text{by (B2)}$$

$$\equiv \sigma_2 \sigma_1 f_2 \sigma_2 \sigma_1^{-1} \quad \text{by (IB4)$''$}$$

$$\equiv \sigma_2 \sigma_1 \sigma_1 f_1 \sigma_1^{-1} \quad \text{by (IB2)$''$}$$

$$\equiv \sigma_2 \sigma_1 \sigma_1 \sigma_1^{-1} \quad \text{by (IB6)$''$}$$

$$\equiv \sigma_2 f_1 \sigma_1 \quad \text{by (F)}.$$  

To show that (D) is in $\equiv$, we must show that $f_i \equiv w_{i,i+1}^{-1} f_1 w_{i,i+1}$ for each $i$. Now if $i = 1$ then there is nothing to prove since $w_{12} = 1$, so we assume that $2 \leq i \leq n - 1$. By (B1) we have $w_{i,i+1} \sim_B w_{i-1,i} \sigma_i \sigma_{i-1}$. Using this we have

$$w_{i,i+1}^{-1} f_1 w_{i,i+1} \equiv \sigma_{i-1}^{-1} \sigma_i^{-1} w_{i,i+1}^{-1} f_1 w_{i-1,i} \sigma_i \sigma_{i-1}$$

$$\equiv \sigma_{i-1}^{-1} \sigma_i^{-1} f_i \sigma_i \sigma_{i-1} \quad \text{by an inductive hypothesis}$$

$$\equiv \sigma_{i-1}^{-1} \sigma_i \sigma_{i-1} f_i \quad \text{by (IB4)$''$}$$

$$\equiv f_i \quad \text{by (F)},$$

completing the proof of the theorem. \qed
Remark 5.19. Relations (IB3)$''$ and (IB4)$''$ form part of the singular braid relations (see [7, 9] and also Section 5.2.3). It is natural then to ask whether the remaining singular braid relations

\[
\sigma_i f_i = f_i \sigma_i \quad \text{for all } i
\]
\[
f_i f_j = f_j f_i \quad \text{if } |i - j| > 1
\]

follow from relations (IB1|IB6)$''$. We will show that the first of the missing singular braid relations does not hold, while the second does. By Theorem 5.18 it suffices to show that $\varsigma_i \theta_i \neq \theta_i \varsigma_i$ for each $i$ (see Figure 5.7), and that $\theta_i \theta_j = \theta_j \theta_i$ whenever $|i - j| > 1$ (see Figure 5.8).

![Figure 5.7. The relation $\sigma_i f_i = f_i \sigma_i$ is not in $(R_{IB}''')^2$ for any $i$.](image)

![Figure 5.8. The relation $f_i f_j = f_j f_i$ is in $(R_{IB}''')^2$ whenever $|i - j| > 1$.](image)

Remark 5.20. If instead we started with the presentation of $\mathcal{B}_n$ in Remark 5.16, and introduced generators $g_1, g_2, \ldots, g_{n-1}$ where $g_i = w_{i,i+1}^{-1} g_1 w_{i,i+1}$ for each $i$, then we obtain the presentation $\langle X_{IB}''' | R_{IB}''' \rangle$, where $X_{IB}''' = X_B \cup X_B^{-1} \cup \{g_1, \ldots, g_{n-1}\}$ and $R_{IB}'''$ is the set of relations obtained from $R_{IB}''$ by changing each $f_i$ to $g_i$ in (IB1)$''$ and (IB3—IB6)$''$ and replacing (IB2)$''$ by

\[
g_i \sigma_i = \sigma_{i+1} g_{i+1} \quad \text{for all } i \leq n - 2.
\]

We denote these modified relations by (IB1—IB6)$''''$. In this presentation the generator $g_i$ is mapped to the partial braid $(\varsigma_i)_{(i+1)^c}$ which is pictured in Figure 5.9.

![Figure 5.9. The partial braid $(\varsigma_i)_{(i+1)^c}$ in $\mathcal{B}_n$.](image)
5.1.3. Presentations of $\mathcal{I}_n$.

Recall that we have a map $\gamma : \mathcal{I}B_n \to \mathcal{I}_n$ which maps a partial braid $\beta \in \mathcal{I}B_n$ to its associated partial permutation $\tilde{\beta} \in \mathcal{I}_n$. In this section we will use the presentations of $\mathcal{I}B_n$ in the previous sections to derive corresponding presentations of $\mathcal{I}_n$. One of these presentations (Theorem 5.22) is due to Popova [55], and has been proved by a number of authors; see for example [20, 21, 45, 54].

We will first prove a general result which will allow us to derive presentations of $\mathcal{I}_n$ from presentations of $\mathcal{I}B_n$. To make the result easier to state, we establish some notation.

Let $X$ be any alphabet disjoint from $X_{B} \cup X_{B}^{-1}$. We say that a word $w \in (X_{B} \cup X_{B}^{-1})^*$ is square-free if $w$ contains no subword from $\{\sigma_1^2, \ldots, \sigma_{n-1}^2\}$. Let $w \in (X_{B} \cup X_{B}^{-1})^*$. We define a word $\tilde{w} \in (X_{B} \cup X_{B}^{-1})^*$ inductively as follows. If $w$ is square-free, put $\tilde{w} = w$. If $w$ is not square-free, let $\tilde{w} = \tilde{w}_0$ where $w_0 \in (X_{B} \cup X_{B}^{-1})^*$ is the word obtained from $w$ by deleting the left-most occurrence of a subword from $\{\sigma_1^2, \ldots, \sigma_{n-1}^2\}$.

Proposition 5.21. Suppose that $\mathcal{I}B_n$ has presentation

\[ \langle X_{B} \cup X_{B}^{-1} \cup X \mid R_B \cup R_F \cup R \rangle \]

via a map $\phi : (X_{B} \cup X_{B}^{-1} \cup X)^* \to \mathcal{I}B_n$ such that

\begin{enumerate}
  \item[(i)] $X$ is disjoint from $X_{B} \cup X_{B}^{-1}$;
  \item[(ii)] $R \subseteq (X_{B} \cup X)^* \times (X_{B} \cup X)^*$; and
  \item[(iii)] $\sigma_i^{\pm 1} \phi = \zeta_i^{\pm 1}$ for each $i$.
\end{enumerate}

Denote by $R_O$ the set of relations

\[ \sigma_i^2 = 1 \quad \text{for each } i, \]

(O)

and let $\tilde{R}$ be the set of relations obtained from $R$ by deleting any relation $(w_1, w_2) \in R$ for which $\tilde{w}_1 = \tilde{w}_2$. Then $\mathcal{I}_n$ has presentation

\[ \langle X_{B} \cup X \mid R_B \cup R_O \cup \tilde{R} \rangle \]

via

\[ x \mapsto \overline{x \phi} \quad \text{for each } x \in X_{B} \cup X. \]

Proof. We first show that $\mathcal{I}_n$ has presentation

\[ \langle X_{B} \cup X_{B}^{-1} \cup X \mid R_B \cup R_F \cup R \cup R_O \rangle \]

via

\[ \phi' : (X_{B} \cup X_{B}^{-1} \cup X)^* \to \mathcal{I}_n : x \mapsto \overline{x \phi} \quad \text{for each } x \in X_{B} \cup X_{B}^{-1} \cup X. \]

Put $\cong = (R_B \cup R_F \cup R)^2$ and $\sim_O = (R_B \cup R_F \cup R \cup R_O)^2$. Now $\phi'$ is an epimorphism since both $\phi$ and $\gamma$ are epimorphisms. Since $s_i^2 = 1$ for each $i$, we see that $\sim_O \subseteq \ker \phi'$. Suppose now that $(w_1, w_2) \in \ker \phi'$, and put $\beta_1 = w_1 \phi$ and $\beta_2 = w_2 \phi$. Then

\[ \tilde{\beta}_1 = w_1 \phi' = w_2 \phi' = \tilde{\beta}_2. \]
Let \( A = \text{dom}(\beta_1) \). We then have
\[
\beta_1 = (\gamma_1)_A = 1_A \gamma_1 \quad \text{and} \quad \beta_2 = (\gamma_2)_A = 1_A \gamma_2
\]
for some \( \gamma_1, \gamma_2 \in B = G(\mathcal{IB}_n) \). Choose \( w_3, w_4 \in (X_B \cup X_B^{-1})^* \) such that
\[
w_3 \phi = \gamma_1 \quad \text{and} \quad w_4 \phi = \gamma_2,
\]
and choose \( w_A \in (X_B \cup X_B^{-1} \cup X)^* \) such that \( w_A \phi = 1_A \). We then have
\[
w_1 \cong w_A w_3 \quad \text{and} \quad w_2 \cong w_A w_4.
\]
Put \( \pi = (w_3 w_4^{-1}) \phi' \in \mathcal{S}_n \). Now for any \( i \in A \) we have
\[
i(w_3 \phi') = i(w_1 \phi') = i(w_2 \phi') = i(w_4 \phi').
\]
Thus \( i \pi = i \), whence \( i \pi^{-1} = i \), for each \( i \in A \). Thus we may write
\[
\pi^{-1} = t_{p_1 q_1} \ldots t_{p_k q_k}
\]
for some \( p_1, \ldots, p_k, q_1, \ldots, q_k \in \mathbf{n} \) with \( p_j < q_j \) and \( p_j, q_j \in A^c \) for each \( j \in \mathbf{k} \). By Lemma 4.6 we have \( \zeta_{p_1 q_1} \cdots \zeta_{p_k q_k} \in B_A \), and it then follows that
\[
1_A = 1_A \zeta_{p_1 q_1} \cdots \zeta_{p_k q_k}
\]
in \( \mathcal{IB}_n \). Thus, by hypothesis, we have
\[
w_A \cong w_A \zeta_{p_1 q_1} \cdots \zeta_{p_k q_k}.
\]
Now \( (\zeta_{p_1 q_1} \cdots \zeta_{p_k q_k}) \phi' = \pi^{-1} = ((w_3 w_4^{-1}) \phi')^{-1} \) and so \( (\zeta_{p_1 q_1} \cdots \zeta_{p_k q_k} w_3 w_4^{-1}) \zeta \in \mathcal{P}_n \).
Thus
\[
\zeta_{p_1 q_1} \cdots \zeta_{p_k q_k} w_3 w_4^{-1} \sim_B \tilde{\alpha}_{u_1 v_1}^{\pm 1} \cdots \tilde{\alpha}_{u_\ell v_\ell}^{\pm 1}
\]
for some \( u_1, \ldots, u_\ell, v_1, \ldots, v_\ell \in \mathbf{n} \) with \( u_j < v_j \) for each \( j \in \mathbf{\ell} \). By (O) and (F) we have
\[
\tilde{\alpha}_{ij}^{\pm 1} = (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^{\pm 2} (\sigma_{i+1} \cdots \sigma_{j-1}) \sim_O 1
\]
for all \( 1 \leq i < j \leq n \), so that
\[
\begin{align*}
w_1 & \sim_O w_A w_3 \\
& \sim_O (w_A w_3 w_4^{-1}) w_4 \\
& \sim_O (w_A \zeta_{p_1 q_1} \cdots \zeta_{p_k q_k} w_3 w_4^{-1}) w_4 \\
& \sim_O (w_A \tilde{\alpha}_{u_1 v_1}^{\pm 1} \cdots \tilde{\alpha}_{u_\ell v_\ell}^{\pm 1}) w_4 \\
& \sim_O w_A w_4 \\
& \sim_O w_2.
\end{align*}
\]
This completes the proof that \( \mathcal{I}_n \) has presentation \( \langle X_B \cup X_B^{-1} \cup X \mid R_B \cup R_F \cup R \cup R_O \rangle \) via \( \phi' \).
It now suffices to show that the presentation we have arrived at is equivalent to the presentation in the statement of the proposition. Now by (F) and (O) we have

$$\sigma_i \sim_O \sigma_i \sigma_i^{-1} \sim_O \sigma_i^{-1}$$

for each $i$. Thus we may remove all generators $\sigma_i^{-1}$, replacing their every occurrence in the remaining relations by $\sigma_i$. Relations (F) become duplicates of (O) and, as such, may be removed. Finally, observe that we may use relations (O) to transform any word $w \in (X_B \cup X)^*$ into $\tilde{w}$, thus allowing us to remove any relation $(w_1, w_2) \in R$ such that $\tilde{w}_1 = \tilde{w}_2$. This completes the proof. \qed

As our first application of Proposition 5.21, we now derive Popova’s presentation of $I_n$.

Put $X_I = \{\sigma_1, \ldots, \sigma_{n-1}, e\} = X_B \cup \{e\}$, and let $R_I$ be the set of relations (B1—B2), (IB1—IB3), and

$$\sigma_i^2 = 1 \quad \text{for all } i. \quad \text{(O)}$$

By Theorem 5.13 and Proposition 5.21 we have the following.

**Theorem 5.22** (Popova [55]). The symmetric inverse semigroup $I_n$ has presentation $\langle X_I | R_I \rangle$ via

$$\sigma_i \mapsto s_i \quad \text{for each } i$$

$$e \mapsto id_{\{1\}}. \quad \square$$

**Remark 5.23.** A similar presentation of $I_n$ was given in [21]. As in Remark 5.14, the role of the generator $e$ was slightly different; in [21], $e$ was mapped to the partial permutation $id_{\{n\}}$.

We may also use Proposition 5.21 to derive a presentation of $I_n$ similar to the presentation of $\mathcal{IB}_n$ in Theorem 5.15. Let $X'_I = \{\sigma_1, \ldots, \sigma_{n-1}, f\} = X_B \cup \{f\}$, and let $R'_I$ be the set of relations (B1—B2), (O), and (IB1—IB4)'.

**Theorem 5.24.** The symmetric inverse semigroup $I_n$ has presentation $\langle X'_I | R'_I \rangle$ via

$$\sigma_i \mapsto s_i \quad \text{for each } i$$

$$f \mapsto s_1|_{\{1\}}. \quad \square$$

**Remark 5.25.** A similar comment to that made in Remark 5.16 may also be made here.
Now put
\[ X''_I = X_B \cup \{f_1, \ldots, f_{n-1}\} \quad \text{and} \quad X'''_I = X_B \cup \{g_1, \ldots, g_{n-1}\}. \]

Let \( R''_I \) denote the set of relations \((B1-B2), (O), \) and \((IB1-IB5)''\), and let \( R'''_I \) denote the set of relations \((B1-B2), (O), \) and \((IB1-IB5)'''\). By Theorem 5.18, Remark 5.20, and Proposition 5.21, we also have the following presentations of \( \mathcal{I}_n \).

**Theorem 5.26.** The symmetric inverse semigroup \( \mathcal{I}_n \) has presentation \( \langle X''_I \mid R''_I \rangle \) via

\[
\sigma_i \mapsto s_i \quad \text{for each } i \\
f_i \mapsto s|_{(i+1)c} \quad \text{for each } i.
\]

**Theorem 5.27.** The symmetric inverse semigroup \( \mathcal{I}_n \) has presentation \( \langle X'''_I \mid R'''_I \rangle \) via

\[
\sigma_i \mapsto s_i \quad \text{for each } i \\
g_i \mapsto s|_{(i+1)c} \quad \text{for each } i.
\]

### 5.2. The Factorisable Braid Monoid

In this section we give presentations of the factorisable braid monoid \( \mathfrak{FB}_n \). We will use much the same method as in the previous section, although things will be a little easier to start with since, by Proposition 4.11, the subgroups \( B_E \) satisfy the condition

\[ B_{E_1} \vee B_{E_2} = B_{E_1 \vee E_2} \quad \text{for all } E_1, E_2 \in \mathfrak{Eq}_n. \quad (B\xi2)' \]

We will then derive presentations of \( \mathfrak{B}^*_n \), the monoid of uniform block bijections on \( n \), including the presentation given by FitzGerald in [32]. We will also explore an interesting connection between \( \mathfrak{FB}_n \) and the singular braid monoid \( SB_n \).

#### 5.2.1. A Presentation of \( \mathfrak{B}B_n \)

We use the notation of Section 4.2, writing \( E = \mathfrak{Eq}_n \) for the join semilattice of equivalence relations on \( n \). Recall that \( X_E = \{\varepsilon_{ij} \mid 1 \leq i < j \leq n\} \) and that \( R_E \) is the set of relations

\[
\varepsilon_{ij}^2 = \varepsilon_{ij} \quad \text{for all } i, j \quad (E1) \\
\varepsilon_{ij} \varepsilon_{kl} = \varepsilon_{kl} \varepsilon_{ij} \quad \text{for all } i, j, k, l \quad (E2) \\
\varepsilon_{ij} \varepsilon_{jk} = \varepsilon_{jk} \varepsilon_{ik} = \varepsilon_{ik} \varepsilon_{ij} \quad \text{for all } i, j, k \quad (E3)
\]

Recall also that for \( 1 \leq i < j \leq n \), we write \( \varepsilon_{ij} \) for the equivalence \( \{(r, s) \mid r = s \} \) or \( \{r, s\} = \{i, j\} \) \( \in E \).
By Theorem 2.8, $E$ has presentation \( \langle X_E \mid R_E \rangle \) via
\[
\varepsilon_{ij} \mapsto \mathcal{E}_{ij} \quad \text{for each } i, j.
\]

To maintain the notation of Section 3.4, and without causing confusion, we denote by \( \eta : X'_E \rightarrow E \) the above map.

We will now gather further information in order to apply Theorem 3.16. Suppose that \( 1 \leq i < j \leq n \) and \( 1 \leq r \leq n - 1 \). It is immediate from the definitions that
\[
\mathcal{E}_{ij}^{\pm 1} = \begin{cases} 
\mathcal{E}_{i-1,j} & \text{if } r = i - 1 \\
\mathcal{E}_{i+1,j} & \text{if } r = i < j - 1 \\
\mathcal{E}_{i,j-1} & \text{if } r = j - 1 > i \\
\mathcal{E}_{i,j+1} & \text{if } r = j \\
\mathcal{E}_{ij} & \text{otherwise.}
\end{cases}
\]

For \( 1 \leq i < j \leq n \), we use the symmetric notation \( \mathcal{E}_{ji} = \mathcal{E}_{ij} \) and \( \varepsilon_{ji} = \varepsilon_{ij} \). We see then that \( \mathcal{E}_{ij}^{\pm 1} = \mathcal{E}_{ir,jsr} \). Thus, we may take \( R_\alpha \) to be the set of relations
\[
\sigma_r^{\pm 1} \varepsilon_{ij} = \varepsilon_{ir,jsr} \sigma_r^{\pm 1} \quad \text{for all } i, j, r.
\]

Let \( R_\sim \) be the set of relations
\[
\varepsilon_{ij} \beta^{-1} \mathcal{E}_{ij} \beta = \varepsilon_{ij} \quad \text{for all } 1 \leq i < j \leq n \text{ and } \beta \in P.
\]

By Theorem 3.16, Lemma 4.10, and \((B_\varepsilon 2)'\) we have the following. (The reader should note that \( R_\sim \) here corresponds to \( R'_\sim \) in the preamble to Theorem 3.16.)

**Corollary 5.28.** The factorisable braid monoid \( \mathfrak{S}_B \) has presentation
\[
\langle X_B \cup X_B^{-1} \cup X_E \mid R_B \cup R_F \cup R_E \cup R_\alpha \cup R_\sim \rangle
\]
via
\[
\sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} \quad \text{for each } i
\]
\[
\varepsilon_{ij} \mapsto [1] \varepsilon_{ij} \quad \text{for each } i, j.
\]

We will now work towards simplifying this presentation. As a first step we will remove a number of the generators. With this in mind, let \( e = \varepsilon_{12} \). By \( R_F \cup R_\alpha \) we see that for any \( 1 \leq i < j \leq n \), the relation
\[
\varepsilon_{ij} = (\sigma_{i-1}^{-1} \cdots \sigma_i^{-1})(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1})e(\sigma_2 \cdots \sigma_{j-1})(\sigma_1 \cdots \sigma_{i-1})
\]
is in \( (R_B \cup R_F \cup R_E \cup R_\alpha \cup R_\sim)^2 \). So we remove the generators \( \varepsilon_{ij} \), replacing their every occurrence in the relations by the word on the right hand side of \((*)\), which we denote by \( e_{ij} \) (noting in particular that \( e_{12} = e \)). We denote the resulting relations by \((E1-E3)'\),
5.2. THE FACTORISABLE BRAID MONOID 77

The entire sets of relations which have been modified in this way will be denoted by $R'_E$, $R'_x$, and $R'_\sim$. Put $X_{FB} = X_B \cup X_B^{-1} \cup \{e\}$.

Corollary 5.29. The factorisable braid monoid $\mathfrak{FB}_n$ has presentation

$$(X_{FB} | R_B \cup R_F \cup R'_E \cup R'_x \cup R'_\sim)$$

via

$$\sigma^\pm_i \mapsto \varsigma^\pm_i \quad \text{for each } i$$
$$e \mapsto [1]_{\eta_{12}}.$$

For the time being, we focus on the relations $R_B \cup R_F \cup R'_E \cup R'_x$. This will allow several calculations of this section to be used in the next section when we consider the permeable braid monoid $\mathfrak{PB}_n$.

Lemma 5.30. The following relations are in $(R_B \cup R_F \cup R'_E \cup R'_x)^2$:

$$e^2 = e \quad \text{(EB1)}$$
$$e\sigma_i = \sigma_i e \quad \text{if } i \neq 2 \quad \text{(EB2)}$$
$$e\sigma_2^{-1}\sigma_2 = \sigma_2^{-1}e\sigma_2e = \sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1e \quad \text{(EB3)}$$
$$e\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}e\sigma_2\sigma_3\sigma_1\sigma_2e \quad \text{(EB4)}$$
$$e\sigma_2^2 = \sigma_2^2 e \quad \text{(EB5)}$$

Proof. Relation (EB1) is part of (E1)', and (EB2) is part of $(\times)'$, while (EB5) follows from two applications of $(\times)'$. Relation (EB3) says $e_{12}e_{13} = e_{13}e_{12} = e_{23}e_{12}$ which follows from (E2)' and (E3)', and (EB4) says $e_{12}e_{34} = e_{34}e_{12}$ which is part of (E2)'. □

Let $R_{EB}$ be the set of relations (F), (B1—B2), and (EB1—EB5), and denote by $\sim_{EB}$ the congruence $R_{EB}^2$. By Lemma 5.30 we have

$$(R_B \cup R_F \cup R'_E \cup R'_x)^2 = (R_{EB} \cup R'_E \cup R'_x)^2.$$  

Our next goal is to show that $R'_E \cup R'_x \subseteq R_{EB}^2$, to conclude that $(R_B \cup R_F \cup R'_E \cup R'_x)^2 = \sim_{EB}$.

Lemma 5.31. If $w \in (X_B \cup X_B^{-1})^*$ and $w \zeta \in \mathcal{P}_n$ then we $\sim_{EB} ew$.

Proof. Suppose that $1 \leq i < j \leq n$. As in the proof of Lemma 5.5, it suffices to show that $e\alpha_{ij}^{\pm 1} \sim_{EB} \hat{\alpha}_{ij}^{\pm 1} e$. If $i \geq 2$, then by relations (F), (EB2), and (EB5) we have $\hat{\alpha}_{ij}^{\pm 1} e \sim_{EB} e\hat{\alpha}_{ij}^{\pm 1}$. If $i = 1$ and $j = 2$ then $\hat{\alpha}_{12}^{\pm 1} = \sigma_1^{\pm 1}$ and we have $\hat{\alpha}_{12}^{\pm 1} e \sim_{EB} e\hat{\alpha}_{12}^{\pm 1}$ by (EB2) and (F).
If \( i = 1 \) and \( j > 2 \), then it is easy to check that \( \hat{\alpha}_{1j}^{\pm 1} \sim_B \sigma_{1}^{-1} \hat{\alpha}_{2j}^{\pm 1} \sigma_{1} \), so that \( \hat{\alpha}_{1j}^{\pm 1} e \sim_{EB} e \hat{\alpha}_{1j}^{\pm 1} \) by (EB2), (F), and the first calculation.

Recall that for \( 1 \leq i < j \leq n \) we defined a word \( w_{ij} = (\sigma_{2} \cdots \sigma_{j-1})(\sigma_{1} \cdots \sigma_{i-1}) \) so that \( e_{ij} = w_{ij}^{-1}e w_{ij} \) by definition.

**Corollary 5.32.** If \( w \in (X_B \cup X_B^{-1})^* \), \( w \zeta \in \mathcal{P}_n \), and \( 1 \leq i < j \leq n \), then \( we_{ij} \sim_{EB} e_{ij}w \).

For convenience, we will write \( e_{ji} = e_{ij} \) for each \( 1 \leq i < j \leq n \).

**Proof.** Now \((w_{ij} w w_{ij}^{-1}) \zeta = (w_{ij} \zeta)(w \zeta)(w_{ij} \zeta)^{-1} \in \mathcal{P}_n \) so that by (F) and Lemma 5.31 we have

\[
we_{ij} \sim_{EB} w_{ij}^{-1}(w_{ij} w w_{ij}^{-1})e w_{ij} \sim_{EB} w_{ij}^{-1}e(w_{ij} w w_{ij}^{-1})w_{ij} \sim_{EB} e_{ij}w,
\]

completing the proof.

**Lemma 5.33.** Suppose that \( 1 \leq i < j \leq n \), \( 1 \leq r \leq n - 1 \), and \( \mu \in \{ \pm 1 \} \). Then

\[ \sigma_{r}^{\mu} e_{ij} \sim_{EB} e_{is_{r}j_{s_{r}}}. \]

**Proof.** First note that \( \sigma_{r}^{-1} e_{ij} \sigma_{r} \sim_{EB} \sigma_{r} \sigma_{r}^{-2} e_{ij} \sigma_{r}^{2} \sigma_{r}^{-1} \sim_{EB} \sigma_{r} e_{ij} \sigma_{r}^{-1} \) by (F) and Corollary 5.32. Thus it suffices to prove the lemma for any choice of \( \mu \).

If \( r = i - 1 \) or \( r = j - 1 > i \) we use \( \mu = 1 \), while if \( r = i < j - 1 \) or \( r = j \) we use \( \mu = -1 \), and the result follows quickly, using (B1) and (F) where appropriate. We now consider the other possibilities for \( r \). Using the braid relations or pictures, it is easy to check that

\[
w_{ij} \sigma_{r} \sim_{B} \begin{cases} 
\sigma_{r+2} w_{ij} & \text{if } r < i - 1 \\
\sigma_{r+1} w_{ij} & \text{if } i < r < j - 1 \\
\sigma_{r} w_{ij} & \text{if } j < r \\
\sigma_{1} w_{ij} & \text{if } r = i = j - 1.
\end{cases}
\]

It now follows that \( \sigma_{r}^{-1} e_{ij} \sigma_{r} = \sigma_{r}^{-1} w_{ij}^{-1} e w_{ij} \sigma_{r} \sim_{EB} e_{ij} \), using (F) and (EB2).

This shows that \( R'_{\zeta} \subseteq R'_{EB} \). By Lemma 5.33 and a simple induction we have the following.

**Corollary 5.34.** If \( w \in (X_B \cup X_B^{-1})^* \) and \( 1 \leq i < j \leq n \), then \( w^{-1} e_{ij} w \sim_{EB} e_{i \overline{\pi} j \overline{\pi}} \).

**Lemma 5.35.** If \( 1 \leq i < j \leq n \), then \( e_{ij}^{2} \sim_{EB} e_{ij} \).

**Proof.** This follows quickly from (F) and (EB1).
Lemma 5.36. If $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, then $e_{ij}e_{kl} \sim_{EB} e_{kl}e_{ij}$.

Proof. First notice that
\[
e_{12}e_{23} = e\sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1 \sim_{EB} \sigma_1^{-1}e\sigma_2^{-1}e\sigma_2\sigma_1 \quad \text{by (EB2) and (F)}
\]
\[
\sim_{EB} \sigma_1^{-1}\sigma_2^{-1}e\sigma_2e\sigma_1 \quad \text{by (EB3)}
\]
\[
\sim_{EB} \sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1 e \quad \text{by (EB2)}
\]
\[
= e_{23}e_{12}.
\]

Notice also that (EB4) says precisely $e_{12}e_{34} \sim_{EB} e_{34}e_{12}$. Suppose now that $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. We first consider the case in which one of $k, l$ is equal to one of $i, j$. By symmetry we may assume that $k = j$. Choose $w \in (X_B \cup X_B^{-1})^*$ such that $(1, 2, 3)^{w} = (i, j, l)$. Then by Corollary 5.34, (F), and the first calculation, we have
\[
e_{ij}e_{jl} = e_{1w}e_{2w}e_{3w}e_{4w} \sim_{EB} w^{-1}e_{12}w^{-1}e_{23}w \sim_{EB} w^{-1}e_{12}e_{23}w \sim_{EB} w^{-1}e_{23}e_{12}w \sim_{EB} e_{jl}e_{ij}.
\]

Finally, if $i, j, k, l$ are all distinct, then we use the same trick as above, choosing a word $w \in (X_B \cup X_B^{-1})^*$ such that $(1, 2, 3, 4)^{w} = (i, j, k, l)$, to show that $e_{ij}e_{kl} \sim_{EB} e_{kl}e_{ij}$. □

Lemma 5.37. If $1 \leq i < j < k \leq n$, then $e_{ij}e_{jk} \sim_{EB} e_{jk}e_{ik} \sim_{EB} e_{ik}e_{ij}$.

Proof. As in the proof of the previous lemma, it suffices to show that the lemma holds when $(i, j, k) = (1, 2, 3)$. First note that $e_{13}e_{23} \sim_{EB} e_{23}e_{12} \sim_{EB} e_{13}e_{12}$ by the last lemma and (EB3). Secondly, we have
\[
e_{23}e_{13} = \sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1\sigma_2^{-1}e\sigma_2
\]
\[
\sim_{EB} \sigma_1^{-1}\sigma_2^{-1}e\sigma_1^{-1}\sigma_2\sigma_1e\sigma_2 \quad \text{by (B2) and (F)}
\]
\[
\sim_{EB} \sigma_1^{-1}\sigma_2^{-1}e\sigma_1^{-1}e\sigma_2\sigma_1e\sigma_2 \quad \text{by (EB2) and (F)}
\]
\[
\sim_{EB} \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1e\sigma_2 \quad \text{by (B2) and (F)}
\]
\[
\sim_{EB} \sigma_2^{-1}e\sigma_2^{-1}e\sigma_2\sigma_2 \quad \text{by (EB3)}
\]
\[
\sim_{EB} \sigma_2^{-1}e\sigma_2^{-1}\sigma_2^2 e \quad \text{by (EB5)}
\]
\[
\sim_{EB} \sigma_2^{-1}e\sigma_2 e \quad \text{by (F)}
\]
\[
= e_{13}e_{12},
\]

and the proof is complete. □

By Lemmas 5.35, 5.36, and 5.37 we see that $R_E^r \subseteq R_{EB}^r$ which, together with the comments after the proofs of Lemmas 5.30 and 5.33, shows that $(R_B \cup R_F \cup R_E^r \cup R_s)^3 = R_{EB}^r$. Incidentally we have the following.
Corollary 5.38. The semidirect product $E \rtimes B$ has presentation $\langle X_{FB} \mid R_{EB} \rangle$ via

\[
\sigma_i^{\pm 1} \mapsto (1, \zeta_i^{\pm 1}) \quad \text{for each } i \\
e \mapsto (E_{12}, 1).
\]

We now return to our task of simplifying the presentation of $\mathfrak{S}B_n$ in Corollary 5.29. Now the relation

\[ e = e\sigma_1 \]

is part of $R'_{\omega}$. Let $R'_{EB}$ denote the set of relations obtained from $R_{EB}$ by replacing $(EB1)$ with

\[ e^2 = e = e\sigma_1, \quad (EB1)' \]

and write $\sim_{FB}$ for the congruence $(R'_{EB})^2$.

Lemma 5.39. Suppose that $\beta \in \mathcal{P}_n$ and $1 \leq i < j \leq n$. Then $e_{ij}\hat{\beta}^{-1}\hat{\zeta}_{ij}\hat{\beta} \sim_{FB} e_{ij}$.

Proof. Under these assumptions we have

\[
e_{ij}\hat{\beta}^{-1}\hat{\zeta}_{ij}\hat{\beta} \sim_{FB} \hat{\beta}^{-1}e_{ij}\hat{\zeta}_{ij}\hat{\beta} \quad \text{by Corollary 5.32} \\
\sim_{FB} \hat{\beta}^{-1}w_{i,j}^{-1}ew_{i,j}w_{i,j}^{-1}\sigma_1w_{i,j}\hat{\beta} \quad \text{by Lemma 5.1} \\
\sim_{FB} \hat{\beta}^{-1}w_{i,j}^{-1}e\sigma_1w_{i,j}\hat{\beta} \quad \text{by (F)} \\
\sim_{FB} \hat{\beta}^{-1}w_{i,j}^{-1}ew_{i,j}\hat{\beta} \quad \text{by (EB1)'} \\
= \hat{\beta}^{-1}e_{ij}\hat{\beta} \quad \text{by Corollary 5.32 and (F)},
\]

and we are done. \qed

Thus $R'_{\omega} \subseteq (R'_{EB})^2$ so that $(R_{EB} \cup R'_{\omega})^2 = (R'_{EB})^2$, and we have proved the following.

Theorem 5.40. The factorisable braid monoid $\mathfrak{S}B_n$ has presentation $\langle X_{FB} \mid R'_{EB} \rangle$ via

\[
\sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} \quad \text{for each } i \\
e \mapsto [1]E_{12}.
\]

We will now show that, apart from (F), the relations in $R'_{EB}$ may be replaced by relations which only involve positive powers of the $\sigma_i$. \boxed
Lemma 5.41. The following relations are in \((R'_{EB})^2\):

\[
\begin{align*}
e^2 &= e = e\sigma_1 & \text{(FB1)} \\
 e\sigma_i &= \sigma_ie & \text{if } i \neq 2 & \text{(FB2)} \\
e\sigma_2e\sigma_2 &= \sigma_2e\sigma_2e & \text{(FB3)} \\
e\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2 &= \sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2e & \text{(FB4)} \\
e\sigma_2^2 &= \sigma_2^2e & \text{(FB5)}
\end{align*}
\]

Proof. Now relations (FB1), (FB2), and (FB5) are in \(R'_{EB}\). To complete the proof, we will show that relations (FB3) and (FB4) are in \(R'_{EB} \subseteq (R_{EB})^2\), a fact which will also be useful later. It suffices, by Corollary 5.38, to show that the relations hold as equations in \(E \rtimes B\) when the generators \(\sigma_i\) and \(e\) are replaced by \((1, \varsigma_i)\) and \((E_{12}, 1)\) respectively. We do this in Figures 5.10 and 5.11 using pictures as described in Section 4.4.

\[
\begin{align*}
\text{Figure 5.10. Relation (FB3): } & xyxy = yxyx \text{ where } x = (E_{12}, 1) \text{ and } y = (1, \varsigma_2).
\end{align*}
\]

\[
\begin{align*}
\text{Figure 5.11. Relation (FB4): } & xyxy = yxyx \text{ where } x = (E_{12}, 1) \text{ and } y = (1, \varsigma_2\varsigma_3\varsigma_1\varsigma_2).
\end{align*}
\]

Let \(R_{FB}\) denote the set of relations (F), (B1—B2), and (FB1—FB5).

Theorem 5.42. The factorisable braid monoid \(\mathfrak{F}B_n\) has presentation \(\langle X_{FB} \mid R_{FB} \rangle\) via

\[
\begin{align*}
\sigma_i^{\pm 1} &\mapsto \varsigma_i^{\pm 1} & \text{for each } i \\
e &\mapsto [1]_{E_{12}}.
\end{align*}
\]
Proof. All that remains to be shown is that relations (EB3) and (EB4) are in $R_{FB}^2$. For the remainder of this proof we denote the congruence $R_{FB}^2$ by $\cong$. Now for (EB3) we have

$$e\sigma_2^{-1}e\sigma_2 \cong e\sigma_2^{-2}\sigma_2 e\sigma_2$$

by (F)

$$\cong \sigma_2^{-2}e\sigma_2 e\sigma_2$$

by (FB5) and (F)

$$\cong \sigma_2^{-2}\sigma_2 e\sigma_2 e$$

by (FB3)

$$\cong \sigma_2^{-1}e\sigma_2 e$$

by (F),

and

$$\sigma_1^{-1}\sigma_2^{-1}e\sigma_2 \sigma_1 e \cong \sigma_1^{-1}\sigma_2^{-1}e\sigma_2 e$$

by (FB2) and (FB1)

$$\cong \sigma_1^{-1}e\sigma_2^{-1}e\sigma_2$$

by the previous calculation

$$\cong \sigma_1^{-1}e\sigma_1\sigma_2^{-1}e\sigma_2$$

by (FB1)

$$\cong e\sigma_2^{-1}e\sigma_2$$

by (FB2) and (F).

For (EB4), first put $u = \sigma_2\sigma_1\sigma_2$. Using the braid relations we have

$$u \sim_B \sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1^{-1}$$

by (F)

$$\sim_B \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_1^{-1}$$

by (B2)

$$\sim_B \sigma_2\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1^{-1}$$

by (B2)

$$= \sigma_2^2\sigma_1\sigma_2^{-1}\sigma_1^{-1}.$$

Now let $w = \sigma_2\sigma_3\sigma_1\sigma_2$. We must show that $ew^{-1}e w \cong w^{-1}e w e$. First note that

$$w^2 = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_1\sigma_3\sigma_2$$

by (F) and (B1)

$$\sim_B \sigma_3^{-1}\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2$$

by (B2)

$$\sim_B \sigma_3^{-1}\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2$$

by (B2)

$$\sim_B \sigma_3^{-1}\sigma_2\sigma_3\sigma_1\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2$$

by (B2)

$$\sim_B \sigma_3^{-1}\sigma_2\sigma_1\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2$$

by (B1) and (B2)

$$\sim_B \sigma_3^{-1}\sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2$$

by (B1)

$$\sim_B \sigma_3^{-1}\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2$$

by (B2)

$$= \sigma_3^{-1}uw\sigma_3u$$

$$\sim_B \sigma_3^{-1}(\sigma_2^2\sigma_1\sigma_2^2\sigma_1^{-1})\sigma_3(\sigma_2^2\sigma_1\sigma_2^2\sigma_1^{-1})$$

by the previous calculation.
It now follows, by (FB2), (FB5), and (F), that $ew^2 \cong w^2 e$. But then

\[
\begin{align*}
ew^{-1}e & \cong ew^{-2}we\quad \text{by (F)} \\
& \cong w^{-2}ew\quad \text{by the previous calculation and (F)} \\
& \cong w^{-2}we\quad \text{by (FB4)} \\
& \cong w^{-1}ew\quad \text{by (F)}.
\end{align*}
\]

This completes the proof of the theorem. \qed

**Remark 5.43.** Since relations (B1—B2) and (FB1—FB5) all involve only positive powers of the generators $\sigma_i$, it is possible to construct a “positive monoid” $\mathcal{B}_n^+$. To do this, we identify $\mathcal{B}_n$ with $X_{FB} / R_{FB}$. Now put $X_{FB}^+ = \{\sigma_1, \ldots, \sigma_{n-1}, e\}$ and let $R_{FB}^+$ be the set of relations (B1—B2) and (FB1—FB5). We define $\mathcal{B}_n^+$ to be $X_{FB}^+ / R_{FB}^+$. We will denote by $[w]$ and $[w]^+$ the $R_{FB}^+$ and $R_{FB}^+$-classes of a word $w \in X_{FB}^+$ respectively. Since $R_{FB} \subseteq R_{FB}$, we may define a map

$$\iota : \mathcal{B}_n^+ \to \mathcal{B}_n : [w]^+ \mapsto [w] \quad \text{for each} \ w \in X_{FB}^+.$$ 

It is natural to ask whether $\iota$ is injective, as a similar map is injective for the braid group (see [34] and [52]), and the singular braid monoid (see [4] and [13]). The author does not currently know the answer to this question.

### 5.2.2. Another Presentation of $\mathcal{B}_n$.

Our goal now is to give a different presentation of $\mathcal{B}_n$. We enlarge the sets of generators and relations, thereby arriving at a presentation which displays more of the symmetry possessed by $\mathcal{B}_n$. This presentation will also highlight an interesting connection between $\mathcal{B}_n$ and $\mathcal{S}_n$, the singular braid monoid (introduced in [7, 9]). We will explore this connection in Section 5.2.3.

We begin with the presentation $\langle X_{FB} | R_{FB} \rangle$ of $\mathcal{B}_n$ given in Theorem 5.42. Recall that for each $1 \leq i < j \leq n$, we defined a word

$$e_{ij} = w_{ij}^{-1}ew_{ij} = (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1})(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1})e(\sigma_2 \cdots \sigma_{j-1})(\sigma_1 \cdots \sigma_{i-1}).$$

We now rename $e = e_1$ and add generators $e_2, \ldots, e_{n-1}$ to the presentation along with the relations

$$e_i = e_{i,i+1} \quad \text{for all} \ i \quad (D)$$

which define them in terms of the original generators. Put $X_{FB}^* = X_B \cup X_{\mathcal{B}}^* \cup \{e_1, \ldots, e_{n-1}\}$. For the remainder of this section we denote by $\cong$ the congruence on $(X_{FB}^*)^*$ generated by $R_{FB}$ and (D).
5.2. THE FACTORISABLE BRAID MONOID

Lemma 5.44. The following relations are in $\cong$:

\begin{align*}
    e_i \sigma_i &= \sigma_i e_i = e_i = e_i^2 & \text{for all } i & \quad \text{(FB1)}' \\
    e_i e_j &= e_j e_i & \text{for all } i, j & \quad \text{(FB2)}' \\
    e_i \sigma_j &= \sigma_j e_i & \text{if } |i - j| > 1 & \quad \text{(FB3)}' \\
    e_i \sigma_j \sigma_i &= \sigma_j \sigma_i e_j & \text{if } |i - j| = 1 & \quad \text{(FB4)}' \\
    e_i \sigma_j^2 &= \sigma_j^2 e_i & \text{if } |i - j| = 1. & \quad \text{(FB5)}'
\end{align*}

Proof. Note first that for each $i$, the map in Theorem 5.42 sends the word $e_{i,i+1} \in X_{FB}^*$ to $[1]_{\mathcal{E}_{i,i+1}} \in \mathcal{FB}_n$. Thus, by Theorem 5.42 and (D), it suffices to show that the stated relations hold as equations in $\mathcal{FB}_n$ when the generators $\sigma_i^{\pm 1}$ and $e_i$ are replaced by $\sigma_i^{\pm 1}$ and $[1]_{\mathcal{E}_{i,i+1}}$ respectively. This may easily be checked with the aid of pictures.

Thus we may add relations (FB1|FB5)' to the presentation. Let $R'_{FB}$ be the set of relations (F), (B1—B2), and (FB1—FB5)', and denote by $\sim_{FB'}$ the congruence $(R'_{FB})^4$ on $(X'_{FB})^*$. Our aim is to show that $\cong = \sim_{FB'}$. Now relations (FB1), (FB2), and (FB5) may clearly be removed since they are part of relations (FB1)', (FB3)', and (FB5)'. Next we will show that relations (FB3), (FB4), and (D) may also be removed.

Lemma 5.45. We have $e_1 \sigma_2 e_1 \sigma_2 \sim_{FB'} \sigma_2 e_1 \sigma_2 e_1$.

Proof. Observe first that if $|i - j| = 1$, then by (F) and (FB4)' we have

$$
\sigma_i e_j \sigma_i^{-1} \sim_{FB'} \sigma_j^{-1} \sigma_j e_j \sigma_i^{-1} \sim_{FB'} \sigma_j^{-1} e_i \sigma_j \sigma_i^{-1} \sim_{FB'} \sigma_j^{-1} e_i \sigma_j.
$$

But then

\begin{align*}
    e_1 \sigma_2 e_1 \sigma_2 &\sim_{FB'} e_1 \sigma_2 e_1 \sigma_2^{-1} \sigma_2^2 & \text{by (F)} \\
    &\sim_{FB'} e_1 \sigma_1^{-1} e_2 \sigma_1 \sigma_2^2 & \text{by the observation} \\
    &\sim_{FB'} e_1 \sigma_1^{-1} e_2 \sigma_1 \sigma_2^2 & \text{by (FB1)' and (F)} \\
    &\sim_{FB'} e_1 \sigma_1^{-1} e_2 \sigma_1 \sigma_2^2 & \text{by (FB2)'} \\
    &\sim_{FB'} e_1 \sigma_1^{-1} e_2 \sigma_1 e_1 \sigma_2^2 & \text{by (FB1)'} \\
    &\sim_{FB'} e_1 \sigma_1^{-1} e_2 \sigma_1 \sigma_2^2 e_1 & \text{by (FB5)'} \\
    &\sim_{FB'} \sigma_2 e_1 \sigma_2^{-1} \sigma_2^2 e_1 & \text{by the observation again} \\
    &\sim_{FB'} \sigma_2 e_1 \sigma_2 e_1 & \text{by (F)},
\end{align*}

and we are done.

Lemma 5.46. We have $e_1 w e_1 w \sim_{FB'} w e_1 w e_1$ where $w = \sigma_2 \sigma_3 \sigma_1 \sigma_2$. 

Proof. We have
\begin{align*}
e_1we_1w &= e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \\
&\sim_{FB'} e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \quad \text{by (B1)} \\
&\sim_{FB'} \sigma_2 \sigma_1 e_2 \sigma_3 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \quad \text{by (FB4)' } \\
&\sim_{FB'} \sigma_2 \sigma_1 \sigma_3 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \quad \text{by (FB4)' } \\
&\sim_{FB'} \sigma_2 \sigma_3 \sigma_1 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \quad \text{by (B1) and (FB2)' } \\
&\sim_{FB'} \sigma_2 \sigma_3 \sigma_1 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \quad \text{by (FB4)' } \\
&\sim_{FB'} \sigma_2 \sigma_3 \sigma_1 \sigma_2 e_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 e_1 \quad \text{by (FB4)' } \\
&= we_1we_1,
\end{align*}
and we are done. \qed

Lemma 5.47. If \(1 \leq i \leq n - 1\), then
\[ e_i \sim_{FB'} w_{i,i+1}^{-1} e_1 w_{i,i+1}. \]

Proof. We prove the lemma by induction on \(i\). If \(i = 1\) then there is nothing to prove, so suppose that \(2 \leq i \leq n - 1\). Then
\begin{align*}
e_i &\sim_{FB'} \sigma_{i-1}^{-1} \sigma_i^{-1} \sigma_i \sigma_{i-1} e_i \quad \text{by (F)} \\
&\sim_{FB'} \sigma_{i-1}^{-1} \sigma_i^{-1} \sigma_i^{-1} e_i \sigma_i - 1 \quad \text{by (FB4)'} \\
&\sim_{FB'} \sigma_{i-1}^{-1} \sigma_i^{-1} w_{i-1,i}^{-1} e_1 w_{i-1,i} \sigma_i \sigma_{i-1} \quad \text{by an inductive hypothesis},
\end{align*}
and we are done since \(w_{i-1,i} \sigma_i \sigma_{i-1} \sim_B w_{i,i+1}\) by (B1). \qed

The last three lemmas have shown that relations (FB3), (FB4), and (D) are in \((R_{FB})^4\).

Thus we have the following.

Theorem 5.48. The factorisable braid monoid \(\mathfrak{FB}_n\) has presentation \(\langle X_{FB}' \mid R_{FB}' \rangle\) via
\[
\begin{align*}
\sigma_i^{\pm 1} &\mapsto \zeta_i^{\pm 1} \quad \text{for each } i \\
e_i &\mapsto [1] e_{i,i+1} \quad \text{for each } i.
\end{align*}
\]

Remark 5.49. As in Remark 5.43, we may use the presentation of \(\mathfrak{FB}_n\) given in Theorem 5.48 to construct an alternative positive monoid \(\mathfrak{FB}_n^+\). It is then natural to ask whether the map \(\mathfrak{FB}_n^+ \rightarrow \mathfrak{FB}_n\) is injective. Again, the answer is currently unknown to the author.
5.2. THE FACTORISABLE BRAID MONOID

5.2.3. Flexible Singular Braids and Relation (FB5)'.

In this section, starting with the observation that the singular braid relations are contained in $R'_{FB}$, we examine an interesting connection between the factorisable braid monoid $\mathcal{F}B_n$ and the singular braid monoid $SB_n$ which was introduced by Baez [7] and Birman [9]. This will lead us to introduce a new (singular) braid monoid $\mathcal{F}SB_n$ called the flexible singular braid monoid. We will see that $\mathcal{F}B_n$ is a homomorphic image of $\mathcal{F}SB_n$ which, in turn, is a homomorphic image of $SB_n$. We will also derive a presentation of $\mathcal{F}SB_n$ (Theorem 5.52) which extends Birman’s presentation of $SB_n$ [9]. This presentation turns out to be almost identical to the presentation of $\mathcal{F}B_n$ in Theorem 5.48; the only difference is the absence of relation (FB5)'. We then show that $\mathcal{F}B_n$ and $\mathcal{F}SB_n$ are not isomorphic, thereby showing that relation (FB5)’ is independent of the other relations in $R'_{FB}$.

A singular braid is a collection of strings, much like a braid, with the exception that there may exist a finite number of double points (or singular points) where a pair of strings intersect. Let $SB_n$ denote the set of all singular braids with $n$ strings. Singular braids are multiplied just like normal braids; if $\beta, \gamma \in SB_n$ then $\beta\gamma$ is the singular braid obtained by joining the bottom of $\beta$ to the top of $\gamma$. Thus $SB_n$ is a groupoid under this product. The singular braid monoid, denoted $SB_n$, is the monoid of rigid-vertex-isotopy classes of singular braids on $n$ strings. See [9] for more details.

In this section it will be useful to draw a clear distinction between a singular braid $\beta \in SB_n$ and its rigid-vertex-isotopy-class which we will denote by $[\beta] \in SB_n$. The singular braid monoid is generated by $[\sigma_1^{\pm 1}], \ldots, [\sigma_{n-1}^{\pm 1}]$ together with the singular generators $[\tau_1], \ldots, [\tau_{n-1}]$. The singular braid $\tau_i \in SB_n$ is pictured in Figure 5.12.

\[
\begin{array}{ccccccc}
& & & \cdots & i & \cdots & \cdots \\
1 & & & & & & n \\
\end{array}
\]

**Figure 5.12.** The singular braid $\tau_i \in SB_n$.

Put $X_{SB} = X'_{FB}$ and denote by $R_{SB}$ the set of relations (F), (B1—B2), and

\[
\begin{align*}
& e_i \sigma_i = \sigma_i e_i & \text{for all } i & \quad \text{(SB1)} \\
& e_i e_j = e_j e_i & \text{if } |i - j| > 1 & \quad \text{(SB2)} \\
& e_i \sigma_j = \sigma_j e_i & \text{if } |i - j| > 1 & \quad \text{(SB3)} \\
& e_i \sigma_j \sigma_i = \sigma_j \sigma_i e_j & \text{if } |i - j| = 1. & \quad \text{(SB4)}
\end{align*}
\]

The following was first proved in [9]; see also [7, 37].
5.2. THE FACTORISABLE BRAID MONOID

Theorem 5.50 (Birman [9]). The singular braid monoid $SB_n$ has monoid presentation $\langle X_{SB} \mid R_{SB} \rangle$ via

$$\phi_{SB} : X_{SB}^* \to SB_n : \left\{ \begin{array}{l}
\sigma_i^{\pm 1} \mapsto [\sigma_i^{\pm 1}] \\
e_i \mapsto [\tau_i]
\end{array} \right. \quad \text{for all } i \right.$$ 

Notice that relations (SB1—SB4) are part of relations (FB1—FB4)’ so that, in particular, $FB_n$ is a homomorphic image of $SB_n$. Write $\sim_{SB} = R_{SB}^\ast$.

If $\beta, \gamma \in SB_n$, then we write

(i) $\beta \succeq_{(i)} \gamma$ if $\beta$ and $\gamma$ are equivalent under rigid-vertex-isotopy;
(ii) $\beta \succeq_{(ii)} \gamma$ if $\beta$ and $\gamma$ are identical except for a neighbourhood which contains any of the fragments catalogued in Figure 5.13;

(iii) $\beta \succeq_{(iii)} \gamma$ if $\beta$ and $\gamma$ are identical except for a neighbourhood which contains any of the fragments catalogued in Figure 5.14.

Remark 5.51. Move (ii) can be thought of as allowing a singular point to “swallow up” or “produce” another twist or singular point directly above or below it, involving the same two strings. Move (iii) can be achieved by allowing a triple singular point (see Figure 5.15) to be momentarily created and then destroyed, as the strings pass from one of the configurations in Figure 5.14 to another.

Figure 5.13. Possible configurations before and after move (ii).

Figure 5.14. Possible configurations before and after move (iii).

Figure 5.15. An intermediate triple singular point created during move (iii).
If $\beta, \gamma \in \mathbf{SB}_n$ are singular braids, then we say that $\beta$ and $\gamma$ are *flexible-vertex-isotopic*, and write $\beta \asymp \gamma$, if there is a sequence of singular braids $\beta = \beta_0, \beta_1, \ldots, \beta_k = \gamma$ such that for each $j$, we have either $\beta_j \asymp_{(i)} \beta_{j+1}$, $\beta_j \asymp_{(ii)} \beta_{j+1}$, or $\beta_j \asymp_{(iii)} \beta_{j+1}$. We denote the $\asymp$-class of a singular braid $\beta \in \mathbf{SB}_n$ by $[\beta]_\asymp$. It is clear that $\asymp$ is a (groupoid) congruence on $\mathbf{SB}_n$, and that $\beta(\gamma \delta) \asymp (\beta \gamma) \delta$ and $1\beta \asymp 1\beta$ for all $\beta, \gamma, \delta \in \mathbf{SB}_n$. Thus we may form the quotient monoid $\tilde{\mathbf{SB}}_n = \mathbf{SB}_n/\asymp = \{ [\beta]_\asymp : \beta \in \mathbf{SB}_n \}$ which we call the *flexible singular braid monoid*. Notice in particular that we have $[\beta]_\asymp = [\beta]$ for each $\beta \in \mathbf{B}_n$. It will be convenient to identify a braid $\beta \in \mathbf{B}_n$ with its $\asymp$-class $[\beta] \in \tilde{\mathbf{SB}}_n$.

Let $R_{\tilde{\mathbf{SB}}}$ be the set of relations (F), (B1—B2), and (FB1—FB4)', and write $\sim_{\tilde{\mathbf{SB}}} = R_{\tilde{\mathbf{SB}}}$.

**Theorem 5.52.** The flexible singular braid monoid $\tilde{\mathbf{SB}}_n$ has a monoid presentation $\langle X_{\tilde{\mathbf{SB}}} \mid R_{\tilde{\mathbf{SB}}} \rangle$ via

$$X_{\tilde{\mathbf{SB}}}^* \to \tilde{\mathbf{SB}}_n : \left\{ \begin{array}{ll}
\sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} & \text{for all } i \\
e_i \mapsto [\tau_i]_\asymp & \text{for all } i.
\end{array} \right.$$}

**Proof.** Denote by $\Phi : X_{\tilde{\mathbf{SB}}}^* \to \tilde{\mathbf{SB}}_n$ the map in the statement of the theorem. We then have $\Phi = \phi_{SB} \nu : X_{\tilde{SB}}^* \to \tilde{\mathbf{SB}}_n$ where $\nu : \mathbf{SB}_n \to \tilde{\mathbf{SB}}_n$ is the natural map defined, for $\beta \in \mathbf{SB}_n$, by $[\beta] \mapsto [\beta]_\asymp$. So $\Phi$ is an epimorphism and since, as may easily be checked diagrammatically, $w_1 \Phi = w_2 \Phi$ for each $(w_1, w_2) \in R_{\tilde{\mathbf{SB}}}$, we have $\sim_{\tilde{\mathbf{SB}}} \subseteq \ker \Phi$. To show the reverse inclusion, suppose that $(w_1, w_2) \in \ker \Phi$. Choose $\beta, \gamma \in \mathbf{SB}_n$ such that $w_1 \phi_{SB} = [\beta]$ and $w_2 \phi_{SB} = [\gamma]$. Then we have $\beta \asymp \gamma$, and we must show that $w_1 \sim_{\tilde{\mathbf{SB}}} w_2$.

By induction it suffices to assume that either $\beta \asymp_{(i)} \gamma$, $\beta \asymp_{(ii)} \gamma$, or $\beta \asymp_{(iii)} \gamma$. If $\beta \asymp_{(i)} \gamma$ then $w_1 \sim_{SB} w_2$. Suppose next that $\beta \asymp_{(ii)} \gamma$. Then there exists $1 \leq i \leq n - 1$ such that $w_1 \sim_{SB} wv$ and $w_2 \sim_{SB} wv'$ for some $w, w' \in X_{SB}^*$ and some

$$u, v \in \{ \sigma_i^{\pm 1}, e_i, \sigma_i, e_i^2, e_i \}.$$

But then we have $w_1 \sim_{FB} w_2$ by (FB1)' and (F). Finally, suppose that $\beta \asymp_{(iii)} \gamma$. Then there exists $1 \leq i, j \leq n - 1$ with $|i - j| = 1$ such that $w_1 \sim_{SB} wv$ and $w_2 \sim_{SB} wv'$ for some $w, w' \in X_{SB}^*$ and some

$$u, v \in \{ e_i e_j, e_j e_i, e_i \sigma_j, e_j \sigma_i e_i \}.$$

To complete the proof we must show that all the words in this set are $\sim_{\tilde{\mathbf{SB}}}$-equivalent.

But this is the case since $e_i e_j \sim_{\tilde{\mathbf{SB}}} e_j e_i$ by (FB2)', while if $\{k, l\} = \{i, j\}$, then

$$e_k \sigma_l^{\pm 1} e_k \sim_{\tilde{\mathbf{SB}}} e_l \sigma_l^{\pm 1} e_k \sim_{\mathbf{SB}} \sigma_i \sigma_i^{\pm 1} e_k \sigma_i^{\pm 1} e_k \sim_{\tilde{\mathbf{SB}}} \sigma_i \sigma_i^{\pm 1} e_k \sigma_i^{\pm 1} e_k \sim_{\tilde{\mathbf{SB}}} e_l e_k$$

by (F)
Since this presentation of $\mathcal{F}\mathcal{S}\mathcal{B}_n$ differs from the presentation of $\mathcal{F}\mathcal{B}_n$ in Theorem 5.48 only by the absence of relation (FB5)', it is natural to wonder whether in fact $\mathcal{F}\mathcal{S}\mathcal{B}_n$ and $\mathcal{F}\mathcal{B}_n$ are isomorphic. The existence of such an isomorphism would be guaranteed if relation (FB5) was in $R^2_{\mathcal{F}\mathcal{S}\mathcal{B}}$ which, by Theorem 5.52, would be equivalent to knowing that $\tau_i \tau_j^2 \sim \tau_j^2 \tau_i$ for each $1 \leq i, j \leq n - 1$ with $|i - j| = 1$. Figure 5.16 gives a good indication that this relation “ought not” to hold, and Proposition 5.53 below proves that it does not.

![Figure 5.16. The relation $\tau_i \tau_j^2 \sim \tau_j^2 \tau_i$ does not appear to hold in $\mathcal{F}\mathcal{S}\mathcal{B}_n$.](image)

**Proposition 5.53.** Suppose that $1 \leq i, j \leq n - 1$ and $|i - j| = 1$. Then $\sigma_i \sigma_j^2 \not\sim_{\mathcal{F}\mathcal{S}\mathcal{B}} \sigma_j^2 \sigma_i$.

**Proof.** This proof makes use of the coset monoid $C(B)$ of the braid group $B = B_n$ (see Sections 3.2 and 4.2.3). We define a homomorphism $\Psi : X_{\mathcal{F}\mathcal{S}\mathcal{B}} \to C(B)$ by

$$\sigma_i^\pm e \Psi = \{1\} \sigma_i^\pm \quad \text{and} \quad e_i \Psi = \langle \sigma_i \rangle \quad \text{for each } i.$$ 

Using the braid relations or pictures, one may easily check that $w_1 \Psi = w_2 \Psi$ for each $(w_1, w_2) \in R_{\mathcal{F}\mathcal{S}\mathcal{B}}$ so that $\sim_{\mathcal{F}\mathcal{S}\mathcal{B}} \subseteq \text{ker } \Psi$. Suppose now that there exists $1 \leq i, j \leq n - 1$ with $|i - j| = 1$ such that $e_i \sigma_j^2 \sim_{\mathcal{F}\mathcal{S}\mathcal{B}} \sigma_j^2 e_i$. Then we must have the coset equality

$$\langle \sigma_j^m \sigma_i^n \mid m \in \mathbb{Z} \rangle = \langle \sigma_i \rangle \sigma_j^m = (e_i \sigma_j^2) \Psi = (\sigma_j^2 e_i) \Psi = \sigma_j^2 \langle \sigma_j \rangle = \langle \sigma_j \sigma_j^2 \sigma_i^m \mid m \in \mathbb{Z} \rangle.$$ 

In particular we have $\sigma_i \sigma_j^2 = \sigma_j \sigma_i^m$ for some $m \in \mathbb{Z}$, and so $\sigma_i \sigma_j^2 \sim_{B} \sigma_j^2 \sigma_i^m$. But since two $\sim_{B}$-equivalent words over $X_B$ have the same exponent sum, we see that $m = 1$ and so $\sigma_i \sigma_j^2 \sim_{B} \sigma_j^2 \sigma_i$. We must then be able to transform $\sigma_i \sigma_j^2$ into $\sigma_j^2 \sigma_i$ using only the “positive” relations $R_B$ (see for example [34] or [52]). But this is clearly impossible, and we have the required contradiction. \hfill \Box

**Remark 5.54.** In light of Theorem 4.12 it is natural to ask whether or not the map $\Psi$ defined in the above proof yields an embedding of $\mathcal{F}\mathcal{S}\mathcal{B}_n$ in $C(B_n)$; that is, whether or not $\text{ker } \Psi = \sim_{\mathcal{F}\mathcal{S}\mathcal{B}}$. Due to obvious similarities to the statement of Birman’s Conjecture (see Chapter 1 and [9, 53]), it is tempting to conjecture that this is the case. If it were true then, in particular, $\mathcal{F}\mathcal{S}\mathcal{B}_n$ would necessarily be a (factorisable) inverse monoid (see [25]). However, the author suspects that the idempotents $[\tau_1]_\omega$ and $[\tau_2^2 \tau_1 \tau_2^2]_\omega$ do not commute, which would of course imply that $\mathcal{F}\mathcal{S}\mathcal{B}_n$ is not an inverse monoid.
5.2. THE FACTORISABLE BRAID MONOID

We now work towards showing that no isomorphism exists from \( \mathfrak{S}B_n \) to \( \mathfrak{S}BS_n \). We first state a well-known result concerning automorphisms of \( B \). For \( \beta \in B \), we denote by \( \chi_\beta \in \text{Aut}(B) \) the inner automorphism defined, for \( \gamma \in B \), by \( \gamma \chi_\beta = \beta^{-1} \gamma \beta \). We also let \( \iota \in \text{Aut}(B) \) be the automorphism of \( B \) determined by \( \zeta_i \iota = \zeta_i^{-1} \) for each \( i \).

**Theorem 5.55** ([17]). We have \( \text{Aut}(B) = \{ \chi_\beta \circ \iota^k \mid \beta \in B, \ k = 0, 1 \} \).

**Lemma 5.56.** Suppose that \( \rho \in \text{Aut}(B) \). Then there exists \( \tilde{\rho} \in \text{Aut}(\mathfrak{S}BS_n) \) such that \( \gamma \tilde{\rho} = \gamma \rho \) for all \( \gamma \in B \).

**Proof.** Suppose first that \( \rho = \chi_\beta \) for some \( \beta \in B \). Then we may take \( \tilde{\rho} \) to be the automorphism of \( \mathfrak{S}BS_n \) defined by \( [\gamma]_x \tilde{\rho} = [\beta^{-1} \gamma \beta]_x \) for all \( [\gamma]_x \in \mathfrak{S}BS_n \). Suppose next that \( \rho = \iota \). Then we define \( \tilde{\rho} : \mathfrak{S}BS_n \rightarrow \mathfrak{S}BS_n \) by \( \zeta_i^{\pm 1} \tilde{\rho} = \zeta_i^{\pm 1} \) and \( [\gamma]_x \tilde{\rho} = [\gamma]_x \) for each \( i \). One may easily check, with the aid of Theorem 5.52, that \( \tilde{\rho} \) is a well-defined homomorphism, which is clearly an involution and hence an automorphism. The result now follows from Theorem 5.55.

To avoid potentially confusing notation, we will identify \( \mathfrak{S}B_n \) with \( (E \rtimes B)/\sim \) for the remainder of this section.

**Corollary 5.57.** If the monoids \( \mathfrak{S}BS_n \) and \( \mathfrak{S}B_n \) are isomorphic, then there is an isomorphism \( \phi : \mathfrak{S}BS_n \rightarrow \mathfrak{S}B_n \) such that \( \beta \phi = [1, \beta] \) for all \( \beta \in B \).

**Proof.** Suppose that \( \psi : \mathfrak{S}BS_n \rightarrow \mathfrak{S}B_n \) is an isomorphism. Since \( \beta \psi \) is invertible for all \( \beta \in B \), we must have \( \beta \psi = [1, \beta \rho] \) for some \( \beta \rho \in B \). But then \( \rho : \beta \mapsto \beta \rho \) is clearly an automorphism of \( B \). By Lemma 5.56, we may extend \( \rho \) to an automorphism \( \tilde{\rho} \) of \( \mathfrak{S}BS_n \) such that \( \gamma \tilde{\rho} = \gamma \rho \) for all \( \gamma \in B \). The result now follows with \( \phi = \tilde{\rho}^{-1} \circ \psi \).

**Theorem 5.58.** The monoids \( \mathfrak{S}BS_n \) and \( \mathfrak{S}B_n \) are not isomorphic.

**Proof.** Suppose that \( \mathfrak{S}BS_n \) and \( \mathfrak{S}B_n \) are isomorphic. Then by Corollary 5.57 there is an isomorphism \( \phi : \mathfrak{S}BS_n \rightarrow \mathfrak{S}B_n \) such that \( \beta \phi = [1, \beta] \) for all \( \beta \in B \). By Theorem 5.52 and Lemma 5.47 (the proof of which uses only the singular braid relations) we have

\[
\tau_i \cong (\zeta_i^{-1} \cdots \zeta_1^{-1})(\zeta_i^{-1} \cdots \zeta_2^{-1})\tau_1 (\zeta_2 \cdots \zeta_i) (\zeta_1 \cdots \zeta_{i-1})
\]

for each \( i \). This shows that \( \phi \) is completely determined by \( [\tau_1]_{x_1} \phi \), and also that \( \mathfrak{S}BS_n \) is generated by \( B \cup \{ [\tau_1]_{x_1} \} \). It follows that \( \mathfrak{S}B_n \) is generated by \( [1, B] \cup \{ [\tau_1]_{x_1} \phi \} \) and, since \( [\tau_1]_{x_1} \phi \) is an idempotent, we must have \( [\tau_1]_{x_1} \phi = [E_{ij}, 1] \) for some \( 1 \leq i < j \leq n \). But then \( [E_{ij}, \zeta_i] = [\tau_1 \zeta_i]_{x_1} \phi = [\tau_1]_{x_1} \phi = [E_{ij}, 1] \) so that \( (i, j) = (1, 2) \). So we must also have

\[
[\tau_1 \zeta_2^2]_{x_1} \phi = ([\tau_1]_{x_1} \phi)(\zeta_2^2 \phi) = [E_{12}, 1][1, \zeta_2] = [1, \zeta_2][E_{12}, 1] = (\zeta_2^2 \phi)([\tau_1]_{x_1} \phi) = [\zeta_2^2 \tau_1]_{x_1} \phi.
\]
5.2. THE FACTORISABLE BRAID MONOID

Since \( \phi \) is an isomorphism, it follows that \( \tau_1 \xi_2^2 \cong \zeta_2^2 \tau_1 \). But then by Theorem 5.52 we have \( e_1 \sigma_2^2 \sim_{FSB} \sigma_2^2 e_1 \), contradicting Proposition 5.53. This completes the proof.

5.2.4. Presentations of \( \mathfrak{F}^*_n \).

Recall that we have an epimorphism

\[
\mathfrak{F}^*_B \rightarrow \mathfrak{F}^*_B : (X_B \cup X)^* \rightarrow \mathfrak{F}^*_B
\]

In this section we will use the presentations of \( \mathfrak{F}^*_B \) given in Theorems 5.42 and 5.48 to obtain presentations of \( \mathfrak{F}^*_B \), the monoid of uniform block bijections on \( n \), including the presentation originally due to FitzGerald [32].

We will first prove a general result analogous to Proposition 5.21. Let \( X \) be any alphabet disjoint from \( X_B \cup X_B^{-1} \). Recall that we say a word \( w \in (X_B \cup X)^* \) is square-free if \( w \) contains no subword from \( \{ \sigma_1^2, \ldots, \sigma_{n-1}^2 \} \). Recall also that for \( w \in (X_B \cup X)^* \) we defined a word \( \tilde{w} \in (X_B \cup X)^* \) which, essentially, is the word obtained from \( w \) by successively removing all subwords from \( \{ \sigma_1^2, \ldots, \sigma_{n-1}^2 \} \).

Proposition 5.59. Suppose that \( \mathfrak{F}^*_B \) has presentation \( \langle X_B \cup X_B^{-1} \cup X \mid R_B \cup R_F \cup R \rangle \) via a map \( \phi : (X_B \cup X_B^{-1} \cup X)^* \rightarrow \mathfrak{F}^*_B \) such that

(i) \( X \) is disjoint from \( X_B \cup X_B^{-1} \);
(ii) \( R \subseteq (X_B \cup X)^* \times (X_B \cup X)^* \); and
(iii) \( \sigma_i^\pm 1 \phi = \xi_i^\pm 1 \) for each \( i \).

Denote by \( R_O \) the set of relations

\[ \sigma_i^2 = 1 \quad \text{for each } i, \]  

and let \( \tilde{R} \) be the set of relations obtained from \( R \) by deleting any relation \( (w_1, w_2) \in R \) for which \( \tilde{w}_1 = \tilde{w}_2 \). Then \( \mathfrak{F}^*_B \) has presentation \( \langle X_B \cup X \mid R_B \cup R_O \cup \tilde{R} \rangle \) via

\[ x \mapsto x\phi \quad \text{for each } x \in X_B \cup X. \]

Proof. We first show that \( \mathfrak{F}^*_B \) has presentation

\[ \langle X_B \cup X_B^{-1} \cup X \mid R_B \cup R_F \cup R \cup R_O \rangle \]

via

\[ \phi' : (X_B \cup X_B^{-1} \cup X)^* \rightarrow \mathfrak{F}^*_B : x \mapsto x\phi \quad \text{for each } x \in X_B \cup X_B^{-1} \cup X. \]

Put \( \cong = (R_B \cup R_F \cup R)^5 \) and \( \sim_O = (R_B \cup R_F \cup R \cup R_O)^5 \). Now \( \phi' \) is an epimorphism since both \( \phi \) and \( \cong \) are epimorphisms. Since \( s_i^2 = 1 \) for each \( i \), we see that \( \sim_O \subseteq \ker \phi' \). Suppose
now that \((w_1, w_2) \in \ker \phi'\). Now we have
\[
w_1 \phi = [\beta_1]_{\mathcal{E}_1} = [1]_{\mathcal{E}_1} \beta_1 \quad \text{and} \quad w_2 \phi = [\beta_2]_{\mathcal{E}_2} = [1]_{\mathcal{E}_2} \beta_2
\]
for some \(\mathcal{E}_1, \mathcal{E}_2 \in E\) and \(\beta_1, \beta_2 \in B\). But since
\[
\mathbf{n}/\mathcal{E}_1 = \text{dom}(w_1 \phi') = \text{dom}(w_2 \phi') = \mathbf{n}/\mathcal{E}_2
\]
we must have \(\mathcal{E}_1 = \mathcal{E}_2\). Choose \(w_3, w_4 \in (X_B \cup X_B^{-1})^*\) such that
\[
w_3 \phi = \beta_1 \quad \text{and} \quad w_4 \phi = \beta_2,
\]
and choose \(w \in (X_B \cup X_B^{-1} \cup X)^*\) such that \(w \phi = [1]_{\mathcal{E}_1}\). Then \(w_1 \phi = (w w_3) \phi\) and \(w_2 \phi = (w w_4) \phi\) and so, by hypothesis, we have
\[
w_1 \cong w w_3 \quad \text{and} \quad w_2 \cong w w_4.
\]
Recall that for \(u \in (X_B \cup X_B^{-1})^*\) we denote by \(\overline{u} \zeta \in S_n\). Put \(\pi = \overline{w_3 w_4^{-1}}\). Now in \(\mathfrak{S}_n\) we have
\[
id_{\mathcal{E}_1} = id_{\mathcal{E}_1} w w_4^{-1} = (w_2 \phi') w w_4^{-1} = (w_1 \phi') w w_4^{-1} = id_{\mathcal{E}_1} w_3 w w_4^{-1} = id_{\mathcal{E}_1} \pi,
\]
and it follows that \((i, i \pi) \in \mathcal{E}_1\), whence \((i, i \pi^{-1}) \in \mathcal{E}_1\), for all \(i \in \mathbf{n}\). Thus we may write
\[
\pi^{-1} = t_{p_1 q_1} \cdots t_{p_k q_k}
\]
for some \(p_1, \ldots, p_k, q_1, \ldots, q_k \in \mathbf{n}\) with \(p_j < q_j\) and \((p_j, q_j) \in \mathcal{E}_1\) for each \(j \in \mathbf{k}\). But then \(\varsigma_{p_1 q_1} \cdots \varsigma_{p_k q_k} \in B_{\mathcal{E}_1}\) by Lemma 4.10, so that
\[
w \phi = [1]_{\mathcal{E}_1} = [1]_{\mathcal{E}_1} \varsigma_{p_1 q_1} \cdots \varsigma_{p_k q_k} = (w \widehat{\varsigma}_{p_1 q_1} \cdots \widehat{\varsigma}_{p_k q_k}) \phi.
\]
By hypothesis we then have
\[
w \cong w \widehat{\varsigma}_{p_1 q_1} \cdots \widehat{\varsigma}_{p_k q_k}.
\]
Now \(\widehat{\varsigma}_{p_1 q_1} \cdots \widehat{\varsigma}_{p_k q_k} = \pi^{-1} = w_3 w w_4^{-1}\), and so \((\widehat{\varsigma}_{p_1 q_1} \cdots \widehat{\varsigma}_{p_k q_k} w_3 w_4^{-1}) \zeta \in \mathcal{P}_n\). Thus
\[
\widehat{\varsigma}_{p_1 q_1} \cdots \widehat{\varsigma}_{p_k q_k} w_3 w_4^{-1} \sim_B \widehat{\alpha}_{u_1 v_1} \cdots \widehat{\alpha}_{u_{v} v_{t}} \pm 1
\]
for some \(u_1, \ldots, u_{\ell}, v_1, \ldots, v_{\ell} \in \mathbf{n}\) with \(u_j < v_j\) for each \(j \in \ell\). By (O) and (F) we have
\[
\widehat{\alpha}_{u_{i} v_{i}} = (\sigma_{j-i} \cdots \sigma_{i+1}) \sigma_{i+1}^+ (\sigma_{i} \cdots \sigma_{j-1}) \sim_O 1
\]
for all \(1 \leq i < j \leq n\), so that
\[
w_1 \sim_O w w_3
\]
\[
\sim_O (w w_3 w_4^{-1}) w_4
\]
\[
\sim_O (w \widehat{\varsigma}_{p_1 q_1} \cdots \widehat{\varsigma}_{p_k q_k} w_3 w_4^{-1}) w_4
\]
\[
\sim_O (w \widehat{\alpha}_{u_{1} v_{1}} \cdots \widehat{\alpha}_{u_{v} v_{t}}) w_4
\]
\[
\sim_O w w_4
\]
\[
\sim_O w_2
\]
The last paragraph of the proof of Proposition 5.21 works unchanged to show that the presentation of $\mathfrak{F}_n^*$ obtained in the previous paragraph is equivalent to the presentation in the statement of the proposition.

As our first application, we will derive the presentation of $\mathfrak{F}_n^*$ given by FitzGerald in [32]. Put $X_{\mathfrak{F}} = \{\sigma_1, \ldots, \sigma_{n-1}, e\} = X_B \cup \{e\}$, and let $R_{\mathfrak{F}}$ be the set of relations (B1—B2), (FB1—FB4), and

$$\sigma_i^2 = 1 \quad \text{for all } i.$$  \hfill (O)

By Theorem 5.42 and Proposition 5.59 we have the following.

**Theorem 5.60** (FitzGerald [32]). The monoid $\mathfrak{F}_n^*$ has presentation $\langle X_{\mathfrak{F}} | R_{\mathfrak{F}} \rangle$ via

$$\sigma_i^{\pm 1} \mapsto s_i \quad \text{for each } i$$

$$e \mapsto \text{id}_{\mathfrak{F}}.$$  \hfill $\Box$

We may also use the presentation of $\mathfrak{F}\mathcal{B}_n$ in Theorem 5.48 to obtain another presentation of $\mathfrak{F}_n^*$. Let $X'_{\mathfrak{F}} = X_B \cup \{e_1, \ldots, e_{n-1}\}$, and denote by $R'_{\mathfrak{F}}$ the set of relations (B1—B2), (FB1—FB4)', and (O). Then by Theorem 5.48 and Proposition 5.59 we have the following.

**Theorem 5.61.** The monoid $\mathfrak{F}_n^*$ has presentation $\langle X'_{\mathfrak{F}} | R'_{\mathfrak{F}} \rangle$ via

$$\sigma_i \mapsto s_i \quad \text{for each } i$$

$$e_i \mapsto \text{id}_{\mathfrak{F}_n}.$$  \hfill $\Box$

### 5.3. The Permeable Braid Monoid

We now consider the permeable braid monoid $\mathfrak{PB}_n$. We follow the same method as in Section 5.2 to find a number of presentations of $\mathfrak{PB}_n$. In fact, many of the calculations required have already been carried out.

#### 5.3.1. A Presentation of $\mathfrak{PB}_n$

We will use the notation of Sections 4.3 and 5.2. Let $R_\approx$ be the set of relations

$$\varepsilon_{ij} \beta^{-1} \xi_{ij}^2 \beta = \varepsilon_{ij} \quad \text{for all } 1 \leq i < j \leq n \text{ and } \beta \in \mathcal{P}_n.$$  \hfill ($\approx$)
Now $\mathfrak{PB}_n$ satisfies the condition
\[
B^{(2)}_E = \bigvee_{(i,j) \in \mathcal{E}} B^{(2)}_{E_{ij}} \quad \text{for all } \mathcal{E} \in \mathcal{E} \quad (B^{(2)}_E)''
\]
by definition. Thus, by Theorem 3.17 and Lemma 4.18, we have the following.

**Corollary 5.62.** The permeable braid monoid $\mathfrak{PB}_n$ has presentation
\[
\langle X_B \cup X_B^{-1} \cup X_E \mid R_B \cup R_F \cup R_E \cup R_\infty \cup R_\approx \rangle
\]
via
\[
\sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1} \quad \text{for each } i
\]
\[
\varepsilon_{ij} \mapsto [1] \varepsilon_{ij} \quad \text{for each } i, j. \quad \square
\]

We now work towards simplifying this presentation. Let $e = \varepsilon_{12}$. By $R_F \cup R_\infty$ again we see that for any $1 \leq i < j \leq n$, the relation
\[
\varepsilon_{ij} = (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1})(\sigma_j^{-1} \cdots \sigma_{j-1}^{-1})e(\sigma_2 \cdots \sigma_j)(\sigma_1 \cdots \sigma_{i-1})
\]
is in $(R_B \cup R_F \cup R_E \cup R_\infty \cup R_\approx)^2$. As before, we remove the generators $\varepsilon_{ij}$, replacing their every occurrence in the relations by the word on the right hand side of ($\ast$), which we denote by $e_{ij}$. We denote the resulting relations by $(E_1|E_3)'$, $(\times)'$, and $(\approx)'$. The entire sets of relations which have been modified in this way will be denoted by $R'_E$, $R'_\times$, and $R'_\approx$. Put $X_{PB} = X_B \cup X_B^{-1} \cup \{e\}$.

**Corollary 5.63.** The permeable braid monoid $\mathfrak{PB}_n$ has presentation
\[
\langle X_{PB} \mid R_B \cup R_F \cup R'_E \cup R'_\times \cup R'_\approx \rangle
\]
via
\[
\sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1} \quad \text{for each } i
\]
\[
e \mapsto [1] \varepsilon_{12}. \quad \square
\]

Recall that $R_{EB}$ is the set of relations (F), (B1—B2), and (EB1—EB5); see Lemma 5.30.

In Section 5.2.1, we showed that $(R_B \cup R_F \cup R'_E \cup R'_\times)^2 = R''_{EB}$; see the comments after the proof of Lemma 5.37. In particular we have $(R_B \cup R_F \cup R'_E \cup R'_\times \cup R'_\approx)^2 = (R_{EB} \cup R'_\approx)^2$. Now let $R''_{EB}$ be the set of relations obtained from $R_{EB}$ by replacing (EB1) by
\[
e^2 = e = e\sigma_i^2. \quad (EB1)''
Since the relation $e = e\sigma_1^2$ is in $R''_E$, we have $(R''_{EB} \cup R''_E)^2 = (R_{EB} \cup R''_E)^2$. Write $\sim_{PB}$ for the congruence $(R''_{EB})^\dagger$. The next lemma may be proved in an almost identical fashion to Lemma 5.39.

**Lemma 5.64.** Suppose that $\beta \in \mathcal{P}_n$ and $1 \leq i < j \leq n$. Then $e_{ij}\beta^{-1}s_{ij}^2\beta \sim_{PB} e_{ij}$. \hfill \Box

It now follows that $R''_E \subseteq (R''_{EB})^\dagger$ and we have the following.

**Theorem 5.65.** The permeable braid monoid $\mathcal{PB}_n$ has presentation $\langle X_{PB} \mid R''_{EB} \rangle$ via

$$
\begin{align*}
\sigma_i^{\pm 1} &\mapsto s_i^{\pm 1} \quad \text{for each } i \\
e &\mapsto [1]s_{12}.
\end{align*}
$$

We now work towards finding a set of relations involving only words over $X_B \cup \{e\}$ which, together with $R_B \cup R_F$, generate $(R''_{EB})^\dagger$.

**Lemma 5.66.** The following relations are in $(R''_{EB})^2$:

$$
\begin{align*}
e^2 &= e = e\sigma_1^2 \quad \text{(PB1)} \\
e\sigma_i &= \sigma_ie \quad \text{if } i \neq 2 \quad \text{(PB2)} \\
e\sigma_2e\sigma_2 &= \sigma_2e\sigma_2e = \sigma_1\sigma_2e\sigma_2\sigma_1e \quad \text{(PB3)} \\
e\sigma_2\sigma_1\sigma_2e\sigma_2\sigma_3\sigma_1\sigma_2 &= \sigma_2\sigma_3\sigma_1\sigma_2e\sigma_2\sigma_3\sigma_1\sigma_2e \quad \text{(PB4)} \\
e\sigma_2^2 &= \sigma_2^2e. \quad \text{(PB5)}
\end{align*}
$$

**Proof.** Now relations (PB1), (PB2), and (PB5) are in $R''_{EB}$. It was shown in the proof of Lemma 5.41 that relation (PB4) and the relation $e\sigma_2e\sigma_2 = \sigma_2e\sigma_2e$ are in $R''_{EB} \subseteq (R''_{EB})^\dagger$. To complete the proof, it suffices to show that

$$(E_{12} \vee E_{12}^{\varsigma_2}, \varsigma_2^2) = (E_{12}, \varsigma_2)^2 \approx (1, \varsigma_1\varsigma_2)(E_{12}, 1)(1, \varsigma_2\varsigma_1)(E_{12}, 1) = (E_{12}^{\varsigma_1\varsigma_2} \vee E_{12}^{\varsigma_1\varsigma_2\varsigma_1}, \varsigma_1\varsigma_2\varsigma_1).$$

Now $E_{12} \vee E_{12}^{\varsigma_2} = E_{12} \vee E_{13} = E_{23} \vee E_{12} = E_{12}^{\varsigma_1\varsigma_2} \vee E_{12}^{\varsigma_1\varsigma_2\varsigma_1}$ and, writing $E = E_{12} \vee E_{13}$, we show in Figure 5.17 that $\varsigma_2^2 \approx_{E} \varsigma_1\varsigma_2\varsigma_2\varsigma_1$.

![Figure 5.17. An $(E, 2)$-homotopy from $\varsigma_2^2$ to $\varsigma_1\varsigma_2\varsigma_2\varsigma_1$.](image-url)
In the diagram, strings 4, . . . , n have been omitted as they play no part in the homotopy, and a grey circle indicates that a pair of strings are about to permeate.

Let $R_{PB}$ denote the set of relations (F), (B1—B2), and (PB1—PB5).

**Theorem 5.67.** The permeable braid monoid $\mathcal{PB}_n$ has presentation $(X_{PB} \mid R_{PB})$ via

$\sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1}$ for each $i$

$e \mapsto [1]_{\zeta_{12}}$.

**Proof.** All that remains to be shown is that relations (EB3) and (EB4) are in $R_{PB}^\sharp$. For the remainder of this proof we denote by $\cong$ the congruence $R_{PB}^\sharp$. Denote also by $\cong' \subseteq \cong$ the congruence on $X_{PB}^*$ generated by (F), (B1—B2), and (PB2—PB5). In the proof of Theorem 5.42, we showed that relation (EB4) is in $\cong'$, and that $\sigma_2^{-1}e\sigma_2e \cong \sigma_2^{-1}e\sigma_2$. To complete the proof, observe first that

$$
\begin{align*}
\sigma_2^{-1}e\sigma_2e &\approx \sigma_2^{-1}\sigma_2^{-1}\sigma_2e\sigma_2e & \text{by (F)} \\
&\approx \sigma_2^{-1}\sigma_2^{-1}\sigma_1\sigma_2e\sigma_2\sigma_1e & \text{by (PB3)} \\
&\approx \sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}e\sigma_2\sigma_1e & \text{by (B2), (F), and (PB1)} \\
&\approx \sigma_1\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_2\sigma_1e & \text{by (B2), (F), and (PB2)} \\
&\approx \sigma_1\sigma_2e\sigma_2\sigma_1e & \text{by (F)} \\
&\approx \sigma_2e\sigma_2e & \text{by (PB3)}.
\end{align*}
$$

Using this observation, and the above-mentioned fact that $\sigma_2^{-1}e\sigma_2e \approx e\sigma_2^{-1}e\sigma_2$, we see that

$$
\begin{align*}
\sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1e &\approx \sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1 & \text{by (PB2)} \\
&\approx \sigma_1^{-1}e\sigma_2^{-1}e\sigma_2\sigma_1 & \text{by (PB2)} \\
&\approx e\sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1 & \text{by (PB2) and (F)} \\
&\approx e\sigma_1\sigma_2^{-1}e\sigma_2\sigma_1 & \text{by (PB1) and (F)} \\
&\approx \sigma_1\sigma_2^{-1}e\sigma_2\sigma_1 & \text{by (PB2)} \\
&\approx \sigma_1\sigma_2e\sigma_2\sigma_1 & \text{by (PB2)} \\
&\approx \sigma_1\sigma_2e\sigma_2\sigma_1e & \text{by (PB2)} \\
&\approx \sigma_2e\sigma_2e & \text{by (PB3)} \\
&\approx \sigma_2^{-1}e\sigma_2e,
\end{align*}
$$

showing that (EB3) is in $R_{PB}^\sharp$. $\square$
5.3. THE PERMEABLE BRAID MONOID

5.3.2. Another Presentation of $\mathfrak{PB}_n$.

We now rename $e = e_1$ and add generators $e_2, \ldots, e_{n-1}$ to the presentation of $\mathfrak{PB}_n$ in Theorem 5.67, along with the relations

$$e_i = e_{i,i+1} \quad \text{for all } i$$

which define them in terms of the original generators. Put $X'_{PB} = X_B \cup X_B^{-1} \cup \{e_1, \ldots, e_{n-1}\}$, and denote by $\congruence$ the congruence on $(X'_{PB})^*$ generated by $R_{PB}$ and (D).

Lemma 5.68. The following relations are in $\congruence$:

- $e_i^2 = e_i = e_i\sigma_i^2$ for all $i$ (PB1)$'$
- $e_i e_j = e_j e_i$ for all $i, j$ (PB2)$'$
- $e_i \sigma_j = \sigma_j e_i$ if $|i - j| \neq 1$ (PB3)$'$
- $e_i \sigma_j \sigma_i = \sigma_j \sigma_i e_j$ if $|i - j| = 1$ (PB4)$'$
- $e_i \sigma_j^2 = \sigma_j^2 e_i$ if $|i - j| = 1$ (PB5)$'$
- $e_i \sigma_j e_j = e_j \sigma_j e_i$ if $|i - j| = 1$. (PB6)$'$

Proof. Again the relations may be established by drawing pictures. Figure 5.18 below shows that relation (PB6)$'$ holds in the case $j = i + 1$.

![Figure 5.18. Relation (PB6)$'$: $xyz = yzx$ where $x = [1]_{e_{i,i+1}}$, $y = \varsigma_j$, and $z = [1]_{\varsigma_{j,j+1}}$ in the case $j = i + 1$.](image)

Again we have omitted all but the $i$th, $(i + 1)$th, and $(i + 2)$th strings. \hfill $\square$

Thus we may add relations (PB1—PB6)$'$ to the presentation. Let $R'_{PB}$ be the set of relations (F), (B1—B2), and (PB1—PB6)$'$, and write $\sim_{PB'} = (R'_{PB})^2$. Our aim is to show that $\congruence = \sim_{PB'}$. Now relations (PB1), (PB2), and (PB5) may be removed since they are part of relations (PB1)$'$, (PB3)$'$, and (PB5)$'$. The proofs of Lemmas 5.46 and 5.47 work unchanged to show that relations (PB4) and (D) are in $(R'_{PB})^2$. Thus we may remove relations (PB4) and (D) from the presentation. Also, the proof of Lemma 5.45 works unchanged to show that $e_1 \sigma_2 e_1 \sigma_2 \sim_{PB'} \sigma_2 e_1 \sigma_2 e_1$. 

5.3. THE PERMEABLE BRAID MONOID

Theorem 5.69. The permeable braid monoid \( \mathcal{PB}_n \) has presentation \( \langle X_{\mathcal{PB}}^i | R_{\mathcal{PB}} \rangle \) via

\[
\sigma_i^\pm 1 \mapsto \varepsilon_i^\pm 1 \quad \text{for each } i \\
e_i \mapsto [1]e_{i,i+1} \quad \text{for each } i.
\]

Proof. It remains only to show that \( \sigma_2 e_1 \sigma_2 e_1 \sim_{\mathcal{PB}} \sigma_1 \sigma_2 e_1 \sigma_2 \sigma_1 e_1 \). To do this, we will show that both expressions are \( \sim_{\mathcal{PB}} \)-equivalent to \( e_1 e_2 \). Now in the proof of Lemma 5.45 it was shown that

\[
\sigma_i \sigma_j \sigma_i^{-1} \sim_{\mathcal{PB}} \sigma_j^{-1} \sigma_i \sigma_j
\]

whenever \( |i - j| = 1 \). Using this, and the fact that \( e_1 \sigma_2 e_1 \sigma_2 \sim_{\mathcal{PB}} e_1 \sigma_2 e_1 \), we have

\[
\sigma_2 e_1 \sigma_2 e_1 \sim_{\mathcal{PB}} e_1 \sigma_2 e_1 \\
\sim_{\mathcal{PB}} e_1 \sigma_2 e_1 \sigma_1^{-1} \sigma_2 
\text{ } \text{ by (F)}
\]

\[
\sim_{\mathcal{PB}} e_1 \sigma_1^{-1} e_2 \sigma_1 \sigma_2 
\text{ } \text{ by (PB1)' and (F)}
\]

\[
\sim_{\mathcal{PB}} e_1 \sigma_1 e_2 \sigma_1 \sigma_2 
\text{ } \text{ by (PB6)'}
\]

\[
\sim_{\mathcal{PB}} e_2 \sigma_1 e_1 \sigma_1 \sigma_2 
\text{ } \text{ by (PB3)'}
\]

\[
\sim_{\mathcal{PB}} e_2 e_1 \sigma_2 
\text{ } \text{ by (PB1)'}
\]

\[
\sim_{\mathcal{PB}} e_1 e_2 \sigma_2 
\text{ } \text{ by (PB2)'}
\]

\[
\sim_{\mathcal{PB}} e_2 e_2 
\text{ } \text{ by (PB1)'}
\]

and

\[
\sigma_1 \sigma_2 e_1 \sigma_2 \sigma_1 e_1 \sim_{\mathcal{PB}} \sigma_1 \sigma_2 e_1 \sigma_2 e_1 \sigma_1 
\text{ } \text{ by (PB3)'}
\]

\[
\sim_{\mathcal{PB}} \sigma_1 e_1 e_2 \sigma_1 
\text{ } \text{ by the previous calculation}
\]

\[
\sim_{\mathcal{PB}} e_1 \sigma_1 e_2 \sigma_1 
\text{ } \text{ by (PB3)'}
\]

\[
\sim_{\mathcal{PB}} e_2 \sigma_1 e_1 \sigma_1 
\text{ } \text{ by (PB6)'}
\]

\[
\sim_{\mathcal{PB}} e_2 e_1 \sigma_1 \sigma_1 
\text{ } \text{ by (PB3)'}
\]

\[
\sim_{\mathcal{PB}} e_2 e_1 
\text{ } \text{ by (PB1)'}
\]

\[
\sim_{\mathcal{PB}} e_1 e_2 
\text{ } \text{ by (PB2)'}
\]

and the proof is complete. \( \square \)

Remark 5.70. The presentations of \( \mathcal{PB}_n \) given in Theorems 5.67 and 5.69 both allow the construction of a positive monoid \( \mathcal{PB}_n^+ \) in a similar way to Remarks 5.43 and 5.49, and a similar, currently unresolved, embedding question may be asked.
CHAPTER 6

Pure Factorisable Inverse Braid Monoids

We have seen that the braid monoids $IB_n$, $F_B^n$, and $P_B^n$ are braid analogues of the transformation semigroups $I_n$, $F^n$, and $S_n$ respectively since there are maps $IB_n \rightarrow I_n$, $F_B^n \rightarrow F^n$, and $P_B^n \rightarrow S_n$ which naturally extend the permutation map $\gamma : B_n \rightarrow S_n$. In this chapter we define pure submonoids of $IB_n$, $F_B^n$, and $P_B^n$. These monoids are designed to be analogues of the pure braid group $P_n$ which is the kernel of the map or, phrased differently, the preimage of $\{1\} = E(S_n)$. Thus we will define submonoids $IP_n \subset IB_n$, $FP_n \subset F_B^n$, and $P_P^n \subset P_B^n$ to be the preimages of $E(I_n)$, $E(F^n)$, and $E(S_n)$ under the above maps. We will see that these monoids decompose as the union of their maximal subgroups, each of which is isomorphic to a pure braid group $P_k$ for some $0 \leq k \leq n$ (in the case of $IP_n$), or a quotient of the pure braid group $P_n$ (in the case of $FP_n$ and $P_P^n$). We obtain presentations of these quotients and semidirect product decompositions analogous to those of the pure braid group. Understanding the structure of these quotients allows us to give algorithms which determine $A$-, $E$-, and $E$-equivalence of braids. Various results of this chapter appear in [23, 24, 27].

We fix a positive integer $n$ for the remainder of this chapter, and we denote by $B = B_n$, $P = P_n$, $T = 2^n$, and $E = Eq_n$ the braid group on $n$, the pure braid group on $n$, the intersection semilattice of subsets of $n$, and the join semilattice of equivalence relations on $n$. Throughout this chapter it will be convenient to identify $IB_n$, $F_B^n$, and $P_B^n$ with the quotients $(T \times B)/\sim$, $(E \times B)/\sim$, and $(E \times B)/\approx$.

Recall that $X_P = \{a_{ij} | 1 \leq i < j \leq n \}$, and that $R_P$ denotes the set of relations

\begin{align*}
    a_{rs}a_{ij}a_{rs}^{-1} &= a_{ij} & \text{if } i < r \text{ or } i > s & \text{(P1)} \\
    a_{rs}a_{sj}a_{rs}^{-1} &= (a_{sj}^{-1}a_{rj})a_{sj}(a_{rj}a_{sj}) & \text{(P2)} \\
    a_{rs}a_{rj}a_{rs}^{-1} &= a_{sj}^{-1}a_{rj}a_{sj} & \text{(P3)} \\
    a_{rs}a_{ij}a_{rs}^{-1} &= (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) & \text{if } r < i < s, \text{ (P4)}
\end{align*}

where in each relation we have $s < j$. Thus, by Theorem 2.13, $P$ has group presentation $\langle X_P \mid R_P \rangle$ via

$F(X_P) \rightarrow P : a_{ij} \mapsto \alpha_{ij}$ for each $i, j$. 99
All presentations in this chapter are group presentations. We denote by $\sim_P$ the congruence $R^2_P$ on $F(X_P)$.

Before moving on, we will provide a presentation of a special family of subgroups of the braid group $\mathcal{B}$. For $\mathcal{E} \in E$ we define a subgroup

$$P_\mathcal{E} = \{ \beta \in B \mid (i, i\beta) \in \mathcal{E} \ (\forall i \in n) \} \subseteq B.$$ 

Suppose now that $\mathcal{E} \in E$ and that $n/\mathcal{E} = \{N_1, \ldots, N_k\}$ where $\min(N_1) < \cdots < \min(N_k)$. Suppose also that $\mathcal{E}$ is convex, by which we mean that $i < j$ whenever $i \in N_r$ and $j \in N_s$ with $r < s$. For $i \in k$ let $N^i_1 = N_i \setminus \{\max(N_i)\}$, and put

$$n^b = N^1_1 \cup \cdots \cup N^k_1 = \{i \in n \mid (i, i+1) \in \mathcal{E}\}.$$

Now put

$$X_{P_\mathcal{E}} = X_P \cup \{\sigma_i \mid i \in n^b\},$$

and let $R_{P_\mathcal{E}} = R_B^r \cup R_P \cup R_D \cup R_C$ where $R_B^r$ is the restriction of the braid relations $R_B$ to $\{\sigma_i \mid i \in n^b\}$, $R_D$ is the set of relations

$$a_{ij} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^{-1}\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \quad \text{if} \ (i, j) \in \mathcal{E} \ \text{and} \ i < j, \quad (D)$$

and $R_C$ is the set of relations

$$\sigma_ra_{ij}\sigma_r^{-1} = \begin{cases} a_{i,j}^{-1}a_{i-1,j}a_{ij} & \text{if} \ r = i - 1 \\ a_{i+1,j} & \text{if} \ r = i < j - 1 \\ a_{ij}^{-1}a_{i,j-1}a_{ij} & \text{if} \ r = j - 1 > i \\ a_{i,j+1} & \text{if} \ r = j \\ a_{ij} & \text{otherwise,} \end{cases} \quad (C)$$

where $1 \leq i < j \leq n$ and $r \in n^b$ for each relation in $R_C$. Our goal is to show that $P_\mathcal{E}$ has presentation $(X_{P_\mathcal{E}} \mid R_{P_\mathcal{E}})$.

**Lemma 6.1.** Suppose that $\beta \in P_\mathcal{E}$. Then $\beta = s_{i_1} \cdots s_{i_\ell}$ for some $i_1, \ldots, i_\ell \in n^b$.

**Proof.** Suppose that $c = (c_1, \ldots, c_r)$ is a cycle from the cycle decomposition of $\beta$. Now $c = t_{c_{i-1}c_i} \cdots t_{c_1c_2}$, and we have $c_1, \ldots, c_r \in N_j$ for some $j \in k$ since $(i, i\beta) \in \mathcal{E}$ for all $i \in n$.

Now for each $i \in \{1, \ldots, r-1\}$ we have

$$t_{c_ic_{i+1}} = \begin{cases} (s_{c_i} \cdots s_{c_{i+1}-2})s_{c_{i+1}-1}(s_{c_{i+1}-2} \cdots s_{c_i}) & \text{if} \ c_i < c_{i+1} \\ (s_{c_{i+1}} \cdots s_{c_2})s_{c_1}(s_{c_2} \cdots s_{c_{i+1}}) & \text{if} \ c_i > c_{i+1}. \end{cases}$$

Each of the subscripts in this expression are in $n^b$ since $(c_i, c_{i+1}) \in \mathcal{E}$ and $\mathcal{E}$ is convex. \qed

**Lemma 6.2.** The subgroup $P_\mathcal{E}$ is generated (as a group) by the set

$$\{\alpha_{ij} \mid 1 \leq i < j \leq n\} \cup \{s_i \mid i \in n^b\}.$$
Proof. Let $Z$ denote the set in the statement of the lemma. We clearly have $Z \subseteq P_E$ so that $\langle Z \rangle \subseteq P_E$. To prove the reverse inclusion, suppose that $\beta \in P_E$. Then by Lemma 6.1 we have $\beta = s_{i_1} \cdots s_{i_\ell}$ for some $i_1, \ldots, i_\ell \in n^\delta$. Putting $\gamma = \varsigma_{i_1} \cdots \varsigma_{i_\ell}$ we have $\beta \gamma^{-1} \in P$, from which it follows that $\beta \gamma^{-1} = \alpha_{p_{i_1}q_{i_1}}^\pm \cdots \alpha_{p_{i_\ell}q_{i_\ell}}^\pm$ for some $p_{i_1}, \ldots, p_{i_\ell}, q_{i_1}, \ldots, q_{i_\ell} \in n$ with $p_j < q_j$ for each $j \in h$. It then follows that $\beta = (\beta \gamma^{-1}) \gamma = \alpha_{p_{i_1}q_{i_1}}^\pm \cdots \alpha_{p_{i_\ell}q_{i_\ell}}^\pm \varsigma_{i_1} \cdots \varsigma_{i_\ell} \in \langle Z \rangle$, completing the proof.

Theorem 6.3. The group $P_E$ has presentation $\langle X_P | R_P \rangle$ via

\[
\begin{align*}
\alpha_{ij} &\mapsto a_{ij} & \text{for each } i, j \\
\varsigma_i &\mapsto \varsigma_i & \text{for each } i.
\end{align*}
\]

Proof. Denote by $\phi : F(X_P) \rightarrow P_E$ the map in the statement of the theorem and write $\cong = R_P^\phi$. One may easily check that $R_P \subseteq \ker \phi$ so that $\cong \subseteq \ker \phi$. To show the reverse inclusion suppose that $w \in F(X_P)$ and $w \phi = 1$. It is enough to show that $w \cong 1$. Now by $R_C$ and the relations in $R_D$ of the form

\[
a_{i,i+1} = \sigma_i^2 \quad \text{for } i \in n^\delta,
\]

we have

\[
w \cong w_1 w_2
\]

for some $w_1 \in F(X_P)$ and $w_2 \in F(\{\varsigma_i | i \in n^\delta\})$. Using the commuting relations from $R_B^\phi$, we have

\[
w_2 \cong u_1 \cdots u_k
\]

where each $u_i$ is some word over $\{\varsigma_j | j \in N^\delta_i\}$. But $\cong \subseteq \ker \phi$ and so

\[
1 = 1 = \underbrace{w \phi}_{\text{w \phi}} = (w_1 w_2) \phi = \underbrace{w_1 \phi}_{\text{w_1 \phi}} \underbrace{w_2 \phi}_{\text{w_2 \phi}} = \underbrace{u_1 \phi}_{\text{u_1 \phi}} \cdots \underbrace{u_k \phi}_{\text{u_k \phi}}.
\]

If $i \in k$, then $\underbrace{u_i \phi}_{\text{u_i \phi}}$ fixes $n \setminus N_i$ pointwise and so $\underbrace{u_k \phi}_{\text{u_k \phi}} = 1$ for each $i$. Thus, using only relations from $R_B^\phi \cup R_D$ which involve subscripts from $N_i$, we see, by Theorems 2.10 and 2.13, that for each $i$ we have $u_i \cong v_i$ for some word $v_i$ over $\{a_{jl} | j, l \in N_i, j < l\}$. Thus,

\[
w \cong w_1 w_2 \cong w_1 u_1 \cdots u_k \cong w_1 v_1 \cdots v_k.
\]

Now $(w_1 v_1 \cdots v_k) \phi = w \phi = 1$ and so, by Theorem 2.13, we have

\[
w_1 v_1 \cdots v_k \sim_P 1
\]

so that $w \cong 1$ and we are done. \qed

Remark 6.4. Digne and Gomi [15] obtained presentations of a special family of subgroups of an arbitrary Artin group; the subgroups $P_E$ of $B$ are a special case. Their techniques (applying the Reidemeister-Schreier method [46]) and presentations differ from ours.
Also of use will be the following two general results concerning group presentations. A proof of the first may be found in \cite{46}.

**Lemma 6.5.** Suppose that $G$ is a group with presentation $\langle X \mid R \rangle$ via $f$. Suppose also that $W \subseteq F(X)$ is a set of words such that $N$ is the normal closure of $Wf$ in $G$. Then $G/N$ has presentation $\langle X \mid R \cup R_W \rangle$ via $f' : F(X) \to G/N : w \mapsto N(wf)$, where $R_W$ is the set of relations

$$w = 1 \quad \text{for all } w \in W.$$ \hfill $\Box$

**Lemma 6.6.** Let $X$ and $Y$ be two disjoint sets, and define $\sim : F(X \cup Y) \to F(X)$ by $\bar{x} = x$ and $\bar{y} = 1$ for each $x \in X$ and $y \in Y$. Suppose that $G$ is a group with presentation $\langle X \cup Y \mid R \cup S \rangle$ via $f$, where $R \subseteq F(X) \times F(X)$ and $\overline{S} \subseteq R^2$. Then $H = (F(X))f$ has presentation $\langle X \mid R \rangle$ via $\phi = f|_{F(X)} : F(X) \to H : w \mapsto wf$.

**Proof.** Write $\sim_R = R^2$. Now by assumption we know that $\phi$ is an epimorphism, and that $\sim_R \subseteq \ker \phi$. To prove the reverse inclusion, suppose that $(w, w') \in \ker \phi$. Then there is a sequence of words $w = w_1, w_2, \ldots, w_k = w' \in F(X \cup Y)$ such that, for each $i$, $w_i = x_iy_i$ and $w_{i+1} = x_iy_i$ for some $x_i, y_i \in F(X \cup Y)$ and $(u, v) \in R \cup S$. Since $\bar{w} = w$ and $\bar{w}' = w'$, the result will follow if we can show that $\bar{w}_i \sim_R \bar{w}_{i+1}$ for each $i$. Now if $(u, v) \in R$ then

$$\bar{w}_{i+1} = \bar{x}_i \bar{u} \bar{y}_i \sim_R \bar{x}_i \bar{v} \bar{y}_i = \bar{w}_{i+1},$$

while if $(u, v) \in S$, then $\bar{u} \sim_R \bar{v}$ by assumption, so that

$$\bar{w}_i = \bar{x}_i \bar{u} \bar{y}_i \sim_R \bar{x}_i \bar{v} \bar{y}_i = \bar{w}_{i+1},$$

completing the proof. \hfill $\Box$

### 6.1. The Pure Inverse Braid Monoid

In this section we define the pure inverse braid monoid $\mathcal{IP}_n$. We show that $\mathcal{IP}_n$ is the union of its maximal subgroups each of which is isomorphic to a pure braid group $\mathcal{P}_k$ for some $0 \leq k \leq n$. This leads to an algorithm which determines $\sim_A$-equivalence of braids.

In Section 4.1.1 we defined an epimorphism $\sim : \mathcal{IB}_n \to \mathcal{I}_n$. Under the identification of $\mathcal{IB}_n$ with $(T \rtimes B)/\sim$, we see that $\sim$ maps $[A, \beta]$ to $[\overline{A}, \overline{\beta}] = \overline{\beta} A$ for each $A \in T$ and $\beta \in B$. We now define the **pure inverse braid monoid**

$$\mathcal{IP}_n = \{ [A, \beta] \in \mathcal{IB}_n \mid \overline{A, \beta} = \text{id}_A \} = \{ [A, \beta] \in \mathcal{IB}_n \mid i\overline{\beta} = i \quad (\forall i \in A) \}.$$ 

Identifying $[A, \beta]$ with $\beta_A$, for $A \in T$ and $\beta \in B$, we see that $\mathcal{IP}_n$ is the set of all (homotopy classes of) partial braids whose associated partial permutation is the identity map on its domain.
Suppose that $A \in T$. We define a subgroup

$$P_A = \{ \beta \in B \mid i \beta = i \ (\forall i \in A) \} = \{ \beta \in B \mid [A, \beta] \in \mathcal{IP}_n \} \subseteq B,$$

and write $[A, P_A] = \{ [A, \beta] \mid \beta \in P_A \}$. It is easy to check that the map

$$P_A \to \mathcal{IP}_n : \beta \mapsto [A, \beta]$$

is a group homomorphism with kernel $B_A$ and image $[A, P_A]$. In particular, $[A, P_A]$ is a group and is isomorphic to the quotient $P_A/B_A$. Thus

$$\mathcal{IP}_n = \bigsqcup_{A \in T} [A, P_A]$$

is the disjoint union of the subgroups $[A, P_A]$ which are therefore its maximal subgroups.

It is these subgroups, or rather the corresponding quotients $P_A/B_A$, which we will study in order to understand the structure of $\mathcal{IP}_n$.

Suppose now that $A \in T$ with $|A| = k$. Choose $\gamma \in B$ such that $A\gamma = k$. It is easy to check that the map

$$[A, B_A] \to [k, B_k] : [A, \beta] \mapsto [k, \gamma\beta\gamma^{-1}] \quad \text{for each } \beta \in B_A$$

is an isomorphism. Thus we will have a good understanding of the structure of $\mathcal{IP}_n$ if we can understand the quotients $P_k/B_k$ for $0 \leq k \leq n$.

For the rest of this section we fix $0 \leq k \leq n$. Define an equivalence

$$\mathcal{E}_k = \{(i, j) \mid i = j \text{ or } i, j > k \} \in E.$$

We see then that $P_k = P_{\mathcal{E}_k}$ in the notation introduced at the beginning of this chapter. Put $X_k = X_{P_{\mathcal{E}_k}} = X_P \cup \{ \sigma_i \mid i > k \}$ and let $R_k = R_{P_{\mathcal{E}_k}} = R_B^r \cup R_P \cup R_D \cup R_C$ where $R_B^r$ is the restriction of $R_B$ to $\{ \sigma_i \mid i > k \}$, $R_D$ is the set of relations

$$a_{ij} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}) \quad \text{if } k < i < j \leq n,$$

and $R_C$ is the set of relations

$$\sigma_r a_{ij} \sigma_r^{-1} = \begin{cases} a_{ij} & \text{if } r = i - 1 \\ a_{i+1, j} & \text{if } r = i < j - 1 \\ a_{j, i-1} a_{ij} & \text{if } r = j - 1 > i \\ a_{i, j+1} & \text{if } r = j \\ a_{ij} & \text{otherwise,} \end{cases}$$

where $1 \leq i < j \leq n$ and $r > k$ for each relation in $R_C$. By Theorem 6.3 we have the following.
Lemma 6.7. The subgroup $P_k$ has presentation $\langle X_k \mid R_k \rangle$ via

\[
\begin{align*}
& a_{ij} \mapsto \alpha_{ij} \quad \text{for each } i, j \\
& \sigma_i \mapsto \varsigma_i \quad \text{for each } i.
\end{align*}
\]

\[\Box\]

Lemma 6.8. The subgroup $B_k$ is the normal closure in $P_k$ of the set

\[\{\varsigma_i \mid i > k\} \cup \{\alpha_{ij} \mid 1 \leq i < j \leq n, \; j > k\} \]

Proof. Let $Z$ be the set in the statement of the lemma. Since $Z \subseteq B_k$ and $B_k$ is normal in $P_k$, we see that the normal closure of $Z$ is contained in $B_k$. Conversely, by Lemma 4.6, we know that $B_k$ is generated by elements of the form

\[
\begin{align*}
& \beta^{-1} \varsigma_{ij} \beta \quad \text{with } \beta \in P \text{ and } i > k \\
& \beta^{-1} \alpha_{ij} \beta \quad \text{with } \beta \in P \text{ and } j > k.
\end{align*}
\]

Now the generators of the second kind are clearly in the normal closure of $Z$. For the first kind, if $\beta \in P$ and $i > k$, then

\[ \beta^{-1} \varsigma_{ij} \beta = (\varsigma_{i+1} \cdots \varsigma_{j-1} \beta)^{-1} \varsigma_i (\varsigma_{i+1} \cdots \varsigma_{j-1} \beta) \]

and the proof is complete since $\varsigma_{i+1} \cdots \varsigma_{j-1} \beta \in P_k$. \[\Box\]

For $\beta \in P_k$, we will denote by $[\beta]_k$ the coset $B_k \beta$. There is no conflict with our earlier use of this notation in Section 4.1 since, by Theorem 4.4, the set of braids which are $\sim_k$-equivalent to $\beta$ is precisely the coset $B_k \beta$. Let $R_O$ denote the set of relations

\[
\begin{align*}
& a_{ij} = 1 \quad \text{if } j > k \\
& \sigma_i = 1 \quad \text{if } i > k.
\end{align*}
\]

(O1) \quad (O2)

The next transitional result follows immediately from Lemmas 6.5, 6.7, and 6.8.

Corollary 6.9. The quotient $P_k/B_k$ has presentation $\langle X_k \mid R_k \cup R_O \rangle$ via

\[
\begin{align*}
& a_{ij} \mapsto [\alpha_{ij}]_k \quad \text{for each } i, j \\
& \sigma_i \mapsto [\varsigma_i]_k \quad \text{for each } i.
\end{align*}
\]

\[\Box\]

Geometrically, we expect $P_k/B_k$ to be isomorphic to $P_k$, so the presentation in Corollary 6.9 should simplify. Now by $R_O$ we may remove the generators $a_{ij}$ with $j > k$, and $\sigma_i$ with $i > k$, replacing their every occurrence in the relations by 1. Relations $R_O$ may now be removed, along with $R_B^0 \cup R_D \cup R_C$ which are now trivial. We also remove any of relations (P1—P4) which involve a subscript greater than $k$. The remaining generators are
thus \{a_{ij} \mid 1 \leq i < j \leq k\} and the remaining relations are all the relations from \(R_P\) which involve only generators \(a_{ij}\) with \(j \leq k\). By Theorem 2.13 we have the following.

**Theorem 6.10.** The quotient \(P_k/B_k\) is isomorphic to \(\mathcal{P}_k\), the pure braid group on \(k\), via

\[ [\alpha_{ij}]_k \mapsto \alpha_{ij} \quad \text{for each } 1 \leq i < j \leq k. \]

The next result follows immediately.

**Theorem 6.11.** Let \(A = \{x_1, \ldots, x_k\} \subseteq \mathfrak{n}\) with \(x_1 < \cdots < x_k\). Then the quotient \(P_A/B_A\) is isomorphic to \(\mathcal{P}_k\), the pure braid group on \(k\), via

\[ [\alpha_{x_ix_j}]_A \mapsto \alpha_{ij} \quad \text{for each } 1 \leq i < j \leq k. \]

As such, \(\mathcal{IP}_n\) is the union of groups each of which is isomorphic to a pure braid group \(\mathcal{P}_k\) for some \(0 \leq k \leq n\). For each \(k\) there are precisely \(\binom{n}{k}\) such subgroups.

### 6.1.1. Decision Problems.

We now provide an algorithm which will determine, given two braids \(\beta_1, \beta_2 \in B\) and two subsets \(A_1, A_2 \in T\), whether or not \([A_1, \beta_1] = [A_2, \beta_2]\). In particular, this algorithm may also be used to determine equality (\(~\)-equivalence) of partial braids.

**Theorem 6.12.** The problem of deciding whether two elements \([A_1, \beta_1], [A_2, \beta_2] \in \mathcal{IB}_n\) are equal or not is decidable.

**Proof.** Now \([A_1, \beta_1] = [A_2, \beta_2]\) if and only if \(A_1 = A_2\) and \(\beta_1\beta_2^{-1} \in B_{A_1}\). An algorithm to determine whether this is the case follows.

(i) If \(A_1 \neq A_2\) then \([A_1, \beta_1] \neq [A_2, \beta_2]\). If \(A_1 = A_2\) then go to step (ii).

(ii) Choose \(\gamma \in B\) such that \(A_1^\gamma = k\) for some \(0 \leq k \leq n\), and put \(\beta = \gamma \beta_1 \beta_2^{-1} \gamma^{-1}\). Then \(\beta_1\beta_2^{-1} \in B_{A_1}\) if and only if \(\beta \in \gamma B_{A_1} \gamma^{-1} = B_k\). Now if \(\beta \notin P_k\) then \(\beta \notin B_k\), and so \([A_1, \beta_1] \neq [A_2, \beta_2]\). If \(\beta \in P_k\) then go to step (iii).

(iii) Now \(\beta = \alpha_{p_1q_1}^{\pm 1} \cdots \alpha_{p_rq_r}^{\pm 1} s_{i_1} \cdots s_{i_s}\) for some \(p_1, \ldots, p_r, q_1, \ldots, q_r \in \mathfrak{n}\) with \(p_j < q_j\) for each \(j \in r\), and some \(i_1, \ldots, i_s \in \{k + 1, \ldots, n - 1\}\). But then

\[ \beta \sim_k \alpha_{p_1q_1}^{\pm 1} \cdots \alpha_{p_rq_r}^{\pm 1} \]

where \(j_1 < \cdots < j_t\) and \(\{j_1, \ldots, j_t\} = \{j \in r \mid q_j \leq k\}\). Now put

\[ w = a_{p_1q_1}^{\pm 1} \cdots a_{p_rq_r}^{\pm 1} \in F(X_{P_k}). \]
Then, by Theorems 2.13 and 6.10, \( \beta \in B_k \) if and only if \((w,1) \in R_k^x\). This may be decided using any solution to Problem 2.15.

6.2. The Pure Factorisable Braid Monoid

In this section we define the pure factorisable braid monoid \( \mathfrak{FP}_n \). We show that \( \mathfrak{FP}_n \) is the union of its maximal subgroups each of which is isomorphic to a quotient of a certain subgroup of the braid group. In the next chapter we will show (Lemma 7.7) that these maximal subgroups are isomorphic to quotients of the pure braid group, a fact which is also implicit in Theorem 6.22 below. We find presentations of these quotients and obtain a semidirect product decomposition theorem similar to that of \( P \) given in Theorem 2.14. Again, this leads to an algorithm which determines \( \sim_{\mathcal{E}} \)-equivalence of braids.

In Section 4.2.1 we defined an epimorphism \( \mathfrak{F}B_n \to \mathfrak{F}^* \). Under the identification of \( \mathfrak{F}B_n \) with \( (E \times B)/\sim \), we see that \( \gamma \) maps \([E, \beta] = \text{id}_E \beta \) for each \( E \in E \) and \( \beta \in B \). We now define the pure factorisable braid monoid

\[
\mathfrak{FP}_n = \left\{ [E, \beta] \in \mathfrak{F}B_n \mid [E, \beta] \text{ id}_E \beta \right\} = \left\{ [E, \beta] \in \mathfrak{F}B_n \mid (i, i\beta) \in E \ (\forall i \in n) \right\}.
\]

Recall that for \( E \in E \) we defined a subgroup

\[
P_E = \left\{ \beta \in B \mid (i, i\beta) \in E \ (\forall i \in n) \right\} \subseteq B.
\]

We write \( [E, P_E] = \{ [E, \beta] \mid \beta \in P_E \} \). It is easy to check that the map

\[
P_E \to \mathfrak{FP}_n : \beta \mapsto [E, \beta]
\]

is a group homomorphism with kernel \( B_E \) and image \( [E, P_E] \). In particular, \([E, P_E]\) is a group and is isomorphic to the quotient \( P_E/B_E \). Thus

\[
\mathfrak{FP}_n = \bigsqcup_{E \in E} [E, P_E]
\]

is the disjoint union of the subgroups \([E, P_E]\) which are therefore its maximal subgroups. It is these subgroups, or rather the corresponding quotients \( P_E/B_E \), which we will study in order to understand the structure of \( \mathfrak{FP}_n \).

We say that two equivalences \( E_1, E_2 \in E \) are conjugate if \( E_1^\beta = E_2 \) for some \( \beta \in B \).

**Lemma 6.13.** If \( E_1, E_2 \in E \) are conjugate, then \([E_1, P_{E_1}] \cong [E_2, P_{E_2}]\).

**Proof.** If \( E_1^\beta = E_2 \) for some \( \beta \in B \), then it is easy to check that the map

\[
[E_1, \gamma] \mapsto [E_2, \beta \gamma \beta^{-1}] \quad \text{for each } \gamma \in B
\]

defines a group isomorphism \([E_1, P_{E_1}] \to [E_2, P_{E_2}]\). \( \square \)
Suppose now that $\mathcal{E} \in E$ and that $\mathbf{n}/\mathcal{E} = \{N_1, \ldots, N_k\}$ with $\min(N_1) < \cdots < \min(N_k)$. Put $\lambda_i = |N_i|$ for each $i$. Recall that $\mathcal{E}$ is convex if $r < s$ whenever $r \in N_i$ and $s \in N_j$ with $1 \leq i < j \leq k$. If $\mathcal{E}$ is convex, then we say that $\mathcal{E}$ is standard if we also have $\lambda_1 \leq \cdots \leq \lambda_k$. Note that every equivalence $\mathcal{E} \in E$ is conjugate to a (unique) standard equivalence $\mathcal{E}'$ and so we have $P_{\mathcal{E}}/B_{\mathcal{E}} \cong P_{\mathcal{E}'}/B_{\mathcal{E}'}$. From now on we fix $\mathcal{E} \in E$, a standard equivalence, with the $N_i$ and $\lambda_i$ as defined above. The remainder of this section will be devoted to analysing the structure of the quotient $P_{\mathcal{E}}/B_{\mathcal{E}}$. Recall that for $i \leq j$ we write $N_i[N_j] = N_i \max(\{N_i\})$, and $N^\circ = N^h_1 \cup \cdots \cup N^h_k$.

Lemma 6.14. The subgroup $B_{\mathcal{E}}$ is the normal closure in $P_{\mathcal{E}}$ of the set $\{s_i \mid i \in \mathbf{n}^h\}$.

Proof. Since $\{s_i \mid i \in \mathbf{n}^h\} \subseteq B_{\mathcal{E}}$ and $B_{\mathcal{E}}$ is normal in $P_{\mathcal{E}}$, we see that the normal closure of $\{s_i \mid i \in \mathbf{n}^h\}$ is contained in $B_{\mathcal{E}}$. Conversely, by Lemma 4.10 we know that $B_{\mathcal{E}}$ is generated by elements of the form $\beta^{-1}s_{ij}\beta$ with $\beta \in P$, $(i, j) \in \mathcal{E}$, and $i < j$. In particular, we have $i, i + 1, \ldots, j - 1 \in \mathbf{n}^h$. Now

$$\beta^{-1}s_{ij}\beta = (s_{i+1} \cdots s_{j-1}\beta)^{-1}s_i(s_{i+1} \cdots s_{j-1}\beta),$$

and we are done since $s_{i+1} \cdots s_{j-1}\beta \in P_{\mathcal{E}}$. \hfill \Box

For $\beta \in B$ we will denote by $[\beta]_{\mathcal{E}}$ the coset $B_{\mathcal{E}}\beta$. There is no conflict with our earlier use of this notation since, by Theorem 4.8, $B_{\mathcal{E}}\beta$ is the set of all braids which are $\sim_{\mathcal{E}}$-equivalent to $\beta$.

Now let $R_O$ denote the set of relations

$$\sigma_i = 1 \quad \text{for each } i \in \mathbf{n}^h.$$  \hfill (O)

There should be no confusion with our earlier use of the notation $R_O$. By Theorem 6.3 and Lemmas 6.5 and 6.14 we have the following transitional result.

Corollary 6.15. The quotient $P_{\mathcal{E}}/B_{\mathcal{E}}$ has presentation $\langle X_{P_{\mathcal{E}}} \mid R_{P_{\mathcal{E}}} \cup R_O \rangle$ via

$$a_{ij} \mapsto [a_{ij}]_{\mathcal{E}} \quad \text{for all } i, j$$

$$\sigma_i \mapsto [s_i]_{\mathcal{E}} \quad \text{for all } i. \hfill \Box$$

We now examine the manner in which this presentation simplifies. For the remainder of this section we denote by $\equiv$ the congruence $(R_{P_{\mathcal{E}}} \cup R_O)^\circ$ on $F(X_{P_{\mathcal{E}}}^\circ)$.

Lemma 6.16. If $(r, s) \in \mathcal{E}$ with $r < s$, then

(i) $a_{ir} \equiv a_{is}$ for all $1 \leq i < r$;

(ii) $a_{sj} \equiv a_{sj}$ for all $s < j \leq n$.  

\hfill \Box
**Proof.** To prove (i), notice that since $E$ is convex we must have $r, \ldots, s - 1 \in n^+$ and it follows, by (O) and (C), that

$$a_{ir} \approx (\sigma_{s-1} \cdots \sigma_r)a_{ir}(\sigma_r^{-1} \cdots \sigma_{s-1}^{-1}) \approx a_{is}.$$ 

Statement (ii) is proved in an almost identical manner.

**Corollary 6.17.** If $(i, j), (r, s) \in E$ with $i < r$ and $j < s$, then $a_{ir} \approx a_{js}$.

**Proof.** Using the previous lemma we have $a_{ir} \approx a_{js}$ if $j < r$, and $a_{ir} \approx a_{is} \approx a_{js}$ if $r \leq j$.

**Lemma 6.18.** If $(s, j) \in E$ with $s < j$, then

$$a_{ij}a_{rs} \approx a_{rs}a_{ij} \quad \text{for all } 1 \leq i < j \text{ and } 1 \leq r < s.$$ 

**Proof.** First observe that by (O) and (D), we also have $a_{pq} \approx 1$ if $(p, q) \in E$ with $p < q$. Now if $i < r$ or $i > s$, then the commuting relation already exists by (P1). If $i = s$, then $a_{ij} = a_{sj} \approx 1$ by the observation, and the relation follows trivially. If $i = r$, then by (P3) and the observation we have $a_{rs}a_{rj}a_{rs}^{-1} \approx a_{sj}^{-1}a_{rj}a_{sj} \approx a_{rj}$. If $r < i < s$, then using (P4) and the observation again, we have

$$a_{rs}a_{ij}a_{rs}^{-1} \approx (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) \approx (a_{rj}^{-1}a_{rj})a_{ij}(a_{rj}^{-1}a_{rj}) \approx a_{ij},$$

and the proof is complete.

**Corollary 6.19.** If $1 \leq i < r < j$ and $j \in N_\ell$ for some $\ell \in k$ with $\lambda_\ell > 1$, then

$$a_{ij}a_{rj} \approx a_{rj}a_{ij}.$$ 

**Proof.** Choose $s \in N_\ell \setminus \{j\}$. If $r < s$, then by Lemmas 6.16(i) and 6.18 we have $a_{ij}a_{rj} \approx a_{ij}a_{rs} \approx a_{rs}a_{ij} \approx a_{rj}a_{ij}$. If $s \leq r$, then since $E$ is convex we must have $(r, j) \in E$ so that $a_{rj} \approx 1$ and the commuting relation follows trivially.

**Corollary 6.20.** If $1 \leq i < j$, $1 \leq r < s < j$, and $j \in N_\ell$ for some $\ell \in k$ with $\lambda_\ell > 1$, then

$$a_{rs}a_{ij}a_{rs}^{-1} \approx a_{ij}.$$ 

**Proof.** By $R_P$, we have $a_{rs}a_{ij}a_{rs}^{-1} \approx w_{aij}w^{-1}$ for some word $w$ in the $a_{hj}^{\pm 1}$. By Corollary 6.19, we have $w_{aij}w^{-1} \approx a_{ij}ww^{-1} \approx a_{ij}$ and we are done.

For $i \in k$ let $\mu_i = \min(N_i)$, and denote by $k_0 \in k$ the index such that

- $\lambda_j = 1$ for all $1 \leq j \leq k_0$; and
- $\lambda_j > 1$ for all $k_0 < j \leq k$. 


Notice that $\mu_j = j$ if $1 \leq j \leq k_0 + 1$, while $\mu_j > j$ if $k_0 + 1 < j \leq k$.

**Corollary 6.21.** If $1 \leq r < s \leq k$, $1 \leq i < j \leq k$, $s < j$, and $j > k_0$, then

$$a_{\mu_r \mu_i} a_{\mu_i \mu_j} a_{\mu_j \mu_s}^{-1} \cong a_{\mu_i \mu_j}.$$  

We now return to the task of simplifying the presentation in Corollary 6.15. By Corollary 6.17, and the observation in the proof of Lemma 6.18, we have

$$a_{rs} \cong \begin{cases} 1 & \text{if } (r, s) \in \mathcal{E} \text{ and } r < s \\ a_{\mu_i \mu_j} & \text{if } r \in N_i \text{ and } s \in N_j \text{ with } 1 \leq i < j \leq k. \end{cases}$$

So we remove all generators $a_{rs} \in X_{P_{\mathcal{E}}}$ unless $r = \mu_i$ and $s = \mu_j$ for some $i, j$. We replace any occurrence of $a_{rs}^{\pm 1}$ in the relations by $a_{\mu_i \mu_j}^{\pm 1}$ if $r \in N_i$ and $s \in N_j$ with $i \neq j$, or by 1 if $(r, s) \in \mathcal{E}$. By (O) we may remove each $\sigma_i \in X_{P_{\mathcal{E}}}$ with $i \in \mathbb{N}_0^+$, replacing every occurrence of $\sigma_i^{\pm 1}$ in the relations by 1. We also remove relations $R_O \cup R_B \cup R_D \cup R_C$ which are now trivial. In the presence of the relations stated in Corollary 6.21, we may also remove any relations from $R_P$ which involve a subscript larger than $k_0$.

Put $X_{\mathcal{E}} = \{a_{\mu_i \mu_j} \mid 1 \leq i < j \leq k\}$, and let $R_{\mathcal{E}}$ be the set of relations

$$a_{rs} a_{ij} a_{rs}^{-1} = a_{ij} \quad \text{if } i < r \text{ or } i > s \quad (R_{\mathcal{E}}1)$$

$$a_{rs} a_{sj} a_{rs}^{-1} = (a_{sj}^{-1} a_{rj}^{-1}) a_{sj} (a_{rj} a_{sj}) \quad (R_{\mathcal{E}}2)$$

$$a_{rs} a_{rj} a_{rs}^{-1} = a_{sj}^{-1} a_{rj} a_{sj} \quad (R_{\mathcal{E}}3)$$

$$a_{rs} a_{ij} a_{rs}^{-1} = (a_{sj}^{-1} a_{rj}^{-1} a_{sj} a_{rj}) a_{ij} (a_{rj}^{-1} a_{sj}^{-1} a_{rj} a_{sj}) \quad \text{if } r < i < s, \quad (R_{\mathcal{E}}4)$$

with $1 \leq r < s \leq k_0$, $1 \leq i < j \leq k_0$, and $s < j$ in each case, together with

$$a_{\mu_r \mu_i} a_{\mu_i \mu_j} a_{\mu_j \mu_s}^{-1} = a_{\mu_i \mu_j} \quad \text{if } j > k_0 \quad (R_{\mathcal{E}}5)$$

with $1 \leq r < s \leq k$, $1 \leq i < j \leq k$, and $s \leq j$ in each case. By the arguments in the previous paragraph we have the following.

**Theorem 6.22.** The quotient $P_{\mathcal{E}} / B_{\mathcal{E}}$ has presentation $\langle X_{\mathcal{E}} \mid R_{\mathcal{E}} \rangle$ via

$$a_{\mu_i \mu_j} \mapsto [a_{\mu_i \mu_j}]_{\mathcal{E}} \quad \text{for each } 1 \leq i < j \leq k. \qed$$

For $2 \leq \ell \leq k$ let $U_{N_{\ell}}$ be the subgroup of $P_{\mathcal{E}} / B_{\mathcal{E}}$ generated by $\{[a_{\mu_i \mu_j}]_{\mathcal{E}} \mid 1 \leq i < \ell\}$, and let $(P_{\mathcal{E}} / B_{\mathcal{E}})^{\prime}$ be the subgroup generated by $\{[a_{\mu_i \mu_j}]_{\mathcal{E}} \mid 1 \leq i < j \leq k - 1\}$.

**Lemma 6.23.** We have the semidirect product decomposition $P_{\mathcal{E}} / B_{\mathcal{E}} = U_{N_k} \rtimes (P_{\mathcal{E}} / B_{\mathcal{E}})^{\prime}$. 
Proof. Now if $\lambda_k = 1$ then $P_\xi/B_\xi \cong P$ and the result follows from Theorem 2.14. So suppose that $\lambda_k > 1$. It is immediate from Theorem 6.22 that $U_{N_k}$ is normal (indeed central) in $P_\xi/B_\xi$, and $P_\xi/B_\xi$ is clearly generated by $U_{N_k} \cup (P_\xi/B_\xi)'$. Suppose now that $\beta \in P_\xi$ such that $[\beta]_\xi \in U_{N_k} \cap (P_\xi/B_\xi)'$. Since $[\beta]_\xi \in U_{N_k}$, and since $U_{N_k}$ is commutative by $(R_\xi 5)$, we have

$$[\beta]_\xi = [\alpha_{\mu_1 \mu_k}^{m_1} \cdots \alpha_{\mu_{k-1} \mu_k}^{m_{k-1}}]_\xi$$

for some $m_1, \ldots, m_{k-1} \in \mathbb{Z}$. Now by Theorem 6.22 we see that for each $1 \leq i < j \leq k$ there is a well-defined homomorphism

$$\exp_{ij} : P_\xi/B_\xi \to (\mathbb{Z}, +) : [\alpha_{\mu_{r \mu_s}}]_\xi \mapsto \begin{cases} 1 & \text{if } r = i \text{ and } s = j \\ 0 & \text{otherwise}. \end{cases}$$

Since $[\beta]_\xi \in (P_\xi/B_\xi)'$, we have $m_i = \exp_{ik} ([\beta]_\xi) = 0$ for each $1 \leq i < k$ so that $[\beta]_\xi = [1]_\xi$. This shows that $U_{N_k} \cap (P_\xi/B_\xi)' = \{ [1]_\xi \}$ and completes the proof. \hfill $\square$

Lemma 6.24. Let $E' \in \mathcal{E}_{n-\lambda_k}$ be the equivalence relation such that

$$\{1, \ldots, n-\lambda_k\}/E' = \{N_1, \ldots, N_{k-1}\}.$$ 

Then $(P_\xi/B_\xi)' \cong P_{E'}/B_{E'}$ where here we regard $P_{E'}$ and $B_{E'}$ as subgroups of $B_{n-\lambda_k}$.

Proof. This follows from Lemma 6.6 and Theorem 6.22. \hfill $\square$

The following result is an analogue of Theorem 2.14.

Theorem 6.25. We have the semidirect product decomposition

$$P_\xi/B_\xi = U_{N_k} \rtimes (U_{N_{k-1}} \rtimes (\cdots \rtimes (U_{N_2} \rtimes U_{N_2}) \cdots)).$$

Further,

(i) if $\lambda_i = 1$, then $U_{N_i}$ is a free group of rank $i-1$;

(ii) if $\lambda_i > 1$, then $U_{N_i}$ is a free abelian group of rank $i-1$.

Proof. The semidirect product decomposition follows from Lemmas 6.23 and 6.24 and a simple induction.

(i) If $\lambda_i = 1$, then by Lemma 6.6 and Theorems 2.13 and 6.22, the subgroup of $P_\xi/B_\xi$ generated by $\{ [\alpha_{\mu_{r \mu_s}}]_\xi \mid 1 \leq r < s \leq i \}$ is isomorphic to $P_\xi$ via the map

$$[\alpha_{\mu_{r \mu_s}}]_\xi \mapsto \alpha_{\mu_{r \mu_s}} = \alpha_{rs} \quad \text{for each } 1 \leq r < s \leq i.$$ 

The image of $U_{N_i}$ under this isomorphism is $U_i$, the subgroup of $P_i$ generated by $\alpha_{1i}, \ldots, \alpha_{i-1,i}$ which, by Theorem 2.14, is a free group of rank $i-1$. 

(ii) If $\lambda_i > 1$, then by Lemma 6.6 and Theorem 2.13, $U_{N_i}$ is a free abelian group of rank $i-1$. 

(ii) If \( \lambda_i > 1 \), then by \((R_5 \varepsilon)\) the map \( \mathcal{U}_N \rightarrow (\mathbb{Z}^{i-1}, +) \) defined by
\[
[\beta]_\varepsilon \mapsto (\exp_{\varepsilon}((\beta]_\varepsilon), \ldots, \exp_{i-1, \varepsilon}((\beta]_\varepsilon)) \quad \text{for each } [\beta]_\varepsilon \in \mathcal{U}_N,
\]
is clearly an isomorphism. \( \square \)

### 6.2.1. Decision Problems.

We now provide an algorithm which will determine, given two braids \( \beta_1, \beta_2 \in B \) and two equivalences \( \varepsilon_1, \varepsilon_2 \in E \), whether or not \([\varepsilon_1, \beta_1] = [\varepsilon_2, \beta_2] \). The substance of this algorithm is to determine \( \sim_\varepsilon \)-equivalence of braids.

**Theorem 6.26.** The problem of deciding whether two elements \([\varepsilon_1, \beta_1], [\varepsilon_2, \beta_2] \in \mathfrak{B}_n \) are equal or not is decidable.

**Proof.** Now \([\varepsilon_1, \beta_1] = [\varepsilon_2, \beta_2] \) if and only if \( \varepsilon_1 = \varepsilon_2 \) and \( \beta_1 \beta_2^{-1} \in B_{\varepsilon_1} \). An algorithm to determine whether this is the case follows.

(i) If \( \varepsilon_1 \neq \varepsilon_2 \) then \([\varepsilon_1, \beta_1] \neq [\varepsilon_2, \beta_2] \). If \( \varepsilon_1 = \varepsilon_2 \) then go to step (ii).

(ii) Choose \( \gamma \in B \) such that \( \varepsilon = \varepsilon_1 \gamma \) is a standard equivalence, and put \( \beta = \gamma \beta_1 \beta_2^{-1} \gamma^{-1} \). Then \( \beta_1 \beta_2^{-1} \in B_{\varepsilon_1} \) if and only if \( \beta \in \gamma B_{\varepsilon_1} \gamma^{-1} = B_{\varepsilon_1} \). Now if \( \beta \notin P_{\varepsilon} \) then \( \beta \notin B_{\varepsilon_1} \) and so \([\varepsilon_1, \beta_1] \neq [\varepsilon_2, \beta_2] \). If \( \beta \in P_{\varepsilon} \) then go to step (iii).

(iii) Again, we write \( n/\varepsilon = \{N_1, \ldots, N_k\} \) where \( \mu_1 = \min(N_1) < \cdots < \mu_k = \min(N_k) \).

Now by the proof of Lemma 6.2, we have \( \beta = \alpha_{p_1 q_1}^{\pm 1} \cdots \alpha_{q_r s_t}^{\pm 1} \cdot s_{r} \cdots s_{s} \) with some \( p_1, \ldots, p_r, q_1, \ldots, q_r \in \mathbb{N} \) with \( p_j < q_j \) for each \( j \in \mathbb{R} \), and some \( i_1, \ldots, i_s \in \mathbb{N}^+ \). For each \( i \in \mathbb{R} \) we have \( p_i \in N_{u_i} \) and \( q_i \in N_{v_i} \) for some \( u_i, v_i \in \mathbb{K} \). Write \( \{i \in \mathbb{R} | u_i \neq v_i\} = \{x_1, \ldots, x_t\} \) with \( x_1 < \cdots < x_t \). Then by the comments after Corollary 6.21, we have
\[
\beta \sim \varepsilon \alpha_{\mu_{x_1 \mu_{x_2} \cdots \mu_{x_t}}}^{\pm 1} \cdot \alpha_{\mu_{x_{t+1}} \mu_{x_{t+2}} \cdots \mu_{x_t}}^{\pm 1}.
\]
Now put \( w = a_{\mu_{x_1} \mu_{x_2} \cdots \mu_{x_t}}^{\pm 1} \in F(X_\varepsilon) \). Then, writing \( \sim_{R_\varepsilon} = R_\varepsilon^2 \), we see by Theorem 6.22 that \( \beta \in B_{\varepsilon} \) if and only if \( w \sim_{R_\varepsilon} 1 \). By Theorems 6.22 and 6.25, we have
\[
w \sim_{R_\varepsilon} w_k \cdots w_2
\]
where each \( w_j \) is a word over \( \{a_{\mu_{x_j}} \mid 1 \leq i < j\} \). By Theorem 6.25, \( w \sim_{R_\varepsilon} 1 \) if and only if each word \( w_j \) either freely reduces to the empty word (in the case \( j \leq k_0 \)), or has a zero exponent sum for each \( a_{\mu_{x_j}} \) (in the case \( j > k_0 \)). \( \square \)

### 6.3. The Pure Permeable Braid Monoid

In this section we define the pure permeable braid monoid \( \mathfrak{P}_n \). We study \( \mathfrak{P}_n \) in much the same way as we studied \( \mathfrak{F}_n \) in the previous section. It is again true that \( \mathfrak{P}_n \) is the union
of its maximal subgroups, and that each of these subgroups is isomorphic to a quotient of the pure braid group. We find presentations of these quotients; the presentations are easier to obtain (see Lemma 6.27 and Corollary 6.28), although harder to study, than those associated with $\mathcal{P}_n$. We obtain a semidirect product decomposition analogous to that of the pure braid group in Theorem 2.14. However, the subgroups in this decomposition are not as easy to understand; indeed they are neither free nor free abelian in general (see Remark 6.37). We conclude the section by giving an algorithm which determines $\approx_{\mathcal{E}}$-equivalence of braids.

Recall that in Section 4.3.1 we defined an epimorphism $\varphi : \mathcal{PB}_n \to \mathcal{S}_n$. Under the identification of $\mathcal{PB}_n$ with $(E \times B)/\approx$, we see that $\varphi$ maps $[\mathcal{E}, \beta]$ to $[\mathcal{E}, \beta]$ for each $\mathcal{E} \in E$ and $\beta \in B$. We now define the pure permeable braid monoid

$$\mathcal{P}_n = \{ [\mathcal{E}, \beta] \in \mathcal{PB}_n \mid \beta = 1 \} = \{ [\mathcal{E}, \beta] \mid \mathcal{E} \in E, \beta \in P \}.$$

Let $\mathcal{E} \in E$. Using the fact that $B^{(2)}_{\mathcal{E}} \subseteq P$, we see that the map

$$P \to \mathcal{P}_n : \beta \mapsto [\mathcal{E}, \beta] \quad \text{for each } \beta \in P$$

is a group homomorphism with kernel $B^{(2)}_{\mathcal{E}}$ and image $[\mathcal{E}, P] = \{ [\mathcal{E}, \beta] \mid \beta \in P \}$. In particular, $[\mathcal{E}, P] \cong P/B^{(2)}_{\mathcal{E}}$ is a group. Thus

$$\mathcal{P}_n = \bigsqcup_{\mathcal{E} \in E} [\mathcal{E}, P]$$

is the disjoint union of the groups $[\mathcal{E}, P]$ which are therefore its maximal subgroups. We will study these groups, or rather the quotients $P/B^{(2)}_{\mathcal{E}}$ which, for simplicity, we will denote by $P/\mathcal{E}$. The following is simply a restatement of Lemma 4.19.

**Lemma 6.27.** The subgroup $B^{(2)}_{\mathcal{E}}$ is the normal closure in $P$ of the set $\{ \alpha_{ij} \mid (i, j) \in \mathcal{E} \}$. □

Again without causing confusion, let $R_O$ denote the set of relations

$$a_{ij} = 1 \quad \text{if } (i, j) \in \mathcal{E} \text{ and } i < j.$$  \hspace{1cm} (O)

For $\beta \in P$ and $\mathcal{E} \in E$, we denote by $[\beta]_\mathcal{E}$ the coset $B^{(2)}_{\mathcal{E}} \beta$. Again, there is no conflict with our earlier use of this notation. By Theorem 2.13 and Lemmas 6.5 and 6.27 we have the following.

**Corollary 6.28.** If $\mathcal{E} \in E$, then the quotient $P/\mathcal{E}$ has presentation $\langle X_P \mid R_P \cup R_O \rangle$ via

$$a_{ij} \mapsto [\alpha_{ij}]_\mathcal{E} \quad \text{for each } i, j.$$ □
We now examine the manner in which this presentation simplifies. Recall that two equivalences \( E_1, E_2 \in E \) are *conjugate* if \( E_1^\beta = E_2 \) for some \( \beta \in B \). The next lemma may be proved in a similar fashion to Lemma 6.13.

**Lemma 6.29.** If \( E_1, E_2 \in E \) are conjugate, then \( [E_1, P_{E_1}] \cong [E_2, P_{E_2}] \).

So again it suffices to study the quotients \( P/E \) where \( E \in E \) is an arbitrary *convex* equivalence. For the rest of this section we fix a convex equivalence \( E \in E \). Again we write \( n/E = \{N_1, \ldots, N_k\} \) where \( r < s \) whenever \( r \in N_i \) and \( s \in N_j \) with \( i < j \). It is important to note that we have not assumed that \( E \) is standard.

Denote by \( \equiv \) the congruence \((R_P \cup R_O)^\sharp\) on \( F(X_P) \). To make the statements and proofs of the following results more concise, we will write \( w_1 \equiv w_2 \) if \( w_1, w_2 \in F(X_P) \) and \( w_1w_2 \equiv w_2w_1 \).

**Lemma 6.30.** If \( (j, s) \in E \) with \( s < j \), then

(i) \( a_{ij} \equiv a_{rs} \) for all \( 1 \leq i < j \) and \( 1 \leq r < s \);

(ii) \( a_{ji} \equiv a_{sr} \) for all \( j < i \leq n \) and \( s < r \leq n \).

**Proof.** The proof of Lemma 6.18 works almost unchanged to prove part (i), using (O) instead of the initial observation of that proof, so it suffices to prove (ii). To prove (ii) we first define an antihomomorphism \( \theta : F(X_P) \to F(X_P) \) by \( a_{ij} \theta = a_{n+1-j, n+1-i} \) for each \( 1 \leq i < j \leq n \). Then, noting that \( \theta \) preserves relations (P1) and (P3), we see that the arguments in the proof of Lemma 6.18 also prove all cases of (ii) except, due to the asymmetry of (P4), for the case in which \( j < r < i \). For this final case, using the already established fact that \( a_{ji} \equiv a_{si} \) (the \( i = r \) case of (ii)), we have

\[
\begin{align*}
a_{ji} \equiv & \ a_{si}a_{ji}a_{si}^{-1} \\
\equiv & \ a_{si}a_{sr}^{-1}a_{ji}a_{sr}^{-1}a_{sr}a_{si}^{-1} \\
\equiv & \ a_{si}a_{sr}^{-1}(a_{ri}^{-1}a_{si}a_{ri}a_{sj})a_{ji}(a_{si}^{-1}a_{ri}a_{si})a_{sr}a_{si}^{-1} \\
\equiv & \ a_{si}a_{sr}^{-1}(a_{sr}a_{si}^{-1}a_{si}^{-1}a_{sr}a_{si})a_{ji}(a_{si}^{-1}a_{sr}a_{si}^{-1}a_{sr})a_{si}^{-1} \\
\equiv & \ a_{sr}^{-1}a_{ji}a_{sr} \
\end{align*}
\]

since \( a_{ji} \equiv a_{si} \)

and the proof is complete.

The proof of the next lemma makes use of the fact that the relation

\[
a_{r_2}^{-1}a_{ij}a_{r_1} = a_{r_2}a_{ij}a_{r_1}^{-1} \quad \text{if } r < i < j
\]

is in \( R_P^\sharp \); see for example [8], or draw pictures and apply Theorem 2.13.
Lemma 6.31. Suppose that $1 \leq r < i < s < j \leq n$ and $(i, s) \in \mathcal{E}$. Then $a_{rs} \equiv a_{ij}$.

Proof. Observe first that by Lemma 6.30 (ii) we have $a_{sj} \equiv a_{ij}$. Using this fact, together with (P1) and (P2)', we have

$$a_{sj} \equiv a_{ri}^{-1}a_{sj}a_{ri} \equiv a_{ri}^{-1}a_{ij}a_{ri} \equiv a_{rj}a_{ij}^{-1}.$$  

It then follows that $a_{rj}^{-1}a_{sj}a_{rj} \equiv a_{ij}$. Using this, together with (P4) and $a_{ij} \equiv a_{sj}$ again, we have

$$a_{rs}a_{ij}^{-1} \equiv (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}a_{rj}a_{sj}) \equiv a_{sj}^{-1}a_{ij}a_{sj} \equiv a_{ij},$$

completing the proof. □

Corollary 6.32. If $1 \leq r < s \leq n$, $1 \leq i < j \leq n$, and one of $r, s$ is $\mathcal{E}$-related, but not equal, to one of $i, j$, then $a_{rs} \equiv a_{ij}$. □

Let $X^{(2)}_{\mathcal{E}} = \{a_{ij} \mid (i, j) \notin \mathcal{E}\}$, and let $R^{(2)}_{\mathcal{E}} = R^\circ \cup R_Z$ where $R^\circ$ is the restriction of $R_P$ to $X^{(2)}_{\mathcal{E}}$, and $R_Z$ is the set of relations

$$a_{rs}a_{ij} = a_{ij}a_{rs} \quad \text{if one of } r, s \text{ is } \mathcal{E}-\text{related, but not equal, to one of } i, j.$$  

Theorem 6.33. The quotient $P/\mathcal{E}$ has presentation $\langle X^{(2)}_{\mathcal{E}} \mid R^{(2)}_{\mathcal{E}} \rangle$ via

$$a_{ij} \mapsto [a_{ij}]_{\mathcal{E}} \quad \text{for each } 1 \leq i < j \leq n \text{ with } (i, j) \notin \mathcal{E}.$$  

Proof. By Corollaries 6.28 and 6.32, $P/\mathcal{E}$ has presentation $\langle X_P \mid R_P \cup R_O \cup R^{(2)}_{\mathcal{E}} \rangle$ via

$$a_{ij} \mapsto [a_{ij}]_{\mathcal{E}} \quad \text{for each } 1 \leq i < j \leq n.$$  

Denote by $\rho$ the congruence $(R_O \cup R^{(2)}_{\mathcal{E}})^\sharp$ on $F(X_P)$. We first show that $R_P \subseteq \rho$. Since $R^{(2)}_{\mathcal{E}} \subseteq \rho$, it suffices to consider only relations from $R_P$ for which at least one letter has $\mathcal{E}$-related subscripts.

(P1): If $(r, s) \in \mathcal{E}$ or $(i, j) \in \mathcal{E}$ then by $R_O$ we have

$$a_{rs}a_{ij}^{-1} \rho a_{ij}.$$  

(P2): If $(s, j) \in \mathcal{E}$ then by $R_O$ we have

$$a_{rs}a_{sj}^{-1} \rho (a_{sj}^{-1}a_{rj})a_{sj}(a_{rj}a_{sj}).$$

If $(r, j) \in \mathcal{E}$ (resp. $(r, s) \in \mathcal{E}$) then by $R^{(2)}_E$ (resp. $R_O$) and $R_O$ (resp. $R^{(2)}_E$) we have

$$a_{rs}a_{sj}^{-1} \rho a_{sj} \rho (a_{sj}^{-1}a_{rj})a_{sj}(a_{rj}a_{sj}).$$
We now work towards finding a semidirect product decomposition for the presentation in the statement of the theorem. If \((r, j) \in \mathcal{E}\) then by \(R_O\) we have
\[ a_{rs} a_{rz} a_{rs}^{-1} \rho a_{sj}^{-1} a_{rj} a_{sj}. \]

If \((r, s) \in \mathcal{E}\) (resp. \((s, j) \in \mathcal{E}\)) then by \(R_O\) (resp. \(R_E^{(2)}\)) and \(R_E^{(2)}\) (resp. \(R_O\)) we have
\[ a_{rs} a_{rz} a_{rs}^{-1} \rho a_{rj} \rho a_{sj}^{-1} a_{rj} a_{sj}. \]

(P4): If \((i, j) \in \mathcal{E}\) then again we are done by \(R_O\). If \((s, j) \in \mathcal{E}\) or \((r, j) \in \mathcal{E}\) then by \(R_E^{(2)}\) and \(R_O\) we have
\[ a_{rs} a_{ij} a_{rs}^{-1} \rho a_{ij} \rho (a_{sj}^{-1} a_{rj} a_{sj}) a_{ij} (a_{rj}^{-1} a_{sj}^{-1} a_{rj} a_{sj}). \]

Finally, if \((r, s) \in \mathcal{E}\), then by \(R_O\) and (P3) we have
\[ a_{rs} a_{ij} a_{rs}^{-1} \rho a_{ij} \rho (a_{rs} a_{rj} a_{rs}^{-1} a_{rj}) a_{ij} (a_{rj}^{-1} a_{sj}^{-1} a_{rj} a_{sj}) \rho (a_{sj}^{-1} a_{rj}^{-1} a_{sj} a_{rj}) a_{ij} (a_{rj}^{-1} a_{sj}^{-1} a_{rj} a_{sj}). \]

This completes the proof that \(R_P \subseteq \rho\) so that the above presentation simplifies to
\[ \langle X_P \mid R_O \cup R_E^{(2)} \rangle. \]

Now removing all generators \(a_{ij}\) with \((i, j) \in \mathcal{E}\), along with relations \(R_O\), leaves us with the presentation in the statement of the theorem. \(\square\)

We now work towards finding a semidirect product decomposition for \(P/\mathcal{E}\). For \(A \subseteq \mathbb{N}\), let \(U_A^{(2)}\) be the subgroup of \(P/\mathcal{E}\) generated by \(\{[a_{ij}]_\mathcal{E} \mid 1 \leq i < j, j \in A\}\). Let \((P/\mathcal{E})'\) be the subgroup generated by \(\{[a_{ij}]_\mathcal{E} \mid 1 \leq i < j \leq \min(N_k)\}\). Also, let \(\mathcal{E}' \subseteq \mathbb{E}_{a_{n-\lambda_k}}\) be the equivalence for which \(\{1, \ldots, n-\lambda_k\}/\mathcal{E}' = \{N_1, \ldots, N_{k-1}\}\).

Lemma 6.34. We have the semidirect product decomposition \(P/\mathcal{E} = U_{N_k}^{(2)} \rtimes (P/\mathcal{E})'\).

Proof. It is immediate from Theorem 6.33, and the nature of the relations in \(R_E^{(2)}\), that \(U_{N_k}^{(2)}\) is normal in \(P/\mathcal{E}\), and \(P/\mathcal{E}\) is clearly generated by \(U_{N_k}^{(2)} \cup (P/\mathcal{E})'\). Suppose now that \(\beta \in P\) such that \([\beta]_\mathcal{E} \in U_{N_k}^{(2)} \cap (P/\mathcal{E})'\). Since \([\beta]_\mathcal{E} \in U_{N_k}^{(2)}\), we have
\[ [\beta]_\mathcal{E} = [\alpha_{i_1j_1} \cdots \alpha_{i_rj_r}]_\mathcal{E} \]
for some \(i_1, \ldots, i_r \in \mathbb{N} \setminus N_k\) and \(j_1, \ldots, j_r \in N_k\). Since \([\beta]_\mathcal{E} \in (P/\mathcal{E})'\), we have
\[ [\beta]_\mathcal{E} = [\alpha_{u_1v_1} \cdots \alpha_{u_kv_k}]_\mathcal{E} \]
for some \(u_1, \ldots, u_s, v_1, \ldots, v_s \in \mathbb{N} \setminus N_k\) with \(u_t < v_t\) for each \(t \in s\). Put
\[ \beta_1 = \alpha_{i_1j_1}^{\pm 1} \cdots \alpha_{i_rj_r}^{\pm 1} \quad \text{and} \quad \beta_2 = \alpha_{u_1v_1}^{\pm 1} \cdots \alpha_{u_kv_k}^{\pm 1}. \]

Since \([\beta_1]_\mathcal{E} = [\beta_2]_\mathcal{E}\), there is an \((\mathcal{E}, 2)\)-homotopy \(H\) from \(\beta_1\) to \(\beta_2\). Let \(\beta_1', \beta_2' \in B_{n-\lambda_k}\) be the braids obtained by removing the last \(\lambda_k\) strings from \(\beta_1\) and \(\beta_2\) (respectively). Then \(H\) may be modified to give an \((\mathcal{E}', 2)\)-homotopy \(H'\) from \(\beta_1'\) to \(\beta_2'\). Now \(\beta_1' = 1_{n-\lambda_k}\), the identity braid in \(B_{n-\lambda_k}\); since \(j_1, \ldots, j_r \in N_k\). Thus, \(H'\) is an \((\mathcal{E}', 2)\)-homotopy from \(1_{n-\lambda_k}\) to \(\beta_2'\).
Let $\beta_1', \beta_2'' \in \mathcal{B}_n$ be the braids obtained by replacing the remaining $\lambda_k$ strings to $\beta_1'$ and $\beta_2''$ (respectively) as straight strings which do not interact with the first $n - \lambda_k$ strings. Then, applying $H'$ to the first $n - \lambda_k$ strings of $\beta_1''$, and keeping the last $\lambda_k$ strings out of the way, we have an $(\mathcal{E},2)$-homotopy from $\beta_1'' = 1 \in \mathcal{B}_n$ to $\beta_2''$. But $\beta_2'' = \beta_2$ since $v_1, \ldots, v_s \notin N_k$. Thus, $\beta \approx_{\mathcal{E}} \beta_2 \approx_{\mathcal{E}} 1$ showing that $[\beta]_{\mathcal{E}} = [1]_{\mathcal{E}}$ and completing the proof. \hfill \Box

**Lemma 6.35.** We have $(P/\mathcal{E})' \cong \mathcal{P}_{n-\lambda_k}/\mathcal{E}' = \mathcal{P}_{n-\lambda_k}/B_{\mathcal{E}'}^{(2)}$ where we regard $B_{\mathcal{E}'}^{(2)}$ as a subgroup of $\mathcal{B}_{n-\lambda_k}$.

**Proof.** This follows from Lemma 6.6 and Theorem 6.33. \hfill \Box

**Theorem 6.36.** We have the semidirect product decomposition

$$P/\mathcal{E} = U_{N_k}^{(2)} \rtimes (U_{N_{k-1}}^{(2)} \rtimes (\cdots \rtimes (U_{N_3}^{(2)} \times U_{N_2}^{(2)}) \cdots)).$$

**Proof.** This follows from Lemmas 6.34 and 6.35 and a simple induction. \hfill \Box

**Remark 6.37.** Notice that, unlike Theorem 6.25, we have not given any additional information about the structure of the subgroups $U_{N_i}^{(2)}$. The reason for this is that these subgroups are neither free nor free abelian in general. For example, consider the convex equivalence $\mathcal{E} = \mathcal{E}_{23} \in \mathcal{E}_{4}$. In the notation of this section we have

$$\{1, 2, 3, 4\}/\mathcal{E} = \{N_1, N_2, N_3\} \quad \text{where} \quad N_1 = \{1\}, N_2 = \{2, 3\}, N_3 = \{4\},$$

and $\mathcal{P}_4/\mathcal{E} = U_{N_3}^{(2)} \rtimes U_{N_2}^{(2)}$ where

$$U_{N_2}^{(2)} = \langle [\alpha_{12}], [\alpha_{13}] \rangle \quad \text{and} \quad U_{N_3}^{(2)} = \langle [\alpha_{14}], [\alpha_{24}], [\alpha_{34}] \rangle.$$ 

Now by $R_Z$, which includes the relation $a_{12}a_{13} = a_{13}a_{12}$ since $(2,3) \in \mathcal{E}$, it is clear that $U_{N_2}^{(2)}$ is a free abelian group of rank 2. Using the presentation of $\mathcal{P}_4/\mathcal{E}$ in Theorem 6.33 and the Reidemeister-Schreier method (see for example [46]), one may show that $U_{N_3}^{(2)}$ has presentation

$$\langle a_{14}, a_{24}, a_{34} \mid a_{14}^i a_{24} a_{34}^{-i} = a_{14} a_{34} a_{14}^{-1} \ (\forall i, j \in \mathbb{Z}) \rangle$$

via $a_{14} \mapsto [\alpha_{14}], a_{24} \mapsto [\alpha_{24}], a_{34} \mapsto [\alpha_{34}]$. Here again we have used the “$\Rightarrow$” symbol to signify a commuting relation. To describe the structure of $U_{N_3}^{(2)}$, we will artificially construct a group with the same presentation. With this in mind, let $X$, $Y$, and $Z$ be free groups freely generated by $\{x_i \mid i \in \mathbb{Z}\}$, $\{y_i \mid i \in \mathbb{Z}\}$, and $\{z\}$ respectively. Define an (anti-)homomorphism $\varphi : Z \to \text{Aut}(X \times Y) : z \mapsto \varphi_z$ where

$$\varphi_z : X \times Y \to X \times Y : \begin{cases} (x_i, 1) \mapsto (x_{i+1}, 1) & \text{for all } i \in \mathbb{Z} \\ (1, y_i) \mapsto (1, y_{i+1}) & \text{for all } i \in \mathbb{Z}. \end{cases}$$
We may then form the semidirect product
\[(X \times Y) \rtimes Z = (X \times Y) \rtimes_\varphi Z = \{(x, y, z^i) \mid x \in X, y \in Y, i \in \mathbb{Z}\}\]
with multiplication defined, for \(x, x' \in X, y, y' \in Y,\) and \(i, j \in \mathbb{Z},\) by
\[(x, y)(x', y') = ((x, y)((x', y')\varphi_z^i), z^{i+j}).\]
We see then, by \([42, 42]\) for example, that \((X \times Y) \rtimes Z\) has presentation
\[
\left\langle a, b, c, \ i \in \mathbb{Z} \ \mid \right. \\
a_i \mapsto b_j \quad \forall i, j \in \mathbb{Z} \\
ca_i c^{-1} = a_{i+1} \quad \forall i \in \mathbb{Z} \\
bc_i c^{-1} = b_{i+1} \quad \forall i \in \mathbb{Z} \left. \right\rangle
\]
via
\[
\begin{align*}
a_i &\mapsto ((x_i, 1), 1) \quad \text{for each } i \in \mathbb{Z} \\
b_i &\mapsto ((1, y_i), 1) \quad \text{for each } i \in \mathbb{Z} \\
c &\mapsto ((1, 1), z). 
\end{align*}
\]
Putting \(a = a_0\) and \(b = b_0,\) it is then easy to see that this presentation simplifies to
\[
\langle a, b, c \mid c^i ac^{-i} = c^j bc^{-j} \ (\forall i, j \in \mathbb{Z}) \rangle.
\]
Thus \(U_{N_3}^{(2)} \cong (X \times Y) \rtimes Z,\) the latter group clearly being neither free nor free abelian.

We now give a little more information about the internal structure of the subgroups \(U_{N_j}^{(2)} .\)

**Lemma 6.38.** If \((i, j) \in \mathcal{E}\) and \(2 \leq i < j,\) then \(U_{i}^{(2)} \cong U_{j}^{(2)}.\)

**Proof.** Define \(\psi : P/\mathcal{E} \rightarrow P/\mathcal{E}\) by \([\alpha_{rs}]_\mathcal{E}\psi = [\alpha_{rt_{ij}, st_{ij}}]_\mathcal{E}\) for \(1 \leq r < s \leq n\) with \((r, s) \notin \mathcal{E}\) where, as usual, \(t_{ij}\) denotes the transposition which interchanges \(i\) and \(j.\) By Theorem 6.33, and a straightforward check of the relations, we see that \(\psi\) is a well-defined homomorphism. Since \(\psi\) is clearly an involution, it is therefore an isomorphism. It then follows that the restriction of \(\psi\) to \(U_{i}^{(2)}\) is an isomorphism onto its image \(U_{j}^{(2)}.\) \(\square\)

**Lemma 6.39.** Suppose that \(j \in k\) and \(N_j = \{i_1, \ldots, i_\ell\}\) where \(\ell = \lambda_j\) and \(i_1 < \cdots < i_\ell.\) Then \(U_{N_j}^{(2)} = U_{i_\ell}^{(2)} \times \cdots \times U_{i_1}^{(2)} .\)

**Proof.** If \(\ell = 1\) then there is nothing to show, so suppose that \(\ell \geq 2.\) By \(R_Z\) we see that all elements of \(U_{i_\ell}^{(2)}\) commute with all elements of \(U_{i_t}^{(2)}\) for \(1 \leq s < t \leq \ell\) so that
6.3. THE PURE PERMEABLE BRAID MONOID

We now provide an algorithm which will determine, given two braids \( \beta \) and \( \gamma \), whether or not \( \beta \approx \gamma \). Assume that \( \beta, \gamma \in B \) and two equivalences \( \mathcal{E}_1, \mathcal{E}_2 \in E \), whether or not \( [\mathcal{E}_1, \beta] = [\mathcal{E}_2, \gamma] \). The substance of this algorithm is to determine \( \approx_{\mathcal{E}} \)-equivalence of braids. To make the proof simpler, we first prove an intermediate result.

\[ \mathcal{U}^{(2)}_{N_j} = \mathcal{U}^{(2)}_{u_1} \cdots \mathcal{U}^{(2)}_{u_t}. \]

Suppose now that \( s \in \mathcal{L} \), and write

\[ G = \prod_{i \in \mathcal{E} \setminus \{s\}} \mathcal{U}^{(2)}_{u_i}. \]

Choose \( \beta \in P \) such that \( [\beta]_{\mathcal{E}} \in \mathcal{U}^{(2)}_{u_i} \cap G \). The proof will be complete if we can show that \( [\beta]_{\mathcal{E}} = [1]_{\mathcal{E}} \). Since \( \beta \in \mathcal{U}^{(2)}_{u_i} \cap G \), we have

\[ [\beta]_{\mathcal{E}} = [\alpha_{p_{1t}}^{\pm 1} \cdots \alpha_{p_{ut}}^{\pm 1}]_{\mathcal{E}} = [\alpha_{q_{1t}}^{\pm 1} \cdots \alpha_{q_{vt}}^{\pm 1}]_{\mathcal{E}} \]

for some \( p_1, \ldots, p_u, q_1, \ldots, q_v \in N_1 \cup \cdots \cup N_j \) and \( r_1, \ldots, r_v \in \mathcal{L} \setminus \{s\} \). Put

\[ \beta_1 = \alpha_{p_{1t}}^{\pm 1} \cdots \alpha_{p_{ut}}^{\pm 1} \quad \text{and} \quad \beta_2 = \alpha_{q_{1t}}^{\pm 1} \cdots \alpha_{q_{vt}}^{\pm 1}. \]

Now there is an \((\mathcal{E}, 2)\)-homotopy \( H \) from \( \beta_1 \) to \( \beta_2 \). Let \( \beta'_1 \) and \( \beta'_2 \) be the braids obtained by removing strings \( i_1, \ldots, s_{i-1}, i_{s+1}, \ldots, i_t \) from \( \beta_1 \) and \( \beta_2 \) (respectively) and replacing them with straight strings which pass in front of the remaining strings. Then, by keeping strings \( i_1, \ldots, s_{i-1}, i_{s+1}, \ldots, i_t \) out of the way, and applying \( H \) to the other strings, we have an \((\mathcal{E}, 2)\)-homotopy from \( \beta'_1 \) to \( \beta'_2 \). But \( \beta'_1 = \beta_1 \) and \( \beta'_2 = 1 \), and the proof is complete. \( \square \)

6.3.1. Decision Problems.

We now provide an algorithm which will determine, given two braids \( \beta_1, \beta_2 \in B \) and two equivalences \( \mathcal{E}_1, \mathcal{E}_2 \in E \), whether or not \( [\mathcal{E}_1, \beta_1] = [\mathcal{E}_2, \beta_2] \). The substance of this algorithm is to determine \( \approx_{\mathcal{E}} \)-equivalence of braids. To make the proof simpler, we first prove an intermediate result.

Theorem 6.40. Given a convex equivalence \( \mathcal{E} \in E \) and a pure braid \( \beta \in P \), it is decidable to determine whether or not \( \beta \approx_{\mathcal{E}} 1 \).

Proof. Again we write \( n/\mathcal{E} = \{N_1, \ldots, N_k\} \) with \( i < j \) whenever \( i \in N_r \) and \( j \in N_s \) with \( r < s \). We also put \( \lambda_i = |N_i| \) for each \( i \in k \).

We prove the theorem by induction on \( n \). Now if \( n = 1 \) then the result is trivial, so suppose that \( n > 1 \). If \( \mathcal{E} = 1 \), then \( \beta \approx_{\mathcal{E}} 1 \) if and only if \( \beta = 1 \) and we are done, using a solution to Problem 2.15, so suppose that \( \mathcal{E} \neq 1 \). By conjugating \( \mathcal{E} \) and \( \beta \) (if necessary) we may also assume that \( \lambda_k > 1 \).

By Theorem 2.14, \( \beta = u_n \cdots u_2 \) for some (unique) \( u_2 \in \mathcal{U}_2, \ldots, u_n \in \mathcal{U}_n \). Now \( \beta \approx_{\mathcal{E}} 1 \) if and only if \( [u_n]_{\mathcal{E}} \cdots [u_2]_{\mathcal{E}} = [\beta]_{\mathcal{E}} = [1]_{\mathcal{E}} \). By Theorem 6.36 and Lemma 6.39, we see that this is the case if and only if \( [u_n]_{\mathcal{E}} = \cdots = [u_2]_{\mathcal{E}} = [1]_{\mathcal{E}} \). Recall that \( \mathcal{E}' \in \mathcal{E}_{n-\lambda_k} \) is the equivalence for which \( \{1, \ldots, n-\lambda_k\}/\mathcal{E}' = \{N_1, \ldots, N_{k-1}\} \) and that, by Lemma 6.35, the
subgroup \((P/\mathcal{E})'\) of \(P/\mathcal{E}\) generated by \(\{[\alpha_{ij}]_{\mathcal{E}} \mid 1 \leq i < j \leq n, \ j \notin N_k\}\) is isomorphic to \(\mathcal{P}_{n-l_k}/\mathcal{E}'\) via the map
\[
[\alpha_{ij}]_{\mathcal{E}} \mapsto [\alpha'_{ij}]_{\mathcal{E}'} \quad \text{for each } 1 \leq i < j \leq n - l_k
\]
where, for \(\gamma \in P\), we have written \(\gamma' \in \mathcal{P}_{n-l_k}\) for the braid obtained by removing the last \(l_k\) strings from \(\gamma\). Thus, we see that \([u_{n-l_k}]_{\mathcal{E}} = \cdots = [u_2]_{\mathcal{E}} = [1]_{\mathcal{E}}\) if and only if \([u'_{n-l_k}]_{\mathcal{E}'} = \cdots = [u'_2]_{\mathcal{E}'} = [1']_{\mathcal{E}'}\) in \(\mathcal{P}_{n-l_k}/\mathcal{E}'\). By induction it is decidable to determine whether or not this is the case. Thus it remains to determine whether or not \([u_n]_{\mathcal{E}} = \cdots = [u_{n-l_k+1}]_{\mathcal{E}} = [1]_{\mathcal{E}}\). Put \(m = n - l_k + 1\). Now for \(m < j \leq n\), we define a map
\[
\psi_j : U_j^{(2)} \to U_m^{(2)} : [\alpha_{ij}]_{\mathcal{E}} \mapsto [\alpha_{im}]_{\mathcal{E}} \quad \text{for each } 1 \leq i < m
\]
which, by the proof of Lemma 6.38, is an isomorphism. Then \([u_n]_{\mathcal{E}} = \cdots = [u_m]_{\mathcal{E}} = [1]_{\mathcal{E}}\) if and only if \([u_n]_{\mathcal{E}} \psi_n = \cdots = [u_{m+1}]_{\mathcal{E}} \psi_{m+1} = [u_m]_{\mathcal{E}} = [1]_{\mathcal{E}}\). Thus it suffices to determine whether or not \([u]_{\mathcal{E}} = [1]_{\mathcal{E}}\) for arbitrary \(u \in U_m\).

Now let \(\mathcal{E}'' \subseteq \mathcal{E}_{q_m}\) be the equivalence for which \(m/\mathcal{E}'' = \{N_1, \ldots, N_{k-1}, \{m\}\}\), and let \((P/\mathcal{E})''\) be the subgroup of \(P/\mathcal{E}\) generated by \(\{[\alpha_{ij}]_{\mathcal{E}} \mid 1 \leq i < j \leq m\}\). By another application of Lemma 6.6 and Theorem 6.33 we have an isomorphism \(\psi : (P/\mathcal{E})'' \to \mathcal{P}_m/\mathcal{E}''\). We then have \([u]_{\mathcal{E}} = [1]_{\mathcal{E}}\) if and only if \([u]_{\mathcal{E}} \psi = [1]_{\mathcal{E}''}\) in \(\mathcal{P}_m/\mathcal{E}''\). By induction it is decidable to determine whether or not this is the case, and the proof is complete. \(\Box\)

**Theorem 6.41.** The problem of deciding whether two elements \([\mathcal{E}_1, \beta_1], [\mathcal{E}_2, \beta_2] \in \mathcal{PB}_n\) are equal or not is decidable.

**Proof.** Now \([\mathcal{E}_1, \beta_1] = [\mathcal{E}_2, \beta_2]\) if and only if \(\mathcal{E}_1 = \mathcal{E}_2\) and \(\beta_1 \beta_2^{-1} \in B^{(2)}_{\mathcal{E}_1}\). An algorithm to determine whether this is the case follows.

(i) If \(\mathcal{E}_1 \neq \mathcal{E}_2\) then \([\mathcal{E}_1, \beta_1] \neq [\mathcal{E}_2, \beta_2]\). If \(\mathcal{E}_1 = \mathcal{E}_2\) then go to step (ii).

(ii) Choose \(\gamma \in B\) such that \(\mathcal{E} = \mathcal{E}'\) is a convex equivalence, and put \(\beta = \gamma \beta_1 \beta_2^{-1} \gamma^{-1}\). Then \(\beta_1 \beta_2^{-1} \in B^{(2)}_{\mathcal{E}_1}\) if and only if \(\beta \in \gamma B^{(2)}_{\mathcal{E}_1} \gamma^{-1} = B^{(2)}_{\mathcal{E}}\). Now if \(\beta \notin P\) then \(\beta \notin B^{(2)}_{\mathcal{E}}\), and so \([\mathcal{E}_1, \beta_1] \neq [\mathcal{E}_2, \beta_2]\). If \(\beta \in P\) then go to step (iii).

(iii) Now \(\beta \in B^{(2)}_{\mathcal{E}}\) if and only if \(\beta \approx_{\mathcal{E}} 1\), and we are done by Theorem 6.40. \(\Box\)
Presentations of Pure Factorisable Inverse Braid Monoids

The presentations of the braid monoids $\mathcal{IB}_n$, $\mathbb{F}B_n$, and $\mathbb{P}B_n$ obtained in Chapter 5 all extend Artin’s presentation of the braid group $\mathcal{B}_n$. In this chapter we consider the pure braid monoids $\mathbb{IP}_n$, $\mathbb{FP}_n$, and $\mathbb{PP}_n$ introduced in the previous chapter. In particular, we give presentations of these monoids, all of which extend Artin’s presentation of the pure braid group $\mathcal{P}_n$. Since these monoids are factorisable, it is possible to apply the results of Section 3.4. However, the results we have obtained thus far allow us to obtain the presentations directly. The substance of our method in the case of $\mathbb{IP}_n$ and $\mathbb{FB}_n$ is to find generating sets for the subgroups $\mathcal{P}_n \cap \mathcal{B}_A$ and $\mathcal{P}_n \cap \mathcal{B}_E$ for arbitrary $A \subseteq 2^n$ and $E \subseteq \mathcal{E}_n$. In contrast, the presentation of $\mathbb{PP}_n$ is rather easy to derive since $\mathcal{B}^{(2)}(2) \subseteq \mathcal{P}_n$ for every $E \subseteq \mathcal{E}_n$.

Fix a positive integer $n$ for the remainder of this chapter, and write $T = 2^n$, $E = \mathcal{E}_n$, $B = \mathcal{B}_n$, and $P = \mathcal{P}_n$. Recall that $X_P = \{a_{ij} \mid 1 \leq i < j \leq n\}$ and that $R_P$ is the set of relations

$$a_{rs}a_{ij}a_{rs}^{-1} = a_{ij} \quad \text{if } i < r \text{ or } i > s \quad (P1)$$
$$a_{rs}a_{sj}a_{rs}^{-1} = (a_{sj}^{-1}a_{rj}^{-1})a_{sj}(a_{rj}a_{sj}) \quad (P2)$$
$$a_{rs}a_{rj}a_{rs}^{-1} = a_{sj}^{-1}a_{rj}a_{sj} \quad (P3)$$
$$a_{rs}a_{ij}a_{rj}a_{rs}^{-1} = (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) \quad \text{if } r < i < s, \quad (P4)$$

where in each relation we have $s < j$. Let $R_F$ denote the set of relations

$$a^{\pm 1}_{ij}a^{\pm 1}_{ij} = 1 \quad \text{for all } i, j. \quad (F)$$

Then by Theorem 2.13, $P$ has monoid presentation $\langle X_P \cup X_P^{-1} \mid R_P \cup R_F \rangle$ via

$$\phi_P : (X_P \cup X_P^{-1})^* \to P : a_{ij}^{\pm 1} \mapsto a_{ij}^{\pm 1} \quad \text{for each } i, j.$$ 

All presentations in this chapter are monoid presentations unless specified otherwise. We write $\sim_P = (R_P \cup R_F)^\sharp$ and choose a set of words $\{\widehat{\beta} \mid \beta \in P\} \subseteq (X_P \cup X_P^{-1})^*$ such that $\widehat{\beta}\phi_P = \beta$ for each $\beta \in P$. If $w = a^{\pm 1}_{i_1j_1} \cdots a^{\pm 1}_{i_rj_r} \in (X_P \cup X_P^{-1})^*$, we will write $w^{-1} = a^{-\pm 1}_{i_rj_r} \cdots a^{-\pm 1}_{i_1j_1}$.

Throughout this chapter it will be convenient to use the symmetric notation $a_{ji} = a_{ij}$ and $\alpha_{ji} = \alpha_{ij}$ for all $1 \leq i < j \leq n$. 

120
Recall that \( X_T = \{\varepsilon_1, \ldots, \varepsilon_n\} \), that \( R_T \) is the set of relations
\[
\varepsilon_i^2 = \varepsilon_i \quad \text{for all } i \quad (T1)
\]
\[
\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad \text{for all } i, j \quad (T2)
\]
and, by Theorem 2.7, that \( T \) has presentation \( \langle X_T \mid R_T \rangle \) via
\[
\phi_T : X_T^* \to T : \varepsilon_i \mapsto n \setminus \{i\} \quad \text{for each } i.
\]
Recall also that \( X_E = \{\varepsilon_{ij} \mid 1 \leq i < j \leq n\} \), that \( R_E \) is the set of relations
\[
\varepsilon_{ij}^2 = \varepsilon_{ij} \quad \text{for all } i, j \quad (E1)
\]
\[
\varepsilon_{ij} \varepsilon_{kl} = \varepsilon_{kl} \varepsilon_{ij} \quad \text{for all } i, j, k, l \quad (E2)
\]
\[
\varepsilon_{ij} \varepsilon_{jk} = \varepsilon_{jk} \varepsilon_{ij} \quad \text{for all } i, j, k \quad (E3)
\]
and, by Theorem 2.8, that \( E \) has presentation \( \langle X_E \mid R_E \rangle \) via
\[
\phi_E : X_E^* \to E : \varepsilon_{ij} \mapsto \mathcal{E}_{ij} \quad \text{for all } i, j.
\]
Choose sets of words \( \{\hat{A} \mid A \in T\} \subseteq X_T^* \) and \( \{\hat{E} \mid E \in E\} \subseteq X_E^* \) such that \( \hat{A} \phi_T = A \) and \( \hat{E} \phi_E = E \) for each \( A \in T \) and \( E \in E \).

### 7.1. The Pure Inverse Braid Monoid

In this section we give a presentation of \( IP_n \), the pure inverse braid monoid. A key step in the derivation is Lemma 7.1 below, which gives a generating set for the subgroup \( P \cap B_A \).

The reader is reminded that we are using symmetric notation for the \( \varepsilon_{ij} \).

#### Lemma 7.1.
If \( A \in T \), then \( P \cap B_A \) is generated by \( \{\gamma^{-1} \alpha_{ij} \gamma \mid \gamma \in P, \ i \in A^c\} \).

**Proof.** By Lemma 4.6, the set in the statement of the lemma is contained in \( P \cap B_A \). Conversely, suppose that \( \beta \in P \cap B_A \). Choose \( \delta \in B \) such that \( A^\delta = r \) where \( |A| = r \). Then \( \delta \beta \delta^{-1} \in P \cap B_r \) and so, by the fifth paragraph of the proof of Theorem 4.4, we see that
\[
\delta \beta \delta^{-1} = \alpha_{i_1 j_1}^{\pm 1} \cdots \alpha_{i_s j_s}^{\pm 1}
\]
for some \( i_1, \ldots, i_s, j_1, \ldots, j_s \in n \) with \( i_t < j_t \) and \( r < j_t \) for each \( t \in s \). But then
\[
\beta = (\delta^{-1} \alpha_{i_1 j_1}^{\pm 1} \delta) \cdots (\delta^{-1} \alpha_{i_s j_s}^{\pm 1} \delta).
\]
By Lemma 4.2, there exists \( \gamma_1, \ldots, \gamma_s \in P \) such that \( \delta^{-1} \alpha_{i_t j_t} \delta = \gamma_t \alpha_{i_t \delta j_t \delta} \gamma_t \) for each \( t \in s \). Thus
\[
\beta = (\gamma_1^{-1} \alpha_{i_1 j_1} \delta \gamma_1) \cdots (\gamma_s^{-1} \alpha_{i_s j_s} \delta \gamma_s)
\]
and, since \( j_t \delta \in A^c \) for each \( t \in s \), we are done. \( \square \)
Now put \( X_{IP} = X_P \cup X_P^{-1} \cup X_T \) and denote by \( R_{IP} \) the set of relations \( R_P \cup R_F \cup X_T \) together with

\[
\varepsilon_i a_{rs} = a_{rs} \varepsilon_i \quad \text{for each } i, r, s \quad \text{(IP1)}
\]
\[
\varepsilon_i a_{ij} = \varepsilon_i \quad \text{for each } i, j. \quad \text{(IP2)}
\]

Notice that by symmetry, relation (IP2) includes both relations

\[
\varepsilon_i a_{ij} = \varepsilon_i \quad \text{if } 1 \leq i < j \leq n
\]
\[
\varepsilon_j a_{ij} = \varepsilon_j \quad \text{if } 1 \leq i < j \leq n.
\]

**Theorem 7.2.** The pure inverse braid monoid \( \mathcal{IP}_n \) has presentation \( \langle X_{IP} \mid R_{IP} \rangle \) via

\[
\begin{align*}
\alpha_{ij}^{\pm 1} & \mapsto [n, \alpha_{ij}^{\pm 1}] \equiv \alpha_{ij}^{\pm 1} \quad \text{for each } i, j \\
\varepsilon_i & \mapsto [i]^c, 1 \equiv 1_{\{i\}} c \quad \text{for each } i.
\end{align*}
\]

**Proof.** Let \( \Psi \) denote the map in the statement of the theorem, and write \( \sim_{IP} = R_{IP}^\circ \).

To show that \( \Psi \) is surjective, suppose that \( [A, \beta] \in \mathcal{IP}_n \). Then \( i \beta = i \), whence \( i \beta^{-1} = i \), for all \( i \in A \) and so we may write \( \beta^{-1} = t_{x_1y_1} \cdots t_{x_ly_l} \) for some \( x_1, \ldots, x_t, y_1, \ldots, y_t \in A^c \). Now put \( \gamma = \varepsilon_{x_1y_1} \cdots \varepsilon_{x_ly_l} \) so that \( \gamma \in B_A \) by Lemma 4.6. We then have \( \bar{\gamma} = \bar{\beta}^{-1} \) so that \( \gamma / \beta \in P \). But then \( \beta(\gamma \beta)^{-1} = \gamma^{-1} \in B_A \) so that \( [A, \beta] = [A, \gamma \beta] = [A, 1][n, \gamma \beta] \in \text{im} \Psi \).

Next, it is easy to check diagrammatically that \( R_{IP} \subseteq \ker \Psi \). It remains to show that \( \ker \Psi \subseteq \sim_{IP} \), so suppose that \((w_1, w_2) \in \ker \Psi \). Now by (IP1) and \( R_P \cup R_F \cup X_T \) we have

\[
w_1 \sim_{IP} \hat{A}_1 \hat{\beta}_1 \quad \text{and} \quad w_2 \sim_{IP} \hat{A}_2 \hat{\beta}_2
\]

for some \( A_1, A_2 \subseteq T \) and \( \beta_1, \beta_2 \in P \). We then have

\[
[A_1, \beta_1] = [A_1, 1][n, \beta_1] = w_1 \Psi = w_2 \Psi = [A_2, 1][n, \beta_2] = [A_2, \beta_2]
\]

from which it follows that \( A_1 = A_2 \) and \( \beta_1 \beta_2^{-1} \in B_{A_1} \cap P \). By Lemma 7.1, we have

\[
\beta_1 \beta_2^{-1} = (\gamma_1^{-1} \alpha_{i_1j_1}^{\pm 1} \gamma_1) \cdots (\gamma_s^{-1} \alpha_{i_sz_s}^{\pm 1} \gamma_s)
\]

for some \( \gamma_1, \ldots, \gamma_s \in P \) and \( i_1, \ldots, i_s, j_1, \ldots, j_s \in n \) with \( i_t \neq j_t \) and \( i_t \in A_1^c \) for each \( t \in s \). Observe that if \( \gamma \in P \) and \( i \neq j \) with \( i \in A_1^c \) then

\[
\begin{align*}
\hat{A}_1 \hat{\gamma}^{-1} a_{ij} \hat{\gamma} & \sim_{IP} \hat{A}_1 \hat{\gamma}^{-1} \hat{A}_1 a_{ij} \hat{\gamma} \quad \text{by (IP1) and (F)} \\
\sim_{IP} \hat{A}_1 \hat{\gamma}^{-1} a_{ij} \hat{\gamma} & \sim_{IP} \hat{A}_1 \hat{\gamma}^{-1} a_{ij} \hat{\gamma} \quad \text{by } R_T \\
\sim_{IP} \hat{A}_1 \hat{\gamma}^{-1} a_{ij} \hat{\gamma} & \sim_{IP} \hat{A}_1 \hat{\gamma}^{-1} a_{ij} \hat{\gamma} \quad \text{by (IP2)} \\
\sim_{IP} \hat{A}_1 & \sim_{IP} \hat{A}_1 \quad \text{by } R_T, (\text{IP1}), \text{ and (F)}.
\end{align*}
\]
We then have
\[ w_1 \sim_{IP} \hat{A}_1 \hat{\beta}_1 \]
\[ \sim_{IP} \hat{A}_1 \hat{\beta}_1 \hat{\beta}_2^{-1} \hat{\beta}_2 \]
\[ \sim_{IP} \hat{A}_1 (\hat{\gamma}_1^{-1} a_{i_1 j_1} \hat{\gamma}_1) \cdots (\hat{\gamma}_s^{-1} a_{i_s j_s} \hat{\gamma}_s) \hat{\beta}_2 \]
by (F)
\[ \sim_{IP} \hat{A}_1 \hat{\beta}_2 \]
by the observation and (F)
\[ = \hat{A}_2 \hat{\beta}_2 \]
\[ \sim_{IP} w_2, \]
completing the proof.

\[ 7.2. \text{The Pure Factorisable Braid Monoid} \]

In this section we give a presentation of \( \mathcal{F} P_n \), the pure factorisable braid monoid. Again, the key step in deriving this presentation is to find a generating set for \( P \cap B_E \) for an arbitrary equivalence \( E \in E \). In order to do this, it will be necessary to prove some preliminary results which extend Lemma 4.2.

For \( 1 \leq i_1 < i_2 < i_3 \leq n \), define
\[ \beta_{i_1 i_2 i_3} = (s_{i_1 - 1} \cdots s_{i_1})(s_{i_2 - 1} \cdots s_{i_2})(s_{i_3 - 1} \cdots s_{i_3}) \in B. \]
See Figure 7.1 for an illustration.

![Figure 7.1](image_url)

**Figure 7.1.** The braid \( \beta_{i_1 i_2 i_3} \in B \).

The proof of the next result makes use of the fact, which may easily be checked diagrammatically with the aid of Figure 7.1, that
\[ \alpha_{i_r i_s} = \beta_{i_1 i_2 i_3} \alpha_{r s} \beta_{i_1 i_2 i_3}^{-1} \]
for any \( 1 \leq r < s \leq 3 \).

**Lemma 7.3.** Suppose that \( 1 \leq i < j < k \leq n \) and \( \delta \in B \). Then there exists \( \gamma \in P \) such that
\[ \delta^{-1} \alpha_{i k} \alpha_{j k}^{-1} \delta = \gamma^{-1} (\alpha_{i j}, \beta \alpha_{j k}, \delta) \gamma. \]
7.2. THE PURE FACTORISABLE BRAID MONOID

Proof. First, it is easy to check diagrammatically (or algebraically, using the braid relations) that if $\delta$ is one of

$$1, \varsigma_1, \varsigma_2\varsigma_1, \varsigma_2\varsigma_2, \varsigma_1, \varsigma_1\varsigma_2,$$

then $\delta^{-1}\alpha_{13}\alpha_{23}^{-1}\delta = \gamma^{-1}\alpha_{15}\alpha_{33}^{-1}\gamma$ where $\gamma \in P$ is equal to

$$1, 1, 1, \alpha_{12}\alpha_{23}, \alpha_{13}\alpha_{12}, \alpha_{13}\alpha_{23},$$

respectively. See for example Figure 7.2 for a proof in the case $\delta = \varsigma_1$.

Figure 7.2. An example calculation:

$$\varsigma_1^{-1}\alpha_{13}\alpha_{23}^{-1}\varsigma_1 = \alpha_{12}^{-1}\alpha_{13}^{-1}\alpha_{23}\alpha_{12} = \alpha_{12}^{-1}\alpha_{13}^{-1}(\alpha_{23}\alpha_{13}^{-1})\alpha_{13}\alpha_{12}.$$ 

Suppose now that $1 \leq i < j < k \leq n$ and $\delta \in B$, and put $x = \delta^{-1}\alpha_{ijk}\alpha_{jk}^{-1}\delta$. Write

$$\{i, j, k\}\delta = \{i_1, i_2, i_3\} \quad \text{and} \quad n \setminus \{i_1, i_2, i_3\} = \{\ell_4, \ldots, \ell_n\}$$

where $i_1 < i_2 < i_3$ and $\ell_4 < \cdots < \ell_n$. Then, writing $\delta_1 = \beta_{ijk}^{-1}\delta$, we have

$$x = \delta^{-1}(\beta_{ijk}\alpha_{13}\beta_{ijk}^{-1})(\beta_{ijk}\alpha_{23}\beta_{ijk}^{-1})\delta = \delta_1^{-1}\alpha_{13}\alpha_{23}^{-1}\delta_1$$

and $(1, 2, 3)\delta_1 = (i, j, k)\delta$. Choose $\delta_2 \in \langle \varsigma_4, \ldots, \varsigma_{n-1} \rangle$ such that

$$(4, \ldots, n)\delta_2 = (\ell_4, \ldots, \ell_n)\delta_1^{-1}.$$ 

Then $\delta_2$ clearly commutes with $\alpha_{13}\alpha_{23}^{-1} \in \langle \varsigma_1, \varsigma_2 \rangle$ and so, writing $\delta_3 = \delta_2\delta_1$, we have

$$x = \delta_1^{-1}\delta_2^{-1}\alpha_{13}\alpha_{23}^{-1}\delta_2\delta_1 = \delta_3^{-1}\alpha_{13}\alpha_{23}^{-1}\delta_3,$$

and

$$(1, 2, 3)\delta_3 = (1, 2, 3)\delta_2\delta_1 = (1, 2, 3)\delta_1 = (i, j, k)\delta,$$

$$(4, \ldots, n)\delta_3 = (4, \ldots, n)\delta_2\delta_1 = (\ell_4, \ldots, \ell_n)\delta_1^{-1}\delta_1 = (\ell_4, \ldots, \ell_n).$$

Now choose $\delta_4 \in \{1, \varsigma_2, \varsigma_2\varsigma_1, \varsigma_2\varsigma_2, \varsigma_1, \varsigma_1\varsigma_2\}$ such that

$$\delta_4 = (i_1\delta_4, i_2\delta_4, i_3\delta_4),$$

and put $\delta_5 = \delta_4^{-1}\delta_3$. Then

$$x = \delta_3^{-1}\delta_4\delta_4^{-1}\alpha_{13}\alpha_{23}^{-1}\delta_4\delta_4^{-1}\delta_3 = \delta_5^{-1}(\delta_4^{-1}\alpha_{13}\alpha_{23}^{-1}\delta_4)\delta_5,$$
and
\[(1, 2, 3)\tilde{\delta}_5 = (1, 2, 3)\tilde{\delta}_3^{-1} = (1\tilde{\delta}_4^{-1}, 2\tilde{\delta}_4^{-1}, 3\tilde{\delta}_4^{-1})\tilde{\delta}_3 = (i_{1\delta_4^{-1}}, i_{2\delta_4^{-1}}, i_{3\delta_4^{-1}}) = (i_1, i_2, i_3), \]
\[(4, \ldots, n)\tilde{\delta}_5 = (4, \ldots, n)\tilde{\delta}_3^{-1} = (\ell_4, \ldots, \ell_n). \]

In particular, \(\tilde{\delta}_5 = \tilde{\beta}_i^{-1} \tilde{\beta}_j \tilde{\gamma}_i \tilde{\gamma}_j.\) By the calculations at the beginning of this proof, there exists \(\gamma_1 \in P\) such that
\[\tilde{\delta}_4^{-1} \alpha_{13} \alpha_{23}^{-1} \tilde{\delta}_4 = \gamma_1^{-1} \alpha_{1\delta_4,3\delta_4} \alpha_{2\delta_4,3\delta_4}^{-1} \gamma_1.\]

Now put \(\gamma = \beta_{i_1 i_2} \gamma_1 \tilde{\delta}_5.\) Then
\[\gamma = \tilde{\beta}_{i_1 i_2} \gamma_1 \tilde{\delta}_5 = \tilde{\beta}_{i_1 i_2} \tilde{\beta}_i^{-1} \tilde{\beta}_j^{-1} = 1, \]
so that \(\gamma \in P.\) We also have
\[x = \delta_5^{-1} (\gamma_1^{-1} \alpha_{1\delta_4,3\delta_4} \alpha_{2\delta_4,3\delta_4}^{-1} \gamma_1) \delta_5 = (\delta_5^{-1} \gamma_1^{-1} \beta_{i_1 i_2}) (\beta_{i_1 i_2} \alpha_{1\delta_4,3\delta_4} \alpha_{2\delta_4,3\delta_4}^{-1} \beta_{i_1 i_2}^{-1}) (\beta_{i_1 i_2} \gamma_1 \delta_5) = \gamma^{-1} \alpha_{i_1 \delta_4, i_3 \delta_4} \alpha_{i_3 \delta_4, i_2 \delta_4}^{-1} \gamma, \]
and we are done since \((i_{1\delta_4}, i_{2\delta_4}, i_{3\delta_4}) = (1, 2, 3)\tilde{\delta}_3 = (i, j, k).\) \(\Box\)

The next two lemmas follow by almost identical proofs.

**Lemma 7.4.** Suppose that \(1 \leq i < j < k \leq n\) and \(\delta \in B.\) Then there exists \(\gamma \in P\) such that \(\delta^{-1} \alpha_{ij} \alpha_{jk}^{-1} \delta = \gamma^{-1} \alpha_{i\delta, j3} \alpha_{j\delta, k3}^{-1} \gamma.\) \(\Box\)

**Lemma 7.5.** Suppose that \(1 \leq i < j < k \leq n\) and \(\delta \in B.\) Then there exists \(\gamma \in P\) such that \(\delta^{-1} \alpha_{ij} \alpha_{ik}^{-1} \delta = \gamma^{-1} \alpha_{i\delta, j\delta} \alpha_{j\delta, k\delta}^{-1} \gamma.\) \(\Box\)

Also of use will be the following result concerning group presentations which is almost certainly well-known. In a sense this result is the converse to Lemma 6.5.

**Lemma 7.6.** Suppose that \(G\) is a group with group presentation \(\langle X \mid R \rangle\) via \(\phi.\) Suppose also that \(N\) is a normal subgroup of \(G\) and that \(W \subseteq F(X)\) is a set of words such that
(i) \(W \phi \subseteq N;\)
(ii) \(G/N\) has group presentation \(\langle X \mid R \cup R_W \rangle\) via
\[\Phi : F(X) \to G/N : w \mapsto N(w \phi) \quad \text{for each } w \in F(X), \]
where \(R_W\) is the set of relations
\[w = 1 \quad \text{for all } w \in W.\]

Then \(N\) is the normal closure in \(G\) of \(W \phi.\)
Proof. Let $N_W$ denote the normal closure in $G$ of $W\phi$ and write $\sim_R = R^2$. Since $W\phi \subseteq N$ and $N$ is normal, it follows that $N_W \subseteq N$. Conversely, suppose that $g \in N$, and choose a word $w \in F(X)$ such that $w\phi = g$. Then we have a sequence of words $w_1, \ldots, w_k \in F(X)$ such that $w_1 = w_1 = w, w_k = 1$, and each $w_{j+1}$ is obtained from $w_j$ by a single use of a relation from $R \cup R_W$. Let $\ell$ be the number of times a relation from $R_W$ is used in this sequence. If $\ell = 0$, then $w \sim_R 1$ so that $g = w\phi = 1 \in N_W$, so suppose now that $\ell \geq 1$. Denote by $i$ the first index such that $w_{i+1}$ is obtained from $w_i$ by using a relation from $R_W$. Then there exist words $u, v \in F(X)$ and $w \in W$ such that either

(i) $w_i = uwv$ and $w_{i+1} = uv$; or

(ii) $w_i = uw$ and $w_{i+1} = uwv$.

Now we have $w \sim_R w_i$ so that $g = w\phi = w_i\phi$ and, by an induction hypothesis, we have $w_{i+1}\phi \in N_W$. Now if (i) holds, then

$$g = w_i\phi = (uwv)^{-1}uv\phi = ((u\phi)(w\phi)(u\phi)^{-1})(uv)\phi,$$

and we are done since $(u\phi)(w\phi)(u\phi)^{-1} \in N_W$ and $(uv)\phi = w_{i+1}\phi \in N_W$. On the other hand, if (ii) holds, then

$$g = w_i\phi = (uwv)^{-1}uv\phi = ((u\phi)(w\phi)(u\phi)^{-1})(uv)\phi,$$

and we are done since $(u\phi)(w\phi)(u\phi)^{-1} \in N_W$ and $(uv)\phi = w_{i+1}\phi \in N_W$.

We are now ready to work towards our presentation of $\mathcal{P}_n$. Let $E \in E$. Now $P$ and $B_E$ are both normal subgroups of $P_E$, and so $P \cap B_E$ is a normal subgroup of $P$. Thus we may consider the quotient $P/(P \cap B_E)$. For $\beta \in P$ we will denote by $[\beta]_E^P$ the coset $(P \cap B_E)\beta$ which, by Theorem 4.8, is the set of all pure braids which are $\sim_E$-equivalent to $\beta$.

Lemma 7.7. Suppose that $E \in E$. Then $P_E = PB_E$, and we have an isomorphism

$$\theta_E : P/(P \cap B_E) \rightarrow P_E/B_E : [\beta]_E^P \rightarrow [\beta]_E \quad \text{for each } \beta \in P.$$

Proof. Since $P$ and $B_E$ are both contained in $P_E$, we have $PB_E \subseteq P_E$. Conversely, suppose that $\beta \in P_E$. Then we have

$$\beta = t_{i_1,j_1} \cdots t_{i_r,j_r}$$

for some $(i_1, j_1), \ldots, (i_r, j_r) \in E$ with $i_t < j_t$ for each $t \in r$. Now put $\gamma = \varsigma_{i_1,j_1} \cdots \varsigma_{i_r,j_r} \in B_E$. Then since $\beta = \gamma$ we have $\beta\gamma^{-1} \in P$, whence

$$\beta = (\beta\gamma^{-1})\gamma \in PB_E.$$

By the Second Isomorphism Theorem (see for example [16]), $\theta_E$ is an isomorphism

$$P/(P \cap B_E) \rightarrow (PB_E)/B_E = P_E/B_E,$$

and the proof is complete. \qed
Suppose now that $E \in E$ is a standard equivalence. Recall that $X_{P_E} = X_P \cup \{\sigma_i \mid i \in n^\flat\}$ where $n^\flat = \{i \in n \mid (i, i + 1) \in E\}$, and that $R_{P_E} = R_B^E \cup R_P \cup R_D \cup R_C$ where $R_B^E$ is the restriction of the braid relations $R_B$ to $\{\sigma_i \mid i \in n^\flat\}$, $R_D$ is the set of relations $R_D$ is the set of relations

$$a_{ij} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{i+1} \cdots \sigma_{j-1}^{-1}) \quad \text{if } (i, j) \in E \text{ and } i < j,$$

and $R_C$ is the set of relations

$$\sigma_r a_{ij} \sigma_r^{-1} = \begin{cases} a_{ij}^{-1}a_{i-1,j}a_{ij} & \text{if } r = i - 1 \\ a_{i+1,j} & \text{if } r = i < j - 1 \\ a_{ij}^{-1}a_{i,j-1}a_{ij} & \text{if } r = j - 1 > i \\ a_{i,j+1} & \text{if } r = j \\ a_{ij} & \text{otherwise,} \end{cases}$$

where $1 \leq i < j \leq n$ and $r \in n^\flat$ for each relation in $R_C$. Let $R_O$ denote the set of relations

$$\zeta_i = 1 \quad \text{for each } i \in n^\flat.$$ 

By Corollary 6.15 and Lemma 7.7, the quotient $P/(P \cap B_E)$ has group presentation

$$\langle X_{P_E} \mid R_{P_E} \cup R_O \rangle$$

via

$$a_{ij} \mapsto [\alpha_{ij}]_E^{-1} = [\alpha_{ij}]_E^P \quad \text{for each } i, j$$

$$\sigma_i \mapsto [\zeta_i]_E^{-1} = [1]_E \theta_E^{-1} = [1]_E^P \quad \text{for each } i.$$

In Section 6.2 we simplified this presentation in order to describe the maximal subgroups of $\mathcal{G}P_n$. Here it will be convenient to simplify the presentation in a different way. Let $R'_E$ denote the set of relations $R_P$ and

$$a_{ij} = 1 \quad \text{if } (i, j) \in E$$

$$a_{ik} = a_{jk} \quad \text{if } (i, j) \in E \text{ and } k \in \{i, j\}^c.$$ 

The reader is reminded again that we are using symmetric notation for the $a_{ij}$ in relation $(R_E 2)'$.

**Theorem 7.8.** Suppose that $E \in E$ is standard. Then the quotient $P/(P \cap B_E)$ has group presentation $\langle X_P \mid R'_E \rangle$ via

$$a_{ij} \mapsto [\alpha_{ij}]_E^P \quad \text{for each } i, j.$$

**Proof.** We start with the presentation $\langle X_{P_E} \mid R_{P_E} \cup R_O \rangle$ above. Now by relation (O) we may remove the generators $\sigma_i$ with $i \in n^\flat$ along with relations $R_B^E$. Relation (D) reduces
to relation \((R_{E}1)\)', and relation (C) becomes

\[
a_{ij} = \begin{cases} 
    a_{ij}^{-1}a_{i-1,j}a_{ij} & \text{if } (i-1, i) \in E \\
    a_{i+1,j} & \text{if } (i, i+1) \in E \text{ and } i < j - 1 \\
    a_{ij}^{-1}a_{i,j-1}a_{ij} & \text{if } (j-1, j) \in E \text{ and } i < j - 1 \\
    a_{i,j+1} & \text{if } (j, j+1) \in E \\
    a_{ij} & \text{otherwise.}
\end{cases}
\]

Removing trivial relations and simplifying yields

\[
a_{ij} = \begin{cases} 
    a_{i-1,j} & \text{if } (i-1, i) \in E \\
    a_{i+1,j} & \text{if } (i, i+1) \in E \text{ and } i < j - 1 \\
    a_{i,j-1} & \text{if } (j-1, j) \in E \text{ and } i < j - 1 \\
    a_{i,j+1} & \text{if } (j, j+1) \in E.
\end{cases}
\]

All of these relations are contained in \((R_{E}2)\)', and we may deduce the remaining relations in \((R_{E}2)\)' as simple consequences. \(\square\)

By Lemma 7.6 and Theorem 7.8 we have the following.

**Lemma 7.9.** If \(E \in E\) is standard, then \(P \cap B_{E}\) is the normal closure in \(P\) of the set

\[
\{ \alpha_{ij} \mid (i, j) \in E \} \cup \{ \alpha_{ik}\alpha_{jk}^{-1} \mid (i, j) \in E, k \in \{i, j\}^{c} \}.
\]

We will now generalise this result to an arbitrary equivalence \(E \in E\).

**Lemma 7.10.** If \(E \in E\), then \(P \cap B_{E}\) is the normal closure in \(P\) of the set

\[
\{ \alpha_{ij} \mid (i, j) \in E \} \cup \{ \alpha_{ik}\alpha_{jk}^{-1} \mid (i, j) \in E, k \in \{i, j\}^{c} \}.
\]

**Proof.** Choose \(\delta \in B\) such that \(E^{\delta}\) is standard. Then

\[
P \cap B_{E} = (\delta^{-1}P\delta) \cap (\delta^{-1}B_{E}\delta) = \delta^{-1}(P \cap B_{E^{\delta}})\delta,
\]

and the result now follows quickly from Lemmas 4.2, 7.3, 7.4, 7.5, and 7.9. \(\square\)

Let \(X_{FP} = X_{P} \cup X_{P}^{-1} \cup X_{E}\), and let \(R_{FP}\) denote the set of relations \(R_{P} \cup R_{F} \cup R_{E}\) along with

\[
\varepsilon_{ij}a_{rs} = a_{rs}\varepsilon_{ij} \quad \text{for each } i, j, r, s \quad (FP1)
\]
\[
\varepsilon_{ij}a_{ij} = \varepsilon_{ij} \quad \text{for each } i, j \quad (FP2)
\]
\[
\varepsilon_{ij}a_{ik} = \varepsilon_{ij}a_{jk} \quad \text{for each } i, j, k. \quad (FP3)
\]
Theorem 7.11. The pure factorisable braid monoid $\mathfrak{FP}_n$ has presentation $\langle X_{FP} \mid R_{FP} \rangle$ via
\[
\begin{align*}
    a_{ij}^{\pm 1} &\mapsto [1, a_{ij}^{\pm 1}] \equiv a_{ij}^{\pm 1} & \text{for each } i, j \\
    \varepsilon_{ij} &\mapsto [\varepsilon_{ij}, 1] \equiv [1] \varepsilon_{ij} & \text{for each } i, j.
\end{align*}
\]

Proof. Let $\Psi$ denote the map in the statement of the theorem and write $\sim_{FP} = R_{FP}^*$. To show that $\Psi$ is surjective, suppose that $[\mathcal{E}, \beta] \in \mathfrak{FP}_n$. Then we have $\beta \in P_\mathcal{E}$ and, since $P_\mathcal{E} = PB_\mathcal{E} = B_\mathcal{E} P$ by Lemma 7.7, it follows that $\beta = \beta_1 \beta_2$ for some $\beta_1 \in B_\mathcal{E}$ and $\beta_2 \in P$. But then $\beta \beta_2^{-1} = \beta_1 \in B_\mathcal{E}$, so that $[\mathcal{E}, \beta] = [\mathcal{E}, \beta_2] = [\mathcal{E}, 1][1, \beta_2] \in \text{im} \Psi$.

Next, it is easy to check diagrammatically that $R_{FP} \subseteq \ker \Psi$. It remains to show that $\ker \Psi \subseteq \sim_{FP}$, so suppose that $(w_1, w_2) \in \ker \Psi$. Now by (FP1) and $R_P \cup R_E \cup R_E$ we have
\[
w_1 \sim_{FP} \widehat{\mathcal{E}}_1 \beta_1 \quad \text{and} \quad w_2 \sim_{FP} \widehat{\mathcal{E}}_2 \beta_2
\]
for some $\mathcal{E}_1, \mathcal{E}_2 \in E$ and $\beta_1, \beta_2 \in P$. Thus
\[
[\mathcal{E}_1, \beta_1] = [\mathcal{E}_1, 1][1, \beta_1] = w_1 \Psi = w_2 \Psi = [\mathcal{E}_2, 1][1, \beta_2] = [\mathcal{E}_2, \beta_2]
\]
from which it follows that $\mathcal{E}_1 = \mathcal{E}_2$ and $\beta_1 \beta_2^{-1} \in P \cap B_\mathcal{E}_1$. Now
\[
w_1 \sim_{FP} \widehat{\mathcal{E}}_1 \beta_1 \sim_{FP} \widehat{\mathcal{E}}_1 \beta_1 \beta_2^{-1} \beta_2
\]
by (F) and, since $\tilde{\mathcal{E}}_1 = \mathcal{E}_2$, it suffices to prove that $\widehat{\mathcal{E}}_1 \beta_1 \beta_2^{-1} \sim_{FP} \tilde{\mathcal{E}}_1$. By Lemma 7.10 and (F), it is enough to show that $\tilde{\mathcal{E}}_1 \gamma^{-1} x \sim_{FP} \tilde{\mathcal{E}}_1$ for any $\gamma \in P$ and $x \in \{a_{ij} \mid (i, j) \in \mathcal{E}_1\} \cup \{a_{ik} a_{jk}^{-1} \mid (i, j) \in \mathcal{E}_1, k \in \{i, j\}^c\}$.

Now let $\gamma \in P$ and $(i, j) \in \mathcal{E}_1$ with $i < j$. Then
\[
\begin{align*}
    \tilde{\mathcal{E}}_1 \gamma^{-1} a_{ij} \gamma &\sim_{FP} \gamma^{-1} \tilde{\mathcal{E}}_1 a_{ij} \gamma & \text{by (FP1) and (F)} \\
    \sim_{FP} \gamma^{-1} \tilde{\mathcal{E}}_1 \varepsilon_{ij} a_{ij} \gamma &\sim_{FP} R_E \\
    \sim_{FP} \gamma^{-1} \tilde{\mathcal{E}}_1 \varepsilon_{ij} \gamma &\sim_{FP} (FP2) \\
    \sim_{FP} \tilde{\mathcal{E}}_1 &\sim_{FP} R_E, (FP1), \text{ and (F),}
\end{align*}
\]
and if, in addition, $k \in \{i, j\}^c$, then
\[
\begin{align*}
    \tilde{\mathcal{E}}_1 \gamma^{-1} a_{ik} a_{jk}^{-1} \gamma &\sim_{FP} \gamma^{-1} \tilde{\mathcal{E}}_1 a_{ik} a_{jk}^{-1} \gamma & \text{by (FP1) and (F)} \\
    \sim_{FP} \gamma^{-1} \tilde{\mathcal{E}}_1 \varepsilon_{ij} a_{ik} a_{jk}^{-1} \gamma &\sim_{FP} R_E \\
    \sim_{FP} \gamma^{-1} \tilde{\mathcal{E}}_1 \varepsilon_{ij} a_{ik} a_{jk}^{-1} \gamma &\sim_{FP} (FP3) \\
    \sim_{FP} \tilde{\mathcal{E}}_1 &\sim_{FP} R_E, (FP1), \text{ and (F).}
\end{align*}
\]
This completes the proof. $\square$
7.3. The Pure Permeable Braid Monoid

We now give a presentation of \( \mathbb{P}_n \), the pure permeable braid monoid. This presentation will be much easier to derive since \( B_E^{(2)} \subseteq P \) for each \( E \in E \). Let \( X_{PP} = X_P \cup X_P^{-1} \cup X_E \), and let \( R_{PP} \) denote the set of relations \( R_P \cup R_F \cup R_E \) along with

\[
\varepsilon_{ij}a_{rs} = a_{rs}\varepsilon_{ij} \quad \text{for each } i, j, r, s \quad (PP1)
\]

\[
\varepsilon_{ij}a_{ij} = \varepsilon_{ij} \quad \text{for each } i, j. \quad (PP2)
\]

**Theorem 7.12.** The pure permeable braid monoid \( \mathbb{P}_n \) has presentation \( \langle X_{PP} \mid R_{PP} \rangle \) via

\[
a_{ij}^{\pm 1} \mapsto [1, \alpha_{ij}^{\pm 1}] \equiv \alpha_{ij}^{\pm 1} \quad \text{for each } i, j
\]

\[
\varepsilon_{ij} \mapsto [\varepsilon_{ij}, 1] \equiv [1]\varepsilon_{ij} \quad \text{for each } i, j.
\]

**Proof.** Let \( \Psi \) denote the map in the statement of the theorem and write \( \sim_{PP} = R_{PP}^4 \). Then \( \Psi \) is clearly surjective, and it is easy to check diagrammatically that \( R_{PP} \subseteq \ker \Psi \). It remains to show that \( \ker \Psi \subseteq \sim_{PP} \), so suppose that \( (w_1, w_2) \in \ker \Psi \). Now by (PP1) and \( R_P \cup R_F \cup R_E \) we have

\[
w_1 \sim_{PP} \hat{E}_1\beta_1 \quad \text{and} \quad w_2 \sim_{PP} \hat{E}_2\beta_2
\]

for some \( E_1, E_2 \in E \) and \( \beta_1, \beta_2 \in P \). We then have

\[
[E_1, \beta_1] = [E_1, 1][1, \beta_1] = w_1\Psi = w_2\Psi = [E_2, 1][1, \beta_2] = [E_2, \beta_2]
\]

from which it follows that \( E_1 = E_2 \) and \( \beta_1\beta_2^{-1} \in B_E^{(2)} \). Now

\[
w_1 \sim_{PP} \hat{E}_1\hat{\beta}_1 \sim_{PP} \hat{E}_1\hat{\beta}_1\hat{\beta}_2^{-1}\hat{\beta}_2
\]

by (F) and again it suffices to prove that \( \hat{E}_1\hat{\beta}_1\hat{\beta}_2^{-1} \sim_{PP} \hat{E}_1 \). By Lemma 4.19 and (F), it is enough to show that \( \hat{E}_1\gamma^{-1}a_{ij}\gamma \sim_{PP} \hat{E}_1 \) for any \( \gamma \in P \) and \( (i, j) \in E_1 \). But in this case we have

\[
\hat{E}_1\gamma^{-1}a_{ij}\gamma \sim_{PP} \hat{E}_1\gamma^{-1}\hat{E}_1a_{ij}\gamma \quad \text{by (PP1) and (F)}
\]

\[
\sim_{PP} \hat{E}_1\varepsilon_{ij}a_{ij}\gamma \quad \text{by } R_E
\]

\[
\sim_{PP} \hat{E}_1\varepsilon_{ij}\gamma \quad \text{by (PP2)}
\]

\[
\sim_{PP} \hat{E}_1 \quad \text{by } R_E, (\text{PP1}), \text{ and (F)},
\]

and the proof is complete. \( \Box \)

**Remark 7.13.** In the proof of Theorem 4.14 it was shown that \( \mathfrak{B}_n \) is a homomorphic image of \( \mathbb{P}_n \). By comparing the presentations of \( \mathfrak{P}_n \) and \( \mathbb{P}_n \) in Theorems 7.11 and 7.12, it is clear also that \( \mathfrak{P}_n \) is a homomorphic image of \( \mathbb{P}_n \).
CHAPTER 8

Applications of Order-Preserving Partial Permutations

In Chapter 5 we found presentations of the braid monoids $IB_n$, $\mathfrak{B}_n$, and $\mathfrak{P}B_n$. Factorisability was the key feature which allowed us to obtain the initial presentations of these monoids in Corollaries 5.2, 5.28, and 5.62. In this chapter we examine different methods of obtaining presentations of the inverse braid monoid $IB_n$. We begin by investigating $POI_n$, the monoid of all order-preserving partial permutations on $n$. This monoid has been investigated by several authors including Fernandes, who gave a presentation of $POI_n$ in [31]. We find two (anti-isomorphic) submonoids $L_n$ and $R_n$ of $POI_n$, and show that there is a natural factorisation $POI_n = L_nR_n$. We make use of this factorisation, and presentations of $L_n$ and $R_n$ to give a new presentation of $POI_n$. We then exhibit a submonoid of $IB_n$ which is naturally isomorphic to $POI_n$. Identifying $POI_n$ with this submonoid, we show that the inverse braid monoid has a natural factorisation $IB_n = L_nB_nR_n$, and then use our presentation of $POI_n$, together with Artin’s presentation of $B_n$, to give a new presentation of $IB_n$. As usual, we may then derive a corresponding presentation of the symmetric inverse semigroup $I_n$.

We also define a monoid $POIB_n \subseteq IB_n$ of order-preserving partial braids. We use a factorisation $POIB_n = L_nP_nR_n$, together with our presentation of $POI_n$ and Artin’s presentation of $P_n$, to give a presentation of $POIB_n$.

We then discuss, without proofs, further applications of the techniques of this chapter towards finding semigroup presentations of the semigroups $IB_n \setminus B_n$ (resp. $POIB_n \setminus P_n$) of strictly-partial (resp. order-preserving strictly-partial) braids. See also [29] where a semigroup presentation is obtained for the semigroup $I_n \setminus S_n$ of stricty-partial permutations on $n$. The chapter concludes with some comments regarding a possible application of order-preserving partial permutations towards finding a presentation of the dual symmetric inverse semigroup $I_n^\ast$, as well as a discussion about “why” this chapter does not feature the other braid monoids $\mathfrak{B}_n$ and $\mathfrak{P}B_n$.

Fix a positive integer $n$. We identify a partial braid with its homotopy-class and, since we will have no need to refer to a homotopy of partial braids, we may use the symbol $\sim$ for a
different purpose. Similarly, we will use the symbols $\approx$ and $\cong$ in different contexts to their previous uses in earlier chapters.

8.1. The Monoid of Order-Preserving Partial Permutations

A partial permutation $\alpha \in \mathcal{I}_n$ is said to be order-preserving if we have $i\alpha < j\alpha$ whenever $i, j \in \text{dom}(\alpha)$ and $i < j$. The set of all order-preserving partial permutations on $n$ is a submonoid of $\mathcal{I}_n$ and is denoted $\mathcal{POI}_n$. For more details see [31], and see [14] for an explanation of the notation. The results of this section will appear in [29].

8.1.1. The Monoids $\mathcal{L}_n$ and $\mathcal{R}_n$.

An element $\alpha \in \mathcal{POI}_n$ is completely determined by $\text{dom}(\alpha)$ and $\text{im}(\alpha)$. For $A \subseteq n$ with $|A| = k$, let $\tilde{\lambda}_A$ and $\tilde{\rho}_A$ be the (unique) elements of $\mathcal{POI}_n$ such that

$$
\begin{align*}
\text{dom}(\tilde{\lambda}_A) &= A, & \text{dom}(\tilde{\rho}_A) &= k, \\
\text{im}(\tilde{\lambda}_A) &= k, & \text{im}(\tilde{\rho}_A) &= A,
\end{align*}
$$

noting that $\tilde{\rho}_A = \tilde{\lambda}_A^{-1}$. See Figure 8.1 for an example.

![Figure 8.1](image)

**Figure 8.1.** The maps $\tilde{\lambda}_A, \tilde{\rho}_A \in \mathcal{POI}_8$ where $A = \{2, 3, 5, 8\}$.

The reason for our choice of notation will become clear shortly, but for the moment it suffices to notice that in Figure 8.1, the lines in the picture of $\tilde{\lambda}_A$ (resp. $\tilde{\rho}_A$) tend to the left (resp. right) as they tend downwards.

**Lemma 8.1.** If $A, B \subseteq n$, then there exists $C \subseteq n$ such that

(i) $\tilde{\lambda}_A \tilde{\lambda}_B = \tilde{\lambda}_C$;
(ii) $\tilde{\rho}_B \tilde{\rho}_A = \tilde{\rho}_C$.

**Proof.** We only prove (i), since (ii) will then follow by inversion. Since $\tilde{\lambda}_A \tilde{\lambda}_B \in \mathcal{POI}_n$, it is sufficient to show that $\text{im}(\tilde{\lambda}_A \tilde{\lambda}_B) = k$ for some $0 \leq k \leq n$. Suppose that $|A| = s$ and $B = \{b_1, \ldots, b_t\}$ with $b_1 < \cdots < b_t$. Then $s \cap B = \{b_1, \ldots, b_k\}$ for some $0 \leq k \leq t$ and so

$$
\text{im}(\tilde{\lambda}_A \tilde{\lambda}_B) = (\text{im}\tilde{\lambda}_A \cap \text{dom}\tilde{\lambda}_B) \tilde{\lambda}_B = (s \cap B) \tilde{\lambda}_B = k,
$$

and we are done. \qed
This lemma allows us to define submonoids $\mathcal{L}_n$ and $\mathcal{R}_n$ of $\mathcal{POI}_n$ by

$$
\mathcal{L}_n = \{ \lambda_A \mid A \subseteq n \}
\quad \text{and} \quad
\mathcal{R}_n = \{ \rho_A \mid A \subseteq n \}.
$$

Since the inversion map $\text{inv} : I_n \rightarrow I_n$ sends $\lambda_A$ to $\rho_A$, the restriction of $\text{inv}$ to $\mathcal{L}_n$ yields an anti-isomorphism $\mathcal{L}_n \rightarrow \mathcal{R}_n$ under which $\lambda_A \mapsto \rho_A$. This duality will allow us to deduce information about $\mathcal{R}_n$ from information known about $\mathcal{L}_n$ (or vice versa). For $i \in n$ let $i = \lambda_{(i)}$ and $i = \rho_{(i)}$, and put $\Sigma_L = \{ \lambda_1, \ldots, \lambda_n \}$ and $\Sigma_R = \{ \rho_1, \ldots, \rho_n \}$. See Figure 8.2 for an example.

![Figure 8.2. The maps $\lambda_4 \in \mathcal{L}_8$ and $\rho_4 \in \mathcal{R}_8$.](image)

**Lemma 8.2.** Let $A \subseteq n$ and suppose that $A^c = \{i_1, \ldots, i_k\}$ where $i_1 < \cdots < i_k$. Then

(i) $\lambda_A = \lambda_{i_k} \cdots \lambda_{i_1}$

(ii) $\rho_A = \rho_{i_1} \cdots \rho_{i_k}$.

In particular, $\mathcal{L}_n$ and $\mathcal{R}_n$ are generated by $\Sigma_L$ and $\Sigma_R$ respectively.

**Proof.** We only prove (i), since (ii) will follow by duality, and the final statement is clear. Now by Lemma 8.1, we have $\lambda_{i_k} \cdots \lambda_{i_1} = \lambda_B$ for some $B \subseteq n$ so it suffices to show that

$$
\text{dom}(\lambda_{i_k} \cdots \lambda_{i_1}) = A.
$$

This is obviously true if $k = 0$, so suppose that $k \geq 1$. Then by an inductive hypothesis we have

$$
\text{dom}(\lambda_{i_k} (\lambda_{i_{k-1}} \cdots \lambda_{i_1})) = (\text{dom}(\lambda_{i_{k-1}} \cdots \lambda_{i_1})) \lambda^{-1}_{i_k}
$$

$$
= (\{i_1, \ldots, i_{k-1}\}^c) \lambda^{-1}_{i_k}
$$

$$
= \{i_1, \ldots, i_k\}^c
$$

$$
= A,
$$

and we are done. \[\square\]

Let $X_L = \{ \lambda_1, \ldots, \lambda_n \}$ and $X_R = \{ \rho_1, \ldots, \rho_n \}$ be sets of letters in one-one correspondence with $\Sigma_L$ and $\Sigma_R$ respectively. Define a homomorphism $\phi : (X_L \cup X_R)^* \rightarrow \mathcal{POI}_n$ by

$$
\lambda_i \phi = \lambda_i \quad \text{and} \quad \rho_i \phi = \rho_i \quad \text{for each} \ i,
$$

and denote the restrictions $\phi|_{X^*_L}$ and $\phi|_{X^*_R}$ by $\phi_L$ and $\phi_R$ respectively. Then by Lemma 8.2,

$$
\phi_L : X^*_L \rightarrow \mathcal{L}_n \quad \text{and} \quad \phi_R : X^*_R \rightarrow \mathcal{R}_n
$$
are epimorphisms. Let $R_L$ and $R_R$ denote the set of relations
\begin{align*}
\lambda_i \lambda_j &= \lambda_{j+1} \lambda_i \quad \text{if } 1 \leq i \leq j \leq n - 1 \quad (L1) \\
\lambda_i \lambda_n &= \lambda_i \quad \text{if } 1 \leq i \leq n \quad (L2)
\end{align*}
and
\begin{align*}
\rho_i \rho_i &= \rho_i \rho_{j+1} \quad \text{if } 1 \leq i \leq j \leq n - 1 \quad (R1) \\
\rho_i \rho_i &= \rho_i \quad \text{if } 1 \leq i \leq n \quad (R2)
\end{align*}
respectively. Write $\tilde{L} = R_L$ and $\tilde{R} = R_R$. We aim to show that
\begin{align*}
\sim_L &= \ker \phi_L \quad \text{and} \quad \sim_R = \ker \phi_R
\end{align*}
so that $\mathcal{L}_n$ and $\mathcal{R}_n$ have monoid presentations
\begin{align*}
\langle X_L \mid R_L \rangle \quad \text{and} \quad \langle X_R \mid R_R \rangle
\end{align*}
via $\phi_L$ and $\phi_R$ respectively.

By the symmetry of the relations, there is a natural duality between $\sim_L$ and $\sim_R$. To be precise, define an anti-isomorphism $\tilde{\phi} : X_L^* \to X_R^*$ by $\lambda_i = \rho_i$ for each $i \in \mathbf{n}$. Then for any $w_1, w_2 \in X_L^*$, we have $w_1 \sim_L w_2$ if and only if $\tilde{w}_1 \sim_R \tilde{w}_2$.

**Lemma 8.3.** We have the inclusions $\sim_L \subseteq \ker \phi_L$ and $\sim_R \subseteq \ker \phi_R$.

**Proof.** We must simply show that relations (L1—L2) and (R1—R2) hold with each generator $\lambda_i$ or $\rho_i$ replaced by $\lambda_i$ or $\rho_i$ respectively. We do this for (L1—L2) with the aid of pictures in Figures 8.3 and 8.4.

![Figure 8.3](image1)

**Figure 8.3.** Relation (L1): $\lambda_i \lambda_j = \lambda_{j+1} \lambda_i$ if $1 \leq i \leq j \leq n - 1$.

![Figure 8.4](image2)

**Figure 8.4.** Relation (L2): $\lambda_i \lambda_n = \lambda_i$ if $1 \leq i \leq n$.

Relations (R1—R2) follow by duality (or alternatively by turning the (L1—L2) pictures upside down).

**Lemma 8.4.** Suppose that $k \geq 1$ and that $i_1, \ldots, i_k \in \mathbf{n}$. Then there exists $1 \leq s \leq k$ and $j_1, \ldots, j_s \in \mathbf{n}$ such that $\min(i_1, \ldots, i_k) = j_1 < \cdots < j_s \leq n$ and
8.1. THE MONOID OF ORDER-PRESERVING PARTIAL PERMUTATIONS

(i) \( \lambda_i \cdots \lambda_k \sim_L \lambda_j \cdots \lambda_l \);
(ii) \( \rho_i \cdots \rho_k \sim_R \rho_j \cdots \rho_l \).

**Proof.** We prove (i) by induction on \( k \), and (ii) will follow by duality. Let \( w = \lambda_i \cdots \lambda_k \).

If \( k = 1 \) then the result is clearly true. So suppose that \( k \geq 2 \). Then by an inductive hypothesis,

\[
\lambda_i \cdots \lambda_{k-1} \sim_L \lambda_j \cdots \lambda_l
\]

for some \( 1 \leq s \leq k - 1 \), and \( \min(i_1, \ldots, i_{k-1}) = j_1 < \cdots < j_s \leq n \). So

\[
w \sim_L \lambda_{j_s} \cdots \lambda_{j_1} \lambda_i \lambda_k.
\]

We are done already if \( i_k < j_1 \), while if \( i_k = n \) then we are done after an application of (L2).

So suppose that \( j_1 \leq i_k < n \). Then by (L1) we have

\[
w \sim_L \lambda_{j_s} \cdots \lambda_{j_1} \lambda_k + 1 \lambda_j.
\]

Since \( 1 \leq s \leq k - 1 \), our inductive hypothesis implies that

\[
\lambda_j \cdots \lambda_{j_1} \lambda_{k+1} \sim_L \lambda_{h_t} \cdots \lambda_{h_1}
\]

for some \( 1 \leq t \leq s \), and \( \min(i_k + 1, j_2, \ldots, j_s) = h_1 < \cdots < h_t \leq n \). In particular, \( 1 \leq t \leq k - 1 \) and \( j_1 < h_1 \) so that

\[
w \sim_L \lambda_{h_t} \cdots \lambda_{h_1} \lambda_j,
\]

with all conditions met. This completes the proof. \( \Box \)

**Theorem 8.5.** The monoids \( \mathcal{L}_n \) and \( \mathcal{R}_n \) have monoid presentations

\[
\langle X_L \mid R_L \rangle \quad \text{and} \quad \langle X_R \mid R_R \rangle
\]

via \( \phi_L \) and \( \phi_R \) respectively.

**Proof.** We prove the \( \mathcal{L}_n \) statement, and the \( \mathcal{R}_n \) statement will follow by duality. By Lemmas 8.2 and 8.3 it suffices to prove that \( \ker \phi_L \subseteq \sim_L \), so suppose that \( (w_1, w_2) \in \ker \phi_L \).

By Lemma 8.4, there exist \( 0 \leq k, h \leq n \) and \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_k \leq n \) such that

\[
w_1 \sim_L \lambda_{i_k} \cdots \lambda_{i_1} \quad \text{and} \quad w_2 \sim_L \lambda_{j_k} \cdots \lambda_{j_1}.
\]

Put \( A = \{i_1, \ldots, i_k\}^c \) and \( B = \{j_1, \ldots, j_k\}^c \). Then by Lemma 8.2,

\[
\tilde{\lambda}_A = \tilde{\lambda}_{i_k} \cdots \tilde{\lambda}_{i_1} = w_1 \phi_L = w_2 \phi_L = \tilde{\lambda}_{j_k} \cdots \tilde{\lambda}_{j_1} = \tilde{\lambda}_B
\]

from which it follows that \( A = B \), and so \( k = h \) and \( i_s = j_s \) for each \( s \in k \). Thus

\[
w_1 \sim_L \lambda_{i_k} \cdots \lambda_{i_1} = \lambda_{j_k} \cdots \lambda_{j_1} \sim_L w_2
\]

and the theorem is proved. \( \Box \)
Remark 8.6. By Lemma 8.4, we see that
\[ |X_L^*/\sim_L| \leq \sum_{k=0}^{n} \binom{n}{k} = 2^n = |L_n|. \]
This observation, together with the fact that \( \phi_L \) is surjective and \( \sim_L \subseteq \ker \phi_L \), is enough to prove Theorem 8.5 (see for example [56]).

8.1.2. A Presentation of \( \POI_n \).

In this section we use the generators introduced earlier to obtain a presentation of \( \POI_n \).

Our presentation differs from that given by Fernandes in [31] in several ways; we use \( 2n \) generators (Fernandes’ presentation uses only \( n \)), although our relations involve words containing at most three letters (some relations in Fernandes’ presentation involve words of length \( n + 1 \)). The normal forms we use (see Corollary 8.16) involve at most \( 2n \) letters, while the normal forms used in Fernandes’ proof can be quite long.

We begin with a simple result concerning the elements of \( \POI_n \).

Lemma 8.7. Let \( \alpha \in \POI_n \). Then \( \alpha = \lambda A \beta B \) where \( A = \text{dom}(\alpha) \) and \( B = \text{im}(\alpha) \). In particular, \( \POI_n = L_n R_n \) is generated by \( \Sigma_L \cup \Sigma_R \).

Proof. The first statement is clear. The second statement follows from the first and Lemma 8.2. \( \square \)

As an immediate consequence, we see that \( \phi : (X_L \cup X_R)^* \to \POI_n \) defined earlier by
\[ \lambda_i \phi = \lambda_i \quad \text{and} \quad \rho_i \phi = \rho_i \quad \text{for each} \ i \]
is an epimorphism. Denote by \( R_{LR} \) the set of relations \( R_L \cup R_R \) together with
\[ \rho_i \lambda_j = \begin{cases} 
\lambda_n \lambda_{j-1} \rho_i & \text{if } 1 \leq i < j \leq n \\
\lambda_n = \rho_n & \text{if } 1 \leq i = j \leq n \\
\lambda_n \lambda_j \rho_{i-1} & \text{if } 1 \leq j < i \leq n, 
\end{cases} \quad \text{(RL1—RL3)} \]
and let \( \sim \) denote the congruence \( R_{LR}^4 \) on \( (X_L \cup X_R)^* \). Our aim is to show that \( \sim = \ker \phi \) so that \( \POI_n \) has monoid presentation
\[ \langle X_L \cup X_R \mid R_{LR} \rangle \]
via \( \phi \).

Lemma 8.8. We have the inclusion \( \sim \subseteq \ker \phi \).

Proof. By Lemma 8.3 it suffices to check that relations (RL1—RL3) are preserved by \( \phi \). This may be done using pictures similar to those in the proof of Lemma 8.3. \( \square \)
The next lemma shows that \((X_L \cup X_R)^* \sim X_L^* X_R^*\) by which we mean that words over \(X_L \cup X_R\) have a factorisation into a “left word” (over \(X_L\)) and a “right word” (over \(X_R\)).

**Lemma 8.9.** Suppose that \(w \in (X_L \cup X_R)^*\). Then \(w \sim w_L w_R\) for some \(w_L \in X_L^*\) and \(w_R \in X_R^*\).

**Proof.** If \(w \in X_L^*\) then there is nothing to prove. Otherwise, we may use (RL1—RL3) to move the right-most generator \(\rho_i\) to the right. The result now follows by a simple induction. 

The next result follows immediately from Lemmas 8.4 and 8.9.

**Corollary 8.10.** Suppose that \(w \in (X_L \cup X_R)^*\). Then there exist \(0 \leq s, t \leq n\), and integers \(1 \leq i_1 < \cdots < i_s \leq n\), \(1 \leq j_1 < \cdots < j_t \leq n\) such that

\[
w \sim \lambda_{i_1} \cdots \lambda_{i_s} \rho_{j_1} \cdots \rho_{j_t}.
\]

We desire to improve the result of Corollary 8.10 by showing that the factorisation may be chosen in such a way that \(s = t\) (see Corollary 8.16). Roughly speaking, we will first show that the longer of the two words \(\lambda_{i_1} \cdots \lambda_{i_s}\) or \(\rho_{j_1} \cdots \rho_{j_t}\) may be used to introduce some extra generators (of the opposite “kind”) to the middle of this expression (Corollary 8.14). We then show that these extra generators may be absorbed by the shorter word (Lemma 8.15) resulting in two words of equal length. Before moving on, we will prove a duality result, which will allow us to simplify some of the proofs.

**Lemma 8.11.** Define an anti-isomorphism \(\sim : (X_L \cup X_R)^* \rightarrow (X_L \cup X_R)^*\) by \(\lambda_i \mapsto \rho_i\) and \(\rho_i \mapsto \lambda_i\) for each \(i \in n\). Then for any \(w_1, w_2 \in (X_L \cup X_R)^*\), we have \(w_1 \sim w_2\) if and only if \(\hat{w}_1 \sim \hat{w}_2\).

**Proof.** It suffices to show that \(\sim\) is generated by a set of relations which is invariant under \(\sim\). Now \(\sim\) interchanges (L1—L2) and (R1—R2), and swaps (RL1—RL3) with

\[
\rho_j \lambda_i = \begin{cases} 
\lambda_i \rho_{j-1} \rho_n & \text{if } 1 \leq i < j \leq n \\
\rho_n = \lambda_n & \text{if } 1 \leq i = j \leq n \\
\lambda_{i-1} \rho_j \rho_n & \text{if } 1 \leq j < i \leq n.
\end{cases}
\]

We now show that relations (RL1)—(RL3) are contained in \(\sim\). Now (RL2) is exactly the same as (RL2). Suppose next that \(1 \leq i < j \leq n\). Then by applying (RL3), (L1), (L2), (RL1), and then (RL2), we have

\[
\rho_j \lambda_i \sim \lambda_n \lambda_i \rho_{j-1} \sim \lambda_i \lambda_{n-1} \rho_{j-1} \sim \lambda_i \lambda_n \lambda_{n-1} \rho_{j-1} \sim \lambda_i \rho_{j-1} \lambda_n \sim \lambda_i \rho_{j-1} \rho_n
\]
so that (RL1) is contained in ∼. Finally suppose that 1 ≤ j < i ≤ n. Then by applying (RL1), (L1), (L2), (RL1), and then (RL2), we have

\[ \rho_j \lambda_i \sim \lambda_n \lambda_{i-1} \rho_j \sim \lambda_{i-1} \lambda_n \lambda_{i-1} \rho_j \sim \lambda_{i-1} \rho_n \lambda_i \sim \lambda_{i-1} \rho_j \rho_n. \]

Thus (RL3) is contained in ∼, and the proof is complete. □

**Lemma 8.12.** Suppose that 1 ≤ k ≤ n and 1 ≤ i_1 < \cdots < i_k ≤ n. Then

(i) \( \lambda_{i_k} \cdots \lambda_{i_1} \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{n-k+1} \); 
(ii) \( \rho_{i_1} \cdots \rho_{i_k} \sim \lambda_{n-k+1} \rho_{i_1} \cdots \rho_{i_k} \).

**Proof.** We prove (i) by induction on \( k \), and (ii) will follow by duality. Now if \( k = 1 \), then (i) holds since \( \lambda_{i_1} \sim \lambda_{i_1} \lambda_n \sim \lambda_{i_1} \rho_n \) by (L2) and (RL2), so suppose that 2 ≤ k ≤ n. Since 1 ≤ i_1 < \cdots < i_k ≤ n we have i_1 < n - k + 2. But then

\[ \lambda_{i_k} \cdots \lambda_{i_2} \lambda_{i_1} \sim \lambda_{i_k} \cdots \lambda_{i_2} \rho_{n-k+2} \lambda_{i_1} \quad \text{by an inductive hypothesis} \]

\[ \sim \lambda_{i_k} \cdots \lambda_{i_2} \lambda_n \lambda_{i_1} \rho_{n-k+1} \quad \text{by (RL3)} \]

\[ \sim \lambda_{i_k} \cdots \lambda_{i_2} \lambda_{i_1} \rho_{n-k+1} \quad \text{by (L2)}, \]

and we are done. □

**Lemma 8.13.** Suppose that 1 ≤ i ≤ n and 1 ≤ s ≤ n - i + 1. Then

(i) \( \lambda_i^s \sim_L \lambda_{i+s-1} \cdots \lambda_{i+1} \lambda_i \);
(ii) \( \rho_i^s \sim_R \rho_i \rho_{i+1} \cdots \rho_{i+s-1} \).

**Proof.** We only prove (i) since (ii) will follow by duality. If \( s = 1 \) then (i) is true trivially, so suppose that 2 ≤ s ≤ n - i + 1. Then

\[ \lambda_i^s = \lambda_i^{s-1} \lambda_i \]

\[ \sim_L \lambda_{i+s-1} \lambda_i^{s-1} \quad \text{by (L1) applied } s - 1 \text{ times} \]

\[ \sim_L \lambda_{i+s-1} \lambda_{i+s-2} \cdots \lambda_{i+1} \lambda_i \quad \text{by an inductive hypothesis}, \]

and we are done. □

**Corollary 8.14.** Suppose that 1 ≤ s ≤ k ≤ n, and 1 ≤ i_1 < \cdots < i_k ≤ n. Then

(i) \( \lambda_{i_k} \cdots \lambda_{i_1} \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{n-k+1} \cdots \rho_{n-k+s} \);
(ii) \( \rho_{i_1} \cdots \rho_{i_k} \sim \lambda_{n-k+s} \cdots \lambda_{n-k+1} \rho_{i_1} \cdots \rho_{i_k} \).

**Proof.** We prove (i), and then (ii) will follow by duality. Now

\[ \lambda_{i_k} \cdots \lambda_{i_1} \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{n-k+1}^s \quad \text{by Lemma 8.12(i) applied } s \text{ times} \]

\[ \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{n-k+1} \cdots \rho_{n-k+s} \quad \text{by Lemma 8.13(ii)}, \]
and we are done.

Lemma 8.15. Suppose that $0 \leq k \leq n - 1$, $1 \leq s \leq n - k$, and $1 \leq i_1 < \cdots < i_k \leq n$. Then there exist integers $1 \leq j_1 < \cdots < j_{k+s} \leq n$ such that

(i) $(\lambda_{i_k} \cdots \lambda_{i_1}) \lambda_{n-k} \cdots \lambda_{n-k-s+1} \sim_L \lambda_{j_{k+s}} \cdots \lambda_{j_1}$;

(ii) $\rho_{n-k-s+1} \cdots \rho_{n-k} (\rho_{i_1} \cdots \rho_{i_k}) \sim_R \rho_{j_1} \cdots \rho_{j_{k+s}}$.

Proof. We only prove (i) since (ii) will follow by duality. By Lemma 8.2, $\text{im}(\lambda_{i_k} \cdots \lambda_{i_1}) = \{1, \ldots, n-k\}$, and $\lambda_{n-k} \cdots \lambda_{n-k-s+1} = \lambda_A$ where

$A = \{n-k-s+1, \ldots, n-k\}^c = \{1, \ldots, n-k-s, n-k+1, \ldots, n\}$.

Thus

$$\text{im}((\lambda_{i_k} \cdots \lambda_{i_1}) \lambda_{n-k} \cdots \lambda_{n-k-s+1}) = (\text{im}(\lambda_{i_k} \cdots \lambda_{i_1}) \cap \text{dom} \lambda_A) \lambda_A$$

$$= (\{1, \ldots, n-k\} \cap A) \lambda_A$$

$$= \{1, \ldots, n-k-s\} \lambda_A$$

$$= \{1, \ldots, n-k-s\},$$

so that $(\lambda_{i_k} \cdots \lambda_{i_1}) \lambda_{n-k} \cdots \lambda_{n-k-s+1} = \lambda_B$ for some $B \subseteq \{1, \ldots, n-k-s\}$. But then $B^c = \{j_1, \ldots, j_{k+s}\}$ for some $1 \leq j_1 < \cdots < j_{k+s} \leq n$ and so

$$(\lambda_{i_k} \cdots \lambda_{i_1}) \lambda_{n-k} \cdots \lambda_{n-k-s+1} = \lambda_{j_{k+s}} \cdots \lambda_{j_1}$$

by Lemma 8.2. The result now follows from Theorem 8.5.

Corollary 8.16. Suppose that $w \in (X_L \cup X_R)^*$. Then there exists $0 \leq k \leq n$, and integers $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ such that

$$w \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{j_1} \cdots \rho_{j_k}.$$

Proof. By Corollary 8.10, there exist $1 \leq u_1 < \cdots < u_s \leq n$ and $1 \leq v_1 < \cdots < v_t \leq n$ such that

$$w \sim \lambda_{u_s} \cdots \lambda_{u_1} \rho_{v_t} \cdots \rho_{v_1}.$$

If $s = t$ we are done, so suppose that $s \neq t$. There are two cases to consider:

(i) $s > t$; and

(ii) $s < t$. 


By duality, we need only consider case (i), so suppose that \( s > t \) and put \( h = s - t \). In particular, \( 1 \leq h \leq s \) and
\[
w \sim (\lambda_{u_s} \cdots \lambda_{u_1})\rho_{n-s+1} \cdots \rho_{n-s+h}(\rho_{v_1} \cdots \rho_{v_n}) \quad \text{by Corollary 8.14(i)}
\]
\[
= (\lambda_{u_s} \cdots \lambda_{u_1})\rho_{n-t+h+1} \cdots \rho_{n-t}(\rho_{v_1} \cdots \rho_{v_n})
\]
\[
\sim (\lambda_{u_s} \cdots \lambda_{u_1})(\rho_{j_1} \cdots \rho_{j_{t+h}}) \quad \text{by Lemma 8.15(ii)}
\]
for some \( 1 \leq j_1 < \cdots < j_{t+h} \leq n \), and we are done, noting that \( t + h = s \).

\[\square\]

**Theorem 8.17.** The monoid \( \mathcal{POI}_n \) has monoid presentation
\[
\langle X_L \cup X_R \mid R_{LR} \rangle
\]
via \( \phi \).

**Proof.** By Lemmas 8.7 and 8.8 it suffices to show that \( \ker \phi \subseteq \sim \), so suppose that \((w_1, w_2) \in \ker \phi \). By Corollary 8.16,
\[
w_1 \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{j_1} \cdots \rho_{j_k} \quad \text{and} \quad w_2 \sim \lambda_{u_h} \cdots \lambda_{u_1} \rho_{v_1} \cdots \rho_{v_h}
\]
for some \( 0 \leq k, h \leq n \), and \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_k \leq n \), \( 1 \leq u_1 < \cdots < u_h \leq n \), \( 1 \leq v_1 < \cdots < v_h \leq n \).

Put
\[
A = \{i_1, \ldots, i_k\}^c, \quad B = \{j_1, \ldots, j_k\}^c, \\
C = \{u_1, \ldots, u_h\}^c, \quad D = \{v_1, \ldots, v_h\}^c.
\]

Then by Lemma 8.2,
\[
\hat{\lambda}_A \rho_B = \hat{\lambda}_{i_k} \cdots \hat{\lambda}_{i_1} \hat{\rho}_{j_1} \cdots \hat{\rho}_{j_k} = \hat{w}_1 \phi = \hat{w}_2 \phi = \hat{\lambda}_{u_h} \cdots \hat{\lambda}_{u_1} \hat{\rho}_{v_1} \cdots \hat{\rho}_{v_h} = \hat{\lambda}_C \rho_D.
\]
Since \( |A| = |B| \), we see that \( \text{dom}(\hat{\lambda}_A \rho_B) = A \) and \( \text{im}(\hat{\lambda}_A \rho_B) = B \). Similarly we have \( \text{dom}(\hat{\lambda}_C \rho_D) = C \) and \( \text{im}(\hat{\lambda}_C \rho_D) = D \). But then \( A = C \) and \( B = D \) so that \( k = h \) and \( i_s = u_s, j_s = v_s \) for each \( s \in k \). Thus
\[
w_1 \sim \lambda_{i_k} \cdots \lambda_{i_1} \rho_{j_1} \cdots \rho_{j_k} = \lambda_{u_h} \cdots \lambda_{u_1} \rho_{v_1} \cdots \rho_{v_h} \sim w_2,
\]
and the theorem is proved. \( \square\)

**Remark 8.18.** A similar comment to that in Remark 8.6 may be made here noting that, by Corollary 8.16, we have

\[
|\langle X_L \cup X_R \rangle / \sim | = \sum_{k=0}^{n} \binom{n}{k}^2 = |\mathcal{POI}_n|.
\]
8.2. The Inverse Braid Monoid

In this section we use Theorem 8.17 to derive a new presentation of $\mathcal{IB}_n$. One may also use Tietze transformations to derive this presentation from Theorem 5.13, but our method in this section is direct, and illustrates a completely different technique.

Using a slightly different notation to earlier chapters, put $X_B = \{\sigma_1^\pm, \ldots, \sigma_{n-1}^\pm\}$, and let $R_B$ denote the set of relations

- $\sigma_i^\pm \sigma_i^\mp = 1$ for all $i$ (F)
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$ (B1)
- $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if $|i - j| = 1$. (B2)

So by Theorem 2.10 the braid group $B_n$ has monoid presentation $\langle X_B \mid R_B \rangle$ via

$$\phi_B : X_B^* \to \mathcal{B}_n : \sigma_i^\pm \mapsto \varsigma_i^\pm$$

for each $i$.

We will write $\sim_B = R_B^I$, and $\Sigma_B = \{\varsigma_1^\pm, \ldots, \varsigma_{n-1}^\pm\} = X_B\phi_B$ so that $\mathcal{B}_n$ is generated (as a monoid) by $\Sigma_B$. For $w = \sigma_i^1 \cdots \sigma_{i_k}^1 \in X_B^*$, we will write

$$w^{-1} = \sigma_i^1 \cdots \sigma_{i_k}^1 \text{ and } \overline{w} = s_{i_1} \cdots s_{i_k} = w\phi_B \in S_n.$$

If $Y \subseteq X_B^*$, write $Y^*$ for the submonoid of $X_B^*$ generated by $Y$.

We begin with some observations regarding $\mathcal{IB}_n$. For $A, B \subseteq n$ with $|A| = |B|$, denote by

- $\varphi_{A,B} \in \mathcal{POL}_n$ the order-preserving partial permutation on $n$ with domain $A$ and image $B$; and
- $\beta_{A,B} \in \mathcal{IB}_n$ the (homotopy-class of the) partial braid which consists of straight strings which join upper point $j$ to lower point $j\varphi_{A,B}$ for each $j \in A$.

For $A, B, C, D \subseteq n$ with $|A| = |B|$ and $|C| = |D|$, we have

$$\beta_{A,B} = \beta_{C,D} \iff A = C \text{ and } B = D \iff \varphi_{A,B} = \varphi_{C,D},$$

and

$$\beta_{A,B} \beta_{C,D} = \beta_{E,F} \iff \varphi_{A,B} \varphi_{C,D} = \varphi_{E,F}$$

showing that the map

$$\mathcal{POL}_n \to \mathcal{IB}_n : \varphi_{A,B} \mapsto \beta_{A,B} \text{ for each } A, B \subseteq n \text{ with } |A| = |B|$$

is an embedding. As a result, we may identify $\mathcal{POL}_n$ with the submonoid

$$\{ \beta_{A,B} \mid A, B \subseteq n, |A| = |B| \} \subseteq \mathcal{IB}_n.$$

In particular, for $A \subseteq n$ with $|A| = k$, we will abuse notation and write

$$\lambda_A = \beta_{A,k} \text{ and } \rho_A = \beta_{k,A}.$$
and we will also identify $L_n$ and $R_n$ with submonoids of $\mathcal{IB}_n$.

For $1 \leq k \leq n$, let $\mathcal{B}_k$ be the subgroup of $\mathcal{B}_n$ generated by $\{\xi^\pm_i | 1 \leq i \leq k - 1\}$. We also define $\mathcal{B}_0 = \{1\}$. Thus $\mathcal{B}_k$ is the set of all braids for which strings $k+1, \ldots, n$ are straight and do not interact with strings $1, \ldots, k$. We see then that $\mathcal{B}_k$ is naturally isomorphic to $\mathcal{B}_k$ for each $k$.

**Lemma 8.19.** Suppose that $\beta \in \mathcal{IB}_n$ and that $A = \text{dom}(\bar{\beta})$ and $B = \text{im}(\bar{\beta})$. Then, writing $k = |A| = |B|$, we have

$$\beta = \lambda_A \gamma \tilde{\rho}_B$$

for some unique braid $\gamma \in \mathcal{B}_k$. In particular, $\mathcal{IB}_n = L_n \mathcal{B}_n R_n$ is generated by $\Sigma_L \cup \Sigma_B \cup \Sigma_R$.

**Proof.** Put $\delta = \tilde{\rho}_A \beta \lambda_B$. Then

$$\lambda_A \delta \tilde{\rho}_B = \lambda_A \tilde{\rho}_A \beta \lambda_B \tilde{\rho}_B = 1_A \beta 1_B = \beta.$$

Now $\text{dom}(\delta) = \text{im}(\delta) = k$, so we extend $\delta$ (if necessary) to $\gamma \in \mathcal{B}_k$ by adding the remaining $n - k$ strings as straight strings which do not interact with the first $k$ strings. We then have $\lambda_A \gamma \tilde{\rho}_B = \lambda_A \delta \tilde{\rho}_B = \beta$. To show that $\gamma$ is unique, suppose that $\gamma' \in \mathcal{B}_k$ and that $\beta = \lambda_A \gamma' \tilde{\rho}_B$. Then $\lambda_A \gamma' \tilde{\rho}_B = \lambda_A \gamma \tilde{\rho}_B = \beta$ so that

$$\gamma' = 1_k \gamma' 1_k = \tilde{\rho}_A \lambda_A \gamma \tilde{\rho}_B \lambda_B = \tilde{\rho}_A \beta \lambda_B = \tilde{\rho}_A \lambda \delta \tilde{\rho}_B \lambda_B = 1_k \delta 1_k = \delta,$$

and the first statement holds. The second statement follows from the first, together with Lemma 8.2 and the fact that $\mathcal{B}_n$ is generated by $\Sigma_B$. \qed

Define a homomorphism $\Phi : (X_L \cup X_B \cup X_R)^* \to \mathcal{IB}_n$ by

$$x \Phi = \tilde{x} \quad \text{for each } x \in X_L \cup X_R$$

$$\sigma_i^{\pm 1} \Phi = \xi_i^{\pm 1} \quad \text{for each } i.$$

Then $\Phi$ is an epimorphism by Lemma 8.19. Let $R_{LBR}$ be the set of relations $R_{LR} \cup R_B$ together with

$$\sigma_i \lambda_j = \begin{cases} 
\lambda_j \sigma_i & \text{if } 1 \leq i < j - 1 \leq n - 1 \\
\lambda_j & \text{if } 1 \leq i = j - 1 \leq n - 1 \\
\lambda_{j+1} & \text{if } 1 \leq i = j \leq n - 1 \\
\lambda_j \sigma_{i-1} & \text{if } 1 \leq j < i \leq n - 1 
\end{cases} \quad \text{(BL1—BL4)}$$

$$\rho_j \sigma_i = \begin{cases} 
\sigma_i \rho_j & \text{if } 1 \leq i < j - 1 \leq n - 1 \\
\rho_j & \text{if } 1 \leq i = j - 1 \leq n - 1 \\
\rho_{j+1} & \text{if } 1 \leq i = j \leq n - 1 \\
\sigma_{i-1} \rho_j & \text{if } 1 \leq j < i \leq n - 1, 
\end{cases} \quad \text{(RB1—RB4)}$$
and write $\approx$ for the congruence $R_{LBR}^n$. Our goal in this section is to show that $\ker \Phi = \approx$ so that $\mathcal{IB}_n$ has presentation

$$\langle X_L \cup X_B \cup X_R \mid R_{LBR} \rangle$$

via $\Phi$.

**Lemma 8.20.** We have the inclusion $\approx \subseteq \ker \Phi$.

**Proof.** Again, pictures will suffice to show the relations hold. We show in Figures 8.5 and 8.6 that relations (BL2) and (BL4) hold.

![Figure 8.5. Relation (BL2): $\lambda_i \lambda_j = \lambda_{j-1}$ if $1 \leq i = j - 1 \leq n - 1$.](image)

![Figure 8.6. Relation (BL4): $\lambda_i \lambda_j = \lambda_{j\lambda_i}$ if $1 \leq j < i \leq n - 1$.](image)

The reader is invited to draw pictures to show that relations (BL1) and (BL3) hold. Relations (RB1—RB4) follow by turning the (BL1—BL4) pictures upside down. □

Before moving on we prove a duality result to help shorten proofs.

**Lemma 8.21.** Define an anti-homomorphism

$$\overset{\sim}{\cdot} : (X_L \cup X_B \cup X_R)^* \to (X_L \cup X_B \cup X_R)^*$$

by $\overset{\sim}{\lambda}_i = \rho_i$, $\overset{\sim}{\rho}_i = \lambda_i$, $\overset{\sim}{\sigma}_j^{\pm 1} = \sigma_j^{\mp 1}$ for each $i, j$. Then for each $w_1, w_2 \in (X_L \cup X_B \cup X_R)^*$, we have $w_1 \approx w_2$ if and only if $\overset{\sim}{w}_1 \approx \overset{\sim}{w}_2$.

**Proof.** This is an immediate consequence of the proof of Lemma 8.11 and the fact that $\overset{\sim}{\cdot}$ preserves $R_B$ and swaps relations (BL1—BL4) with (RB1—RB4). □

Our next step is to show that $(X_L \cup X_B \cup X_R)^* \approx X_L^* X_B^* X_R^*$ by which we mean that every word over $X_L \cup X_B \cup X_R$ has a factorisation into a “left word” (over $X_L$), a “right word” (over $X_R$), with a “middle word” (over $X_B$) in-between.
We begin with some simple consequences of the relations. Notice that by (BL1—BL4), (RB1—RB4), and (F) we have

\[
\sigma_i^{-1} \lambda_j = \begin{cases} 
\lambda_j \sigma_i^{-1} & \text{if } 1 \leq i < j - 1 \leq n - 1 \\
\lambda_{j-1} & \text{if } 1 \leq i = j - 1 \leq n - 1 \\
\lambda_{j+1} & \text{if } 1 \leq i = j \leq n - 1 \\
\lambda_j \sigma_{i-1}^{-1} & \text{if } 1 \leq j < i \leq n - 1 
\end{cases} 
\]  

(8L1—BL4')

\[
\rho_j \sigma_i^{-1} = \begin{cases} 
\sigma_i^{-1} \rho_j & \text{if } 1 \leq i < j - 1 \leq n - 1 \\
\rho_{j-1} & \text{if } 1 \leq i = j - 1 \leq n - 1 \\
\rho_{j+1} & \text{if } 1 \leq i = j \leq n - 1 \\
\sigma_{i-1}^{-1} \rho_j & \text{if } 1 \leq j < i \leq n - 1. 
\end{cases} 
\]  

(RB1—RB4')

**Lemma 8.22.** Suppose that \( w \in (X_L \cup X_B)^* \). Then \( w \approx w_L w_B \) for some \( w_L \in X_L^* \) and \( w_B \in X_B^* \).

**Proof.** If \( w \in X_L^* X_B^* \) then there is nothing to show. Otherwise, we may use relations (BL1—BL4) and (BL1—BL4)' to move a generator \( \sigma_i^{\pm 1} \) in \( w \) to the right of generators from \( X_L \) (or make it disappear), and the result follows by a simple induction. \( \square \)

**Corollary 8.23.** Suppose that \( w \in (X_L \cup X_B \cup X_R)^* \). Then \( w \approx w_L w_B w_R \) for some \( w_L \in X_L^*, w_B \in X_B^*, \) and \( w_R \in X_R^* \).

**Proof.** If \( w \in (X_L \cup X_B)^* X_R^* \) then we are done by Lemma 8.22. Otherwise, we may use relations (RL1—RL3), (RB1—RB4), and (RB1—RB4)' to move a generator \( \rho_j \) in \( w \) to the right of generators from \( X_L \cup X_B \) (or make it disappear), and the result follows by a simple induction. \( \square \)

The next result follows immediately from Lemma 8.4 and Corollary 8.23.

**Corollary 8.24.** Suppose that \( w \in (X_L \cup X_B \cup X_R)^* \). Then there exist \( 0 \leq s, t \leq n, 1 \leq i_1 < \cdots < i_s \leq n, 1 \leq j_1 < \cdots < j_t \leq n \), and a word \( w' \in X_B^* \) such that

\[
w \approx \lambda_{i_1} \cdots \lambda_{i_s} w' \rho_{j_1} \cdots \rho_{j_t}.
\]

For \( 1 \leq k \leq n \) define \( X_{B_k} = \{ \sigma_{i_{k+1}}^{\pm 1}, \cdots, \sigma_{i_{k-1}}^{\pm 1} \} \). Also put \( X_{B_0} = \emptyset \). We aim to improve the result of Corollary 8.24 by showing that the factorisation may be achieved with \( s = t \) and \( w' \in X_{B_{n-s}}^* \). Again, our method will involve using the longer of the two words \( \lambda_{i_1} \cdots \lambda_{i_s} \) and \( \rho_{j_1} \cdots \rho_{j_t} \) to lengthen the shorter, and adjust \( w' \) until the factorisation is of the desired form.
For $0 \leq k \leq n$, we define a word $\epsilon_k = \lambda_n \cdots \lambda_{n-k+1}$, and put $\tilde{\epsilon}_k = \epsilon_k \Phi$. In the next series of results we will catalogue some of the useful properties of the words $\epsilon_k$.

**Lemma 8.25.** Suppose that $0 \leq k \leq n$. Then $\epsilon_k \approx \rho_{n-k+1} \cdots \rho_n$.

**Proof.** Interpreting the $\hat{\lambda}_i$ and $\tilde{\rho}_i$ as partial permutations, we see by Lemma 8.2 and the fact that $\{n - k + 1, \ldots, n\}^c = \{1, \ldots, n - k\}$, that

$$\tilde{\epsilon}_k = \lambda_n \cdots \lambda_{n-k+1} = \lambda_{\{1,\ldots,n-k\}} = \tilde{\rho}_{\{1,\ldots,n-k\}} = \tilde{\rho}_{n-k+1} \cdots \tilde{\rho}_n.$$  

But then by Theorem 8.17, we have $\epsilon_k \approx \rho_{n-k+1} \cdots \rho_n$, completing the proof. \hfill \Box

**Corollary 8.26.** Suppose that $1 \leq k \leq n$ and that $1 \leq i_1 < \cdots < i_k \leq n$. Then

(i) $\lambda_{i_k} \cdots \lambda_{i_1} \epsilon_k \approx \lambda_{i_k} \cdots \lambda_{i_1} \epsilon_k$;

(ii) $\rho_{i_1} \cdots \rho_{i_k} \approx \epsilon_k \rho_{i_1} \cdots \rho_{i_k}$.

**Proof.** This follows directly from Corollary 8.14 (with $s = k$) and Lemma 8.25. \hfill \Box

**Lemma 8.27.** Suppose that $0 \leq s \leq t \leq n$ and that $1 \leq i_1 < \cdots < i_s \leq n$. Then there exist integers $1 \leq j_1 < \cdots < j_t \leq n$ such that

(i) $\lambda_{i_s} \cdots \lambda_{i_1} \epsilon_t \approx \lambda_{j_t} \cdots \lambda_{j_1}$;

(ii) $\epsilon_t \rho_{i_1} \cdots \rho_{i_s} \approx \rho_{j_1} \cdots \rho_{j_t}$.

**Proof.** We prove (i), and (ii) will follow by duality and Lemma 8.25. Suppose that

$$A = \{i_1, \ldots, i_s\}^c = \{a_1, \ldots, a_{n-s}\}$$

where $a_1 < \cdots < a_{n-s}$. Let $B = \{a_1, \ldots, a_{n-t}\} \subset A$, and suppose that $B^c = \{j_1, \ldots, j_t\}$ where $j_1 < \cdots < j_t$. Interpreting the $\hat{\lambda}_i$ as partial permutations, we see by Lemma 8.2 that

$$\hat{\lambda}_{i_s} \cdots \hat{\lambda}_{i_1} \epsilon_t = \lambda_A \hat{\lambda}_{\{1,\ldots,n-t\}} = \hat{\lambda}_B = \hat{\lambda}_{j_t} \cdots \hat{\lambda}_{j_1}.$$  

By Theorem 8.17 it follows that $\lambda_{i_s} \cdots \lambda_{i_1} \epsilon_t \approx \lambda_{j_t} \cdots \lambda_{j_1}$, and we are done. \hfill \Box

**Lemma 8.28.** Suppose that $0 \leq k \leq n$ and that $w \in X_{B_{n-k}}^*$. Then $w \epsilon_k \approx \epsilon_k w$.

**Proof.** This follows immediately from (BL1), (BL1)', and the definition of $\epsilon_k$. \hfill \Box

For $1 \leq i \leq j \leq n$, define a word

$$\delta_{ij} = \sigma_{j-1} \cdots \sigma_i \in X_B^*,$$

noting in particular that $j \delta_{ij} = i$. 

---

8.2. THE INVERSE BRAID MONOID

---

145
Lemma 8.29. Suppose that $1 \leq k \leq n$ and that $1 \leq i \leq n - k + 1$. Then

$$\epsilon_k \delta_{i,n-k+1} \approx \epsilon_k \rho_i.$$ 

Proof. We use (backwards) induction on $i$. If $i = n - k + 1$ then we have $\delta_{i,n-k+1} = 1$ and so $\epsilon_k \delta_{i,n-k+1} = \epsilon_k$. Now, interpreting the $\rho_j$ as partial permutations, we see that $\epsilon_k \rho_j \delta_{n-k+1} = \epsilon_k$ since $\rho_j \delta_{n-k+1}$ acts as the identity on $\{1, \ldots, n - k\} = \text{im}(\epsilon_k)$. But then by Theorem 8.17 we have $\epsilon_k \sim \epsilon_k \rho_{n-k+1}$, showing that the lemma holds when $i = n - k + 1$. Suppose now that $1 \leq i \leq n - k$. Then by (RB2) and an induction hypothesis,

$$\epsilon_k \delta_{i,n-k+1} = \epsilon_k \delta_{i+1,n-k+1} \sigma_i \approx \epsilon_k \rho_i \sigma_i \approx \epsilon_k \rho_i,$$

and we are done. \qed

For $1 \leq i < j \leq n$, define a word

$$A_{ij} = (\sigma_{i-1}^{-1} \cdots \sigma_{j-2}^{-1}) \sigma_{j-1}^{-1} (\sigma_{j-2} \cdots \sigma_i) \in X_B^*,$$

noting in particular that $A_{ij} \phi_B = \alpha_{ij}$. For $0 \leq k \leq n$ define $X_{Ak} = \{A_{ij}^{\pm 1} | 1 \leq i < j \leq k\}$. Then for any $w \in X_{B_k}$ with $w \phi_B \in P_n$, we have $w \sim_B w'$ for some $w' \in X_{Ak}$. We now prove a series of results concerning the words $A_{ij}$.

Lemma 8.30. For any $1 \leq k \leq n$ and $w \in X_{B_k}$, there exist words $u \in X_{Ak}^*$ and $v \in X_{B_{k-1}}^*$, and an integer $1 \leq j \leq k$, such that $w \sim_B uv \delta_{jk}$.

Proof. Put $j = k \overline{\pi}$ and let $\pi = \overline{w} \delta_{jk}^{-1}$. Then $i \pi = i$ for all $i \geq k + 1$ since $w, \delta_{jk} \in X_B^*$, and $k \pi = k \overline{w} \delta_{jk}^{-1} = \overline{j} \delta_{jk}^{-1} = k$. Thus we have $\pi = s_{i_1} \cdots s_{i_r}$ for some $1 \leq i_1, \ldots, i_r \leq k - 2$. Now put $v = \sigma_{i_1} \cdots \sigma_{i_r} \in X_{B_{k-1}}^*$. We then have $\overline{v} = \pi = \overline{w} \delta_{jk}^{-1}$ so that $1 = \overline{w} \delta_{jk}^{-1} v^{-1}$, whence $(\overline{w} \delta_{jk}^{-1} v^{-1}) \phi_B \in P_n$. Thus $\overline{w} \delta_{jk}^{-1} v^{-1} \sim_B u$ for some $u \in X_{Ak}^*$, and it follows that $w \sim_B uv \delta_{jk}$, completing the proof. \qed

Lemma 8.31. Suppose that $1 \leq i < j \leq n$ and $k \in \mathbb{N}$. Then

$$\lambda_k A_{ij}^{\pm 1} \approx \begin{cases} A_{ij}^{\pm 1} \lambda_k & \text{if } j < k \\ A_{i+1,j}^{\pm 1} \lambda_k & \text{if } i < k \leq j < n \\ A_{i+1,j+1}^{\pm 1} \lambda_k & \text{if } k \leq i < j < n \\ \lambda_k & \text{if } j = n \end{cases},$$

and

$$A_{ij}^{\pm 1} \rho_k \approx \begin{cases} \rho_k A_{ij}^{\pm 1} & \text{if } j < k \\ \rho_k A_{i+1,j}^{\pm 1} & \text{if } i < k \leq j < n \\ \rho_k A_{i+1,j+1}^{\pm 1} & \text{if } k \leq i < j < n \\ \rho_k & \text{if } j = n. \end{cases}$$
**Proof.** We first consider the relations involving $\lambda_k$ and positive powers of the $A_{ij}$. The first three cases may be easily proved by starting with the word on the right hand side, and then using relations (BL1—BL4) and (BL1—BL4)' to move $\lambda_k$ to the left. For the $j = n$ case we have

$$
\lambda_k A_{in} = \lambda_k (\sigma_i^{-1} \cdots \sigma_{n-2}^{-1}) \sigma_{n-1}^2 (\sigma_{n-2} \cdots \sigma_i) \\
\approx \lambda_k \rho_i (\sigma_i^{-1} \cdots \sigma_{n-2}^{-1}) \sigma_{n-1}^2 (\sigma_{n-2} \cdots \sigma_i) \quad \text{by (L2) and (RL2)} \\
\approx \lambda_k (\sigma_i^{-1} \cdots \sigma_{n-2}^{-1}) \rho_i \sigma_{n-1}^2 (\sigma_{n-2} \cdots \sigma_i) \quad \text{by (RB1)'} \\
\approx \lambda_k (\sigma_i^{-1} \cdots \sigma_{n-2}^{-1}) \rho_{n-1} \sigma_{n-1} (\sigma_{n-2} \cdots \sigma_i) \quad \text{by (RB2)} \\
\approx \lambda_k (\sigma_i^{-1} \cdots \sigma_{n-2}^{-1}) \rho_{n-1} (\sigma_{n-2} \cdots \sigma_i) \quad \text{by (RB3)} \\
\approx \lambda_k (\sigma_i^{-1} \cdots \sigma_{n-2}^{-1}) (\sigma_{n-2} \cdots \sigma_i) \quad \text{by (RB1)'} \\
\approx \lambda_k \quad \text{by (RL2), (L2), and (F)}.
$$

The relations involving $\lambda_k$ and negative powers of the $A_{ij}$ generators now follow by (F).

To complete the proof, we first define a homomorphism $\sim : X_B^* \to X_B^*$ by $\tilde{\sigma_i}^\pm = \sigma_i^\pm$ for each $i$. Now by relations (BL1—BL4) and (BL1—BL4)', together with Lemma 8.21, we see that for any $w, w' \in X_B^*$ and $1 \leq i \leq n$, we have

$$
\lambda_i w \approx w' \lambda_i \iff \lambda \tilde{w} \approx \tilde{w}' \lambda_i \iff \tilde{w} \rho_i \approx \rho_i \tilde{w}'.
$$

The relations involving $\rho_k$ now follow from those involving $\lambda_k$ together with the easily checked fact that $\tilde{A}_{ij}^\pm = A_{ij}^\pm$ for each $1 \leq i < j \leq n$.

**Lemma 8.32.** Suppose that $1 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, and $1 \leq i < j \leq n$ with $j \geq n - k + 1$. Then

(i) $\lambda_{i_k} \cdots \lambda_{i_1} A_{ij}^\pm \approx \lambda_{i_k} \cdots \lambda_{i_1}$;

(ii) $A_{ij}^\pm \rho_{i_k} \cdots \rho_{i_1} \approx \rho_{i_1} \cdots \rho_{i_k}$.

**Proof.** We prove (i) by induction on $k$. Now if $k = 1$ then $j = n$ and we are done by Lemma 8.31 so suppose that $k \geq 2$. If $j = n$ then again we are done by Lemma 8.31, so suppose that $j < n$. Now since $1 \leq i_1 < \cdots < i_k \leq n$, we have $i_1 \leq n - k + 1$ so that $i_1 \leq j < n$. Thus by Lemma 8.31

$$
\lambda_{i_k} \cdots \lambda_{i_2} \lambda_{i_1} A_{ij}^\pm \approx \begin{cases} 
\lambda_{i_k} \cdots \lambda_{i_2} \lambda_{i_1} A_{i_1 j+1}^\pm & \text{if } i < i_1 \\
\lambda_{i_k} \cdots \lambda_{i_2} A_{i_1 j+1}^\pm \lambda_{i_1} & \text{if } i \geq i_1,
\end{cases}
$$

and in either case we are done by an inductive hypothesis.

To deduce (ii) from (i), we use the $\sim$ map and an argument similar to that in the second paragraph of the proof of Lemma 8.31. □
Lemma 8.33. Suppose that $1 \leq k \leq n$ and $w \in X_{A_{n-k+1}}^*$. Then there exists $w' \in X_{A_{n-k}}^*$ such that

(i) $\epsilon_k w \approx \epsilon_k w'$;
(ii) $\hat{w} \epsilon_k \approx \hat{w}' \epsilon_k$.

Proof. We prove (i), and (ii) will follow from Lemma 8.25 and duality. Now by Theorems 2.10, 2.13, and 2.14 we have $w \sim_B w'' w'$ for some $w'' \in (X_{A_{n-k+1}} \setminus X_{A_{n-k}})^*$ and $w' \in X_{A_{n-k}}^*$. But then by Lemma 8.32, we have $\epsilon_k w \approx \epsilon_k w'' w' \approx \epsilon_k w'$.

Corollary 8.34. Suppose that $1 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, and that $w \in X_{B_{n-k+1}}^*$. Then there exists $1 \leq j \leq n - k + 1$ and $w_{n-k} \in X_{B_{n-k}}^*$ such that

(i) $\lambda_{i_k} \cdots \lambda_{i_1} w \approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \rho_j$
(ii) $\hat{w} \rho_{i_1} \cdots \rho_{i_k} \approx \lambda_{j} \hat{w}_{n-k} \rho_{i_1} \cdots \rho_{i_k}$.

Proof. We only prove (i) since (ii) will follow by duality. Now by Lemma 8.30 there exists $1 \leq j \leq n - k + 1$ and words $u \in X_{A_{n-k+1}}^*$ and $v \in X_{B_{n-k}}^*$ such that $w \approx uv \delta_{j,n-k+1}$, and by Lemma 8.33 there exists $u' \in X_{A_{n-k}}^*$ such that $\epsilon_k u \approx \epsilon_k u'$. Put $w_{n-k} = u'v \in X_{B_{n-k}}^*$.

We then have

$$
\lambda_{i_k} \cdots \lambda_{i_1} w \approx \lambda_{i_k} \cdots \lambda_{i_1} uv \delta_{j,n-k+1} \\
\approx \lambda_{i_k} \cdots \lambda_{i_1} \epsilon_k w \delta_{j,n-k+1} \\
\approx \lambda_{i_k} \cdots \lambda_{i_1} \epsilon_k w' \delta_{j,n-k+1} \\
= \lambda_{i_k} \cdots \lambda_{i_1} \epsilon_k w_{n-k} \delta_{j,n-k+1} \\
\approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \epsilon_k \delta_{j,n-k+1} \\
\approx \lambda_{i_k} \cdots \lambda_{i_1} \epsilon_k \delta_{n-k} \rho_j \\
\approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \epsilon_k \rho_j \\
\approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \rho_j \\
= \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \rho_j$$

by Corollary 8.26

and we are done.

Corollary 8.35. Suppose that $1 \leq r \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, and that $w \in X_{B_{n-k+r}}^*$. Then there exist $1 \leq j_1, \ldots, j_r \leq n$ and $w_{n-k} \in X_{B_{n-k}}^*$ such that

(i) $\lambda_{i_k} \cdots \lambda_{i_1} w \approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \rho_{j_1} \cdots \rho_{j_r}$;
(ii) $\hat{w} \rho_{i_1} \cdots \rho_{i_k} \approx \lambda_{j_1} \cdots \lambda_{j_r} \hat{w}_{n-k} \rho_{i_1} \cdots \rho_{i_k}$.

Proof. We prove (i) by induction on $r$, and (ii) will follow by duality. If $r = 1$, then we are done by Corollary 8.34, so suppose that $2 \leq r \leq k$. Then by Corollary 8.34,

$$
\lambda_{i_{k-r+1}} \cdots \lambda_{i_1} w \approx \lambda_{i_{k-r+1}} \cdots \lambda_{i_1} w' \rho_{j_r}
$$
for some $w' \in X_{B_n-k+r-1}$ and some $1 \leq j_r \leq n - k + r - 1$. By an inductive hypothesis,

$$\lambda_{i_k} \cdots \lambda_{i_1} w' \approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \rho_{j_1} \cdots \rho_{j_r-1}$$

for some $w_{n-k} \in X_{B_n-k}$ and some $1 \leq j_1, \ldots, j_{r-1} \leq n$. But then

$$\lambda_{i_k} \cdots \lambda_{i_1} w = (\lambda_{i_k} \cdots \lambda_{i_{k-r+2}})(\lambda_{i_{k-r+1}} \cdots \lambda_{i_1} w)
\approx (\lambda_{i_k} \cdots \lambda_{i_{k-r+1}})(\lambda_{i_{k-r+1}} \cdots \lambda_{i_1} w' \rho_{j_r})
\approx \lambda_{i_k} \cdots \lambda_{i_1} w_{n-k} \rho_{j_1} \cdots \rho_{j_{r-1}} \rho_{j_r},$$

and we are done. □

**Lemma 8.36.** Suppose that $0 \leq s, t \leq n$, $1 \leq i_1 < \cdots < i_t \leq n$, $1 \leq j_1 < \cdots < j_t \leq n$, and that $w \in X_B^*$. Then

(i) if $s \geq t$, there exists $0 \leq t' \leq n$, $1 \leq h_1 < \cdots < h_{t'} \leq n$, and $w_{n-s} \in X_{B_n-s}^*$ such that
$$\lambda_{i_s} \cdots \lambda_{i_1} w \rho_{j_1} \cdots \rho_{j_t} \approx \lambda_{i_s} \cdots \lambda_{i_1} w_{n-s} \rho_{h_1} \cdots \rho_{h_{t'}};$$

(ii) if $s \leq t$, there exists $0 \leq s' \leq n$, $1 \leq h_1 < \cdots < h_{s'} \leq n$, and $w_{n-t} \in X_{B_n-t}^*$ such that
$$\lambda_{i_s} \cdots \lambda_{i_1} w \rho_{j_1} \cdots \rho_{j_t} \approx \lambda_{i_{s'}} \cdots \lambda_{i_1} w_{n-t} \rho_{j_1} \cdots \rho_{j_t}.$$

**Proof.** We only prove (i) since (ii) will follow by duality. Now if $s = 0$, then we also have $t = 0$ and we are done with $t' = 0$ and $w' = w$, so suppose that $s \geq 1$. By Corollary 8.35 (with $k = r = s$) there exists $q_1, \ldots, q_s \in n$ and $w_{n-s} \in X_{B_n-s}^*$ such that
$$\lambda_{i_s} \cdots \lambda_{i_1} w \approx \lambda_{i_s} \cdots \lambda_{i_1} w_{n-s} \rho_{q_1} \cdots \rho_{q_s}.$$

But then by Lemma 8.4 we have
$$\rho_{q_1} \cdots \rho_{q_s} \rho_{j_1} \cdots \rho_{j_t} \approx \rho_{h_1} \cdots \rho_{h_{t'}}$$
for some $t' \in n$ and $1 \leq h_1 < \cdots < h_{t'} \leq n$. Thus
$$\lambda_{i_s} \cdots \lambda_{i_1} w \rho_{j_1} \cdots \rho_{j_t} \approx \lambda_{i_s} \cdots \lambda_{i_1} w_{n-s} \rho_{q_1} \cdots \rho_{q_s} \rho_{j_1} \cdots \rho_{j_t} \approx \lambda_{i_s} \cdots \lambda_{i_1} w_{n-s} \rho_{h_1} \cdots \rho_{h_{t'}},$$

and we are done. □

**Corollary 8.37.** Suppose that $w \in (X_L \cup X_B \cup X_R)^*$. Then there exists $0 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq n$, and $w' \in X_{B_n-k}^*$ such that
$$w \approx \lambda_{i_k} \cdots \lambda_{i_1} w' \rho_{j_1} \cdots \rho_{j_k}.$$

**Proof.** By Corollary 8.24, there exist $0 \leq s, t \leq n$, integers $1 \leq u_1 < \cdots < u_s \leq n$, $1 \leq v_1 < \cdots < v_t \leq n$, and a word $w'' \in X_B^*$ such that
$$w \approx \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{v_1} \cdots \rho_{v_t}.$$
We will first prove that $w$ is $\approx$-equivalent to a word of the form

$$\lambda_{s'} \cdots \lambda_{x} w'' \rho_{y} \cdots \rho_{y''}$$

for some $0 \leq s', t' \leq n$, $1 \leq x_1 < \cdots < x_{s'} \leq n$, $1 \leq y_1 < \cdots < y_{t'} \leq n$, $w''' \in X_{B_{n-m}}^*$, where $m = \max(s', t')$ and either

(i) $s' = t'$ or

(ii) $s' + t' > s + t$.

By duality it suffices to assume that $s \geq t$. Now by Lemma 8.36(i), we have

$$w \approx \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{v_1} \cdots \rho_{v_t} \approx \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{h_1} \cdots \rho_{h_t}$$

for some $0 \leq t' \leq n$, $1 \leq h_1 < \cdots < h_{t'} \leq n$, $w''' \in X_{B_{n-t}}^*$. Now if $t' > s$ then $s + t' > s + s \geq s + t$ and (ii) holds, so suppose that $t' \leq s$. Then

$$w \approx \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{h_1} \cdots \rho_{h_t}$$

by Corollary 8.26

$$\approx \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{h_1} \cdots \rho_{h_t}$$

by Lemma 8.28

$$\approx \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{j_1} \cdots \rho_{j_s}$$

by Lemma 8.27

for some $1 \leq j_1 < \cdots < j_s \leq n$, and (i) holds.

Returning to the main proof, we iterate the above procedure, noting that after a finite number of steps we must arrive at a word of the form

$$\lambda_{i_k} \cdots \lambda_{i_1} w' \rho_{j_1} \cdots \rho_{j_k}$$

with $0 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq n$, and $w' \in X_{B_{n-k}}^*$, which is $\approx$-equivalent to $w$. This completes the proof. \qed

**Theorem 8.38.** The inverse braid monoid $\mathcal{IB}_n$ has presentation

$$\langle X_L \cup X_B \cup X_R \mid R_{LBR} \rangle$$

via $\Phi$.

**Proof.** By Lemmas 8.19 and 8.20, it suffices to show that $\ker \Phi \subseteq \approx$. So suppose that $(w_1, w_2) \in \ker \Phi$. By Corollary 8.37, we have

$$w_1 \approx \lambda_{i_k} \cdots \lambda_{i_1} w_1' \rho_{j_1} \cdots \rho_{j_k} \quad \text{and} \quad w_2 \approx \lambda_{u_h} \cdots \lambda_{u_1} w_2' \rho_{v_1} \cdots \rho_{v_h}$$

for some $0 \leq k, h \leq n$, $w_1' \in X_{B_{n-k}}^*$, $w_2' \in X_{B_{n-h}}^*$, and

$$1 \leq i_1 < \cdots < i_k \leq n, \quad 1 \leq j_1 < \cdots < j_k \leq n,$$

$$1 \leq u_1 < \cdots < u_h \leq n, \quad 1 \leq v_1 < \cdots < v_h \leq n.$$
Put 
\[ A = \{i_1, \ldots, i_k\}, \quad B = \{j_1, \ldots, j_k\}, \quad C = \{u_1, \ldots, u_h\}, \quad D = \{v_1, \ldots, v_h\}, \]
and let \( \beta_1 = w'_1 \Phi \in \mathcal{B}_{n-k}, \beta_2 = w'_2 \Phi \in \mathcal{B}_{n-h} \). Then by Lemma 8.2 we have
\[ \lambda_A \beta_1 \hat{\rho}_B = \lambda_{i_k} \cdots \lambda_{i_1} \beta_1 \hat{\rho}_{j_k} \cdots \hat{\rho}_{j_1} = w_1 \Phi = w_2 \Phi = \lambda_{u_h} \cdots \lambda_{u_1} \hat{\beta}_2 \hat{\rho}_{v_h} \cdots \hat{\rho}_{v_1} = \lambda_C \beta_2 \hat{\rho}_D. \]
Since \( \{1, \ldots, n-k\} \beta_1 = \{1, \ldots, n-k\} \), we have \( \text{dom}(\lambda_A \beta_1 \hat{\rho}_B) = A \) and \( \text{im}(\lambda_A \beta_1 \hat{\rho}_B) = B \).

Similarly, \( \text{dom}(\lambda_C \beta_2 \hat{\rho}_D) = C \) and \( \text{im}(\lambda_C \beta_2 \hat{\rho}_D) = D \). Thus \( A = C \) and \( B = D \) so that \( k = h \) and \( i_s = u_s, j_s = v_s \) for each \( 1 \leq s \leq k \). But then we have
\[ \lambda_A \beta_1 \hat{\rho}_B = \lambda_C \beta_2 \hat{\rho}_D = \lambda_A \beta_2 \hat{\rho}_B \]
so that \( \beta_1 = \beta_2 \) by Lemma 8.19. We then have \( w'_1 \sim_B w'_2 \) by Theorem 2.10 so that
\[ w_1 \approx \lambda_{i_k} \cdots \lambda_{i_1} w'_1 \rho_{j_1} \cdots \rho_{j_k} = \lambda_{u_h} \cdots \lambda_{u_1} w'_1 \rho_{v_1} \cdots \rho_{v_h} \approx \lambda_{u_h} \cdots \lambda_{u_1} w'_2 \rho_{v_1} \cdots \rho_{v_h} \approx w_2, \]
and the theorem is proved. \( \square \)

**Remark 8.39.** Of course, since \( \mathcal{I}B_n \) is infinite, a counting argument similar to those in Remarks 8.6 and 8.18 will not suffice to prove Theorem 8.38.

As usual, we may add the relations
\[ \sigma_i^2 = 1 \quad \text{for each} \quad i \tag{O} \]
to obtain a presentation of \( \mathcal{I}_n \). Let \( X''_n = X_L \cup \{\sigma_1, \ldots, \sigma_{n-1}\} \cup X_R \) and let \( R''_n \) denote the set of relations obtained from \( R_{LBR} \) by replacing (F) with (O). By Proposition 5.21 and Theorem 8.38 we have the following.

**Theorem 8.40.** The symmetric inverse semigroup \( \mathcal{I}_n \) has presentation \( \langle X''_n \mid R''_n \rangle \) via
\[ x \mapsto \hat{x} \quad \text{for each} \quad x \in X_L \cup X_R, \quad \sigma_i \mapsto s_i \quad \text{for each} \quad i \] \( \square \)

### 8.3. The Monoid of Order-Preserving Partial Braids

We say that a partial braid \( \beta \in \mathcal{I}B_n \) is order-preserving if \( \tilde{\beta} \in \mathcal{POI}_n \). The set \( \mathcal{POI}_n \) of all order-preserving partial braids on \( n \) is a submonoid of \( \mathcal{I}B_n \) since it is the preimage of the submonoid \( \mathcal{POI}_n \subseteq \mathcal{I}_n \) under the \( - \) map. In this section, we give a presentation of \( \mathcal{POI}_n \), making use of our presentation of \( \mathcal{POI}_n \) and Artin’s presentation of the pure braid group \( \mathcal{P}_n \). The method we use is similar to that of the previous section; in particular, we make use of a factorisation \( \mathcal{POI}_n = \mathcal{L}_n \mathcal{P}_n \mathcal{R}_n \).
Again using a slightly different notation to earlier chapters, put \( X_P = \{a_{ij}^{\pm 1} \mid 1 \leq i < j \leq n\} \), and let \( R_P \) denote the set of relations

\[
\begin{align*}
a_{ij}^{\pm 1}a_{ij}^{\mp 1} &= 1 & \text{for all } i, j & (F) \\
ar_{rs}a_{ij}a_{rs}^{-1} &= a_{ij} & \text{if } i < r \text{ or } i > s & (P1) \\
ar_{rs}a_{sj}a_{rs}^{-1} &= (a_{sj}^{-1}a_{rj}^{-1})a_{sj}(a_{rj}a_{sj}) & (P2) \\
ar_{rs}a_{rj}a_{rs}^{-1} &= a_{sj}^{-1}a_{rj}a_{sj} & (P3) \\
ar_{rs}a_{ij}a_{rs}^{-1} &= (a_{sj}^{-1}a_{rj}^{-1}a_{rj}a_{ij})(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) & \text{if } r < i < s, & (P4)
\end{align*}
\]

where in each of relations (P1—P4) we have \( s < j \). Then by Theorem 2.13, \( \mathcal{P}_n \) has monoid presentation \( \langle X_P \mid R_P \rangle \) via

\[
\phi_P : X_P^* \rightarrow \mathcal{P}_n : a_{ij}^{\pm 1} \mapsto \alpha_{ij}^{\pm 1} \quad \text{for each } i, j.
\]

We will write \( \sim_P \) for the congruence \( R_P^e \). Also, put \( \Sigma_P = \{a_{ij}^{\pm 1} \mid 1 \leq i < j \leq n\} = X_P\phi_P \) so that \( \mathcal{P}_n \) is generated by \( \Sigma_P \). If \( w = a_{i_1j_1}^{\pm 1} \cdots a_{i_nj_n}^{\pm 1} \in X_P^* \), we will write \( w^{-1} = a_{i_nj_n}^{\mp 1} \cdots a_{i_1j_1}^{\mp 1} \).

For \( 0 \leq k \leq n \) let \( \mathcal{P}_k \) be the subgroup of \( \mathcal{P}_n \) generated by \( \{a_{ij}^{\pm 1} \mid 1 \leq i < j \leq k\} \) so that \( \mathcal{P}_k \cong \mathcal{P}_k \) for each \( k \). We begin with a simple consequence of Lemma 8.19.

**Lemma 8.41.** Suppose that \( \beta \in \mathcal{POIB}_n \) and that \( A = \text{dom}(\bar{\beta}) \) and \( B = \text{im}(\bar{\beta}) \). Then, writing \( |A| = |B| = k \), we have

\[
\beta = \lambda_A \gamma \bar{\rho}_B
\]

for some unique pure braid \( \gamma \in \mathcal{P}_k \). In particular, \( \mathcal{POIB}_n = L_n\mathcal{P}_nR_n \) is generated by \( \Sigma_L \cup \Sigma_P \cup \Sigma_R \).

**Proof.** By Lemma 8.19 we have \( \beta = \lambda_A \gamma \bar{\rho}_B \) for some uniquely determined \( \gamma \in \mathcal{B}_k \). Write \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_k\} \) where \( a_1 < \cdots < a_k \) and \( b_1 < \cdots < b_k \). Since \( \bar{\beta} \in \mathcal{POI}_n \) and \( \gamma \in \mathcal{B}_k \), we see that for each \( i \in k \), we have

\[
b_i = a_i \bar{\beta} = a_i \lambda_A \gamma \bar{\rho}_B = i \bar{\gamma} \bar{\rho}_B = b_i \bar{\gamma}.
\]

This shows that \( i \bar{\gamma} = i \) for all \( i \in k \), whence \( \gamma \in \mathcal{P}_k \). The final statement follows from the first, together with Lemma 8.2 and the fact that \( \mathcal{P}_n \) is generated by \( \Sigma_P \).

Define a homomorphism \( \Psi : (X_L \cup X_P \cup X_R)^* \rightarrow \mathcal{POIB}_n \) by

\[
x\Psi = \bar{x} \quad \text{for each } x \in X_L \cup X_R \\
a_{ij}^{\pm 1} \Psi = \alpha_{ij}^{\pm 1} \quad \text{for each } i, j.
\]
Then $\Psi$ is an epimorphism by Lemma 8.41. Let $R_{LPR}$ be the set of relations $R_{LR}$ together with $R_P$ and

$$a_{ij} \lambda_k = \begin{cases} 
\lambda_k a_{ij} & \text{if } 1 \leq i < j < k \leq n \\
\lambda_k a_{i,j-1} & \text{if } 1 \leq i < k < j \leq n \\
\lambda_k a_{i-1,j-1} & \text{if } 1 \leq k < i < j \leq n \\
\lambda_k & \text{if } 1 \leq i < j \leq n \text{ and } k \in \{i, j\}
\end{cases} \quad (PL1\text{—}PL4)$$

$$\rho_k a_{ij} = \begin{cases} 
\rho_k a_{ij} & \text{if } 1 \leq i < j < k \leq n \\
\rho_k a_{i,j-1} & \text{if } 1 \leq i < k < j \leq n \\
\rho_k a_{i-1,j-1} & \text{if } 1 \leq k < i < j \leq n \\
\rho_k & \text{if } 1 \leq i < j \leq n \text{ and } k \in \{i, j\},
\end{cases} \quad (RP1\text{—}RP4)$$

and write $\cong$ for the congruence $R_{LPR}^\ast$. Our goal in this section is to show that $\ker \Psi = \cong$ so that $POIB_n$ has presentation

$$\langle X_L \cup X_P \cup X_R \mid R_{LPR} \rangle$$

via $\Psi$.

**Lemma 8.42.** We have the inclusion $\cong \subseteq \ker \Psi$.

**Proof.** We show in Figure 8.7 that relation (PL3) holds.

![Figure 8.7. Relation (PL3): $a_{ij} \lambda_k = \lambda_k a_{i-1,j-1}$ if $1 \leq k < i < j \leq n$.](image)

The reader is invited to draw pictures for the remaining relations. □

Notice that by (PL1—PL4), (RP1—RP4), and (F) we have

$$a_{ij}^{-1} \lambda_k \cong \begin{cases} 
\lambda_k a_{ij}^{-1} & \text{if } 1 \leq i < j < k \leq n \\
\lambda_k a_{i,j-1}^{-1} & \text{if } 1 \leq i < k < j \leq n \\
\lambda_k a_{i-1,j-1}^{-1} & \text{if } 1 \leq k < i < j \leq n \\
\lambda_k & \text{if } 1 \leq i < j \leq n \text{ and } k \in \{i, j\}
\end{cases} \quad (PL1\text{—}PL4)'$$

$$\rho_k a_{ij}^{-1} \cong \begin{cases} 
\rho_k a_{ij}^{-1} & \text{if } 1 \leq i < j < k \leq n \\
\rho_k a_{i,j-1}^{-1} & \text{if } 1 \leq i < k < j \leq n \\
\rho_k a_{i-1,j-1}^{-1} & \text{if } 1 \leq k < i < j \leq n \\
\rho_k & \text{if } 1 \leq i < j \leq n \text{ and } k \in \{i, j\},
\end{cases} \quad (RP1\text{—}RP4)'$$
The proofs of Lemma 8.22 and Corollaries 8.23 and 8.24, may be adapted to prove the following factorisation result for words over $X_L \cup X_P \cup X_R$.

**Lemma 8.43.** Suppose that $w \in (X_L \cup X_P \cup X_R)^*$. Then there exist $0 \leq s, t \leq n$, $1 \leq i_1 < \cdots < i_s \leq n$, $1 \leq j_1 < \cdots < j_t \leq n$, and a word $w' \in X_P^*$ such that

$$w \approx \lambda_{i_s} \cdots \lambda_{i_1} w' \rho_{j_t} \cdots \rho_{j_1}. \quad \square$$

**Lemma 8.44.** Suppose that $1 \leq i \leq n - 1$ and $1 \leq k \leq n$. Then

(i) $\lambda_k a_{in}^{\pm 1} \approx \lambda_k$;

(ii) $a_{in}^{\pm 1} \rho_k \approx \rho_k$.

**Proof.** By (L2), (RL2), (RP4), and (RP4)' we have

$$\lambda_k a_{in}^{\pm 1} \approx \lambda_k \lambda_n a_{in}^{\pm 1} \approx \lambda_k \rho_n a_{in}^{\pm 1} \approx \lambda_k \rho_n \approx \lambda_k \lambda_n \approx \lambda_k,$$

showing that statement (i) holds. Statement (ii) is proved analogously. \square

**Lemma 8.45.** Suppose that $1 \leq i < j \leq n$ and $1 \leq k \leq n$. Then

$$\lambda_k a_{ij}^{\pm 1} \approx \begin{cases} a_{ij}^{\pm 1} \lambda_k & \text{if } j < k \\ a_{i+1,j+1}^{\pm 1} \lambda_k & \text{if } i < k \leq j < n \\ a_{i+1,j+1}^{\pm 1} \lambda_k & \text{if } k \leq i < j < n \\ \lambda_k & \text{if } j = n \end{cases}$$

$$a_{ij}^{\pm 1} \rho_k \approx \begin{cases} \rho_k a_{ij}^{\pm 1} & \text{if } j < k \\ \rho_k a_{i+1,j+1}^{\pm 1} & \text{if } i < k \leq j < n \\ \rho_k a_{i+1,j+1}^{\pm 1} & \text{if } k \leq i < j < n \\ \rho_k & \text{if } j = n. \end{cases}$$

**Proof.** This follows quickly from (PL1—PL4), (RP1—RP4), (PL1—PL4)', (RP1—RP4)', and Lemma 8.44. \square

**Lemma 8.46.** Suppose that $1 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, and $1 \leq i < j \leq n$ with $j > n - k$. Then

(i) $\lambda_{i_k} \cdots \lambda_{i_1} a_{ij}^{\pm 1} \approx \lambda_{i_k} \cdots \lambda_{i_1}$;

(ii) $a_{ij}^{\pm 1} \rho_{i_1} \cdots \rho_{i_k} \approx \rho_{i_1} \cdots \rho_{i_k}$.

**Proof.** The arguments from the first paragraph of Lemma 8.32 work almost unchanged to prove (i); we apply Lemma 8.45 rather than Lemma 8.31. Statement (ii) is proved analogously. \square
Lemma 8.47. Suppose that \( w \in (X_L \cup X_P \cup X_R)^* \). Then there exists \( 0 \leq k \leq n \), \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_k \leq n \), and \( w' \in X_P^* \) such that
\[
    w \cong \lambda_{i_k} \cdots \lambda_{i_1} w' \rho_{j_1} \cdots \rho_{j_k}.
\]

Proof. By Lemma 8.43 we have
\[
w \cong \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{v_1} \cdots \rho_{v_t},
\]
for some \( 0 \leq s, t \leq n \), \( 1 \leq u_1 < \cdots < u_s \leq n \), \( 1 \leq v_1 < \cdots < v_t \leq n \), and \( w'' \in X_P^* \). If \( s = t \) then we are done so suppose that \( s \neq t \). By Lemma 8.45 there is a word \( w''' \in X_P^* \) such that
\[
    \lambda_{u_s} \cdots \lambda_{u_1} w'' \cong w''' \lambda_{u_s} \cdots \lambda_{u_1}.
\]
By Corollary 8.16, there exists \( 0 \leq k \leq n \) and \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_k \leq n \) such that
\[
    \lambda_{u_s} \cdots \lambda_{u_1} \rho_{v_1} \cdots \rho_{v_t} \cong \lambda_{i_k} \cdots \lambda_{i_1} \rho_{j_1} \cdots \rho_{j_k}.
\]
By \( (PL1|PL4) \) and \( (PL1|PL4)' \), there exists \( w' \in X_P \) such that
\[
    w''' \lambda_{i_k} \cdots \lambda_{i_1} \cong \lambda_{i_k} \cdots \lambda_{i_1} w'.
\]
Thus
\[
w \cong \lambda_{u_s} \cdots \lambda_{u_1} w'' \rho_{v_1} \cdots \rho_{v_t} \\
\cong w''' \lambda_{u_s} \cdots \lambda_{u_1} \rho_{v_1} \cdots \rho_{v_t} \\
\cong w''' \lambda_{i_k} \cdots \lambda_{i_1} \rho_{j_1} \cdots \rho_{j_k} \\
\cong \lambda_{i_k} \cdots \lambda_{i_1} w' \rho_{j_1} \cdots \rho_{j_k},
\]
and we are done.

For \( 0 \leq k \leq n \) put \( X_{P_k} = \{ a_{ij}^{\pm 1} \mid 1 \leq i < j \leq k \} \). For \( Y \subseteq X_P^* \) let \( Y^\ast \) denote the submonoid of \( X_P^* \) generated by \( Y \).

Corollary 8.48. Suppose that \( w \in (X_L \cup X_P \cup X_R)^* \). Then there exists \( 0 \leq k \leq n \), \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_k \leq n \), and \( w' \in X_P^* \) such that
\[
w \cong \lambda_{i_k} \cdots \lambda_{i_1} w' \rho_{j_1} \cdots \rho_{j_k}.
\]

Proof. By Lemma 8.47 we have
\[
w \cong \lambda_{i_k} \cdots \lambda_{i_1} w'' \rho_{j_1} \cdots \rho_{j_k}.
\]
for some \( 0 \leq k \leq n \), \( 1 \leq i_1 < \cdots < i_k \leq n \), \( 1 \leq j_1 < \cdots < j_k \leq n \), and \( w'' \in X_P^* \). Now by Theorems 2.13 and 2.14 we have
\[
w'' \sim_P w'' w'
\]
for some \( w'' \in (X_P \setminus X_{P_{n-k}})^* \) and \( w' \in X_{P_{n-k}}^* \). But then by Lemma 8.46, we have
\[
w \approx \lambda_{i_k} \cdots \lambda_{i_1} w'' \rho_{j_1} \cdots \rho_{j_k} \approx \lambda_{i_k} \cdots \lambda_{i_1} w'' w' \rho_{j_1} \cdots \rho_{j_k} \approx \lambda_{i_k} \cdots \lambda_{i_1} w' \rho_{j_1} \cdots \rho_{j_k},
\]
and the proof is complete. \( \square \)

**Theorem 8.49.** The monoid \( \text{POIB}_n \) of order-preserving partial braids has presentation
\[
\langle X_L \cup X_P \cup X_R | R_{LPR} \rangle
\]
via \( \Psi \).

**Proof.** By Lemmas 8.41 and 8.42, it suffices to show that \( \ker \Psi \subseteq \approx \). So suppose that \( (w_1, w_2) \in \ker \Psi \). By Corollary 8.48, we have

\[
w_1 \approx \lambda_{i_k} \cdots \lambda_{i_1} w'_1 \rho_{j_1} \cdots \rho_{j_k} \quad \text{and} \quad w_2 \approx \lambda_{u_h} \cdots \lambda_{u_1} w'_2 \rho_{v_1} \cdots \rho_{v_h}
\]
for some \( 0 \leq k, h \leq n, w'_1 \in X_{P_{n-k}}^*, w'_2 \in X_{P_{n-h}}^* \), and

\[
1 \leq i_1 < \cdots < i_k \leq n, \quad 1 \leq j_1 < \cdots < j_k \leq n, \\
1 \leq u_1 < \cdots < u_h \leq n, \quad 1 \leq v_1 < \cdots < v_h \leq n.
\]

Put

\[
A = \{i_1, \ldots, i_k\}^c, \quad B = \{j_1, \ldots, j_k\}^c, \\
C = \{u_1, \ldots, u_h\}^c, \quad D = \{v_1, \ldots, v_h\}^c,
\]

and let \( \beta_1 = w'_1 \Psi \in \mathcal{P}_{n-k}, \beta_2 = w'_2 \Psi \in \mathcal{P}_{n-h} \). Then by Lemma 8.2 we have

\[
\tilde{\lambda}_A \beta_1 \tilde{\rho}_B = \tilde{\lambda}_{i_k} \cdots \tilde{\lambda}_{i_1} \beta_1 \tilde{\rho}_{j_1} \cdots \tilde{\rho}_{j_k} = w_1 \Psi = w_2 \Psi = \tilde{\lambda}_{u_h} \cdots \tilde{\lambda}_{u_1} \beta_2 \tilde{\rho}_{v_1} \cdots \tilde{\rho}_{v_h} = \tilde{\lambda}_C \beta_2 \tilde{\rho}_D.
\]

Since \( \beta_1 \in \mathcal{P}_n \), we have \( \text{dom}(\tilde{\lambda}_A \beta_1 \tilde{\rho}_B) = A \) and \( \text{im}(\tilde{\lambda}_A \beta_1 \tilde{\rho}_B) = B \). Similarly, we see that \( \text{dom}(\tilde{\lambda}_C \beta_2 \tilde{\rho}_D) = C \) and \( \text{im}(\tilde{\lambda}_C \beta_2 \tilde{\rho}_D) = D \). Thus \( A = C \) and \( B = D \) so that \( k = h \) and \( i_s = u_s, j_s = v_s \) for each \( 1 \leq s \leq k \). But then we have

\[
\tilde{\lambda}_A \beta_1 \tilde{\rho}_B = \tilde{\lambda}_C \beta_2 \tilde{\rho}_D = \tilde{\lambda}_A \beta_2 \tilde{\rho}_B
\]
so that \( \beta_1 = \beta_2 \) by Lemma 8.41. We then have \( w'_1 \sim_P w'_2 \) by Theorem 2.13 so that

\[
w_1 \approx \lambda_{i_k} \cdots \lambda_{i_1} w'_1 \rho_{j_1} \cdots \rho_{j_k} = \lambda_{u_h} \cdots \lambda_{u_1} \rho_{v_1} \cdots \rho_{v_h} \approx \lambda_{u_h} \cdots \lambda_{u_1} w'_2 \rho_{v_1} \cdots \rho_{v_h} \approx w_2,
\]
and the theorem is proved. \( \square \)
8.4. The Singular Part of the Inverse Braid Monoid

In this section we describe a further application of our presentation of $POI_n$ from Section 8.1. In [29], the author studied the semigroup $I_n \setminus S_n$ of all strictly-partial permutations on $n$. In particular, a semigroup presentation of $I_n \setminus S_n$ was given. Use was made of a factorisation

$$I_n \setminus S_n = L_n^\circ \mathcal{S}_{n-1} R_n^\circ$$

where, defining $n^\circ = n \setminus \{n\}$, we have written

- $L_n^\circ = L_n \setminus \{1\}$,
- $R_n^\circ = R_n \setminus \{1\}$,
- $\mathcal{S}_{n-1} = \{ \alpha \in I_n \mid \text{dom}(\alpha) = \text{im}(\alpha) = n^\circ \} = \{ \pi |_{n^\circ} \mid \pi \in S_n, \ n\pi = n \}$.

Here $\mathcal{S}_{n-1}$ is a maximal subgroup of $I_n \setminus S_n$, with identity $\lambda_n = \rho_n = \text{id}_{n^\circ}$, which is naturally isomorphic to the symmetric group $S_{n-1}$ on $n^\circ$.

It is possible to combine the techniques used in [29] and Section 8.2 to study the semigroup $IB_n \setminus B_n$ of all strictly-partial braids on $n$. In particular, we may define a maximal subgroup

$$\mathfrak{B}_{n-1} = \{ \beta \in IB_n \mid \text{dom}(\bar{\beta}) = \text{im}(\bar{\beta}) = n^\circ \} = \{ \beta |_{n^\circ} \mid \beta \in B_n, \ n\bar{\beta} = n \}$$

of $IB_n \setminus B_n$ which is naturally isomorphic to the braid group $B_{n-1}$ on $n^\circ$. This subgroup is also the preimage, under the map $\bar{\cdot} : IB_n \to I_n$, of the subgroup $\mathcal{S}_{n-1} \subseteq I_n \setminus S_n$ described in the previous paragraph. Again, identifying $POI_n$ with a submonoid of $IB_n$, the identity element of $\mathfrak{B}_{n-1}$ is $\tilde{\lambda}_n = \tilde{\rho}_n = 1_{n^\circ}$ and we have a factorisation

$$IB_n \setminus B_n = L_n^\circ \mathfrak{B}_{n-1} R_n^\circ.$$

We now describe a presentation of $IB_n \setminus B_n$. For the sake of brevity, and because the calculations are almost identical to those of Section 8.2, we simply state the results.

Let $X_B^\circ = \{ \sigma_1^\pm, \ldots, \sigma_{n-2}^\pm \}$ and let $R_B^\circ \subseteq (X_B^\circ \cup \{\lambda_n\})^+ \times (X_B^\circ \cup \{\lambda_n\})^+$ be the set of relations

$$\begin{align*}
\sigma_i^\pm \lambda_n = \lambda_n \sigma_i^\pm &= \sigma_i^\pm \\
\sigma_i^\pm \sigma_j^\pm &= \lambda_n & \quad \text{for all } i, \\
\sigma_i \sigma_j = \sigma_j \sigma_i &= \sigma_i \sigma_j & \quad \text{for all } i, \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \quad \text{if } |i - j| > 1, \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \quad \text{if } |i - j| = 1.
\end{align*}$$

Then by Theorem 2.10, $\mathfrak{B}_{n-1}$ has semigroup presentation $\langle X_B^\circ \cup \{\lambda_n\} \mid R_B^\circ \rangle$ via

$$(X_B^\circ \cup \{\lambda_n\})^+ \to \mathfrak{B}_{n-1} : \begin{cases}
\sigma_i^\pm \mapsto (\sigma_i^\pm)_{n^\circ} & \text{for each } i
\end{cases} \quad \lambda_n \mapsto \tilde{\lambda}_n.$$
Let \( R_{LBR} \subseteq (X_L \cup X_B \cup X_R)^+ \times (X_L \cup X_B \cup X_R)^+ \) denote the set of relations \( R_{LR} \) together with \( R_B^0 \) and

\[
\sigma_i \lambda_j = \begin{cases} 
\lambda_n \lambda_j \sigma_i & \text{if } 1 \leq i < j - 1 \leq n - 2 \\
\lambda_n \lambda_{j-1} & \text{if } 1 \leq i = j - 1 \leq n - 2 \\
\lambda_n \lambda_{j+1} & \text{if } 1 \leq i = j \leq n - 2 \\
\lambda_n \lambda_j \sigma_{i-1} & \text{if } 1 \leq j < i \leq n - 2 
\end{cases} \tag{BL1—BL4}^b
\]

\[
\rho_j \sigma_i = \begin{cases} 
\sigma_i \rho_j \rho_n & \text{if } 1 \leq i < j - 1 \leq n - 2 \\
\rho_{j-1} \rho_n & \text{if } 1 \leq i = j - 1 \leq n - 2 \\
\rho_{j+1} \rho_n & \text{if } 1 \leq i = j \leq n - 2 \\
\sigma_{i-1} \rho_j \rho_n & \text{if } 1 \leq j < i \leq n - 2 
\end{cases} \tag{RB1—RB4}^b
\]

Now define

\[
\Omega : (X_L \cup X_B^2 \cup X_R^2)^+ \to \mathcal{I}_n \setminus \mathcal{B}_n : \begin{cases} 
x \mapsto \tilde{x} \\
\sigma_i^{\pm 1} \mapsto (\xi_i^{\pm 1})_n
\end{cases} \quad \text{for each } x \in X_L \cup X_R
\]

It is easy to check diagrammatically that \( R_{LBR}^0 \subseteq \ker \Omega \). With more effort it may also be shown that \( \ker \Omega \subseteq (R_{LBR}^0)^2 \), which proves the following.

**Theorem 8.50.** The semigroup \( \mathcal{I}_n \setminus \mathcal{B}_n \) of all strictly-partial braids on \( n \) has semigroup presentation

\[
\langle X_L \cup X_B^2 \cup X_R^2 \mid R_{LBR}^0 \rangle
\]

via \( \Omega \).

Using a similar proof to that of Proposition 5.21, we may derive the presentation of \( \mathcal{I}_n \setminus \mathcal{S}_n \) proved in [29]. Let \( (R_{LBR}^0)' \subseteq (X_L \cup \{\sigma_1, \ldots, \sigma_{n-2}\} \cup X_R)^+ \times (X_L \cup \{\sigma_1, \ldots, \sigma_{n-2}\} \cup X_R)^+ \) denote the set of relations obtained from \( R_{LBR}^0 \) by replacing (B1)$^b$ and (B2)$^b$ by

\[
\sigma_i \lambda_n = \lambda_n \sigma_i = \sigma_i \quad \text{for all } i
\]

\[
\sigma_i^2 = \lambda_n \quad \text{for all } i,
\]

and define

\[
\Omega' : (X_L \cup \{\sigma_1, \ldots, \sigma_{n-2}\} \cup X_R)^+ \to \mathcal{I}_n \setminus \mathcal{S}_n : \begin{cases} 
x \mapsto \tilde{x} \\
\sigma_i \mapsto s_i^{\pm 1}_n
\end{cases} \quad \text{for each } x \in X_L \cup X_R
\]

**Theorem 8.51 ([29]).** The semigroup \( \mathcal{I}_n \setminus \mathcal{S}_n \) of all strictly-partial permutations on \( n \) has semigroup presentation

\[
\langle X_L \cup \{\sigma_1, \ldots, \sigma_{n-2}\} \cup X_R \mid (R_{LBR}^0)' \rangle
\]

via \( \Omega' \).
It is also possible to combine the techniques of [29] and Section 8.3 to study the semigroup $\POIB_n \setminus \mathcal{P}_n$ of all order-preserving strictly-partial braids on $n$. Again, we obtain a factorisation

$$\POIB_n \setminus \mathcal{P}_n = \mathcal{L}_n \mathfrak{P}_{n-1} \mathcal{R}_n$$

where

$$\mathfrak{P}_{n-1} = \{ \beta \in \POIB_n \mid \text{dom}(\beta) = \text{im}(\beta) = n^\beta \} = \{ \beta^\mathfrak{p} \mid \beta \in \mathcal{P}_n \}$$

is a maximal subgroup of $\POIB_n \setminus \mathcal{P}_n$, with identity $\tilde{\lambda}_n = \tilde{\rho}_n$, which is naturally isomorphic to the pure braid group $\mathcal{P}_{n-1}$ on $n$.

Let $X^\beta_p = \{ a_{ij} \mid 1 \leq i < j \leq n - 1 \}$ and let $R^\beta_p \subseteq \left( X^\beta_p \cup \{ \lambda_n \} \right)^+ \times \left( X^\beta_p \cup \{ \lambda_n \} \right)^+$ be the set of relations

$$a_{ij}^{\pm 1} \lambda_n = \lambda_n a_{ij}^{\pm 1} = a_{ij}^{\pm 1} \quad \text{for all } i, j \tag{P1}$$

$$a_{ij}^{\pm 1} a_{ij}^{\pm 1} = \lambda_n \quad \text{for all } i, j \tag{P2}$$

$$a_{rs} a_{ij} a_{rs}^{-1} = a_{ij} \quad \text{if } i < r \text{ or } i > s \tag{P3}$$

$$a_{rs} a_{sj} a_{rs}^{-1} = (a_{sj}^{-1} a_{rj}^{-1}) a_{sj} (a_{rj} a_{sj}) \quad \text{if } r < i < s, \tag{P4}$$

$$a_{rs} a_{rj} a_{rs}^{-1} = a_{s j}^{-1} a_{rj} a_{sj} \quad \text{if } r < i < s, \tag{P5}$$

$$a_{rs} a_{ij} a_{rs}^{-1} = (a_{sj}^{-1} a_{rj}^{-1} a_{rj} a_{sj}) a_{ij} (a_{rj}^{-1} a_{sj}^{-1} a_{rj} a_{sj}) \tag{P6}$$

where in relations (P3—P6) we always have $s < j$. Then by Theorem 2.13, $\mathfrak{P}_{n-1}$ has semigroup presentation $\langle X^\beta_p \cup \{ \lambda_n \} \mid R^\beta_p \rangle$ via

$$(X^\beta_p \cup \{ \lambda_n \})^+ \rightarrow \mathfrak{P}_{n-1} : \left\{ \begin{array}{l}
\lambda_n \mapsto \tilde{\lambda}_n. \\
\end{array} \right.$$
It is easy to check diagrammatically that $R_{LPR}^\oplus \subseteq \ker \Gamma$. Again it may also be shown that $\ker \Gamma \subseteq (R_{LPR}^\oplus)^\dagger$, which proves the following.

**Theorem 8.52.** The semigroup $\mathcal{POIB}_n \setminus \mathcal{P}_n$ of all order-preserving strictly-partial braids on $n$ has semigroup presentation

$$\langle X_L \cup X_P \cup X_R \mid R_{LPR}^\oplus \rangle$$

via $\Gamma$. \hfill $\square$

### 8.5. The Dual Symmetric Inverse Semigroup

If $A, B \subseteq n$ we will write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. Recall that an equivalence $E \in \mathcal{Eq}_n$ is *convex* if we can write $n/E = \{N_1, \ldots, N_k\}$ where $N_1 < \cdots < N_k$. If $E \in \mathcal{Eq}_n$ is convex, then we will also say that the quotient $n/E$ is convex. We define a block bijection $\theta \in \mathcal{I}_n^*$ to be *planar* if $\text{dom}(\theta)$ and $\text{im}(\theta)$ are both convex, and $A \theta < B \theta$ whenever $A, B \in \text{dom}(\theta)$ and $A < B$. Notice that a planar block bijection $\theta$ is completely determined by $\text{dom}(\theta)$ and $\text{im}(\theta)$. The reason for the choice of terminology is that $\theta \in \mathcal{I}_n^*$ is planar if and only if a picture of $\theta$ may be drawn with no intersecting lines. For example, the planar block bijection $\theta \in \mathcal{I}_8^*$ with

$$\text{dom}(\theta) = \{\{1, 2, 3\}, \{4\}, \{5, 6\}, \{7, 8\}\} \quad \text{and} \quad \text{im}(\theta) = \{\{1\}, \{2, 3\}, \{4, 5, 6, 7\}, \{8\}\}$$

is pictured in Figure 8.8.

![Figure 8.8. A planar block bijection in $\mathcal{I}_8^*$](image)

It is clear diagrammatically that the product of two planar block bijections is also planar. Thus, the set of all planar block bijections on $n$ is a submonoid of $\mathcal{I}_n^*$. We denote this submonoid by $\mathcal{POI}_n^*$. As our choice of notation suggests, there is a strong connection between planar block bijections and order-preserving partial permutations. Given an order-preserving partial permutation $\alpha \in \mathcal{POI}_{n-1}$, we may construct a planar block bijection $\theta_\alpha \in \mathcal{POI}_n^*$ as follows. Suppose that $\alpha \in \mathcal{POI}_{n-1}$ with

$$\text{dom}(\alpha) = \{a_1, \ldots, a_k\} \quad \text{and} \quad \text{im}(\alpha) = \{b_1, \ldots, b_k\}$$
where \( a_1 < \cdots < a_k \) and \( b_1 < \cdots < b_k \). We then define \( \theta_\alpha \in \mathcal{POI}_n^* \) to be the planar block bijection with

\[
\text{dom}(\theta_\alpha) = \{A_1, \ldots, A_{k+1}\} \quad \text{and} \quad \text{im}(\theta_\alpha) = \{B_1, \ldots, B_{k+1}\}
\]

where \( A_j = \{a_{j-1} + 1, \ldots, a_j\} \) and \( B_j = \{b_{j-1} + 1, \ldots, b_j\} \) for each \( 1 \leq j \leq k + 1 \); here we interpret \( a_0 = b_0 = 0 \) and \( a_{k+1} = b_{k+1} = n \). Conversely, given \( \theta \in \mathcal{POI}_n^* \) with

\[
\text{dom}(\theta) = \{A_1, \ldots, A_k\} \quad \text{and} \quad \text{im}(\theta) = \{B_1, \ldots, B_k\}
\]

where \( A_1 < \cdots < A_k \) and \( B_1 < \cdots < B_k \), we define \( \alpha_\theta \in \mathcal{POI}_{n-1} \) to be the order-preserving partial permutation with

\[
\text{dom}(\alpha) = \{a_1, \ldots, a_{k-1}\} \quad \text{and} \quad \text{im}(\alpha) = \{b_1, \ldots, b_{k-1}\}
\]

where \( a_j = \max(A_j) \) and \( b_j = \max(B_j) \) for each \( 1 \leq j \leq k - 1 \). The most informative way to understand these correspondences is to superimpose pictures of \( \alpha \in \mathcal{POI}_{n-1} \) and \( \theta_\alpha \in \mathcal{POI}_n^* \) in such a way that upper (resp. lower) vertex \( i \) in \( \alpha \) is midway between upper (resp. lower) vertex \( i \) and \( i + 1 \) in \( \theta_\alpha \); see Figure 8.9 for an example.

![Figure 8.9](image)

**Figure 8.9.** A picture of \( \alpha \in \mathcal{POI}_8 \) (grey) and \( \theta_\alpha \in \mathcal{POI}_9 \) (black).

It is then easy to see that the maps

\[
f : \mathcal{POI}_{n-1} \to \mathcal{POI}_n^* : \alpha \mapsto \theta_\alpha \quad \text{for each } \alpha \in \mathcal{POI}_{n-1}
\]

\[
g : \mathcal{POI}_n^* \to \mathcal{POI}_{n-1} : \theta \mapsto \alpha_\theta \quad \text{for each } \theta \in \mathcal{POI}_n^*
\]

are mutually inverse bijections. It is a pleasant exercise to show that \( f \) (and hence also \( g \)) is in fact an isomorphism. This may be done directly or, using Theorem 8.17, by showing that relations \( R_{LR} \) are preserved by \( f \); the images under \( f \) of the generators \( \tilde{\lambda}_i, \tilde{\rho}_i \in \mathcal{POI}_{n-1} \) are pictured in Figure 8.10.

![Figure 8.10](image)

**Figure 8.10.** The images \( \tilde{\lambda}_i f \) (left) and \( \tilde{\rho}_i f \) (right) of the generators of \( \mathcal{POI}_{n-1} \).

Thus we may define submonoids \( \mathcal{L}_n^* = \mathcal{L}_{n-1} f \) and \( \mathcal{R}_n^* = \mathcal{R}_{n-1} f \) of \( \mathcal{POI}_n^* \), and we have the factorisation \( \mathcal{POI}_n^* = \mathcal{L}_n^* \mathcal{R}_n^* \). Now since the domain (and image) of an arbitrary block bijection need not be convex we do not have \( \mathcal{I}_n^* = \mathcal{L}_n^* S_n \mathcal{R}_n^* \). However, it is easy to see that
we do have $I_n^* = S_n\mathcal{P}\mathcal{O}T_n^*S_n = S_n\mathcal{L}_n^*\mathcal{R}_n^*S_n$. This observation forms the basis of a current attempt [18] towards finding a presentation of $I_n^*$.

8.6. The Factorisable and Permeable Braid Monoids

We conclude this chapter with a brief explanation as to why we have not applied similar techniques to those of Section 8.2 to investigate the factorisable and permeable braid monoids $\mathfrak{F}B_n$ and $\mathfrak{P}B_n$. Now a planar block bijection is uniform if and only if it is equal to $\text{id}_E$ for some convex equivalence $E \in \mathcal{E}q_n$. Let $\mathcal{P}\mathcal{O}\mathfrak{F}_n^* = \mathcal{P}\mathcal{O}I_n^* \cap \mathfrak{F}_n^*$. We may identify $\mathcal{P}\mathcal{O}\mathfrak{F}_n^*$ with the submonoid

$\{[1]_E \mid E \in \mathcal{E}q_n \text{ is convex}\}$

of $\mathfrak{F}B_n$. With this identification we then have

$\mathfrak{F}B_n = B_n\mathcal{P}\mathcal{O}\mathfrak{F}_n^*B_n$.

(A similar comment may be made regarding $\mathfrak{P}B_n$.) Now it is easy to see that $\mathcal{P}\mathcal{O}\mathfrak{F}_n^*$ is a free semilattice with basis $\text{id}_{E_{12}}, \ldots, \text{id}_{E_{n-1,n}}$. Thus $\mathcal{P}\mathcal{O}\mathfrak{F}_n^*$ has presentation $\langle e_1, \ldots, e_{n-1} \mid R \rangle$ where $R$ consists of the relations

$e_i^2 = e_i$ for all $i$

$e_ie_j = e_je_i$ for all $i, j$.

While the author believes it would be possible to combine this presentation with Artin’s presentation of $B_n$ to yield presentations of $\mathfrak{F}B_n$ and $\mathfrak{P}B_n$, it is likely that we would simply recover the presentations of those monoids given in Theorems 5.48 and 5.69.
APPENDIX A

Catalogue of Presentations

This appendix contains a series of tables which catalogue most of the presentations derived in the body of the thesis. We fix a positive integer \( n \) for the entire appendix.

A.1. Braid Monoids

Let \( X_B = \{ \sigma_1^\pm, \ldots, \sigma_{n-1}^\pm \} \) and denote by \( R_B \) the set of relations

\[
\begin{align*}
\sigma_i^{\pm 1} \sigma_i^{\mp 1} &= 1 & \text{for all } i \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{if } |i - j| > 1 \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{if } |i - j| = 1.
\end{align*}
\]

Each entry in the following table corresponds to a monoid presentation of a braid monoid \( M \in \{ \mathcal{I}B_n, \mathcal{F}B_n, \mathcal{P}B_n \} \). Each presentation is of the form

\[
\langle X_B \cup X \mid R_B \cup R \rangle
\]

via a map

\[
f : (X_B \cup X)^* \to M
\]

under which \( \sigma_i^{\pm 1} \mapsto \zeta_i^{\pm 1} \) for each \( i \). Each entry specifies

- the additional generators \( X \),
- the additional relations \( R \), and
- the restriction \( f|_X \) of \( f \) to \( X \).

We also include a reference to the location in which the presentation appears in the body of the thesis.

| Presentations of \( \mathcal{I}B_n \) | \( X \) | \( R \) | \( f|_X \) | Reference |
|---|---|---|---|---|
| \( e \) | \( e^2 = e \) | \( e \sigma_i = \sigma_i e \) if \( i \neq 1 \) | \( e \mapsto 1_{\{1\}}e \) | Theorem 5.13; see also [21] |
Presentations of $\mathcal{T}B_n$ (continued)

| $X$ | $R$ | $f|_X$ | Reference |
|-----|-----|--------|-----------|
| $f$ | $f \sigma_1 f = f,$  
$\quad f \sigma_1 \sigma_2 = \sigma_2 f \sigma_1$  
$\quad f \sigma_i = \sigma_i f$ if $i \geq 3$  
$\quad f^2 \sigma_1 = f^2 = \sigma_1 f^2$  
$\quad f \sigma_{i}^{2} = f = \sigma_{i}^{2} f$ | $f \mapsto (\sigma_{1})_{\{1\}}^{c}$ | Theorem 5.15 |
| $g$ | $g \sigma_1 g = g,$  
$\quad \sigma_2 \sigma_1 g = \sigma_1 \sigma_2 g\sigma_1$  
$\quad g \sigma_i = \sigma_i g$ if $i \geq 3$  
$\quad g^2 \sigma_1 = g^2 = \sigma_1 g^2$  
$\quad g \sigma_{i}^{2} = g = \sigma_{i}^{2} g$ | $g \mapsto (\sigma_{1})_{\{2\}}^{c}$ | Remark 5.16 |
| $f_1, \ldots, f_{n-1}$ | $f_i \sigma_i f_i = f_i$ for all $i$  
$\quad \sigma_i f_i = f_{i+1} \sigma_{i+1}$ for all $i \leq n - 2$  
$\quad \sigma_i f_j = f_j \sigma_i$ if $|i - j| > 1$  
$\quad \sigma_i \sigma_j f_i = f_j \sigma_i \sigma_j$ if $|i - j| = 1$  
$\quad \sigma_i f_{i}^{2} = f_{i}^{2} = f_{i}^{2} \sigma_i$ for all $i$  
$\quad \sigma_{i}^{2} f_i = f_i = f_i \sigma_{i}^{2}$ for all $i$ | $f_i \mapsto (\sigma_{i})_{\{i\}}^{c}$ for all $i$ | Theorem 5.18 |
| $g_1, \ldots, g_{n-1}$ | $g_i \sigma_i g_i = g_i$ for all $i$  
$\quad g_i \sigma_i = \sigma_{i+1} g_{i+1}$ for all $i \leq n - 2$  
$\quad \sigma_i g_j = g_j \sigma_i$ if $|i - j| > 1$  
$\quad \sigma_i \sigma_j g_i = g_j \sigma_i \sigma_j$ if $|i - j| = 1$  
$\quad \sigma_i g_{i}^{2} = g_{i}^{2} = g_{i}^{2} \sigma_i$ for all $i$  
$\quad \sigma_{i}^{2} g_i = g_i = g_i \sigma_{i}^{2}$ for all $i$ | $g_i \mapsto (\sigma_{i})_{\{i+1\}}^{c}$ for all $i$ | Remark 5.20 |
| $\lambda_1, \ldots, \lambda_n,$  
$\quad \rho_1, \ldots, \rho_n$ | $\lambda_i \lambda_j = \lambda_{j+1} \lambda_i$ if $i \leq j$  
$\quad \lambda_i \lambda_n = \lambda_i$ for all $i$  
$\quad \rho_j \rho_i = \rho_i \rho_{j+1}$ if $i \leq j$  
$\quad \rho_n \rho_i = \rho_i$ for all $i$  
$\quad \rho_i \lambda_j = \begin{cases}  
\lambda_n \lambda_{j+1} \rho_i & \text{if } i < j  
\lambda_n & \text{if } i = j  
\lambda_n \lambda_{j+1} \rho_i & \text{if } j < i  
\lambda_j \sigma_i & \text{if } i < j - 1  
\lambda_{j+1} & \text{if } i = j  
\lambda_{j+1} \sigma_i & \text{if } j < i  
\end{cases}$  
\quad $\lambda_i \mapsto \tilde{\lambda}_i$  
\quad $\rho_i \mapsto \tilde{\rho}_i$ | | Theorem 8.38 |

| $\sigma_i \lambda_j = \begin{cases}  
\lambda_{j-1} & \text{if } i = j - 1  
\lambda_{j+1} & \text{if } i = j  
\lambda_{j+1} \sigma_i & \text{if } j < i  
\end{cases}$  
\quad $\rho_j \sigma_i = \begin{cases}  
\sigma_i \rho_j & \text{if } i < j - 1  
\rho_{j-1} & \text{if } i = j - 1  
\rho_{j-1} \sigma_i & \text{if } i = j  
\sigma_{i-1} \rho_j & \text{if } j < i  
\end{cases}$ | | |
### A.2. Transformation Semigroups

Let $X_S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ and denote by $R_S$ the set of relations

\[
\begin{align*}
\sigma_i^2 &= 1 & \text{for all } i \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{if } |i - j| > 1 \\
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{if } |i - j| = 1.
\end{align*}
\]

More presentations of $\mathcal{IB}_n$, $\mathfrak{B}_n$, and $\mathfrak{PB}_n$ may be found in Corollaries 5.2, 5.3, 5.28, 5.29, 5.62, and 5.63, and Theorems 5.40 and 5.65. See also Theorem 8.50 for a presentation of the semigroup $\mathcal{IB}_n \setminus B_n$ of strictly-partial braids on $n$.

#### Presentations of $\mathfrak{B}_n$

| $X$ | $R$ | $f|_X$ | Reference |
|-----|-----|--------|-----------|
| $e$ | $e^2 = e = e \sigma_1$ | $e \mapsto [1] e_{12}$ | Theorem 5.42; see also [27] |

| $x_1, \ldots, x_{n-1}$ | $x_i^2 = x_i = x_i x_i$ | $x \mapsto [1] x_{i,i+1}$ | Theorem 5.48; see also [27] |

#### Presentations of $\mathfrak{PB}_n$

| $X$ | $R$ | $f|_X$ | Reference |
|-----|-----|--------|-----------|
| $e$ | $e^2 = e = e \sigma_1^2$ | $e \mapsto [1] e_{12}$ | Theorem 5.67 |

| $x_1, \ldots, x_{n-1}$ | $x_i^2 = x_i = x_i x_i$ | $x \mapsto [1] x_{i,i+1}$ | Theorem 5.69 |

| $x_i$ | $x_i x_j = x_j x_i$ | $x \mapsto [1] x_{i,i+1}$ | Theorem 5.69 |

| $x_i$ | $x_i x_j = x_j x_i$ | $x \mapsto [1] x_{i,i+1}$ | Theorem 5.69 |

| $x_i$ | $x_i x_j = x_j x_i$ | $x \mapsto [1] x_{i,i+1}$ | Theorem 5.69 |
Each entry in the following table corresponds to a monoid presentation of a transformation semigroup $M \in \{I_n, S_n^1\}$. Each presentation is of the form

$$\langle X_S \cup X \mid R_S \cup R \rangle$$

via a map

$$f : (X_S \cup X)^* \rightarrow M$$

under which $\sigma_i \mapsto s_i$ for each $i$. Each entry specifies

- the additional generators $X$,
- the additional relations $R$, and
- the restriction $f|_X$ of $f$ to $X$.

We also include a reference to the location in which the presentation appears in the body of the thesis.

| $X$ | $R$ | $f|_X$ | Reference |
|-----|-----|------|-----------|
| $e$ | $e^2 = e$ | $e \mapsto \text{id}_{\{1\}^e}$ | Theorem 5.22; see also [55] |
| $f$ | $f \sigma_1 f = f$ | $f \mapsto s_1|_{\{1\}^e}$ | Theorem 5.24 |
| $g$ | $g \sigma_1 g = g$ | $g \mapsto s_1|_{\{2\}^e}$ | Remark 5.25 |
| $f_1, \ldots, f_{n-1}$ | $f_i \sigma_i f_i = f_i$ for all $i$ | $f_i \mapsto s_{i|\{i\}^e}$ for all $i$ | Theorem 5.26 |
| $g_1, \ldots, g_{n-1}$ | $g_i \sigma_i g_i = g_i$ for all $i$ | $g_i \mapsto s_{i|\{i+1\}^e}$ for all $i$ | Theorem 5.27 |
### A.2. TRANSFORMATION SEMIGROUPS

#### Presentations of $\mathcal{I}_n$ (continued)

| $X$ | $R$ | $f|_X$ | Reference |
|-----|-----|--------|-----------|
| $\lambda_i \lambda_j = \lambda_{j+1} \lambda_i$ | if $i \leq j$ | | |
| $\lambda_i \lambda_n = \lambda_i$ | for all $i$ | | |
| $\rho_j \rho_i = \rho_i \rho_{j+1}$ | if $i \leq j$ | | |
| $\rho_n \rho_i = \rho_i$ | for all $i$ | | |
| $\rho_i \lambda_j = \begin{cases} 
\lambda_n \lambda_{j-1} \rho_i & \text{if } i < j \\
\lambda_n = \rho_n & \text{if } i = j \\
\lambda_n \lambda_j \rho_{i-1} & \text{if } j < i 
\end{cases}$ | | | Theorem 8.40 |
| $\sigma_i \lambda_j = \begin{cases} 
\lambda_j \sigma_i & \text{if } i < j - 1 \\
\lambda_{j-1} & \text{if } i = j - 1 \\
\lambda_j + 1 & \text{if } i = j \\
\lambda_j \sigma_{i-1} & \text{if } j < i 
\end{cases}$ | | | for all $i$ |
| $\rho_j \sigma_i = \begin{cases} 
\sigma_i \rho_j & \text{if } i < j - 1 \\
\rho_{j-1} & \text{if } i = j - 1 \\
\rho_j + 1 & \text{if } i = j \\
\sigma_{i-1} \rho_j & \text{if } j < i 
\end{cases}$ | | | |

#### Presentations of $\mathfrak{S}_n^*$

| $X$ | $R$ | $f|_X$ | Reference |
|-----|-----|--------|-----------|
| $e$ | $e^2 = e = e \sigma_1$ | $e \mapsto \text{id}_{E_{12}}$ | Theorem 5.60; see also [27, 32] |
| | | | |
| $e_i^2 = e_i = e_i \sigma_i = \sigma_i e_i$ | for all $i$ | | |
| | $e_i e_j = e_j e_i$ | for all $i, j$ | |
| | $e_i \sigma_j = \sigma_j e_i$ | if $|i-j| > 1$ | | Theorem 5.61; see also [27] |
| | $e_i \sigma_j \sigma_i = \sigma_j \sigma_i e_j$ | if $|i-j| = 1$ | |

We also include in this section presentations of the monoids $\mathcal{L}_n$, $\mathcal{R}_n$, and $\mathcal{POI}_n$. The entries in the following table are self contained. Each entry describes a monoid presentation $\langle X \mid R \rangle$, via a map $f$, of a monoid $M \in \{\mathcal{L}_n, \mathcal{R}_n, \mathcal{POI}_n\}$. Each entry specifies the generators $X$, the relations $R$, and the map $f : X^* \rightarrow M$, as well as a reference to the location in which the presentation appears in the body of the text.

### Presentation of $\mathcal{L}_n$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$R$</th>
<th>$f$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1, \ldots, \lambda_n$</td>
<td>$\lambda_i \lambda_j = \lambda_{j+1} \lambda_i$ if $i \leq j$</td>
<td>$\lambda_i \mapsto \lambda_i$</td>
<td>Theorem 8.5; see also [29]</td>
</tr>
<tr>
<td></td>
<td>$\lambda_i \lambda_n = \lambda_i$ for all $i$</td>
<td>for each $i$</td>
<td></td>
</tr>
</tbody>
</table>
A.3. Pure Braid Monoids

Let $X_P = \{a_{ij} \mid 1 \leq i < j \leq n\}$ and denote by $R_P$ the set of relations
\[
a^\pm_1 \overline{a^\pm_1} = 1 \\
ar_s a_{ij} a_{rs}^{-1} = a_{ij} \quad \text{if } i < r \text{ or } i > s \\
a_r a_{s_j} a_{rs}^{-1} = (a_{s_j}^{-1} a_{r_j}^{-1}) a_{s_j} (a_{r_j} a_{s_j}) \\
a_r a_{r_j} a_{rs}^{-1} = a_{s_j}^{-1} a_{r_j} a_{s_j} \\
a_r a_{s_j} a_{rs}^{-1} = (a_{s_j}^{-1} a_{r_j}^{-1} a_{s_j} a_{r_j}) a_{ij} (a_{r_j}^{-1} a_{s_j}^{-1} a_{r_j} a_{s_j}) \quad \text{if } r < i < s,
\]
where in each case we have $s < j$. Each entry in the following table corresponds to a monoid presentation of a pure braid monoid $M \in \{\mathcal{I}P_n, \mathcal{P}P_n, \mathcal{P}P_n\}$. Each presentation is of the form
\[
\langle X_P \cup X \mid R_P \cup R \rangle
\]
via a map
\[
f : (X_P \cup X)^* \to M
\]
under which $a^\pm_1 \mapsto a^\pm_1$ for each $i, j$. Each entry specifies
- the additional generators $X$,
- the additional relations $R$, and
- the restriction $f|_X$ of $f$ to $X$. 

See also Theorem 8.51 for a presentation of the semigroup $\mathcal{I}_n \setminus \mathcal{S}_n$ of all strictly-partial permutations on $n$. 

### A.3. PURE BRAID MONOIDS

#### Presentation of $\mathcal{R}_n$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$R$</th>
<th>$f$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1, \ldots, \rho_n$</td>
<td>$\rho_i \rho_j = \rho_i \rho_{j+1}$ if $i \leq j$</td>
<td>$\rho_i \mapsto \rho_i$ for all $i$</td>
<td>Theorem 8.5; see also [29]</td>
</tr>
<tr>
<td></td>
<td>$\rho_i \rho_i = \rho_i$</td>
<td>$\rho_i \mapsto \rho_i$ for each $i$</td>
<td></td>
</tr>
</tbody>
</table>

#### Presentation of $\mathcal{POI}_n$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$R$</th>
<th>$f$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1, \ldots, \lambda_n, \rho_1, \ldots, \rho_n$</td>
<td>$\lambda_i \lambda_j = \lambda_{j+1} \lambda_i$ if $i \leq j$</td>
<td>$\lambda_i \mapsto \lambda_i$ for all $i$</td>
<td>Theorem 8.17; see also [29]</td>
</tr>
<tr>
<td></td>
<td>$\lambda_i \lambda_n = \lambda_n$</td>
<td>$\lambda_i \mapsto \lambda_i$ for all $i$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho_i \rho_i = \rho_i \rho_{j+1}$ if $i \leq j$</td>
<td>$\rho_i \mapsto \rho_i$ for each $i$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho_i \rho_i = \rho_i$ for all $i$</td>
<td>$\rho_i \mapsto \rho_i$ for each $i$</td>
<td></td>
</tr>
</tbody>
</table>
|     | $\rho_i \rho_j = \begin{cases} 
\lambda_n \lambda_{j-1} \rho_i & \text{if } i < j \\
\lambda_n = \rho_n & \text{if } i = j \\
\lambda_n \lambda_j \rho_{i-1} \rho_i & \text{if } j < i 
\end{cases}$ | $\lambda_i \mapsto \lambda_i$ for all $i$ | |
We also include a reference to the location in which the presentation appears in the body of the thesis.

<table>
<thead>
<tr>
<th>Presentation of $\mathcal{IP}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>$\varepsilon_{1, \ldots, \varepsilon_n}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Presentation of $\mathfrak{IP}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Presentation of $\mathfrak{BP}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

See also Theorem 8.49 for a presentation of $\mathcal{POIB}_n$, the monoid of order-preserving partial braids. Theorem 8.52 contains a presentation of $\mathcal{POIB}_n \setminus \mathcal{P}_n$, the semigroup of order-preserving strictly-partial braids.
Bibliography


Index of Notation

$2^n$, 17
$\lor$, 18, 32
$\sim_A$, 47
$\sim_{\varepsilon}$, 49
$\approx_{\varepsilon}$, 55
$\cong(i)$, $\cong(ii)$, $\cong(iii)$, 87
$\preceq$, 88
$\succeq$, 113

$\alpha_{ij}$, 24
$\alpha_{ij}$, 60

$\beta$, 20, 44
$\beta_A$, 47
$[\beta]_A$, 47, 104
$[\beta]_\varepsilon$, 50, 107
$\beta^{P}$, 126
$[\beta]_\varepsilon$, 50
$[\beta]_\varepsilon$, 55, 112
$\beta_{ij}$, 56

$\varepsilon_i$, 17
$\varepsilon_{ij}$, 18

$\theta_i$, 69

$\lambda_A$, 132
$\lambda_i$, 133
$\lambda_i$, 133

$\tilde{\rho}_A$, 132
$\tilde{\rho}_i$, 133
$\rho_i$, 133

$\zeta$, 23
$\zeta_{ij}$, 23

$\tilde{\zeta}_{ij}$, 60
$\sigma_i$, 23
$\sigma_{ij}$, 24
$\Sigma_B$, 141
$\Sigma_L$, 133
$\Sigma_P$, 152
$\Sigma_R$, 133

$\tau_i$, 86

$\phi_L$, 133
$\phi_R$, 133
$\phi_B$, 141
$\phi_P$, 152

$A^\beta$, 46
$[A, \beta]$, 47
$[A, \beta]$, 102
$[A, B_A]$, 103
$A'$, 46
$A_{ij}$, 25
$A_{ij}$, 146

$B_A$, 46
$(B_A1\ldots B_A3)$, 47
$B_{\varepsilon}$, 51
$(B_{\varepsilon}1\ldots B_{\varepsilon}3)$, 52
$(B_{\varepsilon}2)'$, 53, 75
$B_{\varepsilon}^{(2)}$, 56
$(B_{\varepsilon}^{(2)}1\ldots B_{\varepsilon}^{(2)}3)$, 56
$(B_{\varepsilon}^{(2)}2)'$, 94

$\mathcal{B}_n$, 20
$B_n$, 1, 22
$\mathcal{B}_{\kappa}$, 142
$\mathcal{B}_{n-1}$, 157
INDEX OF NOTATION 174

\( \mathcal{C}, 32 \)
\( C(G), 32 \)

\( \text{dom}(\alpha), 11 \)
\( \text{dom}(\theta), 28 \)

\( \mathcal{E}^\beta, 51 \)
\( [\mathcal{E}, \beta], 52 \)
\( [\mathcal{E}, \beta], 106 \)
\( [\mathcal{E}, P], 106 \)
\( [\mathcal{E}, \beta], 56 \)
\( [\mathcal{E}, \beta], 112 \)
\( [\mathcal{E}, P], 112 \)

\( e_i, 62 \)
\( e_{ij}, 76, 94 \)
\( \mathcal{E}_n, 18 \)
\( \mathcal{E}_X, 18, 28 \)

\( \mathcal{E}(S), 9 \)
\( \exp_{ij}, 25, 110 \)

\( \mathfrak{F}_X, 28 \)
\( \mathfrak{F}_X, 30 \)
\( \mathfrak{F}_n, 30 \)
\( \mathfrak{F}_B, 50 \)
\( \mathfrak{F}_B^*, 83, 85 \)
\( \mathfrak{F}^*, 106 \)
\( \mathfrak{F}S_B, 88 \)

\( \mathcal{F}(M), 27 \)

\( F(X), 16 \)

\( G(M), 9 \)
\( G_e, 27 \)
\( (G_e1—G_e3), 36 \)
\( (G_e2)', 39 \)
\( (G_e2)'', (G_e2)', 40 \)

\( I_X, 11, 28 \)
\( I_n, 12 \)
\( I_n \setminus S_n, 157 \)
\( I_X, 29 \)
\( I_n', 29 \)

\( \mathfrak{I}B_n, 44 \)
\( \mathfrak{I}B_n, 3, 44 \)
\( \mathfrak{I}B_n \setminus B_n, 157 \)
\( \mathfrak{I}P_n, 102 \)

\( \text{im}(\alpha), 11 \)
\( \text{im}(\theta), 28 \)
\( (i_1, \ldots, i_k)\pi, 50 \)

\( \ker(\varphi), 10 \)

\( \mathcal{L}_n, 133 \)
\( \mathcal{L}_n', 157 \)

\( \mathbb{N}, 8 \)
\( n, 8 \)
\( n^k, 100, 157 \)
\( N_1, \ldots, N_k, 100 \)
\( N_k^0, 100 \)

\( P, P', 19 \)
\( P_n, 22 \)
\( P_k, 152 \)
\( P_{M, n-1}, 159 \)
\( P_B, 55 \)
\( P_B^*, 98 \)
\( P_F, 112 \)
\( P^T, 11 \)
\( P^T_n, 12 \)
\( P^T_B, 132 \)
\( P^T_B, 151 \)
\( P^T_B \setminus P_n, 159 \)
\( P_A, 103 \)
\( P_k/B_k, 103 \)
\( P/E, 100 \)
\( P/E, 106 \)
\( (P/E)', 109 \)
\( P/E, 112 \)
\( (P/E)', 115 \)

\( \mathbb{R}, 8 \)
\( R_n, 38 \)
\( R_n, 38 \)
\( R^k, 14, 15, 16 \)
\( \mathbb{R}_n, 133 \)
\( \mathbb{R}_n, 157 \)
\( \mathcal{S}_X, 12 \)
\( S_n, 12 \)
\( \mathcal{S}_{n-1}, 157 \)
s_r, 62
S(G), 32
\textbf{SB}_n, 86
\textbf{SB}_n, 2, 86
\text{Stab}(A), 33
\text{Stab}(\mathcal{E}), 33
S/\sim, 10

T_X, 11
T_n, 12
t_{xy}, 33

U_j, 25
U_{N_i}, 109
U^{(2)}(A), 115
U^{(2)}_{N_i}, 109
U^{(2)}(\mathcal{E}), 117

\hat{w}, 134, 137, 143

X^*, 14
X^+, 14
X/\mathcal{E}, 28
(X|R), 13, 15

Z, 8
Z^+, 8
Index

Alexander’s Theorem, 1
   singular version, 1

Birman’s Conjecture, 2
   block bijection, 28
   planar, 160
   uniform, 29
   braid, 1, 20
      closed, 1
      equivalence, 21
      identity, 22
      inverse, 22
      product, 20
   braid group, 1, 22

   congruence, 10
   coset monoid, 31

   dual symmetric inverse semigroup, 28

   embedding, 10
   epimorphism, 10
   equivalence relation
      conjugate, 106
      convex, 100
      semilattice, 18
      standard, 107

   factorisable braid monoid, 49
      algebraic construction, 51
      decision problems, 111
      geometric definition, 49
      in the coset monoid of the braid group, 53
      the pure factorisable braid monoid, 106
   factorisable inverse monoid, 27
   factorisable part

   of an inverse monoid, 27
   of the dual symmetric inverse semigroup, 30
   of the symmetric inverse semigroup, 28
   flexible singular braid monoid, 88
   flexible-vertex-isotopy, 88
   free group, 16
   free monoid, 14
   free semigroup, 14
   Fundamental Homomorphism Theorem, 10
      group, 9
         of units, 9
      groupoid, 9

   homomorphism, 10
      defined using a presentation, 17

   inverse braid monoid, 43
      algebraic construction, 46
      decision problems, 105
      geometric definition, 43
      the pure inverse braid monoid, 102
      the singular part of the inverse braid monoid,
         157
   inverse semigroup, 10
   isomorphism, 10

   join
      of equivalence relations, 18
      of subgroups, 32

   kernel, 10
   knot, 1
      singular, 1
   link, 1

176
merge and part, 49
monoid, 9

partial braid, 43
  empty braid, 43
  equivalence, 44
  multiplication, 44
  order-preserving, 151
  strictly-, 157
partial permutation, 11
  order-preserving, 132
  strictly-, 157
uniform, 28
partial transformation semigroup, 11
permeable braid monoid, 54
  algebraic construction, 56
  decision problems, 118
  geometric definition, 54
  the pure permeable braid monoid, 111
permeate, 54
power set, 17
presentation
  of a factorisable inverse monoid, 38, 39
  of a group, 16
  of a monoid, 15
  of a semigroup, 15
pure braid group, 22

quotient groupoid, semigroup, etc., 10

semigroup, 9
semilattice, 9
  of idempotents, 10
singular braid, 1, 86
  closed, 2
singular braid monoid, 2, 86
string, 20
  product of strings, 20
symmetric group, 12
symmetric inverse semigroup, 11, 28

Tietze transformations, 16
transformation semigroup, 11