CHAPTER 1

Coxeter Groups and Iwahori-Hecke algebras

In this chapter we begin with some background on Coxeter groups and Iwahori-Hecke algebras. All of these results are well known and can be found in Kazhdan & Lusztig [**11**] and Humphreys [**9**], among others. We will also give a short survey of Clifford theory following Curtis & Reiner [**3**].

1.1. Coxeter Groups

We will describe a Coxeter group in a purely abstract form by generators and relations. By $(ab\ldots)_m$, for m a positive integer, we mean that the pattern within the brackets is repeated until there are m symbols present. For example, $(ab\ldots)_5 = ababa$. In chapter 3 we will also need the case when m is a negative integer. In this case the pattern starts with 'b'. So $(ab\ldots)_{-5} = babab$.

Definition 1.1. *A* **Coxeter system**, (W, S), *consists of a group,* W*, and a subset* $S \subset W$*. The group* W *is generated by the elements of* S *subject to the relations:*

$$(rs)^{m_{rs}} = 1 \quad \textit{for all } r, s \in S$$

where $m_{ss} = 1$ *and* $m_{rs} = m_{sr} \in \{2, 3, \ldots, \infty\}$ *if* $r \neq s$*. If there is no relation between* r *and* s *then, by convention, we write* $m_{rs} = \infty$*.*

Note that $m_{ss} = 1$ implies that $s = s^{-1}$ for all $s \in S$. Consequently, we may rewrite the relations as $(rs\ldots)_{m_{rs}} = (sr\ldots)_{m_{rs}}$ and then we call the relations the **braid relations**. When it is not important to distinguish the generating set S, we will just talk about the Coxeter group W.

We can write $w \in W$ as a product of generators, $w = s_1 \ldots s_n$ for s_i in S. This product may not be unique. We define the **length** of w, $\ell(w)$, to be the smallest n such that $w = s_1 \ldots s_n$. So $\ell(s_i) = 1$ for all $s_i \in S$ and we define $\ell(1) = 0$. When w is written as a product of $\ell(w)$ generators we say that this expression for w is **reduced**.

Example 1.2. The symmetric group, $\mathfrak{S}_n$, is a Coxeter group that has generators $s_i = (i\; i+1)$ for $i \in [1, \ldots, n-1]$ and relations:

$$\begin{aligned} s_i^2 &= 1 && \text{for } i \in [1, \ldots, n-1], \\ s_i s_j &= s_j s_i && \text{whenever } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } i \in [1, \ldots, n-2]. \end{aligned}$$

Let $w = s_1 s_2 \ldots s_n$ be a reduced expression for w in a Coxeter group W. A **subexpression** of w is any expression obtained from it by removing some generators. To be precise, a subexpression of w is an expression of the form $s_{i_1} s_{i_2} \ldots s_{i_k}$ where $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$.

There is a partial order on a Coxeter group that is very useful. See, for example, Humphreys [**9**, §5.10]. This order is the **Bruhat order** and we denote it by $\leq$. Let $w, w' \in W$. Then $w' \leq w$ if a reduced expression for w' is a subexpression of a reduced expression for w. If $w' \leq w$ and $w' \neq w$ then we write $w' < w$.

Proposition 1.3. *Suppose (W, S) is a Coxeter system. For $s \in S$ and $w \in W$ we have*

$$\ell(sw) = \begin{cases} \ell(w) + 1, & \text{if } sw > w, \\ \ell(w) - 1, & \text{if } sw < w. \end{cases}$$

For a proof see Humphreys [**9**] (Proposition 5.2 and Theorem 5.10).

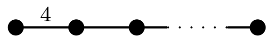
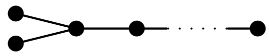
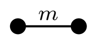
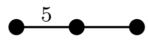
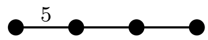
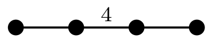
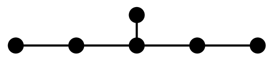
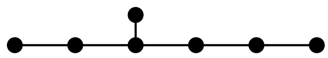
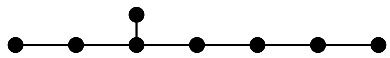
Suppose (W, S) is a Coxeter system. We define the **Coxeter diagram** of (W, S) to be the labelled graph $\Gamma(W, S)$ with a node for every s in the generating set S, and an edge between nodes r and s if $m_{rs} \geq 3$. We label this edge with m_{rs}. So, for $r, s \in S$ and $r \neq s$, if node r and node s are not joined by an edge then r and s commute. By convention, we omit the label if $m_{rs} = 3$.

Definition 1.4. *A Coxeter system (W, S) is* **reducible** *if $S = S_1 \coprod S_2$ and $m_{rs} = 2$ whenever $r \in S_1$ and $s \in S_2$. We say (W, S) is* **irreducible** *if it is not reducible.*

Note that, from this definition, a Coxeter system is irreducible if and only if its Coxeter diagram is connected.

If W_1 is the subgroup of W generated by S_1 and similarly W_2 is the subgroup of W generated by S_2, then (W_1, S_1) and (W_2, S_2) are Coxeter systems and $W = W_1 \times W_2$. See Humphreys [**9**, Proposition 2.2]. So to classify the finite Coxeter groups we need only consider the irreducible finite Coxeter groups.

Theorem 1.5. *The finite irreducible Coxeter groups are precisely the groups with the following Coxeter diagrams.*

Type	*Diagram*	*Rank*	*Order*
A_n		n	$(n+1)!$
$B_n = C_n$		n	$2^n n!$
D_n		n	$2^{n-1}n!$
$I_2(m)$		2	$2p$
H_3		3	$2^3 \cdot 3 \cdot 5$
H_4		4	$2^6 \cdot 3^2 \cdot 5^2$
F_4		4	$2^7 \cdot 3^2$
E_6		6	$2^7 \cdot 3^4 \cdot 5$
E_7		7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8		8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

For a proof see Humphreys [**9**, §2.3-2.7].

The **sign representation** of a Coxeter group, W, is the group homomorphism

$$\begin{array}{rccl} \epsilon: & W & \to & C_2 \\ & w & \mapsto & (-1)^{\ell(w)} \end{array}$$

where $C_2 = \{\pm 1\}$ is the cyclic group of order 2. The **alternating subgroup**, $\mathscr{A}(W)$, of W is the kernel of this map. Therefore it consists of all elements of even length. So

$$\mathscr{A}(W) := \{w \in W \mid \ell(w) \equiv 0 \pmod 2\}.$$

In this thesis we will construct and describe a q-analogue of $\mathscr{A}(W)$.

1.2. Iwahori-Hecke algebras

Definition 1.6. *Suppose that (W, S) is a Coxeter system and let R be a ring which contains an invertible element q. The* **Iwahori-Hecke algebra** *$\mathcal{H} = \mathcal{H}_{R,q}(W, S)$ is the associative R-algebra generated by the elements $\{T_s \mid s \in S\}$, with identity $T_1 = 1$ and relations:*

$$\begin{array}{rl} (T_s - q)(T_s + 1) = 0 & \text{for all } s \in S, \\ (T_r T_s \ldots)_{m_{rs}} = (T_s T_r \ldots)_{m_{rs}} & \text{for all } r \neq s \in S, \end{array}$$

where, for r and s in S, m_{rs} is the order of rs in W.

We will use $\mathcal{H}$ for $\mathcal{H}_{R,q}(W,S)$ when there is no ambiguity.

The first relation says that $T_s^2 = (q-1)T_s + q$ and, since q is invertible, we have $T_s^{-1} = q^{-1}T_s - q^{-1}(q-1)$.

We define $T_w := T_{s_1}T_{s_2}\dots T_{s_n}$ where $w = s_1s_2\dots s_n$ is a reduced expression for $w \in W$. This product is well defined by Lemma 1.7 a) (see Geck & Pfeiffer [**6**] Lemma 4.4.3 and Theorem 4.4.6).

Lemma 1.7. *Suppose* $w \in W$.

a) *If* $w = r_1r_2\dots r_n$ *is another reduced expression for* w*, where* $r_i \in S$*, then* $T_w = T_{r_1}T_{r_2}\dots T_{r_n}$.

b) *Multiplication in* $\mathcal{H}$ *is determined as follows:*

$$T_sT_w = \begin{cases} T_{sw}, & \text{if } sw > w, \\ qT_{sw} + (q-1)T_w, & \text{if } sw < w. \end{cases}$$

c) $\mathcal{H}$ *is free as an* R*-module with basis* $\{T_w \mid w \in W\}$.

The Iwahori-Hecke algebra has two 1-dimensional representations which we will need. The first is called the **sign representation** and is analogous to the sign representation of a Coxeter group. Let $\varepsilon(T_w) = (-1)^{\ell(w)}$. The other representation is the analogue of the **trivial representation** in a Coxeter group. Let $\iota(T_w) = q^{\ell(w)}$. To simplify notation, we will often write ε_w for $\varepsilon(T_w)$ and q_w for $\iota(T_w)$.

1.3. Clifford Theory

Suppose A is an algebra with subalgebra B. Clifford theory describes how irreducible A-modules restrict to B-modules, given certain restrictions on B. We define a Clifford system and give a survey of the relevant results following Curtis and Reiner [**3**, §11C].

Definition 1.8. *Let* C *be a finite group,* R *a commutative ring, and* A *an* R*-algebra that is finitely generated as an* R*-module. A* C**-graded Clifford system** *in* A *is a family of* R*-submodules* $\{A_c\}_{c\in C}$ *of* A*, such that the following conditions are satisfied:*

a) $A = \bigoplus_{c\in C} A_c$.

b) $A_cA_d = A_{cd}$.

c) *For each* $c \in C$ *there exists a unit* $a_c \in A$ *such that* $A_c = A_1a_c = a_cA_1$.

d) $1 \in A_1$.

*The algebra A is called a C-**graded** R-**algebra**.*

Since $A_1A_1 = A_1$ (by definition) we see that A_1 is a subalgebra of A and that we can take $a_1 = 1$ if we wish. If V is a right A-module, let V_{A_1} be its restriction to an A_1-module. If U is a right A_1-module, we define U^A to be the induced A-module $U \otimes_{A_1} A$. Then we have

$$U^A = U \otimes_{A_1} A = \bigoplus_{c \in C} U \otimes_{A_1} A_1 a_c = \bigoplus_{c \in C} U \otimes_{A_1} a_c$$

since A is free as a left A_1-module with basis $\{a_c\}_{c \in C}$. Now $U \otimes_{A_1} a_c$ is a right A_1-module and we call any module of this form a **conjugate** of U in U^A.

Theorem 1.9. *Suppose that A is a C-graded Clifford K-algebra where K is a field. Let V be an irreducible A-module and let U be an irreducible A_1-submodule of V_{A_1}. Then V_{A_1} is a semisimple A_1-module, whose simple summands are isomorphic to conjugates $\{U \otimes_{A_1} a_c \mid c \in C\}$ of U.*

In the proof of this theorem (see Curtis and Reiner [**3**, p. 273]), we see that $\sum_{c \in C} U a_c$ is an A-submodule of V. Hence $V = \sum_{c \in C} U a_c$ since V is an irreducible A-module. The result follows after noticing that $U a_c \cong U \otimes a_c$ as A_1-modules.

Example 1.10. To illustrate the power of these results, let us now consider the case of the symmetric group $\mathfrak{S}_n$ and its alternating subgroup $\mathscr{A}(\mathfrak{S}_n)$.

Let $A = K\mathfrak{S}_n$ be the group algebra of the symmetric group over a field K, and similarly let $A_1 = K\mathscr{A}(\mathfrak{S}_n)$ be the group algebra of $\mathscr{A}(\mathfrak{S}_n)$. Let $C = \{1, -1\}$ be the cyclic group of order 2. Then $K\mathfrak{S}_n$ is a C-graded Clifford K-algebra since $\mathfrak{S}_n = \mathscr{A}(\mathfrak{S}_n) \coprod \mathscr{A}(\mathfrak{S}_n)(12)$, where (12) is the transposition swapping 1 and 2. We have set $a_1 = 1$ and $a_{-1} = (12)$.

Suppose that V is an irreducible $K\mathfrak{S}_n$-module. Theorem 1.9 tells us that $V_{A_1} = U + U(12)$ for some irreducible $A_1 = K\mathscr{A}(\mathfrak{S}_n)$-submodule U of V_{A_1}. Therefore, either V is irreducible as an A_1-module or $V_{A_1} = U \oplus U(12)$ for some irreducible A_1-submodule U of V_{A_1}.

CHAPTER 2

Presentations of Algebras and Subalgebras

The Reidemeister-Schreier rewriting process is used to find a presentation for a particular subgroup, given a presentation for the parent group. In this chapter we generalise part of this process to algebras. Suppose we have an algebra A. We would like to find a presentation for any subalgebra of A that is generated by a set of words in the generators of A. First let us consider the group case.

2.1. The Reidemeister-Schreier Rewriting Process for Groups

We follow the treatment of Johnson [**10**]. Suppose we have a subgroup H of finite index in a group G. Let $U = \{u_1, \ldots, u_n\}$ be a set of left coset representatives for H in G. So $G = \coprod_{i=1}^{n} u_i H$. Without loss of generality we assume $u_1 = 1$, the identity element of G. If $g \in G$, let $\overline{g} = u_i$ if g lies in the left coset $u_i H$. Hence $gH = \overline{g}H$. Note that for all $g \in G$ we have $gH \cap \{u_1, \ldots, u_n\} = \{\overline{g}\}$. Moreover, $\overline{g} = \overline{gh}$ for any $h \in H$.

Now let X be some set. A **word** in X is a product of elements from X. That is, $x_1 \ldots x_n$ is a word for $x_i \in X$ and $n \in \mathbb{N}$. Define $\overline{X}$ to be the monoid generated by X. So $\overline{X}$ contains all words in X with multiplication by concatenation and the empty word, denoted 1, as its multiplicative identity. Now let $\widetilde{X} := \{\tilde{x} \mid x \in X\}$ be a set in bijection with X. Define a relation, $\equiv$, where for all $x \in X$, $x\tilde{x} \equiv 1$ and $\tilde{x}x \equiv 1$. This relation is a congruence relation. Hence we may define the **free group generated by** X to be the group

$$F(X) := \overline{X \cup \widetilde{X}} / \equiv .$$

That is, the quotient of $\overline{X \cup \widetilde{X}}$ by the congruence relation $\equiv$. So effectively $\widetilde{X} = X^{-1} = \{x^{-1} \mid x \in X\}$, the set of formal inverses of elements in X. Then we can write any element in $F(X)$ uniquely as a word in $X \cup X^{-1}$.

The following lemma gives us generators for a subgroup of $F(X)$. See Johnson [**10**, Chapter 2, Lemma 4].

Lemma 2.1. *Suppose $F(X)$ is the free group generated by the set X. If K is a subgroup of $F(X)$ then, the set*

$$Z := \{\overline{xu_i}^{-1} x u_i \mid x \in X,\ u_i \in U,\ xu_i \notin U\}$$

generates K, where U is a set of left coset representatives for K in $F(X)$.

Now suppose R is a subset of a group G. Let $\langle R\rangle_G$ be the normal closure of R in G. We say that G is **generated by** X **with relations** R if $G \cong F(X)/\langle R\rangle_{F(X)}$.

We now describe how to find the relations for a subgroup K of a general group F. The following two results are Lemma 1 and Proposition 1 of Johnson [**10**, §9.1].

Lemma 2.2. *Suppose K is a subgroup of a group F. Let R be a subset of F such that $\langle R\rangle_F$ is inside K. Let $\widehat{R} := \{u^{-1}ru \mid u \in U, r \in R\}$. Then*

$$\langle R\rangle_F = \langle \widehat{R}\rangle_K$$

where U is a set of left coset representatives for K in F.

Now let $\widehat{S}$ be the same set as $\widehat{R}$ except that its elements are written as words in $Z \cup Z^{-1}$, using Lemma 2.1.

Proposition 2.3. *Let H be a subgroup of a group G where G is generated by a subset X with relations R. Then, using the notation above, H is generated by Z with relations $\widehat{S}$.*

The proof of this proposition uses the natural quotient map $\nu : F(X) \longrightarrow G$. We first apply Lemma 2.1 to find generators for $K = \nu^{-1}(H) = \{x \in F(X) \mid \nu(x) \in H\}$. To this end we need left coset representatives for $\nu^{-1}(H)$ in $F(X)$. Suppose U is a set of left coset representatives for H in G. To get a set of coset representatives of $\nu^{-1}(H)$ in $F(X)$ we can choose a set of preimages of U under ν. We will abuse notation by using U for the coset representatives of $\nu^{-1}(H)$ in $F(X)$. Then it follows from Lemma 2.1 that Z generates $\nu^{-1}(H)$.

Now, $H \cong \nu^{-1}(H)/\langle R\rangle_G$ by the first isomorphism theorem, and if we apply Lemma 2.2 with $K = \nu^{-1}(H)$ and $F = F(X)$ we see that $H \cong \nu^{-1}(H)/\langle \widehat{R}\rangle_{\nu^{-1}(H)}$. That is, the set $\widehat{S}$ gives us the relations for H.

Example 2.4. Consider the Symmetric group $\mathfrak{S}_n$. We know that it is generated by elements $s_i = (i\ i+1)$ for $i \in\ [1, \dots, n-1]$ with relations:

$$\begin{aligned} s_i^2 &= 1 \quad \text{for } i \in [1, \dots, n-1] \\ (s_i s_j)^2 &= 1 \quad \text{whenever } |i-j| > 1 \\ (s_i s_{i+1})^3 &= 1 \quad \text{for } i \in [1, \dots, n-2]. \end{aligned}$$

Let $H = \mathscr{A}(\mathfrak{S}_n)$. So H is the subgroup of $\mathfrak{S}_n$ consisting of words of even length. Then $U = \{1, s_1\}$ is a set of left coset representatives for H in $\mathfrak{S}_n$ and $1H$ contains all the elements of even length while $s_1 H$ contains all the elements of odd length. Then the generators for H are:

$$\begin{aligned}
Z &= \{\overline{xu_i}^{-1}xu_i \mid x \in X,\ u_i \in U,\ xu_i \notin U\} \\
&= \{\overline{s_i}^{-1}s_i \mid 1 \le i \le n-1, s_i \notin U\} \\
&\qquad\qquad \cup \{\overline{s_is_1}^{-1}s_is_1 \mid 1 \le i \le n-1, s_is_1 \notin U\} \\
&= \{s_1^{-1}s_i \mid 1 \le i \le n-1, s_i \notin U\} \\
&= \qquad\qquad \cup\{s_is_1 \mid 1 \le i \le n-1, s_is_1 \notin U\} \\
&= \{s_1s_i, s_is_1 \mid 2 \le i \le n-1\}
\end{aligned}$$

We may narrow this down further since s_is_1 is the inverse of s_1s_i. So H is generated by the set

$$\{x_i \mid 1 \le i \le n-2\}$$

where $x_i = s_1s_{i+1}$ for $1 \le i \le n-2$.

The relations in H are given by

$$\begin{aligned}
\widehat{R} &= \{u_i^{-1}ru_i \mid u \in U, r \in R\} \\
&= \{r, s_1rs_1 \mid r \in R\} \\
&= \{s_i^2, s_1s_i^2s_1 \mid 1 \le i \le n-1\} \cup \{(s_is_j)^2, s_1(s_is_j)^2s_1 \mid |i-j| > 1\} \\
&\quad \cup\{(s_is_{i+1})^3, s_1(s_is_{i+1})^3s_1 \mid 1 \le i \le n-2\}.
\end{aligned}$$

After removing the equivalent relations and the trivial relations and rewriting in terms of the generators in Z we have

$$\widehat{S} = \{(x_ix_j^{-1})^2 \mid |i-j| > 1\} \cup \{((s_1s_i)(s_{i+1}s_1))^3 \mid 1 \le i \le n-1\}.$$

Then $\mathscr{A}(\mathfrak{S}_n)$ is generated by x_i for $2 \le i \le n-1$ and has relations:

$$\begin{aligned}
x_1^3 &= 1, \\
x_i^2 &= 1 \quad \text{for } i \in [2, \dots, n-2], \\
(x_ix_j^{-1})^2 &= 1 \quad \text{whenever } |i-j| > 1, \\
(x_ix_{i+1}^{-1})^3 &= 1 \quad \text{for } i \in [1, \dots, n-3].
\end{aligned}$$

2.2. Free Algebras and Presentations of Algebras

The results in this section are well known and are included here for completeness and because we could not find a suitable reference.

Definition 2.5. *Suppose R is a ring and F is an R-algebra. Then F is* **freely generated by** $X \subseteq F$ *if every map $f : X \longrightarrow A$, where A is an R-algebra, extends uniquely to an algebra homomorphism $\phi_f : F \longrightarrow A$ such that $\phi_f(x) = f(x)$ for all x in X. We say that F is* **free** *and has* **rank** $|X|$.

Note that this definition does not guarantee the existence of a free algebra. So we will construct one explicitly. Suppose R is a ring (with identity) and X is a set. Recall that $\overline{X}$ is the free monoid generated by X. Let $R\overline{X}$ be the monoid algebra over R.

Proposition 2.6. *The monoid algebra $R\overline{X}$ is freely generated by X.*

Proof. Suppose that we have some map $f : X \longrightarrow A$ where A is an R-algebra. Let $f' : R\overline{X} \longrightarrow A$ be the R-linear map such that $f'(x_1 \dots x_n) = f(x_1) \dots f(x_n)$ for $x_i \in X$. Then f' is an algebra homomorphism which extends f.

We claim that this is the unique algebra homomorphism extending f. To see this, consider an element $m \in R\overline{X}$ and an algebra homomorphism $\phi : R\overline{X} \longrightarrow A$ that extends f. We can write $m = \sum_{w \in \overline{X}} \lambda_w w$ where w is a word in $\overline{X}$ and λ_w is in R. So $\phi(m) = \phi(\sum_{w \in \overline{X}} \lambda_w w) = \sum_{w \in \overline{X}} \lambda_w \phi(w)$. Now if $w = x_1 \dots x_m$ for some $x_i \in X$ then $\phi(w) = \phi(x_1) \dots \phi(x_m) = f(x_1) \dots f(x_m)$. So $\phi(m)$ is completely determined by the $f(x)$ for $x \in X$. Hence $\phi = f'$ is unique. □

The following proposition tells us that free algebras with the same rank are isomorphic.

Proposition 2.7. *Suppose F and G are algebras freely generated by subsets X and Y respectively. If $|X| = |Y|$ then $F \cong G$.*

Proof. Since $|X| = |Y|$ we can define a bijection $f : X \longrightarrow Y$ and its inverse $f^{-1} : Y \longrightarrow X$. Consider f as a mapping into G and f^{-1} mapping into F. Then f and f^{-1} extend uniquely to algebra homomorphisms $\phi_f : F \longrightarrow G$ and $\phi_{f^{-1}} : G \longrightarrow F$ respectively. Then $\phi_{f^{-1}}\phi_f : F \longrightarrow F$ is an algebra homomorphism which is the identity on X and $\phi_f\phi_{f^{-1}} : G \longrightarrow G$ is an algebra homomorphism which is the identity on Y.

Also, the identity maps $1_F : F \longrightarrow F$ and $1_G : G \longrightarrow G$ extend the identity maps on X and Y. Therefore, by uniqueness, we have $1_F = \phi_{f^{-1}}\phi_f$ and $1_G = \phi_f\phi_{f^{-1}}$. Hence ϕ_f (or indeed $\phi_{f^{-1}}$) gives us an isomorphism.

□

Since any free algebras that have the same rank are isomorphic, any free algebra is isomorphic to $R\overline{X}$ for some set X. So we can think of the free algebras as these monoid algebras.

Proposition 2.8. *Suppose that F is an R-algebra freely generated by a subset X. Then X generates F.*

Proof. Let $\langle X \rangle$ be the algebra generated by X in F. That is,

$$\langle X \rangle := \bigcap_{\substack{A \leq F \\ X \subseteq A}} A$$

where A runs over the subalgebras of F which contain X. Let $f : X \longrightarrow \langle X \rangle$ be the inclusion map. Then we have another map $\phi_f : F \longrightarrow \langle X \rangle$ which extends f. Now let $\iota\colon\langle X\rangle \longrightarrow F$ be the inclusion map. Then $\iota\phi_f(x) = \iota f(x)$ for all $x \in X$. So $\iota\phi_f\colon F \longrightarrow F$ is a map extending ιf.

Let 1_F be the identity map on F. Then $1_F(x) = x = \iota f(x)$ for all $x \in X$. So 1_F is also a map extending ιf. Hence $1_F = \iota\phi_f$. This implies that $F = \mathrm{Im}(1_F) = \mathrm{Im}(\iota\phi_f) = \mathrm{Im}(\phi_f) \subseteq \langle X\rangle$. Hence $F = \langle X\rangle$. □

Now suppose A is an R-algebra and X is a subset of A. A **relation** in A is a linear combination $\sum_{w\in\overline{X}} \lambda_w w$, for some $\lambda_w \in R$, such that $\sum_{w\in\overline{X}} \lambda_w w = 0$ in A.

Definition 2.9. *Suppose R is a ring and X a set. Let $R\overline{X}$ be the monoid algebra of $\overline{X}$ and let $\mathscr{R}$ be a subset of $R\overline{X}$. Write $\langle\mathscr{R}\rangle_{R\overline{X}}$ for the ideal generated by $\mathscr{R}$ in $R\overline{X}$. Then $\langle X \mid \mathscr{R}\rangle := R\overline{X}/\langle\mathscr{R}\rangle_{R\overline{X}}$ is the R-**algebra generated by** X **with relations** $\mathscr{R}$.*

Note that when $\mathscr{R}$ is empty, $\langle X \mid \mathscr{R}\rangle \cong R\overline{X}$ is free.

Let B be an R-algebra and suppose we have a map $f\colon X \longrightarrow B$. If $w \in \overline{X}$ we may write $w = x_{w,1}\ldots x_{w,n}$ for some $x_{w,i} \in X$. The map f **respects the relations** in A if $\sum_{w\in\overline{X}} \lambda_w f(x_{w,1})\ldots f(x_{w,n}) = 0$ for each relation $\sum_{w\in\overline{X}} \lambda_w w$.

Proposition 2.10. *Let $A = \langle X \mid \mathscr{R}\rangle$ be an R-algebra generated by $X \subseteq A$ with relations $\mathscr{R}$. Then every map $f\colon X \longrightarrow B$ (where B is any R-algebra), which respects the relations in $\mathscr{R}$, extends uniquely to an algebra homomorphism $\phi_f\colon A \longrightarrow B$ such that $\phi_f(x) = f(x)$ for all x in X.*

Proof. Let $f\colon X \longrightarrow B$ be a map into an R-algebra B that respects the relations in $\mathscr{R}$. Then since $R\overline{X}$ is free, f extends to an algebra homomorphism $f'\colon R\overline{X} \longrightarrow B$. Since f respects the relations, $f'(r) = 0$ for all $r \in \mathscr{R}$. Therefore $\langle\mathscr{R}\rangle_{R\overline{X}}$ lies in the kernel of f'.

Define $\phi_f\colon A \longrightarrow B$ to be the map sending $a + \langle\mathscr{R}\rangle_{R\overline{X}} \mapsto f'(a)$ for all $a \in A$. We claim that this map is well-defined. To see this, suppose that, for $a, b \in A$, $a + \langle\mathscr{R}\rangle_{R\overline{X}} = b + \langle\mathscr{R}\rangle_{R\overline{X}}$. Hence $b = a + r$ for some $r \in \langle\mathscr{R}\rangle_{R\overline{X}}$. So

$$\begin{aligned}\phi_f(b + \langle\mathscr{R}\rangle_{R\overline{X}}) &= \phi_f(a + r + \langle\mathscr{R}\rangle_{R\overline{X}})\\ &= f'(a + r)\\ &= f'(a)\\ &= \phi_f(a + \langle\mathscr{R}\rangle_{R\overline{X}}).\end{aligned}$$

Now ϕ_f is an algebra homomorphism since f' is. Also, ϕ_f is the unique algebra homorphism extending f because it is completely determined by the fact that ϕ_f must respect the relations and be an algebra homomorphism.

□

2.3. Applying the Reidemeister-Schreier Rewriting Algorithm to Algebras

Let R be a ring and let G be the free group generated by a set X. Let RG be the group algebra of G and let $\mathscr{R}$ be a subset of RG. Consider an algebra, $A = RG/\langle\mathscr{R}\rangle_{RG}$, where $\langle\mathscr{R}\rangle_{RG}$ is the (2-sided) ideal of RG generated by $\mathscr{R}$. Let Y be a subset of G and let H be the subgroup of G generated by Y. We are interested in the subalgebra B of A that is generated by Y. Now $B = (RH + \langle\mathscr{R}\rangle)/\langle\mathscr{R}\rangle_{RG} = RH/(RH \cap \langle\mathscr{R}\rangle_{RG})$. So we may use Lemma 2.1 to obtain generators for H and hence for B.

CHAPTER 3

The Alternating Hecke Algebra

This chapter introduces the main subject of this thesis: the alternating Hecke algebra. In Section 3.2 we define the alternating Hecke algebra as the fixed point algebra under a certain involution. We give various bases for the algebra in Sections 3.2 & 3.3. In Section 3.5 we give the generators for the algebra and some relations.

3.1. An Automorphism of the Hecke Algebra

Here, we introduce a map $\#$ which we use in the next section to define the alternating Hecke algebra.

Definition 3.1. *Let R be a ring containing an invertible element q, and let (W, S) be a Coxeter system. Define a linear map $\#$ as follows:*

$$\begin{array}{rccc} \#: & \mathcal{H}_{R,q}(W) & \to & \mathcal{H}_{R,q}(W) \\ & T_w & \mapsto & \varepsilon_w q_w T_{w^{-1}}^{-1}, \end{array}$$

for $w \in W$.

Note that, for $s \in S$, $T_s^{\#} = -qT_s^{-1} = -T_s + q - 1$.

Lemma 3.2. *The map $\#$ extends to an algebra homomorphism of $\mathcal{H}_{R,q}(W)$.*

Proof. We need only prove that $\#$ respects the relations of the Iwahori-Hecke algebra.

For $s \in S$ we have

$$\begin{aligned} (T_s^{\#})^2 &= (-T_s + q - 1)^2 \\ &= T_s^2 - 2T_s(q-1) + (q-1)^2 \\ &= (q-1)T_s + q - 2T_s(q-1) + (q-1)^2 \\ &= -(q-1)T_s + (q-1)^2 + q \\ &= (q-1)(-T_s + q - 1) + q \\ &= (q-1)T_s^{\#} + q. \end{aligned}$$

We also have, for $r, s \in S$,

$$\begin{aligned}(T_r^{\#}T_s^{\#}\dots)_{m_{rs}} &= (-q)^{m_{rs}}(T_r^{-1}T_s^{-1}\dots)_{m_{rs}} \\ &= (-q)^{m_{rs}}(\underbrace{\dots T_sT_r}_{m_{rs}})^{-1} \\ &= (-q)^{m_{rs}}(\underbrace{\dots T_rT_s}_{m_{rs}})^{-1} \\ &= (T_s^{\#}T_r^{\#}\dots)_{m_{rs}}.\end{aligned}$$

□

Lemma 3.3. *The function $\#$ is an involution and hence an automorphism.*

Proof. Since $\#$ is a homomorphism and

$$\begin{aligned}(T_s^{\#})^{\#} &= (-T_s+q-1)^{\#} \\ &= -(-T_s+q-1)+q-1 \\ &= T_s,\end{aligned}$$

$\#$ is an involution.

If h is in $\mathcal{H}_{R,q}(W)$, $h^{\#}$ is the pre-image of h. So $\#$ is surjective. If $h^{\#} = h'^{\#}$, for $h, h' \in \mathcal{H}_{R,q}(W)$, then $(h^{\#})^{\#} = (h'^{\#})^{\#}$. Hence $h = h'$ and so $\#$ is injective and an automorphism. □

Kazhdan and Lusztig [**11**, §2] described the inverse of T_w as follows:

(3.4) $$T_{w^{-1}}^{-1} = \varepsilon_w q_w^{-1} \sum_{x \le w} \varepsilon_x R_{x,w}(q) T_x$$

where $R_{x,w}(q)$ is a polynomial in $\mathbb{Z}[q]$ of degree $\ell(w) - \ell(x)$ and $R_{w,w}(q) = 1$ for all $w \in W$. The polynomial $R_{x,w}(q)$ is defined to be zero whenever $x \not\le w$.

We can now write an expression for $T_w^{\#}$ as a linear combination of the basis elements.

Lemma 3.5. *For $w \in W$, $T_w^{\#} = \sum_{x \le w} \varepsilon_x R_{x,w} T_x$.*

Proof. We have

$$\begin{aligned}T_w^{\#} &= \varepsilon_w q_w T_{w^{-1}}^{-1} \\ &= \varepsilon_w q_w(\varepsilon_w q_w^{-1} \sum_{x \le w} \varepsilon_x R_{x,w} T_x) \\ &= \sum_{x \le w} \varepsilon_x R_{x,w} T_x.\end{aligned}$$

□

3.2. The Alternating Hecke Algebra

In this section we define the alternating Hecke algebra over a ring, R, containing $1/2$. We require that 2 is invertible to construct our bases and presentation for $\mathcal{A}_{R,q}(W)$. Note that since $1/2 \in R$, the characteristic of R is odd. This is essential because in Chapter 5 we use Clifford theory to analyse the characters of $\mathcal{A}_{R,q}(W)$; to apply Clifford theory, the characteristic of R must not divide the index of $\mathcal{A}_{R,q}(W)$ in $\mathcal{H}_{R,q}(W)$.

Definition 3.6. *Suppose that R is a ring containing $1/2$ and an invertible element q. Let W be a Coxeter group. The* **alternating Hecke algebra** *$\mathcal{A}_{R,q}(W)$ is the subalgebra of $\mathcal{H}_{R,q}(W)$ which is fixed by $\#$. That is,*

$$\mathcal{A}_{R,q}(W) := \{h \in \mathcal{H}_{R,q}(W) \mid h^{\#} = h\}.$$

We will sometimes use $\mathcal{A}$ for $\mathcal{A}_{R,q}(W)$ when there is no ambiguity.

Definition 3.7. *For $w \in \mathscr{A}(W)$ let*

$$B_w := \frac{1}{2}(T_w + T_w^{\#}) = T_w + \frac{1}{2}\sum_{x<w} \varepsilon_x R_{x,w} T_x.$$

Example 3.8. Recall that $R_{w,w} = 1$ for all $w \in W$. For $s \in S$ we have $T_s^{-1} = q^{-1}T_s - q^{-1}(q-1)$. Now, for $r, s \in S$, $r \neq s$ we have

$$\begin{aligned} T_{(rs)^{-1}}^{-1} &= T_{sr}^{-1} \\ &= T_r^{-1}T_s^{-1} \\ &= q^{-2}T_{rs} - q^{-2}(q-1)(T_r + T_s) + q^{-2}(q-1)^2. \end{aligned}$$

Comparing this with Equation 3.4 gives $R_{1,rs} = (q-1)^2$ and $R_{r,rs} = R_{s,rs} = q-1$.

Then we have $B_1 = 1$ and

$$\begin{aligned} B_{rs} &= T_{rs} + \frac{1}{2}\sum_{x<rs} \varepsilon_x R_{x,rs} T_x \\ &= T_{rs} - \frac{1}{2}R_{r,rs}T_r - \frac{1}{2}R_{s,rs}T_s + \frac{1}{2}R_{1,rs} \\ &= T_{rs} - \frac{1}{2}(q-1)(T_r + T_s) + \frac{1}{2}(q-1)^2. \end{aligned}$$

By definition, $B_w \in \mathcal{A}_{R,q}(W)$ for each $w \in \mathscr{A}(W)$. We will show that the set $\{B_w \mid w \in \mathscr{A}(W)\}$ is a basis for the alternating Hecke algebra. In preparation for this we will need the following lemma.

Lemma 3.9. *Suppose $h \in \mathcal{A}_{R,q}(W)$. We can write $h = \sum_{w\in W} a_w T_w$ for some a_w in R. If $y \in W$ such that a_y is non-zero and $a_w = 0$ whenever $l(w) > l(y)$, then $y \in \mathscr{A}(W)$.*

Proof. Write $h = a_y T_y + \sum_{\substack{w\in W\\ w\neq y}} a_w T_w$. Then

$$\begin{aligned} h^\# &= a_y T_y^\# + \sum_{\substack{w\in W\\ w\neq y}} a_w T_w^\# \\ &= a_y(\varepsilon_y T_y + \sum_{x<y} \varepsilon_x R_{x,y} T_x) + \sum_{\substack{w\in W\\ w\neq y}} a_w(\varepsilon_w T_w + \sum_{x<w} \varepsilon_x R_{x,w} T_x) \\ &= a_y \varepsilon_y T_y + a_y \sum_{x<y} \varepsilon_x R_{x,y} T_x + \sum_{\substack{w\in W\\ w\neq y}} a_w \varepsilon_w T_w \\ &\qquad + \sum_{\substack{w\in W\\ w\neq y}} a_w \sum_{x<w} \varepsilon_x R_{x,w} T_x. \end{aligned}$$

Consider the coefficient of T_y. We have $a_y\varepsilon_y$ from the first term. The rest of the sum contains no T_y. This is obvious in the second and third terms. If T_y was in the last term then $y < w$ implying that a_w is zero. Now the coefficient of T_y in h is a_y and since $h^\# = h$ we must have $a_y\varepsilon_y = a_y$. Since $a_y \neq 0$ we have $\varepsilon_y = 1$; ie., $y \in \mathscr{A}(W)$. ☐

Theorem 3.10. *The alternating Hecke algebra, $\mathcal{A}_{R,q}(W)$, is free as an R-module with basis $\{B_w \mid w \in \mathscr{A}(W)\}$.*

Proof. First we show that $\mathcal{A}_{R,q}(W)$ is spanned by $\{B_w \mid w \in \mathscr{A}(W)\}$.

Suppose that h is a non zero element in $\mathcal{A}_{R,q}(W)$ and $h = \sum_{w\in W} a_w T_w$ for some a_w in R. Then there exists some $y \in W$ of maximal length such that $a_y \neq 0$. We will use induction on the length of this element y to show that h is in $\operatorname{Span}\{B_w \mid w \in \mathscr{A}(W)\}$.

If $l(y) = 0$ then $h = a_1 T_1 = a_1 B_1$. For the inductive step, notice that $y \in \mathscr{A}(W)$ by Lemma 3.9. Let

$$h' = h - \sum_{\substack{w\in\mathscr{A}(W)\\ l(w)=l(y)}} a_w B_w.$$

Then $h' \in \mathcal{A}_{R,q}(W)$ and the only non-zero terms that appear in h' are scalar multiples of T_w where $l(w) < l(y)$. Therefore by the inductive hypothesis, $h' \in \operatorname{Span}\{B_w \mid w \in \mathscr{A}(W)\}$, and hence $h \in \operatorname{Span}\{B_w \mid w \in \mathscr{A}(W)\}$.

We will now show that the B_w are linearly independent. Suppose that $\sum_{w\in\mathscr{A}(W)} b_w B_w = 0$ for some b_w in R. Let y be an element of maximal length in $\mathscr{A}(W)$. We argue by induction on $l(y)$.

Suppose $l(y) = 0$. So $0 = b_1 B_1 = b_1 T_1$. Therefore $b_1 = 0$ since T_1 is as basis element in $\mathcal{H}_{R,q}(W)$. Now suppose $l(y) > 0$. Then

$$\begin{aligned} 0 &= \sum_{w \in \mathscr{A}(W)} b_w B_w \\ &= \sum_{\substack{w \in \mathscr{A}(W) \\ l(w)=l(y)}} b_w B_w + \sum_{\substack{w \in \mathscr{A}(W) \\ l(w)<l(y)}} b_w B_w \\ &= \sum_{\substack{w \in \mathscr{A}(W) \\ l(w)=l(y)}} b_w T_w + \sum_{\substack{w \in \mathscr{A}(W) \\ l(w)=l(y)}} \frac{b_w}{2} \sum_{x<w} \varepsilon_x R_{x,w} T_x + \sum_{\substack{w \in \mathscr{A}(W) \\ l(w)<l(y)}} b_w B_w. \end{aligned}$$

The coefficients b_w in the first sum are zero because $\{T_w\}_{w \in W}$ is a basis for the Hecke algebra and this sum is the only one with T_w's in it which have $l(w) = l(y)$. This forces the second part of the sum to be zero. The coefficients in the last term are zero from the inductive hypothesis.

□

We now introduce a nice and very 'natural' basis for $\mathcal{A}_{R,q}(W)$.

Proposition 3.11. *$\mathcal{A}_{R,q}(W)$ has a unique basis $\{B'_w \mid w \in \mathscr{A}(W)\}$ such that*

$$B'_w = T_w + \sum_{\substack{x<w \\ x \notin \mathscr{A}(W)}} b_x T_x$$

for some $b_x \in R$.

Proof.
<u>Existence.</u> By Theorem 3.10, $\{B_w \mid w \in \mathscr{A}(W)\}$ is a basis with

$$B_w = T_w + \sum_{y<w} a_y T_y$$

for certain $a_y \in R$. Set $B'_1 = T_1$, and define

$$B'_w := B_w - \sum_{\substack{y<w \\ y \in \mathscr{A}(W)}} a_y B'_y$$

for $w \in \mathscr{A}(W)$. We will show by induction on $\ell(w)$ that

$$B'_w = T_w + \sum_{\substack{x<w \\ x \notin \mathscr{A}(W)}} b_x T_x$$

for some $b_x \in R$.

We have

$$B'_w = B_w - \sum_{\substack{y<w \\ y\in\mathscr{A}(W)}} a_y B'_y$$
$$= T_w + \sum_{y<w} a_y T_y - \sum_{\substack{y<w \\ y\in\mathscr{A}(W)}} a_y B'_y$$
$$= T_w + \sum_{y<w} a_y T_y - \sum_{\substack{y<w \\ y\in\mathscr{A}(W)}} a_y (T_y + \sum_{\substack{x<y \\ x\notin\mathscr{A}(W)}} b_x T_x)$$

(by the inductive assumption)

$$= T_w + \sum_{\substack{y<w \\ y\notin\mathscr{A}(W)}} a_y T_y - \sum_{\substack{x<y<w \\ y\in\mathscr{A}(W) \\ x\notin\mathscr{A}(W)}} a_y b_x T_x.$$

Uniqueness. Suppose D_w is another element such that

$$D_w = T_w + \sum_{\substack{x<w \\ x\notin\mathscr{A}(W)}} \beta_x T_x$$

for some $\beta_x \in R$. Then

$$B'_w - D_w = \sum_{\substack{x<w \\ x\notin\mathscr{A}(W)}} (b_x - \beta_x) T_x$$

and $B'_w - D_w \in \mathcal{A}_{R,q}(W)$. If $B'_w - D_w \neq 0$ then $b_x - \beta_x \neq 0$ for some $x \notin \mathscr{A}(W)$ which contradicts Lemma 3.9. Hence $D_w = B'_w$.

□

3.3. Kazhdan-Lusztig Bases for $\mathcal{A}_{R,q}(W)$

There are some other bases for the alternating Hecke algebra that both require us to have $q^{1/2}$ and $q^{-1/2}$ in our ring R. For the rest of this section, suppose that we are working over the ring $R = \mathbb{Z}[\frac{1}{2}, q^{\pm 1/2}]$.

Kazhdan and Lusztig introduce an involution in [**11**, p 166] which is known as the **bar involution**. It is defined as follows. Let $^{-} : R \to R$ be the map sending $q^{1/2} \mapsto q^{-1/2}$. This map extends to a ring automorphism (and involution) on $\mathcal{H}_{R,q}(W)$ such that $\overline{\sum_{w\in W} \lambda_w T_w} = \sum_{w\in W} \overline{\lambda_w} T_{w^{-1}}^{-1}$ for some λ_w in R (see Humphreys [**9**, §7.7] for a proof). Kazhdan and Lusztig then go on to give two bases whose elements are invariant under the bar involution (these are stated below). In this section, we give some Kazhdan-Lusztig Bases for $\mathcal{A}_{R,q}(W)$. That is, we give some bases whose elements are fixed under the bar involution.

Theorem 3.12. *For each $w \in W$ there is a unique element, $C_w \in \mathcal{H}_{R,q}(W)$ such that*

- $\overline{C_w} = C_w$
- $C_w = \varepsilon_w q_w^{1/2} \sum_{y \le w} \varepsilon_y q_y^{-1} \overline{P_{y,w}} T_y$

where $P_{y,w}$ is a polynomial in q of degree less than or equal to $\frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$, and $P_{w,w} = 1$.

This theorem tells us that the elements $\{C_w \mid w \in W\}$ form a basis for $\mathcal{H}$. Suppose that $j : \mathcal{H} \longrightarrow \mathcal{H}$ is the ring involution defined by $j(\sum a_w T_w) = \sum \overline{a_w} \varepsilon_w q_w^{-1} T_w$. Then j and $^-$ commute, (see, for example, Humphreys [**9**, Exercise 7.7]) which proves the following theorem.

Theorem 3.13. *For any $w \in W$ there is a unique element, $C'_w \in \mathcal{H}_{R,q}(W)$ such that*

- $\overline{C'_w} = C'_w$
- $C'_w = q_w^{-1/2} \sum_{y \le w} P_{y,w} T_y$

where $P_{y,w}$ is a polynomial in q of degree less than or equal to $\frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$, and $P_{w,w} = 1$.

So, the elements $\{C'_w \mid w \in W\}$ also form a basis for $\mathcal{H}$ and we have the following relation between the two bases: $C'_w = \varepsilon_w j(C_w)$.

The polynomials $P_{y,w}$ are called the **Kazhdan-Lusztig polynomials** and are defined to be zero whenever $y \not\le w$.

We can slightly modify the basis elements B_w, so that they are invariant under the map $^-$. Towards this goal, consider the following lemma.

Lemma 3.14. *The bar involution commutes with $\#$.*

Proof. Let $s \in S$. Then

$$\begin{aligned} \overline{T_s^{\#}} &= \overline{-qT_s^{-1}} \\ &= -q^{-1}T_s \\ &= \left(T_s^{-1}\right)^{\#} \\ &= \left(\overline{T_s}\right)^{\#}, \end{aligned}$$

and the result follows.

□

Corollary 3.15. *The set* $\{q_w^{-1/2}B_w \mid w \in \mathscr{A}(W)\}$ *is a basis for* $\mathcal{A}_{R,q}(W)$ *that is invariant under the bar involution.*

Proof. By Theorem 3.10, the set $\{q_w^{-1/2}B_w \mid w \in \mathscr{A}(W)\}$ is a basis for $\mathcal{A}_{R,q}(W)$. It remains to show that it is bar invariant. Suppose $w \in \mathscr{A}(W)$. Then

$$\begin{aligned}
\overline{q_w^{-1/2}B_w} &= q_w^{1/2}\overline{\frac{1}{2}\left(T_w + T_w^{\#}\right)} \\
&= \frac{1}{2}q_w^{1/2}\left(\overline{T_w} + \overline{q_w T_{w^{-1}}^{-1}}\right) \\
&= \frac{1}{2}q_w^{1/2}\left(T_{w^{-1}}^{-1} + q_w^{-1}T_w\right) \\
&= \frac{1}{2}q_w^{-1/2}\left(q_w T_{w^{-1}}^{-1} + T_w\right) \\
&= \frac{1}{2}q_w^{-1/2}\left(T_w^{\#} + T_w\right) \\
&= q_w^{-1/2}B_w.
\end{aligned}$$

□

There is also another basis for $\mathcal{A}_{R,q}(W)$ (which is again invariant under the bar involution) involving the C and C' bases. First note that, for $x, w \in W$ and $x \leq w$, we have

$$P_{x,w} = q_w \varepsilon_x \sum_{\substack{y \in W \\ x \leq y \leq w}} \varepsilon_y q_y^{-1} R_{x,y} \overline{P_{y,w}}. \tag{3.16}$$

This is just a modified version of equation 2.2.a in [**11**, §2.2].

We will use this result to show that, for all w in $\mathscr{A}(W)$, $C_w^{\#} = C'_w$ and $C_w'^{\#} = C_w$.

Lemma 3.17. *For* w *in* W, $C_w^{\#} = \varepsilon_w C'_w$ *and* $C_w'^{\#} = \varepsilon_w C_w$.

Proof. Suppose that $w \in W$. Then

$$\begin{aligned}
C_w^{\#} &= \varepsilon_w q_w^{1/2} \sum_{y \leq w} \varepsilon_y q_y^{-1} \overline{P_{y,w}} T_y^{\#} \\
&= \varepsilon_w q_w^{1/2} \sum_{y \leq w} \varepsilon_y q_y^{-1} \overline{P_{y,w}} \sum_{x \leq y} \varepsilon_x R_{x,y} T_x \\
&= \varepsilon_w q_w^{1/2} \sum_{\substack{x,y \in W \\ x \leq y \leq w}} \varepsilon_x \varepsilon_y q_y^{-1} R_{x,y} \overline{P_{y,w}} T_x \\
&= \varepsilon_w q_w^{-1/2} \sum_{\substack{x \in W \\ x \leq w}} P_{x,w} T_x \\
&= \varepsilon_w C'_w.
\end{aligned}$$

The other part now follows by taking $\#$ of this equation. □

Theorem 3.18. *The set* $\{\frac{1}{2}(C_w + C'_w) \mid w \in \mathscr{A}(W)\}$ *is a basis for* $\mathcal{A}_{R,q}(W)$ *that is invariant under the bar involution.*

Proof. Since both the C and C' bases are defined to be invariant under $^-$, and because $^-$ is a ring homomorphism, we see that each element in the set is also invariant.

If $w \in \mathscr{A}(W)$ then, by the previous lemma,

$$\begin{aligned}
\left[\frac{1}{2}(C_w + C'_w)\right]^{\#} &= \frac{1}{2}(C_w^{\#} + C_w'^{\#}) \\
&= \frac{1}{2}(C'_w + C_w).
\end{aligned}$$

Therefore we have the right number of hash invariant elements. Now, for $w \in W$, $C_w = \sum_{x \leq w} \alpha_{x,w} T_x$ for some $\alpha_{x,w} \in R$, so

$$\begin{aligned}
\frac{1}{2}(C_w + C'_w) &= \frac{1}{2}(C_w + C^{\#}_w) \\
&= \frac{1}{2}\left(\sum_{x \leq w} \alpha_{x,w} T_x + \sum_{x \leq w} \alpha_{x,w} T^{\#}_x\right) \\
&= \frac{1}{2}\sum_{x \leq w} \alpha_{x,w}(T_x + T^{\#}_x) \\
&= \sum_{\substack{x \leq w \\ x \in \mathscr{A}(W)}} \alpha_{x,w} B_x + \frac{1}{2} \sum_{\substack{x < w \\ x \notin \mathscr{A}(W)}} \alpha_{x,w}(T_x + T^{\#}_x) \\
&= \sum_{\substack{x \leq w \\ x \in \mathscr{A}(W)}} \alpha_{x,w} B_x + \frac{1}{2} \sum_{\substack{x < w \\ x \notin \mathscr{A}(W)}} \alpha_{x,w}(T_x + \sum_{y \leq x} \varepsilon_y R_{y,x} T_y) \\
&= \sum_{\substack{x \leq w \\ x \in \mathscr{A}(W)}} \alpha_{x,w} B_x + \frac{1}{2} \sum_{\substack{x < w \\ x \notin \mathscr{A}(W)}} \alpha_{x,w} \sum_{y < x} \varepsilon_y R_{y,x} T_y.
\end{aligned}$$

Now the second summand above is in $\mathcal{A}_{R,q}(W)$ so it must be a linear combination of basis elements B_z for some $z \in \mathscr{A}(W)$, where $\ell(z) < \ell(w)$. Hence the transition matrix between the set $\{\frac{1}{2}(C_w + C'_w) \mid w \in \mathscr{A}(W)\}$ and the basis elements B_w is triangular, implying that the set $\{\frac{1}{2}(C_w + C'_w) \mid w \in \mathscr{A}(W)\}$ is a basis for $\mathcal{A}_{R,q}(W)$. □

3.4. A Presentation for the Hecke Algebra

In order to give generators and relations for $\mathcal{A}_{R,q}(W)$ we will need a different presentation for $\mathcal{H}_{R,q}(W)$ than that given in Section 1.2. In this section we give a presentation for $\mathcal{H}_{R,q}(W)$ that generalises the work of Mitsuhashi for Coxeter groups of type A_n and B_n, see **[16]** & **[17]**. Let R be a ring containing $1/2$ and invertible elements $q \neq -1$ and $q + 1$.

For each $s \in S$ set

$$f_s = \frac{2T_s - (q-1)}{q+1}.$$

Proposition 3.19. *Let* (W, S) *be a Coxeter system. For each* $s \in S$, f_s *is self inverse.*

Proof. Let $s \in S$. Then

$$\begin{aligned} f_s^2 &= \left[\frac{2T_s - (q-1)}{q+1}\right]^2 \\ &= \frac{4T_s^2 - 4(q-1)T_s + (q-1)^2}{(q+1)^2} \\ &= \frac{4((q-1)T_s + q) - 4(q-1)T_s + (q-1)^2}{(q+1)^2} \\ &= \frac{4q + (q-1)^2}{(q+1)^2}. \\ &= 1 \end{aligned}$$

□

We will find that the elements f_s, for $s \in S$, generate the Hecke algebra. Our aim is to rewrite the relations of the Hecke algebra in terms of these new generators. The next three results will help us investigate this.

Recall that if a and b are elements in $\mathcal{H}_{R,q}(W)$, then $(ab\ldots)_m = \underbrace{abab\ldots}_{m}$ for m a positive integer and $(ab\ldots)_m = \underbrace{baba\ldots}_{-m}$ if m is a negative integer. Also, set $(ab\ldots)_0 = 1$. Note that $(ab\ldots)_{-m} = (ba\ldots)_m$.

Lemma 3.20. *Suppose $r, s \in S$, $r \neq s$ and let $\alpha \in R$. There exist non-negative integers $a^m_{k,l}$ such that*

$$((f_r - \alpha)(f_s - \alpha)\ldots)_m = \sum_{l=0}^{m} (-\alpha)^{m-l} \sum_{\substack{-l \le k \le l \\ k \equiv l \text{ (mod 2)}}} a^m_{k,l} (f_r f_s \ldots)_k, \tag{3.21}$$

whenever $m \geq 1$.

Moreover, the elements $a^m_{k,l}$ are uniquely determined by the recurrence relations:

a) $a^m_{k,l} = 0$ *unless* $0 \le |k| \le l \le m$, $k \equiv l$ *(mod 2),*
b) $a^1_{0,0} = 1$, $a^1_{1,1} = 1$ *and* $a^1_{-1,1} = 0$,
c) $a^m_{k,l} = a^{m-1}_{-k,l} + a^{m-1}_{k-1,l-1}$ *for* $m > 1$.

Proof. The proof of the recurrence relation will be by induction on m. First note that

$$f_r(f_sf_r\ldots)_k = \begin{cases} f_r(\underbrace{f_sf_r\ldots}_{k}) & if \quad k>0, \\ f_r & if \quad k=0, \\ f_r(\underbrace{f_rf_s\ldots}_{-k}) & if \quad k<0. \end{cases}$$
$$= \begin{cases} (\underbrace{f_rf_s\ldots}_{k+1}) & if \quad k>0, \\ f_r & if \quad k=0, \\ (\underbrace{f_sf_r\ldots}_{-k-1}) & if \quad k<0. \end{cases}$$
$$= (f_rf_s\ldots)_{k+1}.$$

When $m=1$, Equation 3.21 is satisfied if we set $a^1_{0,0}=1$, $a^1_{-1,1}=0$, $a^1_{1,1}=1$, and $a^1_{k,l}=0$ for any other values of k and l.

For the inductive step we are assuming that

$$((f_s-\alpha)(f_r-\alpha)\ldots)_m = \sum_{l=0}^{m}(-\alpha)^{m-l}\sum_{\substack{-l\le k\le l\\ k\equiv l \ (\mathrm{mod}\ 2)}} a^m_{k,l}(f_sf_r\ldots)_k.$$

So

$$\begin{aligned} & ((f_r-\alpha)(f_s-\alpha)\ldots)_{m+1} \\ = \ & (f_r-\alpha)\sum_{l=0}^{m}(-\alpha)^{m-l}\sum_{\substack{-l\le k\le l\\ k\equiv l \ (\mathrm{mod}\ 2)}} a^m_{k,l}(f_sf_r\ldots)_k \\ = \ & \sum_{l=0}^{m}(-\alpha)^{m-l}\sum_{\substack{-l\le k\le l\\ k\equiv l \ (\mathrm{mod}\ 2)}} a^m_{k,l}(f_rf_s\ldots)_{k+1} \\ & \qquad +\sum_{l=0}^{m}(-\alpha)^{m+1-l}\sum_{\substack{-l\le k\le l\\ k\equiv l \ (\mathrm{mod}\ 2)}} a^m_{k,l}(f_sf_r\ldots)_k \end{aligned}$$

After replacing k with $k-1$ in the first summand and k with $-k$ in the second, we have

$$((f_r-\alpha)(f_s-\alpha)\dots)_{m+1} = \sum_{l=0}^{m}(-\alpha)^{m-l} \sum_{\substack{-l+1\le k\le l+1\\ k\not\equiv l \text{ (mod 2)}}} a^m_{k-1,l}(f_r f_s\dots)_k$$

$$+\sum_{l=0}^{m}(-\alpha)^{m+1-l} \sum_{\substack{-l\le k\le l\\ k\equiv l \text{ (mod 2)}}} a^m_{-k,l}(f_r f_s\dots)_k$$

$$= \sum_{l=1}^{m+1}(-\alpha)^{m+1-l} \sum_{\substack{-l+2\le k\le l\\ k\equiv l \text{ (mod 2)}}} a^m_{k-1,l-1}(f_r f_s\dots)_k \tag{3.22}$$

$$+\sum_{l=0}^{m}(-\alpha)^{m+1-l} \sum_{\substack{-l\le k\le l\\ k\equiv l \text{ (mod 2)}}} a^m_{-k,l}(f_r f_s\dots)_k \tag{3.23}$$

after replacing l with $l-1$ in the first summand.

For $m>1$ set

$$a^m_{k,l} = \begin{cases} a^{m-1}_{-k,l} + a^{m-1}_{k-1,l-1}, & \text{if } 0\le |k|\le l\le m,\ k\equiv l (\text{mod } 2),\\ 0, & \text{otherwise.}\end{cases}$$

Then in the first summand above, (3.22), we might as well include the case where $l=0$ since $a^m_{k-1,-1}=0$. Note also, that we can include the case where $k=-l$ since $a^m_{-l-1,l-1}=0$ (the case when $k=-l+1$ never occurs since, in the sum, $k\equiv l$). Similarly, in the second summand, (3.23), we can include the $l=m+1$ case. Therefore

$$\begin{aligned}
&((f_r-\alpha)(f_s-\alpha)\dots)_{m+1}\\
&= \sum_{l=0}^{m+1}(-\alpha)^{m+1-l} \sum_{\substack{-l\le k\le l\\ k\equiv l \text{ (mod 2)}}} a^m_{k-1,l-1}(f_r f_s\dots)_k\\
&\quad+\sum_{l=0}^{m+1}(-\alpha)^{m+1-l} \sum_{\substack{-l\le k\le l\\ k\equiv l \text{ (mod 2)}}} a^m_{-k,l}(f_r f_s\dots)_k\\
&= \sum_{l=0}^{m+1}(-\alpha)^{m+1-l} \sum_{\substack{-l\le k\le l\\ k\equiv l \text{ (mod 2)}}} (a^m_{k-1,l-1}+a^m_{-k,l})(f_r f_s\dots)_k\\
&= \sum_{l=0}^{m+1}(-\alpha)^{m+1-l} \sum_{\substack{-l\le k\le l\\ k\equiv l \text{ (mod 2)}}} a^{m+1}_{k,l}(f_r f_s\dots)_k.
\end{aligned}$$

□

Here are some useful results that follow directly from the definition of the elements $a^m_{k,l}$.

Corollary 3.24.

a) $a^m_{0,0} = 1$ *for* $m \geq 1$,
b) $a^m_{k,m} = \delta_{k,m}$ *for* $|k| \leq m$,
c) $a^m_{l,l} = \binom{l + \lfloor \frac{m-l}{2} \rfloor}{l}$, *for all* $0 \leq l \leq m$.

Proof. The first property is clear from Equation 3.21 - the coefficient of $(-\alpha)^m$ on the left-hand side of Equation 3.21 is 1.

For property b) consider $a^m_{k,m}$ in Equation 3.21. This is the coefficient of $(-\alpha)^0(f_r f_s \ldots)_k$ on the right-hand side. The only way to get a non-zero term $(-\alpha)^0(f_r f_s \ldots)_k$ on the left-hand side of Equation 3.21 is when $k = m$.

Property c) will be proved by induction on m. From Lemma 3.20 b), we have $a^1_{0,0} = 1$, but $\binom{0 + \lfloor \frac{1-0}{2} \rfloor}{0} = \binom{0}{0} = 1$. Also from Lemma 3.20 b), we have $a^1_{1,1} = 1$, and $\binom{1 + \lfloor \frac{1-1}{2} \rfloor}{1} = \binom{1}{1} = 1$.

Now suppose $m > 1$. First consider the case when $m - l \geq 2$. Then

$$\begin{aligned}
a^m_{l,l} &= a^{m-1}_{-l,l} + a^{m-1}_{l-1,l-1} \\
&= a^{m-2}_{l,l} + a^{m-2}_{-l-1,l-1} + a^{m-1}_{l-1,l-1} \\
&= a^{m-2}_{l,l} + a^{m-1}_{l-1,l-1} \\
&= \binom{l + \lfloor \frac{m-l}{2} \rfloor - 1}{l} + \binom{l - 1 + \lfloor \frac{m-l}{2} \rfloor}{l-1} \\
&= \binom{l + \lfloor \frac{m-l}{2} \rfloor}{l}.
\end{aligned}$$

Now if $m - l = 1$, then

$$\begin{aligned}
a^m_{l,l} &= a^m_{m-1,m-1} \\
&= a^{m-1}_{-m+1,m-1} + a^{m-1}_{m-2,m-2} \\
&= a^{m-1}_{m-2,m-2} \\
&= \binom{m - 2 + \lfloor \frac{m-1-(m-2)}{2} \rfloor}{m-2} \qquad \text{(by the inductive assumption)} \\
&= 1
\end{aligned}$$

and $\binom{m-1+\lfloor\frac{m-(m-1)}{2}\rfloor}{m-1} = 1$.

Lastly, if $m-l=0$ then by part b) of this lemma we have $a^m_{l,l} = a^m_{m,m} = 1$. However, $\binom{m+\lfloor\frac{m-m}{2}\rfloor}{m} = 1$. □

The following lemma gives a closed form for the elements, $a^m_{k,l}$, in the recurrence relation of Lemma 3.20.

Lemma 3.25. *For* $0 \le |k| \le l < m$ *and* $k \equiv l$ *(mod 2),*

$$a^m_{k,l} = \begin{cases} \binom{\frac{m-2-k}{2}}{\frac{l-k}{2}}\binom{\frac{m+k}{2}}{\frac{l+k}{2}}, & \textit{if } l \equiv m, \\ \binom{\frac{m-1-k}{2}}{\frac{l-k}{2}}\binom{\frac{m-1+k}{2}}{\frac{l+k}{2}}, & \textit{if } l \not\equiv m. \end{cases}$$

Proof. The proof will be by induction on m.

First we consider $m = 1$. Since we must have $0 \le |k| \le l < 1$ we only need to consider $a^1_{0,0}$. In this case $l \not\equiv m$ and the formula gives $\binom{\frac{1-1-0}{2}}{\frac{0-0}{2}}\binom{\frac{1-1+0}{2}}{\frac{0+0}{2}} = 1$, which agrees with the value of $a^1_{0,0}$ given in Lemma 3.20 b).

<u>Case 1. $m \equiv l$</u>

Now for the inductive step. By Lemma 3.20 c) we have

$$a^m_{k,l} = a^{m-1}_{-k,l} + a^{m-1}_{k-1,l-1}.$$

Since $|k| \le l$ we have $-l-1 \le k-1 \le l-1$. So we have two possibilities: $|k-1| \le l-1$ or $k-1 = -l-1$, ie., $k = -l$ (note that we can't have $k-1 = -l$ since $k \equiv l$).

Suppose $k = -l$. In this case $a^{m-1}_{k-1,l-1} = 0$. So $a^m_{k,l} = a^{m-1}_{-k,l}$ and we can use the inductive hypothesis:

$$\begin{aligned} a^m_{k,l} &= a^{m-1}_{l,l} \\ &= \binom{l+\lfloor\frac{m-1-l}{2}\rfloor}{l} && \text{by Corollary 3.24 c)} \\ &= \binom{l+\frac{m-l}{2}-1}{l} && \text{since } m-l \equiv 0 \\ &= \binom{l+\frac{m-l-2}{2}}{l}. \end{aligned}$$

However,

$$\begin{aligned}\binom{\frac{m-2-k}{2}}{\frac{l-k}{2}}\binom{\frac{m+k}{2}}{\frac{l+k}{2}} &= \binom{\frac{m-2+l}{2}}{l}\binom{\frac{m-l}{2}}{0} \\ &= \binom{\frac{m-2+l}{2}}{l}.\end{aligned}$$

which is the equation for $a_{k,l}^m$ when $k = -l$.

Now suppose that $|k-1| \leq l-1$. We can use the inductive hypothesis on both summands:

$$\begin{aligned}a_{k,l}^m &= a_{-k,l}^{m-1} + a_{k-1,l-1}^{m-1} \\ &= \binom{\frac{m-1-1+k}{2}}{\frac{l+k}{2}}\binom{\frac{m-1-1-k}{2}}{\frac{l-k}{2}} + \binom{\frac{m-1-2-k+1}{2}}{\frac{l-1-k+1}{2}}\binom{\frac{m-1+k-1}{2}}{\frac{l-1+k-1}{2}} \\ &= \binom{\frac{m-2+k}{2}}{\frac{l+k}{2}}\binom{\frac{m-2-k}{2}}{\frac{l-k}{2}} + \binom{\frac{m-2-k}{2}}{\frac{l-k}{2}}\binom{\frac{m-2+k}{2}}{\frac{l-2+k}{2}} \\ &= \binom{\frac{m-2-k}{2}}{\frac{l-k}{2}}\left[\binom{\frac{m-2+k}{2}}{\frac{l+k}{2}} + \binom{\frac{m-2+k}{2}}{\frac{l-2+k}{2}}\right] \\ &= \binom{\frac{m-2-k}{2}}{\frac{l-k}{2}}\binom{\frac{m+k}{2}}{\frac{l+k}{2}}.\end{aligned}$$

Case 2. $m \not\equiv l$

Again we must consider the cases when $k = -l$ and $|k-1| \leq l-1$. First consider the case when $k = -l$. Then

$$\begin{aligned}a_{k,l}^m &= a_{-l,l}^m \\ &= a_{l,l}^{m-1} \\ &= \binom{l + \lfloor\frac{m-1-l}{2}\rfloor}{l} \quad \text{by Corollary 3.24 c)} \\ &= \binom{l + \frac{m-1-l}{2}}{l} \quad \text{since } m-l \equiv 1 \\ &= \binom{\frac{m-1+l}{2}}{l}.\end{aligned}$$

Now,

$$\begin{aligned}\binom{\frac{m-1-k}{2}}{\frac{l-k}{2}}\binom{\frac{m-1+k}{2}}{\frac{l+k}{2}} &= \binom{\frac{m-1+l}{2}}{l}\binom{\frac{m-1-l}{2}}{0} \\ &= \binom{\frac{m-1+l}{2}}{l},\end{aligned}$$

which is the equation for $a^m_{k,l}$ when $k = -l$.

Now consider the case when $|k-1| \leq l-1$. This will be similar to the previous case as we are again using the inductive assumption.

$$
\begin{aligned}
a^m_{k,l} &= a^{m-1}_{-k,l} + a^{m-1}_{k-1,l-1} \\
&= \binom{\frac{m-1-2+k}{2}}{\frac{l+k}{2}}\binom{\frac{m-1-k}{2}}{\frac{l-k}{2}} + \binom{\frac{m-1-1-k+1}{2}}{\frac{l-1-k+1}{2}}\binom{\frac{m-1-1+k-1}{2}}{\frac{l-1+k-1}{2}} \\
&= \binom{\frac{m-3+k}{2}}{\frac{l+k}{2}}\binom{\frac{m-1-k}{2}}{\frac{l-k}{2}} + \binom{\frac{m-1-k}{2}}{\frac{l-k}{2}}\binom{\frac{m-3+k}{2}}{\frac{l+k-2}{2}} \\
&= \binom{\frac{m-1-k}{2}}{\frac{l-k}{2}}\left[\binom{\frac{m-3+k}{2}}{\frac{l+k}{2}} + \binom{\frac{m-3+k}{2}}{\frac{l+k-2}{2}}\right] \\
&= \binom{\frac{m-1-k}{2}}{\frac{l-k}{2}}\binom{\frac{m-1+k}{2}}{\frac{l+k}{2}}.
\end{aligned}
$$

□

Corollary 3.26. *If* $0 \leq |k| \leq l < m$,

$$
a^m_{-k,l} = \begin{cases} a^m_{k,l}, & \text{when } m \not\equiv l \text{ (mod 2)}, \\ \frac{m-k}{m+k}\, a^m_{k,l}, & \text{when } m \equiv l \text{ (mod 2)}. \end{cases}
$$

Proof. This comes directly from Lemma 3.25. □

Recall that for $s \in S$, $f_s = \dfrac{2T_s - (q-1)}{q+1}$.

Theorem 3.27. *The set* $\{f_s \mid s \in S\}$ *generates* $\mathcal{H}_{R,q}(W)$ *with the relations:*

- $f_s^2 = 1$ *for* $s \in S$.
- $\sum\limits_{\substack{l=0 \\ l \equiv m_{rs}}}^{m_{rs}} \left(\frac{q-1}{q+1}\right)^{m_{rs}-l} \sum\limits_{\substack{0<k\leq l \\ k\equiv m_{rs}}} \frac{2k}{m_{rs}+k}\, a^{m_{rs}}_{k,l}\big((f_r f_s \ldots)_k - (f_s f_r \ldots)_k\big) = 0$ *for* $r, s \in S$, $r \neq s$.

Proof. For $s \in S$, $T_s = \frac{q+1}{2}(f_s + \frac{q-1}{q+1})$. So $\mathcal{H}_{R,q}(W)$ is generated by the elements $\{f_s \mid s \in S\}$.

Since $(T_s - q)(T_s + 1) = 0$ for each $s \in S$, we have

$$
\left(\frac{q+1}{2} f_s + \frac{q-1}{2} - q\right)\left(\frac{q+1}{2} f_s + \frac{q-1}{2} + 1\right) = 0,
$$

that is,

$$
\left((q+1)f_s - (q+1)\right)\left((q+1)f_s + (q+1)\right) = 0,
$$

which simplifies to $f_s^2 = 1$ (confirming our result from Proposition 3.19).

To simplify calculations set $\alpha = \frac{q-1}{q+1}$. We know that $(T_rT_s\ldots)_{m_{rs}} = (T_sT_r\ldots)_{m_{rs}}$ for all $r, s \in S$, $r \neq s$. This implies that

$$\Big(\big(f_r+\alpha\big)\big(f_s+\alpha\big)\ldots\Big)_{m_{rs}} = \Big(\big(f_s+\alpha\big)\big(f_r+\alpha\big)\ldots\Big)_{m_{rs}}.$$

We can now use Lemma 3.20 to give

$$\sum_{l=0}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{-l\le k\le l\\ k\equiv l}} a_{k,l}^{m_{rs}} (f_rf_s\ldots)_k = \sum_{l=0}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{-l\le k\le l\\ k\equiv l}} a_{k,l}^{m_{rs}} (f_sf_r\ldots)_k.$$

or

$$\sum_{l=0}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{-l\le k\le l\\ k\equiv l}} a_{k,l}^{m_{rs}} \big((f_rf_s\ldots)_k - (f_sf_r\ldots)_k\big) = 0.$$

Then

$$\begin{aligned}
0 &= \sum_{l=0}^{m_{rs}} \alpha^{m_{rs}-l} \Bigg[\sum_{\substack{0<k\le l\\ k\equiv l}} a_{k,l}^{m_{rs}} \big((f_rf_s\ldots)_k - (f_sf_r\ldots)_k\big) \\
&\qquad\qquad + \sum_{\substack{-l\le k<0\\ k\equiv l}} a_{k,l}^{m_{rs}} \big((f_rf_s\ldots)_k - (f_sf_r\ldots)_k\big) \Bigg] \\
&= \sum_{l=0}^{m_{rs}} \alpha^{m_{rs}-l} \Bigg[\sum_{\substack{0<k\le l\\ k\equiv l}} a_{k,l}^{m_{rs}} \big((f_rf_s\ldots)_k - (f_sf_r\ldots)_k\big) \\
&\qquad\qquad + \sum_{\substack{0<p\le l\\ p\equiv l}} a_{-p,l}^{m_{rs}} \big((f_sf_r\ldots)_p - (f_rf_s\ldots)_p\big) \Bigg] \\
&= \sum_{l=0}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{0<k\le l\\ k\equiv l}} (a_{k,l}^{m_{rs}} - a_{-k,l}^{m_{rs}}) \big((f_rf_s\ldots)_k - (f_sf_r\ldots)_k\big) \\
&= \sum_{\substack{l=0\\ l\equiv m_{rs}}}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{0<k\le l\\ k\equiv l}} (a_{k,l}^{m_{rs}} - \tfrac{m_{rs}-k}{m_{rs}+k} a_{k,l}^{m_{rs}}) \big((f_rf_s\ldots)_k - (f_sf_r\ldots)_k\big)
\end{aligned}$$

(by Corollary 3.26)

$$= \sum_{\substack{l=0 \\ l\equiv m_{rs}}}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{0<k\leq l \\ k\equiv m_{rs}}} \tfrac{2k}{m_{rs}+k}\, a_{k,l}^{m_{rs}} \big((f_r f_s \ldots)_k - (f_s f_r \ldots)_k\big).$$

□

Combining Theorem 3.27 with the following examples, we have the presentations for the Hecke algebras of the Weyl groups.

Examples 3.28.

- $f_r f_s - f_s f_r = 0$ whenever $m_{rs} = 2$.
- $f_r f_s f_r - f_s f_r f_s + \left(\frac{q-1}{q+1}\right)^2 (f_r - f_s) = 0$ whenever $m_{rs} = 3$.
- $f_r f_s f_r f_s - f_s f_r f_s f_r + 2\left(\frac{q-1}{q+1}\right)^2 (f_r f_s - f_s f_r) = 0$ whenever $m_{rs} = 4$.
- $f_r f_s f_r f_s f_r - f_s f_r f_s f_r f_s + 3\left(\frac{q-1}{q+1}\right)^2 (f_r f_s f_r - f_s f_r f_s)$ $+ \left(\left(\frac{q-1}{q+1}\right)^2 + \left(\frac{q-1}{q+1}\right)^4\right)(f_r - f_s) = 0$ whenever $m_{rs} = 5$.
- $f_r f_s f_r f_s f_r f_s - f_s f_r f_s f_r f_s f_r + 4\left(\frac{q-1}{q+1}\right)^2 (f_r f_s f_r f_s - f_s f_r f_s f_r)$ $+ \left(2\left(\frac{q-1}{q+1}\right)^2 + 3\left(\frac{q-1}{q+1}\right)^4\right)(f_r f_s - f_s f_r) = 0$ whenever $m_{rs} = 6$.

3.5. Generators and Relations for the Alternating Hecke Algebra

We would like to give a presentation for the alternating Hecke algebra $\mathcal{A}_{R,q}(W)$. To this end we give the generators and some relations. The next few results prepare us for this. Recall that R is a ring containing $1/2$ and invertible elements $q \neq -1$ and $q+1$.

Lemma 3.29. *Suppose that (W, S) is a Coxeter system. Let G be the free group generated by elements $\dot{f}_s$ for $s \in S$ and let H be the subgroup of G which is generated by pairs $\dot{f}_r \dot{f}_s$ for $r, s \in S$, $r \neq s$. Then*

$$G = H \coprod \dot{f}_t H$$

for any t in S.

Proof. Let $X = \{\dot{f}_s \mid s \in S\}$ and let X^{-1} be the set of formal inverses of the elements in X. Then elements in G are words in the set $X \cup X^{-1}$. Similarly let $Y = \{\dot{f}_r \dot{f}_s \mid r, s \in S, r \neq s\}$, and Y^{-1} be the set of inverses of elements in Y. Then H consists of all words in $Y \cup Y^{-1}$.

Choose an arbitrary $t \in S$. Elements in G are words with either an even or odd number of elements from $X \cup X^{-1}$. If an element in G is a word of even length then it lies in H. Suppose $g \in G$ has odd length. Then $g = \dot{f}_t(\dot{f}_t^{-1} g)$ where $\dot{f}_t^{-1} g \in H$. So $g \in \dot{f}_t H$. □

The following nice result will be useful in later calculations and gives us an idea of how $\mathcal{A}_{R,q}(W)$ relates to $\mathcal{H}_{R,q}(W)$. In fact, we use this result in Section 5.1 to prove that, when R is a commutative ring, $\mathcal{H}_{R,q}(W)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded R-algebra.

Lemma 3.30. *Let $t \in S$. Then $\mathcal{H}_{R,q}(W) = \mathcal{A}_{R,q}(W) \oplus f_t \mathcal{A}_{R,q}(W)$ as R-modules.*

Proof. Suppose $h \in \mathcal{A} \cap f_t \mathcal{A}$. Then $h = f_t h'$ for some $h' \in \mathcal{A}$. Now $h^\# = h$ since $h \in \mathcal{A}$. We also have $f_t^\# = -f_t$ so $h^\# = f_t^\# h'^\# = -f_t h' = -h$. Therefore $h = 0$.

Now let h be in $\mathcal{H}_{R,q}(W)$ and let $h_1 = \frac{1}{2}(h + h^\#)$ and $h_2 = \frac{1}{2}(h - h^\#)$. Then $h = h_1 + h_2$. Now $h_1^\# = h_1$ so $h_1 \in \mathcal{A}$. Also, $(f_t h_2)^\# = f_t h_2$ so $f_t h_2 \in \mathcal{A}$ and therefore $h_2 = f_t^2 h_2 \in f_t \mathcal{A}$.

□

Lemma 3.31. *For $s \in S$, $f_s^\# = -f_s$.*

Proof. Let $s \in S$. Then

$$\begin{aligned} f_s^\# &= \left(\frac{2T_s - (q-1)}{q+1}\right)^\# \\ &= \frac{2(-T_s + q - 1) - (q-1)}{q+1} \\ &= \frac{-(2T_s - (q-1))}{q+1} = -f_s. \end{aligned}$$

□

Lemma 3.32. *The alternating Hecke algebra is generated by elements of the form $f_r f_s$ for r, s in S, $r \neq s$.*

Proof. By Lemma 3.31, an even product of generators is fixed under $\#$. Therefore any even product belongs to $\mathcal{A}_{R,q}(W)$.

Let $a \in \mathcal{A}_{R,q}(W)$. Then by Theorem 3.27, a can be written as a linear combination of words in the f_s for $s \in S$. Write $a = a_0 + a_1$ where a_0 is a linear combination of words of even length and a_1 is a linear combination of words of odd length. Then $a = a^\# = a_0 - a_1$, by Lemma 3.31. Therefore $a_1 = 0$. That is, a is linear combination of even products only. □

Let $X = \{f_s \mid s \in S\}$ and $Y = \{f_r f_s \mid r, s \in S, r \neq s\}$. The following theorem gives the generators and some relations for the alternating Hecke algebra.

Theorem 3.33. *Fix $t \in S$ and let $y_s = f_s f_t$ for all $s \in S$, $s \neq t$. Then the alternating Hecke algebra has generators y_s for $s \in S\backslash\{t\}$ and the following relations hold.*

- $\displaystyle\sum_{\substack{l=0 \\ l\equiv m_{rs}}}^{m_{rs}} \left(\tfrac{q-1}{q+1}\right)^{m_{rs}-l} \sum_{\substack{0<k\leq l \\ k\equiv m_{rs}}} \tfrac{2k}{m_{rs}+k}\, a_{k,l}^{m_{rs}} \left((y_r y_s^{-1} \ldots)_k - (y_s y_r^{-1} \ldots)_k\right) = 0$
  for $r, s \in S$, $r \neq s$.
- $\displaystyle\sum_{\substack{l=0 \\ l\equiv m_{rt}}}^{m_{rt}} \left(\tfrac{q-1}{q+1}\right)^{m_{rt}-l} \sum_{\substack{0<k\leq l \\ k\equiv m_{rt}}} \tfrac{2k}{m_{rt}+k}\, a_{k,l}^{m_{rt}} \left(y_r^{k/2} - y_r^{-k/2}\right) = 0$
  for $r \in S$, $r \neq t$ and m_{rt} even.
- $\displaystyle\sum_{\substack{l=0 \\ l\equiv m_{rt}}}^{m_{rt}} \left(\tfrac{q-1}{q+1}\right)^{m_{rt}-l} \sum_{\substack{0<k\leq l \\ k\equiv m_{rt}}} \tfrac{2k}{m_{rt}+k}\, a_{k,l}^{m_{rt}} \left(y_r^{(k+1)/2} - y_r^{-(k-1)/2}\right) = 0$
  for $r \in S$, $r \neq t$ and m_{rt} odd.

Proof. Let G and H be the free groups generated by the sets X and Y (as above) respectively. Then $\mathcal{H}_{R,q}(W) = RG/\langle \mathscr{R} \rangle_{RG}$ where $\mathscr{R}$ is the set containing the relations $f_s^2 = 1$ for $s \in S$, and

$$\sum_{\substack{l=0 \\ l\equiv m_{rs}}}^{m_{rs}} \left(\tfrac{q-1}{q+1}\right)^{m_{rs}-l} \sum_{\substack{0<k\leq l \\ k\equiv m_{rs}}} \tfrac{2k}{m_{rs}+k}\, a_{k,l}^{m_{rs}} \left((f_r f_s \ldots)_k - (f_s f_r \ldots)_k\right) = 0$$

for $r, s \in S$, $r \neq s$. By Lemma 3.29 we have coset representatives 1 and f_t for H in G.

Lemma 2.1 tells us that the set

$$\begin{aligned} Z &= \{\overline{f_s}^{-1} f_s, \overline{f_s f_t}^{-1} f_s f_t, \overline{f_s^{-1}}^{-1} f_s^{-1}, \overline{f_s^{-1} f_t}^{-1} f_s^{-1} f_t \mid s \in S\} \\ &= \{f_t^{-1} f_s, f_s f_t, f_t^{-1} f_s^{-1}, f_s^{-1} f_t \mid s \in S\} \\ &= \{f_t f_s, f_s f_t \mid s \in S\} \end{aligned}$$

generates $\mathcal{A}_{R,q}(W)$. Let $y_s = f_s f_t$ and $y'_s = f_t f_s$ for all $s \in S$. Then $Z = \{y_s, y'_s \mid s \in S\}$. We will write $y'_s = y_s^{-1}$ since $y_s y'_s = y'_s y_s = 1$.

From the presentation of the Hecke algebra we have the relations

$$\sum_{\substack{l=0 \\ l\equiv m_{rs}}}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{0<k\leq l \\ k\equiv m_{rs}}} \tfrac{2k}{m_{rs}+k}\, a_{k,l}^{m_{rs}} \left((f_r f_s \ldots)_k - (f_s f_r \ldots)_k\right) = 0 \tag{3.34}$$

for $r, s \in S$, $r \neq s$. We need to rewrite these relations in terms of the generators for the alternating Hecke algebra.

Case 1. m_{rs} is even

First consider the case when $s \neq t, r \neq t$. We have, for $k > 0$,

$$\begin{aligned}(f_r f_s \ldots)_k &= \underbrace{f_r f_s \ldots f_r f_s}_{k} \\ &= \underbrace{(f_r f_t)(f_t f_s) \ldots (f_r f_t)(f_t f_s)}_{2k} \\ &= (y_r y_s^{-1} \ldots)_k.\end{aligned}$$

So we have the relations

$$\sum_{\substack{l=0 \\ l \equiv m_{rs}}}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{0<k\leq l \\ k \equiv m_{rs}}} \tfrac{2k}{m_{rs}+k}\, a_{k,l}^{m_{rs}} \big((y_r y_s^{-1} \ldots)_k - (y_s y_r^{-1} \ldots)_k\big) = 0,$$

when m_{rt} is even.

Now consider the case when $s = t, r \neq t$. Then

$$(f_r f_t \ldots)_k = \underbrace{f_r f_t \ldots f_r f_t}_{k} = y_r^{k/2} \text{ and}$$

$$(f_t f_r \ldots)_k = y_r^{-k/2}$$

when $k > 0$. So, when m_{rt} is even, we have

$$\sum_{\substack{l=0 \\ l \equiv m_{rt}}}^{m_{rt}} \alpha^{m_{rt}-l} \sum_{\substack{0<k\leq l \\ k \equiv m_{rt}}} \tfrac{2k}{m_{rt}+k}\, a_{k,l}^{m_{rt}} \big(y_r^{k/2} - y_r^{-k/2}\big) = 0.$$

Note that we needn't consider the case when $r = t, s \neq t$, since it is equivalent to this case.

Case 2. m_{rs} is odd

When m_{rs} is odd the relation (3.34) doesn't lie in the alternating Hecke algebra. So we will multiply both sides of this equation on the right by f_t. Hence

$$\sum_{\substack{l=0 \\ l \equiv m_{rs}}}^{m_{rs}} \alpha^{m_{rs}-l} \sum_{\substack{0<k\leq l \\ k \equiv m_{rs}}} \tfrac{2k}{m_{rs}+k}\, a_{k,l}^{m_{rs}} \big((f_r f_s \ldots)_k f_t - (f_s f_r \ldots)_k f_t\big) \quad = \quad 0$$

for $r, s \in S$, $r \neq s$.

Suppose that $s \neq t, r \neq t$. We have, for $k > 0$,

$$\begin{aligned}(f_r f_s \ldots)_k f_t &= \underbrace{f_r f_s \ldots f_r}_{k} f_t \\ &= \underbrace{(f_r f_t)(f_t f_s) \ldots (f_r f_t)}_{2k} \\ &= (y_r y_s^{-1} \ldots)_k\end{aligned}$$

which is the same result that we arrived at in the case where m_{rs} was even. Therefore we have the same relations when m_{rs} is odd.

Now suppose $s = t, r \neq t$. Then, for $k > 0$ we have following:

$$\begin{aligned}(f_r f_t \ldots)_k f_t &= y_r^{(k+1)/2},\\ (f_t f_r \ldots)_k f_t &= y_r^{-(k-1)/2}.\end{aligned}$$

when m_{rt} is odd. Therefore we have the relations

$$\sum_{\substack{l=0\\ l\equiv m_{rt}}}^{m_{rt}} \alpha^{m_{rt}-l} \sum_{\substack{0<k\leq l\\ k\equiv m_{rt}}} \tfrac{2k}{m_{rt}+k}\, a_{k,l}^{m_{rt}} \left(y_r^{(k+1)/2} - y_r^{-(k-1)/2}\right) = 0.$$

□

The following are examples for the alternating Hecke algebras of the Weyl groups.

Examples 3.35.

- $y_r^{-1} = y_r$ if $m_{rt} = 2$
- $y_r^{-1} = y_r^2 + \left(\frac{q-1}{q+1}\right)^2 (y_r - 1)$ if $m_{rt} = 3$
- $y_r^{-1} = y_r^3 + 2\left(\frac{q-1}{q+1}\right)^2 (y_r^2 - 1)$ if $m_{rt} = 4$
- $y_r^{-1} = y_r^4 + 3\left(\frac{q-1}{q+1}\right)^2 (y_r^3 - 1) + \left[\left(\frac{q-1}{q+1}\right)^2 + \left(\frac{q-1}{q+1}\right)^4\right] (y_r^2 - y_r)$ if $m_{rt} = 5$
- $y_r^{-1} = y_r^5 + 4\left(\frac{q-1}{q+1}\right)^2 (y_r^4 - 1) + \left[2\left(\frac{q-1}{q+1}\right)^2 + 3\left(\frac{q-1}{q+1}\right)^4\right] (y_r^3 - y_r)$ if $m_{rt} = 6$
- $y_r y_s^{-1} = y_s y_r^{-1}$ if $m_{rs} = 2$
- $y_r y_s^{-1} y_r = y_s y_r^{-1} y_s + \left(\frac{q-1}{q+1}\right)^2 (y_s - y_r)$ if $m_{rs} = 3$.
- $y_r y_s^{-1} y_r y_s^{-1} = y_s y_r^{-1} y_s y_r^{-1} + 2\left(\frac{q-1}{q+1}\right)^2 (y_s y_r^{-1} - y_r y_s^{-1})$ if $m_{rs} = 4$.
- $y_r y_s^{-1} y_r y_s^{-1} y_r = y_s y_r^{-1} y_s y_r^{-1} y_s + 3\left(\frac{q-1}{q+1}\right)^2 (y_s y_r^{-1} y_s - y_r y_s^{-1} y_r) + \left[\left(\frac{q-1}{q+1}\right)^2 + \left(\frac{q-1}{q+1}\right)^4\right] (y_s - y_r)$ if $m_{rs} = 5$.
- $y_r y_s^{-1} y_r y_s^{-1} y_r y_s^{-1} = y_s y_r^{-1} y_s y_r^{-1} y_s y_r^{-1} + 4\left(\frac{q-1}{q+1}\right)^2 (y_s y_r^{-1} y_s y_r^{-1} - y_r y_s^{-1} y_r y_s^{-1}) + \left[2\left(\frac{q-1}{q+1}\right)^2 + 3\left(\frac{q-1}{q+1}\right)^4\right] (y_s y_r^{-1} - y_r y_s^{-1})$ if $m_{rs} = 6$.

CHAPTER 4

An Isomorphism Theorem

Tits' deformation theorem gives a general method to show that two deformations of a semisimple algebra are isomorphic. The method involves showing that the two algebras have the same Wedderburn decomposition. We will follow the treatment of Geck and Pfeiffer [**6**].

4.1. Valuation Rings and Grothendieck Groups

In this section we outline some basic definitions. Recall that a **valuation ring** $\mathcal{O}$ is an integral domain such that, for each x in the field of fractions of $\mathcal{O}$, we have

$$x \notin \mathcal{O} \Rightarrow x^{-1} \in \mathcal{O}.$$

Therefore $\mathcal{O} \cup \mathcal{O}^{-1}$ is the field of fractions of $\mathcal{O}$, where $\mathcal{O}^{-1}$ is the set of inverses of the non-zero elements in $\mathcal{O}$. In fact, a valuation ring is a local ring with maximal ideal

$$\mathcal{J}(\mathcal{O}) := \{x \in \mathcal{O} \mid x^{-1} \notin \mathcal{O}\} \cup \{0\}.$$

See Lang [**12**, p 481].

We will require various results on valuation rings which we will only state here. Let R be an integral domain contained in a field K.

a) If $I \subset R$ is a prime ideal then there exists a valuation ring $\mathcal{O} \subseteq K$ such that $R \subseteq \mathcal{O}$ and $\mathcal{J}(\mathcal{O}) \cap R = I$.

b) Every finitely generated torsion-free module over a valuation ring in K is free.

c) The intersection of all valuation rings $\mathcal{O} \subseteq K$ with $R \subseteq \mathcal{O}$ is the integral closure of R in K. Each valuation ring itself is integrally closed in K.

For proofs of a) and b) see Goldschmidt [**7**, §5] and for c) see Matsumura [**15**, Theorem 10.4].

We will also need to know something about Grothendieck groups. The following definitions and results can be found in Curtis & Reiner [**3**, §16B].

Let H be an algebra over some field and let $\mathcal{C}$ be the category of finitely generated right H-modules. For $M \in \mathcal{C}$ we write (M) for the class of modules isomorphic to M. Let F be the (additive) free abelian group with generators (M), for (M) an isomorphism class of modules in $\mathcal{C}$. Let F_0 be the subgroup of F generated by all expressions $(M) - (M_1) - (M_2)$, for each short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

of R-modules. Then the **Grothendieck group**, $G_0(H)$, of $\mathcal{C}$ is defined to be the quotient group

$$G_0(H) := F/F_0.$$

Let $[M]$ be the image of (M) under this quotient, so $[M] := (M) + F_0$. Then we have relations $[M] = [M_1] + [M_2]$ for each short exact sequence as above.

We note that the isomorphism classes of simple modules form a basis for $G_0(H)$. To see this, consider $M \in \mathcal{C}$ where M is not irreducible. We can find a maximal submodule N of M and construct a short exact sequence

$$0 \longrightarrow N \overset{f}{\hookrightarrow} M \xrightarrow{g} M/N \longrightarrow 0$$

where f is the inclusion map and g the natural quotient map. Thus we have the relation $[M] = [N] + [M/N]$, where M/N is irreducible. Iterating this procedure on N gives us the required result.

Lastly, let $G_0^+(H)$ be the subset of $G_0(H)$ consisting of elements $[M]$ where $M \in \mathcal{C}$. So $G_0^+(H)$ consists of positive linear combinations of classes of simple modules. Note that $G_0^+(H)$ is a monoid with identity $[\{0\}]$.

4.2. Tits' Deformation Theorem

In this section we set up the framework required for Tits' deformation theorem, following Geck and Pfeiffer [**6**, §7.3,7.4].

Suppose R is an integral domain which is integrally closed in a field K. Let H be an R-algebra which is finitely generated and free over R. Then we can construct the K-algebra $H^K := H \otimes_R K$. Let $\theta : R \to L$ be a ring homomorphism such that L is a field and the field of fractions of $\mathrm{Im}(\theta)$ is contained in L. We will call such a map a **specialisation** of R. We may regard L as a left R-module via θ. Therefore we can construct $H^L := H \otimes_R L$, the algebra H over the field L.

Let X be an indeterminate over K and let Maps$(H, K[X])$ be the K-algebra of maps from H to $K[X]$, with multiplication of maps given by pointwise multiplication. Fix a basis of H. Now suppose V is a H^K-module of finite dimension n, and let $\psi_V \in$ Maps$(H, K[X])$ be the map which sends $h \in H$ to the characteristic polynomial of $\rho_V(h \otimes 1)$, where ρ_V is the representation afforded by V. So

$$\begin{array}{rcl} \psi_V : H & \to & K[X] \\ h & \mapsto & \mathrm{Det}(\rho_V(h \otimes 1) - XI) \end{array}$$

where I is the identity matrix. Then define a map

$$\begin{array}{rcl} \mathfrak{p}_K : G_0^+(H^K) & \to & \mathrm{Maps}(H, K[X]) \\ [V] & \mapsto & \psi_V. \end{array}$$

The map $\mathfrak{p}_K$ is well defined and when we consider Maps$(H, K[X])$ as a semigroup under multiplication, $\mathfrak{p}_K$ is a semigroup homomorphism. Define $\mathfrak{p}_L$ in a similar manner. So

$$\begin{array}{rcl} \mathfrak{p}_L : G_0^+(H^L) & \to & \mathrm{Maps}(H, L[X]) \\ [V] & \mapsto & \psi_V \end{array}$$

where now $\psi_V \in$ Maps$(H, L[X])$.

Note that $\mathrm{Im}(\theta)$ is an integral domain since it is a subring of the field L. Therefore Ker(θ) is a prime ideal since $R/$Ker$(\theta) \cong$ Im(θ). Hence, by property a) in Section 4.1, there exists some valuation ring $\mathcal{O} \subseteq K$ such that $R \subseteq \mathcal{O}$ and $\mathcal{J}(\mathcal{O}) \cap R =$ Ker(θ).

Let k be the field $\mathcal{O}/\mathcal{J}(\mathcal{O})$ and let $\pi : \mathcal{O} \to k$ be the canonical epimorphism. Note that we can consider $\mathrm{Im}(\theta)$ as a subset of k via the following:

$$\mathrm{Im}(\theta) \cong R/\mathrm{Ker}(\theta) = R/(\mathcal{J}(\mathcal{O}) \cap R) \cong (\mathcal{J}(\mathcal{O}) + R)/\mathcal{J}(\mathcal{O}) \subseteq k,$$

by the first and second isomorphism theorems. So for $r \in R$, we consider $\theta(r)$ as the the element $r + \mathcal{J}(\mathcal{O})$ in k. Now, since localisation respects inclusion, the field of fractions of $\mathrm{Im}(\theta)$ is a subfield of the field of fractions of k. But this means that $L \subseteq k$.

Hence we have the following commutative diagram.

$$\begin{array}{ccccc} R & \subseteq & \mathcal{O} & \subseteq & K \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\pi} & & \\ L & \subseteq & k & & \end{array}$$

The importance of working with valuation rings can be seen in the fact that it allows us to find a basis for V such that $\rho_V(h \otimes 1) \in \mathrm{Mat}_n(\mathcal{O})$ for all $h \in H$, where $n = \dim_K V$ (for a proof of this see [**6**], Section 7.3.7). When this happens there exists a finitely generated $H^{\mathcal{O}}$-module $\widetilde{V}$ such that $V \cong \widetilde{V}^K$. Since $\widetilde{V}$ sits inside the K-vector space V it must be torsion-free. Hence $\widetilde{V}$ is free, by b) in section 4.1. Also, $\widetilde{V}^k$ is a right H^k-module with action $(v \otimes 1)(h \otimes 1) = vh \otimes 1$ for $v \in V$ and $h \in H$. We call this module the **modular reduction** of $\widetilde{V}$.

Theorem 7.4.3 of Geck & Pfeiffer asserts the existence of decomposition maps:

Theorem 4.1 ([**6**], Theorem 7.4.3)**.** *Suppose R is an integral domain and that $\theta : R \to L$ is a ring homomorphism into the field of fractions, L, of $\mathrm{Im}(\theta)$. Assume that we have chosen a valuation ring $\mathcal{O}$ as above and that H^L is split. Then the following hold.*

a) *The modular reduction of $\widetilde{V}$ induces an additive map*

$$d_\theta : G_0^+(H^K) \to G_0^+(H^L)$$

such that $d_\theta([\widetilde{V}^K]) = [\widetilde{V}^k]$, where $\widetilde{V}$ is a $H^{\mathcal{O}}$-lattice and $[\widetilde{V}^k]$ is regarded as an element of $G_0^+(H^L)$ via the identification above. The map d_θ is the **decomposition map** *associated to the specialisation θ.*

b) *The map $\mathfrak{p}_K$ has image contained in Maps$(H, R[X])$ and we have the following commuatative diagram*

$$\begin{array}{ccc} G_0^+(H^K) & \xrightarrow{\mathfrak{p}_K} & \textit{Maps}(H, R[X]) \\ {\scriptstyle d_\theta}\downarrow & & \downarrow{\scriptstyle t_\theta} \\ G_0^+(H^L) & \xrightarrow[\mathfrak{p}_L]{} & \textit{Maps}(H, L[X]) \end{array}$$

where t_θ is the map induced by θ.

c) *The map d_θ is uniquely determined by the commutativity of the diagram in b). In particular, d_θ depends only on θ and not on the choice of $\mathcal{O}$.*

Suppose we have an R-linear map $\lambda : H \to R$. Then we will define a new L-linear map

$$\begin{array}{rccc} \lambda^L : & H^L & \to & L \\ & h \otimes 1 & \mapsto & \theta(\lambda(h)) \end{array}$$

for each $h \in H$. Let $\chi : H^K \to K$ be the character of the H^K-module V. Note that $\mathfrak{p}_K$ has image contained in Maps$(H, R[X])$ and, since a character value appears as the coefficient of X^{n-1} in the corresponding characteristic polynomial, we have $\chi(h) \in R$ for all $h \in H^K$. Therefore we can restrict χ to an R-linear map $\dot{\chi} : H \to R$. Recall that a linear map $\tau : H \to R$ is a **trace function** if $\tau(hh') = \tau(h'h)$ for all $h, h' \in H$. Now $\dot{\chi}$ is a trace function and hence, by an easy calculation, $\dot{\chi}^L$ is also a trace function. Moreover, by Theorem 4.1, part b), and using the fact that the character is a coefficient in the characteristic polynomial, $\dot{\chi}^L$ is the character of $d_\theta([V])$.

We now have enough information to state Tits' deformation theorem. For an algebra, A, we let Irr(A) be the set of irreducible right A-characters.

Theorem 4.2 (Tits' Deformation Theorem)**.** *Assume that H^K and H^L are split and that H^L is semisimple. Then the algebra H^K is semisimple and the decomposition map d_θ is an isomorphism which preserves isomorphism classes of simple modules. In particular, the map*

$$\begin{array}{rcl} \mathrm{Irr}(H^K) & \to & \mathrm{Irr}(H^L) \\ \chi & \mapsto & \dot{\chi}^L \end{array}$$

is a bijection.

4.3. An Isomorphism Theorem

Throughout this section let W be a finite Coxeter group and let $\mathcal{A} = \mathcal{A}_{R,q}(W)$ be the alternating Hecke algebra over the ring $R = \mathbb{Z}[q^{\pm 1}, 1/2]$. Let $E = \mathbb{Q}(q)$. Consider E as a left R-module and define $\mathcal{A}^E = \mathcal{A} \otimes_R E$. Then $\mathcal{A}^E$ is isomorphic to the alternating Hecke algebra over E, by Theorem 3.10.

If R is a ring we write $\mathscr{A}^R$ for the group algebra of the alternating group $\mathscr{A}(W)$. We will follow Geck & Pfeiffer by using Tit's deformation theorem to show that, for some sufficiently large field K, the group algebra $\mathscr{A}^K$ is isomorphic to $\mathcal{A}^K$. We will need to show that $\mathcal{A}^E$ is a separable algebra. However, first we must prove the following preliminary results.

Lemma 4.3. *Let $\phi : \mathscr{A}^{\mathbb{Q}} \to \mathbb{Q}$ be the character of the regular representation of the group algebra $\mathscr{A}^{\mathbb{Q}}$. Then the bilinear form $\Phi : \mathscr{A}^{\mathbb{Q}} \times \mathscr{A}^{\mathbb{Q}} \to \mathbb{Q}$, defined by $\Phi(h, h') = \phi(hh')$ for all h, h' in $\mathscr{A}^{\mathbb{Q}}$, is non-degenerate.*

Proof. We have, for x in $\mathscr{A}(W)$,

$$\phi(x) = \begin{cases} |\mathscr{A}(W)| & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $h \in \mathscr{A}^{\mathbb{Q}}$. So $h = \sum_{w \in \mathscr{A}(W)} \lambda_w w$ for some λ_w in $\mathbb{Q}$. Suppose $\phi(hh') = 0$ for all h' in $\mathscr{A}^{\mathbb{Q}}$. That is, $\phi(\sum_{w \in \mathscr{A}(W)} \lambda_w w h') = 0$ for all h' in $\mathscr{A}^{\mathbb{Q}}$. Then certainly $\phi(\sum_{w \in \mathscr{A}(W)} \lambda_w w x) = 0$ for all x in $\mathscr{A}(W)$. Since ϕ is linear we have $\sum_{w \in \mathscr{A}(W)} \lambda_w \phi(wx) = 0$. However $\phi(wx)$ is 0 if $w \neq x^{-1}$ so $|\mathscr{A}(W)|\lambda_{x^{-1}} = 0$ for all $x \in \mathscr{A}(W)$. Therefore $\lambda_x = 0$ for all $x \in \mathscr{A}(W)$ and so $h = 0$. Hence Φ is non-degenerate. □

Let $\theta : R \to \mathbb{Q}$ be the ring homomorphism sending $q \mapsto 1$. Then the group algebra $W^{\mathbb{Q}}$ is an R-module via θ. Since the structure constants in $W^{\mathbb{Q}}$ are obtained by applying θ to the structure constants in $\mathcal{H}$, θ induces an algebra homomorphism from $\mathcal{H} \to W^{\mathbb{Q}}$ sending $q \mapsto 1$ and $T_w \mapsto w$. Now if $w \in \mathscr{A}(W)$ then $B_w = \frac{1}{2}(T_w + T_w^{\#}) = \frac{1}{2}(T_w + \varepsilon_w q_w T_{w^{-1}}^{-1})$ goes to $\frac{1}{2}(w + (w^{-1})^{-1}) = w$ under this map and, since the map is an algebra homomorphism, the structure constants in $\mathscr{A}^{\mathbb{Q}}$ are obtained by setting $q = 1$ in the structure constants of $\mathcal{A}$; that is, by applying θ.

Lemma 4.4. *Consider $\mathcal{A}^E$ as a right module over itself and let $\psi : \mathcal{A}^E \to E$ be the character of the regular representation. Then the bilinear form $\Psi : \mathcal{A}^E \times \mathcal{A}^E \to E$, given by $\Psi(h, h') = \psi(hh')$ for h, h' in $\mathcal{A}^E$, is non-degenerate.*

Proof. Fix a basis $\{B_w \mid w \in \mathscr{A}(W)\}$ of $\mathcal{A}$. Then $\{B_w \otimes 1 \mid w \in \mathscr{A}(W)\}$ is a basis for $\mathcal{A}^E$ over E. Let ρ be the regular representation of $\mathcal{A}^E$ with respect to this basis. We have $\psi(h) = \mathrm{Tr}(\rho(h))$ for all h in $\mathcal{A}^E$. Let ϕ and Φ be defined as in Lemma 4.3. We will use the fact that Φ is non-degenerate to deduce that Ψ is non-degenerate.

First we prove a relationship between ϕ and ψ. Suppose that in $\mathcal{A}$ we have $B_x B_y = \sum_{z \in \mathscr{A}(W)} a_z B_z$ for some a_z in R where the a_z depend on x and y. Then the structure constants in $\mathcal{A}^E$ are also in R, since $(B_x \otimes 1)(B_y \otimes 1) = B_x B_y \otimes 1$. Therefore $\rho(B_w \otimes 1)$ is in R for each w in $\mathscr{A}(W)$ and so $\psi(B_w \otimes 1) \in R$ for each basis element $B_w \otimes 1$ of $\mathcal{A}^E$. Thus we can restrict ψ to a trace function on $\mathcal{A}$. That is, let $\psi^R : \mathcal{A} \to R$ be the R-linear map sending $B_w \mapsto \psi(B_w \otimes 1)$. This map is a trace function because ψ is.

Now we form $\mathcal{A}^{\mathbb{Q}}$ via the specialisation $\theta : R \to \mathbb{Q}$ which sends $q \mapsto 1$. This is isomorphic to the alternating Hecke algebra over $\mathbb{Q}$. We use the map ψ^R to form a trace function on $\mathcal{A}^{\mathbb{Q}}$ as follows. Define $\psi^{\mathbb{Q}} : \mathcal{A}^{\mathbb{Q}} \to \mathbb{Q}$ to be the $\mathbb{Q}$-linear map which sends $B_w \otimes 1 \mapsto \theta(\psi^R(B_w))$ for all $w \in \mathscr{A}(W)$. A simple calculation shows that $\psi^{\mathbb{Q}}$ is also a trace function.

We claim that $\psi^{\mathbb{Q}}$ and ϕ agree on the corresponding basis elements. Suppose $w \in \mathscr{A}(W)$ and let $\rho(B_w \otimes 1) = (m_{x,y})_{x,y \in \mathscr{A}(W)}$ for some $m_{x,y}$ in R. Then $\phi(w) = \mathrm{Tr}(\theta(m_{x,y}))_{x,y \in \mathscr{A}(W)}$ since the structure constants in $\mathcal{A}$ are the same as the structure constants in $\mathcal{A}^E$. Then

$$
\begin{aligned}
\psi^{\mathbb{Q}}(B_w \otimes 1) &= \theta(\psi^R(B_w)) \\
&= \theta(\psi(B_w \otimes 1)) \\
&= \theta(\mathrm{Tr}(\rho(B_w \otimes 1))) \\
&= \theta(\mathrm{Tr}(m_{x,y})) \\
&= \mathrm{Tr}(\theta(m_{x,y})) \\
&= \phi(w)
\end{aligned}
$$

for each w in $\mathscr{A}(W)$.

Now we show that Ψ is non-degenerate. Let $B_x B_y = \sum_{z \in \mathscr{A}(W)} a_z B_z$ for some $a_z \in R$ (where the a_z depend on x and y). Then we must have

$xy = \sum_{z \in \mathscr{A}(W)} \theta(a_z) z$. Now consider the matrix $(\Phi(x,y))_{x,y \in \mathscr{A}(W)}$.

$$\begin{aligned}
(\Phi(x,y))_{x,y\in\mathscr{A}(W)} &= (\phi(xy))_{x,y\in\mathscr{A}(W)} \quad \text{(the definition of } \Phi\text{)} \\
&= \left(\phi\left(\sum_{z\in\mathscr{A}(W)} \theta\left(a_z\right) z\right)\right)_{x,y\in\mathscr{A}(W)} \\
&= \left(\sum_{z\in\mathscr{A}(W)} \theta(a_z)\phi(z)\right)_{x,y\in\mathscr{A}(W)} \\
&= \left(\sum_{z\in\mathscr{A}(W)} \theta(a_z)\psi^{\mathbb{Q}}(B_z\otimes 1)\right)_{x,y\in\mathscr{A}(W)} \\
&= \left(\sum_{z\in\mathscr{A}(W)} \theta(a_z)\theta\left(\psi^{R}(B_z)\right)\right)_{x,y\in\mathscr{A}(W)} \\
&= \left(\sum_{z\in\mathscr{A}(W)} \theta(a_z)\theta\left(\psi(B_z\otimes 1)\right)\right)_{x,y\in\mathscr{A}(W)} \\
&= \left(\theta\left(\sum_{z\in\mathscr{A}(W)} a_z\psi(B_z\otimes 1)\right)\right)_{x,y\in\mathscr{A}(W)} \\
&= \left(\theta\left(\psi\left(\sum_{z\in\mathscr{A}(W)} a_z B_z\otimes 1\right)\right)\right)_{x,y\in\mathscr{A}(W)} \\
&= \Big(\theta\Big(\psi((B_x\otimes 1)(B_y\otimes 1))\Big)\Big)_{x,y\in\mathscr{A}(W)} \\
&= \Big(\theta\Big(\Psi(B_x\otimes 1, B_y\otimes 1)\Big)\Big)_{x,y\in\mathscr{A}(W)}
\end{aligned}$$

Now taking determinants we have

$$\begin{aligned}
\mathrm{Det}(\Phi(x,y))_{x,y\in\mathscr{A}(W)} &= \mathrm{Det}\left(\theta\Big(\Psi(B_x\otimes 1, B_y\otimes 1)\Big)\right)_{x,y\in\mathscr{A}(W)} \\
&= \theta\left(\mathrm{Det}\left(\Psi(B_x\otimes 1, B_y\otimes 1)\right)_{x,y\in\mathscr{A}(W)}\right)
\end{aligned}$$

The determinant of the matrix $(\Phi(x,y))_{x,y\in\mathscr{A}(W)}$ must be a unit in $\mathbb{Q}$ since Φ is non-degenerate. Hence

$$\theta\left(\mathrm{Det}\left(\Psi(B_x\otimes 1, B_y\otimes 1)\right)_{x,y\in\mathscr{A}(W)}\right)\neq 0. \tag{4.5}$$

Now $\mathrm{Det}\left(\Psi(B_x\otimes 1, B_y\otimes 1)\right)_{x,y\in\mathscr{A}(W)}\neq 0$, since otherwise

$$\theta\left(\mathrm{Det}\left(\Psi(B_x\otimes 1, B_y\otimes 1)\right)_{x,y\in\mathscr{A}(W)}\right)=\theta(0)=0$$

contradicting equation (4.5). Therefore $\mathrm{Det}\left(\Psi(B_x\otimes 1, B_y\otimes 1)\right)_{x,y\in\mathscr{A}(W)}\neq 0$. That is, $\mathrm{Det}\left(\Psi(B_x\otimes 1, B_y\otimes 1)\right)_{x,y\in\mathscr{A}(W)}$ is a unit in $\mathbb{Q}\subset E$. Hence Ψ is non-degenerate.

□

Lemma 4.6. *$\mathcal{A}_{R,q}(W)^E$ is a separable algebra.*

Proof. Suppose that K is some field containing E. We need to show that $\mathcal{A}^K := \mathcal{A}_{R,q}(W)^K$ is semisimple. Let $\mathcal{J}=\mathcal{J}(\mathcal{A}^K)$ be the Jacobson radical of $\mathcal{A}^K$. We will use the bilinear form of Lemma 4.4 to show that $\mathcal{J}=\{0\}$.

As in Lemma 4.4, let ψ be the character of the regular representation of $\mathcal{A}^E$ and similarly let ψ^K be the character of the regular representation of $\mathcal{A}^K$. Note that ψ^K is the K-linear map sending $B_w\otimes 1\mapsto\psi(B_w\otimes 1)$. Let $\Psi^K:\mathcal{A}^K\times\mathcal{A}^K\to K$ be the associated bilinear form. So $\Psi^K(h,h')=\psi^K(hh')$ for all $h,h'\in\mathcal{A}^K$.

Since Ψ is non-degenerate then $\mathrm{Det}(\Psi(B_x\otimes 1,B_y\otimes 1))_{x,y\in\mathscr{A}(W)}$ is a unit in E. Now the matrices $(\Psi(B_x\otimes 1,B_y\otimes 1))_{x,y\in\mathscr{A}(W)}$ and $(\Psi^K(B_x\otimes 1,B_y\otimes 1))_{x,y\in\mathscr{A}(W)}$ are the same, so $\mathrm{Det}(\Psi^K(B_x\otimes 1,B_y\otimes 1))_{x,y\in\mathscr{A}(W)}$ is also a unit in $E\subseteq K$. Therefore Ψ^K is non-degenerate.

Suppose that ρ' is the regular representation of $\mathcal{A}^K$. Suppose $h\in\mathcal{J}$ and $h'\in\mathcal{A}^K$. Then $hh'\in\mathcal{J}$. Now $\mathcal{J}$ is a nilpotent ideal, so we have $(hh')^n=0$ for some positive integer n. Therefore $0=\rho'(0)=\rho'((hh')^n)=(\rho'(hh'))^n$. Recall that if a matrix is nilpotent, then it's trace is zero. Hence $\Psi^K(h,h')=\psi^K(hh')=\mathrm{Tr}(\rho'(hh'))=0$. Since this is true for any $h'\in\mathcal{A}^K$ and since Ψ^K is non-degenerate we have $h=0$. Therefore $\mathcal{J}=\{0\}$ and hence $\mathcal{A}^K$ is semisimple. □

Define $\theta:R\to\mathbb{C}$ to be a ring homomorphism sending $q\mapsto c$ for some non-zero c in $\mathbb{C}$. We think of $\mathbb{C}$ as a left R-module via θ and define $\mathcal{A}^{\mathbb{C}}$ to be the specialised algebra $\mathcal{A}\otimes_R\mathbb{C}$. Let $R_{\mathbb{C}}:=\mathbb{C}[q^{\pm1}]$ and define

$\theta_{\mathbb{C}} : R_{\mathbb{C}} \to \mathbb{C}$ to be the ring homomorphism which extends the map θ. Note that $\mathrm{Im}(\theta_{\mathbb{C}}) = \mathbb{C}$. Suppose that K is an extension field of $\mathbb{C}(q)$ and let $R^*_{\mathbb{C}}$ be the integral closure of $R_{\mathbb{C}}$ in K. Then $\theta_{\mathbb{C}}$ extends to a ring homomorphism $\theta^*_{\mathbb{C}} : R^*_{\mathbb{C}} \to \mathbb{C}$. See Section 8.1.6 of Geck and Pfieffer [**6**].

Theorem 4.7. *Let K be an extension field of $\mathbb{C}(q)$ and let W be a finite Coxeter group. If $\mathcal{A}_{R^*_{\mathbb{C}},q}(W)^K$ is split and $\mathcal{A}_{R^*_{\mathbb{C}},q}(W)^{\mathbb{C}}$ is semisimple then $d_{\theta^*_{\mathbb{C}}}$ is an isomorphism preserving isomorphism classes of simple modules. In particular, the map $\mathrm{Irr}(\mathcal{A}^K) \to \mathrm{Irr}(\mathcal{A}^{\mathbb{C}})$ sending $\chi \mapsto \dot{\chi}^{\mathbb{C}}$ is a bijection.*

Proof. The result follows immediately after applying Tit's deformation theorem (Theorem 4.2) with $H = \mathcal{A}_{R^*_{\mathbb{C}},q}(W)$, $L = \mathbb{C}$ and $\theta = \theta^*_{\mathbb{C}}$. Note that $\mathcal{A}^{\mathbb{C}}$ is split since $\mathbb{C}$ is algebraically closed. □

We are now equipped to prove the isomorphism theorem.

Theorem 4.8. *There exists a field $K \supseteq \mathbb{C}(q)$ such that $\mathcal{A}^K \cong \mathscr{A}^K$ as K-algebras. Moreover, there is a bijection*

$$\mathrm{Irr}(\mathcal{A}^K) \to \mathrm{Irr}(\mathscr{A}^{\mathbb{C}}).$$

Proof. We know that $\mathcal{A}^E$ is separable (Lemma 4.6). Then, since $E \subset \mathbb{C}(q)$, $\mathcal{A}^{\mathbb{C}(q)}$ is also separable. Hence there exists a field $K \supseteq \mathbb{C}(q)$ such that K is a splitting field for $\mathcal{A}^{\mathbb{C}(q)}$ over $\mathbb{C}(q)$ (see Curtis & Reiner [**3**], Proposition 7.25). That is, $\mathcal{A}^K$ is split.

Let $\theta : R \to \mathbb{C}$ be the map sending $q \mapsto 1$. Then $\mathcal{A}^{\mathbb{C}} \cong \mathscr{A}^{\mathbb{C}}$ which is semisimple. By Theorem 4.7 the map $d_{\theta^*_{\mathbb{C}}} : G_0^+(\mathcal{A}^K) \to G_0^+(\mathcal{A}^{\mathbb{C}})$ is an isomorphism preserving isomorphism classes of simple modules. Moreover, the map $\mathrm{Irr}(\mathcal{A}^K) \to \mathrm{Irr}(\mathcal{A}^{\mathbb{C}})$ sending $\chi \mapsto \dot{\chi}^{\mathbb{C}}$ is a bijection.

Since $\mathbb{C}$ is algebraically closed, $\mathscr{A}^{\mathbb{C}}$ is split and so by Theorem 4.1 b) we have

$$t_{\theta^*_{\mathbb{C}}} \circ \mathfrak{p}_K([V]) = \mathfrak{p}_{\mathbb{C}} \circ d_{\theta^*_{\mathbb{C}}}([V])$$

for all $\mathcal{A}^K$-modules V. In particular,

$$t_{\theta^*_{\mathbb{C}}} \circ \mathfrak{p}_K([V])(1) = \mathfrak{p}_{\mathbb{C}} \circ d_{\theta^*_{\mathbb{C}}}([V])(1).$$

That is

$$(1-X)^{\dim_K V} = (1-X)^{\dim_{\mathbb{C}}(d_{\theta^*_{\mathbb{C}}}([V]))}$$

for some indeterminate X. Therefore we have $\dim_K V = \dim_{\mathbb{C}}(d_{\theta^*_{\mathbb{C}}}([V]))$ for all $\mathcal{A}^K$-modules V.

Now $\mathcal{A}^K$ is semisimple by Lemma 4.6. So using Wedderburn's theorem we have

$$\mathcal{A}^K \cong \bigoplus_D \mathrm{Mat}_{n_D}(K)$$

where D runs over the simple $\mathcal{A}^K$-modules (up to isomorphism) and n_D is the dimension of D, and similarly

$$\mathscr{A}^{\mathbb{C}} \cong \bigoplus_{D'} \mathrm{Mat}_{n_{D'}}(\mathbb{C})$$

where D' runs over the simple $\mathscr{A}^{\mathbb{C}}$-modules and $n_{D'}$ is the dimension of D'. Now each summand in $\mathcal{A}^K$ corresponds to one in $\mathscr{A}^{\mathbb{C}}$ by the bijection and they have the same dimensions.

Note that the group algebra $\mathscr{A}^K$ is semisimple so

$$\mathscr{A}^K \cong \bigoplus_{D''} \mathrm{Mat}_{n_{D''}}(K)$$

where D'' runs over the simple $\mathscr{A}^K$-modules (up to isomorphism) and $n_{D''}$ is the dimension of D''. Now since $\mathscr{A}^{\mathbb{C}}$ is split, for each simple $\mathscr{A}^{\mathbb{C}}$-module D' we have a simple $\mathscr{A}^K$-module $D' \otimes_{\mathbb{C}} K$. So there are at least as many simple $\mathscr{A}^K$-modules as there are simple $\mathscr{A}^{\mathbb{C}}$-modules. Also,

$$\begin{aligned}
\mathscr{A}^K &\cong \mathscr{A}^{\mathbb{C}} \otimes_{\mathbb{C}} K \\
&\cong \left(\bigoplus_{D'} \mathrm{Mat}_{n_{D'}}(\mathbb{C}) \right) \otimes_{\mathbb{C}} K \\
&\cong \bigoplus_{D'} \left(\mathrm{Mat}_{n_{D'}}(\mathbb{C}) \otimes_{\mathbb{C}} K \right) \\
&\cong \bigoplus_{D'} \mathrm{Mat}_{n_{D'}}(K).
\end{aligned}$$

So there are exactly the same number of simple $\mathscr{A}^K$-modules as simple $\mathscr{A}^{\mathbb{C}}$-modules and hence simple $\mathcal{A}^K$-modules. Also, the modules correspond in that they have the same dimensions. Therefore $\mathcal{A}^K \cong \mathscr{A}^K$. □

CHAPTER 5

Some Character Theory

In this chapter we use Clifford theory to discover under what conditions irreducible $\mathcal{H}_{K,q}(W)$-modules stay irreducible when we restrict them to $\mathcal{A}_{K,q}(W)$-modules. Then we list these conditions for the Hecke algebras associated to the classical Coxter groups A_n, B_n and D_n.

5.1. Applying Clifford Theory to the Alternating Hecke Algebra

We begin by showing that the Hecke algebra has a Clifford system.

Proposition 5.1. *Let R be a commutative ring containing invertible elements 2, $q \neq -1$ and $q+1$. Also, let (W,S) be a Coxeter system and let $C = \{1,-1\}$ be the cyclic group of order 2. Fix $t \in S$. Then $\mathcal{H}_{R,q}(W)$ is a C-graded R-algebra, with $A_1 = \mathcal{A}_{R,q}(W)$ and $A_{-1} = \mathcal{A}_{R,q}(W)f_t = f_t\mathcal{A}_{R,q}(W)$.*

Proof. For $\mathcal{H}_{R,q}(W)$ to be a C-graded R-algebra we need the conditions of Definition 1.8 to be satisfied. Let $\mathcal{A} = \mathcal{A}_{R,q}(W)$. We will first show that $f_t\mathcal{A} = \mathcal{A}f_t$. By Lemma 3.32, $\mathcal{A}$ is generated by elements of the form $f_{r_1}f_{r_2}\dots f_{r_{2n}}$ for $r_i \in S$. Then, if $f_tf_{r_1}f_{r_2}\dots f_{r_{2n}} \in f_t\mathcal{A}$, we can write $f_tf_{r_1}f_{r_2}\dots f_{r_{2n}} = (f_tf_{r_1}f_{r_2}\dots f_{r_{2n}}f_t)f_t \in \mathcal{A}f_t$, showing that $f_t\mathcal{A} \subseteq \mathcal{A}f_t$. Containment in the other direction is similar.

We have shown in Lemma 3.30 that $\mathcal{H}_{R,q}(W) = \mathcal{A}_{R,q}(W) \oplus f_t\mathcal{A}_{R,q}(W)$, for any ring R containing $1/2$ and invertible elements $q \neq -1$ and $q+1$. Also $1^{\#} = 1$, so $1 \in \mathcal{A}$. Lastly, we have $(\mathcal{A}f_t)(\mathcal{A}f_t) = \mathcal{A}f_t^2\mathcal{A} = \mathcal{A}\mathcal{A} = \mathcal{A}$, $\mathcal{A}(\mathcal{A}f_t) = \mathcal{A}f_t$ and $(\mathcal{A}f_t)\mathcal{A} = \mathcal{A}\mathcal{A}f_t = \mathcal{A}f_t$. ☐

Let K be an extension field of $\mathbb{C}(q^{1/2})$. Therefore $q \neq -1$, and hence Theorem 4.8 can be applied. Now suppose that V is a $\mathcal{H}_{K,q}(W)$-module. Theorem 1.9 tells us that $V = U + Uf_t$ for some fixed $t \in S$ and some irreducible $\mathcal{A}$-submodule U of $V_{\mathcal{A}}$. Therefore either V is irreducible as an $\mathcal{A} = \mathcal{A}_{K,q}(W)$-module or V is a direct sum of two irreducible $\mathcal{A}$-modules. That is, $V_{\mathcal{A}} = U \oplus Uf_t$. We now begin to describe a criteria that enables us to distinguish between these two cases.

Definition 5.2. *Suppose that V is an $\mathcal{H}_{K,q}(W)$-module. Define a $\mathcal{H}_{K,q}(W)$-module $V^{\#} = \{v^* \mid v \in V\}$ that has the same K-module structure as V but* *with a new action, $\circ$, given by*

$$v^* \circ h := (vh^\#)^*$$

for $v^* \in V^\#$ and $h \in \mathcal{H}$.

Lemma 5.3. *An $\mathcal{H}_{K,q}(W)$-module V is irreducible if and only if $V^\#$ is irreducible.*

Proof. Note that, since $\#$ is an involution, $(V^\#)^\# \cong V$ for any $\mathcal{H}_{K,q}(W)$-module V. So once we have proved one direction of the lemma, the other direction follows.

Let V be an irreducible $\mathcal{H}_{K,q}(W)$-module and suppose that there exists a non-trivial proper submodule U of $V^\#$. Then $U^\#$ is a non-trivial proper submodule of V, contradicting the assumption that V is irreducible. Therefore $V^\#$ is irreducible. □

Lemma 5.4. *Let V be an irreducible $\mathcal{H}_{K,q}(W)$-module and let $V \cong V^\#$ as $\mathcal{H}$-modules. Then $V_{\mathcal{A}} = V^+ \oplus V^-$ for some irreducible $\mathcal{A}$-modules $V^\pm$. Moreover, $V^+ = V^- f_t$ for some $t \in S$.*

Proof. Suppose $V \cong V^\#$ and let θ be an isomorphism from $V^\#$ to V. Define a map

$$\begin{array}{rcl} \#' : V & \to & V \\ v & \mapsto & \theta(v^*). \end{array}$$

We claim that $\#'_{\mathcal{A}}$ is an $\mathcal{A}$-module automorphism of V. Since θ is K-linear, $\#'$ is K-linear. Also, $\#'$ is a bijection because θ is a bijection. Now we need to show that $\#'$ respects the $\mathcal{A}$ action. Let $h \in \mathcal{H}$ and $v \in V$. Then

$$\begin{array}{rcl} \#'(vh) & = & \theta((vh)^*) \\ & = & \theta(v^* \circ h^\#) \\ & = & \theta(v^*)h^\# \\ & = & \#'(v)h^\#. \end{array}$$

So if $h \in \mathcal{A}$ then $\#'$ respects the $\mathcal{A}$-action. That is, $\#'$ is an $\mathcal{A}$-module automorphsim. Moreover $\#'^2$ is an $\mathcal{H}$-module automorphism.

Now $\mathcal{H}$ splits over K so, without loss of generality, we may assume that $\#'^2$ is the identity map on V, by Schur's Lemma. Therefore the only possible eigenvalues for the matrix of $\#'$ are 1 and -1.

Let $V^\pm$ be the ± 1-eigenspaces of $\#'$, respectively. We will show that $V^\pm$ are non-empty irreducible $\mathcal{A}$-submodules of $V_{\mathcal{A}}$. A simple calculation shows that $V^\pm$ are $\mathcal{A}$-submodules of $V_{\mathcal{A}}$. They are irreducible by the discussion following Proposition 5.1. Now the Jordan canonical form of the matrix of $\#'$ must have only 1's or -1's on the diagonal and zeros everywhere else (if there were any non-zero entries off the diagonal then $\#'^2$

would not be the identity map). Hence $\#'$ is diagonalisable and so the set of eigenvectors is a basis of V.

Suppose that $\mathcal{H}$ is associated to a Coxeter system (W, S) and choose a particular $t \in S$. We claim that $V^- f_t = V^+$. Let $v \in V^-$. Then $\#'(vf_t) = \#'(v)f_t^{\#} = -\#'(v)f_t = vf_t$. So $V^- f_t \subseteq V^+$. Now if $v \in V^+$, then $v = (vf_t)f_t$ and $\#'(vf_t) = \#'(v)f_t^{\#} = -\#'(v)f_t = -vf_t$. So $V^+ \subseteq V^- f_t$ and hence $V^+ = V^- f_t$.

Hence $V^{\pm}$ are non-empty $\mathcal{A}$-submodules of $V_{\mathcal{A}}$. So we have $V_{\mathcal{A}} = V^+ \oplus V^-$ where the set of 1–eigenvectors is a basis for V^+ and the set of -1–eigenvectors is a basis for V^-.

□

Lemma 5.5. *Let V be an irreducible $\mathcal{H}_{K,q}(W)$-module and suppose that, for $t \in S$, $V_{\mathcal{A}} = U \oplus U f_t$ for some irreducible $\mathcal{A}$-submodule U of $V_{\mathcal{A}}$. Then $V \cong V^{\#}$ as $\mathcal{H}$-modules.*

Proof. Suppose that $V_{\mathcal{A}} = U \oplus U f_t$, for some irreducible $\mathcal{A}$-submodule U of $V_{\mathcal{A}}$. Then $V = U \oplus U f_t$ as K-modules. We define the following map:

$$\begin{array}{rcl} \Phi : V^{\#} & \to & V \\ (u + u' f_t)^* & \mapsto & u - u' f_t \end{array}$$

for $u, u' \in U$. We claim that this map is a $\mathcal{H}$-isomorphism.

Let $u, u' \in U$ and let $h \in \mathcal{H}$, so $h = a + f_t a'$ for some $a, a' \in \mathcal{A}$, by Proposition 5.1. Now Φ is clearly K-linear and

$$\begin{aligned} \Phi((u + u' f_t)^* \circ h) &= \Phi((u + u' f_t)^* \circ (a + f_t a')) \\ &= \Phi((u + u' f_t)(a - f_t a')) \\ &= \Phi(ua - u f_t a' + u' f_t a - u' a'). \end{aligned}$$

Now, ua and $u'a'$ are in U and, since $\mathcal{A} f_t = f_t \mathcal{A}$, we conclude that $u f_t a'$ and $u' f_t a$ are both in $U f_t$. Therefore

$$\begin{aligned} \Phi((u + u' f_t)^* \circ h) &= ua + u f_t a' - u' f_t a - u' a' \\ &= (u - u' f_t)(a + f_t a') \\ &= \Phi((u + u' f_t)^*)h. \end{aligned}$$

Therefore Φ is a $\mathcal{H}$-homomorphism.

To show that Φ is injective, consider the kernel.

$$\begin{aligned} \mathrm{Ker}(\Phi) &= \{(u + u' f_t)^* \mid u - u' f_t = 0, u, u' \in U\} \\ &= \{(u + u' f_t)^* \mid u = 0, u' f_t = 0, u, u' \in U\} \\ &= \{0^*\}. \end{aligned}$$

For surjectivity, notice that $\Phi((u - u' f_t)^*) = u + u' f_t$.

So we have that Φ is a $\mathcal{H}$-isomorphism. Therefore $V \cong V^{\#}$ as $\mathcal{H}$-modules. $\square$

Theorem 5.6. *Let V be an irreducible $\mathcal{H}_{K,q}(W)$-module. Then $V \cong V^{\#}$ as $\mathcal{H}$-modules if and only if $V_{\mathcal{A}} = V^{+} \oplus V^{-}$ for some irreducible $\mathcal{A}$-submodules $V^{\pm}$ of $V_{\mathcal{A}}$, where $V^{+} = V^{-} f_t$ for some $t \in S$.*

Proof. This result is a direct corollary of Lemmas 5.4 and 5.5. $\square$

5.2. The Irreducibles in Types A_n, B_n and D_n

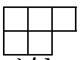
In this section we review the irreducible $\mathcal{H}_{K,q}(W)$-modules for the Coxeter groups of types A_n, B_n and D_n. Then we give the criteria that governs whether they are irreducible as $\mathcal{A}_{K,q}(W)$-modules or if they break up as a direct sum of two irreducible $\mathcal{A}_{K,q}(W)$-modules.

Let χ_q be the character of an irreducible $\mathcal{H}_{K,q}(W)$-module V, and let $\chi_q^{\#}$ be the character of $V^{\#}$. Then let χ_1 be the specialisation achieved by setting $q = 1$ in χ. So the irreducible characters of a Coxeter group W are related to the irreducibles of the Iwahori-Hecke algebra by the following proposition, Geck and Pfeiffer [**6**], Proposition 9.4.1 (b).

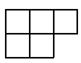
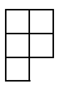
Proposition 5.7. *Let χ_q be an irreducible $\mathcal{H}_{K,q}(W)$-character. Then there is a bijection*

$$\begin{array}{ccc} \mathrm{Irr}\,(\mathcal{H}) & \longleftrightarrow & \mathrm{Irr}\,(W) \\ \chi_q & \leftrightarrow & \chi_1. \end{array}$$

Moreover, $(\chi_q^{\#})_1 = \chi_1 \otimes \varepsilon$, where ε is the sign character of W.

This allows us to relate the hash map $\#$ to the sign representation in the following Lemma. First though, some notation. Suppose that n is a positive integer. Then λ is a **partition** of n if λ is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$ and $\lambda_{i+1} \geq \lambda_i$ for all $1 \leq i \leq k-1$. We write $\lambda \vdash n$ when λ is a partition of n. The **diagram** of λ is the set $\{(r,c) \mid 1 \leq r \leq k, 1 \leq c \leq \lambda_r\}$. Alternatively, we may consider the diagram of λ as an array of boxes where the (r,c)-th box is in the r-th row and c-th column. For example, if $\lambda = (3,2)$, then the diagram of λ is .

Let λ' be the partition which is the transpose of λ. In other words, if the diagram of λ contains the box (i,j) then the diagram of λ' contains the box (j,i). For example, if $\lambda = (3,2)$ then $\lambda' = (2,2,1)$.

$\lambda =$ , $\qquad \lambda' =$ 

Lemma 5.8. *Let V be an irreducible $\mathcal{H}_{K,q}(W)$-module with corresponding character χ_q. Then*

$$V^{\#} \cong V \iff \chi_1 \otimes \varepsilon = \chi_1.$$

Proof. Let V be an irreducible $\mathcal{H}_{K,q}(W)$-module with corresponding character χ_q. Then

$$\begin{aligned} V^{\#} \cong V &\iff \chi_q^{\#} = \chi_q \\ &\iff (\chi_q^{\#})_1 = \chi_1 \\ &\iff \chi_1 \otimes \varepsilon = \chi_1. \end{aligned}$$

□

Case A_n

In the case of the symmetric group, $A_{n-1} = \mathfrak{S}_n$, the irreducible $\mathfrak{S}_n$-modules are parametrised by the partitions of n. Therefore, so are the irreducible $\mathcal{H}_{K,q}(\mathfrak{S}_n)$-modules, see for example Geck and Pfeiffer [**6**, Theorem 8.1.7]. Let $S(\lambda)$ denote a $\mathcal{H}_{K,q}(\mathfrak{S}_n)$-module corresponding to the partition λ, and let χ^λ be the corresponding character. Now Corollary 5.4.9 of Geck and Pfeiffer [**6**] tells us that $\chi_1^\lambda \otimes \varepsilon = \chi_1^{\lambda'}$. Hence

$$\begin{aligned} \lambda = \lambda' &\iff \chi_1^\lambda = \chi_1^{\lambda'} \\ &\iff \chi_1^\lambda = \chi_1^\lambda \otimes \varepsilon \\ &\iff S(\lambda) \cong S(\lambda)^{\#}. \end{aligned}$$

Let $S(\lambda)_{\mathcal{A}}$ be the restriction of $S(\lambda)$ to an $\mathcal{A}$-module. Then we have two cases:

- If $\lambda = \lambda'$ then $S(\lambda)_{\mathcal{A}}$ is a direct sum of two irreducible $\mathcal{A}$-modules.
- If $\lambda \neq \lambda'$ then $S(\lambda)$ is irreducible as an $\mathcal{A}$-module.

Case B_n

The irreducibles in this case are indexed by pairs of partitions λ, μ such that $|\lambda| + |\mu| = n$. Let $S(\lambda, \mu)$ be such an irreducible with corresponding character $\chi^{\lambda,\mu}$. We have $\chi_1^{\lambda,\mu} \otimes \varepsilon = \chi_1^{\mu',\lambda'}$, from Geck and Pfeiffer [**6**, Theorem 5.5.6 (c)]. Therefore

$$\begin{aligned} \lambda = \mu' &\iff \chi_1^{\lambda,\mu} = \chi_1^{\mu',\lambda'} \\ &\iff \chi_1^{\lambda,\mu} = \chi_1^{\lambda,\mu} \otimes \varepsilon \\ &\iff S(\lambda,\mu) \cong S(\lambda,\mu)^{\#}. \end{aligned}$$

Therefore we again have two cases:

- If $\lambda = \mu'$ then $S(\lambda, \mu)_{\mathcal{A}}$ is a direct sum of two irreducible $\mathcal{A}$-modules.

- If $\lambda \neq \mu'$ then $S(\lambda, \mu)$ is irreducible as an $\mathcal{A}$-module.

Case D_n

This case is a little more complicated. Recall that the characters of D_n are indexed in two ways (see, for example, Geck and Pfeiffer [**6**, 5.6.1]), and hence so are the characters of $\mathcal{H}_{K,q}(D_n)$. First let λ and μ be partitions with $|\lambda| + |\mu| = n$. If $\lambda \neq \mu$ we have an irreducible $\mathcal{H}_{K,q}(D_n)$-module, $S(\lambda, \mu)$, for each unordered pair of partitions (λ, μ). Note that since the pair is unordered, $S(\lambda, \mu)$=$S(\mu, \lambda)$. Otherwise, in the case where $\lambda = \mu$ (so n is even), we have two non-isomorphic irreducibles for each $\lambda \vdash n/2$, $S(\lambda, +)$ and $S(\lambda, -)$.

First let us consider the irreducibles $S(\lambda, \mu)$ when $\lambda \neq \mu$. Let $\chi^{\lambda,\mu}$ the corresponding character. Then from Remark 5.6.5 in [**6**] we have that $\chi_1^{\lambda,\mu} \otimes \varepsilon = \chi_1^{\lambda',\mu'}$ where ε is the sign character of D_n. Therefore

$$\begin{aligned} \lambda = \mu' \quad &\iff \quad \chi_1^{\lambda,\mu} = \chi_1^{\mu',\lambda'} \\ &\iff \quad \chi_1^{\lambda,\mu} = \chi_1^{\mu,\lambda} \otimes \varepsilon = \chi_1^{\lambda,\mu} \otimes \varepsilon \\ &\iff \quad S(\lambda, \mu) \cong S(\lambda, \mu)^{\#}. \end{aligned}$$

Now let us consider the case when $\lambda = \mu$. So we have irreducibles $S(\lambda, \pm)$ for each partition $\lambda \vdash n/2$. Let $\chi^{\lambda,\pm}$ be the corresponding characters. In this case we have $\chi_1^{\lambda,\pm} \otimes \varepsilon = \chi_1^{\lambda',\pm}$, by Remark 5.6.5 in [**6**]. Hence

$$\begin{aligned} \lambda = \lambda' \quad &\iff \quad \chi_1^{\lambda,\pm} = \chi_1^{\lambda',\pm} \\ &\iff \quad \chi_1^{\lambda,\pm} = \chi_1^{\lambda,\pm} \otimes \varepsilon \\ &\iff \quad S(\lambda, \pm) \cong S(\lambda, \pm)^{\#}. \end{aligned}$$

In summary,

- If $\lambda \neq \mu$, $\lambda = \mu'$ then $S(\lambda, \mu)_{\mathcal{A}}$ is a direct sum of two irreducible $\mathcal{A}$-modules.
- If $\lambda \neq \mu$, $\lambda \neq \mu'$ then $S(\lambda, \mu)$ is irreducible as an $\mathcal{A}$-module.
- If $\lambda = \lambda'$ (and $\lambda = \mu$) then $S(\lambda, \pm)_{\mathcal{A}}$ both break up as a direct sum of two irreducible $\mathcal{A}$-modules.
- If $\lambda \neq \lambda'$ (and $\lambda = \mu$) then $S(\lambda, \pm)$ are both irreducible as $\mathcal{A}$-modules.

CHAPTER 6

The Characters of $\mathcal{A}_{K,q}(\mathfrak{S}_n)$

In this chapter we start by revising the ordinary irreducible representations of the finite Coxeter groups and the Iwahori-Hecke algebras associated to them. Then we give the characters for the irreducibles of the alternating Hecke algebra of the symmetric group.

6.1. The Characters of $\mathcal{A}_{K,q}(\mathfrak{S}_n)$

In this section we begin to describe the irreducible representations of $\mathcal{A}_{K,q}(\mathfrak{S}_n)$, where K is an extension field of $\mathbb{C}(q^{1/2})$. We will first need some notation.

For $\lambda \vdash n$, a λ-**tableau** is a bijection from the diagram of λ into the set of integers $\{1, 2, \ldots, n\}$. We can consider a λ-tableau as a way of placing the numbers 1 up to n in the boxes of the diagram of λ. For example, for $\lambda = (3, 2)$, the following diagrams are all λ-tableau.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 5&4&3\\\hline 1&2\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 3&2&4\\\hline 5&1\\\cline{1-2}\end{array}$$

If $\mathfrak{t}$ is a λ-tableau, we write $\mathfrak{t}(r, c) = i$ if i appears in row r and column c of $\mathfrak{t}$. A **standard tableau** is one in which the entries in each box increase along each row and column. That is, $\mathfrak{t}(r, c+1) > \mathfrak{t}(r, c)$ and $\mathfrak{t}(r+1, c) > \mathfrak{t}(r, c)$ for all applicable r and c. We write $\mathrm{Std}(\lambda)$ for the set of standard λ-tableaux. The following are examples of standard $(3, 2)$-tableaux.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$$

If $\mathfrak{t}(r, c) = i$ then the **content** of i in $\mathfrak{t}$ is $c_{\mathfrak{t}}(i) = c - r$. The following diagram is a handy reference as each box contains its content.

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&\cdot\\\hline -1&0&1&2&\cdot\\\hline -2&-1&0&1&\cdot\\\hline -3&-2&-1&0&\cdot\\\hline \cdot&\cdot&\cdot&\cdot&\cdot\\\hline\end{array}$$

Given two integers $i < j$ in $\mathfrak{t}$ then the **axial distance** from i to j is

$$d_{\mathfrak{t}}(i, j) := \; c_{\mathfrak{t}}(j) - \; c_{\mathfrak{t}}(i).$$

Suppose that $\lambda \vdash n$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableaux. We will need a partial order on the set of standard λ–tableau. First define $\mathfrak{t}^{\lambda}$ to be the standard λ–tableau in which the integers 1 to n are placed in increasing order from left to right along the rows. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 \\ \cline{1-1}\end{array}\,.$$

Also, define $\mathfrak{t}_\lambda$ to be the λ–tableau in which the integers 1 to n are entered in increasing order from top to bottom along the columns. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array}\,.$$

Define $d(\mathfrak{t})$ to be the unique permutation such that $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$. The partial order, denoted $\trianglerighteq$, is defined as follows. We write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and say that $\mathfrak{s}$ **dominates** $\mathfrak{t}$ if $d(\mathfrak{t}) = d(\mathfrak{s})w$ for some w such that $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{s})) + \ell(w)$. We also write $\mathfrak{s} \triangleright \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$.

Let $s_i = (i, i+1)$ be the permutaion swapping i and $i+1$ and suppose that $\mathfrak{t}s_i \in \mathrm{Std}(\lambda)$. Then $\mathfrak{t} \triangleright \mathfrak{t}s_i$ if and only if i is in a higher row than $i+1$ in $\mathfrak{t}$. For example,

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline & & & \mathrm{i} \\ \hline & & \\ \cline{1-3} & \mathrm{i{+}1} \\ \cline{1-2}\end{array} \qquad \mathfrak{t}s_i = \begin{array}{|c|c|c|c|}\hline & & & \mathrm{i{+}1} \\ \hline & & \\ \cline{1-3} & \mathrm{i} \\ \cline{1-2}\end{array}\,.$$

In fact, $\mathfrak{t}^\lambda$ is the unique maximal standard λ–tableau under this partial order, and $\mathfrak{t}_\lambda$ is the unique minimal standard λ–tableau. See Mathas [**14**].

We define the quantum integers $[d]$, for $d \in \mathbb{Z}$,

$$[d] = \begin{cases} 1 + q + q^2 + \ldots + q^{d-1}, & \text{if } d \geq 0, \\ -q^d[-d], & \text{if } d < 0. \end{cases}$$

Notice that this definition works for any value of q, but it simplifies to $[d] = \frac{q^d-1}{q-1}$ when $q \neq 1$. From the definition we see that, for any integer d,

$$[-d] = -q^{-d}[d].$$

In our discussion of the characters of the alternating Hecke algebra, we will first look at the irreducible representations of $\mathcal{H}_{K,q}(\mathfrak{S}_n)$, the Specht modules. The action in these modules will require that we fix arbitrary square roots $\sqrt{-1}$, $\sqrt{q}$ and $\sqrt{[d]} \in K$ for $d \in \mathbb{N}$.

Definition 6.1. *We define, for $d > 0$,*

$$\sqrt{[-d]} := \sqrt{-1}(\sqrt{q})^{-d}\sqrt{[d]}.$$

For convenience, if $k \in \mathbb{Z}$ we set

$$\alpha(k) = \begin{cases} \frac{\sqrt{-1}\sqrt{q}\sqrt{[k+1]}\sqrt{[k-1]}}{[k]} & \text{if } k > 1 \\ -\alpha(-k) & \text{if } k < 1 \\ 1 & \text{if } k = 0 \end{cases}$$

The case $k = 0$ is included for computation purposes only. We omit the cases where $k = \pm 1$ because they never occur in the Specht module.

Recall that the irreducible representations of $\mathcal{H}_{K,q}(\mathfrak{S}_n)$, the Specht modules, are indexed by partitions of n. The following Lemma describes the Specht module $S(\lambda)$ of the partition $\lambda \vdash n$. It is the corrected version of Theorem 3.36 of Mathas [**14**]. Note that Mathas uses ρ for the axial distance from i to $i+1$ in $\mathfrak{t}$, whereas the axial distance here is defined to be $d_i = -\rho$.

Lemma 6.2. *Let λ be a partition of n. The $\mathcal{H}_{K,q}(\mathfrak{S}_n)$-module $S(\lambda)$ has basis $\{ f_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda) \}$ and , for $s_i \in S$, $T_i := T_{s_i}$ acts as follows:*

$$f_\mathfrak{t} T_i = \begin{cases} q f_\mathfrak{t}, & \text{if } i \text{ and } i+1 \text{ are in the same row of } \mathfrak{t}, \\ -f_t, & \text{if } i \text{ and } i+1 \text{ are in the same column of } \mathfrak{t}, \\ \frac{-1}{[-d_i]} f_\mathfrak{t} + f_{\mathfrak{t}s_i}, & \text{if } \mathfrak{t}s_i \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{t} \rhd \mathfrak{t}s_i, \\ \frac{-1}{[-d_i]} f_\mathfrak{t} + \frac{q[d_i+1][d_i-1]}{[d_i]^2} f_{\mathfrak{t}s_i}, & \text{if } \mathfrak{t}s_i \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{t}s_i \rhd \mathfrak{t}, \end{cases}$$

where $d_i = d_\mathfrak{t}(i, i+1)$ is the axial distance from i to $i+1$ in $\mathfrak{t}$.

Note that i and $i+1$ are in the same row of $\mathfrak{t}$ if and only if $d_i = 1$ (in which case $\frac{-1}{[-d_i]} = q$). Similarly i and $i+1$ are in the same column of $\mathfrak{t}$ if and only if $d_i = -1$ (in which case $\frac{-1}{[-d_i]} = -1$). It follows that the trivial representation of $\mathcal{H}$ corresponds to the case when $\lambda = (n)$ and the sign representation of $\mathcal{H}$ corresponds to the Specht module of $\lambda = (1^n)$.

Also note that $\frac{q[d_i+1][d_i-1]}{[d_i]^2} = -\alpha(d_i)^2$.

We will use the following modified seminormal form which is obtained by rescaling the basis given in Lemma 6.2.

Proposition 6.3. *Let λ be a partition of n. Then $S(\lambda)$ has a basis*

$$\{ v_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda) \}$$

such that

$$v_\mathfrak{t} T_i = \begin{cases} \frac{-1}{[-d_i]} v_\mathfrak{t}, & \text{if } |d_i| = 1, \\ \frac{-1}{[-d_i]} v_\mathfrak{t} + \alpha(d_i) v_{\mathfrak{t}s_i}, & \text{if } |d_i| > 1, \end{cases}$$

where $d_i = d_\mathfrak{t}(i, i+1)$ is the axial distance from i to $i+1$ in $\mathfrak{t}$.

Proof. Fix $\mathfrak{t} \in \mathrm{Std}(\lambda)$ and let $w = s_{i_1} s_{i_2} \dots s_{i_k}$ be the permutation such that $\mathfrak{t} = \mathfrak{t}_\lambda w$, where $s_{i_1} s_{i_2} \dots s_{i_k}$ is a reduced expression for w. We define the standard λ–tableaux $\mathfrak{t}_j := \mathfrak{t}_\lambda s_{i_1} s_{i_2} \dots s_{i_j}$ for $0 \leq j \leq k$. Therefore $\mathfrak{t}_\lambda = \mathfrak{t}_0 \lhd \mathfrak{t}_1 \lhd \dots \lhd \mathfrak{t}_k = \mathfrak{t}$.

Define

$$\alpha(\mathfrak{t}) := \prod_{j=1}^{k} \alpha(d_{\mathfrak{t}_j}(i_j, i_j + 1)).$$

We claim that $\alpha(\mathfrak{t})$ is well-defined. That is, $\alpha(\mathfrak{t})$ is independant of the choice of reduced expression for w. To see this note that

$$f_{\mathfrak{t}_\lambda} T_w = \prod_{j=1}^{k} -\alpha(d_{t_{j-1}}(i_j, i_j+1))^2 f_{\mathfrak{t}} + \sum_{\mathfrak{s} \lhd \mathfrak{t}} r_{\mathfrak{s}} f_{\mathfrak{s}}$$

for some $r_{\mathfrak{s}} \in K$. Now

$$\begin{aligned}\prod_{j=1}^{k} -\alpha(d_{t_{j-1}}(i_j, i_j+1))^2 &= (-1)^k \prod_{j=1}^{k} \alpha(d_{t_{j-1}}(i_j, i_j+1))^2 \\ &= (-1)^k \prod_{j=1}^{k} \left(-\alpha(d_{t_j}(i_j, i_j+1))\right)^2 \\ &= (-1)^k \prod_{j=1}^{k} \alpha(d_{t_j}(i_j, i_j+1))^2.\end{aligned}$$

So $(-1)^k \prod_{j=1}^{k} \alpha(d_{t_j}(i_j, i_j+1))^2$ and hence $\prod_{j=1}^{k} \alpha(d_{t_j}(i_j, i_j+1))^2$ are independant of the choice of reduced expression for w. However, $d_{\mathfrak{t}_j}(i_j, i_j+1)$ is positive for $1 \leq j \leq k$. Therefore $\prod_{j=1}^{k} \alpha(d_{t_j}(i_j, i_j+1))$ is independant of the choice of reduced expression for w. That is, $\alpha(\mathfrak{t})$ is well-defined.

Define a new basis via

$$v_{\mathfrak{t}} := \prod_{j=1}^{k} \alpha(\mathfrak{t}) f_{\mathfrak{t}},$$

for $\mathfrak{t} \in \mathrm{Std}(\lambda)$. Note that

$$\alpha(\mathfrak{t}) = \begin{cases} \alpha(\mathfrak{t}s_i)\alpha(d_i), & \text{if } \mathfrak{t} \rhd \mathfrak{t}s_i \\ -\dfrac{\alpha(\mathfrak{t}s_i)}{\alpha(d_i)}, & \text{if } \mathfrak{t}s_i \rhd \mathfrak{t} \end{cases}$$

where $d_i = d_{\mathfrak{t}}(i, i+1) = -d_{\mathfrak{t}s_i}(i, i+1)$.

Then

$$
\begin{aligned}
v_{\mathfrak{t}}T_i &= \alpha(\mathfrak{t})f_{\mathfrak{t}}T_i \\
&= \begin{cases} q\alpha(\mathfrak{t})f_{\mathfrak{t}} & \text{if } i \text{ and } i+1 \text{ are the same row of } \mathfrak{t} \\ -\alpha(\mathfrak{t})f_{\mathfrak{t}} & \text{if } i \text{ and } i+1 \text{ are the same column of } \mathfrak{t} \\ \frac{-1}{[-d_i]}\alpha(\mathfrak{t})f_{\mathfrak{t}} + \alpha(\mathfrak{t})f_{\mathfrak{t}s_i} & \text{if } \mathfrak{t} \rhd \mathfrak{t}s_i \\ \frac{-1}{[-d_i]}\alpha(\mathfrak{t})f_{\mathfrak{t}} - \alpha(d_i)^2\alpha(\mathfrak{t})f_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \rhd \mathfrak{t} \end{cases} \\
&= \begin{cases} qv_{\mathfrak{t}} & \text{if } i \text{ and } i+1 \text{ are the same row of } \mathfrak{t} \\ -v_{\mathfrak{t}} & \text{if } i \text{ and } i+1 \text{ are the same column of } \mathfrak{t} \\ \dfrac{-1}{[-d_i]}v_{\mathfrak{t}} + \dfrac{\alpha(\mathfrak{t})}{\alpha(\mathfrak{t}s_i)}v_{\mathfrak{t}s_i} & \text{if } \mathfrak{t} \rhd \mathfrak{t}s_i \\ \dfrac{-1}{[-d_i]}v_{\mathfrak{t}} - \dfrac{\alpha(d_i)^2\alpha(\mathfrak{t})}{\alpha(\mathfrak{t}s_i)}v_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \rhd \mathfrak{t} \end{cases} \\
&= \begin{cases} qv_{\mathfrak{t}} & \text{if } i \text{ and } i+1 \text{ are the same row of } \mathfrak{t} \\ -v_{\mathfrak{t}} & \text{if } i \text{ and } i+1 \text{ are the same column of } \mathfrak{t} \\ \frac{-1}{[-d_i]}v_{\mathfrak{t}} + \alpha(d_i)v_{\mathfrak{t}s_i} & \text{if } \mathfrak{t} \rhd \mathfrak{t}s_i \\ \frac{-1}{[-d_i]}v_{\mathfrak{t}} + \alpha(d_i)v_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \rhd \mathfrak{t}. \end{cases}
\end{aligned}
$$

□

We would like to see what happens to $S(\lambda)$ when we restrict it to an $\mathcal{A}_{K,q}(\mathfrak{S}_n)$-module. To this end we need to understand $S(\lambda)^{\#}$, which the following Proposition will enable us to do.

First, if $\mathfrak{t}$ is a λ-tableau, then $\mathfrak{t}'$ is the λ'-tableau with $\mathfrak{t}'(i,j) = \mathfrak{t}(j,i)$. For example, if $\lambda = (3,2)$ and $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$, then $\mathfrak{t}' = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$.

Now define a map $\tau : S(\lambda) \longrightarrow S(\lambda')$ to be the K–linear map given by $v_{\mathfrak{t}} \mapsto v_{\mathfrak{t}'}$, for all $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Note that τ depends on the partition λ; its meaning, however, should always be clear from the context.

Proposition 6.4. *Suppose that* $w \in \mathfrak{S}_n$. *Then* $v_{\mathfrak{t}}T_w^{\#} = \tau(v_{\mathfrak{t}'}T_w)$, *for all* $\mathfrak{t} \in \operatorname{Std}(\lambda)$.

Proof. We first show that $v_{\mathfrak{t}}T_i^{\#} = \tau(v_{\mathfrak{t}'}T_i)$ for $1 \le i \le n-1$. Consider the case when $|d_i| = |d_{\mathfrak{t}}(i,i+1)| = 1$. We have

$$\begin{aligned} v_{\mathfrak{t}}T_i^{\#} &= v_{\mathfrak{t}}(-T_i + q - 1) \\ &= \Big(\frac{1}{[-d_i]} + q - 1\Big)v_{\mathfrak{t}} \\ &= \frac{q^{-d_i}}{[-d_i]}v_{\mathfrak{t}} \\ &= \frac{q^{-d_{\mathfrak{t}}(i,i+1)}}{[-d_{\mathfrak{t}}(i,i+1)]}v_{\mathfrak{t}}. \end{aligned}$$

Now $-d_{\mathfrak{t}}(i,i+1) = d_{\mathfrak{t}'}(i,i+1)$, so

$$\begin{aligned} v_{\mathfrak{t}}T_i^{\#} &= \frac{q^{d_{\mathfrak{t}'}(i,i+1)}}{[d_{\mathfrak{t}'}(i,i+1)]}v_{\mathfrak{t}} \\ &= -\frac{1}{[-d_{\mathfrak{t}'}(i,i+1)]}v_{\mathfrak{t}} \end{aligned}$$

as required.

The case when $|d_i| = |d_{\mathfrak{t}}(i,i+1)| > 1$ is similar:

$$v_{\mathfrak{t}}T_i^{\#} = v_{\mathfrak{t}}(-T_i+q-1) = \Big(\frac{1}{[-d_i]}+q-1\Big)v_{\mathfrak{t}}-\alpha(d_i)v_{\mathfrak{t}s_i} = -\frac{1}{[d_i]}v_{\mathfrak{t}}+\alpha(-d_i)v_{\mathfrak{t}s_i}$$

where we have used the fact that $\alpha(-k) = -\alpha(k)$.

Now, let $w = r_1r_2\dots r_k$ be a reduced expression for $w \in \mathfrak{S}_n$. We will use induction on the length of w.

$$\begin{aligned} v_{\mathfrak{t}}T_w^{\#} &= v_{\mathfrak{t}}T^{\#}_{r_1\dots r_{k-1}}T^{\#}_{r_k} \\ &= \tau\big(v_{\mathfrak{t}'}T_{r_1\dots r_{k-1}}\big)T^{\#}_{r_k} \\ &= \tau\Big(\sum_{\mathfrak{s}\in\mathrm{Std}(\lambda')}\gamma_{\mathfrak{s}}v_{\mathfrak{s}}\Big)T^{\#}_{r_k} \\ &= \sum_{\mathfrak{s}\in\mathrm{Std}(\lambda')}\gamma_{\mathfrak{s}}v_{\mathfrak{s}'}T^{\#}_{r_k} \\ &= \sum_{\mathfrak{s}\in\mathrm{Std}(\lambda')}\gamma_{\mathfrak{s}}\tau(v_{\mathfrak{s}}T_{r_k}) \\ &= \tau\Big(\sum_{\mathfrak{s}\in\mathrm{Std}(\lambda')}\gamma_{\mathfrak{s}}v_{\mathfrak{s}}T_{r_k}\Big) \\ &= \tau\big(v_{\mathfrak{t}'}T_{r_1r_2\dots r_{k-1}}T_{r_k}\big) \\ &= \tau(v_{\mathfrak{t}'}T_w) \end{aligned}$$

where the $\gamma_{\mathfrak{s}}$ depend on $\mathfrak{t}'$ and $r_1r_2\dots r_{k-1}$. □

Consequently, the map τ is a $\mathcal{A}_{K,q}(\mathfrak{S}_n)$-module isomorphism.

Corollary 6.5. *Let λ be a partition. Then $\tau : S(\lambda) \longrightarrow S(\lambda')$ is an isomorphism of $\mathcal{A}$–modules.*

Proof. Since we defined τ to be K-linear, we only need to show that τ respects the $\mathcal{A}$–action. Suppose that $a \in \mathcal{A}$ and let $\mathfrak{t} \in \mathrm{Std}(\lambda)$. Then, by Proposition 6.4 we have $\tau(v_{\mathfrak{t}}a) = v_{\mathfrak{t}'}a^{\#} = v_{\mathfrak{t}'}a = \tau(v_{\mathfrak{t}})a.$ □

We will see that $S(\lambda)^{\#} \cong S(\lambda')$ as $\mathcal{H}_{K,q}(W)$-modules. So to investigate the characters of the alternating Hecke algebra we will investigate how the characters of $S(\lambda')$ behave.

Proposition 6.6. *The $\mathcal{H}_{K,q}(\mathfrak{S}_n)$-modules $S(\lambda)^{\#}$ and $S(\lambda')$ are naturally isomorphic via the map*

$$\begin{aligned} \theta : S(\lambda)^{\#} &\to S(\lambda') \\ v_{\mathfrak{t}}^{*} &\mapsto v_{\mathfrak{t}'}. \end{aligned}$$

Proof. Define a K-linear map $\theta : S(\lambda)^{\#} \longrightarrow S(\lambda')$ sending $v_{\mathfrak{t}}^{*} \mapsto v_{\mathfrak{t}'}$. Then for $h \in \mathcal{H}$ we have

$$\begin{aligned} \theta(v_{\mathfrak{t}}^{*} \circ h) &= \theta((v_{\mathfrak{t}}h^{\#})^{*}) \\ &= \theta(\tau(v_{\mathfrak{t}'}h)^{*}) \\ &= \theta\Big(\tau\Big(\sum_{\mathfrak{s}' \in \mathrm{Std}(\lambda')} \gamma_{\mathfrak{s}'}v_{\mathfrak{s}'}\Big)^{*}\Big) \\ &= \theta\Big(\Big(\sum_{\mathfrak{s}' \in \mathrm{Std}(\lambda')} \gamma_{\mathfrak{s}'}v_{\mathfrak{s}}\Big)^{*}\Big) \\ &= \sum_{\mathfrak{s}' \in \mathrm{Std}(\lambda')} \gamma_{\mathfrak{s}'}v_{\mathfrak{s}'} \\ &= v_{\mathfrak{t}'}h \\ &= \theta(v_{\mathfrak{t}}^{*})h \end{aligned}$$

for some $\gamma_{\mathfrak{s}'} \in K$ depending on $\mathfrak{t}'$ and h. □

Let χ^{λ} be the character of the $\mathcal{H}$-module $S(\lambda)$. Then the next result follows immediately from Proposition 6.6.

Corollary 6.7. *Suppose that $w \in \mathfrak{S}_n$. Then $\chi^{\lambda}(T_w^{\#}) = \chi^{\lambda'}(T_w)$.*

6.2. The case in which $S(\lambda)$ and $S(\lambda)^{\#}$ are inequivalent $\mathcal{H}$-modules

Let us consider the case when $\lambda \neq \lambda'$. Since the irreducible $\mathcal{H}_{K,q}(\mathfrak{S}_n)$-modules are indexed by the partitions of n and they are pairwise non-isomophic then $S(\lambda)$ and $S(\lambda)^{\#}$ are not isomorphic as $\mathcal{H}$-modules. Let

$S(\lambda)_{\mathcal{A}}$ be the restriction of the Specht module $S(\lambda)$ to an $\mathcal{A}_{K,q}(\mathfrak{S}_n)$-module, and let $\chi^\lambda_{\mathcal{A}}$ be the corresponding restricted character.

In Section 5.2, we saw that $S(\lambda)$ is irreducible as an $\mathcal{A}$-module. Therefore $\chi^\lambda_{\mathcal{A}}(a) = \chi^\lambda(a)$ for all $a \in \mathcal{A}_{K,q}(\mathfrak{S}_n)$.

Proposition 6.8. *Suppose* $\lambda \neq \lambda'$ *and* $w \in \mathscr{A}(W)$. *Then*

$$\chi^\lambda_{\mathcal{A}}(B_w) = \frac{1}{2}(\chi^\lambda(T_w) + \chi^{\lambda'}(T_w)).$$

Proof. Let $w \in \mathscr{A}(W)$. Then

$$\begin{aligned}\chi^\lambda_{\mathcal{A}}(B_w) &= \frac{1}{2}\left(\chi^\lambda_{\mathcal{A}}(T_w) + \chi^\lambda_{\mathcal{A}}(T_w^{\#})\right) \\ &= \frac{1}{2}\left(\chi^\lambda(T_w) + \chi^\lambda(T_w^{\#})\right) \\ &= \frac{1}{2}\left(\chi^\lambda(T_w) + \chi^{\lambda'}(T_w)\right) \quad \text{by Corollary 6.7.}\end{aligned}$$

□

6.3. The case in which $S(\lambda)$ and $S(\lambda)^{\#}$ are equivalent $\mathcal{H}$-modules

For the rest of this chapter we will consider the case when $\lambda = \lambda'$. Proposition 6.6 tells us that $S(\lambda)$ and $S(\lambda)^{\#}$ are isomorphic $\mathcal{H}$-modules. From Section 5.1, we see that $S(\lambda)$ breaks up into a direct sum of two irreducible $\mathcal{A}$-modules $S(\lambda)^{\pm}$, where $S(\lambda)^{\pm}$ are the ± 1-eigenspaces under the map $\#'$ (see the proof of Lemma 5.4). Using θ from the proof of Proposition 6.6 we find that $\#' = \tau$ (where τ is defined in Section 6.1). Hence we need to describe the ± 1-eigenspaces of τ.

Define $\mathrm{Std}^+(\lambda)$ to be the set of standard λ-tableaux which have 2 in their first row. Also, let $B^{\pm} := \{v_{\mathfrak{t}} \pm v_{\mathfrak{t}'} \mid \mathfrak{t} \in \mathrm{Std}^+(\lambda)\}$. Then for each $\mathfrak{t} \in \mathrm{Std}^+(\lambda)$, we have $v_{\mathfrak{t}} = \frac{1}{2}(v_{\mathfrak{t}} + v_{\mathfrak{t}'}) + \frac{1}{2}(v_{\mathfrak{t}} - v_{\mathfrak{t}'})$ and $v_{\mathfrak{t}'} = \frac{1}{2}(v_{\mathfrak{t}} + v_{\mathfrak{t}'}) - \frac{1}{2}(v_{\mathfrak{t}} - v_{\mathfrak{t}'})$. Using this fact it follows that $B^+ \cup B^-$ is a basis for $S(\lambda)$, and we have the following result.

Proposition 6.9. *The Specht module,* $S(\lambda)$, *has the following* $\mathcal{A}_{K,q}(W)$-*module decomposition.*

$$S(\lambda)_{\mathcal{A}} = S(\lambda)^+ \oplus S(\lambda)^-$$

Moreover, the sets $B^{\pm}$ *are bases for* $S(\lambda)^{\pm}$, *respectively.*

Proof. The $\mathcal{A}$-module decomposition comes from the preceding discussion and Lemma 5.4. Let $\mathfrak{t} \in \mathrm{Std}^+(\lambda)$. Then $\tau(v_{\mathfrak{t}} + v_{\mathfrak{t}'}) = v_{\mathfrak{t}'} + v_{\mathfrak{t}}$ and $\tau(v_{\mathfrak{t}} - v_{\mathfrak{t}'}) = -v_{\mathfrak{t}'} + v_{\mathfrak{t}} = -(v_{\mathfrak{t}} - v_{\mathfrak{t}'})$. That is, each $v_{\mathfrak{t}} \pm v_{\mathfrak{t}'}$ sits inside the ± 1-eigenspace of τ, respectively. Therefore the sets $B^{\pm}$ defined above are bases for $S(\lambda)^{\pm}$ respectively. □

Let $\chi_{\mathcal{A}}^{\lambda\pm}$ be the characters of the $\mathcal{A}$-modules $S(\lambda)^{\pm}$, respectively. We want to compute $\chi_{\mathcal{A}}^{\lambda\pm}(a)$, for $a \in \mathcal{A}$. The next result shows that we can reduce the calculation of these characters to what is essentially a computation inside $\mathcal{H}$. Before we can state this result we need some more notation.

If $h \in \mathcal{H}$ then right multiplication by h gives a K–endomorphism ρ_h of $S(\lambda)$; let $h \circ \tau = \rho_h \circ \tau$ be the K–endomorphism obtained by composing this map with τ. Then $\chi^\lambda(h \circ \tau)$ is the trace of $\rho_h \circ \tau$ acting on $S(\lambda)$.

Proposition 6.10. *Suppose that* $a \in \mathcal{A}$. *Then*

$$\chi_{\mathcal{A}}^{\lambda\pm}(a) = \frac{1}{2}\big(\chi^\lambda(a) \pm \chi^\lambda(a \circ \tau)\big).$$

Proof. First note that since $S(\lambda) = S(\lambda)^+ \oplus S(\lambda)^-$, we must have

$$\chi^\lambda(a) = \chi^{\lambda+}(a) + \chi^{\lambda-}(a). \tag{6.11}$$

By ordering the basis $B^+ \cup B^-$ of $S(\lambda)$ such that the basis elements B^+ are first, followed by the basis elements B^-, then τ acts via the following matrix.

$$\begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && \ddots &&&& \\ &&& 1 &&& \\ &&&& -1 && \\ &&&&& \ddots & \\ &&&&&& -1 \end{pmatrix}$$

Therefore, since $\chi^{\lambda\pm}(a \circ \tau)$ is the trace of the matrix of $a \circ \tau$, we have that $\chi^{\lambda\pm}(a \circ \tau) = \pm\chi^{\lambda+}(a)$. Hence

$$\begin{aligned} \chi^\lambda(a \circ \tau) &= \chi^{\lambda+}(a \circ \tau) + \chi^{\lambda-}(a \circ \tau) \\ &= \chi^{\lambda+}(a) - \chi^{\lambda-}(a). \end{aligned} \tag{6.12}$$

Then combining equations 6.11 and 6.12, we have $\chi_{\mathcal{A}}^{\lambda\pm}(a) = \frac{1}{2}\big(\chi^\lambda(a) \pm \chi^\lambda(a \circ \tau)\big)$ as required. □

Recall that for $w \in \mathscr{A}(W)$ we have $B_w = \frac{1}{2}(T_w + T_w^{\#})$, and that $\{\, B_w \mid w \in \mathscr{A}(\mathfrak{S}_n) \,\}$ is a basis of $\mathcal{A}_{K,q}(\mathfrak{S}_n)$. So we need to calculate the values $\chi_{\mathcal{A}}^{\pm\lambda}(B_w)$ for each $\lambda \vdash n$ such that $\lambda = \lambda'$ and each $w \in \mathscr{A}(\mathfrak{S}_n)$.

Corollary 6.13. *Suppose that* $w \in \mathscr{A}(\mathfrak{S}_n)$. *Then*

$$\chi_{\mathcal{A}}^{\lambda\pm}(B_w) = \frac{1}{2}\big(\chi^\lambda(T_w) \pm \chi^\lambda(T_w \circ \tau)\big).$$

Proof. By Proposition 6.10 and Corollary 6.7,

$$\begin{aligned}\chi_{\mathcal{A}}^{\lambda\pm}(B_w) &= \frac{1}{2}\chi_{\mathcal{A}}^{\lambda\pm}(T_w + T_w^{\#}) \\ &= \frac{1}{4}\big(\chi^\lambda(T_w + T_w^{\#}) \pm \chi^\lambda((T_w + T_w^{\#}) \circ \tau)\big) \\ &= \frac{1}{2}\chi^\lambda(T_w) \pm \frac{1}{4}\big(\chi^\lambda(T_w \circ \tau) + \chi^\lambda(T_w^{\#} \circ \tau)\big).\end{aligned}$$

We will now show that $\chi^\lambda(T_w \circ \tau) = \chi^\lambda(T_w^{\#} \circ \tau)$. Then the result follows.

Write $v_{\mathfrak{t}}T_w = \sum_{\mathfrak{s}\in\mathrm{Std}(\lambda)} a_{\mathfrak{ts}}v_{\mathfrak{s}}$, for some $a_{\mathfrak{ts}}$ (which depend on w). Then $v_{\mathfrak{t}}(T_w \circ \tau) = \sum_{\mathfrak{s}} a_{\mathfrak{ts}}v_{\mathfrak{s}'}$. Consequently, $\chi^\lambda(T_w \circ \tau) = \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} a_{\mathfrak{tt}'}$.

From Proposition 6.4 we see that $v_{\mathfrak{t}}(T_w^{\#}\circ\tau) = v_{\mathfrak{t}'}T_w = \sum_{\mathfrak{s}\in\mathrm{Std}(\lambda')} a_{\mathfrak{t}'\mathfrak{s}}v_{\mathfrak{s}}$. So $\chi^\lambda(T_w^{\#} \circ \tau) = \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} a_{\mathfrak{tt}'} = \chi^\lambda(T_w \circ \tau)$. □

By Geck and Pfeiffer [**6**, Theorem 10.2.5] the character values $\chi^\lambda(T_w)$ are known for $w \in \mathfrak{S}_n$, so we are reduced to computing $\chi^\lambda(T_w\circ\tau)$, for $w \in \mathscr{A}(\mathfrak{S}_n)$. To this end, for each standard λ–tableau $\mathfrak{t}$ define $\gamma_{\mathfrak{t}}(w)$ to be the coefficient of $v_{\mathfrak{t}'}$ in $v_{\mathfrak{t}}T_w$. Then $\chi^\lambda(T_w \circ \tau) = \sum_{\mathfrak{t}} \gamma_{\mathfrak{t}}(w)$. Thus, to determine the characters of $S(\lambda)^\pm$ it is sufficient to determine the polynomials $\gamma_{\mathfrak{t}}(w)$, for $\mathfrak{t} \in \mathrm{Std}(\lambda)$ and $w \in \mathscr{A}(\mathfrak{S}_n)$.

Write $v_{\mathfrak{t}}T_w = \sum_{\mathfrak{s}} a_{\mathfrak{ts}}v_{\mathfrak{s}}$, for some $a_{\mathfrak{ts}}$ (which depend on w). So $v_{\mathfrak{t}'}T_w = \sum_{\mathfrak{s}} a_{\mathfrak{t}'\mathfrak{s}}v_{\mathfrak{s}}$. Therefore $\chi^{\lambda'}(T_w) = \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} a_{\mathfrak{t}'\mathfrak{t}'}$. Also, by Proposition 6.4, $v_{\mathfrak{t}}T_w^{\#} = \tau(v_{\mathfrak{t}'}T_w) = \sum_{\mathfrak{s}} a_{\mathfrak{t}'\mathfrak{s}}v_{\mathfrak{s}'}$. Therefore $\chi^\lambda(T_w^{\#}) = \sum_{\mathfrak{t}\in\mathrm{Std}(\lambda)} a_{\mathfrak{t}'\mathfrak{t}'}$. Hence $\chi^\lambda(T_w^{\#}) = \chi^{\lambda'}(T_w)$

Suppose that $\mathfrak{t}$ is a λ–tableau. Then we write $[\lambda]$ for the diagram of λ. Also, let $\mathrm{diag}(\mathfrak{t})$ be the set of integers $\{i \mid i = \mathfrak{t}(r,r) \text{ for some } (r,r) \in [\lambda]\}$.

Definition 6.14 (Headley [**8**])**.** *Suppose that $w \in \mathfrak{S}_n$ and that $\mathfrak{t}$ is a standard λ–tableau. Then $\mathfrak{t}$ is w–**transposable** if whenever $(r,c) \in [\lambda]$ with $r \neq c$ then*

- $\mathfrak{t}(r,c) = \mathfrak{t}(c,r) \pm 1$ *and*
- $\mathfrak{t}(r,c)$ *and* $\mathfrak{t}(c,r)$ *are in the same w–orbit.*

We will say that the integers i, j in $\mathfrak{t}$ are **diagonally opposite** if $\mathfrak{t}(r,c) = i$ and $\mathfrak{t}(c,r) = j$, for some $(r,c) \in [\lambda]$.

Suppose that $\mathfrak{t}$ is a w–transposable tableau and that $i \notin \mathrm{diag}(\mathfrak{t})$, for some i with $1 \leq i \leq n$. Then $c_{\mathfrak{t}}(i) = -c_t(i \pm 1)$; that is, i is diagonally opposite to either $i+1$ or $i-1$ in $\mathfrak{t}$. Consequently, the cases in the next definition are mutually exclusive (and exhaustive).

Definition 6.15. *Suppose that $\mathfrak{t}$ is a w–transposable standard λ–tableau where w has a reduced expression of the form $w = s_{i_1} \dots s_{i_k}$, with $1 \leq i_1 < \dots < i_k < n$. For $j = 1, \dots, k$ define*

$$\gamma_{\mathfrak{t}}(i_j) = \begin{cases} \frac{-1}{[-d_j]}, & \text{if } i_j \in \operatorname{diag}(\mathfrak{t}), \\ \frac{-1}{[-d'_j]}, & \text{if } c_{\mathfrak{t}}(i_j) = -c_{\mathfrak{t}}(i_j - 1), \\ \alpha(d_j), & \text{if } c_{\mathfrak{t}}(i_j) = -c_{\mathfrak{t}}(i_j + 1), \end{cases}$$

where $d_j = c_{\mathfrak{t}}(i_j + 1) - c_{\mathfrak{t}}(i_j)$ and $d'_j = c_{\mathfrak{t}}(i_j + 1) - c_{\mathfrak{t}}(i_j - 1)$.

The following example illustrates where this definition comes from and will be helpful in understanding the proof of the following proposition.

Example 6.16. Consider $w = (1, 2, 3, 4, 5, 6) = s_1 s_2 s_3 s_4 s_5 \in \mathfrak{S}_6$ acting on $S(\lambda)$ where $\lambda = (3, 2, 1)$.

Let $\mathfrak{t} = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$. Then $\mathfrak{t}$ is w–transposable. If we wish to find $\gamma_{\mathfrak{t}}(w)$, we need to find the coefficient of $v_{\mathfrak{t}'}$ in $v_{\mathfrak{t}} T_w$. In the following calculations we need only follow the path that leads us to $v_{\mathfrak{t}'}$. That is, we must use the fact that $\mathfrak{t}' = \mathfrak{t}(23)(45) = \mathfrak{t} s_2 s_4$.

Let $\mathfrak{t}_1 = \mathfrak{t}$, $\mathfrak{t}_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$, $\mathfrak{t}_3 = \mathfrak{t}_2$, $\mathfrak{t}_4 = \mathfrak{t}' = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$, and $\mathfrak{t}_5 = \mathfrak{t}_4$. Then, by using the action from Proposition 6.3, we have

$$\begin{aligned} v_{\mathfrak{t}} T_w &= v_{\mathfrak{t}} T_{s_1} T_{s_2} T_{s_3} T_{s_4} T_{s_5} \\ &= -v_{\mathfrak{t}_1} T_{s_2} T_{s_3} T_{s_4} T_{s_5} \\ &= \dots + (-1)\alpha(2) v_{\mathfrak{t}_2} T_{s_3} T_{s_4} T_{s_5} + \dots \\ &= \dots + (-1)\alpha(2) \Big(\frac{-1}{[-3]} \Big) v_{\mathfrak{t}_3} T_{s_4} T_{s_5} + \dots \\ &= \dots + (-1)\alpha(2) \Big(\frac{-1}{[-3]} \Big) \alpha(-4) v_{\mathfrak{t}_4} T_{s_5} + \dots \\ &= \dots + (-1)\alpha(2) \Big(\frac{-1}{[-3]} \Big) \alpha(-4)\alpha(-2) v_{\mathfrak{t}_5} + \dots \end{aligned}$$

which coincides with our definition, where $\gamma_{\mathfrak{t}}(i_j)$ is the coefficient that is contributed by the term $T_{s_{i_j}}$.

Using the notation in Definition 6.15 we can give an explicit formula for $\gamma_{\mathfrak{t}}(w)$, for certain w.

Proposition 6.17. *Suppose that $w = s_{i_1}\dots s_{i_k}$ is reduced, where $1 \le i_1 < \dots < i_k < n$. Let $\mathfrak{t}$ be a standard λ–tableau. Then*

$$\gamma_{\mathfrak{t}}(w) = \begin{cases} \gamma_{\mathfrak{t}}(i_1)\dots\gamma_{\mathfrak{t}}(i_k), & \text{if } \mathfrak{t} \text{ is } w\text{–transposable,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Observe that if $\gamma_{\mathfrak{t}}(w) \neq 0$ then there is a sequence $\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_k$ of (not necessarily distinct) standard λ–tableaux such that $\mathfrak{t}_0 = \mathfrak{t}$, $\mathfrak{t}_k = \mathfrak{t}'$ and $v_{\mathfrak{t}_j}$ appears with non–zero coefficient in $v_{\mathfrak{t}_{j-1}}T_{i_j}$, for $j = 1, \dots, k$. By assumption, $1 \le i_1 < \dots < i_k < n$ so $\gamma_{\mathfrak{t}}(w) \neq 0$ only if we can change $\mathfrak{t}$ into $\mathfrak{t}'$ by swapping entries of the form i_j and $i_j + 1$, where $1 \le j \le k$. Hence, if $\gamma_{\mathfrak{t}}(w) \neq 0$, then $\mathfrak{t}$ is w–transposable. That is, if $\mathfrak{t}$ is not w–transposable then $\gamma_{\mathfrak{t}}(w) = 0$.

Suppose now that $\gamma_{\mathfrak{t}}(w) \neq 0$. Then the tableaux $\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_k$ above are uniquely determined because we must follow the unique path from $\mathfrak{t}$ to $\mathfrak{t}'$. Indeed, $\mathfrak{t}_j$ is the unique standard λ–tableau which has all of the numbers greater than $i_j + 1$ in the same positions as they occur in $\mathfrak{t}$ and all numbers less than or equal to $i_j + 1$ in the same positions as in $\mathfrak{t}'$. It follows that if $v_{\mathfrak{t}_j}$ appears with coefficient γ_j in $v_{\mathfrak{t}_{j-1}}T_{i_j}$ then $\gamma_{\mathfrak{t}}(w) = \gamma_1 \dots \gamma_k$. To complete the proof it remains to show that $\gamma_{\mathfrak{t}}(i_j) = \gamma_j$ is the coefficient of $v_{\mathfrak{t}_j}$ in $v_{\mathfrak{t}_{j-1}}T_{i_j}$. There are three possibilities: either i_j is on the diagonal of $\mathfrak{t}$, or i_j is diagonally opposite either $i_j - 1$ or $i_j + 1$.

If $i_j \in \operatorname{diag}(\mathfrak{t})$ then i_j and i_j+1 have not been moved anywhere. That is, they are still in their $\mathfrak{t}$ position. Also, $\mathfrak{t}_j = \mathfrak{t}_{j-1}$ since we don't want to move i_j off the diagonal. Therefore, from the definitions, $\gamma_j = \frac{-1}{[-d_{i_j}]} = \gamma_{\mathfrak{t}}(i_j)$, where $d_{i_j} = c_{\mathfrak{t}}(i_j + 1) - c_{\mathfrak{t}}(i_j)$.

Next, i_j is diagonally opposite $i_j - 1$ if and only if $c_{\mathfrak{t}}(i_j) = -c_{\mathfrak{t}}(i_j - 1)$. In this case, since we are acting by T_{i_j}, $i_j - 1$ must already be in its $\mathfrak{t}'$ position and hence so must i_j. That is, they must have already been swapped. Therefore $\mathfrak{t}_{j-1} = \mathfrak{t}_{j-2}s_{i_j-1}$ and s_{i_j-1} appears in w, ie. $i_{j-1} = i_j - 1$. Now since i_j and $i_j - 1$ have already been swapped we have $d_{\mathfrak{t}_{j-1}}(i_j, i_j + 1) = d_{\mathfrak{t}}(i_j - 1, i_j + 1)$. There fore, $\gamma_j = \frac{-1}{[-d'_{i_j}]} = \gamma_{\mathfrak{t}}(i_j)$.

Finally, i_j and $i_j + 1$ are diagonally opposite in $\mathfrak{t}$ if and only if $c_{\mathfrak{t}}(i_j) = -c_{\mathfrak{t}}(i_j + 1)$. In this case we have $\mathfrak{t}_j = \mathfrak{t}_{j-1}s_{i_j}$, so $\gamma_j = \alpha(d_j)$, where $d_j = c_{\mathfrak{t}}(i_j + 1) - c_{\mathfrak{t}}(i_j)$. Hence, $\gamma(j) = \gamma_{\mathfrak{t}}(i_j)$ as required. □

Recall that the conjugacy classes of $\mathfrak{S}_n$ are indexed by partitions of n, and that for each $\kappa \vdash n$ we can choose a representative of the form in the previous proposition. So for each partition κ we fix a conjugacy class representative $w_\kappa \in \mathfrak{S}_n$ which is of minimal length in its conjugacy class together with a reduced expression of the form $w_\kappa = s_{i_1}\dots s_{i_k}$ such that $1 \le i_1 < i_2 < \dots < i_k < n$.

For example, if $\kappa = (4,2,1) \vdash 7$ we can choose

$$\begin{aligned} w_\kappa &= (1,2,3,4)(5,6)(7) \\ &= s_1 s_2 s_3 s_5. \end{aligned}$$

Proposition 6.17 tells us that we need only consider w-transposable tableaux in what follows. Let $\mathrm{Std}_\kappa(\lambda)$ be the set of w_κ–transposable standard λ–tableaux.

Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n. We define the i^{th} **hook** of λ to be the subset of the diagram of λ that contains the elements $(i,i), (i,i+1), \ldots, (i,\lambda_i)$ and the elements $(i+1,i), (i+2,i), \ldots, (\lambda_i, i)$. We can now compute $\chi^\lambda(T_{w_{(n)}} \circ \tau)$ in the special case where λ is a hook.

Lemma 6.18. *Suppose that* $n = 2k+1$, $\lambda = (k+1, 1^k)$. *Then*

$$\chi^\lambda(T_{w_{(n)}} \circ \tau) = \big(-\sqrt{-1}\sqrt{q}\big)^{\frac{n-1}{2}} \sqrt{[n]}.$$

Proof. We argue by induction on k. When $k = 0$ then $n = 1$, $w_{(n)} = 1$ and $\chi^\lambda(T_{w_{(n)}} \circ \tau) = 1$, so the result holds.

For the inductive step, let $\mu = (k, 1^{k-1}) \vdash n-2$. Hence, $\chi^\mu(T_{w_{(n-2)}} \circ \tau)$ is known by induction.

Notice that there are exactly 2^k $w_{(n)}$–transposable λ–tableaux. These are the λ–tableaux which have the numbers $2i$ and $2i+1$ in positions $(1, i+1)$ and $(i+1, 1)$, for $i = 1, \ldots, k$. Consequently, from each $w_{(n-2)}$–transposable μ–tableau $\mathfrak{t}$ we can construct two $w_{(n)}$–transposable λ–tableaux $\mathfrak{t}'$ and $\mathfrak{t}''$. We construct $\mathfrak{t}'$ by putting $n-1$ and n into the $(1,k)$ and $(k,1)$ positions, respectively, and $\mathfrak{t}'' = \mathfrak{t}' s_{n-1}$. Further, since $w_{(n)} = w_{(n-2)} s_{n-2} s_{n-1}$, we have that $\gamma_{\mathfrak{t}'}(i) = \gamma_{\mathfrak{t}}(i) = \gamma_{\mathfrak{t}''}(i)$, for $1 \le i \le n-3$. Now, if $n-2$ is in the first row of $\mathfrak{t}$ then, by Lemma 6.17,

$$\begin{aligned}
&\gamma_{\mathfrak{t}'}(w_{(n)}) + \gamma_{\mathfrak{t}''}(w_{(n)}) \\
&= \gamma_{\mathfrak{t}}(1) \ldots \gamma_{\mathfrak{t}}(n-3) \Big(\tfrac{-1}{[-(n-2)]} \alpha(-(n-1)) + \alpha(-(n-1)) \Big) \\
&= \gamma_{\mathfrak{t}}(1) \ldots \gamma_{\mathfrak{t}}(n-3) \Big(\tfrac{1}{[-(n-2)]} \alpha(n-1) - \alpha(n-1) \Big) \\
&= \gamma_{\mathfrak{t}}(1) \ldots \gamma_{\mathfrak{t}}(n-3) \alpha(n-1) \Big(\tfrac{1}{[-(n-2)]} - 1 \Big) \\
&= -\gamma_{\mathfrak{t}}(1) \ldots \gamma_{\mathfrak{t}}(n-3) \alpha(n-1) \tfrac{[n-1]}{[n-2]} \\
&= -\gamma_{\mathfrak{t}}(1) \ldots \gamma_{\mathfrak{t}}(n-3) \tfrac{\sqrt{-1}\sqrt{q}\sqrt{[n]}}{\sqrt{[n-2]}} \\
&= -\gamma_{\mathfrak{t}}(w_{(n-2)}) \cdot \frac{\sqrt{-1}\sqrt{q}\sqrt{[n]}}{\sqrt{[n-2]}}.
\end{aligned}$$

A similar calculation shows that we also have

$$\gamma_{\mathfrak{t}'}(w_{(n)}) + \gamma_{\mathfrak{t}''}(w_{(n)}) = -\gamma_{\mathfrak{t}}(w_{(n-2)}) \cdot \frac{\sqrt{-1}\sqrt{q}\sqrt{[n]}}{\sqrt{[n-2]}}$$

when $n-2$ is in the first column of $\mathfrak{t}$. Hence,

$$\begin{aligned}
\chi^\lambda(T_{w_{(n)}} \circ \tau) &= \sum_{\mathfrak{s}\in\mathrm{Std}_{(n)}(\lambda)} \gamma_{\mathfrak{s}}(w_{(n)}) = \sum_{\mathfrak{t}\in\mathrm{Std}_{(n-2)}(\mu)} \Big(\gamma_{\mathfrak{t}'}(w_{(n)}) + \gamma_{\mathfrak{t}''}(w_{(n)})\Big)\\
&= -\frac{\sqrt{-1}\sqrt{q}\sqrt{[n]}}{\sqrt{[n-2]}} \sum_{\mathfrak{t}\in\mathrm{Std}_{(n-2)}(\mu)} \gamma_{\mathfrak{t}}(w_{(n-2)})\\
&= -\frac{\sqrt{-1}\sqrt{q}\sqrt{[n]}}{\sqrt{[n-2]}} \cdot \chi^\mu(T_{w_{(n-2)}} \circ \tau)\\
&= -\frac{\sqrt{-1}\sqrt{q}\sqrt{[n]}}{\sqrt{[n-2]}} \big(-\sqrt{-1}\sqrt{q}\big)^{\frac{n-3}{2}} \sqrt{[n-2]}\\
&= \big(-\sqrt{-1}\sqrt{q}\big)^{\frac{n-1}{2}} \sqrt{[n]}.
\end{aligned}$$

□

This enables us to compute our first general character value. Let h_i be the length of the i^{th} hook of λ. Therefore, $h_i = 2\lambda_i - 2i + 1$. Also, let $d(\lambda)$ be the length of the diagonal of λ (so $d(\lambda) = \max\{\, i \mid \lambda_i \geq i \,\}$), and let $h(\lambda) = (h_1, h_2, \dots, h_{d(\lambda)})$. Thus, $h(\lambda)$ is a partition of n. For example, if $\lambda = (k+1, 1^k)$ - as in Lemma 6.18 - then $h(\lambda) = (n) = (2k+1)$ and $d(\lambda) = 1$.

For the proof of the next result we need some more notation. Suppose that $\mathfrak{t}$ is a λ–tableau and for $i = 1, \dots, d$ let $\mathfrak{t}_i$ be the subtableau of $\mathfrak{t}$ which contains the numbers in the i^{th} hook of $\mathfrak{t}$. We will think of $\mathfrak{t}_i$ as being a tableau of shape $(\frac{h_i+1}{2}, 1^{\frac{h_i-1}{2}})$ in the natural way. Finally, if $k \in \mathbb{Z}$ let $\mathfrak{t} - k$ be the tableau obtained by subtracting k from each of the entries of $\mathfrak{t}$.

Proposition 6.19. *Suppose that κ and λ are partitions of n such that λ is self–conjugate and let $d = d(\lambda)$. Then*

$$\chi^\lambda(T_{w_{h(\lambda)}} \circ \tau) = \big(-\sqrt{-1}\sqrt{q}\big)^{\frac{n-d}{2}} \textstyle\prod_{i=1}^{d} \sqrt{[h_i]},$$

Proof. Write $h(\lambda) = (h_1, \dots, h_d)$ and suppose that $\mathfrak{t}$ is a $w_{h(\lambda)}$–transposable λ–tableau. Set $\bar{h}_0 = 0$ and let $\bar{h}_i = \sum_{j=1}^{i} h_j$, for $1 \leq i \leq d$. Then

$$w_{h(\lambda)} = (\bar{h}_0 + 1, \dots, \bar{h}_1)(\bar{h}_1 + 1, \dots, \bar{h}_2)\dots(\bar{h}_{d-1} + 1, \dots, \bar{h}_d)$$

is the cycle decomposition of $w_{h(\lambda)}$. Notice that there are d cycles in $w_{h(\lambda)}$, all of odd length. Now, an odd length cycle must contain at least one element from the diagonal of $\mathfrak{t}$, since $\mathfrak{t}$ is $w_{h(\lambda)}$–transposable. It follows that the numbers on the diagonal of $\mathfrak{t}$ come from distinct cycles of $w_{h(\lambda)}$. Moreover, since $\mathfrak{t}$ is standard, the number $\mathfrak{t}(i,i)$ must belong to cycle i of $w_{h(\lambda)}$; that is, for $i = 1,\dots,d$ we have $\mathfrak{t}(i,i) \in \{\bar{h}_{i-1}+1,\dots,\bar{h}_i\}$. Using once more the fact that $\mathfrak{t}$ is standard, for $i = 1,\dots,d$ we see that we must have $\mathfrak{t}(i,i) = \bar{h}_{i-1}+1$ and that the numbers contained in the subtableau $\mathfrak{t}_i$ are precisely the numbers $\{\bar{h}_{i-1}+1,\dots,\bar{h}_i\}$.

For $1 \le i \le d$ let $\lambda^i = (\frac{h_i+1}{2}, 1^{\frac{h_i-1}{2}})$. Then λ^i is a self–conjugate partition and $h(\lambda^i) = (h_i)$. Further, by the last paragraph, there is a bijection

$$\begin{aligned} \mathrm{Std}_{h(\lambda)}(\lambda) &\to \prod_{i=1}^{d} \mathrm{Std}_{(h_i)}(\lambda^i) \\ \mathfrak{t} &\mapsto (\mathfrak{t}_1 - \bar{h}_0, \mathfrak{t}_2 - \bar{h}_1, \dots, \mathfrak{t}_d - \bar{h}_{d-1}). \end{aligned}$$

Using this and Proposition 6.17 and Definition 6.15, if $\mathfrak{t}$ is $w_{h(\lambda)}$-transposable then

$$\gamma_{\mathfrak{t}}(w_{h(\lambda)}) = \prod_{i=1}^{d} \prod_{j=\bar{h}_{i-1}+1}^{\bar{h}_i-1} \gamma_{\mathfrak{t}}(j) = \prod_{i=1}^{d} \prod_{j=1}^{h_i-1} \gamma_{\mathfrak{t}_i-\bar{h}_{i-1}}(j) = \prod_{i=1}^{d} \gamma_{\mathfrak{t}_i-\bar{h}_{i-1}}(w_{(h_i)}).$$

Combining the last two statements we find that

$$\begin{aligned} \chi^{\lambda}(T_{w_{h(\lambda)}} \circ \tau) &= \sum_{\mathfrak{t}\in \mathrm{Std}_{h(\lambda)}(\lambda)} \prod_{i=1}^{d} \gamma_{\mathfrak{t}_i-\bar{h}_{i-1}}(w_{(h_i)}) \\ &= \prod_{i=1}^{d} \sum_{\mathfrak{t}_i\in \mathrm{Std}_{(h_i)}(\lambda^i)} \gamma_{\mathfrak{t}_i}(w_{(h_i)}) \\ &= \prod_{i=1}^{d} \chi^{(\lambda^i)}(T_{w_{(h_i)}} \circ \tau) \\ &= \prod_{i=1}^{d} \big(-\sqrt{-1}\sqrt{q}\big)^{\frac{h_i-1}{2}} \sqrt{[h_i]} \end{aligned}$$

by Lemma 6.18. The result follows. □

Next we want to show that $\chi^{\lambda}(T_{w_\kappa} \circ \tau) = 0$ when $\kappa \neq h(\lambda)$. This is much harder and we will need several lemmas.

First note that the set of partitions of the form $h(\lambda)$ is equal to the set of partitions whose components are odd and pairwise distinct. Thus, we must now consider all cases where κ has components that are not odd and distinct. We first consider the following case.

Lemma 6.20 (cf. [**8**, Lemma 3.3])**.** *Suppose that w_κ has more than $d(\lambda)$ cycles of odd length. Then $\chi^{\lambda}(T_{w_\kappa} \circ \tau) = 0$.*

Proof. We know that if $\mathfrak{t}$ is a w–transposable tableau then each odd cycle of w meets the diagonal of $\mathfrak{t}$ at least once. Consequently, there are no transposable w_κ–tableaux since w_κ has more than $d(\lambda)$ odd cycles. Hence, $\chi^\lambda(T_{w_\kappa} \circ \tau) = 0$. □

Before tackling other cycle types we prove a technical lemma.

Lemma 6.21. *Let $\mathfrak{t}$ be a w_κ–transposable tableaux and suppose that $a, b \in \operatorname{diag}(\mathfrak{t})$ with b minimal such that a and b are in the same cycle of w_κ. Let $\mathfrak{t}'' = \mathfrak{t}s_{a+1}s_{a+3}\dots s_{b-2}$. Then $\mathfrak{t}''$ is w_κ–transposable and $\gamma_\mathfrak{t}(w_\kappa)+\gamma_{\mathfrak{t}''}(w_\kappa) = 0$.*

Proof. Since $\mathfrak{t}$ is w_κ–transposable, the numbers $\{a+2i+1, a+2i+2\}$, for $i = 0, \dots, \frac{b-a-3}{2}$, occur in diagonally opposite positions in $\mathfrak{t}$. Consequently, $b-a$ is odd and $\mathfrak{t}''$ is the tableau obtained from $\mathfrak{t}$ by swapping these entries. Hence, $\mathfrak{t}''$ is w_κ–transposable, proving our first claim.

To show that $\gamma_\mathfrak{t}(w_\kappa) + \gamma_{\mathfrak{t}''}(w_\kappa) = 0$, first observe that $\gamma_\mathfrak{t}(i) = \gamma_{\mathfrak{t}''}(i)$ if $i < a$ or if $i \ge b$. Let $\gamma_\mathfrak{t}[a, b) = \prod_{i=a}^{b-1} \gamma_\mathfrak{t}(i)$ and define $\gamma_{\mathfrak{t}''}[a, b)$ similarly. To complete the proof we will show that $\gamma_\mathfrak{t}[a, b) = -\gamma_{\mathfrak{t}''}[a, b)$.

As in Definition 6.15, set $d_i = c_\mathfrak{t}(i+1) - c_\mathfrak{t}(i) = c_{\mathfrak{t}''}(i) - c_{\mathfrak{t}''}(i+1)$ and $d'_i = c_\mathfrak{t}(i+1) - c_\mathfrak{t}(i-1) = c_{\mathfrak{t}''}(i-1) - c_{\mathfrak{t}''}(i+1)$, for $a \le i < b$. Then, for $i = 0, \dots, b-1-a$ we have

$$\gamma_\mathfrak{t}(a+i) = \begin{cases} \frac{-1}{[-d_a]}, & \text{if } i = 0, \\ \frac{-1}{[-d'_{a+i}]}, & \text{if } i > 0 \text{ is even}, \\ \alpha(d_{a+i}), & \text{if } i \text{ is odd}, \end{cases}$$

and

$$\gamma_{\mathfrak{t}''}(a+i) = \begin{cases} \frac{-1}{[d_a]}, & \text{if } i = 0, \\ \frac{-1}{[d'_{a+i}]}, & \text{if } i > 0 \text{ is even}, \\ -\alpha(d_{a+i}), & \text{if } i \text{ is odd}. \end{cases}$$

Therefore

$$\begin{aligned} \gamma_\mathfrak{t}[a, b) &= \prod_{i=a}^{b-1} \gamma_\mathfrak{t}(i) \\ &= \frac{-1}{[-d_a]} \Bigg[\Big(\alpha(d_{a+1})\Big)\Big(\frac{-1}{[-d'_{a+2}]}\Big)\Big(\alpha(d_{a+3})\Big)\Big(\frac{-1}{[-d'_{a+4}]}\Big) \\ &\qquad \dots \Big(\alpha(d_{b-2})\Big)\Big(\frac{-1}{[-d'_{b-1}]}\Big)\Bigg]. \end{aligned}$$

Also,

$$\begin{aligned}\gamma_{\mathfrak{t}''}[a,b] &= \prod_{i=a}^{b-1}\gamma_{\mathfrak{t}''}(i)\\ &= \frac{-1}{[d_a]}\Big[\Big(-\alpha(d_{a+1})\Big)\Big(\frac{-1}{[d'_{a+2}]}\Big)\Big(-\alpha(d_{a+3})\Big)\Big(\frac{-1}{[d'_{a+4}]}\Big)\\ &\qquad\qquad \dots\Big(-\alpha(d_{b-2})\Big)\Big(\frac{-1}{[d'_{b-1}]}\Big)\Big].\end{aligned}$$

After rearranging we have

$$\begin{aligned}\gamma_{\mathfrak{t}''}[a,b] &= \frac{q^{-d_a}}{[-d_a]}\Big[\Big(-\alpha(d_{a+1})\Big)\Big(\frac{q^{-d'_{a+2}}}{[-d'_{a+2}]}\Big)\Big(-\alpha(d_{a+3})\Big)\Big(\frac{q^{-d'_{a+4}}}{[-d'_{a+4}]}\Big)\\ &\qquad\qquad \dots\Big(-\alpha(d_{b-2})\Big)\Big(\frac{q^{-d'_{b-1}}}{[-d'_{b-1}]}\Big)\Big]\\ &= -q^{-(d_a+d'_{a+2}+d'_{a+4}+\dots d'_{b-1})}\gamma_{\mathfrak{t}}[a,b).\end{aligned}$$

But

$$\begin{aligned}& d_a+d'_{a+2}+d'_{a+4}+\dots d'_{b-1}\\ =\;& (c_{\mathfrak{t}}(a+1)-c_{\mathfrak{t}}(a))\\ +\;& (c_{\mathfrak{t}}(a+3)-c_{\mathfrak{t}}(a+1))\\ +\;& (c_{\mathfrak{t}}(a+5)-c_{\mathfrak{t}}(a+3))\\ +\;& \\ \vdots\;& \\ +\;& (c_{\mathfrak{t}}(b)-c_{\mathfrak{t}}(b-2))\\ =\;& c_{\mathfrak{t}}(b))-c_{\mathfrak{t}}(a)\\ =\;& 0\end{aligned}$$

since a and b are on the diagonal of $\mathfrak{t}$; and the result follows. □

Lemma 6.22 (cf. [**8**, Lemma 3.4])**.** *Suppose that the number of cycles in w_κ is less than $d(\lambda)$. Then $\chi^\lambda(T_{w_\kappa}\circ\tau)=0$.*

Proof. Let $\mathfrak{t}$ be a w_κ–transposable tableau. Then, because w_κ has less than $d(\lambda)$ cycles, at least one cycle in w_κ meets the diagonal of $\mathfrak{t}$ more than once. Therefore, we can find integers $a<b$ on the diagonal of $\mathfrak{t}$, with b minimal such that a and b are both in the same cycle of w_κ. Then Lemma 6.21 tells us that for $\mathfrak{t}''=\mathfrak{t}s_{a+1}s_{a+3}\dots s_{b-2}$ we have $\gamma_{\mathfrak{t}}(w_\kappa)+\gamma_{\mathfrak{t}''}(w_\kappa)=0$. Then, since the map sending $\mathfrak{t}$ to $\mathfrak{t}''$ is bijective and an involution, the result follows. □

Lemma 6.23 (cf. [**8**, Lemma 3.5])**.** *Suppose that κ contains an even part. Then $\chi^\lambda(T_{w_\kappa}\circ\tau)=0$.*

Proof. Let $\mathfrak{t}$ be a w_κ–transposable tableau and consider the first even cycle in κ. As $\mathfrak{t}$ is transposable, this cycle meets the diagonal of $\mathfrak{t}$ in an even number of places. If this cycle meets the diagonal at $a<b$, where b is minimal, then $\gamma_{\mathfrak{t}}(w_\kappa)+\gamma_{\mathfrak{t}''}(w_\kappa)=0$ by Lemma 6.21, where $\mathfrak{t}''=\mathfrak{t}s_{a+1}\dots s_{b-2}$. If the first even cycle $(a+1,a+2,\dots,b-1)$ does not meet the diagonal of $\mathfrak{t}$ at all then we again define $\mathfrak{t}''=\mathfrak{t}s_{a+1}\dots s_{b-2}$. By essentially repeating the argument of Lemma 6.21 we find that $\gamma_{\mathfrak{t}}(w_\kappa)+\gamma_{\mathfrak{t}''}(w_\kappa)=0$. Hence, it follows that $\chi^\lambda(T_{w_\kappa}\circ\tau)=0$ as required. □

We are now left with those elements w_κ which have $d(\lambda)$ cycles of odd length, and no even cycles. This case is the most complicated and requires some more notation and the technical Lemma 6.24 below.

Let $(X,<)$ be a poset. Then X is **connected** if its Hasse diagram is connected; otherwise X is **disconnected**. If X contains $m+1$ elements then a **linearisation** of X is a bijection $f\colon X\longrightarrow\{1,\dots,m+1\}$ which respects the ordering in X; thus, $f(x)<f(y)$ whenever $x<y$ for $x,y\in X$. Let $\mathscr{L}(X)$ be the set of linearisations of X. If $f\in\mathscr{L}(X)$ let $f^*\colon\{1,\dots,m+1\}\longrightarrow X$ be its inverse.

Lemma 6.24. *Suppose that $(X,<)$ is a disconnected poset with $m+1$ elements. Let $z\in X$ and suppose that $\{\,c_x\mid x\in X\,\}$ is a set of pairwise distinct integers such that $c_z=0$. Then*

$$0=\sum_{f\in\mathscr{L}(X)}\prod_{i=1}^{f(z)-1}\frac{-1}{[c_{f^*(i+1)}-c_{f^*(i)}]}\prod_{i=f(z)}^{m}\frac{1}{[c_{f^*(i)}-c_{f^*(i+1)}]}.$$

Proof. Let $\{\,Q_x\mid x\in X\,\}$ be a set of indeterminates. We know that

$$0=\sum_{f\in\mathscr{L}(X)}\prod_{i=1}^{m}\frac{1}{Q_{f^*(i+1)}-Q_{f^*(i)}}$$

(see Headley [**8**, Lemma 3.6]). Setting $Q_x=q^{c_x}$, for $x\in X$, and multiplying through by $(q-1)^m$ this implies that

$$\begin{aligned}0&=\sum_{f\in\mathscr{L}(X)}\prod_{i=1}^{m}\frac{q-1}{q^{c_{f^*(i+1)}}-q^{c_{f^*(i)}}}=\sum_{f\in\mathscr{L}(X)}\prod_{i=1}^{m}\frac{(q-1)q^{-c_{f^*(i)}}}{q^{c_{f^*(i+1)}-c_{f^*(i)}}-1}\\&=q^{-\sum_{x\in X}c_x}\sum_{f\in\mathscr{L}(X)}q^{c_{f^*(m+1)}}\prod_{i=1}^{m}\frac{1}{[c_{f^*(i+1)}-c_{f^*(i)}]}.\end{aligned}$$

Hence,

$$\sum_{f\in\mathscr{L}(X)} q^{c_{f^*(m+1)}} \prod_{i=1}^{m} \frac{1}{[c_{f^*(i+1)} - c_{f^*(i)}]} = 0.$$

To complete the proof observe that, for each $f \in \mathscr{L}(X)$,

$$\begin{aligned} q^{c_{f^*(m+1)}} \prod_{i=f(z)}^{m} \frac{1}{[c_{f^*(i+1)} - c_{f^*(i)}]} &= \prod_{i=f(z)}^{m} \frac{q^{c_{f^*(i+1)}-c_{f^*(i)}}}{[c_{f^*(i+1)} - c_{f^*(i)}]} \\ &= \prod_{i=f(z)}^{m} \frac{-1}{[c_{f^*(i)} - c_{f^*(i+1)}]} \end{aligned}$$

where the first equality follows because $c_z = 0$. Combining the last two equations shows that

$$0 = \sum_{f\in\mathscr{L}(X)} \prod_{i=1}^{f(z)-1} \frac{1}{[c_{f^*(i+1)} - c_{f^*(i)}]} \prod_{i=f(z)}^{m} \frac{-1}{[c_{f^*(i)} - c_{f^*(i+1)}]}.$$

Multiplying both sides of this equation by $(-1)^m$ completes the proof. □

We can now complete our calculation of the characters of $\mathcal{A}_{K,q}(\mathfrak{S}_n)$ for the particular conjugacy class representatives w_κ of minimal length.

Proposition 6.25. *Suppose that w_κ is of minimal length in its conjugacy class and that $\kappa \neq h(\lambda)$. Then $\chi^\lambda(T_{w_\kappa} \circ \tau) = 0$.*

Proof. By Lemmas 6.20–6.23 it remains to consider the case when w_κ has exactly $d(\lambda)$ cycles; each of which is of odd length. Our proof follows Headley [**8**, Theorem 3.7], except that we have some additional complications. The reader may find Example 6.28 useful while reading this proof.

Fix for the time being, a w_κ–transposable λ–tableaux $\mathfrak{t}$. We need to attach a lot of combinatorial data to $\mathfrak{t}$. In a similar manner to the proof of Proposition 6.19 we see that the diagonal elements of $\mathfrak{t}$ lie in different cycles of w_κ. It follows that $\mathfrak{t}(i,i)$ belongs to cycle i of w_κ. Moreover, because $\kappa \neq h(\lambda)$ at least one of the cycles of w_κ must meet more than one of the diagonal hook subtableaux $\mathfrak{t}_i$ of $\mathfrak{t}$, for $1 \leq i \leq d(\lambda)$.

Let $z_\mathfrak{t} > 1$ be minimal such that cycle $z_\mathfrak{t}$ of w_κ is not contained in $\mathfrak{t}_{z_\mathfrak{t}}$ and let $\lambda_\mathfrak{t}$ be the shape of the skew subtableaux which contains the numbers in cycle $z_\mathfrak{t}$ of w_κ. Let $X_\mathfrak{t} = \{\, (r,c) \in [\lambda_\mathfrak{t}] \mid c \geq r \,\}$, which we consider as a poset with ordering $(r,c) \leq (r',c')$ if $r = r'$ and $c \leq c'$. Note that $(z_\mathfrak{t}, z_\mathfrak{t}) \in X_\mathfrak{t}$. Moreover $X_\mathfrak{t}$ must be disconnected; since, if it were connected and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{z_\kappa}, \dots)$ then we would have $\kappa_{z_\kappa} > \kappa_{z_\kappa - 1}$, contradicting that κ is a partition.

If $x = (r,c) \in X_\mathfrak{t}$ let $x' = (c,r)$. (So that $x' \in X_\mathfrak{t}$ if and only if $x = (z_\mathfrak{t}, z_\mathfrak{t})$.) Next, define two "sign functions" $\epsilon_\mathfrak{t}, \epsilon'_\mathfrak{t} : X_\mathfrak{t} \longrightarrow \{\pm 1\}$ by

$$\epsilon_\mathfrak{t}(x) = \begin{cases} 1, & \text{if } \mathfrak{t}(x') \geq \mathfrak{t}(x), \\ -1, & \text{if } \mathfrak{t}(x') < \mathfrak{t}(x), \end{cases} \quad \text{and} \quad \epsilon'_\mathfrak{t}(x) = \begin{cases} 1, & \text{if } \mathfrak{t}(x) \geq \mathfrak{t}(z), \\ -1, & \text{if } \mathfrak{t}(x) < \mathfrak{t}(z), \end{cases}$$

where $z = (z_\mathfrak{t}, z_\mathfrak{t})$ and $x \in X_\mathfrak{t}$. Finally, define a linearisation $f_\mathfrak{t} \in \mathscr{L}(X_\mathfrak{t})$ of $X_\mathfrak{t}$ by $f_\mathfrak{t}(p) = \#\{\, x \in X_\mathfrak{t} \mid \mathfrak{t}(p) \geq \mathfrak{t}(x) \,\}$, for $x \in X_\mathfrak{t}$. (Notice that $\epsilon'_\mathfrak{t}(x) = 1$ if and only if $f_\mathfrak{t}(x) \geq f_\mathfrak{t}(z)$.)

Let $f_\mathfrak{t}^*$ be the inverse of the linearisation $f_\mathfrak{t}$ and define $x_1, \dots, x_{m+1} \in X_\mathfrak{t}$ by $x_i = f_\mathfrak{t}^*(i)$, for $1 \leq i \leq m+1$, where $|X_\mathfrak{t}| = m+1$. Let $k = \kappa_1 + \dots + \kappa_{z_\mathfrak{t}-1} + 1$. Then the $z_\mathfrak{t}^{\text{th}}$ cycle of w_κ is $(k, k+1, \dots, k+2m)$ and the contribution of the $z_\mathfrak{t}^{\text{th}}$ cycle of w_κ to $\gamma_\mathfrak{t}(w_\kappa)$ is $\prod_{j=0}^{2m-1} \gamma_\mathfrak{t}(k+j)$. We claim that

$$\prod_{j=0}^{2m-1} \gamma_\mathfrak{t}(k+j) = \frac{\displaystyle\prod_{i=1}^{m+1} \alpha(2\epsilon_\mathfrak{t}(x_i)c(x_i))}{\displaystyle\prod_{i=1}^{m} [\epsilon_\mathfrak{t}(x_i)c(x_i) - \epsilon_\mathfrak{t}(x_{i+1})c(x_{i+1})]}, \tag{6.26}$$

where the content of a node $x = (r,c)$ is $c(x) = c - r$. To prove the claim we will use Proposition 6.17. According to Definition 6.15 there are three cases to consider. While we are proving the claim, suppose that $0 \leq j < 2m$ and let x and y be the unique nodes in $X_\mathfrak{t}$ such that $k+j \in \{\mathfrak{t}(x), \mathfrak{t}(x')\}$ and $k+j+1 \in \{\mathfrak{t}(y), \mathfrak{t}(y')\}$

Case 1. $k+j \in \operatorname{diag}(\mathfrak{t})$. This happens if and only if $x = (z_\mathfrak{t}, z_\mathfrak{t})$, therefore $\epsilon_\mathfrak{t}(x) = 1$. Our aim is to find $c_\mathfrak{t}(k+j)$ and $c_\mathfrak{t}(k+j+1)$. The former is easy since $k+j \in \operatorname{diag}(\mathfrak{t})$. We have $c_\mathfrak{t}(k+j) = 0 = \pm c(x)$.

If $x = x_i$ then $y = x_{i+1} \in X_\mathfrak{t}$ because of the way $f_\mathfrak{t}$ is defined. Now, if $k+j+1 = \mathfrak{t}(y)$ then $c_\mathfrak{t}(k+j+1) = c(x_{i+1})$. Notice that in this situation we have $\mathfrak{t}(y') > \mathfrak{t}(y)$ and hence $\epsilon_\mathfrak{t}(y) = 1$. On the other hand, if $k+j+1 = \mathfrak{t}(y')$ then $c_\mathfrak{t}(k+j+1) = -c(x_{i+1})$ and $\epsilon_\mathfrak{t}(y) = -1$. Hence, in both cases, $c_\mathfrak{t}(k+j+1) = \epsilon_\mathfrak{t}(x_{i+1})c(x_{i+1})$. Therefore,

$$\begin{aligned} \gamma_\mathfrak{t}(k+j) &= \frac{-1}{[c_\mathfrak{t}(k+j) - c_\mathfrak{t}(k+j+1)]} \\ &= \frac{-1}{[\epsilon_\mathfrak{t}(x_i)c(x_i) - \epsilon_\mathfrak{t}(x_{i+1})c(x_{i+1})]} \end{aligned}$$

where $k+j \in \{\mathfrak{t}(x_i), \mathfrak{t}(x'_i)\}$.

Case 2. $c_\mathfrak{t}(k+j) = -c_\mathfrak{t}(k+j-1)$. In this case $k+j$ and $k+j-1$ are diagonally opposite so if $x = x_i$ then $y = x_{i+1}$. Our aim in this case is to find $c_\mathfrak{t}(k+j+1)$ and $c_\mathfrak{t}(k+j-1)$. We have $\mathfrak{t}(x) = k+j-1$ if and only if $\epsilon_{\mathfrak{t}}(x) = 1$, and $\mathfrak{t}(y) = k + j + 1$ if and only if $\epsilon_{\mathfrak{t}}(y) = 1$. Hence, using a similar argument to that in Case 1,

$$\begin{aligned}\gamma_{\mathfrak{t}}(k+j) &= \frac{-1}{[c_{\mathfrak{t}}(k+j-1) - c_{\mathfrak{t}}(k+j+1)]}\\ &= \frac{-1}{[\epsilon_{\mathfrak{t}}(x_i)c(x_i) - \epsilon_{\mathfrak{t}}(x_{i+1})c(x_{i+1})]}.\end{aligned}$$

Case 3. $c_{\mathfrak{t}}(k+j) = -c_{\mathfrak{t}}(k+j+1)$. In this case $k+j$ and $k+j+1$ are diagonally opposite so $x = y$ and, in particular, $x \neq z = (z_{\mathfrak{t}}, z_{\mathfrak{t}})$. If $k + j = \mathfrak{t}(x)$ then $d_{\mathfrak{t}}(k+j, k+j+1) = c_{\mathfrak{t}}(k+j+1) - c_{\mathfrak{t}}(k+j) = -2c_{\mathfrak{t}}(k+j) = -2c(x)$ and $\epsilon_{\mathfrak{t}}(x) = 1$. Alternatively, if $k+j+1 = \mathfrak{t}(x)$ then $d_{\mathfrak{t}}(k+j, k+j+1) = 2c_{\mathfrak{t}}(k+j+1) = 2c(x)$ and $\epsilon_{\mathfrak{t}}(x) = -1$. Write $x = x_i$. Then

$$\gamma_{\mathfrak{t}}(k+j) = \alpha(c_{\mathfrak{t}}(k+j+1) - c_{\mathfrak{t}}(k+j)) = -\alpha(2\epsilon_{\mathfrak{t}}(x_i)c(x_i)).$$

We can now combine Cases 1–3 to prove (6.26). Note that the $2m$ minus signs in Cases 1–3 cancel out. Also, note that the numerator of (6.26) includes the case when $x_i = z$. Technically it should not be there, however, we may leave it in since $\alpha(2\epsilon_{\mathfrak{t}}(z)c(z)) = \alpha(0) = 1$.

We can now begin to prove the proposition. Define an equivalence relation on the set of w_κ–transposable λ–tableaux by declaring that if $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{w_\kappa}(\lambda)$ then $\mathfrak{s} \sim \mathfrak{t}$ if

a) $\lambda_{\mathfrak{s}} = \lambda_{\mathfrak{t}}$ (so that $z_{\mathfrak{s}} = z_{\mathfrak{t}}$ and $X_{\mathfrak{s}} = X_{\mathfrak{t}}$),
b) $\mathfrak{s}$ and $\mathfrak{t}$ agree on $\lambda \setminus \lambda_{\mathfrak{s}}$, and
c) $\epsilon_{\mathfrak{s}}(x)\epsilon'_{\mathfrak{s}}(x) = \epsilon_{\mathfrak{t}}(x)\epsilon'_{\mathfrak{t}}(x)$, for all $x \in X_{\mathfrak{s}}$.

Let $\mathscr{X}$ be an equivalence class of transposable tableaux. Let $\mathfrak{t} \in \mathscr{X}$ and set $z_0 = z_{\mathfrak{t}}$, $X_0 = X_{\mathfrak{t}}$, $\lambda_0 = \lambda_{\mathfrak{t}}$ and $f_0 = f_{\mathfrak{t}}$. Then for any tableaux $\mathfrak{s}$ in $\mathscr{X}$, $z_{\mathfrak{s}} = z_0$, $X_{\mathfrak{s}} = X_0$, $\lambda_{\mathfrak{s}} = \lambda_0$ and $f_{\mathfrak{s}} = f_0$. It is not hard to see that $\mathscr{X}$ contains a tableau $\mathfrak{t}_0$ such that $\mathfrak{t}_0(z_0, z_0) = k$ where, as above, $k = \kappa_1 + \cdots + \kappa_{z_0-1} + 1$. Fix any such tableau $\mathfrak{t}_0 \in \mathscr{X}$ and let $\epsilon_0 = \epsilon_{\mathfrak{t}_0}$ and $\epsilon'_0 = \epsilon'_{\mathfrak{t}_0}$. Then, looking at the definitions, $\epsilon'_0(x) = 1$, for all $x \in X_0$. Therefore, by the definition of $\sim$, we have $\epsilon_{\mathfrak{t}}(x)\epsilon'_{\mathfrak{t}}(x) = \epsilon_0(x)\epsilon'_0(x) = \epsilon_0(x)$, for all $\mathfrak{t} \in X_0$ and all $x \in X_0$. We set $z = (z_0, z_0)$ and define $c_x = \epsilon_0(x)c(x)$, for $x \in X_0$. Then $\epsilon_{\mathfrak{t}}(x)c(x) = \epsilon'_{\mathfrak{t}}(x)\epsilon_0(x)c(x) = \epsilon'_{\mathfrak{t}}(x)c_x$, for all $x \in X_0$.

Recall that the z_0^{th} cycle of w_κ is $(k, \ldots, k+2m)$, where $|X_0| = m + 1$. Also recall that, by Proposition 6.17, if $\mathfrak{s}, \mathfrak{t} \in \mathscr{X}$ then $\gamma_{\mathfrak{t}}(i) = \gamma_{\mathfrak{s}}(i)$ whenever $i < k$ or $i > k + 2m$. Consequently, using (6.26), the sum of the contributions of the z_0^{th} cycle for the tableaux in $\mathscr{X}$ is

$$
\begin{aligned}
\sum_{\mathfrak{t}\in\mathscr{X}}\prod_{j=0}^{2m-1}\gamma_{\mathfrak{t}}(k+j) &= \sum_{\mathfrak{t}\in\mathscr{X}} \frac{\prod_{i=1}^{m+1}\alpha(2\epsilon_{\mathfrak{t}}(x_i)c(x_i))}{\prod_{i=1}^{m}[\epsilon_{\mathfrak{t}}(x_i)c(x_i)-\epsilon_{\mathfrak{t}}(x_{i+1})c(x_{i+1})]}\\
&= \sum_{\mathfrak{t}\in\mathscr{X}} \frac{\prod_{i=1}^{m+1}\alpha(2\epsilon'_{\mathfrak{t}}(x_i)c_{x_i})}{\prod_{i=1}^{m}[\epsilon'_{\mathfrak{t}}(x_i)c_{x_i}-\epsilon'_{\mathfrak{t}}(x_{i+1})c_{x_{i+1}}]}\\
&= \sum_{\mathfrak{t}\in\mathscr{X}} \frac{\prod_{i=1}^{m+1}\epsilon'_{\mathfrak{t}}(x_i)\alpha(2c_{x_i})}{\prod_{i=1}^{m}[\epsilon'_{\mathfrak{t}}(x_i)c_{x_i}-\epsilon'_{\mathfrak{t}}(x_{i+1})c_{x_{i+1}}]} \qquad (6.27)\\
&= \Big(\prod_{x\in X_0}\alpha(2c_x)\Big)\sum_{\mathfrak{t}\in\mathscr{X}}\prod_{i=1}^{m}\frac{\epsilon'_{\mathfrak{t}}(x_i)}{[\epsilon'_{\mathfrak{t}}(x_i)c_{x_i}-\epsilon'_{\mathfrak{t}}(x_{i+1})c_{x_{i+1}}]},
\end{aligned}
$$

since $\epsilon'_{\mathfrak{t}}(x_{m+1})=1$ for all $\mathfrak{t}\in\mathscr{X}$. Note that throughout equation (6.27) the nodes $x_i=f^*_{\mathfrak{t}}(i)$ depend on the tableau $\mathfrak{t}$ in the summation, for $1\le i<m+1$.

To complete the proof we want to apply Lemma 6.24. To do this we first show that $\mathscr{X}\cong\mathscr{L}(X_0)$ and then we reinterpret (6.27) in light of this bijection. Recall that we have a map $\mathscr{X}\longrightarrow\mathscr{L}(X_0)$ given by $\mathfrak{t}\mapsto f_{\mathfrak{t}}$, for $\mathfrak{t}\in\mathscr{X}$. Note that $\mathfrak{t}(z)$ must be one of $k,k+2,\dots,k+2m$. Moreover, if $\mathfrak{t}(z)=k+i$ then $f_{\mathfrak{t}}(z)=i/2+1$. Hence $f_{\mathfrak{t}}(z)=(\mathfrak{t}(z)-k)/2+1$.

Conversely, given $f\in\mathscr{L}(X_0)$ there is a unique tableau $\mathfrak{t}=\mathfrak{t}_f\in\mathscr{X}$ such that $\mathfrak{t}$ agrees with $\mathfrak{t}_0$ on $\lambda\setminus\lambda_0$, $\mathfrak{t}(z)=k+2(f(z)-1)$ and

$$
\{\mathfrak{t}(x),\mathfrak{t}(x')\}=\begin{cases}\{k+2f(x)-2,k+2f(x)-1\}, & \text{if } f(x)<f(z),\\ \{k+2f(x)-3,k+2f(x)-2\}, & \text{if } f(x)>f(z).\end{cases}
$$

To see that the tableau $\mathfrak{t}$ is unique, we must be able to completely determine its values. First note that, since f is a linearisation , if $f(x)<f(z)$ then $\mathfrak{t}(x)<\mathfrak{t}(z)$. Similarly if $f(x)\ge f(z)$ then $\mathfrak{t}(x)\ge\mathfrak{t}(z)$. Therefore if $f(x)<f(z)$ then $\epsilon'_{\mathfrak{t}}(x)=-1$. Now $\epsilon_{\mathfrak{t}}(x)$ can be found since we must have $\epsilon_{\mathfrak{t}}(x)\epsilon'_{\mathfrak{t}}(x)=\epsilon_0(x)$. Hence the positions of $k+2f(x)-2$ and $k+2f(x)-1$ are completely determined. The second case is similar.

It is easy to see that the maps $\mathfrak{t}\mapsto f_{\mathfrak{t}}$ and $f\mapsto\mathfrak{t}_f$ are mutual inverses, so $\mathscr{X}\cong\mathscr{L}(X_0)$ as claimed.

Recalling that $x_j = f^*_{\mathfrak{t}}(j)$, we can now rewrite (6.27) to see that

$$\begin{aligned}
&\sum_{\mathfrak{t}\in\mathscr{X}} \prod_{j=0}^{2m-1} \gamma_{\mathfrak{t}}(k+j) \\
&= \Big(\prod_{x\in X_0} \alpha(2c_x)\Big) \sum_{f\in\mathscr{L}(X_0)} \prod_{i=1}^{m} \frac{\epsilon'_{\mathfrak{t}}(f^*(i))}{[\epsilon'_{\mathfrak{t}}(f^*(i))c_{f^*(i)} - \epsilon'_{\mathfrak{t}}(f^*(i+1))c_{f^*(i+1)}]} \\
&= \Big(\prod_{x\in X_0} \alpha(2c_x)\Big) \sum_{f\in\mathscr{L}(X_0)} \prod_{i=1}^{f(z)-1} \frac{-1}{[c_{f^*(i+1)} - c_{f^*(i)}]} \prod_{i=f(z)}^{m} \frac{1}{[c_{f^*(i)} - c_{f^*(i+1)}]} \\
&= 0,
\end{aligned}$$

where the last equality follows from Lemma 6.24. Therefore, $\sum_{\mathfrak{t}} \gamma_{\mathfrak{t}}(w_\kappa) = 0$ and the proposition is proved. □

Combining Proposition 6.19 and Proposition 6.25 we have now computed the character values $\chi^{\lambda\pm}_{\mathcal{A}}(T_w + T^{\#}_w)$, where w runs over a set of minimal length representatives for the conjugacy classes of the symmetric group which can be written in the form $w = s_{i_1} \dots s_{i_k}$, with $1 \le i_1 < \cdots < i_k < n$.

Example 6.28. Suppose that $\lambda = (6, 3, 2, 1^3)$, so that $h(\lambda) = (11, 3)$ and take $\kappa = (7^2)$. Then $w_\kappa = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12\ 13\ 14) = s_1 \dots s_6 s_8 \dots s_{13}$. The reader may check that there are 384 w_κ–transposable λ–tableaux. We have $z_{\mathfrak{t}} = 2$ and $X_{\mathfrak{t}} = \{(2, 2), (2, 3), (1, 5), (1, 6)\}$. Let $a = (2, 2)$, $b = (2, 3)$, $c = (1, 5)$ and $d = (1, 6)$ be these nodes in $[\lambda]$ and consider an equivalence class $\mathscr{X}$ of w_κ–transposable tableaux for which the sign sequence $\epsilon_{\mathfrak{t}}\epsilon'_{\mathfrak{t}}$ is $--+$. That is, $\epsilon_{\mathfrak{t}}(b)\epsilon'_{\mathfrak{t}}(b) = -1$, $\epsilon_{\mathfrak{t}}(c)\epsilon'_{\mathfrak{t}}(c) = -1$ and $\epsilon_{\mathfrak{t}}(d)\epsilon'_{\mathfrak{t}}(d) = 1$ (we use this shorthand for the sign sequences in the table below). Note that for any tableau $\mathfrak{t}$ we have $\epsilon_{\mathfrak{t}}(a) = 1 = \epsilon'_{\mathfrak{t}}(a)$, so there is no need to specify the value of the sign functions on $a = (z_{\mathfrak{t}}, z_{\mathfrak{t}})$. For these tableaux we have $\lambda_{\mathfrak{t}} = \lambda \setminus \mu$, where $\mu = (4, 1^3)$.

$\mathfrak{t}$	$\epsilon_{\mathfrak{t}}$	$\epsilon'_{\mathfrak{t}}$	$f^*_{\mathfrak{t}}$	$\gamma_{\mathfrak{t}}(8)$	$\gamma_{\mathfrak{t}}(9)$	$\gamma_{\mathfrak{t}}(10)$	$\gamma_{\mathfrak{t}}(11)$	$\gamma_{\mathfrak{t}}(12)$	$\gamma_{\mathfrak{t}}(13)$
· · · · 12 13 / · 8 10 / · 9 / · / 11 / 14	$--+$	$+++$	$abcd$	$\frac{-1}{[0--1]}$	$\alpha(2)$	$\frac{-1}{[-1--4]}$	$\alpha(8)$	$\frac{-1}{[-4-5]}$	$-\alpha(10)$
· · · · 10 13 / · 8 12 / · 11 / · / 9 / 14	$--+$	$+++$	$acbd$	$\frac{-1}{[0--4]}$	$\alpha(8)$	$\frac{-1}{[-4--1]}$	$\alpha(2)$	$\frac{-1}{[-1-5]}$	$-\alpha(10)$
· · · · 10 11 / · 8 14 / · 13 / · / 9 / 12	$--+$	$+++$	$acdb$	$\frac{-1}{[0--4]}$	$\alpha(8)$	$\frac{-1}{[-4-5]}$	$-\alpha(10)$	$\frac{-1}{[5--1]}$	$\alpha(2)$
· · · · 8 13 / · 10 12 / · 11 / · / 9 / 14	$-++$	$+-+$	$cabd$	$-\alpha(8)$	$\frac{-1}{[4-0]}$	$\frac{-1}{[0--1]}$	$\alpha(2)$	$\frac{-1}{[-1-5]}$	$-\alpha(10)$
· · · · 8 11 / · 10 14 / · 13 / · / 9 / 12	$-++$	$+-+$	$cadb$	$-\alpha(8)$	$\frac{-1}{[4-0]}$	$\frac{-1}{[0-5]}$	$-\alpha(10)$	$\frac{-1}{[5--1]}$	$\alpha(2)$
· · · · 8 11 / · 12 14 / · 13 / · / 9 / 10	$-+-$	$+--$	$cdab$	$-\alpha(8)$	$\frac{-1}{[4--5]}$	$\alpha(10)$	$\frac{-1}{[-5-0]}$	$\frac{-1}{[0--1]}$	$\alpha(2)$

In the table we have only given the positions occupied by $8, \dots, 14$ in the tableaux in $\mathscr{X}$ because these are the only entries which matter in the proof of Proposition 6.25. The positions occupied by the dots can be replaced with any $w_{(7)}$—transposable $(4, 1^3)$–tableaux. The only constraint is that the same $(4, 1^3)$–tableau must be used for all of the tableaux in $\mathscr{X}$. For example, we could use the tableau 1 2 4 6 / 3 / 5 / 7 above.

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