

# **The Alternating Hecke algebra and its Representations**

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## Introduction

The representation theory of the Iwahori-Hecke algebras associated to finite Coxeter groups has been studied exhaustively, see for example Geck and Pfeiffer [6], Humphreys [9], Cabanes & Enguehard [1] and Curtis & Reiner [4]. In particular, the representation theory of the Iwahori-Hecke algebra associated to the symmetric group  $\mathfrak{S}_n$  is well known. See, for example, Mathas [14] and Donkin [5]. The alternating group  $\mathcal{A}(\mathfrak{S}_n)$  is the subgroup of  $\mathfrak{S}_n$  containing all even permutations. In fact, considering  $\mathfrak{S}_n$  as a Coxeter group,  $\mathcal{A}(\mathfrak{S}_n)$  is the kernel of the sign representation  $\varepsilon : \mathfrak{S}_n \rightarrow \{\pm 1\}$  which sends each Coxeter generator to  $-1$ . Suppose that  $W$  is a Coxeter group, then we may generalise by defining the alternating subgroup  $\mathcal{A}(W)$  of  $W$  to be the kernel of the sign representation  $\varepsilon : W \rightarrow \{\pm 1\}$ . The Iwahori-Hecke algebras are similar to the group algebras they are associated to except that multiplication is 'twisted' by a parameter  $q$  (when  $q = 1$  the Iwahori-Hecke algebra is isomorphic to its group algebra). Similarly, in this thesis we give a general  $q$ -analogue of the alternating subgroups and describe their representation theory. Mitsuhashi has looked at the specific cases of Coxeter groups of type  $A_n$ , and  $B_n$ . See [16] and [17]. Our aim is to provide a general approach that can be applied to any finite Coxeter group.

We begin in chapter 1, by recalling the basic definitions and results for Coxeter groups and Iwahori-Hecke algebras. Here we also mention Clifford Theory, which is used to determine branching rules from the Iwahori-Hecke algebra to the alternating algebra.

Our first goal was to generalise the Reidemeister-Schreier process from groups to algebras. More precisely, if  $A$  is an algebra that is generated by a set,  $X$ , of invertible elements and  $B$  is subalgebra of  $A$  generated by words in  $X$ , then the Reidemeister-Schreier process gives generators for  $B$ .

In chapter 3 we define the object of our research, the alternating Hecke algebra, as the fixed points under the hash involution. We then give various bases of the alternating Hecke algebra, some of which are Kazhdan-Lusztig bases, in that they are fixed under the bar involution. We then give a presentation of the Iwahori-Hecke algebra and use this, combined with the results of chapter 2, to give generators for the alternating Hecke algebra.

Chapter 4 uses Tits' deformation theorem to prove that, over a large enough field, the alternating Hecke algebra is isomorphic to the group algebra of the corresponding alternating Coxeter group. In particular, there is a bijection between the irreducible representations of the alternating Hecke algebra and the irreducible representations of the alternating subgroup.

The next step is to determine the irreducible representations of the alternating Hecke algebra. In chapter 5 we prove some general results, with the help of Clifford theory, which tell us what happens to an irreducible module for the Iwahori-Hecke algebra when we restrict it to the alternating Hecke algebra. We find that there are only two cases: either it is still irreducible, or it splits into a direct sum of two irreducibles. We end this chapter by giving criteria that determine when these cases occur for the Iwahori-Hecke algebras of types  $A_n$ ,  $B_n$  and  $D_n$ .

In our final chapter, we use the results of chapter 5 to look specifically at the alternating Hecke algebra associated to the symmetric group. We calculate the values of the irreducible characters on a set of minimal length conjugacy class representatives.