On Singular Solutions of Weighted Divergence Operators

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Abstract

We give a complete classification of the isolated singularities for a broad class of nonlinear elliptic equations of the form

$$-\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) + b(x)\,h(u) = 0 \quad \text{in } B^* := B_1 \setminus \{0\}, \tag{1}$$

where B_1 denotes the unit ball centred at 0 in $\mathbb{R}^N$ with $N \geq 2$. We assume that $\mathcal{A} \in C^1(0,1]$, $b \in C(\overline{B_1} \setminus \{0\})$ and $h \in C[0,\infty)$ are positive functions associated with regularly varying functions of index ϑ, σ and q at 0, 0 and ∞ respectively, satisfying $q > p-1 > 0$ and $\vartheta - \sigma < p < N + \vartheta$.

We prove that the condition $b(x)\,h(\Phi) \notin L^1(B_{1/2})$ is sharp for the removability of all singularities at 0 for the positive solutions of (1), where Φ denotes the "fundamental solution" of $-\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla\Phi|^{p-2}\nabla\Phi) = \delta_0$ (the Dirac mass at 0) in B_1, subject to $\Phi|_{\partial B_1} = 0$. If $b(x)\,h(\Phi) \in L^1(B_{1/2})$, we show that any non-removable singularity at 0 for a positive solution of (1) is either *weak* (i.e., $\lim_{|x|\to 0} u(x)/\Phi(|x|) \in (0,\infty)$) or *strong* ($\lim_{|x|\to 0} u(x)/\Phi(|x|) = \infty$). The main difficulty and novelty of this thesis, for which we develop new techniques, come from the explicit asymptotic behaviour of the strong singularity solutions in the critical case, which had previously remained open even for $\mathcal{A} = 1$. We also study the existence and uniqueness of the positive solution of (1) with a prescribed admissible behaviour at 0 and a Dirichlet condition on ∂B_1.

We also classify the behaviour near 0 of the positive solutions with isolated singularities for the weighted p-Laplacian equation

$$-\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B^*, \tag{2}$$

for $1 < p < \infty$. We show that all positive solutions of (2) either has a finite limit at the singularity (and, in certain cases, u can be extended as a continuous solution in the entire ball B_1), or has a weak singularity depending on the range

of p. We note there are no solutions with strong singularities to (2), unlike the case where absorption terms are introduced as in (1).

Statement of Authorship

This is to certify that to the best of my knowledge, the content of this thesis is my own work and that all the assistance received in preparing this thesis and sources have been acknowledged. This thesis has not been submitted for the award of any other degree or diploma.

Chapters 2, 3 and 4 of this thesis contain material as the paper by Chang, T.-Y. and Cîrstea, F. C., Singular solutions for divergence-form elliptic equations involving regular variation theory: Existence and classification, *Annales de l'Institut Henri Poincaré / Analyse non linéaire* (2016), 10.1016/j.anihpc.2016.12.001 *in press*. The majority of the analysis was performed by TYC. Permission to include the published material has been granted by the corresponding author FCC.

Authorship Statement Confirmation by Supervisor

As supervisor for the candidature upon which this thesis is based, I can confirm the authorship attribution statements above are correct.

Florica Corina Cîrstea
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Contents

Notations

$\mathbb{R}^N$: Euclidean space with points $x = (x_1, ..., x_N)$, $x_i \in \mathbb{R}$ and $|x| = (\sum_{i=1}^N x_i^2)^{1/2}$.
B_1: the open unit ball in $\mathbb{R}^N$ centred at 0.
B^*: $B_1 \setminus \{0\}$.
∇u: the gradient of u, that is $(\partial u/\partial x_1, ..., \partial u/\partial x_N)$, the gradient of u.
$\mu(x)$: the fundamental solution of the p-harmonic equation, as in (1.3).
$\Delta_{\mathcal{A},p} u$: the operator $\operatorname{div}(\mathcal{A}(|x|)\, |\nabla u|^{p-2} \nabla u)$.
δ_0: the Dirac mass at 0.
ω_N: the volume of the unit ball in $\mathbb{R}^N$.
Δu: the Laplacian of u, namely $\sum_{i=1}^N \partial^2 u/\partial x_i^2$.
E: the fundamental solution the Laplacian Δu.
G: the fundamental solution of the weighted Laplacian equation, as in (1.23).
$L^1(B_{1/2})$: $\{f$ is measurable on $B_{1/2} : \int_{B_{1/2}} |f(x)|\, dx < \infty\}$.
$L_{\mathcal{A}}$, L_b, L_h: slowly varying parts of regularly varying functions $\mathcal{A}$, b and h.
q_*: critical exponent, see (2.10).
F: a function defined by (2.11).
m_0, m_1, m_2: constants defined in (2.12).
$f(t) \sim g(t)$ as $t \to t_0$ for $t_0 \in \mathbb{R} \cup \{\infty\}$: $\lim_{t \to t_0} f(t)/g(t) = 1$.

1

Introduction

1.1 The Singularity Problem

> *These occurrences of singularities in cosmology and physics suggest that mathematically singularities will continue to be hard and difficult for some time to come.*
>
> – Salomon Bôchner, *Singularities and Discontinuities* [6]

The local behaviour of solutions for second-order, quasi-linear, elliptic, divergence structure, partial differential equations has been studied extensively by many authors. Given an open subset Ω in $\mathbb{R}^N$ and a subset Σ in Ω, the singularity problem seeks to describe the behaviour of all possible solutions u satisfying some prescribed partial differential equation in $\Omega \setminus \Sigma$. More specifically, it aims to answer the following questions: *Is it possible to extend u to the entire Ω such that the new function $\tilde{u}$ satisfies the same equation in Ω (removable singularity)? If no such extension of u is possible, what is the behaviour of u near Σ?*

Since the singularity problem arises from the presence of the singular set Σ, the location and size of Σ thus contribute a key role to the complexity of the prescribed partial differential equation and the techniques used to describe the admissible behaviours of u near Σ. The topic of *isolated singularities*, in particular, represents an extremely active area of research concerning many different classes of nonlinear elliptic equations. Recent contributions include, on the one hand, boundary singularities [38, 41] and, on the other hand, interior singularities for the fractional Laplacian [11, 13], the weighted p-Laplacian [63], non-homogeneous operators in divergence form [39], nonlinear equations with singular potentials [16, 25] or with nonlinearities depending on the gradient [3, 14] to name only a few. In this thesis, we focus mainly on domains with isolated interior singularities, namely the punctured unit ball $B^* := B_1 \setminus \{0\}$ centred at the origin.

The study itself on the behaviour of solutions to partial differential equations near isolated singularities dates back to 1903, when Maxime Bôcher [5] discovered that *any positive harmonic function u in the punctured unit ball $B_1 \setminus \{0\}$ from $\mathbb{R}^N$ (for $N \geq 2$) can be represented as a linear combination of a harmonic function in the whole unit ball with a fundamental solution of the Laplace operator*, that is, u must be of the form

$$\begin{cases} a\log(1/|x|) + g(x) & \text{if } N = 2, \\ a|x|^{2-N} + g(x) & \text{if } N \geq 3, \end{cases} \tag{1.1}$$

where a is a non-negative constant and g is a harmonic function in B_1.

The concept of the fundamental solution plays a significant role in the theory of elliptic second-order partial differential equations. Such a function is a single-valued radial solution of the operator, regular except for the singular point. As the prescribed problem evolves from the Laplace operator into more complex forms – one of which is the class of nonlinear elliptic partial differential equation –, so too does the fundamental solution evolve depending on the operator. We denote by μ the fundamental solution of the p-harmonic equation

$$-\text{div}(|\nabla\mu|^{p-1}\nabla\mu) = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{1.2}$$

the sense of distributions in $\mathbb{R}^N$ and where δ_0 is the Dirac mass at 0. Explicitly, μ is defined as

$$\mu(x) = \begin{cases} \dfrac{p-1}{N-p}(N\omega_N)^{-\frac{1}{p-1}}|x|^{\frac{p-N}{p-1}} & \text{for } 1 < p < N, \\ (N\omega_N)^{-\frac{1}{N-1}}\log\left(\dfrac{1}{|x|}\right) & \text{for } p = N, \end{cases} \tag{1.3}$$

and ω_N denotes, here and throughout, the volume of the unit ball in $\mathbb{R}^N$.

In two celebrated works, Serrin [48, 50] studied *a priori* estimates of solutions, the nature of removable singularities, and the behaviour of a positive solution in the neighbourhood of an isolated singularity for quasi-linear elliptic equations of the general form

$$\text{div}\,\mathbf{A}(x, u, \nabla u) = B(x, u, \nabla u). \tag{1.4}$$

For a domain Ω in $\mathbb{R}^N$ with $0 \in \Omega$, it is assumed that $\mathbf{A}(x, u, \xi)$ and $B(x, u, \xi)$ are, respectively, vector and scalar measurable functions defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$

satisfying the following growth conditions for all $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$:

$$\begin{cases} |\mathbf{A}(x,u,\xi)| \leq \beta_0|\xi|^{p-1} + \beta_1|u|^{p-1} + \beta_2, \\ \xi \cdot \mathbf{A}(x,u,\xi) \geq |\xi|^p - \beta_3|u|^p - \beta_4, \\ |B(x,u,\xi)| \leq \beta_6|\xi|^{p-1} + \beta_3|u|^{p-1} + \beta_5, \end{cases} \tag{1.5}$$

where $1 < p \leq N$ is a fixed exponent, β_0 is a positive constant and β_i for $1 \leq i \leq 6$ are measurable functions on Ω belonging to suitable Lebesgue classes where $\varepsilon > 0$:

$$\begin{cases} \beta_1, \beta_2 \in L^{N/(p-1-\varepsilon)}(\Omega), \\ \beta_j \in L^{N/(p-\varepsilon)}(\Omega) \text{ for } j = 3,4,5 \text{ and} \\ \beta_6 \in L^{N/(1-\varepsilon)}(\Omega). \end{cases} \tag{1.6}$$

Under the above conditions, Serrin was able to prove the following theorem.

Theorem 1.1.1 (see Theorem 1 of [50]). *Let u be a non-negative continuous solution of* (1.4) *in $\Omega \setminus \{0\}$ satisfying the assumption* (1.5) *and* (1.6). *Then the following dichotomy holds:*

(a) *either u has a removable singularity at 0;*

(b) *or there exist positive constants c_1 and c_2 such that*

$$c_1 \leq \frac{u(x)}{\mu(|x|)} \leq c_2 \tag{1.7}$$

in a neighbourhood of zero, where μ is defined by (1.3).

In general, however, even with such an established result as that of Serrin's, there are many difficulties in completely classifying solutions near an isolated singularity for partial differential equations. The difficulty of the singularity problem is due to a lack of an all-encompassing, general theory for a complete description of the behaviour of solutions near an isolated singularity for all nonlinear partial differential equations.

Despite the extensive research on the singularity problem, they generally satisfied the conditions (1.5), where the growth of $\mathbf{A}$ is bigger than that of B. For instance, Serrin's theorem above also covers the case $0 < q \leq p - 1$ for the particular problem

$$\text{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u \quad \text{in } B_1 \setminus \{0\}. \tag{1.8}$$

Corollary 1.1.1. *Let* $0 < q \le p - 1$ *and* u *be a solution of* (1.8). *Then*

(a) *If* $1 < p \le N$ *and* u *is bounded from below but not from above, then* u *satisfies* (1.7) *for some constants* c_1 *and* c_2.

(b) *If* $p > N$ *and* u *is bounded from below or from above, then* u *can be extended as a continuous function in the entire ball* B_1.

Yet, such were the difficulties of the converse condition that the problem (1.8) in the case $q \ge p - 1$, which does *not* satisfy (1.5), was only fully understood two decades later by Friedman–Véron [26] and Vázquez–Véron [58] (see Chapter 1.3 for the complete treatment).

The main goal of this thesis is thus to address the singularity problem for quasi-linear elliptic equations in divergence form related to (1.4) when *the growth of* B *is bigger than that of* $\mathbf{A}$, which is a challenge formulated by Véron [62]. The main difficulty in this case lies in the fact that solutions with strong singularities may appear, that is, the solution of the partial differential equation dominates the fundamental solution of the operator, see (c) of Theorem 1.3.1 for an example.

We aim to obtain a complete understanding of the isolated singularities for nonlinear elliptic equations of the form

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) = b(x)h(u) \quad \text{in } B^* := B_1 \setminus \{0\}, \tag{1.9}$$

where B_1 denotes the open unit ball in $\mathbb{R}^N$ ($N \ge 2$), and $\mathcal{A}$, b and h are under the framework of regular variation theory (see Assumption 1 in Chapter 2 for specificity and Chapter 1.5 definitions and properties of regular varying functions). A prototype for (1.9) is the pure power function model

$$\begin{cases} \mathcal{A}(|x|) = |x|^{\vartheta}, b(x) = |x|^{\sigma} \text{ and } h(t) = |t|^{q-1}t \quad \text{with} \\ q > p - 1 > 0,\ \vartheta - \sigma < p \le N + \vartheta, \end{cases} \tag{1.10}$$

as has been recently done by Song, Yin and Wang [54], where the difficulties that arise from regular variation theory such as integrability conditions for critical indices do not appear.

We seek to continue the recent works of [7, 16, 18] by extending the singularity problem from power-type nonlinearities into the framework of regular variation theory, as introduced by [12, 16, 17]. In particular, we extend the work of [7] from a weighted Laplacian to a weighted p-Laplacian operator, where the reliance on the linearity of the operator in u for many explicit calculations is no longer applicable.

We are only concerned with positive solutions of (1.9) since any non-negative solution of (1.9) is either identically zero or positive in B^* by the strong maximum principle (see [42, Theorem 2.5.1]), see Definition 2.1.1 for a definition of a solution to (1.9). We say that u has a singularity at the origin if u cannot be extended to a solution for all test functions in $C_c^1(B_1)$ on the entire ball B_1; otherwise, we say the origin has a removable singularity at zero.

In this thesis, we are interested in the necessary and sufficient conditions for the removability of a singularity 0, when can the solution be extended to the entire ball B_1? It is well known that the necessary and sufficient conditions for the harmonic function u to have a removable singularity at 0 is

$$u(x) = o(|x|^{2-N}) \text{ as } x \to 0.$$

Until recently such a precise result for quasi-linear equations was known only for positive solutions since the celebrated paper by Serrin [48], under relevant assumptions on the coefficients in terms of L^p-spaces. Our aim is to extend the removability results such as those of Brezis and Véron [9] below in Chapter 1.2.1. We are able to obtain a necessary and sufficient condition for which all solutions to our problem (1.9) are removable, recovering the removability result of Brandolini, Chiacchio, Cîrstea and Trombetti [7].

We also reveal that the behaviour of the solutions of (1.9) depends on the growth of $\mathcal{A}$, b and h, which manifests itself in the interplay between the indices of regular variation for $\mathcal{A}$, b, h and the dimension N, and how the weight $\mathcal{A}$ affects the aforementioned classifications under suitable assumptions. We establish a trichotomy of positive solutions of (1.9) under optimal conditions, generalising and extending previous results by [7, 18, 26], to name just a few. We also give the necessary and sufficient conditions depending on A, b and h for the existence of positive solutions to (1.9) in all categories of such a classification. As we reveal in this thesis, there exists a so-called critical case which is important in the *non-power nonlinearity* case as it represents the threshold between having *a trichotomy* classification (as in Theorem 2.2.1) or *no singularities at all* as in Theorem 2.2.2. Our results complement a series of works on removable and non-removable singularities, such as those of [9, 31, 49, 51, 58].

In the rest of this chapter, we review the two problems, removability and classification, in relation to two models of our problem (1.9), that of the Laplacian in Chapter 1.2 and the p-Laplacian in Chapter 1.3. In Chapter 1.2, we summarise the history of the model $\Delta u = h(u)$ in B^* before diverging into Chapter 1.2.1 and Chapter 1.2.2 the different histories and challenges associated with the removability and classification problems. Assuming the reader is now familiar with the aforementioned histories and challenges, we combine the problems of removability and classification in Chapter 1.3 for the p-Laplacian models. We end Chapter 1 with the theory of regular variation relevant to this thesis.

In this thesis, we will give a complete characterisation of the behaviour of the isolated singularities of solutions to nonlinear elliptic equations of the form

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) = f(x,u) \quad \text{in } B^*. \tag{1.11}$$

This thesis consists of two problems with a common operator, the weighted p-Laplacian. The majority of the thesis, Chapters 2, 3, and 4, is devoted to our first problem $f(x,u) = b(x)\,h(u)$ in (1.11) and Chapter 5 is devoted to our second problem when $f(x,u) \equiv 0$. The rest of this chapter recounts the history of the difficulties associated with classification of singular solutions to related partial differential equations, as well as a brief background of results in the regular variation theory framework essential to our thesis.

In Chapter 2, our objective is to set up a general framework of regular variation in which singular behaviour of solutions can be described. New difficulties arise from in this context of regular variation as it introduces new ideas of critical cases and integral conditions, such as that of the construction of our fundamental solution. The framework allows us to state our classification and existence results. In particular, we are able to give a complete classification of isolated singularities beyond pure power-type nonlinearities. In this new framework, cases of solutions with strong singularities, which does not appear in the pure power-type framework, can be found explicitly. We also provide sharp conditions guaranteeing the removability of all singularities, for a large class of quasilinear equations involving regular variation theory, as well as the sharp condition for the existence of singular solutions. We introduce some ingredients crucial to the analyses of our problem, namely an *a priori* estimate, a Harnack-type inequality, a regularity result, in all of which the technicality of regular variation theory plays an important role. Before proving our main results in Chapters 3 and 4, we supply the reader with examples and applications of our main results, specifically illustrating the criticality of the boundary cases as well as the necessity of our conditions.

Chapter 3 is devoted to the proofs of our main classification result, that is, the behaviour of the solution with removable, weak and strong singularities. Most importantly, we detail the proof of the behaviour of the solution with strong singularity, whose technicality, especially when influenced by regular variation theory, is evident in the number of cases and steps required for the proof. Most importantly here is the difficulty presented by the critical case for q (introduced in Chapter 2), where there lack an intuition as in the power-nonlinearity cases on the asymptotics of the strong singular solutions. It is this case in particular which necessitates a new perturbation technique. The differences in the proofs of the sub-critical and critical cases are compared throughout the proof. With the help of the crucial ingredients we introduced in the previous chapter, we use a range of techniques such as transformations and scaling arguments, adapted to the context of regular variation.

In Chapter 4, we give the proof to our removability result, emphasising the sharp criteria under which all singularities of the positive solutions to our prescribed problem are removable. Prescribing a Dirichlet boundary condition, we also prove our existence and uniqueness theorem, corresponding to the solutions with removable, weak and strong singularities in our classification theorems. We include also a collection of auxiliary results necessary for the proofs of the above.

In Chapter 5, we present analogues of Bôcher's theorem, classifying the behaviour near 0 of the positive solutions to the weighted divergence operator,

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) = 0 \quad \text{in } B^*. \tag{1.12}$$

The absence of solutions with strong singularities contrasts this chapter to its preceding chapters. Instead, we give a classification of solutions to (1.12) for $1 < p < \infty$ and show that either the solution has a finite limit at 0 or the solution has a weak singularity at the origin. In the former case, we prove that either the solution can be extended as a positive continuous solution in B_1 or we can further classify its behaviour. To this end, we introduce two notions of fundamental solutions. Our proofs make use of the adaptability of these fundamental solutions in the different cases.

We now continue with the remainder of this chapter with a history of associated difficulties and results of to (1.9).

1.2 The model $\Delta u = h(u)$ and Generalisations

Having first arose out of necessity as mathematical models in the physical sciences, research interest in singularities has since evolved as a topic in its own right as seemingly specialised questions – in regularity theories and geometric generalisations to manifolds, to name a few – prove to be ripe for analyses. One of the most well-known is the Emden-Fowler equations, that is,

$$-\Delta u + \varepsilon |u|^{q-1} u = 0, \tag{1.13}$$

with $\epsilon = \pm 1$ and $q > 1$. They have been extensively studied since the end of the 19th century as they model many important physical and geometrical phenomena under the assumption of radial solutions (see Emden [21] and Fowler [23]). Our main focus (1.9) is a generalisation of (1.13) with $\varepsilon = 1$ whose salient points of history and difficulties we expound on in this chapter.

For the case $\varepsilon = 1$ in B^*, under the framework of Thomas-Fermi theory of electric field potential determined by the nuclear charge and distribution of electrons in an atom ($N = 3$ and $q = 3/2$), Hille [30] and Sommerfeld [53] obtained the precise asymptotic behaviours of the *radial solutions* near the singularity. The advantage of radial solutions is the adaptability of classical methods of ordinary differential equations theory, such as asymptotic expansion and linearisation. In order to obtain the asymptotic isotropy of the positive *non-radial* solutions of (1.13) with $\varepsilon = 1$, Véron [60] introduced new methods involving *a priori* estimates of Keller-Osserman type to give an upper bound to the Harnack inequality coefficient. Véron was able to give a complete classification (see Theorem 1.2.3) of the behaviour near zero of all positive solutions of (1.13) for $\varepsilon = 1$, which was also proved later with different techniques by Brezis and Oswald [8].

We also briefly mention that the case with the opposite sign, $\varepsilon = -1$, has also been extensively studied (see Lions [34] for $1 < q < N/(N-2)$, Aviles [1] for $q = N/(N-2)$, Gidas and Spruck [27] for $N/(N-2) < q < (N+2)/(N-2)$, and Caffarelli, Gidas and Spruck [10] for $q = (N+2)/(N-2)$ which is relevant to Yang-Mills equation for $N = 4$ and differential geometry for $N > 2$). However, the analyses and proofs here differ from those of the former problem with $\varepsilon = 1$, such as the existence and nonexistence of a comparison principle between their solutions. This simple difference in sign in (1.13) illustrates how a comprehensive theory for partial differential equations is not achievable, but also what a rich mine of research a particular problem can be in and of itself.

Yet, despite the non-trivial difference between the equations (1.13) with $\varepsilon = \pm 1$ and their proofs, a similarity between their results was noticed in the case $N \geq 3$ and $1 < q < N/(N-2)$: there exist solutions u of (1.13) for both $\varepsilon = \pm 1$ satisfying

$$\lim_{x \to 0} |x|^{N-2} u(x) = \gamma \in [0, \infty). \tag{1.14}$$

It was further noted that for u satisfying (1.14), u^q is integrable near 0, opening the way into the study of isolated singularities of positive solutions of

$$\Delta u = h(u) \quad \text{in } B^*. \tag{1.15}$$

We address below in Chapter 1.2.1 one of the main questions in the research of isolated singularities: *what is the necessary and sufficient condition under which the solution of our prescribed problem can be extended to a continuous solution in the entire ball? Moreover, what is the most general framework in which this can be solved?* We reveal that even for (1.15), the problem proposed below by Vázquez and Véron thirty years ago remains open today. We summarise the frameworks in which these question have been answered for (1.15) and its generalisations. Moreover, in Chapter 1.2.2, we reveal that under the converse of the necessary and sufficient condition for removability, complete classifications of the behaviour of the singular solution can be given for the respective problems.

1.2.1 Removability

The question of the removability of singularities of solutions to elliptic partial differential equations has attracted the interests of many authors. For $\Delta u = h(u)$ in B^*, the removability of the singularity at the origin is known to depend on the growth rate of $h(u)$ near infinity relative to the exponent $N/(N-2)$ where $N \geq 3$. A natural question thus arose as to whether a complete removability of the singular behaviour of the solution can be established for general functions $h(u)$, or as formulated by Vázquez and Véron [59]:

Question. *What is the weakest condition on a continuous non-decreasing function h such that any isolated singularity of a non-negative solution of*

$$\Delta u = h(u) \quad \textit{in } B_1 \setminus \{0\} \tag{1.16}$$

with $N \geq 3$ is removable?

One well-studied example of (1.16) is

$$\Delta u = |u|^{q-1}u \quad \text{in } B^*. \tag{1.17}$$

Whereas for $1 < q < N/(N-2)$, it was found that there are solutions of (1.17) with isolated singularities (see Theorem 1.2.3 below), Brezis and Véron were able to show in their celebrated paper [9] the following removability result.

Theorem 1.2.1 (see Brezis–Véron [9])**.** *Let $q \geq N/(N-2)$ and $u \in C^2(B^*)$ be a positive solution of* (1.17)*, then any isolated singularity of* (1.17) *is removable, that is, u can be extended as a classical solution of* (1.17) *in the whole ball B_1.*

The conditions on $h(u)$ for a removability result of (1.16) were steadily generalised from the pure power case of (1.17) as Brezis and Véron asserted that if $N \geq 3$ and h is a non-decreasing function satisfying

$$\liminf_{|t|\to\infty} \frac{h(t)}{t^{N/(N-2)}} > 0 \quad \text{and} \quad \liminf_{|t|\to-\infty} \frac{h(t)}{t^{N/(N-2)}} > 0, \tag{1.18}$$

then any solution $u \in C^1(B^*)$ of (1.16) in $\mathcal{D}'(B^*)$ can be extended into a C^1-solution of (1.16) in $\mathcal{D}'(B)$, that is, the origin is a removable singularity. This results was improved by Vázquez and Véron [58] who showed that the conclusion holds under the weaker assumption

$$\liminf_{|x|\to\infty} \frac{h(x)\log(|x|)}{|x|^{N/(N-2)}} > 0. \tag{1.19}$$

Richard and Véron [44] further proved that if h is nondecreasing and satisfies the weak singularities existence condition

$$\int_0^1 r^{N-1} h(r^{2-N})\, dr < +\infty, \tag{1.20}$$

then any nonnegative $u \in C^2(B^*)$ satisfying (1.16) in B^* is such that $|x|^{N-2}u(x)$ converges to some $\gamma \in \mathbb{R}^+ \cup \{\infty\}$ as x tends to 0.

On the other hand, only recently, Brandolini *et al.* [7] generalised the question of removability to the framework of regular variation theory for the *weighted* Laplacian equation where the right-hand side remains $h(u) = |u|^{q-1}u$,

$$\operatorname{div}(\mathcal{A}(|x|)\,\nabla u) = u^q \quad \text{in } B^*, \tag{1.21}$$

where the function $\mathcal{A} \in C^1(0,1]$ is positive and can be expressed as $\mathcal{A}(t) = t^{\vartheta} L_{\mathcal{A}}(t)$ with $1 < p < N + \vartheta$ and $L_{\mathcal{A}}$ satisfying

$$\lim_{t \to 0^+} \frac{t L'_{\mathcal{A}}(t)}{L_{\mathcal{A}}(t)} = 0. \tag{1.22}$$

Due to the presence of the weight $\mathcal{A}$ in the operator, their notion of fundamental solution is no longer that of the Laplacian (see (1.34)), but G, defined as follows:

$$G(x) = G(|x|) := \frac{1}{N\omega_N} \int_{|x|}^{1} \frac{t^{1-N}}{\mathcal{A}(t)} dt \quad \text{for every } x \in B_1, \tag{1.23}$$

which can be seen as the fundamental solution of

$$-\operatorname{div}(\mathcal{A}(|x|)\nabla G) = \delta_0 \quad \text{in } \mathcal{D}'(B_1) \tag{1.24}$$

with homogenous Dirichlet boundary condition. They were able to obtain a necessary and sufficient condition, $G \notin L^q(B)$, for the following removability result.

Theorem 1.2.2 (see Theorem 2 of [7])**.** *Let $q > 1$. Every positive solution of* (1.21) *can be extended as a positive continuous solution of* (1.21) *in B_1 if and only if $G \notin L^q(B_1)$.*

Despite such generalisations, the problem of removability as formulated by Vázquez and Véron for (1.16) with a general continuous non-decreasing function $h(u)$ remains open. Even now, the removability of the strong singularity solutions is not completely clear even for Laplacian-type equations. In [59, Remark 2.2], Vázquez and Véron showed that for (1.16), there are examples of continuous non-decreasing functions h satisfying

$$\int_1^{\infty} t^{-\frac{2(N-1)}{N-2}} h(t)\, dt = \infty \quad \text{and} \quad \int_1^{\infty} \frac{dt}{\sqrt{th(t)}} = \infty \tag{1.25}$$

for which there exist no positive solutions with a weak singularity at 0, but infinitely many positive solutions with a strong singularity at 0. It is known (see [59, 61]) that a necessary and sufficient condition for the removability of the weak singularities of the positive solutions is that h satisfies the first condition in (1.25).

Indeed, Vázquez and Véron's question remains open. As the pure power case $h(u) = |u|^{q-1}u$ is fully understood for both the Laplacian and the weighted Laplacian, our goal is to generalise and solve the removability problem in the framework of regular variation for weighted *p-Laplacian* operators with absorption terms.

We include below, for the interested reader, an overview of removability results for equations with operators generalised from the Laplacian and weighted-Laplacian that we have seen above. The following selection is by no means comprehensive, but are chosen for their generalisation of the Laplacian operator as well as their analogous removability conditions.

Removable singularities in more general settings

The question of removability of singularities remains the same across the rich range of nonlinear partial differential equations of elliptic type: under what necessary and sufficient condition subject to the problem can the solution be extended to a continuous solution in the entire ball? It is a question that has been addressed by many authors with challenges arising dependent on the framework of the partial differential equation.

Its importance in the area of geometry has attracted much attention from authors such as Bers [2] who proved that every isolated singularity of a minimal surface having a simply covered plan projection is removable and the classical works of Serrin [49, 51] which proved that any solution to the equation in $\Omega \setminus K$ can be extended as a solution the equation in the entire Ω provided the compact set K has a vanishing $(d-1)$-Hausdorff measure.

Moreover, Vázquez and Véron [59] considered in Theorem 1.3.3 equations involving the p-Laplacian operator, while Labutin [32] studied fully nonlinear uniformly elliptic equations, namely the Pucci maximal operator given below in (1.30). In [22], Felmer and Quaas extended the removability results obtained in [32] to a wide class of nonlinear elliptic equations for which a “fundamental” solution can be constructed.

The question has also been generalised to solutions of quasilinear elliptic equations with absorption in [35] and also to the class of degenerating nonlinear elliptic equations [37]. A sufficient condition for the isolated singular point to be removable is found. In the absence of absorption and degeneration, this condition coincides with already known results.

Thereafter these results were generalised for different semi-, quasi- and nonlinear equations and their parabolic counterparts. Since Brezis and Véron, various extensions of this result have been obtained. We refer to the recent works of Liskevich and Skrypnik, [36, 35]. In [36], they study quasi-linear degenerate elliptic partial differential equations in divergence form,

$$-\operatorname{div} A(x,u,\nabla u) = a_0(x,u) \quad \text{in } B^*, \tag{1.26}$$

where A and a_0 satisfies the same conditions as that of (1.5). They established optimal pointwise conditions on u such that 0 is a removable singularity, that is, u can be extended to B_1. Their results do not assume positivity of u and extend several remarkable theorems from the literature. Shortly afterward, they studied in [35] the problem of removability of isolated singularities for a general second-order quasi-linear equation in divergence form in a punctured domain B^* of $\mathbb{R}^N$ for $N \geq 3$, whose model is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(x)|u|^{p-2}u + |u|^{q-1}u = 0, \tag{1.27}$$

where $q > p-1 > 0$, $p < N$, and g belongs to a suitable nonlinear Kato class of functions. With the addition of the nonlinear absorption term $|u|^{q-1}u$ in (1.27), the main result of the paper states that 0 is a removable singularity if

$$q \geq \frac{N(p-1)}{N-p}, \tag{1.28}$$

and extends the well-known result by Brezis and Véron (see Theorem 1.2.1).

It has been proved that these results hold for even more general operators than the Laplacian. For $0 < \lambda \leq \Lambda$, one generalisation of the Laplacian to F, a fully nonlinear uniformly elliptic second-order operator, is defined as

$$F(\Delta u) = \sup_{a \in [\lambda,\Lambda]^N} \sum_{i=1}^{N} a_i \lambda_i, \tag{1.29}$$

where λ_i for $i = 1, 2, ..., N$ are the ordered eigenvalues of the Laplacian. Substituting the Laplacian in (1.29) for any symmetric matrix M, the operator F is what is referred to as the Pucci maximal operator. For $N \geq 2$, Labutin obtained a removability result fully nonlinear uniformly elliptic equations of the form

$$F(D^2u) + f(u) = 0 \quad \text{in } B^*, \tag{1.30}$$

where f is a continuous function assumed to satisfy some sharp growth conditions depending on F. A model case is $f(u) = |u|^{q-1}u$. Then, by denoting with λ and Λ

the ellipticity constants of F, Labutin was able to prove, using comparison principles and the scale invariance of (1.30) as well as some new viscosity techniques, that every viscosity solution $u \in C(B^*)$ is actually a continuous solution in the whole ball if and only if

$$q \geq \frac{\lambda(N-1)+\Lambda}{\lambda(N-1)-\Lambda}. \tag{1.31}$$

This result extends the analogous property previously proved by Brezis and Véron – we see that for $\lambda = \Lambda = 1$ which coincides with the Laplace operator, (1.31) recovers the removability condition $q \geq N/(N-2)$ given in Theorem 1.2.1 –, but it requires new arguments to be used in place of the integral estimates obtained by integrating semilinear equations by parts.

Only recently, Felmer and Quaas [22] extended the results of Brezis–Véron and Labutin to an even large class of operators $M_{\mathcal{C}}(\Delta u) = \sup_{a\in\mathcal{C}} \sum_{i=1}^{N} a_i\lambda_i$ where $\mathcal{C}$ is a closed, convex, bounded subset of $\mathbb{R}^N_+$. The difficulty in Labutin and Felmer–Quaas' generalisation of the operator is that the notion of dimension has to be reconstructed, before a fundamental solution can even be defined. Theirs is yet another example where a slight change of the prescribed problem requires an entire new theory. Instead of the semilinear or quasilinear theory and the notion of distributional or weak solutions, the techniques for the fully nonlinear equations are new and based on the use of the viscosity notion of generalised solution (see [19] or [20] for background).

1.2.2 Classification

In the study of semilinear elliptic partial differential equations such as

$$\Delta u = h(u) \quad \text{in } B^*, \tag{1.32}$$

much research has been conducted towards understanding the role of the nonlinearity $h(u)$ and its interplay with the Laplacian. We have seen in Chapter 1.2.1 how the growth of h affects the condition necessary for all singular solutions to be removable. In this section, we reveal that under the obverse condition, the classification of the singular solutions are no longer only removable, but can also include singularities which we refer to as weak or strong.

As shown in the Emden-Fowler equations (1.13) for $\varepsilon = 1$, the equation (1.32) has been studied extensively in the case $h(u) = |u|^{q-1}u$, namely

$$-\Delta u + |u|^{q-1}u = 0 \quad \text{in } B^*. \tag{1.33}$$

Let us give below the classification result of Véron [60] and denote by $E(x)$ the fundamental solution of $(-\Delta)$ the Laplacian in $\mathbb{R}^N$,

$$E(x) = \begin{cases} \dfrac{1}{N(N-2)\omega_N}|x|^{2-N} & \text{for } N \geq 3, \\ \dfrac{1}{N\omega_N}\ln(1/|x|) & \text{for } N = 2, \end{cases} \tag{1.34}$$

where ω_N and δ_0 denote the volume of B_1 and the Dirac mass at 0, respectively. We note that this corresponds to the fundamental solution μ of the p-Laplacian, given by (1.3) when $p = 2$.

Theorem 1.2.3 (see Theorem 1.1 of Véron [60]). *If $1 < q < N/(N-2)$ and u is a positive solution of* (1.33) *in $C^2(B^*)$, then one of the following holds:*

(a) *u can be extended as a positive C^2-solution of* (1.33) *in B_1;*

(b) $\lim_{|x|\to 0} u(x)/E(x) = \lambda \in (0, \infty)$ *and u satisfies*

$$-\Delta u + |u|^{q-1}u = \lambda\delta_0 \quad \text{in } \mathcal{D}'(B); \tag{1.35}$$

(c) $u(x)/|x|^{-2/(q-1)} \to c_{q,N}$ *where $c_{q,N}$ is given by*

$$c_{q,N} = \left[\frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right]^{\frac{1}{q-1}} \tag{1.36}$$

We point out that the above theorem is in direct contrast to the result of Corollary 1.1.1(a). There, Serrin's classification of the solutions to (1.33) is for the range $0 < q \leq 1$ and, in turn, only consists of the first two behaviours (a) and (b) given in Theorem 1.2.3. The result revealed in (c) of Theorem 1.2.3 is thus a new behaviour which only occurs when the growth of the absorption term is bigger than that of the operator. Moreover, the constant $c_{q,N}$ is the unique positive constant such that $c_{q,N}|x|^{-2/(q-1)}$ satisfies (1.33). When u is not restricted to being a positive function, the limit in (c) becomes $\pm c_{q,N}$ provided that $\frac{N+1}{N-1} \leq q < \frac{N}{N-2}$. It was announced by Brezis and Véron in [9] that for $N \geq 3$ and $q \geq N/(N-2)$, isolated singularities are removable (see § 1.2.1 for further details of their result). We note in the case $q = N/(N-2)$, the constant $c_{q,N}$ in (c) vanishes.

Brezis and Oswald were able to provide a simpler proof of the classification in Theorem 1.2.3 where they employed a scaling argument (see Lemma 5 in [8]) rather than using Fowler's results [23] for the Emden differential equations. Furthermore, by the strict monotonicity of $|u|^{q-1}u$, they proved in [8, Theorem 8]) that the solution to (1.33) can be determined uniquely when prescribed with a suitable boundary condition.

With the complete understanding and classification of $\Delta u = |u|^{q-1}u$, Brandolini *et al.* [7] sought to generalise the problem by introducing a positive weight $\mathcal{A} \in C^1(0,1]$ in the operator in the framework of regular variation theory, where $\mathcal{A}(t) = t^{\vartheta} L_{\mathcal{A}}(t)$ with $1 < p < N + \vartheta$ and $L_{\mathcal{A}}$ satisfies $\lim_{t\to 0^+} tL'_{\mathcal{A}}(t)/L_{\mathcal{A}}(t) = 0$. They were able to extend Theorem 1.2.3 to the nonlinear elliptic equation in divergence form

$$\operatorname{div}\left(\mathcal{A}(|x|)\,\nabla u\right) = u^q \quad \text{in } B^*. \tag{1.37}$$

Not only does the additional weight $\mathcal{A}$ modify the fundamental solution, G, as given previously in (1.23), it also affected the classifications of the solutions to (1.37) as evident in the case $q = N/(N-2+\vartheta)$, which appears only in the framework of regular variation (see Theorem 1.2.4). The complete classification obtained by Brandolini *et al.* [7] makes clear the gap in Cîrstea–Du's classification [18] but also the difficulty in solving the problem in a weighted and nonlinear p-Laplacian setting as the former were able to rely on the linearity of (1.37) and some explicit calculations. Their main classification result, Theorem 1.2.4 is as follows along with their existence result, Theorem 1.2.5, (see also Theorem 1.2.2 for the removability result). We note that the previous inequality conditions on q is replaced here in the regular variation framework by an integrability condition.

Theorem 1.2.4 (see Theorem 3 of Brandolini *et al.* [7])**.** *Let $q > 1$. Assume $G \in L^q(B_1)$. Then for every positive solution u of* (1.37)*, exactly one of the following cases occurs:*

(a) *u can be extended as a positive continuous solution of* (1.37) *in B_1.*

(b) $\lim_{|x|\to 0} u(x)/G(x) = \lambda \in (0,\infty)$ *and u satisfies*

$$-\operatorname{div}(\mathcal{A}(|x|)\nabla u) + u^q = \lambda\delta_0 \quad \textit{in } \mathcal{D}'(B_1). \tag{1.38}$$

(c) $\lim_{|x|\to 0} u(x)/G(x) = \infty$, *in which case we have* $u(x) \sim \tilde{u}(|x|)$ *as* $|x| \to 0$, *where we define* $\tilde{u}(r)$ *for* $r \in (0,1)$ *as follows*

$$\tilde{u}(r) = \begin{cases} \left[\dfrac{(q-1)^2}{N-(N-2+\vartheta)q}\displaystyle\int_0^r \dfrac{t}{\mathcal{A}(t)}dt\right]^{-\frac{1}{q-1}} & \text{if } q < \dfrac{N}{N-2+\vartheta}, \\ (q-1)^{-\frac{1}{q-1}} G(r)\|G\|_{L^q(B_r)}^{-\frac{q}{q-1}} & \text{if } q = \dfrac{N}{N-2+\vartheta}. \end{cases}$$

Theorem 1.2.5 (see Theorem 2 of Brandolini et al [7]). *Let* $q > 1$. *The following assertions are true.*

(a) *There always exist positive continuous solutions of* (1.37) *in* B_1.

(b) *Every positive solution of* (1.37) *can be extended as a positive continuous solution of* (1.37) *in* B_1 *if and only if* $G \notin L^q(B_1)$.

(c) *There exist positive solutions of* (1.37) *such that* $\lim_{|x|\to 0} u(x)/G(x) \in (0,\infty]$ *if and only if* $G \in L^q(B_1)$.

As well as the generalisation of the Laplace operator, there have been attempts to generalise the absorption term $|u|^{q-1}$ in (1.33) for more general functions $h(u)$. One such example is from Cîrstea and Du [17] who extended Véron's classification result to positive solutions of $\Delta u = h(u)$ in B^* for h varying regularly at infinity of index $q \in (1, N/(N-2))$ for $N \geq 3$ (or $q \in (1,\infty)$ for $N = 2$), that is,

$$\lim_{t\to\infty} \frac{h(\lambda t)}{h(t)} = \lambda^q. \tag{1.39}$$

This framework expanded the function h from having a pure power-type behaviour at infinity to that of regular variation, as they demonstrated for $1 < q < N/(N-2)$ with the case

$$h(u) = |u|^{q-1}u\ln(1+|u|).$$

To overcome the lack of homogeneity of h, they used a perturbation method to obtain the precise limiting behaviour of the solutions with strong singularity at zero by constructing sub- and super-solutions.

As we see, the main difficulty for the Laplacian and weighted Laplacian problems such as $\Delta u = h(u)$ and $\operatorname{div}(\mathcal{A}(|x|)\nabla u) = h(u)$ is due to $h(u)$ and its influences on the necessary and sufficient condition for the removability problem and, in turn, the classification theorem. In the next section, we reveal how the generalisations of the Laplacian to p-Laplacians affect the classification and removability theorems, as well as increase the difficulty of their proofs.

1.3 The p-Laplacian Model

The classification theorem of (1.15) by Véron [60] in Theorem 1.2.3, along with many others [8, 9], has prompted much research into partial differential equations with more general operators. The main motivators for our research are the works of Friedman–Veron [26] and Vázquez–Véron [58] (in the case $\mathcal{A} = b = 1$ and $h(u) = |u|^{q-1}u$ of (1.9)). For $1 < p \leq N$, they give the complete profile of all positive solutions of p-Laplacian type equations (1.40) with pure power nonlinearities depending on the position of $q > p-1$ relative to the exponent $N(p-1)/(N-p)$.

Theorem 1.3.1 (see Theorem 2.1 of Friedman–Véron [26])**.** *Suppose* $1 < p \leq N$ *and* u *be a positive solution of*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u \quad \text{in } B^*. \tag{1.40}$$

If $p-1 < q < N(p-1)/(N-p)$ *(and* $q > p-1$ *if* $p = N$*), then exactly one of the following holds :*

(a) u *can be extended as a continuous solution of the same equation in* B_1 *(removable singularity);*

(b) *There exists a positive number* λ *such that* $u(x)/\mu(x) \to \lambda$ *(weak singularity) as* $|x| \to 0$*, and moreover,*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-1}u = \lambda^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

(c) $|x|^{p/(q-p+1)}u(x) \to \gamma_{N,p,q}$ *(strong singularity) as* $|x| \to 0$*, where* $\gamma_{N,p,q}$ *is a positive constant given by*

$$\gamma_{N,p,q} := \left[\left(\frac{p}{q-p+1}\right)^{p-1}\left(\frac{pq}{q-p+1} - N\right)\right]^{1/(q-p+1)}. \tag{1.41}$$

The alternatives (a), (b) and (c) in Theorem 1.3.1 correspond respectively to a positive solution u with $\limsup_{|x|\to 0} u(x)/\mu(x)$ equal to *(a)* zero, *(b)* a positive finite number, or *(c)* infinity. The positive solutions with a strong singularity at 0 are *all* obtained as limits of solutions with a weak singularity at 0. Their proof made use of the homogeneity of the power nonlinearity and various scaling arguments, which can be extended to a more general class of nonlinearities $h(u)$ under the condition (see [26, Remark 2.3])

$$\lim_{t\to\infty} \frac{h(t)}{t^q} = c > 0, \tag{1.42}$$

which limits $h(u)$ to behave like u^q near infinity (see [26, Remark 2.3]). In cases beyond the power nonlinearities, the understanding of strong singularities had until now remained elusive. Our main goal is to extend the classification result for nonlinearities which need not behave like a pure power at infinity.

Furthermore, Friedman–Véron were able to establish the following existence and uniqueness result for their problem.

Theorem 1.3.2 (see Theorem 1.1 and Theorem 1.2 of Friedman–Véron [26])**.** *Let $1 < p \leq N$ and u be a positive solution of* (1.40)*. Suppose $g \in C^1(\partial B_1)$ is a non-negative function and $\lambda \in (0,\infty)$, then the singular Dirichlet problem*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u & \text{in } B_1 \setminus \{0\}, \\ u = g & \text{on } \partial B_1, \\ \lim\limits_{|x|\to 0} \dfrac{u(x)}{\mu(x)} = \lambda, \end{cases} \tag{1.43}$$

admits a unique non-negative solution if and only if $q < N(p-1)/(N-p)$.

A research branch of interest, complementary to the classification results we have seen, is the removability of singularities of solutions to elliptic partial differential equations. Brezis and Véron's groundbreaking paper [9] deduced that (1.17) has the property that any isolated singularity of its solution is removable. This has been expanded by Labutin [32] for fully nonlinear uniformly elliptic equations and extended by Felmer and Quaas [22] to a large class of nonlinear second order elliptic differential operators for which a fundamental solution can be constructed. We illustrate with the following theorem by Vázquez and Véron the completeness of the classification singular solutions u to (1.40) which depends only on the position of q relative to $N(p-1)/(N-p)$.

Theorem 1.3.3 (see Vázquez–Véron [58])**.** *Let u be a positive solution of* (1.40) *for $1 < p < N$. If $q \geq N(p-1)/(N-p)$, then u can be extended as a continuous solution of* (1.40) *in B_1.*

The complete characterisation Friedman–Véron and Vázquez–Véron accomplished above for (1.40) has inspired many, including this thesis for the problem (1.9), to sought out complete classifications for more general problems. Two directions for generalisations have been explored – extension of the operator on the left-hand side (see Theorem 1.2.4) and that of the absorption terms on the right-hand side (see Theorem 1.3.4 below), which we amalgamate as (1.9) to obtain sharp results in the framework of regular variation.

The above results in Theorem 1.3.1 and Theorem 1.3.3 were extended by Cîrstea and Du in [18] to quasilinear elliptic equations of the form

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = b(x)\, h(u) \quad \text{in } B^*, \tag{1.44}$$

where b and h satisfies the following conditions:

(a) The function h is continuous on $\mathbb{R}$ and positive on $(0,\infty)$ with $h(0) = 0$ and $h(t)/t^{p-1}$ bounded for small $t > 0$, whereas b is a positive continuous function on $\overline{B_1} \setminus \{0\}$.

(b) There exist $q, \sigma \in \mathbb{R}$ with $q + 1 > p > -\sigma$ and functions L_h, L_b that are slowly varying at ∞ and at 0 respectively, such that

$$\lim_{t\to\infty} \frac{h(t)}{t^q L_h(t)} = 1 \text{ and } \lim_{|x|\to 0} \frac{b(x)}{|x|^\sigma L_b(|x|)} = 1. \tag{1.45}$$

This is the particular case $\mathcal{A} \equiv 1$ of our problem (1.9), (see Assumption 1 of Chapter 2). Cîrstea and Du overcame the lack of pure-power behaviour of the nonlinearity h by developing new techniques relying on regular variation theory. In particular, the precise limiting behaviour of u with strong singularities is obtained by introducing a new perturbation method for constructing sub- and super-solutions (see Chapter 2 of [18]). Their result is as below.

Theorem 1.3.4 (see Theorem 1.1 of Cîrstea and Du [18])**.** *Let* $1 < p \le N$ *and* $p - 1 < q < (N + \sigma)(p - 1)/(N - p)$*. If* u *is a positive solution of* (1.44)*, then as* $|x| \to 0$*, exactly one of the following applies:*

(a) $u(x)$ *has a finite limit and can be extended as a continuous solution of* (1.44) *in all* B_1*,*

(b) $u(x)/\mu(x)$ *converges to a positive constant* γ *and satisfies*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x)h(u) = \gamma^{p-1}\delta_0 \quad \textit{in } \mathcal{D}'(B_1), \tag{1.46}$$

(c) $|x|^p b(x)\frac{h(u(x))}{u^{p-1}(x)} \to \gamma_{N,p,q,\sigma}$ *as* $|x| \to 0$*, where* $\gamma_{N,p,q,\sigma}$ *is a positive constant given by*

$$\gamma_{N,p,q,\sigma} := \left(\frac{p+\sigma}{q-p+1}\right)^{p-1}\left(\frac{pq}{q-p+1} - N + \frac{(p-1)\sigma}{q-p+1}\right). \tag{1.47}$$

Theorem 1.3.5 (see Theorem 1.2 of Cîrstea and Du [18])**.** *Let* $1 < p \leq N$ *and* $p-1 < q < \frac{(N+\sigma)(p-1)}{N-p}$*. Assume that* $g \in C^1(\partial B_1)$ *is a non-negative function. If* $h(t)/t^{p-1}$ *is non-decreasing for* $t > 0$*, then for every* $\gamma \in [0,\infty) \cup \{+\infty\}$*, the following problem*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = b(x)\,h(u) \quad in\ B^* \\ \lim\limits_{|x|\to 0} \dfrac{u(x)}{\mu(x)} = \gamma, \\ u = g \ \ on\ \partial B_1 \end{cases} \tag{1.48}$$

admits a unique non-negative solution u_γ*, which is in* $C^{1,\alpha}_{\mathrm{loc}}(B^*)$ *for some* $\alpha \in (0,1)$*. Moreover, if* $\gamma \in [0,\infty)$*, then* (1.46) *holds with* $u = u_\gamma$*.*

Interestingly, the inequalities separating the classifications of solutions to (1.40), and other previous works involving pure-power nonlinearities, are no longer as distinct in the framework of regular variation. In fact, in some instances of $q = (N+\sigma)(p-1)/(N-p)$, the above classification by Cîrstea and Du still holds, but the behaviour of the solution u with the strong singularity in (c) remained elusive under the introduced perturbation method. This is due to the reliance of the perturbation method on the strict inequality of q which allows the sub- and super-solution to be perturbed to different regular variation indices (while still satisfying the strict inequality), thus allowing a comparison between their limiting behaviours. This perturbation technique unfortunately does not hold in the equality case, requiring a new perturbation idea which does not affect the indices of variation of the sub- and super-solutions as well as a method of comparing the limiting behaviours of two functions of the same regular variation index. The equality case of q also presents difficulties, requiring extra conditions, in the removability theorem below, which otherwise recovers Vázquez and Véron's result in Theorem 1.3.3.

Theorem 1.3.6 (see Theorem 1.3 of Cîrstea–Du [18])**.** *Let* $1 < p < N$ *and* $q \geq (N+\sigma)(p-1)/(N-p)$*. If* $q = (N+\sigma)(p-1)/(N-p)$*, then we additionally* *assume in addition that*

$$\liminf_{t\to\infty} \frac{h(t)}{t^{\frac{(p-1)(N+\sigma)}{N-p}}} > 0 \quad \textit{and} \quad \liminf_{|x|\to 0} \frac{b(x)}{|x|^\sigma} > 0. \tag{1.49}$$

Then any positive solution of (1.44) *can be extended as a continuous solution of* (1.44) *in the entire ball* B_1.

Our main contribution is a combination of (1.44) and (1.37), where the weight $\mathcal{A}$ again contributes to difficulties to the operator in terms of the definition of the fundamental solution, as well as the fact that the operator now does not satisfy Serrin's conditions in (1.4), a difficulty overcome by Brandolini *et al.*[7] by explicit calculations relying on the linearity of the operator in u, a course no longer available to our problem. Moreover, the interplay between the weight and the diffusion terms $b(x)$ and $h(u)$ more delicate as we see how regular variation theory complicates and generalises the restrictions on the range of the indices. This allowed us to obtain a complete classification, and in fact, push the classification right to the boundary of the critical exponent q^*, which was excluded in the classification of [18].

We were able to find a necessary and sufficient condition for which classifications holds, an analogous extension of the critical exponent inequality condition for which Friedman and Véron [26] find in one case that a trichotomy occurs and Vázquez and Véron [58] in the other that all solutions have removable singularities. The removability of singularities of solutions to elliptic partial differential equations, such as the removability result in the latter, has been a topic of much attention ever since the groundbreaking work of [9]. For readers unfamiliar with the theory of regular variation, we have included the relevant results and ideas used in this thesis in Chapter 1.5, (see [4, 46] for an extensive study of regular variation theory).

1.4 Divergence-form Elliptic Operators

As we saw at the beginning of this thesis, the pioneering Bôcher's Theorem [5], laid the foundation for classification results in the field of isolated singularities with its fundamental result in harmonic analysis. We saw in the preceding sections how the original problem slowly generalised to include more complex operators, such as the Laplacian and even the p-Laplacian, with absorption terms, such as (1.15), (1.40) and, eventually to the main focus of this thesis, (1.9).

Another generalisation of Bôcher's Theorem that we trace in this thesis in Chapter 4 is that of the operator itself. In connection to our study of (1.9), we now turn our attention to singular solutions of its operator, namely,

$$-\text{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B^*, \tag{1.50}$$

in $\mathbb{R}^N$ with $N \geq 2$, where $\mathcal{A}$ is a positive $C^1(0,1]$-function such that

$$\lim_{t\to 0^+} \frac{t\mathcal{A}'(t)}{\mathcal{A}(t)} = \vartheta \text{ for some } \vartheta \in \mathbb{R}. \tag{1.51}$$

The novelty of our work is that we are able to classify the singular solutions of (1.50) for the entire range of $1 < p < \infty$, whereas, with absorption terms as those in (1.9), our focus was on the range $1 < p < N + \vartheta$ (and potentially $p = N + \vartheta$) where the fundamental solution of the operator blows up.

We recall some highlights in the history of such studies into the behaviour of isolated singularities. In addition to studying (1.4), Serrin [52] also generalised Bôcher's theorem for solutions with isolated singularities to divergence equations of the form,

$$\text{div}\, A(x, \nabla u) = 0, \quad \text{in } B^*, \tag{1.52}$$

where $A(x, \nabla u)$ is a given continuous real N-vector valued function satisfying the following conditions for some $a \geq 1$, $b \geq 0$, $p > 1$:

$$\begin{cases} |A(x,\xi)| \leq a|\xi|^{p-1} + b^{p-1} \\ \xi \cdot A(x,\xi) \geq |\xi|^p - b^p, \\ (\xi - \eta)\cdot(A(x,\xi) - A(x,\eta)) > 0, \quad \xi \neq \eta. \end{cases} \tag{1.53}$$

Theorem 1.4.1 (see Serrin [52])**.** *Let u be a positive solution of equation* (1.52) *in B^* satisfying the assumption* (1.53)*.*

(a) *If $p \leq N$, then exactly one of the following happens,*

  (i) *either u has a removable singularity at* 0*;*

  (ii) *or there exist positive constants c_1 and c_2 such that*

$$c_1 \leq \frac{u(x)}{\mu(|x|)} \leq c_2 \tag{1.54}$$

  in a neighbourhood of zero, where μ is the fundamental solution of the *p-Laplacian operator, defined by* (1.3).

(b) *If, however, $p > N$, then u tends to a finite limit u_0 as x tends to zero, and moreover in the neighborhood of the origin one has*

$$c_3|x|^{(p-N)/(p-1)} \leq u - u_0 \leq c_4|x|^{(p-N)/(p-1)}, \tag{1.55}$$

for some positive constants c_3 and c_4.

Serrin's classification result was improved by Kichenassamy and Véron [31] for the p-harmonic function u, that is, a solution of the p-Laplacian,

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B^*. \tag{1.56}$$

Instead of showing the solution u is bounded by some positive constant multiple of the fundamental solution near the singularity, they proved that their *difference* is also locally bounded. Specifically, they proved the following.

Theorem 1.4.2 (see Theorem 1.1 of Kichenassamy–Véron [31])**.** *Let $1 < p \leq N$ and u be a p-harmonic function in B^* such that $u(x)/\mu(x)$ remains bounded in some neighbourhood of* 0*. Then there exists a real number γ such that*

$$u - \gamma\mu \in L^\infty_{\text{loc}}(B_1). \tag{1.57}$$

Moreover, u satisfies the following equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \gamma|\gamma|^{p-2}\delta_0 \tag{1.58}$$

in the sense of distributions in B_1.

Kichenassamy and Véron were able to obtain (1.57) by constructing appropriate sub- and super-solutions to (1.56) with respect to the fundamental solution μ and (1.58) by the method of rescaling, namely

$$u_r(\xi) = \frac{u(r\xi)}{\mu(r)} \quad \text{for } 0 < |\xi| < \frac{1}{r},$$

which is also used in more generalised equations as that of [18, 26]. They further remarked (see [31, Remark 1.6]) that their result can be extended to include the range $p > N$, where it has been proved by Serrin [52] that any bounded solution

u of (1.56) in B^* can be extended as a continuous function $\tilde{u}$ in B_1. Without loss of generality, we can suppose $\tilde{u}(0) = 0$. By the same methods of sub- and super-solutions, and rescaling with

$$\mu(x) = \frac{p-1}{N-p}(N\omega_N)^{-\frac{1}{p-1}}|x|^{\frac{p-N}{p-1}} \quad \text{for } p > N,$$

Kichenassamy and Véron's result can be adapted to the following isotropy result:

Theorem 1.4.3 (see Remark 1.6 of [31])**.** *Let $p > N$ and $\tilde{u}$ be a p-harmonic function in B^* such that $|\tilde{u}(x)| \leq C|x|^{(p-N)/(p-1)}$ for $|x| \leq 1$. Then there exists a real number γ such that* (1.58) *and* (1.57) *hold.*

Similar to the case $p \leq N$ for Serrin and Kichenassamy–Véron, Bôcher's result has been found to hold for a variety of operators. Gilbarg and Serrin considered in [28] the uniformly elliptic equations of the form

$$Lu = \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + c(x)u = 0. \tag{1.59}$$

They proved a general form of Bôcher's theorem for $N = 2$ assuming only boundedness of the coefficients, a_{ik} and b_i, and uniform ellipticity and for $N \geq 2$ assuming the coefficients to be uniformly Hölder continuous. Even in the context of *infinity harmonic functions* u, that is, the solutions of the infinity Laplace equation, which can be viewed in a certain sense as the limit of the p-Laplacian as $p \to \infty$,

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i x_j} = 0, \tag{1.60}$$

Savin [45] proved that either the non-negative u has a removable singularity at 0 or $u(x) = u(0) + c|x| + o(|x|)$ near 0 for some fixed constant $c \neq 0$.

In [33], Labutin characterised isolated singularities of solutions of fully nonlinear, second-order, uniformly elliptic equations

$$F(\Delta u) = 0 \quad \text{in } B^*. \tag{1.61}$$

Under certain conditions on F, he proves that any nonnegative viscosity solution of (1.61) in a punctured ball either extends to the solution in the entire ball or behaves near the center of the ball like a special radial fundamental solution.

Only recently, Mihăilescu [39] considered the generalised weighted operator

$$\operatorname{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \quad \text{in } B^*, \tag{1.62}$$

where φ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ of class C^1 satisfying

$$0 < \delta \le \frac{t\varphi'(t)}{\varphi(t)} \le \varphi_0 < N - 1 \quad \text{for all } t \ge 0, \tag{1.63}$$

where δ and φ_0 are constants such that $0 < \delta \le \varphi_0 < N - 1$. He found that under suitable assumptions on φ, if u is a non-negative solution to (1.62), then

(a) either 0 is a removable singularity of u, or

(b) u behaves near 0 as a fundamental solution of (1.62).

Motivated by the advances on isolated singularities above, in particular the completeness of the range of p by Kichenassamy and Véron, we have sought to generalise these results in Chapter 5 for our generalised operator (1.50) for $1 < p < \infty$.

1.5 Regular Variation Theory

As there is no general theory for the solvability of all partial differential equations, regular variation opens up areas beyond the power nonlinearities, ushering in an extension of the currently limited analytic theory. In this thesis, we explore the behaviour of the singular solutions to our problem

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right)u = b(x)\,h(u) \quad \text{in } B^*$$

where $\mathcal{A}$, b and h have evolved naturally from the framework of power functions as seen in works such as that of Friedman and Véron [26] to the more generalised and less restrictive framework of regular variation.

The regular variation theory initiated by Karamata in the 1930's has been very fruitful in statistics in connection with extreme value theory (statistical estimation of tails, rates of convergence). It also plays a crucial role in probability theory (weak limit theorems such as central limit theorem and the weak law of large numbers; branching processes; stability and domains of attraction; fluctuation theory; renewal theory) and its applications include areas such as analytic number theory, financial engineering and complex analysis (see [4] for a comprehensive treatment of regular variation theory and its applications, also [43, 46]).

We recall below the concepts and properties of regularly varying functions.

Definition 1.5.1 (Regularly varying functions)**.**

(a) A positive measurable function L defined on a neighbourhood of ∞ is called *slowly varying* at ∞ if

$$\lim_{t\to\infty}\frac{L(\xi t)}{L(t)}=1 \quad \text{for every } \xi>0.$$

(b) The function $r\longmapsto L(r)$ is *slowly varying at (the right of) zero* if $t\longmapsto L(1/t)$ is slowly varying at ∞.

(c) A function f is *regularly varying at* ∞ (respectively, 0) with real index m, in short $f\in RV_m(\infty)$ (respectively, $f\in RV_m(0+)$) if $f(t)/t^m$ is slowly varying at ∞ (respectively, 0).

Example 1. A regularly varying function of index zero is called a slowly varying function. Any positive constant function is trivially a slowly varying function. Other non-trivial examples of slowly varying functions at ∞ are given by:

(a) The logarithm $\ln t$, its iterates $\ln_n t$ (defined as $\ln\ln_{n-1} t$) and powers of $\ln_n t$ for any integer $n\geq 1$.

(b) $\exp\left(\frac{\ln t}{\ln\ln t}\right)$.

(c) $\exp((\ln t)^\nu)$ with $\nu\in(0,1)$.

(d) $\exp\{(\ln t)^{1/3}\cos((\ln t)^{1/3})\}$.

Remark 1.5.1. Note that $\lim_{t\to\infty} f(t)=\infty$ (respectively, 0) for any $f\in RV_m(\infty)$ with $m>0$ (respectively, $m<0$). However, the limit at ∞ of a slowly varying function L at ∞ cannot be determined in general, and it may not even exist (see example (d) above for which $\liminf_{t\to\infty} L(t)=0$ and $\limsup_{t\to\infty} L(t)=\infty$).

Proposition 1.5.1 (Uniform Convergence Theorem)**.** *If L is a slowly varying function at zero, then $L(\xi t)/L(t)\to 1$ as $t\to 0$, uniformly on each compact ξ-set in $(0,\infty)$.*

Theorem 1.5.2 (Representation Theorem)**.** *The function L is slowly varying at 0 if and only if we have*

$$L(t) = \eta(t) \exp\left(\int_t^c \frac{\varepsilon(r)}{r}\, dr \right), \qquad 0 < t \le c$$

for some $c > 0$, where η is a measurable function on $(0, c]$ satisfying

$$\lim_{t \to 0^+} \eta(t) = \eta \in (0, \infty)$$

and ε is a continuous function on $(0, c]$ such that $\lim_{t\to 0^+} \varepsilon(t) = 0$.

Remark 1.5.2. If $\eta(t)$ is replaced by a positive constant η, then the new function η is referred to as a *normalised slowly varying function.* In this case,

$$\varepsilon(t) = -\frac{tL'(t)}{L(t)} \quad \text{for } 0 < t \le c.$$

Conversely, any positive function $\tilde{L} \in C^1(0, c]$, satisfying $\lim_{t\to 0^+} t\tilde{L}'(t)/\tilde{L}(t) = 0$, is a normalised slowly varying function.

Remark 1.5.3. Any slowly varying function at zero is asymptotically equivalent to a normalised slowly varying one.

Theorem 1.5.3 (Karamata's Theorem at 0)**.** *Let f vary regularly at zero with index ρ and be locally bounded on $(0, c]$. The following assertions hold:*

(a) *For any $j \le -(\rho + 1)$, we have*

$$\lim_{t\to 0^+} \frac{t^{j+1} f(t)}{\int_t^c r^j f(r)\, dr} = -\left(j + \rho + 1\right);$$

(b) *For any $j > -(\rho + 1)$ (and for $j = -(\rho + 1)$ if $\int_{0^+} r^{-\rho-1} f(r)\, dr < +\infty$), we have*

$$\lim_{t\to 0^+} \frac{t^{j+1} f(t)}{\int_0^t r^j f(r)\, dr} = j + \rho + 1.$$

Proposition 1.5.4 (Karamata's Theorem at ∞)**.** *If $f \in RV_\rho(\infty)$ is locally bounded in $[A, \infty)$, then*

(a) *For any $j \ge -(\rho + 1)$, we have*

$$\lim_{t\to\infty} \frac{t^{j+1} f(t)}{\int_A^t \xi^j f(\xi)\, d\xi} = j + \rho + 1.$$

(b) *For any $j < -(\rho+1)$ (and for $j = -(\rho+1)$ if $\int^{\infty} \xi^{-(\rho+1)} f(\xi)\, d\xi < \infty$), we have*

$$\lim_{t\to\infty} \frac{t^{j+1} f(t)}{\int_t^{\infty} \xi^j f(\xi)\, d\xi} = -(j+\rho+1).$$

As in [43], we denote by $f^{\leftarrow}$ the (left continuous) inverse of a non-decreasing function f on $\mathbb{R}$, namely

$$f^{\leftarrow}(t) = \inf\{s : \quad f(s) \geq t\}.$$

Proposition 1.5.5 (see Proposition 0.8 in [43])**.** *We have*

(a) *If $f \in RV_{\rho}(\infty)$, then $\lim_{t\to\infty} \ln f(t)/\ln t = \rho$.*

(b) *If $f_1 \in RV_{\rho_1}(\infty)$ and $f_2 \in RV_{\rho_2}(\infty)$ with $\lim_{t\to\infty} f_2(t) = \infty$, then*

$$f_1 \circ f_2 \in RV_{\rho_1\rho_2}.$$

(c) *Suppose f is non-decreasing, $f(\infty) = \infty$, and $f \in RV_{\rho}(\infty)$ with $0 < \rho < \infty$. Then*

$$f^{\leftarrow} \in RV_{1/\rho}(\infty).$$

2

Main Results

This chapter is devoted to the properties and assumptions required to state and prove our existence and classification theorems for the weighted p-Laplacian equation with absorption terms.

Our objective is to set up a general framework of regular variation in Chapter 2.1 under which we can introduce our main theorems, Theorem 2.2.1 and Theorem 2.2.2 in Chapter 2.2. In Theorem 2.2.2, we provide the sharp criteria for the removability of all singularities of the positive solutions of our prescribed problem. In the case of non-removable singularities, we give a complete classification of the singularities in Theorem 2.2.1, accompanied by corresponding existence results in Theorem 2.2.3. In Chapter 2.3, we supply the reader with examples and applications of our main results, Theorem 2.2.1 and Theorem 2.2.2, on the Examples of Table 2.1 with the intention of highlighting the criticality of the boundary cases as well as the necessity of our conditions. Note that L_h in Example 1 satisfies

$$t \longmapsto L_h(e^t) \text{ is regularly varying at } \infty \text{ with index } \gamma \in \mathbb{R}. \tag{2.1}$$

In Example 3, we see that $L_{\mathcal{A}}$ and L_b satisfy the following property

$$t \longmapsto \left[L_{\mathcal{A}}(e^{-t})\right]^{-\frac{q}{p-1}} L_b(e^{-t}) \text{ is regularly varying at } \infty \text{ with index } j \in \mathbb{R}. \tag{2.2}$$

Before proving our main results in Chapter 3, we introduce in Chapter 2.5 some ingredients crucial to the analyses of our problem. This includes an *a priori* estimate, a Harnack-type inequality, and a regularity result, whose proofs have had to be adapted due to the technicality of regular variation theory.

2.1 Our Prescribed Problem

We aim to fully classify the isolated singularities for nonlinear elliptic equations of the form

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) = b(x)\,h(u) \tag{2.3}$$

in the punctured unit ball $B^* := B_1 \setminus \{0\}$ in $\mathbb{R}^N$ $(N \geq 2)$ under the following structural conditions:

Assumption 1.

($\mathbf{A}_1$) *The function $\mathcal{A} \in C^1(0,1]$ is positive and can be expressed as $\mathcal{A}(t) = t^{\vartheta} L_{\mathcal{A}}(t)$ with $1 < p < N + \vartheta$ and $L_{\mathcal{A}}$ satisfying*

$$\lim_{t\to 0^+} \frac{t L_{\mathcal{A}}'(t)}{L_{\mathcal{A}}(t)} = 0. \tag{2.4}$$

($\mathbf{A}_2$) *The function h is continuous on $\mathbb{R}$ and positive on $(0,\infty)$ with $h(0) = 0$ and $h(t)/t^{p-1}$ bounded for small $t > 0$, whereas b is a positive continuous function on $\overline{B_1} \setminus \{0\}$.*

($\mathbf{A}_3$) *There exist $q, \sigma \in \mathbb{R}$ with $q + 1 > p > \vartheta - \sigma$ and functions L_h, L_b that are slowly varying at ∞ and at 0 respectively, such that*

$$\lim_{t\to\infty} \frac{h(t)}{t^q L_h(t)} = 1 \ \textit{and} \ \lim_{|x|\to 0} \frac{b(x)}{|x|^{\sigma} L_b(|x|)} = 1. \tag{2.5}$$

(See Chapter 2.4.1 for implications of the above assumptions.)

Definition 2.1.1. A function u is said to be a *solution* (*sub-solution, super-solution*) of (2.3) if $u(x) \in C^1(B^*)$ and

$$\int_{B_1} \mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u \cdot \nabla\phi\,dx + \int_{B_1} b(x)\,h(u)\,\phi\,dx = 0 \qquad (\leq 0,\ \geq 0) \tag{2.6}$$

for all functions (non-negative functions) $\phi(x)$ in $C_c^1(B^*)$, the space of all $C^1(B^*)$-functions with compact support in B^*. Furthermore, a positive solution u of (2.3) is said to be extended as a positive continuous solution of (2.3) in B_1 if there exists $\lim_{|x|\to 0} u(x) \in (0,\infty)$, the function $\mathcal{A}(|x|)\,|\nabla u|^{p-1}$ belongs to $L^1_{\mathrm{loc}}(B_1)$, and (2.6) holds for every $\phi \in C_c^1(B_1)$.

Remark 2.1.1. If u is a positive solution of (2.3) with $\limsup_{|x|\to 0} u(x) < \infty$, then both integrals in (2.6) are well-defined for every $\phi \in C_c^1(B_1)$. Indeed, we have that $b \in L^1_{\text{loc}}(B_1)$ since $\sigma > -N$ (from ($\mathbf{A}_1$) and ($\mathbf{A}_3$)), whereas the gradient estimates in Lemma 2.5.4 give that $\mathcal{A}(|x|)\, |\nabla u|^{p-1} \in L^1_{\text{loc}}(B_1)$ since $t^{1-p}\mathcal{A}(t)$ is regularly varying at 0 with index $\vartheta - p + 1$ (greater than $-N$).

Throughout this thesis, we are concerned with non-negative solutions of (2.3). By the strong maximum principle, any non-negative solution of (2.3) is either identically zero or positive in B^*. Indeed, the conditions in Theorem 2.5.1 of [42] are satisfied on any subset $\Omega \subset\subset B_1 \setminus \{0\}$ with

$$\begin{cases} \tilde{\mathbf{A}}(x, u, \nabla u) = \mathcal{A}(|x|)\, |\nabla u|^{p-2} \nabla u, \\ \tilde{B}(x, u, \nabla u) = -b(x)\, h(u), \end{cases} \tag{2.7}$$

since $\mathcal{A} \in C(0, 1]$ is a positive function, while h and b satisfy the properties in Assumption ($\mathbf{A}_2$).

The study of the local behaviour of singular solutions to nonlinear elliptic equations relies heavily on the properties of what is referred to as the fundamental solutions. In our analysis, instead of $\mu(x)$, the fundamental solution of the p-Laplacian given by (1.3), a crucial role is played by the function Φ given below. This necessary change is akin to the fundamental solution (1.23) of the weighted Laplacian (1.24) given by Brandolini *et al.* [7] as modified from the fundamental solution (1.34) of the Laplacian.

Let $C_{N,p} := (N\omega_N)^{-1/(p-1)}$, where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Assuming ($\mathbf{A}_1$), we can define the "fundamental solution" of the operator $\Delta_{\mathcal{A},p}(\cdot)$ in $\mathcal{D}'(B_1)$, namely

$$\Phi(r) := C_{N,p} \int_r^1 \left(\frac{t^{1-N-\vartheta}}{L_{\mathcal{A}}(t)} \right)^{\frac{1}{p-1}} dt \quad \text{for all } r \in (0, 1]. \tag{2.8}$$

Note that Φ can thus be seen as the fundamental solution of our weighted p-Laplacian operator with homogeneous Dirichlet boundary condition.

$$\begin{cases} -\Delta_{\mathcal{A},p}\Phi = \delta_0 & \text{in } \mathcal{D}'(B_1) \\ \Phi = 0 & \text{on } \partial B_1. \end{cases} \tag{2.9}$$

(See Chapter 2.4.2 for more details.)

Our analysis brings new understanding of the behaviour of the solutions to

(2.3) with strong singularities at zero as the perturbation technique introduced in [18] for the subcritical case is not applicable in the critical case. The main innovation we develop is a perturbation technique which enables us to give precise explicit asymptotic formulas for the behaviour of the strong singular solutions. Our Theorem 2.2.1 and Theorem 2.2.3 extend the corresponding optimal results in Theorem 1.2.4 and Theorem 1.2.2. While the understanding of strong singularity solutions for Laplacian-type equations with power-like nonlinearities in [7] relied on the earlier work of Taliaferro [55], this is no longer possible in our general context of quasi-linear equations such as (2.3).

2.2 Classification and Existence Theorems

In this section, we give our main theorems on the classification of singularities, the removability of singularities as well as the existence of the solutions.

Our central result, Theorem 2.2.1, establishes a complete classification of positive solutions for (2.3) assuming that $p-1<q<q_*$, where q_* shall be henceforth referred to as a *critical exponent* and be defined by

$$q_* = \frac{(N+\sigma)(p-1)}{N+\vartheta-p}. \tag{2.10}$$

We also show that the condition $b(x)\,h(\Phi)\in L^1(B_{1/2})$ is sharp and there exist solutions in each category of Theorem 2.2.1 under suitable regularity and monotonicity assumptions. From a practical viewpoint (see Chapter 2.4.3 for details), we thus need to check $b(x)\,h(\Phi)\in L^1(B_{1/2})$ only for $q=q_*$ (see Chapter 2.4 for more details). In such a critical case, assuming either (2.1) or (2.2), then $b(x)\,h(\Phi)\in L^1(B_{1/2})$ if and only if $F(r)<\infty$, where we define

$$F(r) := \int_0^r \xi^{-1}\,[L_{\mathcal{A}}(\xi)]^{-\frac{q_*}{p-1}}\,L_b(\xi)\,L_h(1/\xi)\,d\xi \quad \text{for } r>0 \text{ small.} \tag{2.11}$$

The function F plays an important role in the asymptotic behaviour at zero for a strong singularity solution of (2.3).

For convenience, we also define three constants, m_0, m_1 and m_2, whose positivity follows from Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$,

$$m_0 := \frac{p+\sigma-\vartheta}{q-p+1}, \qquad m_1 := \frac{q-p+1}{p-1}, \qquad m_2 := \frac{N+\vartheta-p}{p-1}. \tag{2.12}$$

We now state our first main result.

Theorem 2.2.1 (Classification of singularities)**.** *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. If* $b(x)\,h(\Phi) \in L^1(B_{1/2})$*, then for every positive solution* u *of* (2.3)*, exactly one of the following cases occurs:*

(i) *u can be extended as a positive continuous solution of* (2.3) *in the whole ball* B_1 *(in the sense of Definition 2.1.1).*

(ii) *u has a weak singularity at* 0*, that is* $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda \in (0,\infty)$ *and, moreover, u verifies*

$$-\Delta_{\mathcal{A},p} u + b(x)\,h(u) = \lambda^{p-1}\delta_0 \quad \textit{in } \mathcal{D}'(B_1). \tag{2.13}$$

(iii) *u has a strong singularity at* 0*. Moreover,* $\lim_{|x|\to 0} u(x)/\tilde{u}(|x|) = 1$*, where* $\tilde{u}$ *is given by*

$$\int_{\tilde{u}(r)}^{\infty} \frac{t^{-\frac{q+1}{p}}}{[L_h(t)]^{\frac{1}{p}}}\, dt = \int_0^r \left[M \frac{\xi^{\sigma-\vartheta} L_b(\xi)}{L_{\mathcal{A}}(\xi)} \right]^{\frac{1}{p}} d\xi \quad \textit{if } q < q_*, \tag{2.14}$$

where the reciprocal of M is defined as

$$\frac{1}{M} := q - \frac{N+\sigma}{m_0}. \tag{2.15}$$

However, in the critical case $q = q_*$, then $\lim_{|x|\to 0} u(x)/\tilde{u}(|x|) = 1$*, where if* (2.1) *holds, then $\tilde{u}$ is given by*

$$\tilde{u}(r) = \left[m_1 m_0^{\gamma+1-p} F(r) \right]^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0} \quad \textit{or} \tag{2.16}$$

if (2.2) *holds, $\tilde{u}$ is given by*

$$\int_c^{\tilde{u}(r)} [F(1/t)]^{\frac{1}{q_*-p+1}}\, dt = \left(m_1 m_0^{-p-j} \right)^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0}, \tag{2.17}$$

where m_0, m_1 and F are prescribed by (2.12) *and* (2.11)*, respectively. In* (2.17)*, $c > 0$ is a large constant.*

Remark 2.2.1. When $\mathcal{A} = b = 1$ and $h(t) = |t|^{q-1}t$, our Theorem 2.2.1 recovers Theorem 1.3.1. Moreover, Theorem 2.2.1 generalises and sharpens Theorem 1.3.4, which analysed the case $\mathcal{A} = 1$ and $q < q_*$. Our Theorem 2.2.1 is also established under the optimal condition for the existence of solutions with singularities at 0 for (2.3). Even for $\mathcal{A} = 1$, the behaviour of the strong singularity solutions in the critical case $q = q_*$ is new, being obtained via a perturbation technique we devise in this thesis (see Chapter 3.2.1).

Understanding the blow-up singular behaviour at zero for Theorem 2.2.1 (iii) is more intricate than in Friedman–Véron [26] due to the inhomogeneity of h and even Cîrstea–Du [17] and Brandolini *et al.* [7] due to the richness of the weights $\mathcal{A}$ and b respectively.

Theorem 2.2.2 (Sharp Removability)**.** *If $b(x)\,h(\Phi) \notin L^1(B_{1/2})$, then $q \geq q_*$ and every positive solution of* (2.3) *can be extended as a positive continuous solution of* (2.3) *in the whole ball B_1.*

Remark 2.2.2. When $\mathcal{A} = b = 1$ and $h(t) = t^q$, Theorem 2.2.2 recovers Theorem 1.2.1 and Theorem 1.3.3, the removability result of Brezis–Véron $p = 2$ and Vázquez–Véron for $1 < p < N$ respectively. By letting $\mathcal{A} = 1$ in Theorem 2.2.2, we also obtain a sharp version of the result of Cîrstea–Du in Theorem 1.3.6.

A singular behaviour such as described in each case of Theorem 2.2.1 can be prescribed together with a boundary condition, and these uniquely determine the solution. Next, in our second main result, under suitable conditions, we show that there exist positive solutions of (2.3) in any of the categories appearing in the complete classification of Theorem 2.2.1. Furthermore, we obtain a uniqueness result for (2.3) subject to a Dirichlet condition on the boundary ∂B_1 with a prescribed, admissible behaviour at zero.

Theorem 2.2.3 (Existence and uniqueness)**.** *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. Assume that h is a non-decreasing function on $(0,\infty)$ and $g \in C^1(\partial B_1)$ is an arbitrary non-negative function. We consider the following problem*

$$\begin{cases} \Delta_{\mathcal{A},p} u = b(x)\,h(u) \quad \text{in } B^* := B_1 \setminus \{0\}, \\ \lim_{|x|\to 0} \frac{u(x)}{\Phi(x)} = \lambda, \quad u > 0 \quad \text{in } B^* \\ u\big|_{\partial B_1} = g. \end{cases} \tag{2.18}$$

(a) *If $\lambda = 0$ and $g \not\equiv 0$ on ∂B_1, then* (2.18) *has a unique solution.*

(b) *If $\lambda \in (0,\infty]$, then* (2.18) *admits solutions if and only if $b(x)\,h(\Phi)$ is in $L^1(B_{1/2})$.*

(c) *Assume that* $b(x)\,h(\Phi) \in L^1(B_{1/2})$ *and* $h(t)/t^{p-1}$ *is non-decreasing for* $t > 0$.

(i) *For* $\lambda \in (0, \infty)$, *then* (2.18) *has a unique solution. The same conclusion holds for* $\lambda = \infty$ *and* $q < q_*$.

(ii) *For* $\lambda = \infty$ *and* $q = q_*$, *then* (2.18) *has a unique solution provided that either* (2.1) *or* (2.2) *holds.*

Remark 2.2.3. When $b = 1$ and $h(t) = |t|^{q-1}t$, our Theorem 2.2.3 recovers previous results such as that of Friedman–Véron in Theorem 1.3.2 (with $\mathcal{A} = 1$) and Brandolini *et al.* in Theorem 1.2.5 (with $p = 2$). Moreover, in Theorem 2.2.3, we generalise Cîrstea–Du's result in Theorem 1.3.5 (where $\mathcal{A} = 1$) by sharpening the condition under which there exists a unique singular solution to (2.18).

Remark 2.2.4. For this problem (2.3), we focus on the case $p < N + \vartheta$ in Assumption $(\mathbf{A}_1)$. However, we mention that Theorem 2.2.1 and Theorem 2.2.3, apart from (2.13), remain valid also for the case $p = N + \vartheta$ provided that $\limsup_{r\to 0^+} L_{\mathcal{A}}(r) < \infty$ (which ensures that $\Phi(r) \to \infty$ as $r \to 0^+$). Since m_2 in (2.12) becomes zero for $p = N + \vartheta$, we must understand $q_* = \infty$ in connection with the fact that $b(x)\,h(\Phi) \in L^1(B_{1/2})$ holds for any $q \in (p-1, \infty)$ and thus in Theorem 2.2.1 only the assertion of (a) is meaningful in which the strong singularity behaviour of (iii) is given by (2.14).

2.3 Applications

In this section, we give examples and applications of our main results, Theorem 2.2.1 and Theorem 2.2.2. Our first application illustrates how *weighted* divergence-form equations such as (2.3) arise naturally in the study of p-Laplacian type equations in *exterior* domains. In Corollary 2.3.2, we give three examples which emphasise the wide framework of regular variation as an extension of the pure-power functions and, in turn, the difficulty of the critical exponent case $q = q^*$ and the necessity of conditions (2.1) and (2.2). We end the section with two remarks on more general examples applicable to our Theorems and on reformulations of (2.14) related to regular variation theory. Let us begin our first application.

Corollary 2.3.1. *Assuming* $2 \le N \le p < a$ *and* $q > p - 1$. *Let* $v(\tilde{x})$ *be an arbitrary solution of*

$$\Delta_p v(\tilde{x}) = |\tilde{x}|^{-a}[v(\tilde{x})]^q \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}. \tag{2.19}$$

(1) *If $p > N$, then the following classification holds for the positive solutions $v(\tilde{x})$ of* (2.19)*:*

(a) *If $q < \frac{(a-N)(p-1)}{p-N}$, then as $|\tilde{x}| \to \infty$, exactly one of the following holds*

(i) *$v(\tilde{x})$ converges to a positive number;*

(ii) *$|\tilde{x}|^{-\frac{p-N}{p-1}} v(\tilde{x})$ converges to a positive number;*

(iii) $|\tilde{x}|^{-(a-p)/(q-p+1)} v(\tilde{x}) \to \left[\left(\frac{a-p}{q-p+1} \right)^{p-1} \left(\frac{-pq+ap-a}{q-p+1} - N \right) \right]^{1/(q-p+1)}$.

(b) *If, in turn, $q \geq \frac{(a-N)(p-1)}{p-N}$, then for every positive solution of* (2.19)*, only (a)(i) holds.*

(2) *If $p = N$, then for all $q > p - 1$, only (1)(a) holds in which (ii) should read as $\lim_{|\tilde{x}| \to \infty} v(\tilde{x}) / \ln(|\tilde{x}|) \in (0, \infty)$.*

Proof. By a modified Kelvin transform (see [24, Appendix A]) where

$$u(x) = v(\tilde{x}) \quad \text{with } x = \frac{\tilde{x}}{|\tilde{x}|^2},$$

the behaviour near ∞ of the positive solutions of (2.19) can be obtained from the behaviour near 0 of the positive solutions of (2.3) in the form

$$\operatorname{div}(|x|^{2(p-N)} |\nabla u|^{p-2} |\nabla u|) = |x|^{a-2N} [u(x)]^q \quad \text{for } B_1 \setminus \{0\}. \tag{2.20}$$

Hence, by applying our Theorem 2.2.1 and Theorem 2.2.2, we conclude the proof of Corollary 2.3.1. □

For readers with specific examples of $\mathcal{A}$, b and h in mind, we supply below the sharp condition (2.27) for which Theorem 2.2.1 can be applied. From a practical point of view, $\tilde{u}$ in (2.14) can be rewritten asymptotically as in (2.22), provided that (2.1) holds. In Example 1, we choose the prototype model satisfying (2.1) introduced in Remark 2.3.1 and give in Example 2 a special example where (2.2) is satisfied.

Corollary 2.3.2. *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. Suppose that $\alpha, \beta, \gamma \in \mathbb{R}$ and $\nu \in (0, 1/2)$ is arbitrary.*

Ex	$L_{\mathcal{A}}(r)$ *as* $r \to 0^+$	$L_b(r)$ *as* $r \to 0^+$	$L_h(t)$ *as* $t \to \infty$
1	$\left(\ln \frac{1}{r}\right)^{\alpha}$	$\left(\ln \frac{1}{r}\right)^{\beta}$	$(\ln t)^{\gamma}$
2	$\left(\ln \frac{1}{r}\right)^{\alpha}$	$\left(\ln \frac{1}{r}\right)^{\beta}$	$\exp\{-(\ln t)^{\nu}\}$
3	$\left(\ln \frac{1}{r}\right)^{\alpha} \exp\left\{-\frac{p-1}{q}\sqrt{\ln \frac{1}{r}}\right\}$	$\exp\left\{-\sqrt{\ln \frac{1}{r}}\right\}$	$\exp\{-(\ln t)^{\nu}\}$

Table 2.1: Summary

(A) *If $q < q_*$ in Examples 1–3, then for any positive solution u of (2.3) exactly one of the following holds:*

(i) *u can be extended as a positive continuous solution of (2.3) in B_1.*

(ii) *u has a weak singularity at 0, that is $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda \in (0, \infty)$ and, moreover, u satisfies (2.13).*

(iii) *u has a strong singularity at 0 and, moreover, as $|x| \to 0$, the behaviour of u is given by Table 2.2 below.*

Ex	$u(x)$ *is asymptotically equivalent to*
1	$|x|^{-m_0}\left[\frac{m_0^{p-\gamma}}{M}\left(\ln \frac{1}{|x|}\right)^{\alpha-\beta-\gamma}\right]^{\frac{1}{q-p+1}}$
2	$|x|^{-m_0}\left[\frac{m_0^{p}}{M}\left(\ln \frac{1}{|x|}\right)^{\alpha-\beta}\right]^{\frac{1}{q-p+1}} \exp\left\{\frac{1}{q-p+1}\left(m_0 \ln \frac{1}{|x|}\right)^{\nu}\right\}$
3	$|x|^{-m_0}\left[\frac{m_0^{p}}{M}\left(\ln \frac{1}{|x|}\right)^{\alpha}\right]^{\frac{1}{q-p+1}} \exp\left\{\frac{1}{q}\left(\ln \frac{1}{|x|}\right)^{\frac{1}{2}} + \frac{1}{q-p+1}\left(m_0 \ln \frac{1}{|x|}\right)^{\nu}\right\}$

Table 2.2: Strong singularity behaviour for $q < q_*$

(B) *If $q = q_*$ (and, in addition, $\alpha q_*/(p-1) > \beta + \gamma + 1$ for Example 1), then the trichotomy in* **(A)** *remains valid except (iii) which is replaced by the behaviour in Table 2.3 below.*

Ex	*$u(x)$ is asymptotically equivalent to*
1	$\|x\|^{-m_0}\left[\dfrac{m_0^{p-1-\gamma}\left(\frac{\alpha q_*}{p-1}-\beta-\gamma-1\right)}{m_1}\left(\ln\dfrac{1}{\|x\|}\right)^{\alpha-\beta-\gamma-1}\right]^{\frac{1}{q-p+1}}$
2	$\|x\|^{-m_0}\left[\dfrac{\nu\, m_0^{p-1+\nu}}{m_1}\left(\ln\dfrac{1}{\|x\|}\right)^{\alpha-\beta+\nu-1}\right]^{\frac{1}{q-p+1}}\exp\left\{\dfrac{1}{q-p+1}\left(m_0\ln\dfrac{1}{\|x\|}\right)^{\nu}\right\}$
3	$\|x\|^{-m_0}\left[\dfrac{\nu\, m_0^{p-1+\nu}}{m_1}\left(\ln\dfrac{1}{\|x\|}\right)^{\alpha+\nu-1}\right]^{\frac{1}{q-p+1}}\exp\left\{\dfrac{1}{q}\left(\ln\dfrac{1}{\|x\|}\right)^{\frac{1}{2}}+\dfrac{1}{q-p+1}\left(m_0\ln\dfrac{1}{\|x\|}\right)^{\nu}\right\}$

Table 2.3: Strong singularity behaviour for $q = q_*$

(C) *If $q > q_*$, then any positive solution u of* (2.3) *can be extended as a positive continuous solution of* (2.3) *in B_1. For Example 1, this conclusion also holds for $q = q_*$ and $\alpha q_*/(p-1) \le \beta+\gamma+1$.*

Here and throughout, let it be understood that by $f_1(t) \sim f_2(t)$ as $t \to t_0$ for $t_0 \in \mathbb{R}\cup\{\infty\}$, we mean that $\lim_{t\to t_0} f_1(t)/f_2(t) = 1$.

Remark 2.3.1. A prototype model for (2.1) is $L_h(t) \sim (\ln t)^\gamma$ as $t \to \infty$, where $\gamma \in \mathbb{R}$. More generally, we see that (2.1) holds if $L_h(T) \sim \mathcal{L}(T)$ as $T \to \infty$ and $\mathcal{L}(T) = \Pi_{i=1}^{k}(\ln_{m_i} T)^{\beta_i}$ for $T > 0$ large, where k and m_i are positive integers and $\beta_i \in \mathbb{R}$ for every $1 \le i \le k$. We use the notation $\ln_{m_i}$ for the m_i-iterated natural logarithm. Without loss of generality, we can take m_i to be strictly increasing, that is, $1 \le m_1 < m_2 < \ldots < m_k$. Then $t \longmapsto L_h(e^t)$ is regularly varying at ∞ with index equal to β_1 (respectively, 0) if $m_1 = 1$ (respectively, $m_1 > 1$). Similarly, (2.2) is verified if $[L_{\mathcal{A}}(1/T)]^{-\frac{q}{p-1}} L_b(1/T) \sim \mathcal{L}(T)$ as $T \to \infty$.

Remark 2.3.2. In Theorem 2.2.1 for $q < q_*$, the function $\tilde{u}$ in (2.14) is well-defined, regularly varying at 0 with index $-m_0$ and

$$\tilde{u}(r)\,[L_h(\tilde{u}(r))]^{\frac{1}{q-p+1}} \sim \left[\frac{m_0^p}{M}\frac{L_{\mathcal{A}}(r)}{L_b(r)}\right]^{\frac{1}{q-p+1}} r^{-m_0} \quad \text{as } r \to 0^+. \tag{2.21}$$

Indeed, the integral in the left-hand side of (2.14) is well-defined since the integrand is regularly varying at ∞ with index $-(q+1)/p < -1$ from the assumption $q > p-1$. The right-hand side of (2.14) also exists since the integrand is regularly varying at 0^+ with index $(\sigma-\vartheta)/p > -1$ by virtue of $\sigma > \vartheta - p$. By Karamata's Theorem in Chapter 1.5, (2.14) implies (2.21). Furthermore, if (2.1) holds, then

$L_h(\tilde{u}(r)) \sim m_0^\gamma L_h(1/r)$ as $r \to 0^+$ so that (2.21) is refined by

$$\tilde{u}(r) \sim \left[m_0^{\gamma-p} M \frac{L_h(1/r)\, L_b(r)}{L_{\mathcal{A}}(r)} \right]^{-\frac{1}{q-p+1}} r^{-m_0} \quad \text{as } r \to 0^+. \tag{2.22}$$

2.4 Commentary on Our Framework

We place in this section the commentary on our framework whose technicality would have detracted from the introduction of our prescribed problem. We include them here as they are pertinent to the proofs of the lemmas in the next section.

2.4.1 On the assumptions

Remark 2.4.1. If $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold, then by [18, Lemma A.7], there exist continuous functions h_1 and h_2 on $[0,\infty)$, positive on $(0,\infty)$ with $h_1(0) = h_2(0) = 0$ such that

$$\begin{cases} h_1(t) \le h(t) \le h_2(t) \quad \text{for } t \in [0,\infty), \\ h_1(t)/t^{p-1} \text{ and } h_2(t)/t^{p-1} \text{ are both increasing for } t \in (0,\infty), \\ h_1(t) \sim h_2(t) \sim h(t) \ \ \text{as } t \to \infty. \end{cases} \tag{2.23}$$

Therefore, without loss of generality, we can assume that $t \longmapsto t^{q-p+1} L_h(t)$ is increasing on $(0,\infty)$ so that $t^q L_h(t)$ is non-decreasing on $(0,\infty)$. Moreover, as in [16, Section 1.2.4], we can take $L_h \in C^2[t_0,\infty)$ and $L_b \in C^2(0,r_0]$ for some large constant $t_0 > 0$ and $r_0 \in (0,1)$ such that

$$\lim_{t\to\infty} \frac{t L_h'(t)}{L_h(t)} = \lim_{t\to\infty} \frac{t^2 L_h''(t)}{L_h(t)} = 0, \quad \lim_{r\to 0^+} \frac{r L_b'(r)}{L_b(r)} = \lim_{r\to 0^+} \frac{r^2 L_b''(r)}{L_b(r)} = 0. \tag{2.24}$$

The condition in (2.4) implies that $L_{\mathcal{A}}$ is slowly varying at 0 (see Definition 1.5.1 and Remark 1.5.2 in Chapter 1.5). A complete characterisation of slowly varying function at 0 is provided by Theorem 1.5.2. We note that the results of this thesis can be extended for the case $p = N + \vartheta$ for certain cases of $L_{\mathcal{A}}$, see Remark 2.2.4.

2.4.2 On the fundamental solution

Recall the fundamental solution Φ given in (2.8). Since $1 < p < N + \vartheta$, we have that $\lim_{r\to 0^+} \Phi(r) = \infty$. We note that both $r \mapsto \Phi(r)$ and $r \mapsto -r\,\Phi'(r)$

are regularly varying at 0^+ of index $-m_2$, where m_2 is defined in (2.12). Under Assumption $(\mathbf{A}_1)$, using Karamata's Theorem (see Theorem 1.5.3), we find that

$$\lim_{r\to 0^+} \frac{\ln \Phi(r)}{\ln(1/r)} = \lim_{r\to 0^+} \Upsilon(r) = m_2, \tag{2.25}$$

where we define Υ to be

$$\Upsilon(r) := \frac{r\,|\Phi'(r)|}{\Phi(r)} = \frac{C_{N,p} r^{-m_0} \left[L_{\mathcal{A}}(r)\right]^{-\frac{1}{p-1}}}{\Phi(r)} \quad \text{for } r \in (0,1). \tag{2.26}$$

2.4.3 On the necessary and sufficient condition

Remark 2.4.2. Assuming $(\mathbf{A}_1)$–$(\mathbf{A}_3)$, we note that $b(x)\,h(\Phi) \in L^1(B_{1/2})$ is equivalent to

$$\int_{0^+} r^{N-1+\sigma} L_b(r)\, h(\Phi(r))\, dr < \infty. \tag{2.27}$$

The integrand in (2.27) varies regularly at 0 with index $N-1+\sigma-m_2 q$.

Hence, if $q \neq q_*$ then (2.27) holds if and only if $q < q_*$, where q_* is given by (2.10). If $q = q_*$, then (2.27) may hold in some cases and fail in others. For example, if $L_{\mathcal{A}} = L_b = 1$ and $h(t) = t^{q_*}(\ln t)^\alpha$ for $t > 0$ large, then (2.27) holds if and only if $\alpha < -1$.

2.5 Basic Tools

Throughout this section, let Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold. Our aim is to prove the basic tools used in this thesis: *a priori* estimates (Lemma 2.5.2), a spherical Harnack-type inequality (Lemma 2.5.3) and a regularity result (Lemma 2.5.4).

Lemma 2.5.1 (Comparison principle, see Theorem 2.4.1 in [42]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let $u, v \in C^1(\Omega)$ satisfy (in the sense of distributions in $\mathcal{D}'(\Omega)$) the pair of differential inequalities*

$$-\mathrm{div}\, \boldsymbol{A}(x, \nabla u) + B(x,u) \leq 0 \quad \textit{and} \quad -\mathrm{div}\, \boldsymbol{A}(x,\nabla v) + B(x,v) \geq 0 \quad \textit{in } \Omega.$$

Suppose that $\boldsymbol{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is in $L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R}^N)$ and $B : \Omega \times \mathbb{R} \to \mathbb{R}$ is in $L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R})$ such that $B = B(x,z)$ is independent of $\boldsymbol{\xi}$ and non-decreasing in z, whereas $\boldsymbol{A} = \boldsymbol{A}(x,\boldsymbol{\xi})$ is independent of z and monotone in $\boldsymbol{\xi}$, that is

$$\langle \boldsymbol{A}(x,\boldsymbol{\xi}) - \boldsymbol{A}(x,\boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta}\rangle > 0 \quad \textit{when } \boldsymbol{\xi} \neq \boldsymbol{\eta}.$$

If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

In the following lemma, we prove that every positive sub-solution of (2.3) satisfies *a priori* estimates of the form (2.28), which is then used to derive a Harnack inequality for positive solutions of (2.3).

Lemma 2.5.2 (*A priori* estimates)**.** *For any $r_0 \in (0, 1/2)$, there exists a positive constant C, depending on r_0, such that for every positive (sub-)solution of* (2.3)*, we have*

$$\frac{|x|^p b(x)}{\mathcal{A}(|x|)} \frac{h(u(x))}{[u(x)]^{p-1}} \le C \quad \textit{for every } 0 < |x| \le r_0. \tag{2.28}$$

Proof. Fix $x_0 \in \mathbb{R}^N$ with $0 < |x_0| \le r_0$. Denote $\rho := \dfrac{|x_0|}{2}$ and $p' := \dfrac{p}{p-1}$. Let

$$\begin{cases} \zeta(r) := r^{\frac{\sigma-\vartheta+p}{p}} \left[\frac{L_b(r)}{L_{\mathcal{A}}(r)}\right]^{\frac{1}{p}} & \text{for } r \in (0, r_0] \text{ and} \\ f(t) := \dfrac{t^{1-\frac{q+1}{p}} [L_h(t)]^{-\frac{1}{p}}}{\int_t^\infty \xi^{-\frac{q+1}{p}} [L_h(\xi)]^{-\frac{1}{p}} \, d\xi} & \text{for } t > 0 \quad \text{large.} \end{cases} \tag{2.29}$$

Let $c > 0$ be a positive constant. We define $S = S_{x_0} : B_\rho(x_0) \to \mathbb{R}$ by

$$\int_{S(x)}^\infty t^{-\frac{q+1}{p}} [L_h(t)]^{-\frac{1}{p}} \, dt = c\zeta(|x_0|) \left[1 - \left(\frac{|x - x_0|}{\rho}\right)^{p'}\right], \tag{2.30}$$

for every $x \in B_\rho(x_0)$.

Claim. *There exists a small positive constant c depending on r_0, but independent of x_0 such that the function S defined by* (2.30) *is a super-solution of* (2.3) *in $B_\rho(x_0)$, namely for h_1 as in Remark 2.4.1, it holds*

$$\Delta_{\mathcal{A},p} S(x) \le b(x)\, h_1(S(x)) \quad \textit{in } B_\rho(x_0). \tag{2.31}$$

Proof of Claim. By (2.30), we find that

$$|\nabla S(x)|^{p-2} \nabla S(x) = \frac{(cp')^{p-1}}{\rho^p} [\zeta(|x_0|)]^{p-1} \left[S^{q+1}(x)\, L_h(S(x))\right]^{\frac{1}{p'}} (x - x_0) \tag{2.32}$$

in $B_\rho(x_0)$. Using f given by (2.29), we denote by $T_{x_0}(x)$ and $Y_{x_0}(x)$ the following quantities

$$T_{x_0}(x) = \left(\frac{|x-x_0|}{\rho}\right)^{p'} \left(q+1+\frac{S(x)L_h'(S(x))}{L_h(S(x))}\right) \quad \text{and} \tag{2.33}$$
$$Y_{x_0}(x) = f(S(x))\left[1-\left(\frac{|x-x_0|}{\rho}\right)^{p'}\right]\left(N+\frac{|x|\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\,\frac{(x-x_0)\cdot x}{|x|^2}\right).$$

We thus derive that $(p')^{1-p}\,(2c)^{-p}\Delta_{\mathcal{A},p}S(x)$ equals

$$\left(\frac{|x|}{|x_0|}\right)^{\vartheta}\frac{L_{\mathcal{A}}(|x|)}{L_{\mathcal{A}}(|x_0|)}\,|x_0|^{\sigma}L_b(|x_0|)\,[S(x)]^q L_h(S(x))\,[T_{x_0}(x)+Y_{x_0}(x)]. \tag{2.34}$$

By Remark 2.4.1 in Chapter 1.5 and Assumption $(\mathbf{A}_1)$, we have

$$\lim_{t\to\infty}\frac{tL_h'(t)}{L_h(t)} = 0 \quad \text{and} \quad \lim_{r\to 0^+}\frac{r\mathcal{A}'(r)}{\mathcal{A}(r)} = \vartheta.$$

which are to be applied respectively to $T_{x_0}(x)$ and $Y_{x_0}(x)$ in (2.33) as $r\to 0$, with additional help from (2.36) in the latter element. Moreover, by Proposition 1.5.1 in Chapter 1.5, there exist positive constants c_i $(0\le i\le 3)$ depending on r_0, but independent of x_0 such that

$$c_0 \le \frac{L_{\mathcal{A}}(|x|)}{L_{\mathcal{A}}(|x_0|)} \le c_1 \quad \text{and} \quad c_2 \le \frac{L_b(|x_0|)}{L_b(|x|)} \le c_3$$

for every x, x_0 such that $0<|x_0|\le r_0$ and $|x|/|x_0|\in[1/2,3/2]$. Thus, using (2.5) and (2.34), we can take in our definition of $S(x)$ in (2.30) a small constant $c>0$ depending on r_0, but independent of x_0, to conclude (2.31), that $S(x)$ is indeed a super-solution of (2.3) in $B_\rho(x_0)$.

Proof of Lemma 2.5.2 completed. Since $S(x)\to\infty$ as $|x-x_0|\to\rho$, by the comparison principle of Lemma 2.5.1, we find that $u\le S$ in $B_\rho(x_0)$. In particular, we have $u(x_0)\le S(x_0)$. Since ζ is regularly varying at 0^+ with positive index $(p+\sigma-\vartheta)/p$, we have $\lim_{r\to 0^+}\zeta(r)=0$ so that $\sup_{0<r\le r_0}\zeta(r)<\infty$. Since the right-hand side of (2.30) is bounded from above by $c\sup_{0<r\le r_0}\zeta(r)$, for every $M>0$ there exists a small positive constant c (depending on M and r_0) such that $S\ge M$ in $B_\rho(x_0)$ for every $0<|x_0|\le r_0$. Using (2.29) and (2.30), we find that

$$[S(x_0)]^{q-p+1}\,L_h(S(x_0)) = [c\zeta(|x_0|)f(S(x_0))]^{-p}. \tag{2.35}$$

We fix $M > 0$ as large as needed. Let h_1 and h_2 be as in Remark 2.4.1 of Chapter 1.5. We can thus assume that $h_2(t) \leq 2t^q L_h(t)$ for all $t \geq M$. By Karamata's Theorem in Chapter 1.5, we have

$$\lim_{t\to\infty} f(t) = \frac{q-p+1}{p} > 0. \tag{2.36}$$

Since $u(x_0) \leq S(x_0)$, using (2.35) and (2.23), we can find a positive constant $C_1 = C_1(r_0)$ independent of x_0 such that

$$\begin{aligned}\frac{|x_0|^p b(x_0)}{\mathcal{A}(|x_0|)}\,\frac{h(u(x_0))}{[u(x_0)]^{p-1}} &\leq \frac{|x_0|^p b(x_0)}{\mathcal{A}(|x_0|)}\,\frac{h_2(S(x_0))}{[S(x_0)]^{p-1}}\\ &\leq \frac{2}{[cf(S(x_0))]^p}\,\frac{b(x_0)}{|x_0|^\sigma L_b(|x_0|)}\\ &\leq C_1. \end{aligned}\tag{2.37}$$

Since (2.37) holds for every $0 < |x_0| \leq r_0$, we conclude the proof. □

We give below a Harnack-type inequality which relates the values of a positive harmonic function at two points in a plane. It was first introduced by Harnack in [29] to prove the convergence of sequences of harmonic functions but can also be used to show the interior regularity of weak solutions of partial differential equations. The inequality has since its introduction been generalised to solutions of elliptic or parabolic partial differential equations, see Serrin [47] and Moser [40], becoming an important tool in the general theory of harmonic functions and partial differential equations. As linearity is not necessary for Harnack's inequality, Serrin [48] and Trudinger [57], were able to extend Moser's results to the situation of nonlinear elliptic equations of the type (1.4) satisfying (1.5) and also deduce the Hölder regularity of solutions.

Lemma 2.5.3 (Spherical Harnack-type inequality). *Fix $r_0 \in (0, 1/2)$. There exists a positive constant K (depending on p, N and r_0) such that for every positive solution u of (2.3), we have*

$$\max_{|x|=r} u(x) \leq K \min_{|x|=r} u(x) \quad \textit{for all } 0 < r \leq r_0/2. \tag{2.38}$$

Proof. We first observe that (2.3) is equivalent to

$$-\Delta_p u + \frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\,|\nabla u|^{p-2}\frac{\nabla u\cdot x}{|x|} + \frac{b(x)\,h(u)}{\mathcal{A}(|x|)\,u^{p-1}}\,u^{p-1} = 0 \quad \text{in } B^*. \tag{2.39}$$

Let b_1 and b_2 denote two non-negative functions as follows

$$b_1(x) := \frac{|\mathcal{A}'(|x|)|}{\mathcal{A}(|x|)} \quad \text{and} \quad [b_2(x)]^p := \frac{b(x)\, h(u)}{\mathcal{A}(|x|)\, u^{p-1}} \quad \text{for } 0 < |x| \leq r_0. \tag{2.40}$$

By (2.4) and the *a priori* estimate from Lemma 2.5.2, there exists a positive constant C_1, depending on r_0, such that

$$|x|\, b_1(x) \leq C_1 \quad \text{and} \quad |x|\, b_2(x) \leq C_1 \quad \text{for all } 0 < |x| \leq r_0. \tag{2.41}$$

Fix $x_0 \in \mathbb{R}^N$ such that $0 < |x_0| \leq r_0/2$ and set $\rho := |x_0|/2$. We use μ to denote

$$\mu = \mu_{x_0} := \max\{\|b_1\|_{L^\infty(B_\rho(x_0))}, \|b_2\|_{L^\infty(B_\rho(x_0))}\}.$$

Since $\rho \leq |x|$ for every $x \in B_\rho(x_0)$, from (2.41) it follows that

$$\rho\mu \leq C_1 \quad \text{for every } x \in B_\rho(x_0). \tag{2.42}$$

We apply the Harnack inequality of [57, Theorem 1.1] for (2.39) on $B_{|x_0|/2}(x_0)$ where the structure conditions in (1.2) and (1.3) of [57] are satisfied with $a_0 = 1$ and $a_i = b_0 = b_3 = 0$ for $i \in \{1, 2, 3, 4\}$. Hence, there exists a positive constant k, depending only on p, N and $\rho\mu$, such that

$$\sup_{x \in B_{\rho/3}(x_0)} u(x) \leq k \inf_{x \in B_{\rho/3}(x_0)} u(x). \tag{2.43}$$

By the covering argument in [26], any two points x_1 and x_2 in $\mathbb{R}^N$ with the property $0 < |x_1| = |x_2| \leq r_0/2$ can be joined by ten overlapping balls of radius $|x_1|/6$ with centres positioned on $\partial B_{|x_1|}(0)$. Thus, by (2.42) and (2.43), we obtain (2.38) with $K = k^{10}$, a positive constant depending on p, N and r_0. □

Remark 2.5.1. Using (2.38) and the same argument as in [7, Corollary 4] and [16, Corollary 4.5], we can show that

$$\begin{cases} \text{If } \limsup\limits_{|x|\to 0} \dfrac{u(x)}{[\Phi(x)]^j} = \infty, \text{ then } \lim\limits_{|x|\to 0} \dfrac{u(x)}{[\Phi(x)]^j} = \infty \quad \text{for } j \in \{0, 1\}. \\ \text{If } \liminf\limits_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0, \text{ then } \lim\limits_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0. \end{cases} \tag{2.44}$$

Consequently, we either have $\limsup_{|x|\to 0} u(x) < \infty$ or $\lim_{|x|\to 0} u(x) = \infty$. In the latter case, the *a priori* estimate in (2.28), together with Assumptions ($\mathbf{A}_1$) and ($\mathbf{A}_3$), give that

$$\limsup_{|x|\to 0} \frac{L_b(|x|)}{L_{\mathcal{A}}(|x|)}\, |x|^{p+\sigma-\vartheta}\, [u(x)]^{q-p+1}\, L_h(u(x)) < \infty. \tag{2.45}$$

In particular, (2.45) yields that $\limsup_{|x|\to 0} u(x)/T(|x|) < \infty$ for some function T regularly varying at 0 with index $-m_0$. Since $\lim_{r\to 0^+} \ln T(r)/\ln(1/r) = m_0$, we find that $\limsup_{|x|\to 0} \ln u(x)/\ln(1/|x|) \le m_0$. Furthermore, if $q = q_*$, then $m_0 = m_2$ and any positive solution u of (2.3) with a strong singularity at zero satisfies the limit

$$\lim_{|x|\to 0} \frac{\ln u(x)}{\ln(1/|x|)} = m_0. \tag{2.46}$$

Corollary 2.5.1. *Let u be a positive solution of* (2.3)*.*

(a) *If* $\limsup_{|x|\to 0} u(x) = \infty$*, then* $\lim_{|x|\to 0} u(x) = \infty$.

(b) *If* $\limsup_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = \infty$*, then* $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$.

(c) *If* $\liminf_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0$*, then* $\lim_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0$.

Proof. The following proofs are all by contradiction.

(a) Let $l := \liminf_{|x|\to 0} u(x) < \infty$. Then there exists a sequence $(x_n)_{n\ge 1} \in R^N$ decreasing to zero such that $u(x_n) \to l$ as $x_n \to 0$. Without loss of generality, we can take the sequence $(|x_n|)_{n\ge 1}$ to be decreasing to zero and lying inside a ball of radius $r_0/2$, that is, $0 < |x_n| < r_0/2$ for some $r_0 \in (0, 1/2)$. With this in mind, we can apply our Harnack-type inequality (2.38) and use the definition of limits to establish the following chain of inequalities:

$$\max_{|x|=|x_n|} u(x) \le K \min_{|x|=|x_n|} u(x) \le K u(x_n) \le K(l+1)$$

for all $n \ge n_0$ where n_0 is a large positive integer and K is the constant from (2.38). To achieve the desired contradiction, we apply the comparison principle on the annuli $\{x \in \mathbb{R}^N : |x_n| < |x| < |x_{n_0}|\}$ for $n \ge n_0$ to obtain that

$$u(x) \le K(l+1) \quad \text{for } 0 < |x| < |x_{n_0}|.$$

(b) Let $l := \liminf_{|x|\to 0} u(x)/\Phi(x) < \infty$. By definition, there exists a sequence $(x_n)_{n\geq 1} \in R^N$ decreasing to zero such that $u(x_n)/\Phi(x_n) \to l < \infty$ as $x_n \to 0$. Without loss of generality, we take the sequence $(|x_n|)_{n\geq 1}$ to be decreasing to zero and satisfying the condition $0 < |x_n| < r_0/2$ for some $r_0 \in (0, 1/2)$. By the Harnack-type inequality (2.38) and the definition of limits, we obtain the following inequality:

$$\max_{|x|=|x_n|} u(x) \leq K \min_{|x|=|x_n|} u(x) \leq Ku(x_n) \leq K(l+1)\Phi(x_n)$$

for all $n \geq n_0$ where n_0 is a large positive integer and K is the constant from (2.38). To achieve the desired contradiction, we apply the comparison principle on the annuli $\{x \in \mathbb{R}^N : |x_n| < |x| < |x_{n_0}|\}$ for $n \geq n_0$ to obtain that

$$\frac{u(x)}{\Phi(x)} \leq K(l+1) \quad \text{for } 0 < |x| < |x_{n_0}|.$$

(c) Suppose $\limsup_{|x|\to 0} u(x)/\Phi(x) = l \in (0, \infty]$. Let (x_n) be a sequence in $\mathbb{R}^N$ decreasing to zero and satisfying the condition $0 < |x_n| < r_0/2$ for some $r_0 \in (0, 1/2)$ such that $\lim_{n\to\infty} u(x_n)/\Phi(x_n) = 0$. We choose $c > 0$ such that $cK < l$ where K is the Harnack constant and by definition of limits, we have that $u(x_n)/\Phi(x_n) \leq c$ for $n \geq n_0$ where n_0 is a large positive number. We acquire the inequality

$$\max_{|x|=|x_n|} u(x) \leq K \min_{|x|=|x_n|} u(x) \leq Ku(x_n) \leq cK\Phi(x_n)$$

for all $n \geq n_0$. Again, we obtain the contradiction by applying the comparison principle on the annuli $\{x \in \mathbb{R}^N : |x_n| < |x| < |x_{n_0}|\}$ for $n \geq n_0$ to obtain that

$$\frac{u(x)}{\Phi(x)} \leq cK < l \quad \text{for } 0 < |x| < |x_{n_0}|.$$

This completes the proof of Corollary 2.5.1. □

Here, we give the regularity result which is invoked many times in the rest of the thesis. For instance, the technique of combining the regularity result with a rescaling argument has been particular fruitful in proving existence results (see, for example, [26] and [31]).

Lemma 2.5.4 (A regularity result). *Fix $r_0 \in (0, 1/4)$ and $\delta \geq 0$. Let $g \in C(0, 1)$ be a positive function such that g is regularly varying at 0 with index $-\delta$. Suppose* *that u is a positive solution of* (2.3) *and $C_0 > 0$ is a constant such that*

$$0 < u(x) \leq C_0\, g(|x|) \quad \textit{for } 0 < |x| < 2r_0. \tag{2.47}$$

Then there exist positive constants $C > 0$ and $\alpha \in (0,1)$ such that

$$|\nabla u(x)| \leq C\, \frac{g(|x|)}{|x|} \quad \textit{and} \quad |\nabla u(x) - \nabla u(x')| \leq C\, \frac{g(|x|)}{|x|^{1+\alpha}} |x - x'|^{\alpha} \tag{2.48}$$

for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \leq |x'| < r_0$.

Proof. We use an argument close to [18, Lemma 4.1], which is similar to [26, Lemma 1.1] (see also [7, Lemma 3]). There is, however, one essential difference with respect to the derivation of the first inequality in (2.48). We show below the main modifications compared with [18, Lemma 4.1].

Using (2.39) and defining Ψ_β as in (4.5) of [18], that is

$$\Psi_\beta(\xi) := \frac{u(\beta \xi)}{g(\beta)} \quad \text{for } \xi \in \bar{\Gamma}, \tag{2.49}$$

where $\beta \in (0, r_0/6)$ is fixed, we see that Ψ_β satisfies an equation of the form (4.3) of [18], namely

$$-\Delta_p \Psi_\beta + B_\beta = 0 \quad \text{in } \Gamma, \quad \text{where } \Gamma := \{y \in \mathbb{R}^N : 1 < |y| < 7\}. \tag{2.50}$$

However, instead of (4.7) in [18], the expression of B_β is more complicated here, involving a gradient term, namely, for $\xi \in \Gamma$,

$$B_\beta(\xi) := \frac{\beta^p}{[g(\beta)]^{p-1}}\, b(\beta\xi) \frac{h(u(\beta\xi))}{\mathcal{A}(\beta|\xi|)} - \frac{\beta \mathcal{A}'(\beta|\xi|)}{\mathcal{A}(\beta|\xi|)}\, |\nabla \Psi_\beta|^{p-2}\, \frac{\nabla \Psi_\beta(\xi) \cdot \xi}{|\xi|}. \tag{2.51}$$

Claim. *The functions Ψ_β and B_β are in $L^\infty(\Gamma)$ with their L^∞-norms bounded above by a positive constant independent of $\beta \in (0, r_0/6)$.*

Proof of Claim. Using (2.47) and (2.49),

$$|\Psi_\beta(\xi)| \leq C \frac{g(\beta|\xi|)}{g(\beta)} \quad \text{for every } \xi \in \Gamma \text{ and all } \beta \in (0, r_0/6). \tag{2.52}$$

Since $g \in RV_{-\delta}(0^+)$, it can be written as $g(t) = t^{-\delta} L(t)$ for some function L that is continuous on $(0, 2r_0)$ and slowly varying at 0. By Proposition 1.5.1, we have

$$\lim_{\beta \to 0} \frac{L(\beta|\xi|)}{L(\beta)} = 1 \quad \text{uniformly with respect to } \xi \in \Gamma. \tag{2.53}$$

Hence there exists positive constants c_1 and c_2 which depend on r_0 such that

$$c_1 g(\beta) \le g(\beta|\xi|) \le c_2 g(\beta) \quad \text{for every } \beta \in (0, r_0/6) \text{ and every } \xi \in \Gamma. \tag{2.54}$$

By (2.55), we get that $\Psi_\beta \in L^\infty(\Gamma)$ and $\|\Psi_\beta\|_{L^\infty(\Gamma)} \le Cc_2$ for every $\beta \in (0, r_0/6)$.

We now prove the claim for B_β. Assume for now that the first inequality in (2.48) is proved. Then we can infer that

$$|\nabla \Psi_\beta(\xi)| \le C \frac{g(\beta|\xi|)}{g(\beta)} \quad \text{for every } \xi \in \Gamma. \tag{2.55}$$

Using Lemma 2.5.2 and (2.47), jointly with (4.10) in [18], we find that the $L^\infty(\Gamma)$-norm of the *first term* in the right-hand side of (2.51) is bounded above by a constant independent of β.

Hence, using (2.41), as well as (4.10) in [18], we could conclude the claim for B_β given by (2.51).

Since $B \in L^\infty(\Gamma)$ and $\Psi \in L^\infty(\Gamma) \cap W^{1,p}(\Gamma)$ is a weak solution of (2.50), from the $C^{1,\alpha}$-regularity result of Tolksdorf [56], we conclude that there exist constants $\alpha = \alpha(N, p) \in (0, 1)$ and $\tilde{C} = \tilde{C}\left(N, p, \|\Psi\|_{L^\infty(\Gamma)}, \|B\|_{L^\infty(\Gamma)}\right) > 0$ such that

$$\|\nabla \Psi\|_{C^{0,\alpha}(\Gamma^*)} \le \tilde{C}, \quad \text{where } \Gamma^* := \{y \in \mathbb{R}^N : 2 < |y| < 6\}. \tag{2.56}$$

We now use this fact to derive the second inequality in (2.48).

Proof of the second inequality in (2.48). Let $0 < |x| \le |x'| < 2|x|$. Then we have $x'/\beta \in \Gamma^*$. Since (2.49) implies that $\nabla u(x) = [g(\beta)/\beta] \nabla \Psi_\beta(x/\beta)$ for all $x \in \{\beta\xi : \xi \in \Gamma\}$, by (2.56), we obtain that

$$\beta |\nabla u(x) - \nabla u(x')| = g(\beta) |\nabla \Psi_\beta(x/\beta) - \nabla \Psi_\beta(x'/\beta)| \le \tilde{C} g(\beta) \beta^{-\alpha} |x - x'|^\alpha. \tag{2.57}$$

Then the second inequality in (2.48) holds by (2.54) for $0 < |x| \le |x'| < 2|x|$. If $2|x| \le |x'| < r_0$, then

$$|x' - x| \ge |x'| - |x| \ge |x|. \tag{2.58}$$

Since $g(t)/t \in RV_{-\delta-1}(0^+)$, $g(t)/t$ behaves near 0 like a monotone decreasing function. Then by the first inequality in (2.48) and (2.58), we find that

$$\beta|\nabla u(x) - \nabla u(x')| = C\left(\frac{g(|x|)}{|x|} + \frac{g(|x'|)}{|x'|}\right) \leq C' \frac{g(|x|)}{|x|} \leq C' \frac{g(|x|)}{|x|^{\alpha+1}}|x' - x|^{\alpha},$$

where $C' > 0$ denotes a large constant. This completes the proof of the second inequality in (2.48).

Proof of the first inequality in (2.48). Our proof here is different from both [18, Lemma 4.1] and [7, Lemma 3]. We require a new argument to that of [18] as we used the first inequality in (2.48) to derive (2.56). The ideas in [7] work for the special case $p = 2$. In our general situation, we apply Tolksdorf's result in [56, Theorem 1] for the function v in (2.59). More precisely, let $x_0 \in \mathbb{R}^N$ be fixed such that $0 < |x_0| \leq r_0$ and set $\rho := |x_0|/2$. We define $v = v_{x_0} : B_1 \to (0, \infty)$ by

$$v(y) := \frac{u(x_0 + \rho y)}{g(|x_0|)} \quad \text{for every } y \in B_1. \tag{2.59}$$

Since u satisfies (2.39), by using the formula for ∇v derived from (2.59), that is

$$\nabla v(y) = \frac{\rho}{g(|x_0|)}(\nabla u)(x_0 + \rho y) \quad \text{for } y \in B_1, \tag{2.60}$$

we obtain that v is a positive solution of the following equation

$$-\Delta_{\mathcal{A},p} v + \tilde{B}(y, v, \nabla v) = 0 \quad \text{in } B_1,$$

where we define $\tilde{B}(y, v, \nabla v)$ to be the following quantity

$$-\frac{\rho\,\mathcal{A}'(|x_0 + \rho y|)}{\mathcal{A}(|x_0 + \rho y|)}|\nabla v|^{p-2}\frac{\nabla v(y)\cdot(x_0 + \rho y)}{|x_0 + \rho y|} + \rho^p \frac{b(x_0 + \rho y)\,h(v)}{\mathcal{A}(|x_0 + \rho y|)\,v^{p-1}}\,v^{p-1}.$$

Since $|x_0 + \rho y| \in [\rho, 3\rho]$ for all $y \in B_1$, in view of (2.4) and (2.41), we find that

$$|\tilde{B}(y, v, \nabla v)| \leq A_1 |\nabla v|^{p-1} + A_2\, v^{p-1} \tag{2.61}$$

for some positive constants A_1 and A_2, which depend on r_0, but are independent of x_0. Using the assumptions on g, namely g is regularly varying at 0, we obtain (similar to (4.10) in [18]) that

$$\underline{c}\, g(|x_0|) \leq g(|x_0 + \rho y|) \leq \overline{c}\, g(|x_0|) \quad \text{for all } y \in B_1,$$

where $\underline{c}$ and $\overline{c}$ are positive constants, which depend on r_0, but are independent of x_0 satisfying $0 < |x_0| < r_0$. Moreover, from (2.47) and (2.59), we deduce that

$$v(y) \leq \overline{c}\, C_0 \quad \text{for every } y \in B_1.$$

Thus, in view of (2.61), we can find a positive constant $A_3 = A_3(r_0)$, which is independent of x_0 such that

$$|\tilde{B}(y, v, \eta)| \leq A_3(1 + |\eta|)^p \quad \text{for all } y \in B_1 \text{ and } \eta \in \mathbb{R}^N.$$

We can then apply Tolksdorf's result from [56, Theorem 1] to obtain a constant A_4, which depends on N, p and A_3, but is independent of x_0, such that $|\nabla v(0)| \leq A_4$. This, jointly with (2.60), proves that

$$|\nabla u(x_0)| \leq 2A_4 \, \frac{g(|x_0|)}{|x_0|} \quad \text{for every } 0 < |x_0| < r.$$

This completes the proof of Lemma 2.5.4. □

3

Classification of Singularities

In this chapter, we prove our classification result, Theorem 2.2.1, restated here for the reader's convenience,

Theorem 3.0.1 (Classification of singularities)**.** *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. If* $b(x)\,h(\Phi) \in L^1(B_{1/2})$*, then for every positive solution* u *of*

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) = b(x)\,h(u) \quad \text{in } B^*, \tag{3.1}$$

exactly one of the following cases occurs:

(i) *u can be extended as a positive continuous solution of* (3.1) *in the whole ball* B_1 *(in the sense of Definition 2.1.1).*

(ii) *u has a weak singularity at* 0*, that is* $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda \in (0,\infty)$ *and, moreover,* u *verifies*

$$-\Delta_{\mathcal{A},p} u + b(x)\,h(u) = \lambda^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{3.2}$$

(iii) *u has a strong singularity at* 0*. Moreover,* $\lim_{|x|\to 0} u(x)/\tilde{u}(|x|) = 1$*, where* $\tilde{u}$ *is given by*

$$\int_{\tilde{u}(r)}^{\infty} \frac{t^{-\frac{q+1}{p}}}{[L_h(t)]^{\frac{1}{p}}}\,dt = \int_0^r \left[M \frac{\xi^{\sigma-\vartheta} L_b(\xi)}{L_{\mathcal{A}}(\xi)}\right]^{\frac{1}{p}} d\xi \quad \text{if } q < q_*, \tag{3.3}$$

where the reciprocal of M *is defined as*

$$\frac{1}{M} := q - \frac{N+\sigma}{m_0}. \tag{3.4}$$

However, in the critical case $q = q_*$, *then* $\lim_{|x|\to 0} u(x)/\tilde{u}(|x|) = 1$, *where if* (3.37) *holds, then* $\tilde{u}$ *is given by*

$$\tilde{u}(r) = \left[m_1 m_0^{\gamma+1-p} F(r)\right]^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0} \quad or \tag{3.5}$$

if (3.38) *holds,* $\tilde{u}$ *is given by*

$$\int_c^{\tilde{u}(r)} [F(1/t)]^{\frac{1}{q_*-p+1}}\, dt = \left(m_1 m_0^{-p-j}\right)^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0}, \tag{3.6}$$

where m_0, m_1 *and* F *are prescribed by* (2.12) *and* (2.11), *respectively. In* (3.6), $c > 0$ *is a large constant.*

Let us define a nonnegative constant λ as the following

$$\lambda := \limsup_{|x|\to 0} \frac{u(x)}{\Phi(|x|)}.$$

Then the categories (i)–(iii) of Theorem 3.0.1 correspond respectively to:

(i) $\lambda = 0$. Then the assertion of Theorem 3.0.1(i) follows from Lemma 4.1.2 in Chapter 4.1 where it is also used to prove the removability of all singularities in Theorem 2.2.2.

(ii) $\lambda \in (0, \infty)$. One can show that u has a weak singularity at 0 and can verify (2.13) by using the same argument as in [18, Theorem 5.1] (see also [7, Proposition 6]). We show the working in Chapter 3.1.

(iii) $\lambda = \infty$. Then by Corollary 2.5.1(b), we yield that

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi(x)} = \infty.$$

Reasoning as Cîrstea in [16, Lemma 4.12] and Brandolini *et al.* in [7, Lemma 4], we show below how to reduce the proof of Theorem 3.0.1(iii) to the case of strong singularities for radial solutions of an approximate problem (3.34) treated in Theorem 3.2.1. Due to its technicality, we split the proof into two, treating first the critical case in Chapter 3.2.1 and second the subcritical case in Chapter 3.2.2.

A major advance in this thesis is the analysis of the *critical case* and the derivation of the asymptotic behaviour of the strong singularities. Our contribution here is the development of a perturbation technique suitable for the *critical case* $q = q_*$. Unlike the subcritical case, where the power model corresponding to $\mathcal{A} = b = 1$ and $h(t) = |t|^{q-1}t$ was completely understood due to Friedman and Véron [26] (see also Remark 2.2.1), in the critical case we had no model in the literature to provide us with intuition on the asymptotics of strong singularity solutions.

3.1 Weak Singularities

Our aim in this section is to prove that case (ii) of weak singularities in Theorem 3.0.1 occurs for any positive solution of (3.1) satisfying (3.7). More precisely, we prove the following proposition.

Proposition 3.1.1. *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. If* $b(x)\,h(\Phi) \in L^1(B_{1/2})$, *then for any positive solution* u *of* (3.1) *satisfying*

$$\limsup_{|x|\to 0} \frac{u(x)}{\Phi(x)} = \lambda \in (0,\infty), \tag{3.7}$$

we have that

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi(x)} = \lambda \quad \textit{and} \quad \lim_{|x|\to 0} \frac{x\cdot \nabla u(x)}{\Phi(x)} = \lambda \frac{p-N-\vartheta}{p-1}. \tag{3.8}$$

Moreover, u *satisfies* (3.2), *namely*

$$-\Delta_{\mathcal{A},p} u + b(x)\,h(u) = \lambda^{p-1}\delta_0 \quad \textit{in } \mathcal{D}'(B_1). \tag{3.9}$$

Proof. We divide the proof into three steps. In Step 2, we introduce a rescaled function $V_{(r)}$ as in (3.12) and use Step 1 to show that $V_{(r)} \to V$ in $C^1_{\rm loc}(\mathbb{R}^N \setminus \{0\}$ as $r \to 0$ with V given by (3.13). Taking $x = r\xi$ when $|\xi| = 1$, we conclude that (3.8) holds. We then finalise the proof by showing in Step 3 that u satisfies (3.9).

Step 1: *Fix* $r_0 \in (0, 1/2)$. *We show that* $\lim_{r\to 0^+} \tilde{\lambda}(r) = \lambda$ *where we define*

$$\tilde{\lambda}(r) := \sup_{|x|=r} \frac{u(x)}{\Phi(|x|)} \quad \text{for all } r \in (0, 2r_0). \tag{3.10}$$

It is immediate that $\limsup_{r\to 0^+} \tilde{\lambda}(r) = \lambda$. We now show that $\liminf_{r\to 0^+} \tilde{\lambda}(r) = \lambda$. Assume by contradiction that $\liminf_{r\to 0^+} \tilde{\lambda}(r) < \lambda$ and there exists a decreasing sequence of positive real numbers (t_n), converging to zero as $n \to \infty$ and $\varepsilon > 0$

small, such that $\tilde{\lambda}(t_n) \leq \lambda - \varepsilon$ for all $n \geq 1$. Since $\limsup_{r\to 0^+} \tilde{\lambda}(r) = \lambda$, we have

$$\tilde{\lambda}(t_*) > \lambda - \varepsilon \quad \text{for some small } t_* > 0. \tag{3.11}$$

Without loss of generality, let $t_* < t_1 < 1$ and $n_1 > 1$ be large enough such that $t_{n_1} < t_*$. Fix $n \geq n_1$ (so that $t_n \leq t_{n_1}$), and define the annulus

$$\Omega := \{x \in \mathbb{R}^N : t_n < |x| < t_1\}.$$

We infer then that $\max\{\tilde{\lambda}(t_n), \tilde{\lambda}(t_1)\} < \tilde{\lambda}(t_*)$, we find that u/Φ achieves its maximum in the interior of Ω and u/Φ is not a constant in Ω. By applying [42, Theorem 2.5.2] with $v = \beta\Phi$, we see that u/Φ necessarily is constant in Ω. Thus by contradiction, it follows that $\lim_{r\to 0^+} \tilde{\lambda}(r) = \lambda$ holds.

Step 2: *For any fixed $r \in (0, r_0)$, we define*

$$V_{(r)}(\xi) := \frac{u(r\xi)}{\Phi(r)} \quad \textit{for } 0 < |\xi| < \frac{r_0}{r}. \tag{3.12}$$

We show that $V_{(r)} \to V$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ as $r \to 0^+$, where

$$V(\xi) = \lambda |\xi|^{\frac{p-N-\vartheta}{p-1}} \quad \textit{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{3.13}$$

It is enough to show that any sequence $\tilde{r}_n$ decreasing to zero contains a subsequence r_n such that

$$V_{(r)} \to V \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}).$$

From our prescribed problem in (3.1), we see that $V_{(r)}$ satisfies

$$\Delta_p V_{(r)}(\xi) + \frac{r|\xi|\mathcal{A}'(r|\xi|)}{\mathcal{A}(r|\xi|)} |\nabla V_{(r)}(\xi)|^{p-2} \nabla V_{(r)}(\xi) \cdot \frac{\xi}{|\xi|^2} = \frac{r^p b(r\xi) h(u(r\xi))}{\mathcal{A}(r|\xi|)[\Phi(r)]^{p-1}} \tag{3.14}$$

for $0 < |\xi| < r_0/r$.

From (3.7), there exists a positive constant C depending on r_0 such that

$$u(x) \leq C\Phi(|x|) \quad \text{for every } 0 < |x| \leq 2r_0. \tag{3.15}$$

By taking $g(|x|) := \Phi(|x|)$ in Lemma 2.5.4, we find that there exist constants $c > 0$

and $\alpha \in (0,1)$ such that for any x, x' with $0 < |x| \le |x'| < r_0$,

$$|\nabla u(x)| \le C\,\frac{\Phi(|x|)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \le C\,\frac{\Phi(|x|)}{|x|^{1+\alpha}}|x-x'|^{\alpha}. \tag{3.16}$$

By (3.15) and (3.16), it follows from Lemma 2.5.4 that for every compact subset $K \subset \mathbb{R}^N \setminus \{0\}$, there exist positive constants C_1, C depending on K and independent of r, such that for every fixed $r \in (0, r_0)$,

$$\begin{cases} 0 < V_{(r)}(\xi) \le C_1 |\xi|^{\frac{p-N-\vartheta}{p-1}}, \quad |\nabla V_{(r)}(\xi)| \le C|\xi|^{\frac{p-N-\vartheta}{p-1}}, \\ |\nabla V_{(r)}(\xi) - \nabla V_{(r)}(\xi')| \le C|\xi-\xi'|^{\alpha} \frac{1}{|\xi|^{1+\alpha}} |\xi|^{\frac{p-N-\vartheta}{p-1}}, \end{cases} \tag{3.17}$$

for every ξ and ξ' in $\mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$.

We want to show that the right-hand side of (3.14) vanishes as $r \to 0$, namely,

$$\frac{r^p b(r\xi) h(u(r\xi))}{\mathcal{A}(r|\xi|)[\Phi(r)]^{p-1}} \to 0 \quad \text{as } r \to 0 \text{ for every fixed } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{3.18}$$

By the properties of the slowly varying parts of b and h in (2.5), Remark 2.4.1, (3.15) and (2.8), this is equivalent to showing that

$$\tau(r) := \frac{|r\xi|^p}{[\Phi(r|\xi|)]^{p-1}} \frac{|r\xi|^{\sigma} L_b(r|\xi|) h_2(\Phi(r|\xi|))}{\mathcal{A}(r|\xi|)} \to 0 \quad \text{as } r \to 0. \tag{3.19}$$

Indeed, by Karamata's Theorems in Theorem 1.5.3, we deduce that

$$\lim_{r\to 0} \frac{|r\xi|^{p-N}}{\mathcal{A}(r|\xi|)[\Phi(r|\xi|)]^{p-1}} = \left(\frac{N+\vartheta-p}{p-1}\right)^{p-1} \quad \text{and} \tag{3.20}$$

$$\lim_{r\to 0} \frac{|r\xi|^{N+\sigma} L_b(r|\xi|) h_2(\Phi(r\xi))}{\int_0^{r|\xi|} t^{N-1+\sigma} L_b(t) h_2(\Phi(t))\,dt} = N+\sigma-q\left(\frac{N+\vartheta-p}{p-1}\right). \tag{3.21}$$

Since it holds that

$$\lim_{r\to 0} \int_0^{r|\xi|} t^{N-1+\sigma} L_b(t) h_2(\Phi(t))\,dt = 0,$$

we thus obtain (3.19), proving (3.18).

From (3.14), (3.18) and (3.17), we find that for any sequence $\tilde{r}_n$ decreasing to zero, there exists a subsequence r_n such that

$$V_{(r_n)} \to V \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{as } n \to \infty, \tag{3.22}$$

where V satisfies

$$\Delta_p V(\xi) + \vartheta |\nabla V(\xi)|^{p-2} \nabla V(\xi) \cdot \frac{\xi}{|\xi|^2} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}). \tag{3.23}$$

To conclude Step 2, it remains to show that the above V is given by (3.13).

Let ξ_{r_n} be on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ such that $\tilde{\lambda}(r_n) = u(r_n \xi_{r_n})/\Phi(r_n)$. Thus we obtain that for $0 < |\xi| < r_0/r_n$,

$$V_{(r_n)}(\xi) \leq \tilde{\lambda}(r_n|\xi|) \frac{\Phi(r_n|\xi|)}{\Phi(r_n)} \quad \text{and} \quad V_{(r_n)}(\xi_{r_n}) = \tilde{\lambda}(r_n) \frac{\Phi(r_n|\xi_{r_n}|)}{\Phi(r_n)}. \tag{3.24}$$

Since $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$ is compact, we can assume $\xi_{r_n} \to \xi_0$ as $n \to \infty$. Then for every $\xi \in \mathbb{R}^N \setminus \{0\}$, by taking $n \to \infty$ in (3.24) we have

$$V(\xi) \leq \lambda |\xi|^{\frac{p-N-\vartheta}{p-1}} \quad \text{and} \quad V(\xi_0) = \lambda |\xi_0|^{\frac{p-N-\vartheta}{p-1}}.$$

Thus, by [42, Theorem 2.5.2], we find that

$$\lim_{n\to\infty} V_{(r_n)}(\xi) = \lambda |\xi|^{\frac{p-N-\vartheta}{p-1}}, \quad \lim_{n\to\infty} \nabla V_{(r_n)}(\xi) = \lambda \frac{p-N-\vartheta}{p-1} |\xi|^{\frac{1-N-\vartheta}{p-1}} \frac{\xi}{|\xi|} \tag{3.25}$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$. This shows that V in (3.22) is given by (3.13). This completes the proof of Step 2.

Step 3: *Proof of Proposition 3.1.1 completed.* It remains to verify that

$$\int_{B_1} \mathcal{A}(|x|) \, |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx + \int_{B_1} b(x) \, h(u) \psi \, dx = \lambda^{p-1} \psi(0), \tag{3.26}$$

for all $\psi \in C_c^1(B_1)$. For any $\varepsilon \in (0, 1/2)$, we construct a smooth, non-decreasing function w_ε on $(0, \infty)$ as follows

$$\begin{cases} w_\varepsilon(r) = 0, & \text{for } r \in (0, \varepsilon], \\ 0 < w_\varepsilon(r) < 1, & \text{for } r \in (\varepsilon, 2\varepsilon), \\ w_\varepsilon(r) = 1, & \text{for } r \in [2\varepsilon, \infty]. \end{cases}$$

For $\psi \in C_c^1(B_1)$ fixed, we have that $\psi(x) w_\varepsilon(|x|) \in C_c^1(B_1)$. We define $\mathcal{W}_\varepsilon$ by

$$\mathcal{W}_\varepsilon = -\int_{\varepsilon<|x|<2\varepsilon} \psi(x) \mathcal{A}(|x|) |\nabla u|^{p-2} \nabla u \cdot \frac{x}{|x|} w_\varepsilon'(|x|) \, dx. \tag{3.27}$$

By taking ψw_ε in (2.6) (see Definition 2.1.1), we get that

$$\int_{B_1} w_\varepsilon \mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u\cdot\nabla\psi\,dx+\int_{B_1} b(x)h(u)\psi w_\varepsilon\,dx=\mathcal{W}_\varepsilon. \tag{3.28}$$

By (3.8) and (2.25), we find that as $|x|\to 0$,

$$\mathcal{V}(x):=-\mathcal{A}(|x|)|\nabla u|^{p-2}\psi(x)|x|^{N-2}\nabla u\cdot x\to\psi(0)(N\omega_N)^{-1}\lambda^{p-1}. \tag{3.29}$$

Thus for every $\tau>0$, there exists $\varepsilon_0=\varepsilon_0(\tau)\in(0,1/2)$ such that

$$\psi(0)(N\omega_N)^{-1}\lambda^{p-1}-\tau\le\mathcal{V}(x)\le\psi(0)(N\omega_N)^{-1}\lambda^{p-1}+\tau, \tag{3.30}$$

for every $0<|x|\le 2\varepsilon_0$. Moreover, we obtain that

$$\int_{\varepsilon<|x|<2\varepsilon}|x|^{1-N}w_\varepsilon'(|x|)\,dx=N\omega_N\int_\varepsilon^{2\varepsilon}w_\varepsilon'(r)\,dr=N\omega_N. \tag{3.31}$$

Thus, we proved that for every $\tau>0$, there exists $\varepsilon_0=\varepsilon_0(\tau)\in(0,1/2)$ such that

$$N\omega_N\left(\frac{\psi(0)}{N\omega_N}\lambda^{p-1}-\tau\right)\le\mathcal{W}_\varepsilon\le N\omega_N\left(\frac{\psi(0)}{N\omega_N}\lambda^{p-1}+\tau\right), \tag{3.32}$$

for every $0<\varepsilon<\varepsilon_0$. As $\tau>0$ is arbitrary, we get that $\lim_{\varepsilon\to 0}\mathcal{W}_\varepsilon=\lambda^{p-1}\psi(0)$. Thus (3.26) follows by letting $\varepsilon\to 0$ in (3.28). This concludes the proof of Proposition 3.1.1. □

3.2 Strong Singularities

Our aim in this section is to treat the case (iii) of strong singularities in Theorem 3.0.1. To this end, we show that it is reducible to the case of strong singularities for radial solutions of an approximate problem (3.34) treated in Theorem 3.2.1.

We reason as Cîrstea in [16, Lemma 4.12], using our *a priori* estimate in Lemmas 2.5.2 and our regularity result in Lemma 2.5.4 to deduce that for every $\varepsilon\in(0,1)$, there exists $r_\varepsilon\in(0,1)$ and a function v_ε satisfying

$$(1-\varepsilon)\,u\le v_\varepsilon\le(1+\varepsilon)\,u\quad\text{in } B_{r_\varepsilon}^* \tag{3.33}$$

with v_ε a positive solution of

$$-\Delta_{\mathcal{A},p} v + |x|^\sigma\, v^q L_b(|x|)\, L_h(v) = 0 \quad \text{in } B^*_{r_\varepsilon}. \tag{3.34}$$

Moreover, if v is any positive solution of (3.34), then as Brandolini *et al.* in [7, Lemma 4], we can obtain two positive *radial* solutions of (3.34) in $B^*_{r_\varepsilon/2}$, say v_* and v^*, such that for a sufficiently large constant $K > 1$, we have

$$K^{-1} v \le v_* \le v \le v^* \le K v \quad \text{in } B^*_{r_\varepsilon/2}. \tag{3.35}$$

We observe that any positive radial solution of (3.34) in B^* satisfies

$$\frac{d}{dr}\left(r^{N-1+\vartheta} L_{\mathcal{A}}(r) |v'(r)|^{p-2} v'(r)\right) = r^{N-1+\sigma} L_b(r)\, L_h(v(r))\, v^q(r) \tag{3.36}$$

for $r = |x| \in (0,1)$.

In view of (3.35), to conclude the assertion of (iii) in Theorem 3.0.1, it is thus enough to prove Theorem 3.2.1 below. Recall first that in the $q = q_*$ case, we further require either L_h satisfies the property that

$$t \longmapsto L_h(e^t) \text{ is regularly varying at } \infty \text{ with index } \gamma \in \mathbb{R}, \tag{3.37}$$

or we require $L_{\mathcal{A}}$ and L_b *together* to satisfy the property that

$$t \longmapsto \left[L_{\mathcal{A}}(e^{-t})\right]^{-\frac{q}{p-1}} L_b(e^{-t}) \text{ is regularly varying at } \infty \text{ with index } j \in \mathbb{R}. \tag{3.38}$$

Theorem 3.2.1. *Let Assumptions* $(\mathbf{A}_1)$*–*$(\mathbf{A}_3)$ *hold and* $b(x)\, h(\Phi) \in L^1(B_{1/2})$. *Suppose* v *is any positive solution of* (3.36) *with a strong singularity at* 0.

(a) *If* $q < q_*$*, then* $v(r) \sim \tilde{u}(r)$ *as* $r \to 0$*, where* $\tilde{u}$ *is given by*

$$\int_{\tilde{u}(r)}^\infty \frac{t^{-\frac{q+1}{p}}}{[L_h(t)]^{\frac{1}{p}}}\, dt = \int_0^r \left[M \frac{\xi^{\sigma-\vartheta} L_b(\xi)}{L_{\mathcal{A}}(\xi)}\right]^{\frac{1}{p}} d\xi \quad \text{if } q < q_*, \tag{3.39}$$

where the reciprocal of M *is defined as*

$$\frac{1}{M} := q - \frac{N+\sigma}{m_0}. \tag{3.40}$$

(b) *If* $q = q_*$*, then we have* $v(r) \sim \tilde{u}(r)$ *as* $r \to 0$*, where if* (3.37) *holds, then* $\tilde{u}$

is given by

$$\tilde{u}(r) = \left[m_1 m_0^{\gamma+1-p} F(r)\right]^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0} \quad \text{or} \tag{3.41}$$

if, on the other hand, (3.38) *holds, then* $\tilde{u}$ *is given by*

$$\int_c^{\tilde{u}(r)} [F(1/t)]^{\frac{1}{q_*-p+1}}\, dt = \left(m_1 m_0^{-p-j}\right)^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0}, \tag{3.42}$$

where m_0 *and* m_1 *are constants prescribed by* (2.12) *and* F *is defined by*

$$F(r) := \int_0^r \xi^{-1} \left[L_{\mathcal{A}}(\xi)\right]^{-\frac{q_*}{p-1}} L_b(\xi)\, L_h(1/\xi)\, d\xi \quad \text{for } r > 0 \text{ small.} \tag{3.43}$$

In (3.42)*,* $c > 0$ *is a large constant.*

The proofs of Theorem 3.2.1(a) and (b) are intricate, each being composed of three main steps. First, in Chapter 3.2.1, we shall prove Theorem 3.2.1(b), the critical case $q = q_* < \infty$, while also pointing out the major differences between the subcritical and critical cases. We conclude this chapter with the proof of the subcritical case $q < q_*$ in Chapter 3.2.2.

3.2.1 In the critical case

Under the assumptions of Theorem 3.2.1, let v be any positive solution of (3.36) with a strong singularity at 0. A change of variable $y(s) = v(r)$ with $s = \Phi(r)$ moves the singularity from $r = 0$ to $s = \infty$ for the equation

$$(p-1)\left|\frac{dy}{ds}\right|^{p-2} \frac{d^2y}{ds^2} = C_{N,p}^{-p+1}\, r^{N-1+\sigma} L_b(r)\, L_h(y(s))\, [y(s)]^q \left|\frac{dr}{ds}\right| \tag{3.44}$$

for $s \in (0, \infty)$.

Step 1. *Fix* $\eta_0 > 0$ *small. For every* $\varepsilon \in (0,1)$ *small, there exists* $r_\varepsilon \in (0,1)$ *such that* $(1-\varepsilon)\, v_{-\eta}$ *and* $(1+\varepsilon)\, v_\eta$ *is a sub-solution and super-solution of* (3.36) *for* $0 < r < r_\varepsilon$*, respectively, for every* $\eta \in [0, \eta_0]$*. Moreover, it holds that* $\lim_{\eta\to 0^+} v_{\pm\eta}(r) = \tilde{u}(r)$ *for every* $r \in (0, r_\varepsilon]$*, where* $\tilde{u}$ *is as in Theorem 3.2.1.*

The local one-parameter family $v_{\pm\eta}$ of sub- and super-solutions of (3.36) is constructed such that $v_{\pm\eta}(r)$ converges to $\tilde{u}(r)$ as η approaches 0^+. The function $\tilde{u}$ in Theorem 3.2.1 is regularly varying at 0 with index $-m_0$, where m_0 and m_2

are given by (2.12). The definition of $v_{\pm\eta}$ in the subcritical case is different from that of the critical case as follows.

In the subcritical case $q < q_*$, we define $v_{\pm\eta}$ in (3.74) as a regularly varying function at 0 with index $-(1 \pm \eta)\, m_0$ (here $m_0 > m_2$). We shall check the assertion of Step 1 in Chapter 3.2.2.

In the critical case $q = q_* < \infty$, we have $m_0 = m_2$, that is, $\tilde{u}$ has the same index of regular variation at 0 as the fundamental solution Φ in (2.8), namely $-m_2$. In this case, $v_{\pm\eta}$ is defined by (3.50) as a regularly varying function at 0 with index $-m_2$. We shall verify Step 1 with the change of variable $y_{\pm\eta}(s) = v_{\pm\eta}(r)$ where $s = \Phi(r)$. Notice that when either (3.37) holds or (3.38) holds, by the definitions of $\tilde{u}$ in (3.5) or (3.6) respectively and $v_{\pm\eta}$ in (3.50), we infer that

$$\lim_{r\to 0^+} \frac{\tilde{u}(r)}{v_\eta(r)} = 0 \quad \text{and} \quad \lim_{r\to 0^+} \frac{\tilde{u}(r)}{v_{-\eta}(r)} = \infty \quad \text{for every } \eta \in [0, \eta_0]. \tag{3.45}$$

We first give the construction of a local family of sub- and super-solutions of (3.36). Let F be given by (3.43) and $c > 0$ be a large constant. Fix $\eta_0 \in (0,1)$ small. Then for any $\eta \in [0, \eta_0]$, we define $v_{\pm\eta}(r)$ for $r > 0$ small, by (3.50) if (3.37) holds, or as (3.54) if (3.38) holds.

We set

$$y_{\pm\eta}(s) = v_{\pm\eta}(r) \quad \text{with } s = \Phi(r). \tag{3.46}$$

Using $y'_{\pm\eta}(s)$ and $y''_{\pm\eta}(s)$ to denote $dy_{\pm\eta}/ds$ and $d^2y_{\pm\eta}/ds^2$, respectively, then we observe that

$$(p-1)\left(y'_{\pm\eta}(s)\right)^{p-2} y''_{\pm\eta}(s) = \frac{1}{m_1}\left(y'_{\pm\eta}(s)\right)^{q_*} \left|\frac{d}{ds}\left[\left(y'_{\pm\eta}(s)\right)^{-q_*+p-1}\right]\right|. \tag{3.47}$$

Moreover, we obtain the following asymptotic equivalence (uniform with respect to η)

$$\ln y_{\pm\eta}(s) \sim \ln s \quad \text{and} \quad s y'_{\pm\eta}(s) \sim y_{\pm\eta}(s) \ \text{ as } s \to \infty. \tag{3.48}$$

Using Υ given by (2.25), we introduce the notation

$$\mathcal{K}_{\pm\eta}(s) := \frac{\Upsilon(r)}{m_0} \frac{s\, y'_{\pm\eta}(s)}{y_{\pm\eta}(s)}. \tag{3.49}$$

Recall that $m_0 = m_2$ for $q = q_*$. Thus by using (2.25) and (3.48), we infer that $\lim_{s\to\infty} \mathcal{K}_{\pm\eta}(s) = 1$ uniformly with respect to η.

Step 1. *For every $\varepsilon \in (0,1)$ small, there exists $s_\varepsilon > 0$ large such that*

$(1-\varepsilon)\, y_{-\eta}$ and $(1+\varepsilon)\, y_\eta$ are respectively a sub-solution and super-solution of (3.44) for $s > s_\varepsilon$, respectively, for every $\eta \in [0, \eta_0]$.

We divide the proof of Step 1 into two cases.

Case 1. Assume (3.37) holds. We define $y_{\pm\eta}(s)$ by (3.46), where $v_{\pm\eta}(r)$ is given by

$$v_{\pm\eta}(r) := C_{N,p}^{-1} \left(\frac{m_1 m_0^{\gamma-q}}{1 \pm \eta} \right)^{-\frac{1}{q-p+1}} \int_c^{\Phi(r)} \left[F(\Phi^{-1}(t)) \right]^{-\frac{1\pm\eta}{q-p+1}} dt. \tag{3.50}$$

From (3.50), we find that

$$y'_{\pm\eta}(s) = C_{N,p}^{-1} \left(\frac{m_1 m_0^{\gamma-q_*}}{1 \pm \eta} \right)^{-\frac{1}{q_*-p+1}} [F(r)]^{-\frac{1\pm\eta}{q_*-p+1}}. \tag{3.51}$$

From (3.48), we deduce the following asymptotic equivalence as $s \to \infty$ (uniform with respect to η)

$$m_0^\gamma L_h(1/r) \sim L_h(s) \sim L_h(y_{\pm\eta}(s)). \tag{3.52}$$

We also denote $R_{\pm\eta}(s)$ as follows

$$R_{\pm\eta}(s) = \frac{m_0^\gamma\, L_h(1/r)}{L_h(y_{\pm\eta}(s))}\, [F(r)]^{\pm\eta}\, [\mathcal{K}_{\pm\eta}(s)]^{q_*}.$$

Hence, using (3.52), we derive the following asymptotics as $s \to \infty$ (uniform with respect to η)

$$R_{\pm\eta}(s) \sim [F(r)]^{\pm\eta}. \tag{3.53}$$

The right-hand side of (3.47) equals the product between $R_{\pm\eta}(s)$ and the right-hand side of (3.44) for $y = y_{\pm\eta}$. By the definition of F in (3.43), we have $\lim_{r\to 0^+} F(r) = 0$. Since $q > p-1$, using (3.53), we conclude Case 1 for Step 1.

Case 2. Assume (3.38) holds. We define $y_{\pm\eta}(s)$ by (3.46), where $v_{\pm\eta}(r)$ is given by

$$\int_c^{v_{\pm\eta}(r)} [F(1/t)]^{\frac{1\pm\eta}{q_*-p+1}} dt = C_{N,p}^{-1} \left(\frac{m_1 m_0^{-q-1-j}}{1 \pm \eta} \right)^{-\frac{1}{q-p+1}} \Phi(r). \tag{3.54}$$

From (3.50), we find that

$$y'_{\pm\eta}(s) = C_{N,p}^{-1} \left(\frac{m_1 m_0^{-q_*-1-j}}{1 \pm \eta} \right)^{-\frac{1}{q_*-p+1}} [F(1/y_{\pm\eta}(s))]^{-\frac{1\pm\eta}{q_*-p+1}} . \tag{3.55}$$

From (3.48), we deduce the following asymptotic equivalence as $s \to \infty$ (uniform with respect to η)

$$\frac{(m_0)^{-j} [L_{\mathcal{A}}(1/y_{\pm\eta}(s))]^{-\frac{q_*}{p-1}} L_b(1/y_{\pm\eta}(s))}{[L_{\mathcal{A}}(\Phi^{-1}(s))]^{-\frac{q_*}{p-1}} L_b(\Phi^{-1}(s))} \sim 1. \tag{3.56}$$

We also denote $R_{\pm\eta}(s)$ as follows

$$R_{\pm\eta}(s) = m_0^{-j} \left[\frac{L_{\mathcal{A}}(1/y_{\pm\eta}(s))}{L_{\mathcal{A}}(\Phi^{-1}(s))} \right]^{-\frac{q_*}{p-1}} \frac{L_b(1/y_{\pm\eta}(s))}{L_b(\Phi^{-1}(s))} [F(1/y_{\pm\eta}(s))]^{\pm\eta} [\mathcal{K}_{\pm\eta}(s)]^{q_*+1} .$$

Hence, using (3.56), we derive the following asymptotics as $s \to \infty$ (uniform with respect to η)

$$R_{\pm\eta}(s) \sim [F(1/y_{\pm\eta}(s))]^{\pm\eta} . \tag{3.57}$$

The right-hand side of (3.47) equals the product between $R_{\pm\eta}(s)$ and the right-hand side of (3.44) for $y = y_{\pm\eta}$. By the definition of F in (3.43), we have $\lim_{r\to 0^+} F(r) = 0$. Since $q > p - 1$, using (3.57), we conclude Step 1.

Step 2. *The functions* v_η and $v_{-\eta}$ *constructed in Step 1 satisfy the following property:*

$$\lim_{r\to 0^+} \frac{v(r)}{v_\eta(r)} = 0 \quad \text{and} \quad \lim_{r\to 0^+} \frac{v(r)}{v_{-\eta}(r)} = \infty. \tag{3.58}$$

In both the subcritical and critical cases, since v has a strong singularity at 0, that is $v(r)/\Phi(r) \to \infty$ as $r \to 0^+$, then we have $y(s)/s \to \infty$ as $s \to \infty$. Using that $y''(s) \geq 0$, we find that $y'(s)$ is increasing so that $\lim_{s\to\infty} y'(s) = \infty$. As the function $s \longmapsto sy'(s) - y(s)$ is increasing on $(0, \infty)$ and $\lim_{s\to\infty} y(s) = \infty$, we see that

$$\liminf_{s\to\infty} \frac{sy'(s)}{y(s)} \geq 1. \tag{3.59}$$

In the *subcritical* case, we shall use (3.59) in Lemma 3.2.2(b) of Chapter 3.2.2 to improve the behaviour of the solution v of (3.36) from dominating near zero the fundamental solution Φ (of index $-m_2$) to dominating *any* function f regularly varying at zero with index $-\kappa$, where $m_2 < \kappa < m_0$. We deduce (3.58) by using

Lemma 3.2.2 with $f = v_{\pm\eta}$ since the index of regular variation at 0 for the function v_η (respectively, $v_{-\eta}$) is smaller (respectively, bigger) than $-m_0$. We point out that Lemma 3.2.2 relies essentially on the assumption that $q < q_*$ and cannot be adapted to the critical case.

Hence, in the *critical* case, we need a new argument that takes into account that $v_{\pm\eta}$ varies regularly at 0 with the same index as $\tilde{u}$. We now prove Step 2 in the critical case.

Proof of Step 2 for the critical case $q = q_*$.

The main ingredient in the proof of (3.58) is given by the following

$$0 < \liminf_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \leq \limsup_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} < \infty. \tag{3.60}$$

By combining (3.45) and (3.60), we conclude (3.58) in the critical case.

Proof of (3.60). Using (2.45) and (3.59), we infer that

$$\limsup_{s\to\infty} sy''(s)/y'(s) < \infty.$$

Indeed, by (3.44), we have

$$\frac{sy''(s)}{y'(s)} = \frac{1}{p-1}\left[\frac{y(s)}{sy'(s)}\right]^{p-1} [\Upsilon(r)]^{-p}\,\frac{L_b(r)}{L_{\mathcal{A}}(r)}\, r^{p+\sigma-\vartheta} L_h(y(s))\,[y(s)]^{q_*-p+1}\,, \tag{3.61}$$

where Υ is given by (2.25). For $s_0 > 0$, there exists a large constant $C > 0$ so that $s \mapsto sy'(s) - Cy(s)$ is non-increasing for all $s > s_0$. It follows that $\ell = \limsup_{s\to\infty} sy'(s)/y(s) < \infty$. From (3.59), we can take $s_0 > 0$ large such that

$$\frac{1}{2} \leq \frac{s\,y'(s)}{y(s)} \leq 2\ell \quad \text{for all } s \geq s_0. \tag{3.62}$$

In view of (2.46), we find that

$$\ln y(s) \sim \ln s \quad \text{as } s \to \infty. \tag{3.63}$$

Case 1: *Assume that* (3.37) *holds.*

From (3.63), as $s \to \infty$, we obtain that

$$m_0^\gamma L_h(1/r) \sim L_h(s) \sim L_h(y(s)). \tag{3.64}$$

For all $s \geq s_0$, by using (3.62) and (3.64) in (3.44), we find positive constants c_1 and c_2 so that

$$c_1 \frac{d}{dr} F(r) \leq [y'(s)]^{-q_*+p-2}\, y''(s) \left| \frac{ds}{dr} \right| \leq c_2 \frac{d}{dr} F(r) \tag{3.65}$$

where F is defined by (3.43).

Since $y'(s) \to \infty$ as $s \to \infty$, by integrating (3.65), we obtain that

$$c_3 F(\Phi^{-1}(s)) \leq [y'(s)]^{-q_*+p-1} \leq c_4 F(\Phi^{-1}(s)) \quad \text{for all } s > s_0, \tag{3.66}$$

where c_3 and c_4 are positive constants. Using (3.62) in (3.66), then reversing the change of variable $y(s) = v(r)$ with $s = \Phi(r)$, we infer that there exist positive constants c_5 and c_6 such that

$$c_5\, [F(r)]^{-\frac{1}{q_*-p+1}}\, \Phi(r) \leq v(r) \leq c_6\, [F(r)]^{-\frac{1}{q_*-p+1}}\, \Phi(r). \tag{3.67}$$

for all $r \in (0, \Phi^{-1}(s_0))$. Hence, using (2.25) and the definition of $\tilde{u}$ in (3.5), we conclude Step 2 in Case 1.

Remark 3.2.1. Notice that when (3.37) holds, the existence of a solution v of (3.36) with a strong singularity at zero implies that $b(x)\, h(\Phi(|x|)) \in L^1(B_{1/2})$. Indeed, fixing $r_0 \in (0, \Phi^{-1}(s_0))$, then for every $\varepsilon \in (0, r_0)$, by integrating the first inequality in (3.65) with respect to r from ε to r_0, and letting $\varepsilon \to 0$, we conclude the claim (using Remark 2.4.2). A more general statement is proven later in Proposition 4.1.3.

Case 2: *Assume that* (3.38) *holds.*

From (3.63), as $s \to \infty$, we obtain that

$$(m_0)^{-j}\, [L_{\mathcal{A}}(1/y(s))]^{-\frac{q_*}{p-1}}\, L_b(1/y(s)) \sim \left[L_{\mathcal{A}}(\Phi^{-1}(s))\right]^{-\frac{q_*}{p-1}}\, L_b(\Phi^{-1}(s)). \tag{3.68}$$

For all $s \geq s_0$, by using (3.62) and (3.68) in (3.44), we find positive constants c_1 and c_2 so that

$$c_1 \frac{d}{ds}[F(1/y(s))] \leq [y'(s)]^{-q_*+p-2}\, y''(s) \leq c_2 \frac{d}{ds}[F(1/y(s))] \tag{3.69}$$

where F is defined by (3.43). By twice integrating (3.69), we find positive constants

c_3 and c_4 such that

$$c_3 \le \frac{d}{ds}\left(\int_{y(s_0)}^{y(s)} [F(1/t)]^{\frac{1}{q_*-p+1}}\,dt\right) \le c_4 \quad \text{for every } s > s_0.$$

We thus conclude that

$$0 < \liminf_{s\to\infty} \frac{\int_{y(s_0)}^{y(s)} [F(1/t)]^{\frac{1}{q_*-p+1}}\,dt}{s} \le \limsup_{s\to\infty} \frac{\int_{y(s_0)}^{y(s)} [F(1/t)]^{\frac{1}{q_*-p+1}}\,dt}{s} < \infty.$$

This, jointly with (2.25) and the definition of $\tilde{u}$ in (3.6), proves the assertion of Step 2 in Case 2.

Step 3. *Proof of Theorem 3.2.1 concluded.*

Proof of Step 3. The reasoning is the same for the subcritical and critical case. It is based on the previous two steps and the comparison principle Lemma 2.5.1, introduced in Chapter 2.

Let $\varepsilon \in (0,1)$ be small and $r_\varepsilon \in (0,1)$ be as in Step 1. Fix $\eta \in [0,\eta_0]$ arbitrarily. Then, we obtain that $(1+\varepsilon)\,v_\eta(r) + v(r_\varepsilon)$ and $v(r) + \tilde{u}(r_\varepsilon)$ are super-solutions of (3.36) for $r \in (0, r_\varepsilon)$. By (3.58) and Lemma 2.5.1, for all $0 < r \le r_\varepsilon$, we have

$$v(r) \le (1+\varepsilon)\,v_\eta(r) + v(r_\varepsilon) \quad \text{and} \quad (1-\varepsilon)\,v_{-\eta}(r) \le v(r) + \tilde{u}(r_\varepsilon). \tag{3.70}$$

Since r_ε is independent of $\eta \in [0,\eta_0]$, by letting $\eta \to 0^+$ in (3.70), we find that

$$v(r) \le (1+\varepsilon)\,\tilde{u}(r) + v(r_\varepsilon) \quad \text{and} \quad (1-\varepsilon)\,\tilde{u}(r) \le v(r) + \tilde{u}(r_\varepsilon). \tag{3.71}$$

for all $0 < r \le r_\varepsilon$. By letting $r \to 0^+$ in (3.71), we deduce that

$$1-\varepsilon \le \liminf_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \le \limsup_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \le 1+\varepsilon. \tag{3.72}$$

Finally, by passing to the limit $\varepsilon \to 0^+$ in (3.72), we conclude that

$$v(r) \sim \tilde{u}(r) \quad \text{as } r \to 0^+. \tag{3.73}$$

This completes the proof of Theorem 3.2.1 in the critical case $q = q_*$.

3.2.2 In the subcritical case

We need only to justify the first two steps in the outline of the proof of Theorem 3.2.1 as Step 3 is common for both critical and subcritical cases (see Step 3 of Chapter 3.2.1). We shall adapt the perturbation method initiated by Cîrstea and Du in [18]. We construct a local family of sub-and super-solutions of (3.36). Fix $\eta_0 \in (0,1)$ such that $2\eta_0(p-1)M < 1$, where M is the positive constant given by (3.4). For every $\eta \in [0,\eta_0]$, we define the function $v_{\pm\eta}$ and the constant $C_{\pm\eta} > 0$ as

$$v_{\pm\eta}(r) = C_{\pm\eta}[\tilde{u}(r)]^{1\pm\eta} \quad \text{for } r \in (0,1), \tag{3.74}$$

where

$$C_{\pm\eta}^{q-p+1} := (1\pm\eta)^{p-1}\left[1 \pm \eta M(p-1)\right]. \tag{3.75}$$

From this definition, we have that $\lim_{\eta\to 0^+} v_{\pm\eta}(r) = \tilde{u}(r)$ for every $r \in (0,1)$ and $\lim_{\eta\to 0} C_{\pm\eta} = 1$.

Step 1. *For every $\varepsilon \in (0,1)$ small, there exists $r_\varepsilon \in (0,1)$ such that $(1-\varepsilon)\, v_{-\eta}$ and $(1+\varepsilon)\, v_\eta$ are respectively a sub-solution and super-solution of* (3.36) *for $0 < r < r_\varepsilon$, for every $\eta \in [0,\eta_0]$.*

Claim. *We see that $\tilde{u}$ satisfies* (3.36) *asymptotically as $r \to 0^+$.*

Proof of Claim. Let $r_0 \in (0,1)$ be small so that $\tilde{u}(r_0) > t_0$, where t_0 is as in Remark 2.4.1. For all $r \in (0,r_0)$, we set

$$\begin{cases} Q_{\pm\eta}(r) := r^{N-1+\vartheta} L_{\mathcal{A}}(r) \left|v'_{\pm\eta}(r)\right|^{p-2} v'_{\pm\eta}(r), \\ P(r) := M \left[q + 1 + \dfrac{\tilde{u}(r)\, L_h'(\tilde{u}(r))}{L_h(\tilde{u}(r))} - \dfrac{\tilde{u}(r)\, \tilde{u}''(r)}{[\tilde{u}'(r)]^2} \right. \\ \qquad\qquad \left. + \left(N-1+\sigma+\dfrac{rL_b'(r)}{L_b(r)}\right) \dfrac{\tilde{u}(r)}{r\,\tilde{u}'(r)} \right]. \end{cases} \tag{3.76}$$

One can verify that $\lim_{r\to 0^+} P(r) = 1$ using the definition of M in (3.4). By differentiating (2.14), we find that

$$Q_0(r) = M r^{N-1+\sigma} L_b(r) \frac{[\tilde{u}(r)]^{q+1}}{\tilde{u}'(r)} L_h(\tilde{u}(r)) \quad \text{for all } r \in (0,r_0). \tag{3.77}$$

The claim follows since $Q_0'(r)$ equals the product between $P(r)$ in (3.76) and the right-hand side of (3.36) for $v = \tilde{u}$.

By twice differentiating (3.74), we obtain that

$$\begin{cases} Q_{\pm\eta}(r) = [C_{\pm\eta}(1\pm\eta)]^{p-1}\,[\tilde{u}(r)]^{\pm\eta(p-1)}\,Q_0(r), \\ \dfrac{dQ_{\pm\eta}}{dr} = [C_{\pm\eta}(1\pm\eta)]^{p-1}\,[\tilde{u}(r)]^{\pm\eta(p-1)}\times \\ \qquad\qquad \left\{\pm\eta\,(p-1)M\,[\tilde{u}(r)]^q\,L_h(\tilde{u}(r))\,L_b(r)\,r^{N-1+\sigma} + \dfrac{dQ_0}{dr}\right\}. \end{cases} \tag{3.78}$$

Hence, using (3.74) and the above claim, we find the following asymptotics (uniform with respect to η)

$$\frac{dQ_{\pm\eta}}{dr} \sim C^q_{\pm\eta} r^{N-1+\sigma}\,L_b(r)\,L_h(\tilde{u}(r))\,[\tilde{u}(r)]^{q\pm\eta(p-1)} \quad \text{as } r\to 0^+. \tag{3.79}$$

From Remark 2.4.1 in Chapter 2.4.1, the function $t \longmapsto t^{q-p+1}\,L_h(t)$ is increasing on $(0,\infty)$ so that

$$L_h(\tilde{u}^{1-\eta})\,[\tilde{u}(r)]^{-\eta\,(q-p+1)} \leq L_h(\tilde{u}(r)) \leq L_h(\tilde{u}^{1+\eta})\,[\tilde{u}(r)]^{\eta\,(q-p+1)}$$

for every $r\in(0,r_0)$ and all $\eta\in[0,\eta_0]$. This, together with (3.74), implies that for every $r\in(0,r_0)$ and all $\eta\in[0,\eta_0]$

$$\pm C^q_{\pm\eta}\,L_h(\tilde{u}(r))\,[\tilde{u}(r)]^{q\pm\eta(p-1)} \leq \pm L_h(v_{\pm\eta}(r)/C_{\pm\eta})\,[v_{\pm\eta}(r)]^q\,. \tag{3.80}$$

Since $q > p-1$, from (3.79), (3.80) and Proposition 1.5.1 in Chapter 1.5, we conclude the proof of Step 1.

Step 2. *Any positive solution v of* (3.36) *with a strong singularity at* 0 *satisfies* (3.58).

Since $v_{\pm\eta}$ is regularly varying at 0 with index $-(1\pm\eta)\,m_0$, we conclude Step 2 based on Lemma 3.2.2 with $f = v_{\pm\eta}$.

Lemma 3.2.2. *Let* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold and $q < q_*$. Suppose that v is a positive solution of* (3.36) *with a strong singularity at zero. Let f be a regularly varying function at zero with real index $-\kappa$. With m_0 given by* (2.12)*, the following hold:*

(a) *If $\kappa > m_0$, then* $\lim_{r\to 0^+} v(r)/f(r) = 0$.

(b) *If $\kappa < m_0$, then* $\lim_{r\to 0^+} v(r)/f(r) = \infty$.

Proof. We adapt ideas from Cîrstea and Du [18, Theorem 1.4].

(a) The *a priori* estimates in (2.28) show that v is bounded from above near zero by a regularly varying function at 0 with index $-m_0$. The assertion now follows easily since every regularly varying function at 0 with positive (respectively, negative) index must converge to 0 (respectively, ∞).

(b) Since $\kappa < m_0$, we can choose $q_1 \in (q, q_*)$ sufficiently close to q such that $\kappa < (p+\sigma-\vartheta)/(q_1-p+1)$. Then, $\lim_{t\to\infty} t^{q-q_1} L_h(t) = 0$ (see Remark 1.5.1 in Chapter 1.5) and using (3.59), we can let $s_0 > 0$ large and find that

$$L_h(y(s))\,[y(s)]^q \leq [y(s)/2]^{q_1} \leq s^{q_1}[y'(s)]^{q_1} \quad \text{for all } s \geq s_0. \tag{3.81}$$

We set $f_{q_1}(r) := r^{N-1+\sigma} L_b(r)[\Phi(r)]^{q_1}$ for $r \in (0,1)$. Since Φ is regularly varying at 0 with index $-m_2$ (see (2.25)), we find that f_{q_1} is regularly varying at 0 with index $N+\sigma-q_1 m_2 - 1$, which is greater than -1. This gives that

$$\int_{0^+} f_{q_1}(\xi)\,d\xi < \infty.$$

Moreover, the function $F_{q_1}(r) = \int_r^{\Phi^{-1}(s_0)} \left[\int_0^\tau f_{q_1}(\xi)\,d\xi\right]^{-\frac{1}{q_1-p+1}} |\Phi'(\tau)|\,d\tau$ is regularly varying at zero with index $-(p+\sigma-\vartheta)(q_1-p+1)$, which is less than $-\kappa$ from our choice of q_1. We thus have $\lim_{r\to 0^+} F_{q_1}(r)/f(r) = \infty$.

We conclude that $\lim_{r\to 0^+} v(r)/f(r) = \infty$ by showing that

$$\liminf_{r\to 0^+} \frac{v(r)}{F_{q_1}(r)} > 0.$$

Indeed, we see that

$$\liminf_{r\to 0^+} \frac{v(r)}{F_{q_1}(r)} = \liminf_{s\to\infty} \frac{y(s)}{\int_{s_0}^s \left[\int_0^{\Phi^{-1}(t)} f_{q_1}(\xi)\,d\xi\right]^{-\frac{1}{q_1-p+1}} dt}. \tag{3.82}$$

From (3.44) and (3.81), we deduce that

$$[y'(s)]^{p-2-q_1}\, y''(s) \leq -\frac{C_{N,p}^{-p+1}}{p-1} f_{q_1}(\Phi^{-1}(s))\, \frac{d(\Phi^{-1}(s))}{ds} \quad \text{for all } s > s_0. \tag{3.83}$$

Recall that $\lim_{s\to\infty} y'(s) = \infty$ since v has a strong singularity at 0. Thus, by integrating (3.83), we obtain that

$$y'(s) \geq \left[\frac{(q_1 - p + 1)\, C_{N,p}^{-p+1}}{p-1} \int_0^{\Phi^{-1}(s)} f_{q_1}(\xi)\, d\xi \right]^{-\frac{1}{q_1 - p + 1}} \quad \text{for all } s > s_0,$$

which shows that the right-hand side of (3.82) is positive. This concludes the assertion of Lemma 3.2.2(b). □

4

Removability and Existence

In this chapter, we prove removability and existence results regarding our problem

$$\operatorname{div}\left(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u\right) = b(x)\,h(u) \quad \text{in } B^*, \tag{4.1}$$

where $\mathcal{A}$, b and h satisfy Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$. In Chapter 4.1, we prove Theorem 2.2.2 which states that under the necessary and sufficient condition $b(x)h(\Phi(x)) \notin L^1(B_{1/2})$, all singularities at 0 are removable. We then prove the existence and uniqueness of solutions to (4.35) in Chapter 4.2.

4.1 Removable Singularities

This section is dedicated to the proof of Theorem 2.2.2, restated here as Theorem 4.1.1 for the convenience of the reader. Throughout this section, we let Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold.

Theorem 4.1.1 (Sharp Removability). *If $b(x)\,h(\Phi) \notin L^1(B_{1/2})$, then $q \geq q_*$ and every positive solution of* (4.1) *can be extended as a positive continuous solution of* (4.1) *in the whole ball B_1.*

The proof of Theorem 4.1.1 relies on two main ingredients, Proposition 4.1.2 and Proposition 4.1.3.

Proposition 4.1.2. *Suppose u is a positive solution of* (4.1) *such that it satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi(x)} = 0. \tag{4.2}$$

Then it holds that

$$\lim_{|x|\to 0} u(x) \in (0,\infty) \quad \text{and} \quad \lim_{|x|\to 0} |x||\nabla u(x)| = 0. \tag{4.3}$$

Moreover, u can be extended as a continuous positive solution of (4.1) *in B_1.*

This result, which was also invoked in the proof of Theorem 2.2.1(i), generalises [18, Lemma 3.2(ii)] (where Cîrstea–Du investigated (4.1) for $\mathcal{A} = 1$) and [7, Proposition 3] (where Brandolini *et al.* looked at (4.1) in the case $p = 2$, $b = 1$ and $h(u) = u^q$).

Proof of Proposition 4.1.2. For a positive solution u of (4.1) satisfying (4.2), we define

$$\theta := \limsup_{|x|\to 0} u(x). \tag{4.4}$$

We divide the proof into five steps. We prove in Steps 1 and 2 that θ is positive and finite. In Steps 3 and 4, we introduce a rescaled function $U_{(r)}$ and show that it is bounded and converges to some function U which then gives that (4.3) holds. We then show in Step 5 that u can be extended as a continuous positive solution of (4.1) in B_1.

Step 1: *Fix $r_0 \in (0, 1/2)$. We show $\theta > 0$.*

By the comparison principle in Lemma 2.5.1, we follow [18, Lemma 3.2] in finding that $\theta < \infty$. Set $C := \max_{|x|=r_0} u(x)$ where $r_0 \in (0,1)$. For all integers $n \geq 1$, we define a sequence of functions $v_n(x)$ such that

$$v_n(x) = \frac{1}{n}\Phi(x) + C \quad \text{for every } 0 < |x| \leq r_0.$$

Since (4.2) holds for any integer $n \geq 1$, there exists $r_n > 0$ such that

$$u(x) \leq v_n(x) \quad \text{for every } x \in \mathbb{R}^N \text{ with } 0 < |x| \leq r_n. \tag{4.5}$$

We can assume the sequence $\{r_n\}$ decreases to zero and $r_n < r_0$ for every $n \geq 1$. Let us define by $\mathcal{Q}_n$ a sequence of annuli

$$\mathcal{Q}_n := \{x \in \mathbb{R}^N : r_n < |x| < r_0\}.$$

Then we have $u \leq v_n$ on $\partial\mathcal{Q}_n$ and

$$-\Delta_{\mathcal{A},p} u \leq 0 = -\Delta_{\mathcal{A},p} v_n \quad \text{in } \mathcal{Q}_n.$$

By the comparison principle, we find that $u \le v_n$ in $\mathcal{Q}_n$ for all $n \ge 1$. For $x \in \mathbb{R}^N$ with $0 < |x| < r_0$, we have

$$u(x) \le v_n(x) \quad \text{for all } n \ge 1 \text{ sufficiently large.} \tag{4.6}$$

Since $\lim_{n\to\infty} v_n(x) = C$, we conclude that

$$u(x) \le C \quad \text{for } 0 < |x| \le r_0. \tag{4.7}$$

This completes the proof that $\theta < \infty$. Unlike the case $\mathcal{A} = 1$ treated in [18], our general assumption $(\mathbf{A}_1)$ does not satisfy (1.5). Thus we cannot invoke Serrin's result [50, Theorem 1] to conclude the assertions of Proposition 4.1.2.

Step 2: *We show $\theta < \infty$.*

In the special case $p = 2$ and $h(t) = t^q$ of [7], the claim follows by a reduction to radial solutions, coupled with a change of variable and the work of Taliaferro in [55, Theorem 1.1]. For our general divergence-form equation, we require different ideas that are inspired by Cîrstea in [16, Lemma 5.2].

Since Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold and $\theta < \infty$, there exists a positive constant C such that

$$b(x)\, h(u) \le C|x|^\sigma L_b(|x|)\, u^{p-1} \quad \text{for all } 0 < |x| \le 1/2. \tag{4.8}$$

Similar to Step 2 of [16, Lemma 5.2], we construct a positive radial solution v_∞ of

$$-\Delta_{\mathcal{A},p} v + C|x|^\sigma L_b(|x|)\, v^{p-1} = 0 \quad \text{for } 0 < |x| < 1/2 \tag{4.9}$$

such that $v_\infty(|x|) \le u(x)$ for $0 < |x| \le 1/2$. By a contradiction argument and the comparison principle Lemma 2.5.1, we find that the radial solution v_∞ of (4.9) has a non-negative limit at 0.

To construct v_∞, we set, as in Step 2 of the proof of [16, Lemma 5.2], the sequence of annuli $\mathcal{D}_n := \{x \in \mathbb{R}^N : \frac{1}{n} < |x| < \frac{1}{2}\}$, for integers $n \ge 3$. We consider the boundary value problem

$$\begin{cases} -\Delta_{\mathcal{A},p} v + C|x|^\sigma L_b(|x|) v^{p-1} = 0 & \text{in } \mathcal{D}_n, \\ v(x) = \min_{|y|=|x|} u(y) & \text{for } |x| = \frac{1}{n} \text{ and } |x| = \frac{1}{2}. \end{cases} \tag{4.10}$$

We denote by v_n the unique positive C^2-solution of (4.10). The uniqueness follows from the comparison principle (see Lemma 2.5.1). We have that v_n is radially symmetric (by invariance of the operator under rotation, the symmetry of $\mathcal{D}_n$ and the boundary data). From (4.8), we infer that u is a super-solution of (4.10). Therefore, we have that $v_{n+1} \leq v_n \leq u$ in $\mathcal{D}_n$ for every $n \geq 3$. By the regularity results in Lemma 2.5.4, we conclude that for a sequence $n_k \to \infty$, we have $v_{n_k} \to v_\infty$ in $C^1_{\rm loc}(B^*)$ and v_∞ is a non-negative radial solution of (4.10) satisfying $v_\infty \leq u$ for $0 < |x| < \frac{1}{2}$. Moreover, since $v_\infty(1/2) = \min_{|y|=|x|} u(y) > 0$ and from the strong maximum principle, we must have $v_\infty(r) > 0$ for each $r \in (0, 1/2)$.

To conclude that $\theta > 0$, it suffices to show that $\lim_{r\to 0^+} v_\infty(r) > 0$. By assuming that $\lim_{r\to 0^+} v_\infty(r) = 0$, we arrive at a contradiction as follows. We apply on (4.10) the change of variable

$$z(s) = v_\infty(r) \quad \text{with } s = \Phi(r). \tag{4.11}$$

Then, we have $\lim_{s\to\infty} z(s) = 0$. Moreover, z is a positive solution of the ordinary differential equation

$$\left|\frac{dz}{ds}\right|^{p-2} \frac{d^2 z}{ds^2} = C_1 r^{N-1+\sigma} L_b(r)\ [z(s)]^{p-1} \left|\frac{dr}{ds}\right| \quad \text{for } s \in (\Phi(1/2), \infty), \tag{4.12}$$

where C_1 denotes a positive constant. Since $z''(s) > 0$, then $z'(s)$ is increasing on $(\Phi(1/2), \infty)$ with $\lim_{s\to\infty} z'(s) = 0$. Therefore, using (4.12), we find that

$$z(s) = C_2 \int_s^\infty \left(\int_0^{\Phi^{-1}(t)} \xi^{N-1+\sigma} L_b(\xi) \left[z(\Phi(\xi))\right]^{p-1} d\xi \right)^{\frac{1}{p-1}} dt \quad \text{for } s > \Phi(1/2),$$

where C_2 is a positive constant. Since z is decreasing, we infer that

$$1/C_2 \leq \int_s^\infty \left(\int_0^{\Phi^{-1}(t)} \xi^{N-1+\sigma} L_b(\xi)\, d\xi \right)^{\frac{1}{p-1}} dt \quad \text{for every } s > \Phi(1/2). \tag{4.13}$$

Let $V(s)$ denote the right-hand side of (4.13). We claim that $V(s)$ is well-defined and that $V(s) \to 0$ as $s \to \infty$. Indeed, we have $\Phi \in RV_{-m_2}(0+)$ and thus $\Phi^{-1} \in RV_{-1/m_2}(\infty)$. Note that $r \longmapsto \int_0^r \xi^{N-1+\sigma} L_b(\xi)\, d\xi$ is regularly varying at 0^+ with positive index given by $\sigma + N$. Consequently, V is regularly varying at ∞ with *negative* index $(p+\sigma-\vartheta)/(p-N-\vartheta)$ so that the claim follows. Then, (4.13) leads to a contradiction, which proves that $\lim_{r\to 0^+} v_\infty(r) > 0$ and, hence, $\theta > 0$.

Step 3: *Fix $r_0 \in (0, 1/2)$. We show that $\lim_{r\to 0^+} F(r) = \theta$ where we define*

$$F(r) := \sup_{|x|=r} u(x) \quad \text{for all } r \in (0, 2r_0). \tag{4.14}$$

It is clear that $\limsup_{r\to 0} F(r) = \theta$. We now show that $\liminf_{r\to 0} F(r) = \theta$. Assume by contradiction that $\liminf_{r\to 0} F(r) < \theta$. By definition of lim inf, we can find a sequence of positive real numbers (t_n), decreasing to zero as $n \to \infty$ and $\varepsilon > 0$ small, such that $F(t_n) \le \theta - \varepsilon$ for all $n \ge 1$. Since $\limsup_{r\to 0} F(r) = \theta$, then we have

$$F(t_*) > \theta - \varepsilon \quad \text{for some small } t_* > 0. \tag{4.15}$$

Without loss of generality, let $t_* < t_1 < 1$ and $n_1 > 1$ be large enough such that $t_{n_1} < t_*$. Fix $n \ge n_1$ (so that $t_n \le t_{n_1}$, and define the annulus

$$\Omega := \{x \in \mathbb{R}^N : t_n < |x| < t_1\}.$$

Using (4.15), we infer that $\max\{F(t_n), F(t_1)\} \le \theta - \varepsilon < F(t_*)$, we find that u achieves its maximum β in the interior of Ω and u is not a constant in Ω. By applying Theorem 2.5.2 of [42] with $v = \beta$, we see that u necessarily is constant in Ω. Thus by contradiction, it follows that $\lim_{r\to 0^+} F(r) = \theta$ holds.

Step 4: *For any fixed $r \in (0, r_0)$, we define*

$$U_{(r)}(\xi) := u(r\xi) \quad \textit{for every } \xi \in \mathbb{R}^N \textit{ with } 0 < |\xi| < \frac{r_0}{r}. \tag{4.16}$$

We show that $U_{(r)} \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ as $r \to 0^+$, where

$$U(\xi) = \lambda \quad \textit{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{4.17}$$

A direct calculation shows that $U_{(r)}$ satisfies

$$\Delta_p U_{(r)}(\xi) + \frac{r|\xi|\mathcal{A}'(r|\xi|)}{\mathcal{A}(r|\xi|)} |\nabla U_{(r)}(\xi)|^{p-2} \nabla U_{(r)}(\xi) \cdot \frac{\xi}{|\xi|^2} = r^p \frac{b(r\xi) h(U_{(r)}(\xi))}{\mathcal{A}(r|\xi|)} \tag{4.18}$$

for $0 < |\xi| < \frac{r_0}{r}$. We deduce by the properties of slowly varying functions that

$$\lim_{r\to 0} r^p \frac{b(r\xi)}{\mathcal{A}(r|\xi|)} = 0 \quad \text{for every fixed } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{4.19}$$

There exists a positive constant C_1 depending on r_0 such that

$$u(x) \le C_1 \quad \text{for every } 0 < |x| \le 2r_0. \tag{4.20}$$

By taking $g \equiv 1$ in Lemma 2.5.4, we obtain that there exists positive constants $C > 0$ and $\alpha \in (0, 1)$ such that for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \le |x'| < r_0$,

$$|\nabla u(x)| \le \frac{C}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \le C\frac{|x - x'|^\alpha}{|x|^{1+\alpha}}. \tag{4.21}$$

It is enough to show that any sequence $\tilde{r}_n$ decreasing to zero contains a subsequence r_n such that

$$U_{(r_n)} \to U \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}). \tag{4.22}$$

By (4.20) and (4.21), it follows from Lemma 2.5.4 that for every compact subset $K \subset \mathbb{R}^N \setminus \{0\}$, there exists positive constants C_1, C depending on K and independent of r, for every fixed $r \in (0, r_0)$ such that we have

$$\begin{cases} 0 < U_{(r)}(\xi) \le C_1, \quad |\nabla U_{(r)}(\xi)| \le \frac{C}{|\xi|}, \\ |\nabla U_{(r)}(\xi) - \nabla U_{(r)}(\xi')| \le C\frac{|\xi-\xi'|^\alpha}{|\xi|^{1+\alpha}}, \end{cases} \tag{4.23}$$

for every ξ and ξ' in $\mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$.

From (4.23), (4.18) and the Arzelá-Ascoli Theorem, that is, the uniform boundedness and equicontinuity of $U_{(r)}$ from the regularity result in Lemma 2.5.4, any sequence $\tilde{r}_n$ decreasing to zero contains a subsequence r_n such that $U_{(r_n)}$ such that $U_{(r_n)} \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, where U satisfies

$$\Delta_p U + \vartheta|\nabla U|^{p-2}\nabla U \cdot \frac{\xi}{|\xi|^2} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}). \tag{4.24}$$

To conclude Step 4, it remains to show that the above U is given by (4.22).

Let ξ_{r_n} be on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ such that $F(r_n) = u(r_n\xi_{r_n})$. By our definitions of F and U, we obtain

$$U_{(r_n)}(\xi) \le F(r_n|\xi|) \quad \text{for } 0 < |\xi| < \frac{r_0}{r_n} \quad \text{and} \quad U_{(r_n)}(\xi_{r_n}) = F(r_n).$$

Since S^{N-1} is compact, we can assume $\xi_{r_n} \to \xi_0$ as $n \to \infty$. Then for every $\xi \in \mathbb{R}^N \setminus \{0\}$, we have

$$U(\xi) \le \theta \quad \text{and} \quad U(\xi_0) = \theta.$$

We apply the strong maximum principle (see [42, Theorem 2.5.1]) on $U - \lambda$ to conclude $U = \lambda$ in $\mathbb{R}^N \setminus \{0\}$. This gives us that

$$\lim_{n\to\infty} U_{(r_n)}(x) = \lambda \quad \text{and} \quad \lim_{n\to\infty} \nabla U_{(r_n)}(x) = 0$$

for every $x \in \mathbb{R}^N \setminus \{0\}$, that is

$$\lim_{r\to 0} U_{(r)}(x) = \lambda \quad \text{and} \quad \lim_{r\to 0} \nabla U_{(r)}(x) = 0$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. Then for $x = r\xi$ when $|\xi| = 1$, we conclude that $\lim_{|x|\to 0} u(x) = \lambda$ and $\lim_{|x|\to 0} |x||\nabla u(x)| = 0$.

Step 5: *Proof of Proposition 4.1.2 completed.* It remains to show that u can be continuously extended as a positive solution of (4.1). It should be noted that in [18], Cîrstea–Du were able to Serrin's result in [50, Theorem 1.1]. However, in the case of Brandolini *et al.* [7] and our case, the equivalent form of our problem in (2.39) contains $\mathcal{A}'(|x|)/\mathcal{A}(|x|) \sim 1/r$ which is unbounded as $r \to 0$. We construct a smooth function w_ε satisfying the following conditions.

$$\begin{cases} w_\varepsilon(r) = 0 & \text{for } r \in (0, \varepsilon], \\ 0 < w_\varepsilon(r) < 1 & \text{for } r \in (\varepsilon, 2\varepsilon), \\ w_\varepsilon(r) = 1 & \text{for } r \in [2\varepsilon, \infty]. \end{cases}$$

Fix $\psi \in C_c^1(B_1)$, we have that $\psi(x)w_\varepsilon(|x|) \in C_c^1(B_1)$. We define $\mathcal{W}_\varepsilon(x)$ by

$$\mathcal{W}_\varepsilon(x) := -\int_{\varepsilon<|x|<2\varepsilon} \psi(x)\mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot w_\varepsilon'(|x|)\frac{x}{|x|}\, dx.$$

By taking ψw_ε in (2.6) (see Definition 2.1.1) that, we get that

$$\int_{B_1} w_\varepsilon \mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, dx + \int_{B_1} b(x)h(u)\psi w_\varepsilon\, dx = \mathcal{W}_\varepsilon(x). \tag{4.25}$$

We wish to prove that u can be continuously extended as a positive solution of (5.1) in $\mathcal{D}'(B_1)$ by showing that $\lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon = 0$.

Since $\lim_{|x|\to 0} |x||\nabla u(x)| = 0$ holds, then for all $\tau > 0$, there exists $r_\tau \in (0, 1)$ such that

$$|\nabla u| \le \frac{\tau}{|x|} \quad \text{for every } 0 < |x| \le r_\tau. \tag{4.26}$$

Let I_ε denote

$$I_\varepsilon := \int_{\varepsilon<|x|<2\varepsilon} |x|^{1-p}\mathcal{A}(|x|)w_\varepsilon'(|x|)\,dx. \tag{4.27}$$

We find that $\mathcal{W}_\varepsilon$ in (4.25) satisfies

$$|\mathcal{W}_\varepsilon| \le \|\psi\|_{L^\infty(B_1)}\tau^{p-1}I_\varepsilon \quad \text{for any } \varepsilon\in(0,r_\tau/2). \tag{4.28}$$

In view of (4.28), it is enough to conclude Step 5 by showing that $I_\varepsilon \to 0$ as $\varepsilon\to 0^+$.

Since $p<N+\vartheta$, then $r\mapsto r^{N-p}\mathcal{A}(r)$ is regularly varying at 0^+ of positive index $N+\vartheta-p$. Thus it is asymptotically equivalent near 0^+ to a monotone increasing function and $\lim_{r\to 0^+} r^{N-p}\mathcal{A}(r)=0$. Since $w_\varepsilon(2\varepsilon)=1$ and $w_\varepsilon(\varepsilon)=0$, we conclude

$$\int_\varepsilon^{2\varepsilon} r^{N-p}\mathcal{A}(r)w_\varepsilon'(r)\,dr \to 0 \quad \text{as } \varepsilon\to 0, \tag{4.29}$$

which proves that $I_\varepsilon \to 0$ as $\varepsilon\to 0$. This concludes Step 5 and the proof of Proposition 4.1.2.

□

Proposition 4.1.3. *If* (3.36) *has a positive solution with either a weak or a strong singularity at* 0*, then* $b(x)\,h(\Phi)\in L^1(B_{1/2})$.

Proof. We show that $b(x)\,h(\Phi)\in L^1(B_{1/2})$ is a necessary condition for the existence of a positive solution of (3.36) with a weak or strong singularity at 0. Let v be a positive solution of (3.36) with

$$\lim_{r\to 0^+}\frac{v(r)}{\Phi(r)}=\lambda\neq 0. \tag{4.30}$$

First, we consider the case $\lambda\in(0,\infty)$. Let $\Phi^{-1}(t)$ denote the inverse of Φ, which exists for any $t>0$.

Since $v(r)\sim\lambda\Phi(r)$ as $r\to 0^+$, we have $y(s)\sim\lambda s$ as $s\to\infty$. Using that $d^2y/ds^2\ge 0$, we get that dy/ds is increasing on $(0,\infty)$ so that $\lim_{s\to\infty} dy/ds=\lambda$. For $s>0$ large, we define Λ by

$$\Lambda(s) := \frac{C_{N,p}^{-p+1}}{p-1}[\Phi^{-1}(s)]^{N-1+\sigma}L_b(\Phi^{-1}(s))\,L_h(s)\,s^{p-2}\left|\frac{dr}{ds}\right|. \tag{4.31}$$

Since $L_h\in RV_0(\infty)$ and $y(s)\sim\lambda s$ as $s\to\infty$, we have $L_h(y(s))\sim L_h(s)$ as

$s \to \infty$. We apply (3.48) to (3.44) to get that

$$\begin{cases} \dfrac{d^2y}{ds^2} \sim \Lambda(s)[y(s)]^{q-p+2} \quad \text{as } s \to \infty, \\ y'(s) \to \lambda \quad \text{as } s \to \infty. \end{cases} \tag{4.32}$$

By Taliaferro [55, p. 96], we obtain that $\int^\infty t^{q-p+2}\Lambda(t)\,dt < \infty$. By a change of variable $r = \Phi^{-1}(t)$ and Remark 2.4.2, we obtain that $b(x)\,h(\Phi) \in L^1(B_{1/2})$.

Secondly, let $\lambda = \infty$. We adapt ideas from the proof of [16, Lemma 5.8]. Choose $m \in (p-1, q_*)$ and for $t > 0$, and set

$$\chi(t) = t^{q_*-m} L_h(t).$$

By the property of the slowly varying function L_h in (2.24), we have

$$\lim_{t\to\infty} \frac{t\chi'(t)}{\chi(t)} = q_* - m > 0.$$

Hence, $\chi(t)$ is increasing for $t > 0$ sufficiently large. Since $\lim_{r\to 0^+} v_*(r)/\Phi(r) = \infty$, there exists a constant $a_0 > 0$ such that $v_*(r) \geq a_0\Phi(r)$ for all $0 < r \leq 1/2$. Then there exists a constant $c > 0$ such that

$$L_h(v_*)\, v_*^{q_*} \geq c\chi(\Phi(r))\, v_*^m \quad \text{for all } r \in (0, 1/2]. \tag{4.33}$$

Define a function $\tilde{b}(r) := c\, r^\sigma L_b(r)\, \chi(\Phi(r))$ for $r \in (0, 1/2]$. We construct a positive radial solution v_∞ of

$$-\Delta_{\mathcal{A},p} v + \tilde{b}(|x|)\, v^m = 0 \quad \text{in } B_{1/2}^* \tag{4.34}$$

such that $v_* \leq v_\infty$ in $B_{1/2}^*$. Then, v_∞ has a strong singularity at 0^+. Since $\chi \in RV_{q_*-m}(\infty)$, we find that $\tilde{b} \in RV_{\tilde{\sigma}}(0+)$ with $\tilde{\sigma}$ given by $m(N+\sigma)/q_* - N$, which is greater than $\vartheta - p$ from our choice of m. We note that (4.34) corresponds to (3.36) in the *critical* case with $r^\sigma L_b(r) = \tilde{b}(r)$, $L_h \equiv 1$ and $q = m$, where (2.1) holds. Using Remark 3.2.1 on (4.34), and the definition of $\tilde{b}$, we conclude that $b(x)\,h(\Phi) \in L^1(B_{1/2})$. This completes the proof of Proposition 4.1.3. □

Proof of Theorem 4.1.1. We show how to use Proposition 4.1.2 and Proposition 4.1.3 to finish the proof of Theorem 4.1.1. We thus assume that $b(x)\,h(\Phi) \notin L^1(B_{1/2})$ and prove that any positive solution of (4.1) can be extended as a positive solution of (4.1) in B_1. By Remark 2.4.2, we have $q \geq q_*$, with q_* as in (2.10). Our argument is twofold:

Case 1: $q > q_*$.

Since $m_0 < m_2$, the claim follows from Proposition 4.1.2 and the *a priori* estimates in (2.45). Indeed, we have $\limsup_{|x|\to 0} u(x)/T(|x|) < \infty$ for a function T regularly varying at 0 with index $-m_0$. Using that $\Phi \in RV_{-m_2}(0+)$, by Remark 1.5.1 and Definition 1.5.1 in Chapter 1.5, we find that $\lim_{r\to 0^+} T(r)/\Phi(r) = 0$ so that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$ for any positive solution u of (4.1). Then, by Proposition 4.1.2, we conclude the proof of Theorem 4.1.1.

Case 2: $q = q_*$.

The previous argument no longer applies since T and Φ are now regularly varying at 0 with the same index $-m_0$. Hence, T/Φ is slowly varying at 0, whose behaviour at 0 is, in general, undetermined as illustrated by Example 1 in Appendix 1.5. In view of Proposition 4.1.2, we conclude the proof by showing that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$.

Assuming the contrary and using (2.44), we deduce $\lim_{|x|\to 0} u(x) = \infty$. Then there exists $k \in (0, 1/2)$ and a positive solution v_* of (3.36) for $0 < r < k$ such that $C_1 u \leq v_* \leq C_2$ in B_k^*, where C_1 and C_2 are positive constants. Thus, by Proposition 4.1.3, we cannot have $\limsup_{|x|\to 0} u(x)/\Phi(x) \in (0, \infty]$. This completes the proof of Theorem 4.1.1. □

4.2 Existence and Uniqueness

In this section, we give the proof of Theorem 2.2.3, restated below, studying the existence and uniqueness of solutions to (4.35).

Theorem 4.2.1 (Existence and uniqueness)**.** *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. Assume that h is a non-decreasing function on $(0, \infty)$ and $g \in C^1(\partial B_1)$ is an arbitrary non-negative function. We consider the following problem*

$$\begin{cases} \Delta_{\mathcal{A},p} u = b(x)\, h(u) \quad \text{in } B^* := B_1 \setminus \{0\}, \\ \lim_{|x|\to 0} \frac{u(x)}{\Phi(x)} = \lambda, \quad u > 0 \quad \text{in } B^* \\ u\big|_{\partial B_1} = g. \end{cases} \tag{4.35}$$

(a) *If $\lambda = 0$ and $g \not\equiv 0$ on ∂B_1, then (4.35) has a unique solution.*

(b) *If* $\lambda \in (0,\infty]$, *then* (4.35) *admits solutions if and only if* $b(x)\,h(\Phi)$ *is in* $L^1(B_{1/2})$.

(c) *Assume that* $b(x)\,h(\Phi) \in L^1(B_{1/2})$ *and* $h(t)/t^{p-1}$ *is non-decreasing for* $t > 0$.

   (i) *For* $\lambda \in (0,\infty)$, *then* (4.35) *has a unique solution. The same conclusion holds for* $\lambda = \infty$ *and* $q < q_*$.

   (ii) *For* $\lambda = \infty$ *and* $q = q_*$, *then* (4.35) *has a unique solution provided that either* (2.1) *or* (2.2) *holds.*

We treat separately the following cases: $\lambda = 0$, $\lambda \in (0,\infty)$ and $\lambda = \infty$. For the construction of a solution of (4.35), we adapt ideas from Cîrstea–Du [18, Theorem 1.2] (where $\mathcal{A} = 1$), see also Brandolini *et al.* [7, Proposition 5], where $p = 2$, $b = 1$ and $h(t) = t^q$. We denote $C_0 := \max_{|x|=1} g(x)$. For every $n \geq 2$ and $0 \leq \lambda < \infty$, we consider the auxiliary problem

$$\begin{cases} \Delta_{\mathcal{A},p} u = b(x)\,h(u) & \text{in } D_n := B_1 \setminus \overline{B_{1/n}}, \\ u(x) = \lambda\,\Phi(|x|) + C_0 & \text{for } |x| = 1/n, \\ u\big|_{\partial B_1} = g. \end{cases} \tag{4.36}$$

For $\lambda = 0$, we further assume that $g \not\equiv 0$ on ∂B_1. By the method of sub- and super-solutions, and the comparison principle Lemma 2.5.1, the problem (4.36) admits a unique non-negative solution $u_{n,\lambda,g}$, which is continuous on $\overline{D_n}$. For simplicity, whenever λ and g are fixed, we simply write u_n instead of $u_{n,\lambda,g}$. By the strong maximum principle (see [42, Theorem 2.5.1]), we see that u_n positive in D_n. Moreover, by Lemma 2.5.1, we infer that

$$0 < u_{n+1} \leq u_n \leq \lambda\Phi(|x|) + C_0 \quad \text{in } D_n. \tag{4.37}$$

By Lemma 2.5.4, we have that, up to a subsequence, u_n converges to $u_{\lambda,g}$ in $C^1_{\rm loc}(B^*)$. Moreover, for some $\alpha \in (0,1)$, we find that $u_{\lambda,g}$ is a non-negative $C^{1,\alpha}_{\rm loc}(B^*) \cap C(\overline{B_1} \setminus \{0\})$-solution of the problem

$$\begin{cases} \Delta_{\mathcal{A},p} u = b(x)\,h(u) & \text{in } B^* := B_1 \setminus \{0\}, \\ u\big|_{\partial B_1} = g. \end{cases} \tag{4.38}$$

By the strong maximum principle, $u_{\lambda,g}$ is positive in B^* (using here that $g \not\equiv 0$ on ∂B_1 when $\lambda = 0$). From (4.37), we find that $\limsup_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \leq \lambda$. In particular, the problem (4.35) with $\lambda = 0$ admits $u_{\lambda,g}$ as a solution.

4.2.1 Proof of Theorem 4.2.1(a)

It remains to show the uniqueness of the solution of (4.35) with $\lambda = 0$. Let u_1 and u_2 be two solutions of (4.35) with $\lambda = 0$. To show that $u_1 = u_2$ in B^*, we proceed as in [7, Proposition 4] with modifications appearing here due to our more general setting. By Proposition 4.1.2, u_1 and u_2 can be extended by continuity at 0. Since u_1, $u_2 \in C^1(B^*) \cap C(\overline{B_1})$ with $u_1 = u_2 = g$ on ∂B_1, then $u_1 = u_2$ in B_1 would be a consequence of the following claim.

Claim: We have $\nabla(u_1 - u_2)(x) = 0$ for all $x \in B^*$.

Proof of Claim. Assume by contradiction that there exists $x_0 \in B^*$ such that $|\nabla(u_1 - u_2)(x_0)| > 0$. We fix r_0 small such that $0 < r_0 < \min\{1 - |x_0|, |x_0|\}$, which ensures that $\overline{B_{r_0}(x_0)} \subset B^*$. Since $u_1 - u_2 \in C^1(B^*)$, by making r_0 smaller if necessary, we can assume that $|\nabla(u_1 - u_2)(x)| > 0$ on $\overline{B_{r_0}(x_0)}$ and thus

$$|\nabla u_1(x)| + |\nabla u_2(x)| > 0 \quad \text{on } \overline{B_{r_0}(x_0)}.$$

Hence, there exists a positive constant c_0 such that

$$(|\nabla u_1(x)| + |\nabla u_2(x)|)^{p-2} |\nabla(u_1 - u_2)(x)|^2 \geq c_0 \quad \text{for all } x \in \overline{B_{r_0}(x_0)}. \tag{4.39}$$

By [15, Proposition 17.3, on p. 235], there exists a positive constant c_p such that

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_p (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2 \tag{4.40}$$

for every ξ, η in $\mathbb{R}^N$. Let us denote by $\mathcal{H}$ the following quantity,

$$\mathcal{H}(x) := [|\nabla u_1(x)|^{p-2}\nabla u_1(x) - |\nabla u_2(x)|^{p-2}\nabla u_2(x)] \cdot \nabla(u_1 - u_2)(x). \tag{4.41}$$

Thus by (4.39) and (4.40), we find for all $x \in \overline{B_{r_0}(x_0)}$ that $\mathcal{H}(x) \geq c_p c_0$.

For any $\varepsilon \in (0, 1/2)$, we denote $D_\varepsilon := B_1 \setminus \overline{B_\varepsilon}$. Let w_ε be a non-decreasing and smooth function on $(0, \infty)$ such that

$$\begin{cases} w_\varepsilon(r) \in (0,1) & \text{if } \varepsilon < r < 2\varepsilon, \\ w_\varepsilon(r) = 1 & \text{if } r \geq 2\varepsilon, \\ w_\varepsilon(r) = 0 & \text{if } 0 < r \leq \varepsilon. \end{cases} \tag{4.42}$$

We choose $\varepsilon > 0$ such that $2\varepsilon < |x_0| - r_0$, which yields $\overline{B_{r_0}(x_0)} \subseteq D_{2\varepsilon} \subset D_\varepsilon$. Since $w_\varepsilon(|x|) = 1$ for all $x \in D_{2\varepsilon}$, by using (4.41), we arrive at

$$\begin{aligned} \int_{D_\varepsilon} w_\varepsilon(|x|)\,\mathcal{A}(|x|)\,\mathcal{H}(x)\,dx &\geq \int_{B_{r_0}(x_0)} \mathcal{A}(|x|)\,\mathcal{H}(x)\,dx \\ &\geq c_p\,c_0\,\omega_N r_0^N \min_{x \in \overline{B_{r_0}(x_0)}} \mathcal{A}(|x|) := c_{p,\mathcal{A}}. \end{aligned} \tag{4.43}$$

Since $\mathcal{A} \in C(0,1]$ is a positive function and $\overline{B_{r_0}(x_0)} \subset B^*$, we then obtain that $c_{p,\mathcal{A}}$ is a positive constant.

Observe that u_1, u_2 and w_ε belong to $W^{1,p}(D_\varepsilon) \cap L^\infty(D_\varepsilon)$. We define

$$\phi_\varepsilon(x) := (u_1 - u_2)(x)\,w_\varepsilon(|x|) \quad \text{for all } x \in B^*.$$

Since $\phi_\varepsilon|_{\partial D_\varepsilon} = 0$, it follows by the product rule that $\phi_\varepsilon \in W_0^{1,p}(D_\varepsilon)$. Using the density of $C_c^1(D_\varepsilon)$ in $W_0^{1,p}(D_\varepsilon)$, we have

$$\int_{D_\varepsilon} \mathcal{A}(|x|)\,|\nabla u_j|^{p-2}\nabla u_j \cdot \nabla \phi_\varepsilon\,dx + \int_{D_\varepsilon} b(x)\,h(u_j)\,\phi_\varepsilon\,dx = 0. \tag{4.44}$$

with $j = 1, 2$. In particular, by subtracting the relation in (4.44) with $j = 2$ from the one corresponding to $j = 1$, we obtain that

$$\begin{aligned} \int_{D_\varepsilon} w_\varepsilon(|x|)\,\mathcal{A}(|x|)\,\mathcal{H}(x)\,dx + \int_{D_\varepsilon} b(x)\,(h(u_1) - h(u_2))\,(u_1 - u_2)\,w_\varepsilon(|x|)\,dx \\ = -K_\varepsilon, \end{aligned} \tag{4.45}$$

where $\mathcal{H}$ is given by (4.41) and K_ε is defined by

$$\begin{aligned} K_\varepsilon = \int_{\varepsilon<|x|<2\varepsilon} &|x|^\vartheta\,L_{\mathcal{A}}(|x|)\,w'_\varepsilon(|x|)\,(u_1 - u_2) \\ &\times \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot \frac{x}{|x|}\,dx. \end{aligned} \tag{4.46}$$

Since $w_\varepsilon(2\varepsilon) = 1$ and $w_\varepsilon(\varepsilon) = 0$ by definition in (4.42), we observe that

$$\begin{aligned} \mathcal{L}_\varepsilon :&= \int_{\varepsilon<|x|<2\varepsilon} |x|^{\vartheta-p+1} L_{\mathcal{A}}(|x|)\,w'_\varepsilon(|x|)\,dx = |\partial B_1| \int_\varepsilon^{2\varepsilon} r^{\vartheta+N-p} L_{\mathcal{A}}(r)\,w'_\varepsilon(r)\,dr \\ &\leq |\partial B_1| \max_{r\in[\varepsilon,2\varepsilon]} \{r^{\vartheta+N-p}\,L_{\mathcal{A}}(r)\}. \end{aligned}$$

Using that $\vartheta + N - p > 0$ and $L_{\mathcal{A}}$ is slowly varying at zero, we get that

$$\lim_{r\to 0^+} r^{\vartheta+N-p} L_{\mathcal{A}}(r) = 0.$$

By Remark 2.2.4, if $\vartheta+N = p$ and we further assume that $\limsup_{r\to 0^+} L_{\mathcal{A}}(r) < \infty$, then we obtain that $\limsup_{\varepsilon\to 0^+} \mathcal{L}_\varepsilon \in (0,\infty)$.

Thus, using (4.46), jointly with $|x||\nabla u_j(x)| \to 0$ as $|x| \to 0$ for $j = 1, 2$ (see Proposition 4.1.2), we find that

$$|K_\varepsilon| \leq \left(\|u_1\|_{L^\infty(B_1)} + \|u_2\|_{L^\infty(B_1)}\right) \mathcal{L}_\varepsilon \max_{\varepsilon\leq |x|\leq 2\varepsilon} |x|^{p-1} \left(|\nabla u_1(x)|^{p-1} + |\nabla u_2(x)|^{p-1}\right)$$

which tends to 0 as $\varepsilon \to 0^+$. Hence, we can fix $\varepsilon > 0$ small enough to ensure that $|K_\varepsilon| < c_{p,\mathcal{A}}$, where $c_{p,\mathcal{A}}$ is the positive constant appearing in (4.43). Since the second term in the left-hand side of (4.45) is non-negative, from (4.43) and (4.45), we get a contradiction. This proves the claim, which concludes the proof of the uniqueness of the solution of (4.35) with $\lambda = 0$.

4.2.2 Proof of Theorem 4.2.1(b)

If (4.35) has a solution for $\lambda \in (0,\infty]$, then $b(x)\, h(\Phi) \in L^1(B_{1/2})$ from Theorem 4.1.1.

Claim 1: If $b(x)\, h(\Phi) \in L^1(B_{1/2})$, then $u_{\lambda,g}$ constructed above for $\lambda \in (0,\infty)$ is a solution of (4.35).

Proof of Claim 1. We need only to show that

$$\liminf_{|x|\to 0} \frac{u_{\lambda,g}(x)}{\Phi(|x|)} \geq \lambda. \tag{4.47}$$

We note that (2.27) is equivalent to $\int^\infty t^{q-p+2}\Lambda(t)\, dt < \infty$, where Λ is defined by (4.31). Then, by [55, Theorem 2.4], if $R > 0$ is large, there exists a positive proper solution of the following problem

$$\begin{cases} \dfrac{d^2y}{ds^2} = \Lambda(s)[y(s)]^{q-p+2} \quad \text{for } s \in (R,\infty), \\ y'(s) \to \lambda \quad \text{as } s \to \infty \quad \text{and } y(R) \in (0,\infty). \end{cases} \tag{4.48}$$

Using the transformation $w(r) = y(s)$ with $r = \Phi^{-1}(s)$ and Remark 2.4.1, we obtain that

$$\begin{cases} \Delta_{\mathcal{A},p} w \sim b(x)\, h_2(w(|x|)) & \text{as } |x| \to 0^+, \\ w(r) \sim \lambda\Phi(r) & \text{as } r \to 0^+. \end{cases} \tag{4.49}$$

Hence, for every $\varepsilon \in (0,1)$, there exists $r_\varepsilon \in (0, \Phi^{-1}(R))$ such that $(1-\varepsilon)\, w$ is a sub-solution of

$$\Delta_{\mathcal{A},p} v = b(x)\, h_2(v) \quad \text{in } B^*_{r_\varepsilon} := B_{r_\varepsilon} \setminus \{0\}. \tag{4.50}$$

Recall that $u_{n,\lambda,g}$, in short u_n, represents the unique non-negative solution of (4.36). Since $w(r) \sim \lambda\Phi(r)$ as $r \to 0^+$ (see (4.49)), there exists $n_\varepsilon \geq 1$ large such that

$$(1-\varepsilon)\, w(1/n) \leq \lambda\, \Phi(1/n) \leq u_n(x) \quad \text{for every } |x| = 1/n \text{ and all } n \geq n_\varepsilon.$$

Let $C_\varepsilon := w(r_\varepsilon)$. Since u_n is a positive super-solution of (4.50) due to our choice of h_2, by Lemma 2.5.1, we have

$$(1-\varepsilon)\, w \leq u_n + C_\varepsilon \quad \text{for } 1/n < |x| < r_\varepsilon \text{ and all } n \geq n_\varepsilon.$$

By letting $n \to \infty$, we find that $(1-\varepsilon)\, w \leq u_{\lambda,g} + C_\varepsilon$ in $B^*_{r_\varepsilon}$. Hence, we conclude that $\liminf_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \geq (1-\varepsilon)\lambda$. Since $\varepsilon \in (0,1)$ is arbitrary, we obtain that $\liminf_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \geq \lambda$. Since $\limsup_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \leq \lambda$, it follows that $u_{\lambda,g}$ is a solution of (4.35) for $\lambda \in (0,\infty)$.

Claim 2: If $b(x)\, h(\Phi) \in L^1(B_{1/2})$, then (4.35) admits a solution for $\lambda = \infty$.

Proof of Claim 2. Let k be any positive integer and denote by $u_{k,g}$ the solution we constructed earlier for (4.35) with λ replaced by k. Then, by the comparison principle (Lemma 2.5.1), we find that $0 < u_{k,g} \leq u_{k+1,g}$ in B^*. We show that for every fixed $x \in B_1 \setminus \{0\}$, there exists $\lim_{k\to\infty} u_{k,g}(x) \in (0,\infty)$. Indeed, since $|x| > 0$, we can fix $\rho = \rho_x$ such that $0 < \rho < \min\{|x|, 1/4\}$. Hence, by Lemma 2.5.2, there exists $C_\rho > 0$ such that $u_{k,g}(y) \leq C_\rho$ for all $|y| = \rho$ and every $k \geq 1$. By Lemma 2.5.1, it follows that $u_{k,g}(y) \leq \max\{C_0, C_\rho\}$ for all $\rho \leq |y| \leq 1$ and all $k \geq 1$, where $C_0 = \max_{|x|=1} g(x)$. Hence, for all $x \in \overline{B_1} \setminus \{0\}$, we can define $u_{\infty,g}(x) := \lim_{k\to\infty} u_{k,g}(x)$. Moreover, by Lemma 2.5.4, we have that, up to a subsequence, $u_{k,g} \to u_{\infty,g}$ in $C^1_{\text{loc}}(B^*)$ and $u_{\infty,g}$ is a solution of (4.35) with $\lambda = \infty$. This concludes Claim 2 and the proof of Theorem 4.2.1(b).

4.2.3 Proof of Theorem 4.2.1(c)

Assume that $b(x)\,h(\Phi) \in L^1(B_{1/2})$ and $h(t)/t^{p-1}$ is non-decreasing for $t > 0$. We show the uniqueness of the solution of (4.35) in any of the following situations:

(A) $\lambda \in (0, \infty)$;
(B) $\lambda = \infty$ and $q < q_*$;
(C) $\lambda = \infty$ and $q = q_*$, assuming also that either (2.1) or (2.2) holds.

Indeed, if u_1 and u_2 are arbitrary solutions of (4.35) corresponding to the same λ and g, then $\lim_{|x|\to 0} u_1(x)/u_2(x) = 1$. This follows immediately in Case (A) with finite λ. For the Case (B) and Case (C), we use Theorem 2.2.1 to obtain the same asymptotic behaviour near zero for any positive solution of (4.1) with a strong singularity at 0. Let $\varepsilon > 0$ be arbitrary. As $h(t)/t^{p-1}$ is increasing on $(0, \infty)$, we can check that $(1 + \varepsilon)u_i$ are super-solutions of (4.35) for $i = 1, 2$. Applying the comparison principle from Lemma 2.5.1, we find that $u_1 \le (1 + \varepsilon)u_2$ and $u_2 \le (1 + \varepsilon)u_1$ in B^*. Letting ε tend to zero, the uniqueness claim follows and $u_1 = u_2$ in B^*. This completes the proof of Theorem 4.2.1. □

5

Weighted Elliptic Operator

This theorem has the important consequence that the asymptotic character of a fundamental solution is an inherent property of the particular equation.

– David Gilbarg and James Serrin [28]

In this chapter, we give the profile near 0 of all positive solutions to the weighted p-Laplacian equation in (5.1) for the range $1 < p < \infty$. The weight $\mathcal{A}$ in (5.1) is introduced under the framework of regular variation theory, complementing our previous work on $\operatorname{div}(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = b(x)\,h(u)$ in B^* in Chapters 2–4.

In Chapter 5.1, we provide the framework for our problem (5.1) whose positive solutions are classified near 0 in Theorem 5.1.1 and Theorem 5.1.2. We reveal that the profile of any such solution depends on α in (5.5): if $\alpha = \infty$, the classical result stated in the above chapter quote holds (see Theorem 5.1.1), whereas if $\alpha < \infty$, the solution to (5.1) has a *nonnegative* finite limit at 0 but the singularity 0 is not necessarily removable (see Theorem 5.1.2 and Remark 5.1.4). Chapter 5.2 is dedicated to the auxiliary tools used to prove the main theorems. We give the proofs of the main results in Chapter 5.3 and Chapter 5.4.

5.1 A Dichotomous Classification

We classify the behaviour of solutions with isolated singularities to general nonlinear elliptic equations of the type

$$-\operatorname{div}(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B^* := B_1 \setminus \{0\}, \tag{5.1}$$

where B_1 denotes the unit ball centred at 0 in $\mathbb{R}^N$ with $N \geq 2$. We impose $\mathcal{A}$ to be a positive $C^1(0,1]$-function such that

$$\lim_{t\to 0^+} \frac{t\mathcal{A}'(t)}{\mathcal{A}(t)} = \vartheta \text{ for some } \vartheta \in \mathbb{R}. \tag{5.2}$$

Definition 5.1.1. We say that $u \in C^1(B^*)$ a *solution* (*sub-solution, super-solution*) of (5.1) if for all $\psi(x) \in C_c^1(B^*)$, it holds

$$\int_{B_1} \mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot \nabla \psi \, dx = 0 \quad (\leq 0, \ \geq 0). \tag{5.3}$$

We say a positive solution u of (5.1) in B^* can be *extended as a positive continuous solution* of (5.1) in the whole ball B_1, if u admits a positive finite limit at the origin, $\mathcal{A}(|x|)|\nabla u|^{p-1} \in L^1_{\text{loc}}(B_1)$ and satisfies

$$\int_{B_1} \mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot \nabla \psi \, dx = 0, \qquad \text{for all } \psi \in C_c^1(B_1). \tag{5.4}$$

Our classification of the behaviour near the origin of the solutions to (5.1) depends on the finiteness of the following quantity:

$$\alpha := \lim_{r\to 0^+} \int_r^1 \left(\frac{t^{1-N}}{\mathcal{A}(t)} \right)^{\frac{1}{p-1}} dt. \tag{5.5}$$

Remark 5.1.1. Due to the presence of $\mathcal{A}$ in (5.1), the finiteness of (5.5) depends on the dimension N and on the index ϑ of regular variation at 0 for $\mathcal{A}$.

(a) If $p < N + \vartheta$, we note that $\alpha = \infty$ in (5.5).

(b) If $p > N + \vartheta$, then $\alpha < \infty$ in (5.5).

(c) If $p = N + \vartheta$, then $t \mapsto (t^{1-N}/\mathcal{A}(t))^{1/(p-1)}$ is slowly varying at 0^+ (see Definition 1.5.1(b)). Hence, we have that

  (i) in some cases, $\alpha = \infty$ such as when $\mathcal{A}(|x|) = |x|^{\vartheta}$,

  (ii) whereas in other cases, $\alpha < \infty$ such as when $\mathcal{A}(|x|) = |x|^{\vartheta} \ln(1/|x|)^{\frac{p-1}{2}} \exp\{(p-1)[\ln(1/|x|)]^{1/2}\}$.

In summary, if $\alpha = \infty$ (respectively, $\alpha < \infty$) in (5.5), then $p \leq N + \vartheta$ (respectively, $p \geq N + \vartheta$). In relation to our operator in (5.1) and Remark 5.1.1, we introduce two functions Φ_1, $\Phi_2 : (0,1) \to (0,\infty)$ as follows

$$\begin{cases} \Phi_1(r) = C_{N,p} \displaystyle\int_r^1 \left(\frac{t^{1-N}}{\mathcal{A}(t)} \right)^{\frac{1}{p-1}} dt & \text{if } \alpha = \infty, \\ \Phi_2(r) = C_{N,p} \displaystyle\int_0^r \left(\frac{t^{1-N}}{\mathcal{A}(t)} \right)^{\frac{1}{p-1}} dt & \text{if } \alpha < \infty, \end{cases} \tag{5.6}$$

for every $r \in (0,1)$, where $C_{N,p} := (N\omega_N)^{-1/(p-1)}$ and ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. When $\mathcal{A} \equiv 1$, then we recover the p-Laplacian operator in (5.1). In this case, then $\alpha = \infty$ for $1 < p \leq N$ and we can find that

$$\Phi_1(|x|) = \mu(|x|) - \mu(1) \quad \text{for } 0 < |x| \leq 1,$$

where $\mu(|x|)$ is the fundamental solution of the p-Laplacian as defined in (1.3); meanwhile, for $p > N$, then $\alpha < \infty$ and we have instead

$$\Phi_2(|x|) = [(p-1)/(p-N)] C_{N,p} |x|^{(p-N)/(p-1)}.$$

As seen in Chapter 2, we note that Φ_1 satisfies (2.9), that is, $-\Delta_{\mathcal{A},p}\Phi_1 = \delta_0$ in $\mathcal{D}'(B_1)$ and $\Phi_1 = 0$ on ∂B_1. We find that Φ_2 also satisfies $\Delta_{\mathcal{A},p}\Phi_2 = \delta_0$ in $\mathcal{D}'(B_1)$. For this reason, we can think of Φ_1 and Φ_2 as the fundamental solution of the operator $(\Delta_{\mathcal{A},p})$ when $\alpha = \infty$ and $\alpha < \infty$, respectively. Moreover, all positive radial solutions to (5.1) are described in Remark 5.1.4.

We now introduce the main results of this chapter which classify the behaviour of the positive solutions to (5.1) near 0 for the entire range $1 < p < \infty$. We differentiate between the two cases $\alpha = \infty$ and $\alpha < \infty$ in Theorem 5.1.1 and Theorem 5.1.2 respectively. The novelty of the Theorem 5.1.2 is that all positive solutions to (5.1) have a finite limit at the origin and we can distinguish them further depending on whether they vanish at the origin or have a positive limit.

Theorem 5.1.1. *Let* (5.2) *hold. Assume that* $\alpha = \infty$ *in* (5.5) *(additionally, in the case* $p = N + \vartheta$*, we require that* $\limsup_{r\to 0^+} \mathcal{A}(r)/r^{\vartheta} < \infty$*).*

If u *is a positive solution of* (5.1)*, then one of the following holds:*

(a) *either* u *can be extended as a positive continuous solution of* (5.1) *in the whole unit ball* B_1*;*

(b) *or,* $u(x)/\Phi_1(|x|)$ *converges to a positive constant* γ *as* $|x| \to 0$ *and*

$$u - \gamma\Phi_1 \in L^\infty_{\rm loc}(B_1). \tag{5.7}$$

Moreover, if $p < N + \vartheta$*, then* u *satisfies the following equation,*

$$-\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = \gamma^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{5.8}$$

Theorem 5.1.2. *Let* (5.2) *hold and assume that* $\alpha < \infty$ *in* (5.5)*. Then* u *has a nonnegative finite limit at the origin and one of the following holds*

(a) *either* u *converges to a positive finite limit* u_0 *at the origin and it holds that*

$$\frac{u - u_0}{\Phi_2} \in L^\infty_{\rm loc}(B_1); \tag{5.9}$$

(b) *or* $\lim_{|x|\to 0} u(x) = 0$ *and* $u(x)/\Phi_2(|x|)$ *converges to a positive constant* γ *as* $|x| \to 0$*. Moreover, if* $p > N + \vartheta$*, then* u *satisfies*

$$\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = \gamma^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{5.10}$$

Remark 5.1.2. In the case $\mathcal{A} \equiv 1$, our Theorem 5.1.1 recovers Theorem 1.4.2 whereas Theorem 5.1.2 recaptures Theorem 1.4.3. Both results are inspired by the work of Kichenassamy–Véron [31].

Remark 5.1.3. In Theorem 5.1.2(a), we have that $\lim_{|x|\to 0} u(x)$ is positive and finite. However, unlike the case $\alpha = \infty$ in Theorem 5.1.1(a), it does not imply that u can be extended as a continuous solution of (5.1) in $\mathcal{D}'(B_1)$. Indeed, for example, take $u = \gamma\Phi_2 + u_0$ where $\gamma \in (0, \infty)$ and $u_0 > 0$.

Remark 5.1.4. The only positive radial solutions v to (5.1) are

$$v(r) = c_1\Phi^+(r) + c_2\Phi^-(r) \quad \text{for } r \in (0, 1),$$

where $c_1 \geq 0$, $c_2 \in \mathbb{R}$ and $\Phi^\pm$ is defined by

$$\begin{cases} \Phi^+ = \Phi_1 \text{ and } \Phi^- = 1 & \text{on } (0, 1] \quad \text{if } \alpha = \infty, \\ \Phi^+ = 1 \text{ and } \Phi^- = \Phi_2 & \text{on } [0, 1] \quad \text{if } \alpha < \infty. \end{cases} \tag{5.11}$$

Any positive solution u to (5.1) is bounded above and below by radial solutions of (5.1) in a neighbourhood of 0 (see Lemma 5.2.2), so that we have

$$\begin{cases} \text{(a) either} \qquad 0 < \liminf_{|x|\to 0} \dfrac{u(x)}{\Phi^-(|x|)} \le \limsup_{|x|\to 0} \dfrac{u(x)}{\Phi^-(|x|)} < \infty, \\ \text{(b) or} \qquad 0 < \liminf_{|x|\to 0} \dfrac{u(x)}{\Phi^+(|x|)} \le \limsup_{|x|\to 0} \dfrac{u(x)}{\Phi^+(|x|)} < \infty. \end{cases} \tag{5.12}$$

In case (a), we say that u is a *dominated* solution (since $\lim_{|x|\to 0} u(x)/\Phi^+(|x|) = 0$). In the latter case (b), we call u a *dominating* solution.

Our main results establish in particular that for any positive solution u of (5.1)

$$\text{either} \quad \lim_{|x|\to 0} \frac{u(x)}{\Phi^-(|x|)} \in (0,\infty) \quad \text{or} \quad \lim_{|x|\to 0} \frac{u(x)}{\Phi^+(|x|)} \in (0,\infty). \tag{5.13}$$

In Chapter 5.2, we present auxiliary results such as a spherical Harnack-type inequality (and its consequence Lemma 5.2.2 that gives rise to Remark 5.1.4), as well as a regularity result. The main theorems are proved as follows: Chapter 5.3 is dedicated to the proofs of Theorem 5.1.1(a) and Theorem 5.1.2(b) where the solutions are *dominated*; in Chapter 5.4, we prove the assertions of Theorem 5.1.1(b) and Theorem 5.1.2(a) on the *dominating* solutions of (5.1).

5.2 Auxiliary Tools

Assuming (5.2) throughout this section, we give a spherical Harnack-type inequality (Lemma 5.2.1), Lemma 5.2.2, and a regularity result (Lemma 5.2.3).

Lemma 5.2.1 (Harnack-type inequality)**.** *Fix $r_0 \in (0, 1/2)$. There exists a positive constant K (depending on p, N and r_0) such that for every positive solution u of* (5.1)*, we have*

$$\max_{|x|=r} u(x) \le K \min_{|x|=r} u(x) \quad \textit{for all } 0 < r \le r_0/2. \tag{5.14}$$

Proof. We observe that (5.1) is equivalent to

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) - \frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\,|\nabla u|^{p-2}\frac{\nabla u \cdot x}{|x|} = 0 \quad \text{in } B^*. \tag{5.15}$$

By (5.2), there exists a positive constant C_1, depending on r_0, such that

$$|x|\frac{|\mathcal{A}'(|x|)|}{\mathcal{A}(|x|)} \leq C_1 \quad \text{for all } 0 < |x| \leq r_0. \tag{5.16}$$

Fix $x_0 \in \mathbb{R}^N$ such that $0 < |x_0| \leq r_0/2$. We apply the Harnack inequality for (5.15) on $B_{|x_0|/2}(x_0)$ as in Theorem 1.1 of Trudinger [57] where the structure conditions in (1.2) and (1.3) of [57] are satisfied with $a_0 = 1$, $b_1(x) = C_1|x|^{-1}$ and $a_i = b_j = 0$ for $i \in \{1, 2, 3, 4\}$ and $j \in \{0, 2, 3\}$. Hence, there exists a positive constant k, depending only on p, N and r_0, such that

$$\sup_{x\in B_{|x_0|/6}(x_0)} u(x) \leq k \inf_{x\in B_{|x_0|/6}(x_0)} u(x). \tag{5.17}$$

By the covering argument in [26], any two points x_1 and x_2 in $\mathbb{R}^N$ satisfying the condition $0 < |x_1| = |x_2| \leq r_0/2$ can be joined by ten overlapping balls of radius $|x_1|/6$ with centres positioned on $\partial B_{|x_1|}(0)$. Thus, by (5.16) and (5.17), we obtain (5.14) with $K = k^{10}$, where K is a positive constant depending on p, N and r_0. □

Using Lemma 5.2.1, we can show that positive solutions u of (5.1) can always be bounded above and below by radial solutions of (5.1) in a neighbourhood of 0.

Lemma 5.2.2. *Suppose u is a positive solution of* (5.1)*. There exists $K > 1$ and two radial solutions v_* and v^* of* (5.1) *in $B^*_{1/4}$ such that*

$$K^{-1}u \leq v_* \leq u \leq v^* \leq Ku \quad \text{in } B^*_{1/4}. \tag{5.18}$$

Proof. For $n \geq 5$, we define D_n as the following annulus

$$D_n := \{x \in \mathbb{R}^N : 1/n < |x| < 1/4\}.$$

Let v_n denote the positive, radial C^2-solution to the problem

$$\begin{cases} -\text{div}\,(\mathcal{A}(|x|)\,|\nabla v_n|^{p-2}\nabla v_n) = 0 & \text{in } D_n, \\ v_n(x) = \max_{|y|=|x|} u(y) & \text{for } x \in \partial D_n. \end{cases} \tag{5.19}$$

The uniqueness of v_n follows from Lemma 2.5.1. Moreover, we have that $v_n \leq v_{n+1}$ in D_n. By Lemma 5.2.1, there exists a constant $K > 1$ such that

$$K^{-1}v_n(x) \leq u(x) \leq v_n(x) \quad \text{for } x \in \partial D_n, \tag{5.20}$$

since it holds that

$$Ku(x) \geq K \min_{|y|=|x|} u(y) \geq \max_{|y|=|x|} u(y) = v_n(x) \quad \text{for } x \in \partial D_n. \tag{5.21}$$

As Ku is a supersolution of (5.19), we apply the comparison principle on v_n and Ku to obtain that $v_n \leq Ku$ in D_n. Moreover, we can find up to a subsequence v_n (relabelled as v_n), which converges to v^* in $C^1_{\mathrm{loc}}(B^*_{1/4})$ and that

$$K^{-1}v^* \leq u(x) \leq v^* \quad \text{ in } B^*_{1/4}. \tag{5.22}$$

Hence $v^*(x) = \lim_{n\to\infty} v_n(x)$ is a positive radial solution of

$$-\mathrm{div}(\mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } B^*_{1/4}.$$

Changing the boundary condition in (5.19) to $v_n(x) = \min_{|y|=|x|} u(y)$, we obtain the similar results for the left-hand side of (5.18). □

We next state a regularity result whose proof follows that of Lemma 2.5.4.

Lemma 5.2.3 (A regularity result)**.** *Fix $r_0 \in (0, 1/4)$ and $\delta \geq 0$. Let $g \in C(0,1)$ be a positive function such that g is regularly varying at 0 with index $-\delta$. Assume u is a positive solution of (5.1) such that $u(x)/g(|x|)$ remains bounded in some neighbourhood of 0. Then there exists constants $C > 0$ and $\alpha \in (0,1)$ such that for any x, x' satisfying $0 < |x| \leq |x'| \leq r_0$, the following holds:*

$$|\nabla u(x)| \leq C\frac{g(x)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \leq C|x-x'|^{\alpha}\frac{g(x)}{|x|^{1+\alpha}}. \tag{5.23}$$

5.3 The Dominated Solutions

Recall that $\Phi^{\pm}$ is given by (5.11). In this chapter, we classify the behaviour of the positive solutions u of (5.1) *dominated by* Φ^+ near 0, that is,

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi^+(|x|)} = 0. \tag{5.24}$$

Specifically, we prove Theorems 5.1.1(a) and 5.1.2(b) in the following proposition.

Proposition 5.3.1. *Let* (5.2) *hold. If u is a positive solution of* (5.1) *satisfying* (5.24)*, then u has the following limiting behaviours,*

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi^-(|x|)} = \gamma \in (0,\infty), \quad \lim_{|x|\to 0} \frac{x\cdot\nabla u(x)}{\Phi^-(|x|)} = \max\left\{0, \gamma\frac{p-N-\vartheta}{p-1}\right\}. \tag{5.25}$$

Moreover, if $\alpha = \infty$ (additionally, in the case $p = N+\vartheta$, we require that $\limsup_{r\to 0^+} \mathcal{A}(r)/r^{\vartheta} < \infty$), then u can be extended as a positive continuous solution of (5.1) *in B_1. On the other hand, if $p > N+\vartheta$, then u satisfies* (5.10).

Proof. For a positive solution u of (5.1) satisfying (5.24), we define

$$\gamma := \limsup_{|x|\to 0} \frac{u(x)}{\Phi^-(|x|)}. \tag{5.26}$$

By Remark 5.1.4 and (5.24), we see that (a) in (5.12) holds, that is

$$0 < \liminf_{|x|\to 0} \frac{u(x)}{\Phi^-(|x|)} \le \limsup_{|x|\to 0} \frac{u(x)}{\Phi^-(|x|)} =: \gamma < \infty. \tag{5.27}$$

We divide the proof into three steps. In Step 2, we introduce a rescaled function $U_{(r)}$ as in (5.32) and use Step 1 to show that $U_{(r)} \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}$ as $r \to 0$ with U given by (5.31). Taking $x = r\xi$ when $|\xi| = 1$, we conclude that (5.25) holds. We then further divide Step 3 into two cases. We show in Step 3(i) the proof of the case $\alpha = \infty$ where u can be extended as a positive continuous solution of (5.1) in B_1. On the other hand, if $p > N+\vartheta$, then we show in Step 3(ii) that u satisfies (5.10).

Step 1: *Fix $r_0 \in (0,1/2)$. We show that* $\lim_{r\to 0^+} F(r) = \gamma$ *where we define*

$$F(r) := \sup_{|x|=r} \frac{u(x)}{\Phi^-(|x|)} \quad \text{for all } r \in (0, 2r_0). \tag{5.28}$$

It is clear that $\limsup_{r\to 0^+} F(r) = \gamma$. We next show that $\liminf_{r\to 0^+} F(r) = \gamma$. Assume by contradiction that $\liminf_{r\to 0^+} F(r) < \gamma$. Then there exist $\varepsilon > 0$ and a positive decreasing sequence of real numbers (t_n), converging to zero as $n \to \infty$ such that $F(t_n) \le \gamma - \varepsilon$ for all $n \ge 1$. Since $\limsup_{r\to 0^+} F(r) = \gamma$, then we have

$$F(t_*) > \gamma - \varepsilon \quad \text{for some small } t_* > 0. \tag{5.29}$$

Without loss of generality, let $t_* < t_1 < 1$ and $n_1 > 1$ be large enough such that

$t_{n_1} < t_*$. Fix $n \geq n_1$ (so that $t_n \leq t_{n_1}$), and define the annulus

$$\Omega := \{x \in \mathbb{R}^N : t_n < |x| < t_1\}.$$

Using (5.29), we infer that $\max\{F(t_n), F(t_1)\} \leq \gamma - \varepsilon < F(t_*)$, we find that u/Φ^- achieves its maximum β in the interior of Ω and u/Φ^- is not a constant in Ω. By applying [42, Theorem 2.5.2] with $v = \beta\Phi^-$, we see that u/Φ^- necessarily is constant in Ω. Thus by contradiction, it follows that $\lim_{r\to 0^+} F(r) = \gamma$ holds.

Step 2: *For any fixed $r \in (0, r_0)$, we define*

$$U_{(r)}(\xi) := \frac{u(r\xi)}{\Phi^-(r)} \quad \textit{for every } \xi \in \mathbb{R}^N \textit{ with } 0 < |\xi| < \frac{r_0}{r}. \tag{5.30}$$

We show that $U_{(r)} \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $r \to 0^+$, where

$$U(\xi) = \gamma|\xi|^{\max\left\{0, \frac{p-N-\vartheta}{p-1}\right\}} \quad \textit{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.31}$$

It is enough to show that any sequence $\tilde{r}_n$ decreasing to zero contains a subsequence r_n such that

$$U_{(r_n)} \to U \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}). \tag{5.32}$$

A direct calculation shows that $U_{(r)}(\xi)$ satisfies for $0 < |\xi| < r_0/r$

$$\Delta_p U_{(r)}(\xi) + \frac{r|\xi|\mathcal{A}'(r|\xi|)}{\mathcal{A}(r|\xi|)}|\nabla U_{(r)}(\xi)|^{p-2}\nabla U_{(r)}(\xi) \cdot \frac{\xi}{|\xi|^2} = 0. \tag{5.33}$$

From (5.27), there exists a positive constant C_1 depending on r_0 such that

$$u(x) \leq C_1 \Phi^-(|x|) \quad \text{for every } 0 < |x| \leq 2r_0. \tag{5.34}$$

By taking $g \equiv \Phi^-$ in Lemma 5.2.3, we obtain that there exists positive constants $C > 0$ and $\alpha \in (0,1)$ such that for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \leq |x'| < r_0$,

$$|\nabla u(x)| \leq \frac{C\Phi^-(|x|)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \leq C\frac{|x - x'|^\alpha}{|x|^{1+\alpha}}\Phi^-(|x|). \tag{5.35}$$

By (5.34) and (5.35), it follows from Lemma 5.2.3 that for every compact subset $K \subset \mathbb{R}^N \setminus \{0\}$, there exists positive constants C_1, C_2 depending on K and independent of r, for every fixed $r \in (0, r_0)$ such that we have

$$\begin{cases} 0 < U_{(r)}(\xi) \le C_1 h(\xi), \quad |\nabla U_{(r)}(\xi)| \le \frac{C_2 h(\xi)}{|\xi|}, \\ |\nabla U_{(r)}(\xi) - \nabla U_{(r)}(\xi')| \le C_2 \frac{|\xi - \xi'|^\alpha}{|\xi|^{1+\alpha}} h(\xi), \end{cases} \tag{5.36}$$

for every ξ and ξ' in $K \subset \mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$, where

$$h(\xi) = |\xi|^{\max\left\{0, \frac{p-N-\vartheta}{p-1}\right\}}. \tag{5.37}$$

From (5.36), (5.33) and the Arzelá-Ascoli Theorem, that is, the uniform boundedness and equicontinuity of $U_{(r)}$ from Lemma 5.2.3, any sequence $\tilde{r}_n$ decreasing to zero contains a subsequence r_n such that (5.32) holds, where U satisfies

$$\Delta_p U(\xi) + \vartheta |\nabla U(\xi)|^{p-2} \nabla U(\xi) \cdot \frac{\xi}{|\xi|^2} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}). \tag{5.38}$$

To conclude Step 2, it remains to show that the above U is given by (5.31).

By (5.28), there exists ξ_{r_n} on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ such that $F(r_n) = u(r_n \xi_{r_n})/\Phi^-(r_n)$. Thus we obtain that for $0 < |\xi| < r_0/r_n$,

$$U_{(r_n)}(\xi) \le F(r_n|\xi|) \frac{\Phi^-(r_n|\xi|)}{\Phi^-(r_n)} \quad \text{and} \quad U_{(r_n)}(\xi_{r_n}) = F(r_n) \frac{\Phi^-(r_n|\xi_{r_n}|)}{\Phi^-(r_n)}. \tag{5.39}$$

Since $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ is compact, we can assume up to a subsequence, (relabelled ξ_{r_n}), we have $\xi_{r_n} \to \xi_0$ as $n \to \infty$ for some $\xi_0 \in \mathbb{S}^{N-1}$. Then for every $\xi \in \mathbb{R}^N \setminus \{0\}$ fixed, using Step 1 and taking $n \to \infty$ in (5.39), we find that

$$U(\xi) \le \gamma |\xi|^{\max\left\{0, \frac{p-N-\vartheta}{p-1}\right\}} \quad \text{with} \quad U(\xi_0) = \gamma |\xi_0|^{\max\left\{0, \frac{p-N-\vartheta}{p-1}\right\}}.$$

To conclude Step 2, we apply the strong maximum principle [42, Theorem 2.5.1] for $U - \gamma$ when $p \le N + \vartheta$. Otherwise for $p > N + \vartheta$, we apply [42, Theorem 2.5.2] with $v = \gamma |\xi|^{\frac{p-N-\vartheta}{p-1}}$ satisfying $\text{div}(|\xi|^\vartheta |\nabla v(\xi)|^{p-2} \nabla v(\xi)) = 0$ in $\mathcal{D}'(\mathbb{R}^N \setminus \{0\}$.

Step 3: *Proof of Proposition 5.3.1 completed.* For any $\varepsilon \in (0, 1/2)$, we construct a smooth non-decreasing function w_ε on $(0, \infty)$ such that

$$\begin{cases} w_\varepsilon(r) = 0 & \text{for } r \in (0, \varepsilon], \\ 0 < w_\varepsilon(r) < 1 & \text{for } r \in (\varepsilon, 2\varepsilon), \\ w_\varepsilon(r) = 1 & \text{for } r \in [2\varepsilon, \infty). \end{cases} \tag{5.40}$$

For any $\psi \in C_c^1(B_1)$, we have that $\psi w_\varepsilon \in C_c^1(B^*)$. We define J_ε by

$$J_\varepsilon := - \int\limits_{\varepsilon<|x|<2\varepsilon} \psi(x)\mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot \frac{x}{|x|} w_\varepsilon'(|x|)\, dx. \tag{5.41}$$

By taking ψw_ε in (5.3) (see Definition 5.1.1), we get that

$$\int_{B_1} w_\varepsilon \mathcal{A}(|x|)|\nabla u|^{p-2}\nabla u \cdot \nabla \psi \, dx = J_\varepsilon. \tag{5.42}$$

We observe that $\mathcal{A}(|x|)|\nabla u|^{p-1} \in L^1_{\text{loc}}(B_1)$ by using (5.25) when $\alpha = \infty$ and additionally Theorem 1.5.3(b) when $\alpha < \infty$.

Step 3(i): *For $\alpha = \infty$ (additionally, in the case $p = N + \vartheta$, we require that $\limsup_{r\to 0^+} \mathcal{A}(r)/r^\vartheta < \infty$), we prove that u can be continuously extended as a positive solution of* (5.1) *in $\mathcal{D}'(B_1)$ by showing that* $\lim_{\varepsilon\to 0} J_\varepsilon = 0$.

Since (5.25) holds, for all $\tau > 0$, there exists $r_\tau \in (0,1)$ such that

$$|\nabla u| \leq \frac{\tau}{|x|} \quad \text{for every } 0 < |x| \leq r_\tau. \tag{5.43}$$

Let I_ε denote

$$I_\varepsilon := \int_{\varepsilon<|x|<2\varepsilon} |x|^{1-p}\mathcal{A}(|x|) w_\varepsilon'(|x|)\, dx. \tag{5.44}$$

We find that J_ε in (5.41) satisfies

$$|J_\varepsilon| \leq \|\psi\|_{L^\infty(B_1)} \tau^{p-1} I_\varepsilon \quad \text{for any } \varepsilon \in (0, r_\tau/2). \tag{5.45}$$

In view of (5.45), it is enough to conclude Step 3(i) by showing the following holds

$$\begin{cases} I_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0^+ & \text{if } p < N+\vartheta, \\ \varepsilon \mapsto I_\varepsilon \text{ is bounded for } \varepsilon \in (0, r_\tau/2) & \text{if } p = N+\vartheta. \end{cases} \tag{5.46}$$

If $p < N+\vartheta$, then $r \mapsto r^{N-p}\mathcal{A}(r)$ is regularly varying at 0^+ of positive index $N+\vartheta-p$. Thus it is asymptotically equivalent near 0^+ to a monotone increasing function and $\lim_{r\to 0^+} r^{N-p}\mathcal{A}(r) = 0$. Since $w_\varepsilon(2\varepsilon) = 1$ and $w_\varepsilon(\varepsilon) = 0$, we conclude

$$\int_\varepsilon^{2\varepsilon} r^{N-p}\mathcal{A}(r) w_\varepsilon'(r)\, dr \to 0 \quad \text{as } \varepsilon \to 0, \tag{5.47}$$

which proves that $I_\varepsilon \to 0$ as $\varepsilon \to 0$ for $p < N+\vartheta$.

If, however, $p = N+\vartheta$, then $\limsup_{r\to 0^+} \mathcal{A}(r)/r^\vartheta < \infty$ implies that there exists $M := \sup_{r\in(0,1/2)} \mathcal{A}(r)/r^\vartheta \in (0,\infty)$ such that for every $\varepsilon \in (0, r_\tau/2)$,

$$I_\varepsilon = \omega_N \int_\varepsilon^{2\varepsilon} \frac{\mathcal{A}(r)}{r^\vartheta} w_\varepsilon'(r)\, dr \leq M \int_\varepsilon^{2\varepsilon} w_\varepsilon'(r)\, dr = M. \tag{5.48}$$

This concludes the proof of Step 3(i).

Step 3(ii): *If $p > N+\vartheta$, then u satisfies* (5.10), *that is*

$$-\int_{B_1} \mathcal{A}(|x|)\, |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, dx = \gamma^{p-1}\psi(0) \quad \text{for all } \psi \in C_c^1(B_1). \tag{5.49}$$

By the latter term in (5.25) for $\alpha < \infty$, we find that

$$\psi(x)\mathcal{A}(|x|)|\nabla u|^{p-2}|x|^{N-2}\nabla u \cdot x \to \psi(0)(N\omega_N)^{-1}\gamma^{p-1} \quad \text{as } |x| \to 0. \tag{5.50}$$

Thus for every $\tau > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\frac{\psi(0)\gamma^{p-1}}{N\omega_N} - \tau \leq \psi(x)\mathcal{A}(|x|)|\nabla u|^{p-2}|x|^{N-2}\nabla u \cdot x \leq \frac{\psi(0)\gamma^{p-1}}{N\omega_N} + \tau. \tag{5.51}$$

Moreover, we calculate that

$$\int_{\varepsilon<|x|<2\varepsilon} |x|^{1-N} w_\varepsilon'(|x|)\, dx = N\omega_N \int_\varepsilon^{2\varepsilon} w_\varepsilon'(r)\, dr = N\omega_N. \tag{5.52}$$

We obtain by (5.51) and (5.52) that for every $\varepsilon \in (0, \varepsilon_0)$,

$$N\omega_N \left(\frac{\psi(0)\gamma^{p-1}}{N\omega_N} - \tau \right) \leq -J_\varepsilon \leq N\omega_N \left(\frac{\psi(0)\gamma^{p-1}}{N\omega_N} + \tau \right). \tag{5.53}$$

Since $\tau > 0$ is arbitrary, we use (5.53) to get that $\lim_{\varepsilon\to 0} J_\varepsilon = -\gamma^{p-1}\psi(0)$. Letting $\varepsilon \to 0$ in (5.42), we obtain (5.49). This concludes the proof of Step 3(ii) and the proof of Proposition 5.3.1. □

5.4 The Dominating Solutions

Recall that $\Phi^\pm$ is given by (5.11). In this chapter, we classify the behaviour of the positive solutions u of (5.1) *dominating* Φ^- near 0 (which is equivalent to (5.54)). Specifically, we prove Theorem 5.1.1(b) and Theorem 5.1.2(a) in the following proposition.

Proposition 5.4.1. *If u is a positive solution of* (5.1) *satisfying*

$$\limsup_{|x|\to 0} \frac{u(x)}{\Phi^+(|x|)} = \gamma \in (0,\infty), \tag{5.54}$$

then we have that

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi^+(|x|)} = \gamma \quad \textit{and} \quad \lim_{|x|\to 0} \frac{x\cdot\nabla u(x)}{\Phi^+(|x|)} = \min\left\{\gamma\frac{p-N-\vartheta}{p-1}, 0\right\}. \tag{5.55}$$

Moreover, u has the following limiting behaviour

$$\frac{u-\gamma\Phi^+}{\Phi^-} \in L^\infty_{\rm loc}(B_1). \tag{5.56}$$

Furthermore, if $p < N+\vartheta$, then u satisfies (5.8)*.*

Proof. The proof (5.8) for $p < N+\vartheta$ follows Step 3(i) of the proof of Proposition 3.1.1. As the majority of the proof of Proposition 5.4.1 follows that of Proposition 5.3.1 (we point out the differences below), the main purpose of this proof is to prove (5.56), given in Step 3. We recall that Step 1 is used in conjunction with Step 2, jointly with the strong maximum principle, to prove (5.55).

Step 1: *Fix $r_0 \in (0,1/2)$. By the same method in Step 1 of the proof of Proposition 4.1.2, we show that $\lim_{r\to 0^+}\tilde\gamma(r) = \gamma$ where we define*

$$\tilde\gamma(r) := \sup_{|x|=r} \frac{u(x)}{\Phi^+(|x|)}. \tag{5.57}$$

Step 2: *For any fixed $r \in (0,r_0)$, we define*

$$U_{(r)}(\xi) := \frac{u(r\xi)}{\Phi^+(r)} \quad \textit{for every } \xi \in \mathbb{R}^N \textit{ with } 0<|\xi|<\frac{r_0}{r}. \tag{5.58}$$

We show that $U_{(r)} \to U$ in $C^1_{\rm loc}(\mathbb{R}^N\setminus\{0\})$ as $r\to 0^+$, where

$$U(\xi) = \gamma|\xi|^{\min\left\{\frac{p-N-\vartheta}{p-1},0\right\}} \quad \textit{for every } \xi\in\mathbb{R}^N\setminus\{0\}. \tag{5.59}$$

The proof of Step 2 follows by placing $\Phi^+(|x|)$ and $\min\{(p-N-\vartheta)/(p-1),0\}$ in lieu of $\Phi^-(|x|)$ and $\max\{0,(p-N-\vartheta)/(p-1)\}$ respectively in Step 2 of the proof of Proposition 4.1.2.

Step 3: *Proof of Proposition 5.4.1 completed.* We show that (5.56) holds. Let $\varepsilon \in (0,\gamma)$ and $v_\varepsilon^+(x)$ and $v_\varepsilon^-(x)$ be defined as follows for $0 < |x| < 1/2$,

$$v_\varepsilon^+(x) = (\gamma+\varepsilon)\Phi^+(|x|) + \frac{\max_{\partial B_{1/2}} u}{\Phi^-(1/2)}\Phi^-(|x|), \tag{5.60}$$

$$v_\varepsilon^-(x) = (\gamma-\varepsilon)\Phi^+(|x|) + \frac{\min_{\partial B_{1/2}} u}{\Phi^-(1/2)}\Phi^-(|x|). \tag{5.61}$$

Note that $v_\varepsilon^\pm$ are both radial solutions to (5.1) such that

$$\lim_{|x|\to 0} \frac{v_\varepsilon^\pm(x)}{\Phi^+(|x|)} = \gamma \pm \varepsilon. \tag{5.62}$$

Since $v_\varepsilon^-(x) \leq u(x)$ and $v_\varepsilon^+(x) \geq u(x)$ for $x \in \partial B_{1/2}$ and in a neighbourhood of 0, we obtain by way of the comparison principle that

$$v_\varepsilon^-(x) \leq u(x) \leq v_\varepsilon^+(x) \quad \text{for } 0 < |x| < \frac{1}{2}. \tag{5.63}$$

Letting $\varepsilon \to 0$ in (5.63), we get that

$$\frac{\min_{\partial B_{1/2}} u}{\Phi^-(1/2)} \leq \frac{u(x) - \gamma\Phi^+(|x|)}{\Phi^-(|x|)} \leq \frac{\max_{\partial B_{1/2}} u}{\Phi^-(1/2)} \quad \text{in } B^*. \tag{5.64}$$

This completes the proof of (5.56). □

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