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A New Measure of Vector Dependence, with an Application to Financial Contagion*

Ivan Medovikov[†] Artem Prokhorov[‡]

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We propose a new nonparametric measure of association between an arbitrary number of random vectors. The measure is based on the empirical copula process for the multivariate marginals, corresponding to the vectors, and is insensitive to the *within*-vector dependence. It is bounded by the $[0, 1]$ interval, covering the entire range of dependence from vector independence to a vector version of a monotone relationship. We study the properties of the new measure under several well-known copulas and provide a nonparametric estimator of the measure, along with its asymptotic theory, under fairly general assumptions. To illustrate the applicability of the new measure, we use it to assess the degree of interdependence between equity markets in North and South America, Europe and Asia, surrounding the financial crisis of 2008. We find strong evidence of previously unknown contagion patterns, with selected regions exhibiting little dependence before and after the crisis and a lot of dependence during the crisis period.

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[†]Brock University, St. Catharines, ON, Canada; email: imedovikov@brocku.ca

[‡]University of Sydney, NSW, Australia; email: artem.prokhorov@sydney.edu.au

1 Introduction

Measures of multivariate association between $d \geq 2$ scalar components of a random vector $\mathbf{X} = (X_1, \dots, X_d)$ have received substantial attention in the literature. Such measures include Kendall's (1938) τ , Spearman's (1904) ρ , Pearson's (1894) ϕ^2 and its multivariate extension, various multivariate divergence measures such as relative entropy measures of Joe (1987, 1989) and many others (see, e.g., Joe, 1997, for a survey). Most desirable of them are invariant to increasing transformations and therefore capture dependence regardless of specific marginals.

Less attention has been paid to constructing measures of multivariate association between several *vectors*, invariant to dependence between the *within*-vector components, even though such measures are key to multivariate modelling in many fields, for example for the study of contagion in financial markets, where we look for a measure of dependence between entire markets, which is robust to co-movements within them. The reason why such measures are of interest in their own right is that mutual independence between elements of vectors is not implied by pairwise independence.

In a recent paper, Gaißer et al. (2010) generalize nonparametric bivariate dependence measure of Hoeffding (see Hoeffding, 1940 and Fisher and Sen, 1994) to the multivariate case. The measure, termed Multivariate Hoeffding's Φ^2 (or just Multivariate Φ^2), is based on an L^2 -distance between the copula of \mathbf{X} and the independence copula. Mutual independence between the components of \mathbf{X} occurs when $\Phi^2 = 0$, and $\Phi^2 = 1$ represents co-monotonicity, or an increasing deterministic relationship between (X_1, \dots, X_d) . Estimation of Φ^2 is based on the empirical copula and, as noted by Gaißer et al. (2010), carries a low computation cost even when the dimension d is large, making it convenient to use in practice.

The main objective of this paper is to generalize the measure of Gaißer et al.

(2010) to the case of vectors, that is, to the case when association is to be measured between $p \geq 2$ partitions of $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_p)$, where now $\mathbf{X}_k \in \mathbb{R}^{d_k}$, $k = 1, \dots, p$ and $d_1 + d_2 + \dots + d_p = d$. Such problems often arise in practical situations. For example, in finance we often need a measure of dependence between entire classes of assets such as equities or bonds, or a measure of interdependence between regional groups of markets (e.g., between Asian, European and American foreign or stock exchanges). Usually such measures cannot be obtained through a combination of existing scalar statistics, since in general not all information about dependence between sets is contained in dependence structures of the sub-sets (for an interesting counterexample, see Section 1.8 of Romano (1986)). In a closely related paper, Grothe et al. (2014) work out similar generalizations of Spearman's ρ and Kendall's τ . However, their measures do not explicitly account for the case of more than two dependent vectors.

We refer to the proposed measure as Hoeffding's Vector Phi-square (or simply Vector Φ^2) and denote it by $\bar{\Phi}^2$. Unlike the multivariate version of the scalar statistics, $\bar{\Phi}^2$ is designed to distinguish dependence *between* $\mathbf{X}_1, \dots, \mathbf{X}_p$ from dependence *within* them. We show that $\bar{\Phi}^2$ remains unaffected by dependence among the components of individual vectors, and if the partitions are independent, we have $\bar{\Phi}^2 = 0$, even when the *within*-vector components are not independent. More generally, the behavior of $\bar{\Phi}^2$ under a variety of dependence scenarios differs substantially from that of Φ^2 . Using conventional measures of multivariate dependence can lead to erroneous conclusions when interest is in vector dependence.

We design a nonparametric estimator for the new measure and derive its asymptotic distribution, valid both in the iid case and under certain forms of serial dependence. The estimator is based on an empirical copula computed for the multivariate marginals corresponding to the individual vectors. The limiting behavior is therefore

based on a Gaussian process and will usually be bootstrapped.

To illustrate the use of $\bar{\Phi}^2$ in practice we measure the degree of interdependence between stock markets of different regions during the period surrounding the 2008 financial crisis. The conventional version of Φ^2 presents a somewhat distorted picture of contagion due to the dependence between stocks within a regional stock market. Comparisons between Multivariate Φ^2 and Vector Φ^2 show that the actual contagion effects are larger and the intra-regional dependence conceals the true extent of contagion.

The paper is organized as follows. Vector Φ^2 is introduced in Section 2, where we also present results of a few simulations investigating the ability of the new measure to capture *between*-vectors rather than *within*-vector dependence, under several well known copulas. A nonparametric estimator of $\bar{\Phi}^2$ is proposed in Section 3, which also provides the large sample asymptotics of the estimator. Section 4 provides an empirical application of $\bar{\Phi}^2$ to the analysis of contagion between Asian, European and American equity markets during the financial crisis of 2008. Section 5 contains concluding remarks.

2 Hoeffding's Φ^2

Let F be the joint cdf of $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_d)$, $\mathbf{X}_j \in \mathbb{R}, j = 1, \dots, d$, and let F_1, \dots, F_d denote the corresponding univariate marginal cdf's. Following a result by Sklar (1959), the function F can be represented in terms of the marginals F_1, \dots, F_d and the copula C as

$$F(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d)), \quad (x_1, \dots, x_d) \in \mathbb{R}^d. \quad (1)$$

The copula $C : [0, 1]^d \rightarrow [0, 1]$ provides a complete, and in the case of continuous marginals, a unique description of the relationship between X_1, \dots, X_d . Many well-established and new measures of association such as Spearman's ρ , Kendall's τ , mutual information coefficient δ (Blumentritt and Schmid, 2012) are based on C or the corresponding density $c(\mathbf{u}) = \frac{\partial^d}{\partial u_1 \dots \partial u_d} C(\mathbf{u})$, $\mathbf{u} \in [0, 1]^d$. Schmid et al. (2010) provide a survey of copula-based dependence measures.

2.1 Multivariate Hoeffding's Φ^2

It is easy to see that mutual independence between the d scalar components of \mathbf{X} is characterized by *independence copula* $C^\perp(\mathbf{u}) = \prod_{j=1}^d u_j$, $\mathbf{u} \in [0, 1]^d$, which makes it natural to develop measures of association based on the distance $C(\mathbf{u}) - C^\perp(\mathbf{u})$. The measure proposed by Gaißer et al. (2010) can be written as follows

$$\Phi^2 = \frac{\|C - C^\perp\|_2^2}{\|M - C^\perp\|_2^2}, \quad (2)$$

where $\|\cdot\|_2$ denotes the L_2 -norm, and $\|M - C^\perp\|_2^2$ is the normalization factor where the function $M(\mathbf{u}) = \min(u_1, \dots, u_d)$, $\mathbf{u} \in [0, 1]^d$, is the so-called *comonotone copula* representing an almost-sure strictly-increasing functional relationship between all components of \mathbf{X} . Here, M is the upper Frechet-Hoeffding bound, that is, for any valid copula C , $C(\mathbf{u}) \leq M(\mathbf{u})$, $\forall \mathbf{u} \in [0, 1]^d$.

The statistic of Gaißer et al. (2010) represents a multivariate extension of the bivariate measure of association initially proposed by Hoeffding (1940) (also see Fisher and Sen, 1994) – we refer to it as Multivariate Hoeffding's Φ^2 . For any copula C , the measure Φ^2 is constrained to the $[0, 1]$ interval where the case $\Phi^2 = 1$ occurs when there is co-monotonicity between all components of \mathbf{X} , while $\Phi^2 = 0$ corresponds to mutual independence. It is important to note that the converse is generally not

true – it is possible to have $\Phi^2 < 1$ when \mathbf{X} are actually deterministically dependent. This is possible because copula C may be arbitrarily close to C^\perp when it is expressed as a *shuffle* of M . It is therefore possible for the statistic Φ^2 to be made arbitrarily small, while maintaining a deterministic relationship between all components of \mathbf{X} . This implication follows from the copula theory rather than from the definition of Φ^2 , and is a feature of any measure of association that is based on distances between C and C^\perp . For additional details, see Section 3.2.3 in Nelsen (2006).

It is clear that Φ^2 is not well-suited for the measurement of dependence between random vectors since even under mutual independence *between* vectors, Φ^2 may be non-zero due to dependence *within* them. Our aim here is to develop a measure of multivariate association that is invariant to dependence *within* vectors.

2.2 Additional definitions and notation

Before we proceed, we need to introduce some additional notation to assist with partitioning of \mathbf{X} into p sub-vectors $(\mathbf{X}_1, \dots, \mathbf{X}_p)$ of dimension d_1, \dots, d_p , respectively. Define $b_0 = 0$ and, for any integer $k \in \{1, \dots, p\}$, set $b_k = \sum_{\ell=1}^k d_\ell$. Moreover, for $k = 1, \dots, d$ and $\mathbf{u} \in [0, 1]^d$, let

$$\begin{aligned}\mathbf{u}^{\{k\}} &= (u_{b_{k-1}+1}, \dots, u_{b_k}) \in \mathbb{R}^{d_k}, \\ \mathbf{u}^{[k]} &= (\mathbf{1}_{d_1+\dots+d_{k-1}}, \mathbf{u}^{\{k\}}, \mathbf{1}_{d_{k+1}+\dots+d_p}) \in \mathbb{R}^d.\end{aligned}$$

Here, for $m \in \mathbb{N}$, $\mathbf{1}_m$ denotes an m -dimensional vector of ones. For $k = 1, \dots, p$, let

$$C^{\{k\}} : [0, 1]^{d_k} \rightarrow [0, 1], \quad C^{\{k\}}(\mathbf{v}) = C(\mathbf{1}_{d_1+\dots+d_{k-1}}, \mathbf{v}, \mathbf{1}_{d_{k+1}+\dots+d_p})$$

denote the copula of \mathbf{X}_k . Also, define a copula $C^\Pi : [0, 1]^d \rightarrow [0, 1]$ through

$$C^\Pi(\mathbf{u}) = \prod_{k=1}^p C^{\{k\}}(\mathbf{u}^{\{k\}}) = \prod_{k=1}^p C(\mathbf{u}^{[k]}).$$

This copula serves as a vector analogue of the scalar independence copula C^\perp .

2.3 Hoeffding's Vector $\bar{\Phi}^2$

First we note that $\mathbf{X}_1, \dots, \mathbf{X}_p$ are mutually independent if and only if $C = C^\Pi$. In this case, the joint distribution of p vectors is a product of p multivariate marginals, corresponding to the vector dimensions. This observation suggests that we can define a measure of dependence between the vectors $\mathbf{X}_1, \dots, \mathbf{X}_p$ by considering a suitable distance between C and C^Π .

The new measure we propose is a generalization of the scalar version of Φ^2 discussed in the previous section. We follow Gaïffer et al. (2010) and use an L^2 -type distance. The *vector version of Hoeffding's* Φ^2 , denoted by $\bar{\Phi}^2$, is defined as follows

$$\bar{\Phi}^2 := \bar{\Phi}^2(C, d_1, \dots, d_p) := \frac{\|C - C^\Pi\|_2^2}{\|M - C^\Pi\|_2^2},$$

where as before $\|\cdot\|_2$ denotes the L_2 -norm. Now, we use $C^{\{k\}}$, $k \in \{1, \dots, p\}$ in the distance and in the normalizing factor. The scalar version is obtained as a special case when $d_j = 1, j = 1, \dots, p$.

We use the same bound M in our measure as in the scalar case above. There may exist a vector equivalent of this bound. For example, we may consider using $\min(C^{\{1\}}, \dots, C^{\{p\}})$ instead. However, little is known about the properties of this object, specifically we cannot guarantee that $C \leq \min(C^{\{1\}}, \dots, C^{\{p\}})$ for any copula C . What *is* known is that this would be the correct Frechet-Hoeffding bound for

some copula $C_c(C^{\{1\}}, \dots, C^{\{p\}})$ but, unless all margins are scalar distributions, the resulting distribution is a d -variate copula only if $C_c(u_1, \dots, u_p) = C_\perp(u_1, \dots, u_p)$ (see, e.g., Quesada-Molina and Rodriguez-Lallena, 1994; Genest et al., 1995). So this would not be the right bound to use in our case.

Next we can state a few properties of the new measure. Some of them are less obvious than others so we provide a brief discussion.

- (i) *Bounds*: For any copula C and marginals $C^{\{k\}}$, we have that $\bar{\Phi}^2 \in [0, 1]$. $\bar{\Phi}^2 = 1$ when $C = M$ and $\bar{\Phi}^2 = 0$ when $C = C^\Pi$.
- (ii) *Independence of partitions*: If the *partitions* $\mathbf{X}_1, \dots, \mathbf{X}_p$ are independent then $\bar{\Phi}^2 = 0$. The converse is also true.
- (iii) *Vector dependence*: Vectors $\mathbf{X}_1, \dots, \mathbf{X}_p$ are associated when $\bar{\Phi}^2 > 0$. The converse is also true.
- (iv) *Co-monotonicity of components*: For the case of two vectors $p = 2$, when $\bar{\Phi}^2 = 1$, each of the individual components of \mathbf{X} is almost-surely a strictly-increasing function of another. The converse is not true. (There is no equivalent result for counter-monotonicity.)
- (v) *Invariance with respect to partition order*: Given partition sizes d_1, \dots, d_p , for every permutation π of partition order $\{1, \dots, p\}$, we have that $\bar{\Phi}^2(C, d_1, \dots, d_p) = \bar{\Phi}^2(C, d_{\pi(1)}, \dots, d_{\pi(p)})$, as long as the composition of each partition is maintained. This and the next property follow from Fubini's Theorem.
- (vi) *Invariance with respect to ordering of components within partitions*: Given partition sizes d_1, \dots, d_p and partition composition, $\bar{\Phi}^2$ is invariant with respect to permutations of components within each partition.

- (vii) *Invariance with respect to strictly-increasing transformations:* For any $d \geq 2$, we have that $\bar{\Phi}^2$ is invariant with respect to strictly-increasing transformations of one or many components of \mathbf{X} . This is true since the copula C is invariant under such transformations.

2.4 Numerical Examples

In this section, we use Monte-Carlo simulations to calculate and compare the approximate values of $\bar{\Phi}^2$ and Φ^2 for some commonly used multivariate families of copulas, each having a range of dependence parameter values. To keep computational costs low, we set $p = 2$, overall dimension $d = 4$, and partition data into two vectors of equal size, so that $d_1 = d_2 = 2$ in all cases. For each value of the dependence parameter, we evaluate the integral using 100,000 draws.

We begin with the case when C is the equi-correlated Gaussian copula and plot the corresponding values of $\bar{\Phi}^2$ and Φ^2 in the top-left panel of Figure 1 against correlation coefficient ρ . Both statistics are increasing in ρ as expected but the values of $\bar{\Phi}^2$ are substantially smaller than Φ^2 . The difference is caused by the dependence within the partitions. It is perhaps surprising that the gap is narrowing as correlation approaches one as this corresponds to stronger dependence within the partitions (but also between them).

To further illustrate the difference between $\bar{\Phi}^2$ and Φ^2 , the top-right panel of Figure 1 shows the case when C is a product of two equi-correlated bivariate Gaussian copulas, implying independence of partitions. Here, $\bar{\Phi}^2 = 0$ and it remains insensitive to dependence within partitions, regardless of the degree of correlation, while the measure Φ^2 is increasing in ρ . The multivariate measure shows spurious dependence induced by the correlation within the independent bivariate marginals. Clearly, in

this case, the gap between $\bar{\Phi}^2$ and Φ^2 increases as correlation approaches one.

The remaining panels in Figure 1 show the two measures for selected four-dimensional copulas. The classes of copulas we consider include comprehensive, asymmetric, Archimedian and tail-dependent families. The Gumbel, Clayton and Frank copulas reduce to independence when the dependence parameter is at the origin, and this translates into mutual independence between *all* components of \mathbf{X} . When the dependence parameter takes increasing values away from the origin, this corresponds to increasing dependence between all components so that C approaches M . For the t-copula, we vary correlation while keeping the degrees of freedom constant at $\tau = 2$. Therefore, symmetric tail-dependence is generally non-zero in this case, even when $\rho = 0$. So $\rho = 0$ does not imply independence between all components. Even though we do not differentiate the strength of dependence within the marginals in these panels, all the cases we report show that Φ^2 overestimates $\bar{\Phi}^2$ by a large margin.

3 Statistical inference for $\bar{\Phi}^2$

We now consider the formal definition and asymptotic properties of an estimator of $\bar{\Phi}^2$.

3.1 An estimator of $\bar{\Phi}^2$

For a sample $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}$, $\mathbf{X}^{(i)} = (X_1^{(i)}, \dots, X_d^{(i)}) \sim F = C(F_1, \dots, F_d)$ and continuous marginal cdf's F_1, \dots, F_d , let $\hat{\mathbf{U}}^{(i)} = (\hat{U}_1^{(i)}, \dots, \hat{U}_d^{(i)})$ denote *pseudo-observations*

from the copula C defined through

$$\hat{U}_j^{(i)} = \frac{1}{n} \left(\text{rank of } X_j^{(i)} \text{ among } X_j^{(1)}, \dots, X_j^{(n)} \right),$$

for $i = 1, \dots, n$ and $j = 1, \dots, d$. Let $\hat{C}_n : [0, 1]^d \rightarrow [0, 1]$ denote the associated empirical copula, defined for $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$ as

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(\hat{\mathbf{U}}^{(i)} \leq \mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{I}(\hat{U}_j^{(i)} \leq u_j),$$

where $\mathbb{I}(\cdot)$ is the indicator function. The empirical copula easily allows to define a sample version of $\bar{\Phi}^2$ through

$$\hat{\Phi}_n^2 := \bar{\Phi}^2(\hat{C}_n, d_1, \dots, d_p) = \frac{\|\hat{C}_n - \hat{C}_n^\Pi\|_2^2}{\|M - \hat{C}_n^\Pi\|_2^2},$$

where $\hat{C}_n^\Pi(\mathbf{u}) = \prod_{k=1}^p \hat{C}_n(\mathbf{u}^{[k]})$.

The following lemmas are useful in the subsequent proposition. They show that the estimator $\hat{\Phi}_n^2$ can be calculated directly from the pseudo-observations. In part, the proposition relies on calculations provided in Proposition 10 of Kojadinovic and Holmes (2009). For notational convenience, for any positive integer i define $[i] = \{1, 2, \dots, i\}$.

Lemma 1. *For some $a, b \in [0, 1]$ s.t. $b > a$, some $p > 0$, and integer $d > 1$, let $m(a, b, d, p)$ represent the value of the definite integral*

$$\int_a^b \dots \int_a^b \min(\mathbf{u})^p d\mathbf{u}.$$

Then, we can find $m(a, b, d, p)$ as

$$m(a, b, d, p) = d! \frac{b^{d+p} - a^{d+p}}{\prod_{i=1}^d (p+i)} - \sum_{k=1}^{d-1} \frac{\Gamma(d+1)}{\Gamma(d-k+1)} \frac{a^{p+k}}{\prod_{j=1}^k (p+j)} (b-a)^{d-k}, \quad (3)$$

where Γ denotes the Gamma function.

Proof. See Appendix for all proofs. \square

Lemma 2. For some integer $d > 1$, $\mathbf{a} \in [0, 1]^d$, and $p > 0$, let $I(\mathbf{a}, p)$ represent the value of the definite integral

$$I(\mathbf{a}, p) = \int_{a_1}^1 \dots \int_{a_d}^1 \min(u_1, \dots, u_d)^p du_1 \dots du_d, \quad (4)$$

where for $j \in [d]$, a_j denotes the j 'th component of \mathbf{a} . Then, we can find the value of $I(\mathbf{a}, p)$ as

$$I(\mathbf{a}, p) = \sum_{j=1}^{d-1} \sum_{k=1}^j \sum_{l=1}^{\binom{j}{k}} \left(\prod_{n=1}^{d-k} (1 - a_{[A(d,j,k,l,n)]}) \right) m(a_{[j]}, a_{[j+1]}, k, p) + m(a_{[d]}, 1, d, p), \quad (5)$$

where for $j \in [d]$, $a_{[j]}$ denotes the j 'th largest component of \mathbf{a} (so that $a_{[1]} \leq a_{[2]} \leq \dots \leq a_{[d]}$), and the function $A : \mathbb{Z}^{d+} \rightarrow \mathbb{Z}^+$ is defined below.

For some $j, k \in \mathbb{Z}^+$ such that $k \leq j$, let $[\bar{j}_k]$ denote the set of k -component combinations of $[j]$, and let $[\bar{j}_k](s)$ denote its s 'th element. For example, we have that $[\bar{2}_1] = \{\{1\}, \{2\}\}$, $[\bar{3}_2] = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and $[\bar{3}_2](2) = \{1, 3\}$. Further, define a set $J(d, j, k, s) = [d] \setminus [\bar{j}_k](s)$, which is a set of components of $[d]$ not in $[\bar{j}_k](s)$, and let $J(d, j, k, s, n)$ represent its n 'th element. Lastly, for positive integers d, j, k, s, n ,

the function A is defined as follows:

$$\begin{aligned} A(d, j, k, s, n) &= \mathbb{I}(J(d, j, k, s)(n) \in [j])(j+1) \\ &+ \mathbb{I}(J(d, j, k, s)(n) \notin [j])J(d, j, k, s,)(n), \end{aligned} \quad (6)$$

where $\mathbb{I}()$ is an indicator function as before.

Proposition 1. *The numerator and denominator of $\bar{\Phi}_n^2(C)$ can be expressed as*

$$\begin{aligned} \|\hat{C}_n - \hat{C}_n^\Pi\|_2^2 &= \frac{1}{n^2} \sum_{i=1}^n \sum_{l=1}^n \prod_{j=1}^d [1 - \hat{u}_{i,j} \vee \hat{u}_{l,j}] \\ &- \frac{2}{n^{p+1}} \sum_{i=1}^n \prod_{k=1}^p \sum_{l=1}^n \prod_{j=b_{k-1}+1}^{b_k} [1 - \hat{u}_{i,j} \vee \hat{u}_{l,j}] \\ &+ \frac{1}{n^{2p}} \prod_{k=1}^p \sum_{i=1}^n \sum_{l=1}^n \prod_{j=b_{k-1}+1}^{b_k} [1 - \hat{u}_{i,j} \vee \hat{u}_{l,j}], \end{aligned} \quad (7)$$

and

$$\begin{aligned} \|M - \hat{C}_n^\Pi\|_2^2 &= \frac{2}{(d+1)(d+2)} \\ &- \frac{2}{n^p} \sum_{i_1=1}^n \dots \sum_{i_p=1}^n I([\hat{u}_{i_{k(1)}1}, \hat{u}_{i_{k(2)}2}, \dots, \hat{u}_{i_{k(d)}d}], 1) \\ &+ \frac{1}{n^{2p}} \prod_{k=1}^p \sum_{i=1}^n \sum_{l=1}^n \prod_{j=b_{k-1}+1}^{b_k} [1 - \hat{u}_{i,j} \vee \hat{u}_{l,j}], \end{aligned} \quad (8)$$

where for $j \in [d]$, $k(j) : [d] \rightarrow [p]$ is a function such that $b_{k(j)-1} + 1 \leq j \leq b_{k(j)}$, and $I(\mathbf{a}, p)$ is defined in Lemma 2 and $m(a, b, d, p)$ is defined in Lemma 1.

3.2 Asymptotic properties of $\bar{\Phi}_n^2$

In this section we obtain a weak convergence result regarding $\hat{\Phi}_n^2$ using functional weak convergence of the empirical copula process $\mathbb{C}_n = \sqrt{n}(\hat{C}_n - C)$ and of the process \mathbb{C}_n^Π , defined on $\mathbf{u} \in [0, 1]^d$ through

$$\mathbb{C}_n^\Pi(\mathbf{u}) = \sqrt{n}\{\hat{C}_n^\Pi(\mathbf{u}) - C^\Pi(\mathbf{u})\}.$$

Naturally, both \mathbb{C}_n and \mathbb{C}_n^Π can be considered as elements of the space of real-valued, bounded functions on $[0, 1]^d$, denoted by $\ell^\infty([0, 1]^d)$, equipped with the uniform metric induced by the sup-norm $\|f\|_\infty = \sup_{\mathbf{u} \in [0, 1]^d} |f(\mathbf{u})|$. Weak convergence of the empirical copula process \mathbb{C}_n has been investigated by various authors under slightly different assumptions (see, e.g., Rüschendorf, 1976; Gaenssler and Stute, 1987; Fermanian et al., 2004; Segers, 2012; Bücher and Volgushev, 2013). Under appropriate smoothness conditions and under $C = C^\Pi$, Kojadinovic and Holmes (2009, Theorem 3) gave a weak convergence result for $\mathbb{H}_n = \mathbb{C}_n - \mathbb{C}_n^\Pi$. The following result can be regarded as a generalization of their result to multivariate marginals.

Theorem 1. *Let C be a copula such that, for any $j = 1, \dots, d$, the j th first order partial derivative $\dot{C}_j = \partial C / \partial u_j$ exists and is continuous on the set $\{\mathbf{u} \in [0, 1]^d : u_j \in (0, 1)\}$. Let \mathbb{B}_C denote a C -Brownian bridge on $[0, 1]^d$, i.e., a centered Gaussian process with continuous sample paths and covariance*

$$\text{cov}\{\mathbb{B}_C(\mathbf{u}), \mathbb{B}_C(\mathbf{v})\} = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}).$$

Then, $(\mathbb{C}_n, \mathbb{C}_n^\Pi) \rightsquigarrow (\mathbb{C}_C, \mathbb{C}_C^\Pi)$ in $\{\ell^\infty([0, 1]^d)\}^2$, where

$$\mathbb{C}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - \sum_{j=1}^d \dot{C}_j(\mathbf{u}) \mathbb{B}_C(1, \dots, 1, u_j, 1, \dots, 1),$$

with \dot{C}_j defined as 0 wherever it does not exist, and with

$$\mathbb{C}_C^\Pi(\mathbf{u}) = \sum_{k=1}^p \mathbb{C}_C(\mathbf{u}^{[k]}) \prod_{\substack{k'=1 \\ k' \neq k}}^p C(\mathbf{u}^{[k]}).$$

The preceding theorem also holds under many serial dependence scenarios for time series (e.g., under α -mixing), see Bücher and Volgushev (2013). More precisely, provided the weak limit $\tilde{\mathbb{B}}_C$ of the empirical process

$$\mathbf{u} \mapsto \sqrt{n} \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{I}(\mathbf{U}^{(i)} \leq \mathbf{u}) - C(\mathbf{u}) \right\}$$

exists and has continuous sample paths, the statement of Theorem 1 holds with the C -Brownian bridge \mathbb{B}_C replaced by $\tilde{\mathbb{B}}_C$.

Proposition 2. *Under the assumptions of Theorem 1,*

(a) *if $\bar{\Phi}^2 \neq 0$, then*

$$\begin{aligned} \sqrt{n}(\hat{\Phi}_n^2 - \bar{\Phi}^2) &\rightsquigarrow 2 \frac{\int_{[0,1]^d} \{C(\mathbf{u}) - C^\Pi(\mathbf{u})\} \{\mathbb{C}_C(\mathbf{u}) - \mathbb{C}_C^\Pi(\mathbf{u})\} d\mathbf{u}}{\|M - C^\Pi\|_2^2} \\ &\quad + 2 \frac{\|C - C^\Pi\|_2^2 \int \mathbb{C}_C^\Pi(\mathbf{u}) \{M(\mathbf{u}) - C^\Pi(\mathbf{u})\} d\mathbf{u}}{\|M - C^\Pi\|_2^4}. \end{aligned}$$

(b) *if $\bar{\Phi}^2 = 0$, then*

$$n\hat{\Phi}_n^2 \rightsquigarrow \frac{\|\mathbb{C}_C - \mathbb{C}_C^\Pi\|_2^2}{\|M - C^\Pi\|_2^2}.$$

Note that in the case $\bar{\Phi}^2 \neq 0$ the limiting distribution of $\sqrt{n}(\hat{\Phi}_n^2 - \bar{\Phi}^2)$ is Gaussian. This follows from the fact that \mathbb{C}_C is a centered Gaussian process. The limiting variance depends in a complicated way on C and its partial derivatives. Moreover, in practice, the true copula is unknown; and even if it was known, analytical calculations, even approximate, of these quantities can be intractable. Bootstrap distributions will be used instead.

4 Empirical example: global financial crisis

4.1 Literature review

To illustrate the applicability of our proposed measure, we study linkages between the global equity markets before, during and after the financial crisis of 2008. We are particularly interested in financial contagion, that is we are interested in the transmission of shocks sustained by one market to the other markets. We define contagion as a change in strength of cross-market dependence during or after financial shocks and aim to assess global contagion allowing for a partition of markets into regional groups. Specifically, we study linkages between Europe, Asia, North and South America, in the time surrounding the global financial crisis of 2008. Our measure is well-suited for this task since it is not affected by the association between markets belonging to the same region, meaning that financial contagion between, say, Germany and the UK will have no influence on our estimates of contagion between Europe and, say, Asia.

The issue of contagion continues to receive substantial attention in the literature, which is not surprising given the severe economic consequences of financial and currency crises and the resources devoted by policy makers to crisis management.

The evidence to date overwhelmingly supports the presence of contagion effects in financial markets, that is, it shows an increase in cross-market linkages during major financial crises; for an overview see, for example, Forbes and Rigobon (2001). In most cases, contagion is interpreted as an increase in correlation coefficients or other measures of association between cross-country asset returns. For example, in one of earlier attempts at measuring contagion, King and Wadhwani (1990) tested for an increase in correlations between equity returns in the US, UK and Japan and found that correlations rose substantially following a US stock market crisis, interpreting this as contagion. Similarly, Lee and Kim (1993) document the strengthening of co-movements between several stock market indexes in the aftermath of the 1987 stock market crash.

A related strand of literature studies transmission of volatility rather than returns dynamics, using a GARCH framework. For example, Edwards (1998) uses an augmented GARCH model to assess the role of capital control in transmission of shocks following the Mexican peso crisis.

More recently, the copula approach has been adopted to probe for financial contagion that may be nonlinear and non-monotone in nature. For example, using a copula model Rodriguez (2007) documents increased cross-country tail-dependence during crises in Asia, suggesting that transmission of shocks may be asymmetric and generally has a complicated, nonlinear form. Similarly, Chiang and Wang (2011) study volatility transmission using a measure based on a transition copula and document volatility spillovers from the US to the other G7 countries.

Interestingly, few attempts appear to have been made in the literature at measuring cross-region rather than cross-country contagion. One exception is Bae et al. (2003), who study regional contagion using a multinomial logistic regression model which captures probabilities of co-occurrence of extreme returns in pairs of global

regions such as Asia and Europe and find evidence of inter-regional contagion. This approach, however, requires a relatively complicated set up along with fairly strong assumptions about the joint distributions of market returns (specifically, joint normal or multivariate t-distribution), meaning that the evidence of contagion can be equally interpreted as evidence against joint normality or symmetry of the return distributions.

While the issue of regional contagion appears to be largely overlooked, regional partitioning can represent an important factor in the propagation of financial shocks from one country to another. For example, some markets can be closely linked for institutional reasons such as membership in currency or customs union, or common bail-out guarantees such as those issued by the European Central Bank to some Euro-zone members. To this end, the statistic that we propose allows for such partitions, even when there are numerous regions and they have arbitrary many members.

Our contribution to the empirical literature on financial contagion is two-fold. First, this appears to be the first attempt to estimate contagion using a fully-nonparametric copula-based measure that is sensitive to any form of vector dependence, including nonlinear and non-monotone. Second, our estimates seem to be first to encompass more than two regions simultaneously and are therefore closer to what could be viewed as regional contagion in a global sense.

4.2 Estimates of inter-regional contagion

We restrict our attention to the total of 15 national equity market indexes, which we partition into four regional groups as follows:

- *North America:* S&P 500 Index (US), TSX-S&P Composite Index (Canada), IPC Index (Mexico).

- *South America:* HSBC Chile Index, HSBC Colombia Index, Bovespa Index (Brazil), Merval Index (Argentina).
- *Europe:* FTSE 100 Index (UK), Euronext 100 Index (Netherlands, France, Belgium and Portugal), DAX 100 Index (Germany).
- *Asia:* Hang Seng Index (Hong Kong), Shanghai Composite Index (China), Nikkei 225 Index (Japan), ASX Index (Australia), BSE 500 Index (India).

We use monthly closing values for all of the indexes in our sample. The period we consider is January 2004 to January 2014 and the data comes from the Standard and Poor's COMPUSTAT database. The series are plotted in Figure 2. There is some homogeneity among indexes within a region but regional profiles differ substantially in spite of the common trough in late 2008.

Most markets in our sample suffered substantial draw-downs at some point during the 2007 - 2009 period, and we next attempt to establish whether this coincided with an increase in cross-market linkages. To get a comprehensive picture of contagion and to better understand how market linkages differ for the four regions we estimate the values of both Φ^2 and $\bar{\Phi}^2$ for all regional pairs in our sample. The estimates are provided in Tables 1. We estimate the statistics separately for the full sample as well as for the pre-crisis sub-sample which we define to span January 2004 - December 2007, the crisis sub-sample spanning January 2008 - May 2009, and the post-crisis sub-sample which begins in April 2009 and ends in January 2014. For any regional pair, the multivariate measure Φ^2 captures overall association between all countries belonging to either of the two regions, while the vector measure $\bar{\Phi}^2$ captures linkages between the regions as a whole after netting out dependence due to co-movements of markets belonging to the same partition.

Perhaps unsurprisingly, substantial differences emerge in regional linkages as captured by either measure; the strongest connection by far appears to be between North America and Europe, followed by Europe and Asia, North America and Asia, and South America and Europe. Interestingly, the connection between the Americas is among the weakest, stronger only than the South America - Asia link.

Important differences arise when comparing vector Φ^2 with the scalar analogue, which we have been calling multivariate Φ^2 . Multivariate Φ^2 tends to underestimate the degree of *inter*-regional contagion, compared to vector Φ^2 . *Intra*-regional variation seems to have a dampening effect on the estimates of contagion. For example, the strength of the link North America - Europe did not increase substantially during the crisis, when judged by multivariate Φ^2 . It doubled when judged by vector $\bar{\Phi}^2$.

Such discrepancies happen because multivariate Φ^2 tends to overestimate dependence outside the crisis period and underestimate it during the crisis. In order to assess whether the difference between the two measures is statistically significant we also report (in the last column of Table 1) the bootstrap standard errors of the difference. We observe that with the exception of the crisis period, in which $\bar{\Phi}^2$ standard errors are particularly high due to a small number of observations, all the differences are statistically significant.

Pre-crisis global regional links appear to be rather weak for all regional pairs with the exception of North America - Europe, and increase very sharply during the crisis period. The increase in purely regional links captured by $\bar{\Phi}^2$ is particularly dramatic, where in some cases, such as South America - Asia, regions that were almost entirely disconnected before crisis become tightly associated, with the dependence estimate increasing more than tenfold, from 0.0438 in the pre-crisis period to 0.4891 during crisis.

Table 2 reports the results of significance tests for these changes. We test the null of no contagion effects against the alternative of contagion captured by an increase in Φ^2 and $\bar{\Phi}^2$ before the crisis and we also test the same null against the alternative of deteriorating dependence captured by a drop in Φ^2 and $\bar{\Phi}^2$ after the crisis. The four columns contain results of the two tests obtained using either of the two dependence measures. The p-values reported are based on bootstrapped standard errors.

We observe that the p-values are generally lower for the vector statistic and the two statistics reveal different contagion patterns. For the first three pairs, the multivariate statistic shows no significant contagion effects before or after crisis, while the vector statistic rejects the null or comes very close to rejecting the null for all pairs (despite the short crisis sample). For the remaining pairs, both statistics show clear evidence of contagion in the run-up to the crisis, but the vector statistic also detects deteriorating linkages afterwards and the evidence provided by $\bar{\Phi}^2$ against the null is usually stronger. In summary, the multivariate statistic seems to detect three significant contagion patterns, while the vector statistic detects seven. The North America - South America pair is particularly interesting – the conventional Φ^2 for this pair is far from detecting any significance dependence, while the vector Φ^2 firmly detects at least one contagion episode.

Overall, our findings suggest that regional linkages strengthened sharply during the 2007 - 2009 period, which we interpret as evidence of regional contagion. Interestingly, we find that post-crisis, all the regions aside from North America - Europe became once again largely disjoint. For example, in the case of South America, $\bar{\Phi}^2$ dropped to nearly same values as pre-crisis, and in the case of South America - North America – to an even lower value.

5 Concluding Remarks

We have proposed a new vector dependence measure and considered its properties. We have derived its asymptotic distribution, discussed various estimation issues and provided an important application where using our measure we come to conclusions which differ from those available in the literature.

The new measure has important advantages over other available dependence measures. First, it captures the *between*-, rather than *within*-, vector dependence and reflects any form of dependence, not only linear. Second, the number of vectors it can accommodate is not limited to two, and nothing is assumed about their parametric distributions. It also has several drawbacks, including invariance to negative versus positive dependence and a lack of a simple analytical form. However, similar issues arise when using other multivariate dependence measures and so the balance of these characteristics speaks, in our opinion, in favor of the new measure.

We establish empirically that the use of multivariate contagion measures results in highly conservative estimates of contagion effects. Using equity indexes from four regions we demonstrate that invariance to intra-regional dependence leads to much stronger contagion effects than previously thought and we discover new regional contagion patterns.

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6 Appendix: Proofs

Proof of Lemma 1 Note that for any $\mathbf{u} \in [0, 1]^d$, we can expand $\min(\mathbf{u})^p$ as

$$\min(\mathbf{u})^p = \sum_{j=1}^d \left[\prod_{k \in [d], k \neq j} \mathbb{I}(u_j \leq u_k) \right] u_j^p,$$

and for any $j \in [d]$, express the integral of the summand as

$$\begin{aligned} & \int_a^b \dots \int_a^b \left[\prod_{k \in [d], k \neq j} \mathbb{I}(u_j \leq u_k) \right] u_j^p du_j d\mathbf{u}^{[-j]} \\ &= \int_a^b \dots \int_a^b \int_a^{\min(\mathbf{u}^{[-j]})} u_j^p du_j d\mathbf{u}^{[-j]} \\ &= \frac{1}{p+1} \int_a^b \dots \int_a^b \min(\mathbf{u}^{[-j]})^{p+1} d\mathbf{u}^{[-j]} - \frac{a^{p+1}}{p+1} \int_a^b \dots \int_a^b 1 d\mathbf{u}^{[-j]}, \end{aligned}$$

where $\mathbf{u}^{[-j]} \in [0, 1]^{d-1}$ denotes the sub-vector of \mathbf{u} with its j 'th component removed.

Applying induction, it is tedious but relatively straightforward to show that for any $d > 1$, by continuing the expansion for $d-1$ steps, we can write $\int_a^b \dots \int_a^b \min(\mathbf{u})^p d\mathbf{u}$ as the sum containing $d!$ integrals of the form

$$\frac{1}{\prod_{j=1}^{d-1} (p+j)} \int_a^b u^{p+d-1} du,$$

and $\sum_{k=1}^{d-1} b\{k\}$ constants of the form

$$-\frac{a^s}{t} \int_a^b \dots \int_a^b 1 \mathbf{u}^{[-J]},$$

where $b\{k\} = d!/(d-k)! = \Gamma(d+1)/\Gamma(d-k+1)$ is the falling factorial (the Pochhammer function), J is the subset of elements of \mathbf{u} that have been removed,

and s and t are some exponents. The resulting sum of single integrals is then given by

$$\sum_{j=1}^d \frac{1}{\prod_{j=1}^{d-1} (p+j)} \int_a^b u^{p+d-1} du = \frac{d!}{\prod_{j=1}^d (p+j)} (b^{p+d} - a^{p+d}), \quad (9)$$

and the sum of multiple integrals by

$$\sum_{k=1}^{d-1} \sum_{m=1}^{d\{k\}} \frac{a^{p+k}}{\prod_{j=1}^k (p+j)} \int_a^b \dots \int_a^b 1 du_2 \dots du_{d-k+1} \quad (10)$$

$$= \sum_{k=1}^{d-1} \sum_{m=1}^{d\{k\}} \frac{a^{p+k}}{\prod_{j=1}^k (p+j)} \prod_{j=2}^{d-k} (b-a) \quad (11)$$

$$= \sum_{k=1}^{d-1} \frac{\Gamma(d+1)}{\Gamma(d-k+1)} \frac{a^{p+k}}{\prod_{j=1}^k (p+j)} (b-a)^{d-k}, \quad (12)$$

which yields the desired result.

Proof of Lemma 2 First, note that for any $\mathbf{a} \in [0, 1]^d$ we can change integration order by arranging u_1, \dots, u_d according to the values of a_1, \dots, a_d as

$$I(\mathbf{a}, p) = \int_{a_1}^1 \dots \int_{a_d}^1 \min(u_1, \dots, u_d)^p du_1 \dots du_d \quad (13)$$

$$= \int_{a_{[1]}}^1 \dots \int_{a_{[d]}}^1 \min(u_1, \dots, u_d)^p du_{[1]} \dots du_{[d]}, \quad (14)$$

where for $j \in [d]$, $u_{[j]}$ denotes the variable with j 'th lowest integration limit $a_{[j]}$.

Since $a_{[1]} \leq a_{[2]} \leq \dots \leq a_{[d]}$, as a first step, we can write the integral as

$$\int_{a_{[1]}}^1 \dots \int_{a_{[d]}}^1 \min(u_1, \dots, u_d)^p du_{[1]} \dots du_{[d]} \quad (15)$$

$$= \int_{a_{[1]}}^{a_{[2]}} u_{[1]}^p du_{[1]} \int_{a_{[2]}}^1 1 du_{[2]} \dots \int_{a_{[d]}}^1 1 du_{[d]} \quad (16)$$

$$+ \int_{a_{[2]}}^1 \int_{a_{[2]}}^1 \dots \int_{a_{[d]}}^1 \min(u_1, \dots, u_d)^p du_{[1]} \dots du_{[d]} \quad (17)$$

$$= m(a_{[1]}, a_{[2]}, 1, p) \prod_{j=2}^d (1 - a_{[j]}) \quad (18)$$

$$+ \int_{a_{[2]}}^1 \int_{a_{[2]}}^1 \dots \int_{a_{[d]}}^1 \min(u_1, \dots, u_d)^p du_{[1]} \dots du_{[d]}, \quad (19)$$

where $m()$ is the function defined in Lemma 1. Continuing with such expansion for $d - 1$ more steps, we arrive at the final integral in this series which can also be expressed in terms of $m()$ as

$$\int_{a_{[d]}}^1 \dots \int_{a_{[d]}}^1 \min(u_{[1]}, \dots, u_{[d]})^p du_{[1]} \dots du_{[d]} = m(a_{[d]}, 1, d, p). \quad (20)$$

Induction will reveal that for $1 \leq n < d$, step n of the expansion yields a total of $2^n - 1$ new terms of the form $m(\cdot) \prod (1 - a_{[.]})$. Since $2^n - 1 = \sum_{i=1}^n \binom{n}{i}$, the new terms can be collected in n groups containing $\binom{n}{i}$ elements each, for $i = 1, \dots, n$. Further induction yields combinations of indexes to be used in $m(\cdot)$ as well as the function $A(\cdot)$ that generates product indexes in every group.

Proof of Proposition 1 Expression for the numerator is available directly from Proposition 10 of Kojadinovic and Holmes (2009). For the denominator, we have

that

$$\|M - \hat{C}_n^\Pi\|_2^2 = \int_{[0,1]^d} M(\mathbf{u})^2 d\mathbf{u} - 2 \int_{[0,1]^d} M(\mathbf{u}) \prod_{k=1}^p C_n(\mathbf{u}^{[k]}) d\mathbf{u} \quad (21)$$

$$+ \int_{[0,1]^d} \left(\prod_{k=1}^p C_n(\mathbf{u}^{[k]}) \right)^2 d\mathbf{u}. \quad (22)$$

For the first term, using Lemma 2, we have that

$$\int_{[0,1]^d} M(\mathbf{u})^2 d\mathbf{u} = I(\mathbf{0}_d, 2) = \frac{2}{(d+1)(d+2)}. \quad (23)$$

The last term is the same as that in Proposition 10 of Kojadinovic and Holmes (2009). For the middle term, we have that

$$\int_{[0,1]^d} M(\mathbf{u}) \prod_{k=1}^p C_n(\mathbf{u}^{[k]}) d\mathbf{u} \quad (24)$$

$$= \int_{[0,1]^d} M(\mathbf{u}) \prod_{k=1}^p \left(\frac{1}{n} \sum_{i=1}^n \prod_{j=b_{k-1}+1}^{b_k} \mathbb{I}(\hat{u}_{ij} \leq u_j) \right) d\mathbf{u} \quad (25)$$

$$= \frac{1}{n^p} \sum_{i_1=1}^n \dots \sum_{i_p=1}^n \int_{[0,1]^d} \left(M(\mathbf{u}) \prod_{k=1}^p \prod_{j=b_{k-1}+1}^{b_k} \mathbb{I}(\hat{u}_{i_k j} \leq u_j) \right) d\mathbf{u} \quad (26)$$

$$= \frac{1}{n^p} \sum_{i_1=1}^n \dots \sum_{i_p=1}^n I([\hat{u}_{i_k[1]}1, \hat{u}_{i_k[2]}2, \dots, \hat{u}_{i_k[d]}d], 1), \quad (27)$$

where for $j \in [d]$, $k[j] = k \in [p]$ such that $b_{k-1} + 1 \leq j \leq b_k$.

Proof of Theorem 1. The proof is similar to the one of Theorem 3 in Kojadinovic and Holmes (2009). First, define a map $\Psi : \ell^\infty([0,1]^d) \rightarrow \{\ell^\infty([0,1]^d)\}^2$

through

$$\Psi(F)(\mathbf{u}) = \left(F(\mathbf{u}), \prod_{k=1}^p F(\mathbf{u}^{[k]}) \right), \quad \mathbf{u} \in [0, 1]^d$$

and note that

$$\mathbb{H}_n = \sqrt{n} \{ \Psi(\hat{C}_n) - \Psi(C) \}.$$

It follows as in Lemma 2 in Kojadinovic and Holmes (2009) that Ψ is Hadamard-differentiable at any function F with derivative $\Psi'_F : \ell^\infty([0, 1]^d) \rightarrow \{\ell^\infty([0, 1]^d)^2\}$, defined through

$$\Psi'_F(G)(\mathbf{u}) = \left(G(\mathbf{u}), \sum_{k=1}^p G(\mathbf{u}^{[k]}) \prod_{\substack{k'=1 \\ k' \neq k}}^p F(\mathbf{u}^{[k']}) \right), \quad \mathbf{u} \in [0, 1]^d.$$

Therefore, the assertion of the theorem easily follows from an application of the functional delta method, see Theorem 3.9.6 in Van der Vaart and Wellner (1996), observing that, under the assumptions on the partial derivatives of C , $\mathbb{C}_n \rightsquigarrow \mathbb{C}_C$ in $(\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$ from Proposition 3.1 in Segers (2012).

Proof of Proposition 2 Decompose $\hat{\Phi}_n^2 - \bar{\Phi}^2 = A_{n1} + A_{n2}$ where

$$A_{n1} = \frac{1}{\|M - C^\Pi\|_2^2} \left\{ \|\hat{C}_n - \hat{C}_n^\Pi\|_2^2 - \|C - C^\Pi\|_2^2 \right\}$$

$$A_{n2} = \|\hat{C}_n - \hat{C}_n^\Pi\|_2^2 \left\{ \frac{1}{\|M - \hat{C}_n^\Pi\|_2^2} - \frac{1}{\|M - C^\Pi\|_2^2} \right\}$$

Some simple algebra reveals that the expression A_{n1} can be written as

$$\begin{aligned}
& \|M - C^\Pi\|_2^2 A_{n1} \\
&= \int (\hat{C}_n - \hat{C}_n^\Pi)^2 - (C - C^\Pi)^2 d\lambda \\
&= \int (\hat{C}_n - \hat{C}_n^\Pi - C + C^\Pi)^2 d\lambda + 2 \int (C - C^\Pi)(\hat{C}_n - \hat{C}_n^\Pi - C + C^\Pi) d\lambda \\
&= \frac{1}{n} \|\mathbb{C}_n - \mathbb{C}_n^\Pi\|_2^2 + \frac{2}{\sqrt{n}} \int (C - C^\Pi)(\mathbb{C}_n - \mathbb{C}_n^\Pi) d\lambda
\end{aligned}$$

Regarding A_{n2} , we have

$$\frac{A_{n2}}{\|\hat{C}_n - \hat{C}_n^\Pi\|_2^2} = \frac{\|M - C^\Pi\|_2^2 - \|M - \hat{C}_n^\Pi\|_2^2}{\|M - \hat{C}_n^\Pi\|_2^2 \|M - C^\Pi\|_2^2},$$

and the numerator on the right-hand side can be further simplified to

$$\|M - C^\Pi\|_2^2 - \|M - \hat{C}_n^\Pi\|_2^2 = \frac{1}{\sqrt{n}} \int \mathbb{C}_n^\Pi (2M - C^\Pi - \hat{C}_n^\Pi) d\lambda.$$

Assembling terms, we have

$$\begin{aligned}
\hat{\Phi}_n^2 - \bar{\Phi}^2 &= \frac{1}{\|M - C^\Pi\|_2^2} \left\{ \frac{1}{n} \|\mathbb{C}_n - \mathbb{C}_n^\Pi\|_2^2 + \frac{2}{\sqrt{n}} \int (C - C^\Pi)(\mathbb{C}_n - \mathbb{C}_n^\Pi) d\lambda \right\} \\
&\quad + \frac{\|\hat{C}_n - \hat{C}_n^\Pi\|_2^2}{\|M - \hat{C}_n^\Pi\|_2^2 \|M - C^\Pi\|_2^2} \left\{ \frac{1}{\sqrt{n}} \int \mathbb{C}_n^\Pi (2M - C^\Pi - \hat{C}_n^\Pi) d\lambda \right\}.
\end{aligned}$$

This readily implies the assertion for $\bar{\Phi}^2 \neq 0$, i.e., for $C \neq C^\Pi$, by Theorem 1.

Moreover, if $\bar{\Phi}^2 = 0$ and hence $C = C^\Pi$, then

$$n \|\hat{C}_n - \hat{C}_n^\Pi\|_2^2 = \|\mathbb{C}_n - \mathbb{C}_n^\Pi\|_2^2 = O_P(1),$$

which implies the second assertion.

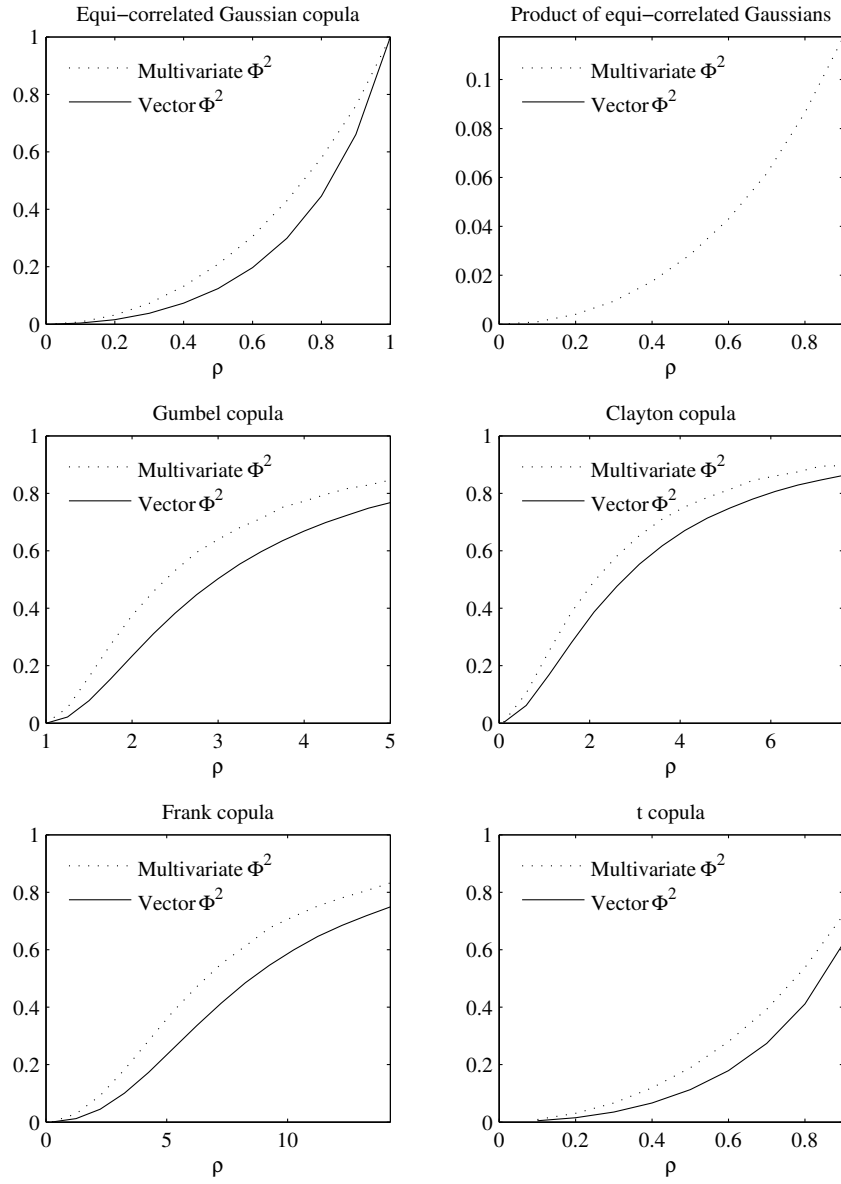


Figure 1: Approximate values of multivariate Φ^2 and vector Φ^2 for selected multivariate copula families obtained using Monte-Carlo integration.

Note: Scalar Φ overestimates Vector $\bar{\Phi}^2$

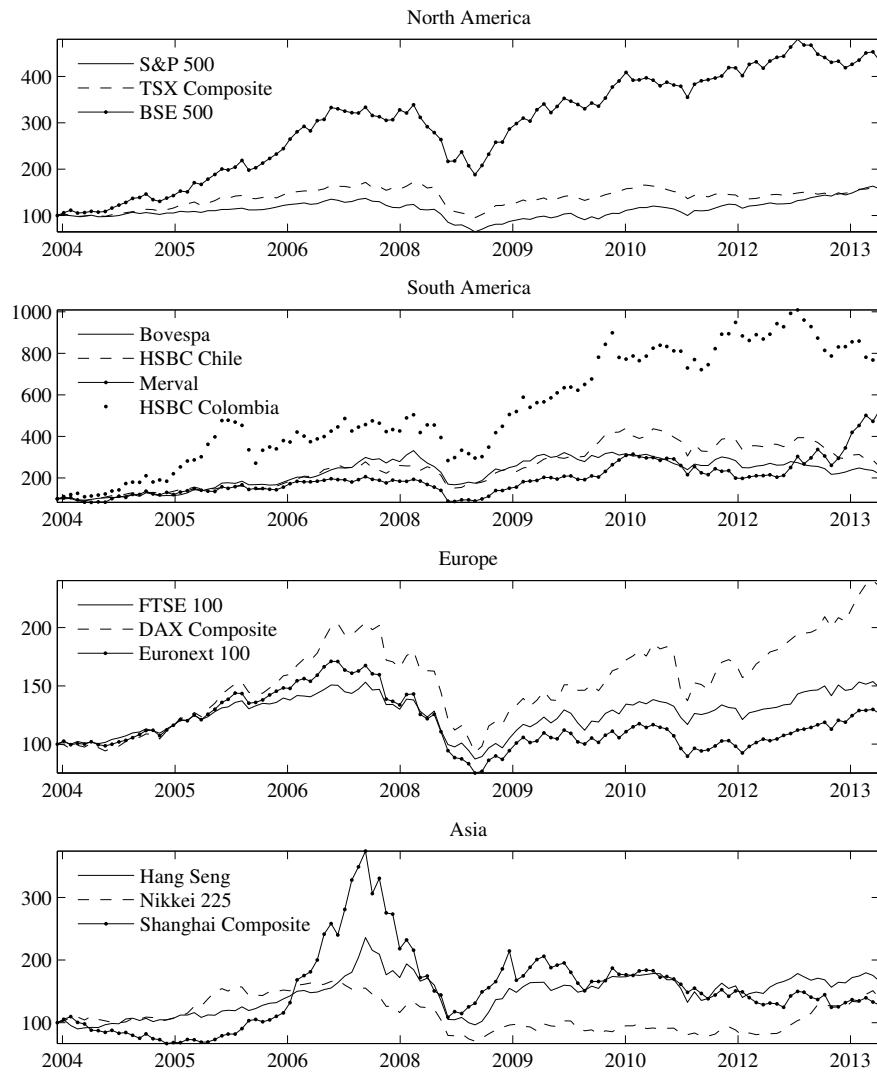


Figure 2: Normalized monthly closing values of leading global stock market indexes.

	Multivariate Φ^2	Vector Φ^2	Difference
Full sample, 2004 - 2014			
North America - Europe	0.5163 (0.051)	0.3814 (0.056)	-0.1349 (0.009)
North America - South America	0.1971 (0.048)	0.1091 (0.039)	-0.0880 (0.011)
North America - Asia	0.2168 (0.051)	0.1287 (0.042)	-0.0881 (0.011)
South America - Europe	0.2307 (0.049)	0.1263 (0.041)	-0.1044 (0.012)
South America - Asia	0.1517 (0.043)	0.0892 (0.033)	-0.0625 (0.011)
Europe - Asia	0.2594 (0.051)	0.1439 (0.042)	-0.1155 (0.011)
Pre-crisis, 2004 - 2007			
North America - Europe	0.4110 (0.072)	0.2943 (0.079)	-0.1167 (0.020)
North America - South America	0.1717 (0.045)	0.1077 (0.041)	-0.0640 (0.012)
North America - Asia	0.1277 (0.051)	0.0819 (0.043)	-0.0458 (0.012)
South America - Europe	0.1591 (0.045)	0.0855 (0.039)	-0.0736 (0.012)
South America - Asia	0.0632 (0.022)	0.0396 (0.019)	-0.0236 (0.006)
Europe - Asia	0.0888 (0.038)	0.0438 (0.031)	-0.0450 (0.012)
Crisis, 2007-2009			
North America - Europe	0.4759 (0.095)	0.6006 (0.212)	0.1247 (0.143)
North America - South America	0.3096 (0.104)	0.3693 (0.281)	0.0597 (0.198)
North America - Asia	0.2282 (0.099)	0.2201 (0.211)	-0.0081 (0.129)
South America - Europe	0.4331 (0.091)	0.6108 (0.258)	0.1777 (0.193)
South America - Asia	0.4005 (0.091)	0.5331 (0.274)	0.1326 (0.213)
Europe - Asia	0.3974 (0.082)	0.4891 (0.194)	0.0917 (0.134)
Post-crisis, 2009-2014			
North America - Europe	0.4615 (0.061)	0.3308 (0.071)	-0.1307 (0.019)
North America - South America	0.1477 (0.057)	0.0747 (0.051)	-0.0730 (0.014)
North America - Asia	0.1990 (0.050)	0.1189 (0.046)	-0.0801 (0.012)
South America - Europe	0.1823 (0.060)	0.0954 (0.052)	-0.0867 (0.016)
South America - Asia	0.1175 (0.041)	0.0667 (0.030)	-0.0508 (0.013)
Europe - Asia	0.2706 (0.063)	0.1691 (0.063)	-0.1085 (0.012)

Table 1: Measures of global market linkages, 2004 - 2014. Numbers in parentheses represent bootstrap standard errors.

	Multivariate Φ^2		Vector Φ^2	
	Increase before crisis	Decrease after crisis	Increase before crisis	Decrease after crisis
North America - Europe	0.5152	0.3535	0.0606	0.1212
North America - South America	0.2121	0.2424	0.0808	0.0202
North America - Asia	0.2323	0.4949	0.1111	0.1212
South America - Europe	0.0202	0.1010	0.0101	0.0101
South America - Asia	0.0010	0.0505	0.0010	0.0010
Europe - Asia	0.0202	0.3232	0.0010	0.0404

Table 2: P-values for tests of significance of contagion effects