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Spherical functions on homogeneous superspaces

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Homogeneous superspaces arising from the general linear supergroup are studied within a Hopf algebraic framework. Spherical functions on homogeneous superspaces are introduced, and the structures of the superalgebras of the spherical functions on classes of homogeneous superspaces are described explicitly. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1868859]

I. INTRODUCTION

We study spherical functions on homogeneous superspaces arising from the complex general linear supergroup. This is the first part of our endeavour to develop a theory of spherical functions on Lie supergroups[8,13] and quantum supergroups.[12,28] The theory of spherical functions on ordinary Lie groups has long reached its maturity (see, e.g., Ref. 26). There also exists extensive literature on spherical functions[9,18,14,3,17] on quantum symmetric spaces.[10,17,3,4,11] However, little seems to be known about spherical functions on Lie supergroups, let alone those on quantum supergroups. On the other hand, supersymmetry and its quantum analogue have become an integral part of modern mathematical physics, and have also permeated many areas of pure mathematics. A good understanding of spherical functions on Lie supergroups and quantum supergroups should facilitate practical means for studying the dynamics of physical systems with classical or quantum supersymmetries.

We choose to work within a Hopf superalgebraic framework to study homogeneous superspaces, as it can incorporate both the Lie supergroup (as defined by Kostant[8]) and quantum supergroup[12,28] cases. Our methodology is similar to that adopted in the literature on quantum homogeneous spaces.[10,17,3,4,11] The starting point is the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of the general linear superalgebra $\mathfrak{g}=\mathfrak{gl}(m|n,\mathbb{C})$, which is a cocommutative Hopf superalgebra.[15] A $\mathbb{Z}_2$-graded subalgebra $\mathbb{C}(G)$ (see Definition 3.1) of the dual of the universal enveloping algebra acquires a Hopf superalgebra structure, from which the general linear supergroup can be reconstructed[21] in a manner similar to the Tanakan–Krein theory for compact Lie groups. The universal enveloping algebra admits many Hopf $*$-superalgebra structures, each corresponding to a real form $\mathfrak{g}^{\sigma,\sqrt{i}}$ (see Sec. IV A for definition) of $\mathfrak{g}$. Each Hopf $*$-superalgebra structure θ of $\mathrm{U}(\mathfrak{g})$ induces a Hopf $*$-superalgebraic structure on $\mathbb{C}(G)$. We fix the θ corresponding to one of the compact real forms of $\mathfrak{g}$ [see Eq. (4.3)]. Let $\mathfrak{p}\subset\mathfrak{g}$ be a parabolic subalgebra with Levi factor $\mathfrak{l}$, and let $\mathfrak{k}=\mathfrak{l}\cap\mathfrak{g}^{\sigma,\sqrt{i}}$ be the real form of $\mathfrak{l}$. Then the $*$-subalgebra $\mathbb{C}(K\backslash G)$ of $\mathbb{C}(G)$ invariant with respect to $\mathfrak{k}$ under the left translation defines a homogeneous superspace[12] in the spirit of noncommutative geometry.[2] We shall call this superalgebra the superalgebra of functions on the homogeneous superspace. Next we consider the subspace $\mathbb{C}(K\backslash G/K)$ of $\mathbb{C}(K\backslash G)$ consisting of elements that are

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invariant with respect to $\mathfrak{k}$ under the right translation. It can be shown that $\mathbb{C}(K\backslash G/K)$ forms a $*$-superalgebra, which will be referred to as the superalgebra of spherical functions on the homogeneous superspace.

Our aim in the present paper is to understand the structures of the superalgebras $\mathbb{C}(K\backslash G)$ and $\mathbb{C}(K\backslash G/K)$. The main results obtained are Theorem 4.2, Lemma 4.5, and Lemma 4.6, which give explicit descriptions of the superalgebra of functions on the homogeneous superspace and the superalgebra of spherical functions. In the case of a homogeneous superspace associated to a maximal rank reductive subgroup of a compact real form of the general linear supergroup, the superalgebra of spherical functions is either the polynomial algebra in one variable or a quotient thereof (Theorems 5.1 and 5.2).

Recall that the space of functions on an ordinary Lie group has another natural algebraic structure with the multiplication defined by convolution. In this context, the counter parts of $\mathbb{C}(K\backslash G)$ and $\mathbb{C}(K\backslash G/K)$ form subalgebras under convolution, where the analogue of $\mathbb{C}(K\backslash G/K)$ is the celebrated Hecke algebra.[26] The Hecke algebras associated with Riemannian symmetric spaces are commutative, and their elements provide the invariant integral operators acting on functions on the symmetric spaces. It is an important problem to develop a theory for such Hecke algebras in the Lie supergroup context, and to investigate properties of supersymmetric spaces from the viewpoint of Hecke algebras. We plan to do this in a future publication, as the problem requires in-depth investigations into the analytical theory of Lie supergroups.

The organization of the paper is as follows. In Sec. II we provide some preliminary material on the complex general linear superalgebra and its invariant theory. In Sec. III we discuss the Hopf superalgebra of functions on the general linear supergroup, and explain how the general linear supergroup itself can be extracted from this Hopf superalgebra.[21] The material in this section is not all new, but it forms the basis for the study of homogeneous superspaces and spherical functions in later sections. Sections IV and V contain the main results of the paper. In Sec. IV A we discuss real forms of the complex general linear superalgebra and general linear supergroup from a Hopf algebraic point of view. The material presented here is largely new, and we believe it to be interesting in its own right. In Sec. IV B we explain the notion of homogeneous superspaces in a Hopf algebraic setting, and in Sec. IV C we investigate the superalgebras of spherical functions on the homogeneous superspaces. In Sec. V we analyze in detail the superalgebras of spherical functions on the projective superspace and other symmetric superspaces arising from maximal rank subgroups of real forms of the general linear supergroup.

II. PRELIMINARIES ON $\mathfrak{gl}(m|n,\mathbb{C})$

We present some background material on the universal enveloping superalgebra of the general linear Lie superalgebra, which will be used later. General references are Refs. 6 and 19.

We shall work on the complex number field $\mathbb{C}$ for simplicity. Let W be a superspace, i.e., a $\mathbb{Z}_2$-graded vector space $W=W_{\bar{0}}\oplus W_{\bar{1}}$, where $W_{\bar{0}}$ and $W_{\bar{1}}$ are the even and odd subspaces, respectively. The elements of $W_{\bar{0}}\cup W_{\bar{1}}$ will be called homogeneous. Define a map $[\ ]: W_{\bar{0}}\cup W_{\bar{1}}\to\mathbb{Z}_2$ by $[w]=\alpha$ if $w\in W_\alpha$. (Quite generally, whenever a symbol like $[w]$ appears in the sequel, it is tacitly assumed that the element w is homogeneous.) The dual superspace ($\mathbb{Z}_2$-graded dual vector space) of W will be denoted by W^*, and the dual space pairing $W^*\otimes W\to\mathbb{C}$ by $\langle,\rangle$.

Denote by $\mathfrak{g}$ the Lie superalgebra $\mathfrak{gl}(m|n,\mathbb{C})$. A standard basis for $\mathfrak{g}$ is $\{E_{ab}|a,b\in\mathbf{I}\}$, where $\mathbf{I}=\{1,2,\ldots,m+n\}$ The element E_{ab} belongs to $\mathfrak{g}_{\bar{1}}$ if $a\leqslant m<b$, or $b\leqslant m<a$, and belongs to $\mathfrak{g}_{\bar{0}}$ otherwise. For convenience, we define the map

$$[\ ]:\mathbf{I}\to\mathbb{Z}_2 \text{ by } [a]=\begin{cases}\bar{0} & \text{if } a\leqslant m,\\ \bar{1} & \text{if } a>m.\end{cases}$$

Then $[E_{ab}]=[a]+[b]$. The supercommutation relations of the Lie superalgebra are given for the basis elements by

$$[E_{ab},E_{cd}] = E_{ad}\delta_{bc} - (-1)^{([a]-[b])([c]-[d])}E_{cb}\delta_{ad}.$$

As usual, we choose the Cartan subalgebra $\mathfrak{h}=\oplus_a \mathbb{C}E_{aa}$. Let $\{\epsilon_a | a \in \mathbf{I}\}$ be the basis of $\mathfrak{h}^*$ such that $\epsilon_a(E_{bb})=\delta_{ab}$. The space h^* is equipped with a bilinear form $(,):\mathfrak{h}^*\times\mathfrak{h}^*\to\mathbb{C}$ such that $(\epsilon_a,\epsilon_b) =(-1)^{[a]}\delta_{ab}$. The roots of $\mathfrak{g}$ are $\epsilon_a-\epsilon_b$, $a\neq b$, where $\epsilon_a-\epsilon_b$ is even if $[a]+[b]=\overline{0}$ and odd otherwise. We choose as positive roots the elements of $\{\epsilon_a-\epsilon_b | a<b\}$, and as simple roots the elements of $\{\epsilon_a-\epsilon_{a+1} | a<m+n\}$.

The enveloping algebra $\mathrm{U}(\mathfrak{gl}(m|n,\mathbb{C}))$ of $\mathfrak{gl}(m|n,\mathbb{C})$ will be denoted by $\mathrm{U}(\mathfrak{g})$. We shall always regard $\mathfrak{g}$ as embedded in $\mathrm{U}(\mathfrak{g})$ in the natural way. As is well known, $\mathrm{U}(\mathfrak{g})$ forms a $\mathbb{Z}_2$-graded cocommutative Hopf algebra (i.e., a Hopf superalgebra) in the sense of Ref. 15, with

$$\text{comultiplication, } \Delta:\mathrm{U}(\mathfrak{g})\to\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g}),\quad \Delta(X)=X\otimes 1+1\otimes X,\quad X\in\mathfrak{g},$$

$$\text{counit,}\quad \epsilon:\mathrm{U}(\mathfrak{g})\to\mathbb{C},\quad \epsilon(X)=0,\quad X\in\mathfrak{g},$$

$$\text{antipode,}\quad S:\mathrm{U}(\mathfrak{g})\to\mathrm{U}(\mathfrak{g}),\quad S(X)=-X,\quad X\in\mathfrak{g}.$$

In particular, this Hopf superalgebra structure allows us to introduce a natural left $\mathrm{U}(\mathfrak{g})$-module structure on the dual superspace W^* of any left $\mathrm{U}(\mathfrak{g})$-module W, with the $\mathrm{U}(\mathfrak{g})$-action given by

$$U(\mathfrak{g})\otimes W^*\to W^*,\quad x\otimes\overline{w}\mapsto x\overline{w},$$

$$\langle x\overline{w},v\rangle := (-1)^{[x][\overline{w}]}\langle\overline{w},S(x)v\rangle,\quad \forall\, v\in W.$$

As it stands, the last equation only makes sense for homogeneous $\overline{w}\in W^*$ and homogeneous $x\in\mathrm{U}(\mathfrak{g})$, but it can be extended to all elements of W^* and $\mathrm{U}(\mathfrak{g})$ linearly.

We shall denote by L_λ the irreducible left $\mathrm{U}(\mathfrak{g})$-module with highest weight $\lambda\in\mathfrak{h}^*$. The module L_λ is finite-dimensional if and only if λ is dominant,[7,19] i.e.,

$$2(\lambda,\epsilon_a-\epsilon_{a+1})/(\epsilon_a-\epsilon_{a+1},\epsilon_a-\epsilon_{a+1})\in\mathbb{Z}_+\quad\forall\, a\neq m. \tag{2.1}$$

A basic problem in the representation theory of Lie superalgebras is to understand the weight space decompositions of the finite dimensional irreducible representations. However, the problem turned out to be unexpectedly difficult, resisting solution for some 20 years. Only a few years ago, Serganova[22] succeeded in developing an algorithm to compute formal characters of irreducible representations.

Of particular importance to us here is the contravariant vector module $V=L_{\epsilon_1}$ of $\mathfrak{g}$. It has the standard basis $\{v_a | a\in\mathbf{I}\}$ such that $E_{ab}v_c=\delta_{bc}v_a$, where v_a is even if $a\leq m$, and odd otherwise. The dual module V^* of V is the covariant vector module with highest weight $-\epsilon_{m+n}$. It has a basis $\{\overline{v_a} | a\in\mathbf{I}\}$ dual to the standard basis of V, i.e., $\langle\overline{v_a},v_b\rangle=\delta_{ab}$. The action of $\mathfrak{g}$ on V^* is given by

$$E_{ab}\overline{v_c} = -(-1)^{[a]+[a][b]}\delta_{ac}\overline{v_b}. \tag{2.2}$$

As the antipode of $\mathrm{U}(\mathfrak{g})$ is of order two, there is a $\mathrm{U}(\mathfrak{g})$-module isomorphism between V and its double dual $V^{**}:=(V^*)^*$,

$$V\cong V^{**},\quad v\mapsto v^{**},$$

$$\langle v^{**},\overline{w}\rangle = (-1)^{[v]}\langle\overline{w},v\rangle,\quad\forall\,\overline{w}\in V^*.$$

Remark 2.1: (Reference 27) For all $d>0$, $V^{\otimes d}$ is a semisimple $\mathrm{U}(\mathfrak{g})$-module, which does not contain any one-dimensional submodule.

Let $\mathfrak{S}_d$ be the symmetric group on d letters. There exists a natural action ρ_d of $\mathfrak{S}_d$ on $V^{\otimes d}$ defined in the following way. Let s_i denote the permutation $(i,i+1)$. Then

$$\rho_d(s_i)(v_{a_1} \otimes \cdots \otimes v_{a_{i-1}} \otimes v_{a_i} \otimes v_{a_{i+1}} \otimes v_{a_{i+2}} \cdots \otimes v_{a_d})$$
$$= (-1)^{[a_i][a_{i+1}]} v_{a_1} \otimes \cdots \otimes v_{a_{i-1}} \otimes v_{a_{i+1}} \otimes v_{a_i} \otimes v_{a_{i+2}} \cdots \otimes v_{a_d}.$$

Let us denote by t^d the representation of $\mathrm{U}(\mathfrak{g})$ in $V^{\otimes d}$, and denote by $\mathbb{C}\mathfrak{S}_d$ the group algebra of $\mathfrak{S}_d$. The following result was first proven by Sergeev[24,25] (see Ref. 1 for a detailed treatment).

Theorem 2.1: *The superalgebras $t^d(\mathrm{U}(\mathfrak{g}))$ and $\rho_d(\mathbb{C}\mathfrak{S}_d)$ are mutual centralizers in* $\mathrm{End}_{\mathbb{C}}(V^{\otimes d})$.

Let W be a finite dimensional $\mathrm{U}(\mathfrak{g})$-module. Let $\pi: \mathrm{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\mathrm{W})$ be the $\mathrm{U}(\mathfrak{g})$-representation furnished by W. Then $\mathrm{End}_{\mathbb{C}}(\mathrm{W})$ acquires a natural $\mathrm{U}(\mathfrak{g})$-module structure under the action

$$\mathrm{U}(\mathfrak{g}) \otimes \mathrm{End}_{\mathbb{C}}(\mathrm{W}) \to \mathrm{End}_{\mathbb{C}}(\mathrm{W}), \quad x \otimes \phi \mapsto \mathrm{Ad}_x(\phi),$$

$$\mathrm{Ad}_x(\phi) := \sum_{(x)} (-1)^{[x_{(2)}][\phi]} \pi(x_{(1)}) \phi \pi(S(x_{(2)})) ,$$

where we have used Sweedler's notation $\Delta(x) = \Sigma_{(x)} x_{(1)} \otimes x_{(2)}$ for the comultiplication of $x \in \mathrm{U}(\mathfrak{g})$. There exists the natural isomorphism $j: W \otimes W^* \cong \mathrm{End}_{\mathbb{C}}(\mathrm{W})$ of $\mathrm{U}(\mathfrak{g})$ modules defined, for any $u \otimes \overline{v} \in W \otimes W^*$ and $w \in W$, by

$$j(u \otimes \overline{v})(w) = \langle \overline{v}, w \rangle u.$$

For any $\mathrm{U}(\mathfrak{g})$-module M, we use the notation $(M)^{\mathrm{U}(\mathfrak{g})}$ to denote the invariant submodule

$$(M)^{\mathrm{U}(\mathfrak{g})} := \{ w \in M | xw = \epsilon(x) w, \quad \forall\, x \in \mathrm{U}(\mathfrak{g}) \}.$$

We have

$$(W \otimes W^*)^{\mathrm{U}(\mathfrak{g})} \cong \mathrm{End}_{\mathrm{U}(\mathfrak{g})}(\mathrm{W}) := \{ \phi \in \mathrm{End}_{\mathbb{C}}(W) | \mathrm{Ad}_x(\phi) = \epsilon(x)\phi, \forall\, x \in \mathrm{U}(\mathfrak{g}) \}. \tag{2.3}$$

Consider $V^{\otimes k} \otimes (V^*)^{\otimes \ell}$ as a $\mathrm{U}(\mathfrak{g})$-module, where the $\mathrm{U}(\mathfrak{g})$-action is defined by using the comultiplication. The element $Z = \Sigma_a E_{aa}$ acts on $V^{\otimes k} \otimes (V^*)^{\otimes \ell}$ by $(k-\ell)\mathrm{id}$. This immediately shows that

$$(V^{\otimes k} \otimes (V^*)^{\otimes \ell})^{\mathrm{U}(\mathfrak{g})} = \{0\} \quad \text{if } k \neq \ell. \tag{2.4}$$

As $(V^{\otimes d})^* \cong (V^*)^{\otimes d}$, we have the $\mathrm{U}(\mathfrak{g})$-module isomorphism

$$j: V^{\otimes d} \otimes (V^*)^{\otimes d} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes d}).$$

It follows from Theorem 2.1 that the even subspace of $(V^{\otimes d} \otimes (V^*)^{\otimes d})^{\mathrm{U}(\mathfrak{g})}$ is isomorphic to $j^{-1} \circ \rho_d(\mathbb{C}\mathfrak{S}_d)$. Let $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ be the maximal even subalgebra of $\mathfrak{g}$. Both V and V^* naturally restrict to $\mathfrak{g}_{\bar{0}}$-modules. By using Weyl's first fundamental theorem for the invariant theory of the general linear group,[5] we easily prove that $(V^{\otimes d} \otimes (V^*)^{\otimes d})^{\mathrm{U}(\mathfrak{g}_{\bar{0}})}$ is contained in the even subspace of $V^{\otimes d} \otimes (V^*)^{\otimes d}$. Since

$$(V^{\otimes d} \otimes (V^*)^{\otimes d})^{\mathrm{U}(\mathfrak{g}_{\bar{0}})} \supset (V^{\otimes d} \otimes (V^*)^{\otimes d})^{\mathrm{U}(\mathfrak{g})},$$

we have

$$(V^{\otimes d} \otimes (V^*)^{\otimes d})^{\mathrm{U}(\mathfrak{g})} = j^{-1} \circ \rho_d(\mathbb{C}\mathfrak{S}_d). \tag{2.5}$$

This result may be stated more explicitly as follows.

Theorem 2.2: (Reference 25) *The vector space $(V^{\otimes d} \otimes (V^*)^{\otimes d})^{\mathrm{U}(\mathfrak{g})}$ is spanned by the following elements:*

$$\sum_{a_1,\ldots,a_d} \mathrm{sgn}(\sigma, a_1, \ldots, a_d) \otimes v_{a_{\sigma(1)}} \otimes v_{a_{\sigma(2)}} \otimes \ldots \otimes v_{a_{\sigma(d)}} \otimes \overline{v_{a_d}} \otimes \overline{v_{a_{d-1}}} \otimes \ldots \overline{v_{a_1}}, \quad \forall\, \sigma \in \mathfrak{S}_d, \tag{2.6}$$

where $\mathrm{sgn}(\sigma, a_1, \ldots, a_d)$ *is a sign factor which is determined by the restriction of* σ *on the subset of odd indices in* $\{a_1, \ldots, a_d\}$ *in such a way that if the restriction is even then* $\mathrm{sgn}(\sigma, a_1, \ldots, a_d)$ *is* 1 *and* -1 *otherwise.*

We shall refer to both Theorems 2.1 and 2.2 as the first fundamental theorem of the invariant theory of the general linear supergroup.

III. SUPERALGEBRAS OF FUNCTIONS ON THE GENERAL LINEAR SUPERGROUP

We examine properties of the Hopf superalgebra of regular functions on the general linear supergroup in this section. The material presented here is of critical importance for setting up the framework for studying spherical functions. Some of the material can be extracted from Refs. 20 and 21.

Let $\mathrm{U}(\mathfrak{g})^0 := \{f \in \mathrm{U}(\mathfrak{g})^* \mid \ker f$ contains a cofinite $\mathbb{Z}_2$- graded ideal of $\mathrm{U}(\mathfrak{g})\}$ be the finite dual[16] of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$. Standard Hopf algebra theory[15,16] asserts that the Hopf superalgebra structure of $\mathrm{U}(\mathfrak{g})$ induces a Hopf superalgebra structure on $\mathrm{U}(\mathfrak{g})^0$. Denote by $m_\circ$, $\Delta_\circ$, $\epsilon_\circ$, and $S_\circ$ the multiplication, comultiplication, counit, and antipode of $\mathrm{U}(\mathfrak{g})^0$, respectively. The maps are defined for all $f,\ \mathfrak{g} \in \mathrm{U}(\mathfrak{g})^0$, and $a,\ b \in \mathrm{U}(\mathfrak{g})$, by

$$\langle m_\circ(f \otimes g), a\rangle = \langle f \otimes g, \Delta(a)\rangle,$$

$$\langle \Delta_\circ(f), a \otimes b\rangle = \langle f, ab\rangle,$$

$$\langle S_\circ(f), a\rangle = \langle f, S(a)\rangle,$$

and $\mathbb{1}_{\mathrm{U}(\mathfrak{g})^0} = \epsilon$, $\epsilon_\circ = \mathbb{1}_{\mathrm{U}(\mathfrak{g})}$. Because $\mathrm{U}(\mathfrak{g})$ is supercocommutative, $\mathrm{U}(\mathfrak{g})^0$ is supercommutative. Recall that $S^2 = \mathrm{id}$ and hence also $S_\circ^2 = \mathrm{id}$. For convenience, we shall drop the subscript 0 from the notations for the multiplication, comultiplication, and antipode of $\mathrm{U}(\mathfrak{g})^0$.

Let π be a $\mathrm{U}(\mathfrak{g})$-representation of dimension $d < \infty$. Now for any $x \in \mathrm{U}(\mathfrak{g})$, $\pi(x)$ is a $d \times d$-matrix. We define a set of elements $\pi_{ij} \in \mathrm{U}(\mathfrak{g})^*$, $i, j = 1, 2, \ldots, d$, by

$$\pi(x) = (\pi_{ij}(x))_{i,j=1}^d, \qquad \forall\, x \in \mathrm{U}(\mathfrak{g}).$$

The π_{ij} will be called the matrix elements of π. It is easy to see that the matrix elements of every finite-dimensional representation of $\mathrm{U}(\mathfrak{g})$ belong to $\mathrm{U}(\mathfrak{g})^0$. Conversely, $\mathrm{U}(\mathfrak{g})^0$ is spanned by the matrix elements of all the finite-dimensional representations of $\mathrm{U}(\mathfrak{g})$. To see this, we only need to consider an arbitrary nonzero element $f \in \mathrm{U}(\mathfrak{g})^0$. Let Ker be a graded cofinite ideal of $\mathrm{U}(\mathfrak{g})$ contained in the kernel of f. Then $\mathrm{U}(\mathfrak{g})/\mathrm{Ker}$ forms a left $\mathrm{U}(\mathfrak{g})$-module,

$$\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})/\mathrm{Ker} \to \mathrm{U}(\mathfrak{g})/\mathrm{Ker},$$

$$y \otimes (x + \mathrm{Ker}) \mapsto yx + \mathrm{Ker}.$$

Let $\{x_i + \mathrm{Ker}\}$ be a basis of $\mathrm{U}(\mathfrak{g})/\mathrm{Ker}$, and denote by f_{ij} the matrix elements of the associated representation relative to this basis. Choose a set of complex numbers $c_i \in \mathbb{C}$ such that $\mathbb{1}_{\mathrm{U}(\mathfrak{g})} + \mathrm{Ker} = \Sigma_i c_i x_i + \mathrm{Ker}$, where the set $\mathbb{1}_{\mathrm{U}(\mathfrak{g})} + \mathrm{Ker}$ is not contained in the kernel of f since $f \neq 0$. Then $f = \Sigma_{i,j} c_i \langle f, x_j\rangle f_{ji}$.

We denote by t the $\mathrm{U}(\mathfrak{g})$-representation associated with the contravariant vector module $V = L_{\epsilon_1}$ in the standard basis, and denote its matrix elements by $t_{ab} \in \mathrm{U}(\mathfrak{g})^0$, $a, b \in \mathbf{I}$, where t_{ab} is even if $[a] + [b] = \bar{0}$, and odd otherwise. Note that

$$t_{ab}(E_{cd}) = \delta_{ac}\delta_{bd}.$$

Denote by $\bar{t}$ the covariant vector representation of $\mathrm{U}(\mathfrak{g})$ relative to the basis $\{\bar{v}_a \mid a \in \mathbf{I}\}$. Let $\bar{t}_{ab} \in \mathrm{U}(\mathfrak{g})^0$, $a, b \in \mathbf{I}$, be the matrix elements of $\bar{t}$. Then

$$\overline{t_{ab}}(E_{cd}) = -(-1)^{[a][b]+[b]}\delta_{bc}\delta_{ad}.$$

Note that $\overline{t_{ab}}$ is even if $[a]+[b]=\overline{0}$, and odd otherwise.

Definition 3.1: (Reference 20) Let $\mathbb{C}(G)$ be the subsuperalgebra of $\mathrm{U}(\mathfrak{g})^0$ generated by $\{t_{ab},\overline{t_{ab}}|a,b\in\mathbf{I}\}$.

The following relations hold in $\mathbb{C}(G)$:

$$\sum_c t_{ac}\overline{t_{bc}}(-1)^{[c][a]+[b]} = \delta_{ab}, \quad \sum_c \overline{t_{ca}}t_{cb}(-1)^{[b][c]+[c]} = \delta_{ab}, \tag{3.1}$$

because t and $\overline{t}$ are dual representations of $\mathrm{U}(\mathfrak{g})$. More precisely, the first relation states that the canonical tensor $\Sigma_c v_c\otimes\overline{v_c}\in V\otimes V^*$ is $\mathrm{U}(\mathfrak{g})$-invariant, while the second relation means that the dual pairing $\langle,\rangle$: $V^*\otimes V\to\mathbb{C}$ is a $\mathrm{U}(\mathfrak{g})$-module homomorphism.

$\mathbb{C}(G)$ has a bisuperalgebra structure, with the comultiplication defined by

$$\Delta(t_{ab}) = \sum_{c\in\mathbf{I}}(-1)^{([c]-[a])([c]-[b])}t_{ac}\otimes t_{cb},$$

$$\Delta(\overline{t_{ab}}) = \sum_{c\in\mathbf{I}}(-1)^{([c]-[a])([c]-[b])}\overline{t_{ac}}\otimes\overline{t_{cb}}.$$

Let us also denote by S the antipode of $\mathrm{U}(\mathfrak{g})^0$. By using the definition of dual modules we can show that

$$S(t_{ab}) = (-1)^{[a][b]+[a]}\overline{t_{ba}}, \quad S(\overline{t_{ab}}) = (-1)^{[a][b]+[b]}t_{ba}. \tag{3.2}$$

The following result was proven in Ref. 20.

Proposition 3.1: (Reference 20) *(1)* $\mathbb{C}(G)$ *forms a Hopf sub-superalgebra of* $\mathrm{U}(\mathfrak{g})^0$. *(2)* $\mathbb{C}(G)$ *is dense in* $\mathrm{U}(\mathfrak{g})^*$ *in the following sense: for every nonzero element* $x\in\mathrm{U}(\mathfrak{g})$, *there exists some* $f\in\mathbb{C}(G)$ *such that* $\langle f,x\rangle\neq 0$.

Let Λ denote a finite-dimensional Grassmann algebra. Recall that the general linear supergroup $\mathrm{GL}(m|n,\Lambda)$ over Λ is the group of even invertible $(m+n)\times(m+n)$-matrices with entries in Λ. It was shown in Ref. 21 that $\mathrm{GL}(m|n,\Lambda)$ can be reconstructed from $\mathbb{C}(G)$ in the following way. The $\mathbb{Z}_2$-graded vector space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(G),\Lambda)$ has a natural superalgebra structure, with the multiplication defined for any ϕ and ψ by

$$(\phi\psi)(f) := \sum_{(f)}(-1)^{[f_{(1)}][\psi]}\phi(f_{(1)})\psi(f_{(2)}), \quad \forall f\in\mathbb{C}(G), \tag{3.3}$$

where we have used Sweedler's notation expressing the comultiplication $\Delta(f)$ of any $f\in\mathbb{C}(G)$ by $\Sigma_{(f)}f_{(1)}\otimes f_{(2)}$.

Theorem 3.1: (Reference 21) *Let* $G_{\mathbb{C}} := \{$*superalgebra homomorphisms* $\mathbb{C}(G)\to\Lambda\}$. *Then with the multiplication defined by (3.3), the set* $G_{\mathbb{C}}$ *forms a group, which is isomorphic to* $\mathrm{GL}(m|n,\Lambda)$.

We shall not repeat the proof of the theorem here, but merely point out that the inverse α^{-1} of any element $\alpha\in G_{\mathbb{C}}$ is given by $\alpha^{-1}(f)=\alpha(S(f))$, for all $f\in\mathbb{C}(G)$.

We shall refer to the elements of $\mathbb{C}(G)$ as the regular functions on the general linear supergroup. We now consider their properties. Note that there exists two natural left actions dR and dL of $\mathrm{U}(\mathfrak{g})$ on $\mathbb{C}(G)$, respectively, corresponding to the left and right translations. For all $x\in\mathrm{U}(\mathfrak{g})$, $f\in\mathbb{C}(G)$,

$$\mathrm{d}R_x(f) = \sum_{(f)}(-1)^{[x][f]}f_{(1)}\langle f_{(2)},x\rangle,$$

$$\mathrm{d}L_x(f)=\sum_{(f)}(-1)^{[x]}\langle f_{(1)},S(x)\rangle f_{(2)}. \tag{3.4}$$

Equivalently, the equations in (3.4) can be rewritten in the form

$$\langle \mathrm{d}R_x(f),y\rangle=(-1)^{[x]([f]+[y])}\langle f,yx\rangle,\quad \langle \mathrm{d}L_x(f),y\rangle=(-1)^{[x][f]}\langle f,S(x)y\rangle, \tag{3.5}$$

for all $x,y\in \mathrm{U}(\mathfrak{g})$ and $f\in\mathbb{C}(G)$. Straightforward calculations show that each of dL and dR indeed converts $\mathbb{C}(G)$ into a (graded) left $\mathrm{U}(\mathfrak{g})$-module. With respect to this module structure the product map of $\mathbb{C}(G)$ is a $\mathrm{U}(\mathfrak{g})$-module homomorphism, and the unit element of $\mathbb{C}(G)$ is $\mathrm{U}(\mathfrak{g})$-invariant. Take dL as an example, we have

$$\sum_{(x)}(-1)^{[x_{(2)}][f]}\mathrm{d}L_{x_{(1)}}(f)\otimes \mathrm{d}L_{x_{(1)}}(g)\mapsto \mathrm{d}L_x(fg),\quad \forall f,g\in \mathrm{U}(\mathfrak{g})^0,\quad x\in\mathrm{U}(\mathfrak{g}),$$

$$\mathrm{d}L_x(\epsilon)=\epsilon(x)\epsilon,\quad \forall x\in \mathrm{U}(\mathfrak{g}). \tag{3.6}$$

This is saying that each of the actions dL and dR converts $\mathbb{C}(G)$ into a left $\mathrm{U}(\mathfrak{g})$-module superalgebra.[16] The two actions supercommute as can be easily checked. Thus $\mathbb{C}(G)$ forms a left $\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})$-module algebra, with the action

$$(x\otimes y)f=\mathrm{d}L_x\,\mathrm{d}R_y(f),\quad \forall x,y\in\mathrm{U}(\mathfrak{g}),\quad f\in\mathbb{C}(G).$$

The fact that the product map in $\mathbb{C}(G)$ is a module homomorphism means that the operators dR_x and dL_x behave as some sort of generalized superderivations. In particular, if $x\in\mathfrak{g}$, they are superderivations.

To better understand the structure of $\mathbb{C}(G)$, we let $X=V\otimes V^*$ and $\overline{X}=V^*\otimes V$. Using the standard bases of V and V^* we manufacture the bases $\{x_{ab}\coloneqq v_b\otimes\overline{v_a}\}$ and $\{\overline{x_{ab}}\coloneqq\overline{v_b}\otimes v_a\}$ for X and $\overline{X}$, respectively. Denote by $\mathbb{C}[X,\overline{X}]$ the $\mathbb{Z}_2$-graded symmetric algebra of $X\oplus\overline{X}$. Then $\mathbb{C}[X,\overline{X}]$ as an associative superalgebra can be described more explicitly as generated by x_{ab}, $\overline{x_{ab}}$, $a,b\in\mathbf{I}$, subject to the relations

$$x_{ab}x_{cd}=(-1)^{([b]-[a])([d]-[c])}x_{cd}x_{ab},$$

$$x_{ab}\overline{x_{cd}}=(-1)^{([b]-[a])([c]-[d])}\overline{x_{cd}}x_{ab},$$

$$\overline{x_{ab}}\,\overline{x_{cd}}=(-1)^{([a]-[b])([c]-[d])}\overline{x_{cd}}\,\overline{x_{ab}}.$$

The generators x_{ab} and $\overline{x_{ab}}$ are even if $[a]+[b]=\overline{0}$, and odd otherwise. Stated differently, the $2(m^2+n^2)$ even generators generate a polynomial algebra, the $4mn$ odd generators generate a Grassmann algebra with the standard grading, and $\mathbb{C}[X,\overline{X}]$ is the tensor product of the two. Let $\mathcal{J}$ be the (graded) ideal of $\mathbb{C}[X,\overline{X}]$ generated by the following elements:

$$\sum_c x_{ac}\overline{x_{bc}}(-1)^{[c][a]+[b]}-\delta_{ab},\quad \sum_c \overline{x_{ca}}x_{cb}(-1)^{[b][c]+[c]}-\delta_{ab},\quad a,b\in\mathbf{I}. \tag{3.7}$$

We have the following theorem.

Theorem 3.2: (Reference 20) *The assignments* $x_{ab}\mapsto t_{ab}$, $\overline{x_{ab}}\mapsto\overline{t_{ab}}$, $a,b\in\mathbf{I}$ *specify a well-defined superalgebra isomorphism* $j:\mathbb{C}[X,\overline{X}]/\mathcal{J}\to\mathbb{C}(G)$.

Define two left $\mathrm{U}(\mathfrak{g})$-actions on $X\oplus\overline{X}$,

$$\Phi:\mathrm{U}(\mathfrak{g})\otimes(X\oplus\overline{X})\to X\oplus\overline{X},\quad u\otimes w\mapsto\Phi(u)w,$$

$$\Psi : \mathrm{U}(\mathfrak{g}) \otimes (X \oplus \overline{X}) \to X \oplus \overline{X}, \quad u \otimes w \mapsto \Psi(u)w,$$

by

$$\Phi(u)(v_b \otimes \overline{v_a}) = (-1)^{[u]} u v_b \otimes \overline{v_a},$$

$$\Psi(u)(v_b \otimes \overline{v_a}) = (-1)^{[u][b]} v_b \otimes u\overline{v_a},$$

$$\Phi(u)(\overline{v_b} \otimes v_a) = (-1)^{[u]} u\overline{v_b} \otimes v_a,$$

$$\Psi(u)(\overline{v_b} \otimes v_a) = (-1)^{[u]([u]+[b])}\overline{v_b} \otimes u v_a, \quad u \in \mathrm{U}(\mathfrak{g}).$$

These actions supercommute, and can both be extended to left $\mathrm{U}(\mathfrak{g})$-actions on $\mathbb{C}[X,\overline{X}]$ by

$$\Phi(x)(p_1 p_2) = \sum (-1)^{[x_{(2)}][p_1]}(\Phi(x_{(1)})p_1)(\Phi(x_{(2)})p_2),$$

$$\Psi(x)(p_1 p_2) = \sum (-1)^{[x_{(2)}][p_1]}(\Psi(x_{(1)})p_1)(\Psi(x_{(2)})p_2),$$

where $p_1, p_2 \in \mathbb{C}[X,\overline{X}]$ and $x \in \mathrm{U}(\mathfrak{g})$. This gives rise to a $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$-module algebra structure on $\mathbb{C}[X,\overline{X}]$. Note that the $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$-action leaves the ideal $\mathcal{J}$ invariant. Thus we have the following proposition.

Proposition 3.2: *The map* $j : \mathbb{C}[X,\overline{X}]/\mathcal{J} \to \mathbb{C}(G)$ *of Theorem 2 is a* $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$*-module algebra isomorphism, with*

$$j((\Psi(x) \otimes \Phi(y))p) = (\mathrm{d}L_x \otimes \mathrm{d}R_y) j(p), \quad \forall\, x,y \in \mathrm{U}(\mathfrak{g}), \quad p \in \mathbb{C}[X,\overline{X}].$$

IV. HOMOGENEOUS SUPERSPACES AND SPHERICAL FUNCTIONS

Recall the following well-known fact in the context of classical homogeneous spaces: if H is a compact semisimple Lie group, and $H_{\mathbb{C}}$ is its complexification, then for any parabolic subgroup Q of $H_{\mathbb{C}}$, we have $H_{\mathbb{C}}/Q = H/R$, where R is the intersection of the Levi factor of Q with H. We shall imitate this construction in the algebraic setting for Lie supergroups. For this we need to discuss real forms of the complex general linear superalgebra and the general linear supergroup.

A. Real forms

Let us begin by briefly discussing the notion of Hopf $*$-superalgebras.[28] A $*$-superalgebraic structure on an associative superalgebra A is a conjugate linear anti-involution θ: $A \to A$: for all $x, y \in A$, $c, c' \in \mathbb{C}$,

$$\theta(cx + c'y) = \overline{c}\theta(x) + \overline{c'}\,\theta(y), \quad \theta(xy) = \theta(y)\,\theta(x), \quad \theta^2(x) = x.$$

Note that the second equation does not involve any sign factors as one would normally expect of superalgebras. We shall sometimes use the notation (A,θ) for the $*$-superalgebra A with the $*$-structure θ. Let (B,θ_1) be another associative $*$-superalgebra. Now $A \otimes B$ has a natural superalgebra structure, with the multiplication defined for any $a, a' \in A$ and $b, b' \in B$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{[b][a']} a a' \otimes b b',$$

where $(-1)^{[b][a']}$ is the usual sign factor required for exchanging positions of odd elements. Furthermore, the following conjugate linear map,

$$\theta\star\theta_1 : a\otimes b \mapsto (1\otimes\theta_1(b))(\theta(a)\otimes 1) = (-1)^{[a][b]}\theta(a)\otimes\theta_1(b), \tag{4.1}$$

defines a $*$-superalgebraic structure on $A\otimes B$.

Let us assume that A is a Hopf superalgebra with comultiplication Δ, counit ϵ and antipode S. If the $*$-superalgebraic structure θ satisfies

$$(\theta\star\theta)\Delta = \Delta\theta, \quad \theta\epsilon = \epsilon\theta,$$

then A is called a Hopf $*$-superalgebra. Now

$$\sigma := S\theta$$

satisfies $\sigma^2 = id_A$, as follows from the definition of the antipode.

Let A^0 denote the finite dual of A, which has a natural Hopf superalgebraic structure. We shall still use Δ and S to, respectively, denote the comultiplication and antipode of A^0, but write its counit as ϵ_o. If A is a Hopf $*$-superalgebra with the Hopf $*$-superalgebraic structure θ, then $\sigma = S\theta$ induces a map ω: $A^0 \to A^0$ defined for any $f\in A^0$ by

$$\langle \omega(f), x\rangle = \overline{\langle f, \sigma(x)\rangle}, \quad \forall\, x\in A. \tag{4.2}$$

Lemma 4.1: The map ω: $A^0\to A^0$ *defined by (4.2) gives rise to a Hopf $*$-superalgebraic structure on* A^0.

Proof: It is clear that ω is conjugate linear. Also, $\sigma^2 = \mathrm{id}_A$ implies $\omega^2 = \mathrm{id}_{A^0}$. For all $f,g\in A^0$, $x,y\in A$, we have

$$\langle \omega(fg), x\rangle = \overline{\langle fg, \sigma(x)\rangle} = \overline{\langle f\otimes g, (S\otimes S)(\theta\star\theta)\Delta'(x)\rangle}$$
$$= (-1)^{[f][g]}\langle \omega(f)\otimes\omega(g), \Delta'(x)\rangle = \langle \omega(g)\omega(f), x\rangle,$$

that is, $\omega(fg) = \omega(g)\omega(f)$. Define $\omega\star\omega$ as in (4.1), we have

$$\langle (\omega\star\omega)\Delta(f), x\otimes y\rangle = (-1)^{[x][y]}\overline{\langle \Delta(f), \sigma(x)\otimes\sigma(y)\rangle} = \overline{\langle f, \sigma(xy)\rangle}$$
$$= \langle \omega(f), xy\rangle = \langle \Delta\omega(f), x\otimes y\rangle,$$

that is $(\omega\star\omega)\Delta(f) = \Delta\omega(f)$. It is easy to show that ω also satisfies all the other minor requirements to qualify as a Hopf $*$-superalgebraic structure on A^0. ■

The universal enveloping algebra of the general linear superalgebra admits many Hopf $*$-superalgebraic structures. Let us fix one Hopf $*$-superalgebraic structure θ: $\mathrm{U}(\mathfrak{g})\to\mathrm{U}(\mathfrak{g})$ here. As $\mathfrak{g}$ is canonically embedded in $\mathrm{U}(\mathfrak{g})$, the restriction of θ to $\mathfrak{g}$ defines a conjugate anti-involution of the Lie superalgebra. Let $\mathfrak{g}_{\bar 0}^{\sigma}$ and $\mathfrak{g}_{\bar 1}^{\sigma}$ be the fixed point sets of $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$ under σ, respectively. Let $\mathfrak{g}^{\sigma,\sqrt{i}}\subset\mathfrak{g}$ be the real span of $\mathfrak{g}_{\bar 0}^{\sigma}\cup\sqrt{i}\mathfrak{g}_{\bar 1}^{\sigma}$. Then $\mathfrak{g}^{\sigma,\sqrt{i}}$ forms a real Lie superalgebra, which is a real form of $\mathfrak{g}$. However, note that the σ-invariants of $\mathfrak{g}$ do not form a real subalgebra of $\mathfrak{g}$ if $\mathfrak{g}_{\bar 1}^{\sigma}$ is nontrivial. This is the reason for us to consider $\mathfrak{g}^{\sigma,\sqrt{i}}$ instead.

Denote by $\mathrm{U}^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}})$ the real universal enveloping algebra of $\mathfrak{g}^{\sigma,\sqrt{i}}$, which is embedded in $\mathrm{U}(\mathfrak{g})$ in the natural way. Furthermore,

$$\mathrm{U}(\mathfrak{g}) = \mathbb{C}\otimes_{\mathbb{R}}\mathrm{U}^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}}).$$

By Lemma 4.1, the Hopf $*$-superalgebraic structure θ induces a Hopf $*$-superalgebraic structure ω: $\mathbb{C}(G)\to\mathbb{C}(G)$ on $\mathbb{C}(G)$. By using the embedding of the real associative superalgebra $\mathrm{U}^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}})$ in $\mathrm{U}(\mathfrak{g})$, we can see that $f\in\mathbb{C}(G)$ vanishes if and only if $\langle f,x\rangle = 0$, for all $x\in\mathrm{U}^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}})$. Therefore elements of $\mathbb{C}(G)$ can be considered as complex valued functionals on the real superalgebra $\mathrm{U}^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}})$. From this point of view, we should interpret $\mathbb{C}(G)$ as the $*$-superalgebra of functions on some real supergroup G. Now let us make this discussion more precise.

Let Λ be the complex Grassmann algebra introduced in Sec. III. Let $^{-}$: $\Lambda\to\Lambda$ be a complex conjugation operation on supernumbers [i.e., $(\Lambda, ^{-})$ is a $*$-superalgebra]. Theorem 3.1 shows that all the superalgebra homomorphisms $\mathbb{C}(G) \to \Lambda$ form a supergroup $G_{\mathbb{C}}$, which is isomorphic to $\mathrm{GL}(m|n,\Lambda)$. A homomorphism α: $\mathbb{C}(G) \to \Lambda$ will be called a $*$-superalgebra homomorphism if it preserves the $*$-superalgebraic structures in the sense that $\alpha(\omega(f)) = \overline{\alpha(f)}$, for all $f \in \mathbb{C}(G)$. The following result can be easily proven.

Lemma 4.2: If an element α of $G_{\mathbb{C}}$ is a $*$-superalgebra homomorphism, then its inverse is also a $*$-superalgebra homomorphism. The product of any two $*$-superalgebra homomorphisms in $G_{\mathbb{C}}$ is again a $*$-superalgebra homomorphism.

Proof: We shall prove the first statement only. The second one can be shown in a similar way. Recall that the inverse of $\alpha \in G_{\mathbb{C}}$ is defined by

$$\langle \alpha^{-1}, f \rangle = \langle \alpha, S(f) \rangle, \quad \forall\, f \in \mathbb{C}(G).$$

Now if α is a $*$-superalgebra homomorphism, then for all $f \in \mathbb{C}(G)$,

$$\langle \alpha^{-1}, \omega(f) \rangle = \langle \alpha, S\omega(f) \rangle = \overline{\langle \alpha, S(f) \rangle} = \overline{\langle \alpha^{-1}, f \rangle}.$$

This shows that α^{-1} is indeed a $*$-superalgebra homomorphism. ■

Introduce the map $\check{\theta}$: $G_{\mathbb{C}} \to G_{\mathbb{C}}$ defined by

$$\langle \check{\theta}(\alpha), f \rangle = \overline{\langle \alpha, \omega S(f) \rangle}, \quad \forall\, f \in \mathbb{C}(G).$$

We need to show that the image of $\check{\theta}$ indeed lies in $G_{\mathbb{C}}$. For any $f, g \in \mathbb{C}(G)$, we have

$$\begin{aligned}
\langle \check{\theta}(\alpha), fg \rangle &= (-1)^{[f][g]} \overline{\langle \alpha, \omega S(f) \omega S(g) \rangle} \\
&= (-1)^{[f][g]} \overline{\langle \alpha \otimes \alpha, \omega S(f) \otimes \omega S(g) \rangle} \\
&= (-1)^{[f][g]} \overline{\langle \alpha, \omega S(f) \rangle \langle \alpha, \omega S(g) \rangle} \\
&= \overline{\langle \alpha, \omega S(g) \rangle \langle \alpha, \omega S(f) \rangle} \\
&= \overline{\langle \alpha, \omega S(f) \rangle} \cdot \overline{\langle \alpha, \omega S(g) \rangle} \\
&= \langle \check{\theta}(\alpha), f \rangle \langle \check{\theta}(\alpha), g \rangle.
\end{aligned}$$

Therefore, $\check{\theta}(\alpha)$ is a superalgebra homomorphism from $\mathbb{C}(G)$ to Λ, thus is indeed an element of $G_{\mathbb{C}}$.

Definition 4.1: $G := \{$*-superalgebra homomorphism $\mathbb{C}(G) \to \Lambda\}$.

Theorem 4.1: *G forms a subgroup of $G_{\mathbb{C}}$. Furthermore, $\check{\theta}(\alpha) = \alpha^{-1}$ for all $\alpha \in G$.*

Proof: The fact that G forms a subgroup immediately follows from the above lemma. If $\alpha \in G$, we have

$$\begin{aligned}
\langle \check{\theta}(\alpha), f \rangle &= \overline{\langle \alpha, \omega S(f) \rangle} = \langle \alpha, S(f) \rangle \\
&= \langle \alpha^{-1}, S(f) \rangle, \quad \forall\, f \in \mathbb{C}(G).
\end{aligned}$$

This confirms the second claim. ■

B. Spherical functions on homogeneous superspaces

Hereafter we fix a Hopf $*$-superalgebraic structure θ for $\mathrm{U}(\mathfrak{g})$, which is defined for all the generators by

$$\theta : E_{ab} \mapsto E_{ba}. \tag{4.3}$$

The associated real form $\mathfrak{gl}(m|n;\mathbb{C})^{\sigma,\sqrt{i}}$ of the general linear superalgebra is one of the compact real forms of the general linear superalgebra, which probably deserves the notation $\mathfrak{u}(m|n)$ because it contains the maximal even subalgebra $\mathfrak{u}(m) \oplus \mathfrak{u}(n)$. (The unitarizable representations of this compact real form comprise the tensor powers of the natural representation, while the unitarizable representations of the other compact real form are the duals of these representations.[27]) Direct calculations can show that the Hopf $*$-superalgebraic structure on $\mathbb{C}(G)$ induced by θ is given by

$$\omega(t_{ab}) = (-1)^{[b]([a]+[b])}\overline{t_{ab}}, \quad \omega(\overline{t_{ab}}) = (-1)^{[b]([a]+[b])}t_{ab}. \tag{4.4}$$

The real supergroup G has body $U(m)\times U(n)$.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. Let $\mathfrak{k}=\mathfrak{l}^{\sigma,\sqrt{i}}$ be the real form of $\mathfrak{l}$, which is a subalgebra of $\mathfrak{g}^{\sigma,\sqrt{i}}$. Denote by $U^{\mathbb{R}}(\mathfrak{k})$ the universal enveloping algebra of $\mathfrak{k}$ over $\mathbb{R}$. Note that $U^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}})$ inherits a real Hopf superalgebra structure from $U(\mathfrak{g})$, and $U^{\mathbb{R}}(\mathfrak{k})$ inherits a real Hopf superalgebra structure from $U^{\mathbb{R}}(\mathfrak{g}^{\sigma,\sqrt{i}})$. Let us introduce the following definition.

Definition 4.2:

$$\mathbb{C}(K\setminus G) := \{f \in \mathbb{C}(G) | dL_k(f) = \epsilon(k)f, \ \forall\, k \in U^{\mathbb{R}}(\mathfrak{k})\}. \tag{4.5}$$

Note the following obvious fact, which will be used immediately below:

$$\mathbb{C}(K\setminus G) := \{f \in \mathbb{C}(G) | dL_k(f) = \epsilon(k)f, \ \forall\, k \in U(\mathfrak{l})\}. \tag{4.6}$$

We have the following lemma.

Lemma 4.3: $\mathbb{C}(K\backslash G)$ forms a $*$-subalgebra of $\mathbb{C}(G)$, which is also a left coideal of $\mathbb{C}(G)$.

Proof: Since $U(\mathfrak{l})$ is a Hopf subalgebra of $U(\mathfrak{g})$, we have $\Delta(k)=\Sigma_{(k)}k_{(1)}\otimes k_{(2)} \in U(\mathfrak{l})\otimes U(\mathfrak{l})$ for all $k\in U(\mathfrak{l})$. If $a,b\in\mathbb{C}(K\backslash G)$, then by (4.6),

$$\begin{aligned} dL_k(ab) &= \sum (-1)^{[a_{(2)}][b_{(1)}]+[k]}\langle a_{(1)}b_{(1)}, S(k)\rangle a_{(2)}b_{(2)} \\ &= \sum (-1)^{[a_{(2)}][k_{(1)}]+[k]}\langle a_{(1)}, S(k_{(2)})\rangle b_{(1)}, S(k_{(1)})a_{(2)}b_{(2)} \\ &= \sum (-1)^{[k]}\epsilon(k_{(1)})\langle a_{(1)}, S(k_{(2)})\rangle a_{(2)}b = \epsilon(k)ab, \quad \forall\, k \in U(\mathfrak{l}). \end{aligned}$$

Thus $ab\in\mathbb{C}(K\backslash G)$.

Given any $f\in\mathbb{C}(K\backslash G)$, we have $dL_k(\omega(f))=\omega(dL_{\theta(k)}(f))(-1)^{[k]([k]+[f])}$ for all $k\in U(\mathfrak{l})$. As $U(\mathfrak{l})$ is θ invariant, we have $dL_{\theta(k)}(f)=\overline{\epsilon(k)}f$. Thus

$$dL_k(\omega(f)) = \epsilon(k)\omega(f), \quad \forall\, k \in U(\mathfrak{l}).$$

Also, a straightforward calculation shows that

$$(dL_k \otimes \mathrm{id})\Delta(f) = \epsilon(k)\Delta(f), \quad \forall\, k \in U(\mathfrak{l}).$$

Thus $\mathbb{C}(K\backslash G)$ is a left coideal. This completes the proof. ■

The subalgebra $\mathbb{C}(K\backslash G)$ consists of the elements of $\mathbb{C}(G)$ which are invariant with respect to $U^{\mathbb{R}}(\mathfrak{k})$ under left translation. Following the general philosophy of noncommutative geometry,[2] we may take the viewpoint that $\mathbb{C}(K\backslash G)$ defines an algebraic homogeneous superspace.[12] We shall refer to $\mathbb{C}(K\backslash G)$ as the superalgebra of functions on the homogeneous superspace. Also a word about the notation $\mathbb{C}(K\backslash G)$: here K is used to indicate some real subsupergroup of G with Lie superalgebra $\mathfrak{k}$.

Remark 4.1: Since $\mathbb{C}(G)$ and $\mathbb{C}(K\backslash G)$ are all $*$-superalgebras, their elements are in general not holomorphic functions on the supergroup. This is a particularly welcome fact, as it indicates that our construction can lead to analogues of compact complex super manifolds like projective superspaces. As is well known from the Gelfand–Naimark theorem, the continuous functions on a compact manifold determine the manifold completely, even when the manifold is complex, where all the holomorphic functions are constants.

Remark 4.2: In the quantum group context, one usually considers left or right coideal subalgebras of the algebra of functions[10,17,3,11] in the place of $\mathbb{C}(K\backslash G)$. By Lemma 4.3 $\mathbb{C}(K\backslash G)$ forms a left coideal subalgebra of $\mathbb{C}(G)$.

Because the two left actions $\mathrm{d}R$ and $\mathrm{d}L$ of $\mathrm{U}(\mathfrak{g})$ on $\mathbb{C}(G)$ supercommute, the subalgebra $\mathbb{C}(K\backslash G)$ of $\mathbb{C}(G)$ forms a left module algebra over $\mathrm{U}(\mathfrak{g})$ under the action $\mathrm{d}R$. We shall study the $\mathrm{d}R(\mathrm{U}^{\mathbb{R}}(\mathfrak{k}))$-invariant subspace of $\mathbb{C}(K\backslash G)$. Let us first generalize the definition of zonal spherical functions[26] to the supergroup setting. We shall refer to elements of the following space as spherical functions on the homogeneous superspace.

Definition 4.3:

$$\mathbb{C}(K\setminus G/K) := \{f \in \mathbb{C}(K\setminus G) | \mathrm{d}R_k(f) = \epsilon(k)f, \quad \forall\, k \in \mathrm{U}^{\mathbb{R}}(\mathfrak{k})\}. \tag{4.7}$$

Similar arguments as those in the proof of Lemma 4.3 show the following.

Lemma 4.4: The subspace $\mathbb{C}(K\backslash G/K)$ *forms a* $*$*-subalgebra of* $\mathbb{C}(K\backslash G)$. Obviously

$$\mathbb{C}(K\setminus G/K) = \{f \in \mathbb{C}(K\setminus G) | \mathrm{d}R_x(f) = \epsilon(x)f,\ \forall\, x \in \mathrm{U}(\mathfrak{l})\}, \tag{4.8}$$

where $\mathfrak{l}$ is the complexification of $\mathfrak{k}$. The fact will be used in the next section to prove Theorem 4.2.

C. Structure of superalgebra of spherical functions

Let $\mathfrak{l}$ be a reductive subalgebra of $\mathfrak{g}$ generated by E_{aa}, $a \in \mathbf{I}$, and $E_{c,c+1}$, $E_{c+1,c}$ with c belonging to some proper subset of $\mathbf{I}\backslash\{m+n\}$. As in the last section, we let

$$\mathfrak{k} = \mathfrak{l}^{\sigma,\sqrt{i}}.$$

See Remark 4.3 for further discussions on this choice of $\mathfrak{k}$. The main result here is Theorem 4.2, which enables us to obtain the superalgebras $\mathbb{C}(K\backslash G)$ and $\mathbb{C}(K\backslash G/K)$ from the invariants of $\mathbb{C}[X,\overline{X}]$. An explicit description of the generators of these superalgebras will also be given in Lemmas 4.5 and 4.6.

Theorem 4.2: *When* $\mathfrak{k}=\mathfrak{l}^{\sigma,\sqrt{i}}$, *we have*

$$\begin{aligned}\mathbb{C}(K\setminus G) &= \{j(p) | p \in \mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))}\},\\ \mathbb{C}(K\setminus G/K) &= \{j(p) | p \in \mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))}\}.\end{aligned} \tag{4.9}$$

The remainder of this section is devoted to the proof of Theorem 4.2. The proof is carried out in two steps. We first show that the theorem holds when $\mathfrak{l}=\mathfrak{k}_{\mathbb{C}}$ is even, that is, when $\mathfrak{l}$ is a reductive Lie subalgebra of $\mathfrak{g}$. Then we use this fact to prove the general case. In the process of proving the theorem, we also establish Lemmas 4.5 and 4.6. We mention that Eqs. (4.6) and (4.8) will be used repeatedly in the proof without further warning.

1. Proof of Theorem 4.2 for $\mathfrak{l}$ *even*

In this case we can find a set of positive integers k_i, $i=1,2,\ldots,r,r+1,\ldots,s$ such that $\Sigma_{i=1}^{r}k_i=m$, $\Sigma_{j=r+1}^{s}k_j=n$, and $\mathfrak{l}=\oplus_{i=1}^{s}\mathfrak{gl}(k_i)$. More explicitly,

$$\mathfrak{l} = \left\{\begin{pmatrix} A_1 & & 0\\ & \ddots & \\ 0 & & A_s\end{pmatrix} \middle| A_i \in \mathfrak{gl}(k_i)\right\} \subset \mathfrak{g}.$$

Proposition 3.2 implies the following short exact sequence:

$$0 \to \mathcal{J} \to \mathbb{C}[X,\overline{X}] \overset{\mathcal{J}}{\to} \mathbb{C}(G) \to 0$$

in the category of $\mathrm{U}(\mathfrak{l})\otimes\mathrm{U}(\mathfrak{l})$-module superalgebras. Since the various $\mathrm{U}(\mathfrak{l})$ and $\mathrm{U}(\mathfrak{l})\otimes\mathrm{U}(\mathfrak{l})$ actions on $\mathcal{J}$, $\mathbb{C}[X,\overline{X}]$ and $\mathbb{C}(G)$ are all semisimple, we have the following short exact sequences of $\mathrm{U}(\mathfrak{l})\otimes\mathrm{U}(\mathfrak{l})$-modules:

$$0 \to \mathcal{J}^{\Psi(\mathrm{U}(\mathfrak{l}))} \to \mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))} \to \mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l})}} \to 0, 0 \to \mathcal{J}^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))} \to \mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))}$$
$$\to \mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l})}\otimes \mathrm{d}R_{\mathrm{U}(\mathfrak{l})}} \to 0,$$

where $\mathbb{C}(K\backslash G)=\mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l})}}$ and $\mathbb{C}(K\backslash G/K)=\mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l})}\otimes \mathrm{d}R_{\mathrm{U}(\mathfrak{l})}}$. These are also short exact sequences of $\mathrm{U}(\mathfrak{l})\otimes\mathrm{U}(\mathfrak{l})$-module algebras, thus they imply the claims of Theorem 4.2 in the case under consideration.

Let us now describe the algebras $\mathbb{C}(K\backslash G)$ and $\mathbb{C}(K\backslash G/K)$ more carefully. Set $l_i=\Sigma_{t=1}^{i}k_t$. Recall that $\mathbb{C}[X,\overline{X}]$ is the symmetric algebra in $X\oplus\overline{X}$ where $X=V\otimes\overline{V}$ and $\overline{X}=\overline{V}\otimes V$. Restricted to a $\mathrm{U}(\mathfrak{l})$-module, V decomposes into

$$V=\oplus_{i=1}^{s}V_i^{(k_i)}.$$

The ideal $\mathfrak{gl}(k_i)$ of $\mathfrak{l}$ acts on $V_i^{(k_i)}$ by the natural representation, and acts on all other submodules trivially. There is also an analogous decomposition of the restriction of $\overline{V}$ to a $\mathrm{U}(\mathfrak{l})$-module. By applying the first fundamental theorem of the invariant theory of the general linear group,[5] we obtain that the subalgebra $\mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))}$ of $\mathbb{C}[X,\overline{X}]$ is generated by

$$\hat{C}_{ab}^{(i)} := \sum_{c=1+l_{i-1}}^{l_i} x_{ca}\overline{x_{cb}}, \quad i=1,2,\ldots,s, \quad a,b\in\mathbf{I}.$$

It then immediately follows that $\mathbb{C}(K\backslash G)$ is generated by

$$C_{ab}^{(i)} := j(\hat{C}_{ab}^{(i)}) = \sum_{c=1+l_{i-1}}^{l_i} t_{ca}\overline{t_{cb}}, \quad i=1,2,\ldots,s, \quad a,b\in\mathbf{I}.$$

Note that the $C_{ab}^{(i)}$ are not algebraically independent, for example, for $a,b\in\mathbf{I}$ the following hold:

$$\sum_{i=1}^{s} C_{ab}^{(i)}(-1)^{[l_i]}=\delta_{ab}, \quad \sum_{a=1}^{m+n} C_{ab}^{(i)}=k_i. \tag{4.10}$$

Thus the elements of the set $\{C_{ab}^{(i)}|i\neq s; a,b\in\mathbf{I}\}$ can also generate $\mathbb{C}(K\backslash G)$. By using the fact that t_{ab} and $\overline{t_{cd}}$ supercommute, one can verify the following proposition easily.

Proposition 4.1: We have

$$C_{ab}^{(i)}C_{cd}^{(j)}=(-1)^{([a]+[b])([c]+[d])}C_{cd}^{(j)}C_{ab}^{(i)}, \tag{4.11}$$

in particular, if $[a]+[b]=1$ *then* $(C_{ab}^{(i)})^2=0$. *Thus for fixed i, there is an onto algebra homomorphism* $\mathbb{C}[X]\to\langle C_{ab}^{(i)}|a,b\in\mathbf{I}\rangle$, *where* $\mathbb{C}[X]$ *is the subalgebra of* $\mathbb{C}[X,\overline{X}]$ *generated by X, and* $\langle C_{ab}^{(i)}|a,b\in\mathbf{I}\rangle$ *is the subalgebra of* $\mathbb{C}(K\backslash G)$ *generated by* $\{C_{ab}^{(i)}|a,b\in\mathbf{I}\}$.

In a similar way we can show that $\mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))}$ is generated by

$$\hat{C}^{(i,j)} := \sum_{a=1+l_{j-1}}^{l_j}\sum_{c=1+l_{i-1}}^{l_i} x_{ca}\overline{x_{ca}}, \quad i,j=1,2,\ldots,s,$$

and $\mathbb{C}(K\backslash G/K)$ is generated by

$$C^{(i,j)} := j(\hat{C}^{(i,j)}) = \sum_{a=1+l_{j-1}}^{l_j}\sum_{c=1+l_{i-1}}^{l_i} t_{ca}\overline{t_{ca}}, \quad i,j=1,2,\ldots,s.$$

Again, the $C^{(i,j)}$ are not algebraically independent, for example,

$$\sum_{i=1}^{s} C^{(i,j)}(-1)^{[l_i]} = k_j, \quad \sum_{j=1}^{s} C^{(i,j)}(-1)^{[l_j]} = k_i. \tag{4.12}$$

Thus the elements of the set $\{C^{(i,j)} | i,j \neq r\}$ generate $\mathbb{C}(K\backslash G/K)$.

2. Proof of Theorem 4.2 for generic $\mathfrak{l}$

The most general form of $\mathfrak{l}$ is as follows. There exists a set of positive integers k_i as in the last section such that

$$\mathfrak{l} = (\oplus_{i=1}^{r-1}\mathfrak{gl}(k_i)) \oplus \mathfrak{gl}(k_r|k_{r+1}) \oplus (\oplus_{j=r+2}^{s}\mathfrak{gl}(k_j)).$$

More explicitly, we have

$$\mathfrak{l} = \left\{ \begin{pmatrix} \ddots & & & & & 0 \\ & A_{r-1} & & & & \\ & & B & & & \\ & & & A_{r+2} & & \\ 0 & & & & \ddots & \\ & & & & & A_s \end{pmatrix} \middle| \begin{array}{lcl} A_i & \in & \mathfrak{gl}(k_i), \\ B & \in & \mathfrak{gl}(k_r|k_{r+1}) \end{array} \right\}.$$

Note that $\mathfrak{l}$ contains the maximal even subalgebra $\mathfrak{l}_0 = \oplus_{i=1}^{s}\mathfrak{gl}(k_i)$.

We first consider the subalgebra $\mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}}$ of $\mathbb{C}(G)$. By using results of the last section, we can immediately see that $\mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}}$ is generated by the elements of $\{C_{ab}^{(i)} | i \neq r; a,b \in \mathbf{I}\}$. Now

$$\mathbb{C}(K\backslash G) = \{f \in \mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}} | \mathrm{d}L_{E_{mm+1}}(f) = \mathrm{d}L_{E_{m+1,m}}(f) = 0\}.$$

We shall show that $\mathbb{C}(K\backslash G)$ is generated by $\{C_{ab}^{(i)} | i \neq r, r+1; a,b \in \mathbf{I}\}$.

Note that all the elements of $\{C_{ab}^{(i)} | i \neq r; a,b \in \mathbf{I}\}$ are annihilated by $\mathrm{d}L_{E_{mm+1}}$ and $\mathrm{d}L_{E_{m+1,m}}$ except for $C_{ab}^{(r+1)}$, for which we have

$$\mathrm{d}L_{E_{mm+1}}(C_{ab}^{(r+1)}) = -(-1)^{[a]+[b]} t_{m+1,a}\overline{t_{mb}}, \tag{4.13}$$

$$\mathrm{d}L_{E_{m+1,m}}(C_{ab}^{(r+1)}) = -(-1)^{[a]+[b]} t_{ma}\overline{t_{m+1,b}}, \quad a,b \in \mathbf{I}. \tag{4.14}$$

Note that as a $\mathrm{U}(\mathfrak{g})$-module, $\mathbb{C}(G)$ has a filtration defined by the degrees of the polynomials in the t_{ab} and the $\overline{t_{ab}}$, and the filtration on the $\mathrm{U}(\mathfrak{l}_0)$-module $\mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}}$ defined by the degrees of the polynomials in the $\{C_{ab}^{(i)} | i \neq r; a,b \in \mathbf{I}\}$ is compatible with this filtration. Thus in order to find those $f \in \mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}}$ such that $\mathrm{d}L_{E_{mm+1}}(f) = L_{E_{m+1,m}}(f) = 0$, by passing through to the associated graded modules defined by these filtrations if necessary, we may assume that f is homogeneous of degree μ in the elements of $\{C_{ab}^{(i)} | i \neq r; a,b \in \mathbf{I}\}$. We consider an element $f \in \mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}}$ as a polynomial in $\{C_{ab}^{(r+1)} | a,b \in \mathbf{I}\}$ with coefficients being polynomials in $\{C_{ab}^{(i)} | i \neq r, r+1; a,b \in \mathbf{I}\}$. Set $C_{ab} = C_{ab}^{(r+1)}$ $(a,b \in \mathbf{I})$. Then by Proposition 4.1, the subalgebra $\langle C_{ab} | a,b \in \mathbf{I}\rangle$ has a basis consists of elements of the form

$$C_{a_1b_1}^{p_1} \cdots C_{a_sb_s}^{p_s} C_{c_1d_1} \cdots C_{c_td_t}, \tag{4.15}$$

with $[a_i]+[b_i]=0(1 \leq i \leq s)$, $[c_j]+[d_j]=1(1 \leq j \leq t)$, and $p_i \geq 0(1 \leq i \leq s)$ are integers. Extend such a basis of $\langle C_{ab} | a,b \in I\rangle$ to a homogeneous basis $\mathbf{B}$ of $\mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}(\mathfrak{l}_0)}}$, so that the elements of $\mathbf{B}$ are of the form

$$CC^{p_1}_{a_1b_1}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots C_{c_td_t}, \tag{4.16}$$

where C is a monomial in $\{C^{(i)}_{ab}|i\neq r,r+1;a,b\in \mathbf{I}\}$. Now let us write $f=\Sigma_{0\leqslant k\leqslant \mu}f_k$, where f_k is a linear combination of the basis elements of (4.16) such that $\Sigma_i p_i+t=k$ and $\deg(C)+k=\deg(f)$. The action of E_{mm+1} (similarly for E_{m+1m}) on the elements of (4.5) can be computed by using (4.13), and we have

$$\mathrm{d}L_{E_{mm+1}}(C^{p_1}_{a_1b_1}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots C_{c_td_t})=-\sum_{i=1}^{s}(-1)^t p_i C^{p_1}_{a_1b_1}\cdots C^{p_i-1}_{a_ib_i}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots C_{c_td_t}t_{m+1a_i}\overline{t_{mb_i}} + C^{p_1}_{a_1b_1}\cdots C^{p_s}_{a_sb_s}\sum_{j=1}^{t}(-1)^j C_{c_1d_1}\cdots \hat{C}_{c_jd_j}\cdots C_{c_td_t}t_{m+1c_j}\overline{t_{md_j}}. \tag{4.17}$$

Since the product map of $\mathbb{C}(G)$ is a $\mathrm{U}(\mathfrak{g})$-module homomorphism [see (3.6)], by (4.17) the action of E_{mm+1} on the elements of (4.16) is given by

$$\begin{aligned}\mathrm{d}L_{E_{mm+1}}&(CC^{p_1}_{a_1b_1}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots C_{c_td_t})\\ &=-(-1)^{[C]}C\sum_{i=1}^{s}(-1)^t p_i C^{p_1}_{a_1b_1}\cdots C^{p_i-1}_{a_ib_i}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots C_{c_td_t}t_{m+1a_i}\overline{t_{mb_i}}\\ &\quad+(-1)^{[C]}CC^{p_1}_{a_1b_1}\cdots C^{p_s}_{a_sb_s}\sum_{j=1}^{t}(-1)^j C_{c_1d_1}\cdots \hat{C}_{c_jd_j}\cdots C_{c_td_t}t_{m+1c_j}\overline{t_{md_j}},\end{aligned} \tag{4.18}$$

where $\hat{C}_{c_jd_j}$ means that the factor $C_{c_jd_j}$ is omitted.

For an element x of the form (4.16), let $x'(ab)$ be

$$CC^{p_1}_{a_1b_1}\cdots C^{p_{i-1}}_{a_{i-1}b_{i-1}}C^{p_i-1}_{a_ib_i}C^{p_{i+1}}_{a_{i+1}b_{i+1}}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots C_{c_td_t},$$

or

$$CC^{p_1}_{a_1b_1}\cdots C^{p_s}_{a_sb_s}C_{c_1d_1}\cdots \hat{C}_{c_jd_j}\cdots C_{c_td_t},$$

depending on whether $(ab)=(a_ib_i)$ or $(ab)=(c_jd_j)$.

Let us make some observations. First note that since

$$\langle m_\circ(h\otimes g),a\rangle=\langle h\otimes g,\Delta(a)\rangle,\quad h,g\in U(\mathfrak{g})^0,\quad a\in U(\mathfrak{g}),$$

if $\{b_i|1\leqslant i\leqslant \ell\}\subset\mathbb{C}(G)$ is a set of linearly independent functions which are constants on $U(\mathfrak{k}_0)$ and $\{t_{m+1a}\overline{t_{mb}}|a,b\in\mathbf{J}\subset\mathbf{I}\}$ is linearly independent, then the set $\{b_it_{m+1a}\overline{t_{mb}}|1\leqslant i\leqslant\ell,a,b\in\mathbf{J}\}$ is linearly independent. Then note that if $S_{cd}\subset\mathbf{B}$ with C_{cd} appearing in every element for a fixed pair c and d, then the set

$$S'_{cd}=\{x'(cd)|x\in S_{cd}\}$$

is linearly independent. In fact the elements of S_{cd} and $C_{cd}S'_{cd}$ are the same up to signs. Finally note that the only relation among the elements in $\{t_{m+1a}\overline{t_{mb}}|a,b\in\mathbf{I}\}$ is [see (3.1)]

$$\sum_{a\in I}t_{m+1a}\overline{t_{ma}}(-1)^{[a]}=\sum_{1\leqslant a\leqslant m}t_{m+1a}\overline{t_{ma}}-\sum_{m+1\leqslant a\leqslant m+n}t_{m+1a}\overline{t_{ma}}=0,$$

and this relation can only come from (via the map $\mathrm{d}L_{E_{mm+1}}$)

$$\sum_{1\leq a\leq m} C_{aa} - \sum_{m+1\leq a\leq m+n} C_{aa} = \sum_{1\leq a\leq m}\sum_{c=m+1}^{l_{r+1}} t_{ca}\overline{t_{ca}} - \sum_{m+1\leq a\leq m+n}\sum_{c=m+1}^{l_{r+1}} t_{ca}\overline{t_{ca}} = -k_{r+1},$$

i.e., a constant.

These observations together with (4.18) imply that $\mathrm{d}L_{E_{mm+1}}(f)=\mathrm{d}L_{E_{m+1,m}}(f)=0$ if and only if $f=f_0$, i.e., f is independent of $C_{ab}^{(r+1)}(a,b\in\mathbf{I})$. Therefore, we have the following.

Lemma 4.5: $\mathbb{C}(K\backslash G)$ *is generated by the elements of*

$$\{C_{ab}^{(i)}| i\neq r,r+1; a,b\in\mathbf{I}\}. \tag{4.19}$$

By Theorem 2.1 and the first fundamental theorem of the invariant theory of the general linear group, $\mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))}$ is generated by $\hat{C}_{ab}^{(i)}(i\neq r,r+1;a,b\in\mathbf{I})$, and $\hat{C}_{ab}^{(r)}-\hat{C}_{ab}^{(r+1)}(a,b\in\mathbf{I})$. We have $j(\hat{C}_{ab}^{(i)})=C_{ab}^{(i)}(i\neq r,r+1;a,b\in\mathbf{I})$, which yield all the elements of (4.19). This establishes the short exact sequence

$$0\rightarrow \mathcal{J}^{\Psi(\mathrm{U}(\mathfrak{l}))}\rightarrow \mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))}\rightarrow \mathbb{C}(K\setminus G)\rightarrow 0$$

of $\mathrm{U}(\mathfrak{l})\otimes\mathrm{U}(\mathfrak{l})$-module algebras, thus proves the first claim of Theorem 4.2.

Let us now consider the subalgebra $\mathbb{C}(K\backslash G)^{\mathrm{d}R_{\mathrm{U}(\mathfrak{l}_0)}}$ of $\mathbb{C}(K\backslash G)$, which is generated by the elements of the set $\{C^{(i,j)}|i\neq r,r+1;j\neq r\}$, as follows from results of the last section. Among all the elements of this set, only $C^{(i,r+1)}$ are not annihilated by $\mathrm{d}R_{E_{mm+1}}$ and $\mathrm{d}R_{E_{m+1,m}}$. Thus similar to the case of the left action, we can prove that $f\in\mathbb{C}(K\backslash G)^{\mathrm{d}R_{\mathrm{U}(\mathfrak{l}_0)}}$ satisfies $\mathrm{d}R_{E_{mm+1}}(f)=0$ and $\mathrm{d}R_{E_{m+1,m}}(f)=0$ if and only if it is independent of the $C^{(i,r+1)}(i\neq r,r+1)$. Observe that

$$\mathbb{C}(K\setminus G/K)=\{f\in\mathbb{C}(K\setminus G)^{\mathrm{d}R_{\mathrm{U}(\mathfrak{l}_0)}}|\mathrm{d}R_{E_{mm+1}}(f)=\mathrm{d}R_{E_{m+1,m}}(f)=0\}.$$

We have

Lemma 4.6: $\mathbb{C}(K\backslash G/K)$ *is generated by the elements of*

$$\{C^{(i,j)}|i,j\neq r,r+1\}. \tag{4.20}$$

By Theorem 2.1 and the first fundamental theorem of the invariant theory of the general linear group, $\mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))}$ is generated by

$$\hat{C}^{(i,j)},\hat{C}^{(r,j)}-\hat{C}^{(r+1,j)},\hat{C}^{(i,r)}-\hat{C}^{(i,r+1)},\quad i,j\neq r,r+1,\quad a,b\in\mathbf{I},$$

and since

$$j(\{\hat{C}^{(i,j)}|i,j\neq r,r+1\})=\{C^{(i,j)}|i,j\neq r,r+1\},$$

we have the following short exact sequence of $\mathrm{U}(\mathfrak{l})\otimes\mathrm{U}(\mathfrak{l})$-module algebras:

$$0\rightarrow \mathcal{J}^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))}\rightarrow \mathbb{C}[X,\overline{X}]^{\Psi(\mathrm{U}(\mathfrak{l}))\otimes\Phi(\mathrm{U}(\mathfrak{l}))}\rightarrow \mathbb{C}(K\setminus G/K)\rightarrow 0,$$

which is equivalent to the second claim of Theorem 4.2.

Remark 4.3: Geometric homogeneous superspaces have been studied since the 1970s, see, for example, Refs. 8 and 13. Symmetric superspaces were also classified by Serganova in Ref. 23 at the level of Lie superalgebras. In relation to our algebraic definition of homogeneous superspaces, one may ask the following question. Let P be the parabolic subgroup of $\mathrm{GL}(m|n,\Lambda)$ with Lie superalgebra $\mathfrak{p}$. We have the homogeneous superspace $\mathrm{GL}(m|n,\Lambda)/P$ (understood as a left coset of P). Now let $\mathfrak{l}$ be the Levi factor of $\mathfrak{p}$ and take $\mathfrak{k}=\mathfrak{l}^{\sigma,\sqrt{i}}$, with θ being the Hopf $*$-superalgebraic structure of $\mathrm{U}(\mathfrak{g})$ corresponding to the compact real form of the general linear superalgebra [defined by (4.3)]. Then the question is whether the homogeneous superspace determined by $\mathbb{C}(K\backslash G)$ is the same as $\mathrm{GL}(m|n,\Lambda)/P$ in some appropriate sense. We expect the answer to be affirmative, but have not been able to locate a reference, which addresses any form of the question, in the literature on supergeometry.

V. SPHERICAL FUNCTIONS ON $\mathbb{C}(K\backslash G)$ WITH MAXIMAL RANK K

We keep notations from the last section. In particular, we fix the $*$-structure θ of $\mathrm{U}(\mathfrak{g})$ given by (4.3), which corresponds to the real form $\mathfrak{u}(m|n)$ for the general linear superalgebra. We use $\mathfrak{l}$ to denote the Levi factor of a parabolic subalgebra of $\mathfrak{g}$, and set $\mathfrak{k}=\mathfrak{l}^{\sigma,\sqrt{i}}$. The homogeneous superspaces studied in this section are all examples of symmetric superspaces in the sense of Ref. 22 (see Tables 2 and 3 in Ref. 23).

A. The case with $\mathfrak{k}=\mathfrak{u}(m|n-1)\oplus\mathfrak{u}(1)$

We first examine in some detail the spherical functions on the homogeneous superspace corresponding to $\mathfrak{k}=\mathfrak{u}(m|n-1)\oplus\mathfrak{u}(1)$, where the complexification $\mathfrak{l}$ of $\mathfrak{k}$ is the subalgebra of $\mathfrak{g}$ spanned by the elements E_{ij}, $i,j\in\mathbf{I}\backslash\{m+n\}$, and $E_{m+n,m+n}$. But before discussing the superalgebra $\mathbb{C}(K\backslash G)$, let us introduce the following superalgebra.

Definition 5.1: $\mathbb{C}(\mathbb{S}^{2n-1|2m}) \coloneqq \mathbb{C}(G)^{\mathrm{d}L_{\mathrm{U}^{\mathbb{R}}(\mathfrak{u}(m|n-1))}}$ relative to $\mathfrak{u}(m|n-1)\subset\mathfrak{k}$.

More explicitly,

$$\mathbb{C}(\mathbb{S}^{2n-1|2m}) = \{f\in\mathbb{C}(G)|\mathrm{d}L_k(f)=\epsilon(k)f,\ \forall\ k\in\mathrm{U}^{\mathbb{R}}(\mathfrak{u}(m|n-1))\}.$$

We can modify the analysis of Sec. IV C to construct $\mathbb{C}(\mathbb{S}^{2n-1|2m})$. With the help of Theorem 2.1 for $\mathfrak{gl}(m|n-1)$, we can show that $\mathbb{C}(\mathbb{S}^{2n-1|2m})$ is generated by

$$z_a \coloneqq t_{m+n,a}, \quad \overline{z_a} \coloneqq \overline{t_{m+n,a}},$$

$$Q_{ab} \coloneqq \sum_{c<m+n} \overline{t_{ca}}t_{cb}(-1)^{[b][c]+[c]}, \quad a\in\mathbf{I},$$

where z_a and $\overline{z_a}$ are odd if $a\leq m$, and even otherwise. The defining relations of $\mathbb{C}(G)$ imply $Q_{ab}=\delta_{ab}\mathbb{1}-z_a\overline{z_b}(-1)^{[b]}$. Thus the z_a and $\overline{z_a}$ generate $\mathbb{C}(\mathbb{S}^{2n-1|2m})$ by themselves. We have the following result.

Lemma 5.1: The subalgebra of $\mathbb{C}(\mathbb{S}^{2n-1|2m})$ of $\mathbb{C}(G)$ is generated by z_a, $\overline{z_a}$, $a\in\mathbf{I}$, which satisfy the following relation:

$$\sum_{a\in\mathbf{I}} \overline{z_a}z_a = \mathbb{1}. \tag{5.1}$$

Remark 5.1: The notation suggests $\mathbb{C}(\mathbb{S}^{2n-1|2m})$ be the superalgebra of functions on the supersphere. This can be understood as follows. Under the $*$-map ω defined by (4.4), we have

$$\omega(z_a)=\overline{z_a}, \quad \omega(\overline{z_a})=z_a.$$

Thus we may interpret $\overline{z_a}$ as the complex conjugate of z_a, and this indeed makes perfect sense when z_a and $\overline{z_a}$ are regarded as functions on G (see Sec. IV A). Thus Eq. (5.1) defines a supersphere in analogy with the embedding of a supersphere $\mathbb{S}^{2n-1|2m}$ in $\mathbb{C}^{n|m}$. This also indicates the importance of the $*$-structure in determining the underlying supermanifold of $\mathbb{C}(K\backslash G)$.

Remark 5.2: When $\mathfrak{k}=\mathfrak{u}(m|n-1)\oplus\mathfrak{u}(1)$, we have $\mathbb{C}(K\backslash G)=\mathbb{C}(\mathbb{S}^{2n-1|2m})^{\mathrm{d}L_{\mathfrak{u}(1)}}$. This superalgebra embedding $\mathbb{C}(K\backslash G)\hookrightarrow\mathbb{C}(\mathbb{S}^{2n-1|2m})$ corresponds to a projection from $\mathbb{S}^{2n-1|m}$ to the symmetric superspace, which is the supergeneralization of the Hopf map $\mathbb{S}^{2n-1}\to\mathbb{CP}^{n-1}$. Therefore, we shall regard the symmetric superspace associated with $\mathbb{C}(K\backslash G)$ as an algebraic analogue of the projective superspace $\mathbb{CP}^{n-1|m}$.

We denote $\mathbb{C}(K\backslash G)$ by $\mathbb{C}(\mathbb{P}^{n-1|m})$ when $\mathfrak{k}=\mathfrak{u}(m|n-1)\oplus\mathfrak{u}(1)$. Lemma 5.1 immediately leads to the following result.

Lemma 5.2: The superalgebra $\mathbb{C}(\mathbb{P}^{n-1|m})$ *is the* $*$*-subalgebra of* $\mathbb{C}(\mathbb{S}^{2n-1|2m})$ *generated by the quadratic elements* $z_a\overline{z_b}$, $a,b\in\mathbf{I}$.

Proof: Since for all a, $\mathrm{d}L_{E_{m+n,m+n}}z_a = z_a$, and $\mathrm{d}L_{E_{m+n,m+n}}\overline{z_a} = -\overline{z_a}$, any $\mathrm{d}L_{\mathfrak{u}(1)}$-invariant element of $\mathbb{C}(\mathbb{S}^{2n-1|2m})$ must be a polynomial in $z_a\overline{z_b}$, $a,b\in\mathbf{I}$. This result can also be obtained in a more direct way by using Theorem 4.2. ■

Remark 5.3: We should emphasize that elements of $\mathbb{C}(\mathbb{P}^{n-1|m})$ are functions on the projective superspace that are not holomorphic in general because $\mathbb{C}(\mathbb{P}^{n-1|m})$ is a $*$-superalgebra.

Now we use Theorem 4.2 to extract the algebra $\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$ of spherical functions on the projective superspace. Let $z:=z_{m+n}$ and $\overline{z}=\overline{z}_{m+n}$. We have the following.

Theorem 5.1: *The algebra of the spherical functions on the projective superspace is generated by* $r:=z\overline{z}$ *as a* $*$*-subalgebra of* $\mathbb{C}(\mathbb{P}^{n-1|m})$. *When* $n>1$, *the spherical functions form a polynomial algebra in one variable. When* $n=1$, *we have* $(1-r)^{m+1}=0$.

Proof: It is an immediate consequence of Theorem 4.2 that the algebra $\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$ of the spherical functions on the projective superspace is indeed generated by the single element r.

When $n=1$, all the z_c, $\overline{z_c}$, $c\le m$, are odd. Thus the $(m+1)$th power of $1-r=\Sigma_{c\le m}z_c\overline{z_c}$ vanishes identically.

To study the case with $n>1$, we first analyze $\mathbb{C}[\mathrm{GL}_n]$, the algebra generated by the matrix elements of the contravariant and covariant vector representations of $\mathfrak{gl}(n)$. Let $\mathfrak{q}=\mathfrak{gl}(n-1)\oplus\mathfrak{gl}(1)$ be the subalgebra of $\mathfrak{gl}(n)$ embedded block diagonally. Set $A=\mathbb{C}[\mathrm{GL}_n]^{\mathrm{d}L_{\mathrm{U}(\mathfrak{q})}\otimes \mathrm{d}R_{\mathrm{U}(\mathfrak{q})}}$. Recall that $\mathbb{C}[\mathrm{GL}_n]$ is semisimple as a left module $\mathrm{U}(\mathfrak{q})$-module under the action $\mathrm{d}L_{\mathrm{U}(\mathfrak{q})}\otimes\mathrm{d}R_{\mathrm{U}(\mathfrak{q})}$. There exists a surjective $\mathrm{d}L_{\mathrm{U}(\mathfrak{q})}\otimes\mathrm{d}R_{\mathrm{U}(\mathfrak{l})}$-module map $\psi:\mathbb{C}[\mathrm{GL}_n]\to \mathrm{A}$. Let $\psi^*,(\mathrm{id}-\psi)^*:\mathrm{U}(\mathfrak{gl}(n))\to\mathrm{U}(\mathfrak{gl}(n))$ be vector space maps defined by

$$\langle f,(\mathrm{id}-\psi)^*(u)\rangle = \langle(\mathrm{id}-\psi)(f),u\rangle,$$

$$\langle f,\psi^*(u)\rangle = \langle\psi(f),u\rangle, \qquad \forall\, u\in\mathfrak{gl}(n),\ f\in\mathbb{C}[\mathrm{GL}_n].$$

Since the dual space pairing between $\mathbb{C}[\mathrm{GL}_n]$ and $\mathrm{U}(\mathfrak{gl}(n))$ is nondegenerate, there is a nondegenerate pairing between A and $\psi^*(\mathrm{U}(\mathfrak{gl}(n)))$. Now as vector spaces,

$$\psi^*(\mathrm{U}(\mathfrak{gl}(n)))\cong\mathrm{U}(\mathfrak{gl}(n))/(\mathfrak{q}\mathrm{U}(\mathfrak{gl}(n))+\mathrm{U}(\mathfrak{gl}(n))\mathfrak{q}),$$

where the right-hand side is clearly infinite dimensional. This in particular implies that the subalgebra A of $\mathbb{C}[\mathrm{GL}_n]$ is infinite dimensional.

Let $\zeta:\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}\to\mathbb{C}[\mathrm{GL}_n]$ be the map defined for any $f\in\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$ and $u\in\mathrm{U}(\mathfrak{gl}(n))$ by $\langle\zeta(f),u\rangle=\langle\zeta(f),i(u)\rangle$, where i is the canonical embedding $\mathrm{U}(\mathfrak{gl}(n))\subset\mathrm{U}(\mathfrak{g})$. Then ζ is an algebra homomorphism, and we have

$$\zeta(\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}})=A.$$

If there existed a nontrivial polynomial $P(r)$ in r which was identically zero as an element of $\mathbb{C}(\mathbb{P}^{n-1|m})$, then $\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$ would have to be finite dimensional over $\mathbb{C}$. This contradicts the fact that A is an infinite dimensional algebra. ■

Let us now study the action of a generalized Laplacian operator on the spherical functions. Recall that the quadratic Casimir of $\mathrm{U}(\mathfrak{g})$ can be expressed as $c=\Sigma_{a,b=1}^{m+n}(-1)^{[b]}E_{ab}E_{ba}$. For any $f\in\mathbb{C}(K\backslash G/K)$, we have $\mathrm{d}R_X\,\mathrm{d}R_c(f)=\mathrm{d}R_c\,\mathrm{d}R_X(f)=0$, $\forall X\in\mathfrak{l}$. That is $\mathrm{d}R_c(f)\in\mathbb{C}(K\backslash G/K)$. Consider the following generalized Laplacian operator on the homogeneous superspace:

$$\nabla^2=-\sum_{i=1}^{m+n-1}E_{i,m+n}E_{m+n,i}.$$

Then the actions of $\mathrm{d}R_{\nabla^2}$ and $\frac{1}{2}\mathrm{d}R_c$ coincide on $\mathbb{C}(K\backslash G/K)$. Thus $\mathrm{d}R_{\nabla^2}$ also maps $\mathbb{C}(K\backslash G/K)$ to itself.

In the case of the projective superspace, we can show that

$$\mathrm{d}R_{\nabla^2}(r^k) = kr^{k-1}[(m-n-k+1)r+k], \quad k=0,1,\ \dots\ . \tag{5.2}$$

Let us now consider eigenfunctions of $\mathrm{d}R_{\nabla^2}$ in $\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$. Things turn out to be quite different for $m-n+1\leqslant 0$ and $m-n+1>0$.

(1) If $m-n+1\leqslant 0$, there exists an eigenfunction $\theta_k \in \mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$ of $\mathrm{d}R_{\nabla^2}$ for each $k\in\mathbb{Z}_+$ with $\mathrm{d}R_{\nabla^2}(\theta_k)=k(m-n-k+1)\theta_k$, where

$$\theta_k = \sum_{i=0}^{k} (-1)^i \binom{n-m+2k-2}{i}\binom{k}{i}^2 (i!)^2 r^{k-i}. \tag{5.3}$$

Furthermore, the θ_k, $k\in\mathbb{Z}_+$, span $\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$.

(2) If $m-n+1>0$, we let $L=m-n+1$, and denote by $[L/2]$ the largest integer $\leqslant L/2$. Then there exists an eigenfunction $\theta_k\in\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$ of $\mathrm{d}R_{\nabla^2}$ for each non-negative integer k satisfying either $k\leqslant[L/2]$ or $k>L$ with $\mathrm{d}R_{\nabla^2}$-eigenvalue $k(L-k)$, where the θ_k are still given by (5.3). However, the θ_k's do not span $\mathbb{C}(\mathbb{P}^{n-1|m})^{\mathrm{d}R_{\mathrm{U}^{\mathbb{R}}(\mathfrak{k})}}$.

Note that if $m-n+1>0$, the operator $\mathrm{d}R_{\nabla^2}$ is not diagonalizable over $\mathbb{C}(K\backslash G/K)$. The simplest illustration comes from the case with $L=1$, where $\mathbb{C}(K\backslash G/K)$ is the direct sum of $\{a+br \mid a,b\in\mathbb{C}\}$ and $\oplus_{k>1}\mathbb{C}\theta_k(r)$. While acting diagonally on the latter subspace, $\mathrm{d}R_{\nabla^2}$ acts on the former subspace by $\mathrm{d}R_{\nabla^2}(a+br)=b$.

Remark 5.4: $\mathbb{C}(G)$ is not semisimple with respect to $\mathrm{d}R_{\mathrm{U}(\mathfrak{g})}$. There exist $\mathrm{d}R_{\mathrm{U}(\mathfrak{g})}$-submodules of $\mathbb{C}(G)$ on which $\mathrm{d}R_c$ cannot be diagonalized. Therefore, $\mathrm{d}R_{\nabla^2}$ is not diagonalizable on $\mathbb{C}(K\backslash G/K)$ in general, and case (2) shows this fact.

B. The other maximal rank *K* cases

We assume that both m and n are greater than 2 in this section, and consider the maximal rank K's that correspond to the subalgebras $\mathfrak{k}_{n,k}:=\mathfrak{l}_{n,k}^{\sigma,\sqrt{i}}$ and $\mathfrak{k}_{m,k}:=\mathfrak{l}_{m,k}^{\sigma,\sqrt{i}}$, where

$$\mathfrak{l}_{n,k} = \mathfrak{gl}(m|n-k)\oplus\mathfrak{gl}(k), \quad 0<k\leqslant n,$$

$$\mathfrak{l}_{m,k} = \mathfrak{gl}(k)\oplus\mathfrak{gl}(m-k|n), \quad 0<k\leqslant m.$$

For the subalgebra $\mathfrak{k}_{n,k}$, by Theorem 4.1, the corresponding homogeneous superspace $\mathbb{C}(K_{n,k}\backslash G)$ is generated by

$$C_{ab} = \sum_{c=m+n-k+1}^{m+n} t_{ca}\overline{t_{cb}}, \quad a,b\in\mathbf{I}.$$

Note that $[c]=1$. As in Theorem 5.1, we can show that $\mathbb{C}[C_{ab}]$ forms a polynomial algebra in one variable if $[a]=[b]=1$; and if $[a]=[b]=0$, then $(C_{ab})^{k+1}=0$ and $(C_{ab})^k\neq 0$. Recall that by Proposition 4.1, we always have $(C_{ab})^2=0$ if $[a]+[b]=1$. The subalgebra of spherical functions $\mathbb{C}[K_{n,k}\backslash G/K_{n,k}]$ is generated by

$$C = \sum_{c,a=m+n-k+1}^{m+n} t_{ca}\overline{t_{ca}},$$

and forms a polynomial algebra in one variable. Similarly, for $\mathfrak{k}_{m,k}$, the symmetric superspace $\mathbb{C}(K_{m,k}\backslash G)$ is generated by

$$C_{ab} = \sum_{c=1}^{k} t_{ca}\overline{t_{cb}}, \quad a,b \in \mathbf{I}.$$

If $[a]=[b]=0$, then $\mathbb{C}[C_{ab}]$ forms a polynomial algebra in one variable, and if $[a]=[b]=1$, then $(C_{ab})^{k+1}=0$ and $(C_{ab})^{k}\neq 0$. The subalgebra of spherical functions $\mathbb{C}(K_{m,k}\backslash G/K_{m,k})$ is generated by

$$C = \sum_{c,a=1}^{k} t_{ca}\overline{t_{ca}},$$

as a polynomial algebra. To summarize, we have

Theorem 5.2: (1) *If $m\leqslant n$, then there is an onto algebra homomorphism,*

$$\phi:\mathbb{C}(K_{n,k}\setminus G) \to \mathbb{C}(K_{m,k}\setminus G)$$

which induces an isomorphism $\mathbb{C}(K_{n,k}\backslash G/K_{n,k})\to\mathbb{C}(K_{m,k}\backslash G/K_{m,k})$.

(2) *For each $1\leqslant k< n$, there is an onto algebra homomorphism,*

$$\phi_{k+1,k}:\mathbb{C}(K_{n,k+1}\setminus G) \to \mathbb{C}(K_{m,k}\setminus G)$$

which induces an isomorphism $\mathbb{C}(K_{n,k+1}\backslash G/K_{n,k+1})\to\mathbb{C}(K_{n,k}\backslash G/K_{n,k})$.

Proof: For (1), we just need to note that any relation among the C_{ab} holds for both algebras by symmetry. For (2), let the generators of $\mathbb{C}(K_{n,k}\backslash G)$ described above be $C_{ab}(k)$ $(a,b\in\mathbf{I},1\leqslant k\leqslant n)$, and define $\phi_{k+1,k}:\mathbb{C}(K_{n,k+1}\backslash G)\to\mathbb{C}(K_{m,k}\backslash G)$ by $\phi_{k+1,k}(C_{ab}(k+1))=[(k+1)/k]C_{ab}(k)$. ■

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