Eur. Phys. J. C (2010) 69: 271–279
DOI 10.1140/epjc/s10052-010-1342-2

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Gravitational collapse of spherically symmetric stars in noncommutative general relativity

Wen Sun[1], **Ding Wang**[2,a], **Naqing Xie**[3,b], **R.B. Zhang**[4,c], **Xiao Zhang**[2,d]

[1]Lianyungang Teachers College, Jiangsu 222006, China
[2]Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[3]Institute of Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, China
[4]School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia

Received: 14 April 2010 / Revised: 30 April 2010 / Published online: 27 May 2010

Abstract Gravitational collapse of a class of spherically symmetric stars is investigated. We quantise the geometries describing the gravitational collapse by a deformation quantisation procedure. This gives rise to noncommutative spacetimes with gravitational collapse.

1 Introduction

Gravitational collapse [1–5] is one of the most dramatic phenomena in the universe. When the pressure is not sufficient to balance the gravitational attraction inside a star, the star undergoes sudden gravitational collapse possibly accompanied by a supernova explosion, reducing to a super dense object such as a neutron star or black hole.

Theoretical investigations predicting gravitational collapse were carried out in the early 1930s in the groundbreaking work of Chandrasekhar [6, 7]. In 1939, Oppenheimer and Snyder [8] investigated the collapse process of ideal spherically symmetric stars equipped with the Tolman metric [9] $-d\tau^2 + e^{\bar{\omega}(\tau,R)}\,dR^2 + e^{\omega(\tau,R)}(d\theta^2 + \sin^2\theta\,d\psi^2)$. When the energy-momentum of an ideal star is assumed to be given by perfect fluids, Tolman's metric allows for the case of dust which has zero pressure. In the dust case, Oppenheimer and Snyder solved the Einstein field equations by further assuming that the energy density is constant. They showed that stars above the Tolman–Oppenheimer–Volkoff mass limit [8, 10] (approximately three solar masses) would collapse into black holes for reasons given by Chandrasekhar. The work of Oppenheimer and Snyder also marked the beginning of the modern theory of black holes.

The purpose of the present paper is to investigate gravitational collapse in noncommutative general relativity. We shall work within the framework of the noncommutative Moyal geometry developed in [11–13]. Gravitational collapse is in principle understood classically, however, quantum effects will become important at the final stage of the collapse, especially if a star collapses into a black hole. As there still lacks a consist theory of quantum gravity, it is useful to incorporate some quantum effects into gravity by deforming general relativity. Much effort has been made in this area in recent years, resulting in several tentative proposals for noncommutative general relativity [11–21]. Much work has been done to investigate noncommutative corrections to black holes, and we refer to [12, 23–29] for details.

On a cautionary note, we should mention that the widely cited papers [16, 17] were shown in [22] to yield results entirely different from the low-energy limit of string theory. If the cause of this is not mathematical imprecisions in [16, 17], then it may be an indication that there are flaws in the rationale of these papers, as the low-energy limit of string theory for an appropriate choice of vacuum is physically realistic.

In the present paper, we first investigate static interior solutions of a class of spherically symmetric stars equipped with the Misner–Zapolsky metric [34] (see (2.1)) and have energy-momentum given by perfect fluids (see (2.2)). The energy densities of the stars are taken to be decreasing functions instead of (physically unrealistic) constant functions. New solutions of the Einstein field equations are obtained, and their singularities signalling gravitational collapse are discussed. We feel that these results are interesting by themselves even from the point of view of classical gravity.

We then quantise the metrics obtained by the deformation quantisation procedure developed in [11–13]. This gives rise to noncommutative spacetimes with static interior singularity.

[a] e-mail: wangding@amss.ac.cn
[b] e-mail: nqxie@fudan.edu.cn
[c] e-mail: ruibin.zhang@sydney.edu.au
[d] e-mail: xzhang@amss.ac.cn

Finally, we quantise the dust solutions studied by Oppenheimer and Snyder [8]. It gives rise to noncommutative dynamical gravitational collapse. As far as we are aware, gravitational collapse in the noncommutative setting has not been studied before.

2 Spherically symmetric stars

Suppose that a spherically symmetric star is equipped with the following metric

$$g = -e^{2\alpha(t,r)}\,dt^2 + e^{2\beta(t,r)}\,dr^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\psi^2\big), \tag{2.1}$$

and its energy-momentum is given by perfect fluids with

$$T_{\mu\nu} = (\varepsilon + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{2.2}$$

where ε is the energy density, p the pressure and the vector U is the 4-velocity of the fluid elements. This metric was studied by Misner and Zapolsky [34] for neutron star models. Its physical properties differ considerably from those of the Tolman metric [9]. In particular, it rules out dust solutions (with $p = 0$ but $\varepsilon \neq 0$). We shall show that, for general spherical symmetric metric (2.1), if the energy density is independent of the time t, then the metric must be static. We shall also study the cases that energy densities are given by step functions and quadratic decay functions.

We mention that in some textbooks, e.g. [2, 3, 5], the metric (2.1) was discussed in the static case with functions α, β depending on r only. The appearance of a curvature singularity was interpreted as an indication of gravitational collapse.

We choose the co-moving coordinates and take the four-velocity U to be pointing in the timelike direction, that is, $U_1 = U_2 = U_3 = 0$. We also normalise U by setting $U^\nu U_\nu = -1$. Now the components of energy-momentum tensor in the coordinates (t, r, θ, ψ) are given by

$$T_{\mu\nu} = \begin{pmatrix} e^{2\alpha}\varepsilon & & & \\ & e^{2\beta}p & & \\ & & r^2 p & \\ & & & r^2\sin^2\theta p \end{pmatrix}.$$

Denote the Einstein tensor by $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$. The Einstein field equations give that [4]

$$G^t_t = e^{-2\beta}\left(\frac{1}{r^2} - \frac{2\partial_r\beta}{r}\right) - \frac{1}{r^2} = -\varepsilon,$$

$$G^r_r = e^{-2\beta}\left(\frac{1}{r^2} + \frac{2\partial_r\alpha}{r}\right) - \frac{1}{r^2} = p,$$

$$G^r_t = e^{-2\beta}\frac{2\partial_t\beta}{r} = 0,$$

$$\begin{aligned} G^\theta_\theta &= G^\psi_\psi \\ &= e^{-2\beta}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 + \frac{1}{r}(\partial_r\alpha - \partial_r\beta) - \partial_r\alpha\partial_r\beta\right) \\ &\quad - e^{-2\alpha}\big(\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\big) = p. \end{aligned}$$

As a consequence of the Bianchi identity, we have the Tolman–Oppenheimer–Volkoff (TOV) equation [1, 9, 10]

$$(\varepsilon + p)\partial_r\alpha + \partial_r p = 0. \tag{2.3}$$

We now show that the metric (2.1) does not allow any dust solution: the vanishing of the pressure p implies the vanishing of the energy density ε. In fact, (2.3) implies either $\varepsilon = 0$ or $\partial_r\alpha = 0$ if $p = 0$. In the latter case the G^r_r equation gives $\beta = 0$, thus the G^t_t equation gives $\varepsilon = 0$. By the Birkhoff theorem, (2.1) must be Schwarzschild in this case. This is different from the Tolman metric, for which the dust solution was constructed by Oppenheimer and Snyder [8].

It is straightforward to show that the G^θ_θ and G^ψ_ψ equations are consequence of those of G^t_t, G^r_r, G^r_t and the TOV equation (2.3) (see, e.g., [2, 3, 5] for the proof of this fact in the static case). In fact, the G^t_t, G^r_r equations give

$$\partial_r\beta = \frac{1}{2r} + \left(\frac{\varepsilon r}{2} - \frac{1}{2r}\right)e^{2\beta},$$

$$\partial_r\alpha = -\frac{1}{2r} + \left(\frac{pr}{2} + \frac{1}{2r}\right)e^{2\beta}.$$

From the G^r_t equation, we obtain

$$\beta = \beta(r).$$

This implies that

$$\varepsilon = \varepsilon(r). \tag{2.4}$$

From the TOV equation (2.3), we obtain

$$\partial_r^2\alpha = e^{4\beta}\left(\frac{\varepsilon p r^2}{4} + \frac{\varepsilon}{4} - \frac{3p}{4} - \frac{1}{2r^2} - \frac{r^2p^2}{4}\right) + e^{2\beta}\left(\frac{5p}{4} + \frac{\varepsilon}{4}\right) + \frac{1}{2r^2}.$$

Substituting these into the G^θ_θ equation, we find that the left hand side is equal to p, so it is an identity.

Now we show that if the state equation $p = p(\varepsilon)$ holds, the metric (2.1) must be static. In fact, from (2.4), we know that $p = p(r)$. Using the G^r_r equation, we obtain

$$\partial_t\partial_r\alpha = 0.$$

This implies

$$\alpha = f(r) + h(t).$$

Replacing t by $\bar{t} = \int e^h dt$, the metric (2.1) can be rewritten as

$$g = -e^{2f(r)}\, d\bar{t}^2 + e^{2\beta(r)}\, dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\psi^2\big). \quad (2.5)$$

Since $G_{tt} = G_{\bar{t}\bar{t}}(\frac{d\bar{t}}{dt})^2 = e^{2\alpha}\varepsilon$ if and only if $G_{\bar{t}\bar{t}} = e^{2f}\varepsilon$, we conclude that the fields are static in the sense that they do not depend on $\bar{t}$. Thus we can assume $\alpha = \alpha(r)$ by replacing f by α in (2.5).

A spherically symmetric star may be assumed to be a ball of radius r_0 centred at $r = 0$. Then the energy density ε and the pressure p must satisfy

$$\varepsilon = 0 \quad \text{if } r > r_0,$$

$$p = 0 \quad \text{if } r \geq r_0.$$

From now on we assume that the pressure of the star depends on r only. Thus the star is static by the above discussion. Let $e^{-2\beta(r)} = 1 - \frac{2m(r)}{r}$ for some $m(r)$, then

$$m(r) = \frac{1}{2}\big(r - re^{-2\beta}\big). \quad (2.6)$$

The G_r^r equation gives

$$\frac{d\alpha}{dr} = \frac{r^3 p + 2m(r)}{2r(r - 2m(r))}, \quad (2.7)$$

and the G_t^t equation leads to

$$\frac{dm}{dr} = \frac{1}{2} r^2 \varepsilon.$$

If $m(0) \neq 0$, then $e^{2\beta(0)} = 0$ by (2.6) and the metric g degenerates at $r = 0$. Exclude this case, we may assume

$$m(0) = 0. \quad (2.8)$$

Finally, in terms of the contracted Einstein equation, we have

$$R(g) = \varepsilon(r) - 3p(r). \quad (2.9)$$

3 Interior solutions

In the original investigation on spherically symmetric stars equipped with the Tolman metric in [8], the energy density was assumed to be a constant. In view of the fact that the energy density should decrease as r becomes large, Gu [35] studied the spherically symmetric dust with the energy density given by a (decreasing) step function. In this case he found that singularity could only appear away from the origin. The complete classification of the spherically symmetric dust was given by Hu [36]. In this section, we analyze the results of Sect. 2 in detail for two classes of spherically symmetric stars which are equipped with the Misner–Zapolsky metric [34] and have decreasing energy density functions.

3.1 Constant and step function energy densities

We first assume that

$$\varepsilon(r) = \varepsilon = \text{constant} > 0 \quad \text{for } r \leq r_0, \quad (3.1)$$

and the mass function is

$$m(r) = \begin{cases} \frac{\varepsilon}{6} r^3 & \text{for } r \leq r_0 \\ \frac{\varepsilon}{6} r_0^3 & \text{for } r > r_0. \end{cases} \quad (3.2)$$

Then the TOV equation (2.3) with boundary condition $p(r_0) = 0$ gives the following explicit formula for the pressure $p(r)$ inside the star

$$p(r) = \varepsilon \frac{\sqrt{3 - \varepsilon r^2} - \sqrt{3 - \varepsilon r_0^2}}{3\sqrt{3 - \varepsilon r_0^2} - \sqrt{3 - \varepsilon r^2}} \quad \text{for } r \leq r_0. \quad (3.3)$$

By Birkhoff's theorem [1], the exterior metric must be a solution of Schwarzschild type with mass $\frac{\varepsilon r_0^3}{6}$:

$$g_{\text{ext}} = -\left(1 - \frac{\varepsilon r_0^3}{3r}\right) dt^2 + \left(1 - \frac{\varepsilon r_0^3}{3r}\right)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\psi^2\big) \quad (3.4)$$

for $r > r_0$. Solving (2.7) with the continuity condition

$$e^{2\alpha(r_0)} = 1 - \frac{\varepsilon r_0^3}{3}, \quad (3.5)$$

we obtain the interior metric

$$g_{\text{int}} = -\left(\frac{3}{2}\sqrt{1 - \frac{\varepsilon}{3} r_0^2} - \frac{1}{2}\sqrt{1 - \frac{\varepsilon}{3} r^2}\right)^2 dt^2 + \left(1 - \frac{\varepsilon}{3} r^2\right)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\psi^2\big) \quad (3.6)$$

for $r \leq r_0$. Now it is possible that there exist spherically symmetric stars whose energy density ε and radius r_0 satisfy

$$\sqrt{\frac{8}{3\varepsilon}} \leq r_0 < \sqrt{\frac{3}{\varepsilon}}. \quad (3.7)$$

From (2.9), we know that the interior metric g_{int} has singularity at

$$r_* = 3\sqrt{r_0^2 - \frac{8}{3\varepsilon}}, \quad (3.8)$$

where the pressure function $p(r)$ and the scalar curvature R blow up. When $r_0 > \sqrt{\frac{8}{3\varepsilon}}$, the singularity appears at $r_* \neq 0$ and the origin $r = 0$ is regular. However, for $r_0 = \sqrt{\frac{8}{3\varepsilon}}$, we have $r_* = 0$ and the singularity appears at the origin.

In [35], Gu used the following step function as energy density to study the gravitational collapse:

$$\varepsilon(r)=\begin{cases}\lambda\varepsilon & \text{for } 0\le r<A,\\ \mu\varepsilon & \text{for } A\le r<B,\\ \varepsilon & \text{for } B\le r<r_0,\\ 0 & \text{for } r\ge r_0,\end{cases}$$

where $0<A<B<r_0$, ε, λ, μ are three positive constants and $\lambda>1$, $0<\mu<1$. The physical interpretation of the above energy density is that the star has a considerable dense shell.

We shall see that, in certain circumstances, there exists a singularity where the energy density is finite but the pressure blows up.

Set $A=\sqrt[3]{\frac{1-\mu}{\lambda-\mu}}B$. Thus we have

$$\frac{1}{2}\int_0^A \lambda\varepsilon r^2\,dr+\frac{1}{2}\int_A^B \mu\varepsilon r^2\,dr=\frac{1}{2}\int_0^B \varepsilon r^2\,dr. \tag{3.9}$$

Therefore, in the region $B<r<r_0$, the star has mass function

$$\begin{aligned} m(r)&=\frac{1}{2}\int_0^r \varepsilon(s)s^2\,ds\\ &=\frac{1}{2}\int_0^A \lambda\varepsilon s^2\,ds+\frac{1}{2}\int_A^B \mu\varepsilon s^2\,ds+\frac{1}{2}\int_B^r \varepsilon s^2\,ds\\ &=\frac{1}{2}\left(\int_0^B+\int_B^r\right)\varepsilon s^2\,ds=\frac{\varepsilon}{6}r^3\end{aligned}$$

and the total mass $M=\frac{\varepsilon}{6}r_0^3$. By the Birkhoff theorem [1], outside the star, the spacetime metric is the Schwarzschild solution (3.4).

Similar to the constant energy density case [8], plugging the mass formula in the region $B<r<r_0$ back into the TOV (2.3) with boundary condition $p(r_0)=0$, we see that the pressure has the same expression (3.3). If

$$r_*=3\sqrt{r_0^2-\frac{8}{3\varepsilon}}>B,$$

there is also a spacetime metric singularity where the energy density is finite but the pressure and the scalar curvature blow up.

3.2 Quadratically decaying energy density functions

In [34], Misner and Zapolsky found an exact solution of the TOV equation (2.3) when the energy density function decays quadratically. However, their solution does not satisfy the zero pressure condition $p(r_0)=0$ on the boundary $r=r_0$ of the star. Aided by information on the Riccati and Bernoulli equations, Zhong [37] obtained an exact interior solution for the perfect fluid sphere. In both of the above cases, the energy density is singular at the origin $r=0$ and the physics for such a case is somewhat unclear. In this subsection, by modifying the energy density function to be finite near the origin, we propose a new model in which the energy density is a decreasing function. We shall see that, inside the star, there exists a real spacetime singularity and thus the origin is not naked.

Let ε and λ be two small positive constants such that $0<\varepsilon<\frac{1}{2}$. Define the energy density function by

$$\varepsilon(r)=\begin{cases}\lambda & \text{for } 0<r\le\sqrt{\frac{3\varepsilon}{\lambda}},\\ \frac{\varepsilon}{r^2} & \text{for } \sqrt{\frac{3\varepsilon}{\lambda}}<r\le r_0,\\ 0 & \text{for } r>r_0.\end{cases}$$

Note that $\varepsilon(r)$ is a decreasing function and

$$\lim_{r\to\sqrt{\frac{3\varepsilon}{\lambda}}^-}\varepsilon(r)=\lambda>\frac{\lambda}{3}=\lim_{r\to\sqrt{\frac{3\varepsilon}{\lambda}}^+}\varepsilon(r).$$

Thus the mass function is

$$m(r)=\begin{cases}\frac{\lambda}{6}r^3 & \text{for } 0<r\le\sqrt{\frac{3\varepsilon}{\lambda}},\\ \frac{\varepsilon}{2}r & \text{for } \sqrt{\frac{3\varepsilon}{\lambda}}<r\le r_0,\\ \frac{\varepsilon r_0}{2} & \text{for } r>r_0.\end{cases}$$

Similar analysis leads to the result that, outside the star (for $r>r_0$), the spacetime metric is the Schwarzschild solution

$$\begin{aligned} g_{\text{ext}}=&-\left(1-\frac{\varepsilon r_0}{r}\right)dt^2+\left(1-\frac{\varepsilon r_0}{r}\right)^{-1}dr^2\\ &+r^2\left(d\theta^2+\sin^2\theta\,d\psi^2\right)\end{aligned} \tag{3.10}$$

with the total mass $\frac{\varepsilon r_0}{2}$. In the region $\sqrt{\frac{3\varepsilon}{\lambda}}<r<r_0$, the metric reads

$$\begin{aligned} g_{\text{int}}=&-e^{2\alpha(r)}\,dt^2+\frac{1}{1-\varepsilon}\,dr^2\\ &+r^2\left(d\theta^2+\sin^2\theta\,d\psi^2\right),\end{aligned} \tag{3.11}$$

where $e^{\alpha(r)}$ can be described as follows. Let

$$\begin{aligned} q_+&=\sqrt{1-\varepsilon}+\sqrt{1-2\varepsilon},\\ q_-&=\sqrt{1-\varepsilon}-\sqrt{1-2\varepsilon}.\end{aligned}$$

Then

$$\begin{aligned} e^{\alpha(r)}=&\frac{r^{1+\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}r_0^{-1+\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}}{4\sqrt{1-2\varepsilon}}\\ &\times\left(q_+^2 r^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}-q_-^2 r_0^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}\right).\end{aligned}$$

Moreover, (2.3) can be integrated with boundary condition $p(r_0)=0$ to yield the pressure

$$p(r)=\frac{\varepsilon^2\left(r^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}-r_0^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}\right)}{r^2\left(q_+^2 r^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}-q_-^2 r_0^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}\right)}, \tag{3.12}$$

which is valid in the region $\sqrt{\frac{3\varepsilon}{\lambda}}<r<r_0$ and is obviously positive. The pressure $p(r)$ has finite limit as $r\to\sqrt{\frac{3\varepsilon}{\lambda}}^{+}$.

The energy density $\varepsilon(r)$ is discontinuous at $r=\sqrt{\frac{3\varepsilon}{\lambda}}$, but we require the pressure be continuous at this point. By contracting the Einstein equation (2.9), we obtain

$$\check{P}_-:=\lim_{r\to\sqrt{\frac{3\varepsilon}{\lambda}}^-}p(r)=\frac{2\lambda}{9}+\lim_{r\to\sqrt{\frac{3\varepsilon}{\lambda}}^+}p(r). \tag{3.13}$$

In the region $0<r<\sqrt{\frac{3\varepsilon}{\lambda}}$, the pressure $p(r)$ satisfies the equation

$$\frac{dp}{(\lambda+p(r))(\lambda+3p(r))}=\frac{dr}{2(\lambda r^2-3)} \tag{3.14}$$

with boundary condition

$$p|_{r=\sqrt{\frac{3\varepsilon}{\lambda}}}=\frac{2\lambda}{9}+\frac{\varepsilon^2\left[\left(\frac{3\varepsilon}{\lambda}\right)^{-\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}-r_0^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}\right]}{\frac{3\varepsilon}{\lambda}\left[q_+^2\left(\frac{3\varepsilon}{\lambda}\right)^{-\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}-q_-^2 r_0^{-2\sqrt{\frac{1-2\varepsilon}{1-\varepsilon}}}\right]}.$$

As in the case of constant energy density, one can obtain an explicit solution of (3.14) with singularity at

$$r_*=\sqrt{\frac{3}{\lambda}-\frac{9(3-3\varepsilon)(\check{P}_-+\lambda)^2}{\lambda(3\check{P}_-+\lambda)^2}}. \tag{3.15}$$

Curvature tensors are finite but discontinuous at $r=\sqrt{\frac{3\varepsilon}{\lambda}}$. So the point $r=\sqrt{\frac{3\varepsilon}{\lambda}}$ is not a spacetime singularity.

Note that the density function in reference [37] is of the form $\rho=\frac{3c^2}{56\pi Gr^2}$ [37, (4)], where c and G denote the speed of light and Newton's constant respectively. Thus, the coefficient of $\frac{1}{r^2}$ is fixed and the only free parameter is the star radius r_0. In our context, both ε and r_0 are free parameters. If we set the parameter $\varepsilon=\frac{3}{7}$, then the exterior Schwarzschild solution (3.10) with the total mass $\frac{\varepsilon r_0}{2}=\frac{3r_0}{14}$ can be glued to the interior solution (3.11) at the boundary $r=r_0$ of the star, i.e.

$$g=-\frac{4}{7}dt^2+\frac{7dr^2}{4}+r^2\left(d\theta^2+\sin^2\theta\, d\psi^2\right).$$

This recovers (27) and (28), the particular case considered in [37].

4 Quantising interior solutions

In this section we quantise the metrics (3.6) (see Sect. 3) following the approach to noncommutative general relativity developed in [11–13]. The purpose of our investigation is to examine possible effects of spacetime noncommutativity by considering simple models.

Note that the interior metrics (3.6) and (3.11) for spherically symmetric stars can both be written as

$$\begin{aligned} g_{\mathrm{int}}&=-a^2(r)\,dt^2+b^2(r)\,dr^2\\ &\quad+r^2\left(d\theta^2+\sin^2\theta\, d\psi^2\right),\quad r\le r_0, \end{aligned} \tag{4.1}$$

for appropriate functions $a(r)$ and $b(r)$ where

$$\begin{aligned} &a(r_*)=0, \quad a'(r_*)\neq 0, \quad a''(r_*)\neq 0,\\ &b(r_*)\neq 0, \quad b'(r_*)\neq 0 \end{aligned} \tag{4.2}$$

for some $r_*<r_0$.

To carry out the deformation quantisation (in the sense of [11–13]) of the metric (4.1), we need first to specify a Moyal algebra. Denote $x^0=t$, $x^1=r$, $x^2=\theta$ and $x^3=\phi$. We deform the algebra of functions in these variables by imposing the Moyal product

$$(f*g)(x)=f(x)\exp^{\frac{\bar{h}}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}}g(x)$$

with the following anti-symmetric matrix

$$\left(\theta^{\mu\nu}\right)_{\mu,\nu=0}^3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}, \tag{4.3}$$

where $\bar{h}$ is the deformation parameter, which may be regarded as related to the Planck constant.

Some comments are in order. Noncommutativity of time coordinate results in violation of unitarity and causality for quantum field theories defined on flat spacetimes as well as spacetimes with compact spatial submanifolds [30] (also see [31]; for a discussion in the context of string theory, see [32]). In order to retain basic principles of quantum physics such as unitarity and causality, we have to keep the time coordinate commutative, and the Ansatz (4.3) enforces this. Note also that the Ansatz (4.3) leads to the simplest possible model of spacetime noncommutativity.

Following [11], we choose the following embedding

$$\begin{aligned} &X^1=a(r)\sin t, \quad X^2=a(r)\cos t,\\ &X^3=f(r), \quad X^4=r\sin\theta\cos\phi,\\ &X^5=r\sin\theta\sin\phi, \quad X^6=r\cos\theta, \end{aligned} \tag{4.4}$$

where $f(r)$ is related to the functions $a(r)$ and $b(r)$ through the differential equation

$$(f')^2 + 1 = (a')^2 + b^2.$$

Here the rationale is much the same as in the classical theory of surfaces, which can be treated as 2-dimensional geometries embedded in 3-dimensional Euclidean space. The surfaces themselves are determined by the embeddings, and their differential geometry can be studied by analysing the embeddings in a very concrete manner using elementary techniques (see, e.g., [33] for an introduction to the theory of surfaces).

The embedding (4.4) determines the noncommutative geometry of the spacetime, which we now analyze. The quantum deformation of the metric (4.1) is defined by

$$\begin{aligned}\mathbf{g}_{\mu\nu} &= -\partial_\mu X^1 * \partial_\nu X^1 - \partial_\mu X^2 * \partial_\nu X^2 \\ &\quad + \partial_\mu X^3 * \partial_\nu X^3 + \partial_\mu X^4 * \partial_\nu X^4 \\ &\quad + \partial_\mu X^5 * \partial_\nu X^5 + \partial_\mu X^6 * \partial_\nu X^6.\end{aligned} \tag{4.5}$$

Note that in the classical commutative limit with $\bar{h} = 0$, (4.5) reduces to

$$g_{\mu\nu} = -\sum_{i=1}^{2} \partial_\mu X^i \partial_\nu X^i + \sum_{j=3}^{6} \partial_\mu X^j \partial_\nu X^j,$$

which indeed recovers the metric (4.1) as one can verify.

Lengthy computations yield the following result for the noncommutative metric (4.5):

$$\begin{aligned}
&\mathbf{g}_{00} = -a^2(r),\\
&\mathbf{g}_{01} = \mathbf{g}_{10} = \mathbf{g}_{02} = \mathbf{g}_{20} = \mathbf{g}_{03} = \mathbf{g}_{30} = 0,\\
&\mathbf{g}_{11} = b^2(r) + (\sin^2\theta - \cos^2\theta)\sinh^2\bar{h},\\
&\mathbf{g}_{12} = \mathbf{g}_{21} = 2r\sin\theta\cos\theta\sinh^2\bar{h},\\
&\mathbf{g}_{13} = -\mathbf{g}_{31} = -2r\sin\theta\cos\theta\sinh\bar{h}\cosh\bar{h},\\
&\mathbf{g}_{22} = r^2\big[1 - (\sin^2\theta - \cos^2\theta)\sinh^2\bar{h}\big],\\
&\mathbf{g}_{23} = -\mathbf{g}_{32} = r^2(\sin^2\theta - \cos^2\theta)\sinh\bar{h}\cosh\bar{h},\\
&\mathbf{g}_{33} = r^2\big[\sin^2\theta + (\sin^2\theta - \cos^2\theta)\sinh^2\bar{h}\big].
\end{aligned} \tag{4.6}$$

Given the general theory of [11], the computation of the connection and Riemannian curvature of the noncommutative metric (4.6) is in principle straightforward though very laborious. We shall not spell out the complete results here. However, we shall need the noncommutative scalar curvature, which can be expressed in the form

$$\mathbf{R} = \frac{A_0}{2r^2A^3} - \frac{a'(r)A_1}{ra(r)A^2} - \frac{a''(r)A_2}{a(r)A}, \tag{4.7}$$

where A, A_0, A_1 and A_2 are given in the Appendix.

For the sake of being concrete, we shall only discuss the noncommutative singularity for quantum analogue of the metric (3.6). As we have seen in Sect. 3, singularity occurs essentially in the region with constant energy density. Similarly in the present setting, if the energy density ε and radius r_0 of a spherically symmetric satisfy (3.7), noncommutative singularity also occurs for the metric (4.1) at

$$r_* = 3\sqrt{r_0^2 - \frac{8}{3\varepsilon}}.$$

Indeed, in this case,

$$\begin{aligned}a(r) &= \frac{3}{2}\sqrt{1 - \frac{\varepsilon}{3}r_0^2} - \frac{1}{2}\sqrt{1 - \frac{\varepsilon}{3}r^2},\\ b(r) &= \left(1 - \frac{\varepsilon}{3}r^2\right)^{-\frac{1}{2}},\end{aligned} \tag{4.8}$$

we obtain

$$\mathbf{R} = B_1 - \frac{B_2}{B}, \tag{4.9}$$

where B, B_1 and B_2 are given in the appendix.

Note that $B|_{r=r_*} = 0$. Inspecting the formulae for $B_1|_{r=r_*}$ and $B_2|_{r=r_*}$ in the Appendix, we see that $B_1|_{r=r_*}$ is finite and $B_2|_{r=r_*} \neq 0$. Thus we have

$$\mathbf{R}(r_*) = \infty, \tag{4.10}$$

and hence $r = r_*$ is a singularity in the noncommutative spacetime.

Remark 4.1 If $r_0^2\varepsilon > \frac{8}{3}$, then $r = 0$ is a regular point of the quantum deformed metric for (3.6). In this case, the noncommutative scalar curvature

$$\mathbf{R} = -\frac{\varepsilon(\cosh 2\bar{h} - 2\sqrt{9 - 3r_0^2\varepsilon} + 3)\operatorname{sech}^4\bar{h}}{\sqrt{9 - 3r_0^2\varepsilon} - 1}$$

is finite. When $r_0 = \sqrt{\frac{8}{3\varepsilon}}$, the origin $r = 0$ is singular.

5 Noncommutative gravitational collapse

In this section, we quantise the dust solutions [8] and study noncommutative gravitational collapse. While our method for quantisation is much the same as in Sect. 4, a new feature is that the quantised dust solutions have an explicit time dependence and their time evolutions are thus clear.

By replacing τ by t, and R by r, the Tolman metric studied in [8] can be written as

$$ds^2 = -dt^2 + (1 - ct)^{4/3}\big[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\big] \tag{5.1}$$

with $c = 3r_0^{\frac{1}{2}} R_b^{-\frac{3}{2}}$, where r_0 is the gravitational radius and R_b is the radius of the star or some micro object. This spacetime can be embedded into a 5-dimensional flat Minkowski spacetime via

$$\begin{aligned}
X^1 &= \frac{9(1-ct)^{4/3}}{32c^2} + \left(\frac{r^2}{4}+1\right)(1-ct)^{2/3},\\
X^2 &= \frac{9(1-ct)^{4/3}}{32c^2} + \left(\frac{r^2}{4}-1\right)(1-ct)^{2/3},\\
X^3 &= (1-ct)^{2/3} r\cos\phi\sin\theta,\\
X^4 &= (1-ct)^{2/3} r\sin\theta\sin\phi,\\
X^5 &= (1-ct)^{2/3} r\cos\theta.
\end{aligned} \tag{5.2}$$

As in Sect. 4, we deform the algebra of functions in the variables r, t, π and θ into a Moyal algebra $\mathcal{A}$ defined by the anti-symmetric matrix (4.3). Now we consider the noncommutative geometry embedded in $\mathcal{A}^5$ by (5.2). The noncommutative metric of the embedded noncommutative geometry (defined in the standard way [11]) yields a quantum deformation of the metric (5.1):

$$\begin{aligned}
\mathbf{g}_{\mu\nu} &= -\partial_\mu X^1 * \partial_\nu X^1 + \partial_\mu X^2 * \partial_\nu X^2\\
&\quad + \partial_\mu X^3 * \partial_\nu X^3 + \partial_\mu X^4 * \partial_\nu X^4\\
&\quad + \partial_\mu X^5 * \partial_\nu X^5,
\end{aligned} \tag{5.3}$$

which can be computed explicitly. We have

$$\mathbf{g}_{11} = -\frac{4c^2 r^2 \cos 2\theta \sinh^2\bar{h}}{9(1-ct)^{2/3}} - 1,$$

$$\mathbf{g}_{12} = \mathbf{g}_{21} = \frac{2}{3} cr(1-ct)^{1/3}\cos 2\theta \sinh^2\bar{h},$$

$$\mathbf{g}_{13} = \mathbf{g}_{31} = -\frac{4}{3} cr^2(1-ct)^{1/3}\cos\theta\sin\theta\sinh^2\bar{h},$$

$$\mathbf{g}_{14} = -\mathbf{g}_{41} = \frac{1}{3} cr^2(1-ct)^{1/3}\sin 2\theta\sinh 2\bar{h},$$

$$\mathbf{g}_{22} = (1-ct)^{4/3}\left(1-\cos 2\theta\sinh^2\bar{h}\right),$$

$$\mathbf{g}_{23} = \mathbf{g}_{32} = r(1-ct)^{4/3}\sin 2\theta\sinh^2\bar{h},$$

$$\mathbf{g}_{24} = -\mathbf{g}_{42} = -2r(1-ct)^{4/3}\cos\theta\cosh\bar{h}\sin\theta\sinh\bar{h},$$

$$\mathbf{g}_{33} = r^2(1-ct)^{4/3}\left(\cos 2\theta\sinh^2\bar{h}+1\right),$$

$$\mathbf{g}_{34} = -\mathbf{g}_{43} = -\frac{1}{2} r^2(1-ct)^{4/3}\cos 2\theta\sinh 2\bar{h},$$

$$\mathbf{g}_{44} = -\frac{1}{2} r^2(1-ct)^{4/3}(\cos 2\theta\cosh 2\bar{h}-1).$$

The noncommutative scalar curvature is given by

$$\mathbf{R} = \frac{4c^2\cosh^2\bar{h}}{(1-ct)^{4/3}}\frac{C_1}{C^3} \tag{5.4}$$

where C and C_1 are given in the Appendix.

We shall consider aspects of the noncommutative spacetime by examining the behaviour of the scalar curvature as time increases by following an approach adopted in [5] in the classical context. When time approaches values where $\mathbf{R}\to\infty$, the radius of the stellar object reduces to zero, and this is an indication of gravitational collapse [5]. Obviously this only provides a snapshot, nevertheless, it enables us to gain some understanding of gravitational collapse in the non-commutative setting. A full treatment of the time evolution of stellar objects ending at gravitational collapses in noncommutative geometry will be given in a future publication.

Let us regard $\bar{h}$ as a real number and make the (physically realistic) assumption that $\bar{h}$ is positive but close to zero. Now if t is significantly smaller than $\frac{1}{c}$ compared to $\bar{h}$, that is, $\frac{1}{c} - t \gg \bar{h}$, both the noncommutative metric and noncommutative scalar curvature $\mathbf{R}$ are finite, and there is non-singularity in the noncommutative spacetime. Thus the stellar object described by the noncommutative geometry behaves much the same as the corresponding classical object.

When $t = t_* := \frac{1}{c}$, we have $\mathbf{R}|_{\mathbf{t}_*=\frac{1}{\mathbf{c}}} = \infty$ and the radius of the stellar object reduces to zero. This is the time when gravitational collapse happens in the usual classical setting.

However, in the noncommutative case, singularities of the scalar curvature already appear before t_*. Indeed, when time reaches

$$\begin{aligned}
t(r,\theta) &= \frac{1}{c} - \frac{\sqrt{8}}{27} c^2 r^3 (2\cos 2\theta + \cosh 2\bar{h} + 3)^{3/2}\frac{\sinh^3\bar{h}}{\cosh^6\bar{h}}\\
&\cong \frac{1}{c} - \frac{8}{27} c^2 r^3 (\cos 2\theta + 2)^{3/2}\bar{h}^3.
\end{aligned}$$

C vanishes and $C_1/(1-ct)^{4/3}$ is finite of order 0 in $\bar{h}$. Thus the scalar curvature tends to infinity for all $t(r,\theta)$ and the noncommutative spacetime becomes singular.

This indicates that in the noncommutative setting, gravitational collapse happens within a certain range of time because of the quantum effects captured by the non-commutativity of spacetime. This is fully consistent with usual expectations of quantum mechanics. However, effect of non-commutativity only starts to appear at third order of $\bar{h}$.

Acknowledgements N. Xie wishes to thank Profs. C.H. Gu and H.S. Hu for their encouragement. X. Zhang wishes to thank the School of Mathematics and Statistics, University of Sydney for the hospitality during his visits when part of this work was carried out. Partial financial support from the Australian Research Council, National Science Foundation of China (grants 10421001, 10725105, 10731080, 10801036), NKBRPC (2006CB805905) and the Chinese Academy of Sciences is gratefully acknowledged.

Appendix: Some formulae used in the main text

This appendix spells out some lengthy formulae which have been used in the main body of the paper. The quantities A, A_0, A_1 and A_2 used in (4.7) are given by

$$\begin{aligned}
A &= 2\big(\cos 2\theta \sinh^2 \bar{h} - 1\big) b(r)^2 \\
&\quad - (2\cos 2\theta + \cosh 2\bar{h} + 3)\sinh^2 \bar{h}, \\
A_0 &= -8\big(3\cos 2\theta \sinh^2 2\bar{h} + 2\cosh 2\bar{h} + 2\cosh 4\bar{h}\big) b(r)^6 \\
&\quad + 2\big[\big(-8\cos 4\theta \sinh^4 \bar{h} + 92\cosh 2\bar{h} \\
&\quad - 19(\cosh 4\bar{h} + 3)\big)\cosh^2 \bar{h} \\
&\quad + 4\cos 2\theta (7 - 4\cosh 2\bar{h})\sinh^2 2\bar{h}\big] b(r)^4 \\
&\quad + 2r\big[\big(8\cos 4\theta \sinh^4 \bar{h} - 28\cosh 2\bar{h} \\
&\quad + \cosh 4\bar{h} - 5\big)\cosh^2 \bar{h} \\
&\quad + 2\cos 2\theta (3\cosh 2\bar{h} - 1)\sinh^2 2\bar{h}\big] b'(r) b(r)^3 \\
&\quad - 4\big[4\big(\cos 2\theta (3\cosh 2\bar{h} - 23) \\
&\quad - 2\cos 4\theta\big)\cosh^2 \bar{h} \sinh^4 \bar{h} \\
&\quad + (-14\cosh 2\bar{h} + \cosh 4\bar{h} + 9)\sinh^2 2\bar{h}\big] b(r)^2 \\
&\quad + r\big[\cos 2\theta (3\cosh 4\bar{h} + 13)\sinh^2 2\bar{h} \\
&\quad - 16(\cos 4\theta - \cosh 2\bar{h} + 2)\cosh^2 \bar{h} \sinh^4 \bar{h}\big] b'(r) b(r) \\
&\quad + 16\big[-7\cos 2\theta - \cos 4\theta \\
&\quad + (3\cos 2\theta + 4)\cosh 2\bar{h} - 5\big]\cosh^2 \bar{h} \sinh^4 \bar{h}, \\
A_1 &= 8\cos 2\theta \cosh^2 \bar{h} \sinh^4 \bar{h} + (3\cosh 2\bar{h} + 1)\sinh^2 2\bar{h} \\
&\quad + 2\big[(6\cosh 2\bar{h} + 2)\cosh^2 \bar{h} + \cos 2\theta \sinh^2 2\bar{h}\big] b(r)^2 \\
&\quad - 8r\big(\cos 2\theta \sinh^2 \bar{h} - 1\big)^2 b(r) b'(r), \\
A_2 &= 4\big(\cos 2\theta \sinh^2 \bar{h} - 1\big).
\end{aligned}$$

The following quantities appeared in (4.9):

$$\begin{aligned}
B &= -3 + \varepsilon r^2 + 3\sqrt{\big(3 - r^2\varepsilon\big)\big(3 - r_0^2\varepsilon\big)}, \\
B_1 &= -\frac{\varepsilon}{2D^3}\big\{\big[-16 r^4 \varepsilon^2 \cos 4\theta \sinh^4 \bar{h} \\
&\quad - 2r^2\varepsilon\big(23\varepsilon r^2 + 9\big) \\
&\quad + \big(\varepsilon r^2\big(68\varepsilon r^2 + 21\big) + 405\big)\cosh 2\bar{h} \\
&\quad + 2\big(\varepsilon r^2\big(9 - 13r^2\varepsilon\big) + 81\big)\cosh 4\bar{h} \\
&\quad + \big(r^2\varepsilon - 3\big)\big(4r^2\varepsilon - 9\big)\cosh 6\bar{h} + 270\big]\cosh^2 \bar{h} \\
&\quad + 2r^2\varepsilon \cos 2\theta (3\cosh 2\bar{h} - 7) \\
&\quad \times \big(\big(r^2\varepsilon - 3\big)\cosh 2\bar{h} - 3 - \varepsilon r^2\big)\sinh^2 2\bar{h}\big\}, \\
B_2 &= -\frac{1}{D^2}\big\{\varepsilon\big(r^2\varepsilon - 3\big)\cosh^2 \bar{h}\big[-8r^2\varepsilon \cos 2\theta \sinh^4 \bar{h} \\
&\quad - r^2\varepsilon + 4\big(\varepsilon r^2 + 9\big)\cosh 2\bar{h} \\
&\quad + \big(9 - 3r^2\varepsilon\big)\cosh 4\bar{h} + 27\big]\big\},
\end{aligned}$$

where

$$D = \big[2r^2\varepsilon \cos 2\theta + \big(r^2\varepsilon - 3\big)(\cosh 2\bar{h} + 3)\big]\sinh^2 \bar{h} - 6.$$

Evaluating D, B_1 and B_2 at

$$r = r_* = 3\sqrt{r_0^2 - \frac{8}{3\varepsilon}},$$

we obtain

$$\begin{aligned}
D|_{r=r_*} &= \big[6\big(3r_0^2\varepsilon - 8\big)\cos 2\theta \\
&\quad + 9\big(r_0^2\varepsilon - 3\big)(\cosh 2\bar{h} + 3)\big]\sinh^2 \bar{h} - 6, \\
B_1|_{r=r_*} &= -\frac{\varepsilon}{2(D|_{r=r_*})^3}\Big\{9\cosh^2 \bar{h} \\
&\quad \times \big[\big(612\varepsilon^2 r_0^4 - 3243\varepsilon r_0^2 + 4341\big)\cosh 2\bar{h} \\
&\quad - 2\big(3\varepsilon\big(39r_0^2\varepsilon - 211\big)r_0^2 + 847\big)\cosh 4\bar{h} \\
&\quad + 3\big(r_0^2\varepsilon - 3\big)\big(12r_0^2\varepsilon - 35\big)\cosh 6\bar{h} \\
&\quad - 2\big(8\big(8 - 3r_0^2\varepsilon\big)^2 \cos 4\theta \sinh^4 \bar{h} \\
&\quad + 3r_0^2\varepsilon\big(69r_0^2\varepsilon - 365\big) + 1433\big)\big] \\
&\quad + 18\sinh^2 2\bar{h}\big[\big(3r_0^2\varepsilon - 8\big)\cos 2\theta (3\cosh 2\bar{h} - 7) \\
&\quad \times \big(-3\varepsilon r_0^2 + 3\big(r_0^2\varepsilon - 3\big)\cosh 2\bar{h} + 7\big)\big]\Big\}, \\
B_2|_{r=r_*} &= -\frac{27\varepsilon(r_0^2\varepsilon - 3)\cosh^2 \bar{h}}{(D|_{r=r_*})^2} \\
&\quad \times \big[8\big(8 - 3r_0^2\varepsilon\big)\cos 2\theta \sinh^4 \bar{h} - 3r_0^2\varepsilon \\
&\quad + 4\big(3r_0^2\varepsilon - 5\big)\cosh 2\bar{h} \\
&\quad - 9\big(r_0^2\varepsilon - 3\big)\cosh 4\bar{h} + 17\big].
\end{aligned}$$

These formulae are used in the derivation of (4.10).

The quantities C and C_1 in (5.4) in the main body of the paper are given by the following formulae:

$$\begin{aligned}
C &= 9(1 - ct)^{2/3}\cosh^4 \bar{h} \\
&\quad - 2c^2 r^2 (2\cos 2\theta + \cosh 2\bar{h} + 3)\sinh^2 \bar{h}, \\
C_1 &= -243(1 - ct)^{4/3}\cosh^8 \bar{h} \\
&\quad + 486(1 - ct)^{4/3}\cosh^6 \bar{h} \\
&\quad - 18c^2 r^2 (1 - ct)^{2/3}\sinh^2 \bar{h} \cosh^4 \bar{h} \\
&\quad \times (2\cos 2\theta - 3\cosh 2\bar{h} - 1) \\
&\quad - 9c^2 r^2 (1 - ct)^{2/3}\sinh^2 \bar{h} \cosh^2 \bar{h}
\end{aligned}$$

$$\times\big(52\cosh 2\bar{h} + 3\cosh 4\bar{h}$$
$$+\cos 2\theta(28\cosh 2\bar{h} + \cosh 4\bar{h} - 13) + 9\big)$$
$$+4c^4r^4\big(4\cos 2\theta(\cosh 2\bar{h} + 15)\sinh^2\bar{h}$$
$$+2\cos 4\theta(\cosh 2\bar{h} - 3) + 38\cosh 2\bar{h}$$
$$+3\cosh 4\bar{h} - 13\big)\sinh^4\bar{h}.$$

References

1. R. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity*, 2nd edn. (McGraw–Hill, New York, 1975)
2. S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, London, 1973)
3. D. Kramer, H. Stephani, E. Herlt, M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, London, 1980)
4. J.L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960)
5. R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984)
6. S. Chandrasekhar, The highly collapsed configurations of a stellar mass. Mon. Not. R. Astron. Soc. **91**, 456–466 (1931)
7. S. Chandrasekhar, The highly collapsed configurations of a stellar mass (second paper). Mon. Not. R. Astron. Soc. **95**, 207–225 (1935)
8. J.R. Oppenheimer, H. Snyder, On continued gravitational contraction. Phys. Rev. **56**, 455–459 (1939)
9. R.C. Tolman, Static solutions of Einstein's field equations for spheres of fluid. Phys. Rev. **55**, 364–373 (1939)
10. J.R. Oppenheimer, G. Volkoff, On massive neutron cores. Phys. Rev. **55**, 374–381 (1939)
11. M. Chaichian, A. Tureanu, R.B. Zhang, X. Zhang, Riemannian geometry of noncommutative surfaces. J. Math. Phys. **49**, 073511 (2008)
12. D. Wang, R.B. Zhang, X. Zhang, Quantum deformations of Schwarzschild and Schwarzschild–de Sitter spacetimes. Class. Quantum Gravity **26**, 085014 (2009), 14 pp.
13. D. Wang, R.B. Zhang, X. Zhang, Exact solutions of noncommutative vacuum Einstein field equations and plane-fronted gravitational waves. Eur. Phys. J. C **64**, 439–444 (2009)
14. A.H. Chamseddine, Commun. Math. Phys. **218**, 283 (2001)
15. A.H. Chamseddine, Phys. Rev. D **69**, 024015 (2004)
16. P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, J. Wess, Class. Quantum Gravity **22**, 3511 (2005)
17. P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Class. Quantum Gravity **23**, 1883 (2006)
18. M. Buric, T. Grammatikopoulos, J. Madore, G. Zoupanos, J. High Energy Phys. **0604**, 054 (2006)
19. M. Burić, J. Madore, Eur. Phys. J. C **58**, 347–353 (2008)
20. S. Marculescu, F. Ruiz Ruiz, Seiberg-Witten maps for SO(1,3) gauge invariance and deformations of gravity. Phys. Rev. D **79**, 025004 (2009)
21. S. Marculescu, F. Ruiz Ruiz, Noncommutative Einstein–Maxwell pp-waves. Phys. Rev. D **74**, 105004 (2006)
22. L. Álvarez-Gaumé, F. Meyer, M.A. Vazquez-Mozo, Nucl. Phys. B **75**, 392 (2006)
23. M. Chaichian, A. Tureanu, G. Zet, Phys. Lett. B **660**, 573 (2008)
24. M. Chaichian, M.R. Setare, A. Tureanu, G. Zet, J. High Energy Phys. **0804**, 064 (2008)
25. B.P. Dolan, K.S. Gupta, A. Stern, Class. Quantum Gravity **24**, 1647 (2007)
26. H.C. Kim, M.I. Park, C. Rim, J.H. Yee, J. High Energy Phys. **10**, 060 (2008)
27. S. Ansoldi, P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. B **645**, 261 (2007)
28. P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. B **632**, 547 (2006)
29. A. Kobakhidze, Phys. Rev. D **79**, 047701 (2009)
30. M. Chaichian, A. Demicheva, P. Presnajder, A. Tureanua, Noncommutative quantum field theory: unitarity and discrete time. Phys. Lett. B **515**, 426–430 (2001)
31. J. Gomis, T. Mehen, Nuclear Phys. B **591**(12), 265–276 (2000)
32. N. Seiberg, L. Susskind, N. Toumbas, J. High Energy Phys. **0006**, 044 (2000)
33. M.P. do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice Hall, Englewood Cliffs, 1976), viii+503 pp.
34. C.W. Misner, H.S. Zapolsky, High-density behavior and dynamical stability of neutron star models. Phys. Rev. Lett. **12**, 635–637 (1964)
35. C.H. Gu, Gravitational collapse of spherical symmetry with non-uniform density. Front. Math. China **2**, 161–168 (2006). Translated from J. Fudan Univ. (Nat. Sci.) **1**, 73–78 (1973)
36. H.S. Hu, Exact solutions of the spherically symmetric gravitational field equations. Front. Math. China **2**, 169–177 (2006). Translated from J. Fudan Univ. (Nat. Sci.) **1**, 92–98 (1974)
37. M.Q. Zhong, Inquiry about the exact interior solution to Einstein field equation for a perfect fluid sphere. Acta Phys. Sin. **52**, 1585–1588 (2003)