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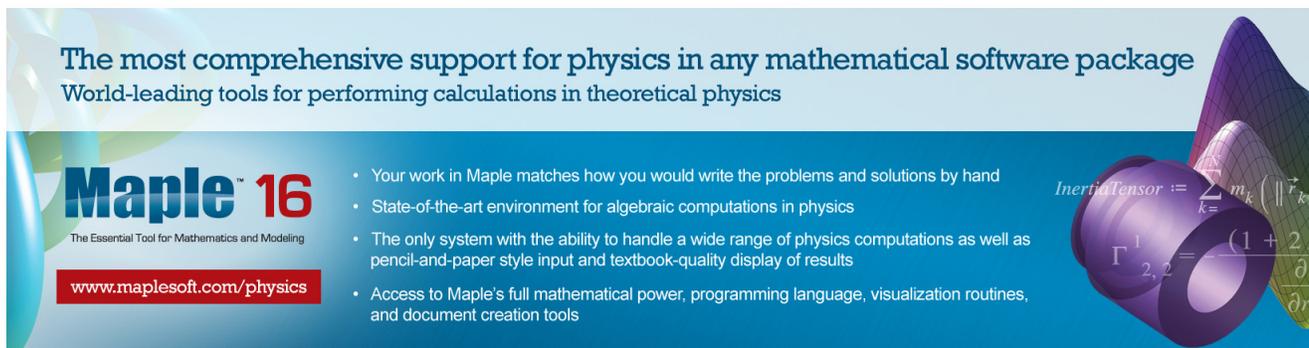
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InertiaTensor := $\sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \right)$
 $\Gamma_{2,2}^1 = \frac{(1 + 2)}{\partial r}$

Extended Poincaré supersymmetry in three dimensions and supersymmetric anyons

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We classify the unitary representations of the extended Poincaré supergroups in three dimensions. Irreducible unitary representations of any spin can appear, which correspond to supersymmetric anyons. Our results also show that all irreducible unitary representations necessarily have physical momenta. This is in sharp contrast to the ordinary Poincaré group that admits in addition irreducible unitary representations with nonphysical momenta, which are discarded on physical grounds. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4732535>]

I. INTRODUCTION

In a series of articles¹⁻⁴ in recent years, Mezincescu and Townsend investigated bosonic strings and superstrings in target spaces of dimension 3, which is the only non-critical dimension where strings can be quantised consistently. They discovered that the spectra of three-dimensional quantum strings always contain anyons (see Ref. 5 for a review). This showed the relevance of anyons (see, e.g., Ref. 6) to string theory for the first time.

The stringy anyons are anyons in the three-dimensional target space, which correspond to unitary representations of the target space Poincaré (super)group with spins which are neither integers nor half integers. Thus, it is essential for the study of stringy anyons to understand such unitary representations. In their papers, Mezincescu and Townsend demonstrated the existence of some unitary representations for the $N = 1$ and $N = 2$ Poincaré supergroups in three dimensions. A systematic understanding of the unitary representations of all the three-dimensional Poincaré supergroups is not only useful for the study of supersymmetric anyons, e.g., in further developing the Mezincescu-Townsend programme, but is also of interest in its own right.

Recall that an influential work⁷ of Strathdee from the 1980s classified unitary representations with integer and half integer spins ≤ 2 of the extended Poincaré supergroups in all dimensions. His aim was to catalogue the supermultiplets relevant to supergravity in dimension 4 and above. From his point of view, the three-dimensional case is degenerate as there is no graviton. It appears that the study of unitary representations of three-dimensional extended Poincaré supergroups is still incomplete.

In this note we systematically investigate the unitary representations of extended Poincaré supergroups in three dimensions. We give a classification of the irreducible unitary representations and construct them using the method of induced representations starting from unitary representations of the underlying Poincaré group.

The irreducible unitary representations of the three-dimensional Poincaré group have been widely studied (see, e.g., Ref. 8). It is known that unitarity occurs for both physical and nonphysical momenta. However, the situation is totally different for Poincaré supergroups: only the positive

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energy unitary representations of the Poincaré group with physical momenta can induce unitary representations of the Poincaré supergroups.

At zero momenta, an irreducible unitary representation of a Poincaré supergroup is nothing more than an irreducible unitary representation of the underlying Poincaré group with the super generators and central charges all act by zero. In the massless case, all central charges must act by zero, and half of the supersymmetries are broken. We gain a complete understanding of the irreducible unitary representations for all N in both cases, so do we also for massive unitary irreducible representations of the $N = 1$ Poincaré supergroup. However, in the massive case, the values of the central charges become crucial in determining the irreducible unitary representations of the extended Poincaré supergroups. For any given spin, we will describe the values which allow unitarity. In particular, for $N = 2$ and 3 , the values of the central charges divided by the mass lie in a unit interval and a unit ball, respectively (see Sec. IV C).

We should point out that a mathematically rigorous (and technically more demanding) treatment of induced representations for Lie supergroups was given in Ref. 9. For general references on the theory of Lie superalgebras, see Refs. 10 and 11.

II. EXTENDED POINCARÉ SUPERGROUPS IN THREE DIMENSIONS

We choose the metric $\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for the three-dimensional Minkowski space M . The Poincaré group is the semi-direct product $O(1, 2) \ltimes \mathbb{R}^{1,2}$ acting on M by

$$(\Lambda, a) : x^\mu \mapsto \Lambda^\mu_\nu x^\nu + a^\mu,$$

for any $x^\mu = (x^0, x^1, x^2) \in M$. Here $\Lambda = (\Lambda^\mu_\nu) \in O(1, 2)$ and $a^\mu = (a^0, a^1, a^2) \in \mathbb{R}^{1,2}$. The action leaves invariant $\|x - y\|^2 = (x - y)^\mu (x - y)_\mu$ for all $x, y \in M$.

For physical applications, we are only interested in the connected component of $O(1, 2)$ containing the identity, defined by $\{\Lambda \in O(1, 2) \mid \det \Lambda = 1, \Lambda^0_0 > 0\}$. This subgroup can be identified with the quotient group $L = SL(2, \mathbb{R})/\mathbb{Z}_2$ of $SL(2, \mathbb{R})$, where $\mathbb{Z}_2 = \{I, -I\}$. Then L acts on $x^\mu \in M$ by

$$\begin{pmatrix} x^0 - x^1 & x^2 \\ x^2 & x^0 + x^1 \end{pmatrix} \mapsto g \begin{pmatrix} x^0 - x^1 & x^2 \\ x^2 & x^0 + x^1 \end{pmatrix} g^T, \quad g \in L. \tag{2.1}$$

Following Ref. 12 we may regard a Lie supergroup as a super Harish-Chandra pair $(G_{\bar{0}}, \mathfrak{g})$, where $G_{\bar{0}}$ is an ordinary Lie group, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra which is a $G_{\bar{0}}$ -module with $\mathfrak{g}_{\bar{0}} = Lie(G_{\bar{0}})$, such that the $\mathfrak{g}_{\bar{0}}$ -action on \mathfrak{g} is the differential of the $G_{\bar{0}}$ -action. Thus, the extended Poincaré supergroup is the super Harish-Chandra pair with $G_{\bar{0}} = L \ltimes \mathbb{R}^{1,2}$ and \mathfrak{g} being the extended super Poincaré algebra in three dimensions.

We largely follow the convention of Ref. 7 for the extended Poincaré superalgebra. Take the following Γ matrices:

$$\Gamma^0 = i\sigma_2, \quad \Gamma^1 = \sigma_1, \quad \Gamma^2 = \sigma_3,$$

which have real entries. Then $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. Let C be the charge conjugation matrix, which may be taken as $C = \Gamma^0$.

We denote by $J_{\mu\nu}$ and P_μ with $\mu, \nu = 0, 1, 2$ the generators of the usual Poincaré algebra in three dimensions, where $J_{\mu\nu}$ are antisymmetric in the indices μ and ν . The commutation relations are

$$[J_{\mu\nu}, J_{\sigma\rho}] = \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\rho} J_{\nu\sigma}, \tag{2.2}$$

$$[J_{\mu\nu}, P_\sigma] = \eta_{\nu\sigma} P_\mu - \eta_{\mu\sigma} P_\nu.$$

In this convention, we have $P_\mu^\dagger = P_\mu$, $J_{0\nu}^\dagger = J_{0\nu}$ and $J_{12}^\dagger = -J_{12}$.

Let $Q_{i,\alpha}$ be the supercharges, where $\alpha = 1, 2$ is the spinor index, and $i = 1, 2, \dots, N$. Denote by Z_{ij} the central charges. Then

$$\{Q_{i,\alpha}, Q_{j,\beta}\} = (\mathcal{P}C)_{\alpha\beta}\delta_{ij} + C_{\alpha\beta}Z_{ij}. \quad (2.3)$$

The central charges are real, i.e., $Z_{ij}^* = Z_{ij}$, and satisfy $Z_{ij} = -Z_{ji}$. The supercharges are also real in the sense that $Q_i = C\bar{Q}_i^T$, where $\bar{Q}_i = Q_i^\dagger\Gamma^0$. This amounts to $Q_{i,\alpha}^* = Q_{i,\alpha}$. Set $\Sigma_{\mu\nu} = \frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$, which are real 2×2 matrices. Then

$$[J_{\mu\nu}, Q_{i\alpha}] = \sum_{\beta=1}^2 (\Sigma_{\mu\nu})_{\alpha\beta} Q_{i,\beta}. \quad (2.4)$$

Furthermore, Z_{ij} , being central, commute with all the elements in the Poincaré superalgebra; the momentum operators commute among themselves and with the supercharges.

It is useful to note that a real 2×2 matrix Ω belongs to $SL(2, \mathbb{R})$ if and only if $CgC^{-1} = g^{-1}$. Furthermore, (2.1) can be rewritten as

$$\not{x}C \mapsto g\not{x}Cg^T.$$

Now, under the identification of the three-dimensional Lorentz group with L , the Lorentz Lie algebra is identified with the real Lie algebra $sl(2)$. Denote by Ad the action of L on \mathfrak{g} . Then for any $X \in sl(2)$, we have $Ad_g(X) = gXg^{-1}$, where $g \in L$. The action on P_μ and $Q_{i,\alpha}$ is given by

$$Ad_g(P_\mu)\Gamma^\mu C = g\not{P}Cg^T, \quad Ad_g(Q_{i,\alpha}) = (gQ_i)_\alpha, \quad (2.5)$$

where Q_i is the column vector with the two entries $Q_{i,1}$ and $Q_{i,2}$. Obviously, L leaves the central charges invariant.

III. INDUCED REPRESENTATIONS AND UNITARITY

The method of constructing unitary representations of the Poincaré superalgebra is a generalisation of the well-known induction method of Wigner and Mackey. A mathematically rigorous treatment of this generalisation was recently given in Ref. 9.

Note that every irreducible unitary representation of the subgroup $\mathbb{R}^{1,2}$ of translations of the Poincaré group is of the form

$$\mathbb{R}^{1,2} \longrightarrow \mathbb{C} \setminus \{0\}, \quad a \mapsto \exp(ip_\mu a^\mu),$$

for any fixed three momentum p_μ in the dual space $(\mathbb{R}^{1,2})^*$ of the subgroup. The differential of the representation is $P_\mu \mapsto p_\mu$. The operator $P_\mu P^\mu$ in this representation takes the value $p_\mu p^\mu$, which may be 0, $-m^2$, or m^2 , with m being any fixed real number.

The Lorentz subgroup acts on the subgroup $\mathbb{R}^{1,2}$ of translations, and thus also on its dual space $(\mathbb{R}^{1,2})^*$. This action changes neither $p_\mu p^\mu$ nor the sign of p^0 . Therefore, each orbit of the Lorentz subgroup in $(\mathbb{R}^{1,2})^*$ is generated by a momentum, which is in one of the following standard forms:

1. $p_\mu p^\mu = 0$:

$$\begin{aligned} p^\mu &= 0, & \text{zero-momentum case,} \\ p^\mu &= (\omega, \omega, 0), & \text{massless case,} \\ p^\mu &= (-\omega, -\omega, 0), \end{aligned}$$

where $\omega > 0$ is fixed.

2. $p_\mu p^\mu = -m^2$, with $m > 0$:

$$\begin{aligned} p^\mu &= (m, 0, 0), & \text{massive case,} \\ p^\mu &= (-m, 0, 0). \end{aligned}$$

3. $p_\mu p^\mu = m^2$, with $m > 0$:

$$p^\mu = (0, 0, \pm m).$$

Given a 3-momentum p^μ in any of the standard forms listed above, we let L_p be the subgroup of L , which fixes p^μ . Let $\mathfrak{l}_p = \text{Lie}(L_p)$, which is a subalgebra of \mathfrak{g} . Then \mathfrak{l}_p , the momentum operators P_μ , the supercharges $Q_{i,\alpha}$ and the central charges Z_{ij} together span a subalgebra \mathfrak{g}_0 of the Poincaré superalgebra \mathfrak{g} . The Harish-Chandra pair $(L_p \times \mathbb{R}^{1,2}, \mathfrak{g}_0)$ is a subsupergroup of the extended Poincaré supergroup.

Let π_0 be an irreducible unitary representation of the subsupergroup $(L_p \times \mathbb{R}^{1,2}, \mathfrak{g}_0)$ furnished by the complex Hilbert superspace $V_0(p, z, h)$, where $z = (z_{ij})$ with z_{ij} being the eigenvalue of Z_{ij} , and h denoting some L_p labels. We construct the Hilbert superspace $V(p, z, h)$ of $V_0(p, z, h)$ -valued L^2 functions on L with the following property: for any $\phi \in V(p, z, h)$,

$$\phi(gh) = \pi_0(h^{-1})\phi(g), \quad \forall g \in L, h \in L_p.$$

The action of the extended Poincaré supergroup on $V(p, z, h)$ is defined by

$$\begin{aligned} (h\phi)(g) &= \phi(h^{-1}g), \quad h \in L, \\ (a\phi)(g) &= \exp(i \text{Ad}_g(p)_\mu a^\mu)\phi(g), \quad a \in \mathbb{R}^{1,2}, \\ (Q_{i,\alpha}\phi)(g) &= \pi_0(\text{Ad}_{g^{-1}}(Q_{i,\alpha}))\phi(g), \\ (Z_{ij}\phi)(g) &= z_{ij}\phi(g), \quad \forall g \in L. \end{aligned} \tag{3.1}$$

We shall denote the associated irreducible representation by π .

Recall that the ordinary Poincaré group admits irreducible unitary representations with nonphysical momenta,⁸ which are discarded on physical grounds. This is in sharp contrast to the present case with supersymmetry, where representations with nonphysical momenta are ruled out by unitarity. More precisely,

Poincaré supergroups only admit unitary representations satisfying either $p_\mu p^\mu \leq 0$ with $p^0 > 0$, or $p^\mu = 0$.

To prove the claim, we need to show that no unitary representations exist if p^μ is one of the standard momenta $(-\omega, -\omega, 0)$, $(-m, 0, 0)$, or $(0, \pm m, 0)$.

If $p^\mu = (-\omega, -\omega, 0)$, we have

$$\{Q_{i,\alpha}, Q_{j,\beta}\} = -2\omega\delta_{\alpha 2}\delta_{\beta 2}\delta_{ij} + C_{\alpha\beta}z_{ij},$$

when we regard the operators as endomorphisms of $V_0(p, z, h)$. In particular,

$$\{Q_{i,2}, Q_{j,2}\} = -2\omega\delta_{ij}. \tag{3.2}$$

For any nonzero vector $|v\rangle$ in $V_0(p, z, h)$, we have

$$\|Q_{i2}|v\rangle\|^2 = \frac{1}{2}\langle v|\{Q_{i,2}, Q_{i,2}\}|v\rangle = -\omega\|v\|^2 < 0,$$

contradicting unitarity.

If $p^\mu = (-m, 0, 0)$, we have

$$\{Q_{i,\alpha}, Q_{j,\beta}\} = -m\delta_{\alpha\beta}\delta_{ij} + C_{\alpha\beta}z_{ij},$$

which, in particular, implies

$$\{Q_{i,\alpha}, Q_{i,\alpha}\} = -m. \tag{3.3}$$

For $p^\mu = (0, \pm m, 0)$, we define the p -dependent sign factor $\varepsilon_p = \pm 1$. Then the commutation relations of the supercharges become

$$\{Q_{i,\alpha}, Q_{j,\beta}\} = \varepsilon_p(-1)^\alpha m\delta_{\alpha\beta}\delta_{ij} + C_{\alpha\beta}z_{ij}.$$

This, in particular, implies $\{Q_{i,1}, Q_{i,1}\} = -\{Q_{i,2}, Q_{i,2}\} = -\varepsilon_p m$. Thus,

$$\begin{aligned} \{Q_{i,1}, Q_{i,1}\} &= -m, \quad \text{for } p^\mu = (0, m, 0), \\ \{Q_{i,2}, Q_{i,2}\} &= -m, \quad \text{for } p^\mu = (0, -m, 0). \end{aligned} \tag{3.4}$$

Applying the same arguments used in the $p^\mu = (-\omega, -\omega, 0)$ case to Eqs. (3.3) and (3.4), one can see that no unitary representation exists for the Poincaré supergroup in these cases.

IV. IRREDUCIBLE UNITARY REPRESENTATIONS

Now we consider the irreducible unitary representations of the Poincaré supergroups with a physical momentum p^μ of the standard form $p^\mu = 0, (\omega, \omega, 0)$, or $(m, 0, 0)$.

A. Zero-momentum case: $p^\mu = 0$

We have

$$\{Q_{i,\alpha}, Q_{j,\alpha}\} = 0, \quad \alpha = 1, 2.$$

For any vector $|v\rangle$ in $V_0(p, z, h)$, we have

$$\|Q_{i,\alpha}|v\rangle\|^2 = \frac{1}{2}\langle v|\{Q_{i,\alpha}, Q_{i,\alpha}\}|v\rangle = 0, \quad \forall i, \alpha. \quad (4.1)$$

Unitarity of π_0 implies $Q_{i,\alpha}|v\rangle = 0$ for all $|v\rangle$. Hence, $\pi_0(Q_{i,\alpha}) = 0$, for all i and α . This also forces the central charges to be zero.

Now the entire group L leaves p^μ invariant. Therefore, every irreducible unitary representation of a Poincaré supergroup is the inflation of an irreducible representation of L by requiring $Q_{i,\alpha}$ and Z_{ij} to act by zero. From the works of Bargmann, Gel'fand, and Naimark, and Harish-Chandra in the 1940s and 1950s, it is known¹³ that $SL(2, \mathbb{R})$ has 3 series (discrete, principal, and complementary series) of irreducible unitary representations, beside the trivial representation. All the irreducible unitary representations are infinite dimensional except the trivial representation, which is of course one-dimensional. Such zero energy-momentum irreducible unitary representations, which have been sometimes called spurions (see the book¹⁴) in imagined analogy with particles, could perhaps be considered as vacuum states. In this sense, there appears an infinite degeneracy of vacuum and its nonuniqueness.

B. Massless case: $p^\mu = (\omega, \omega, 0)$

The subgroup of L_p consists of the elements $\begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}$ in $SL(2, \mathbb{R})$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{R}$. Recall that every irreducible unitary representation of \mathbb{R} is given by $\mathbb{R} \rightarrow \mathbb{C} \setminus \{0\}$, $c \mapsto \exp(itc)$, where t is any fixed real number. Thus, the irreducible unitary representations of L_p are

$$\begin{aligned} \phi^{(+;t)} : L_p &\rightarrow \mathbb{C}, & \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix} &\mapsto \exp(itc), \\ \phi^{(-;t)} : L_p &\rightarrow \mathbb{C}, & \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix} &\mapsto \pm \exp(itc). \end{aligned}$$

A simple calculation yields

$$\{Q_{i,\alpha}, Q_{j,\beta}\} = 2\omega\delta_{\alpha 2}\delta_{\beta 2}\delta_{ij} + C_{\alpha\beta}z_{ij}. \quad (4.2)$$

This, in particular, implies that

$$\{Q_{i,2}, Q_{j,2}\} = 2\omega\delta_{ij}, \quad \{Q_{i,1}, Q_{i,1}\} = 0. \quad (4.3)$$

A calculation similar to (4.1) shows that $\pi_0(Q_{i,1}) = 0$ for all i . Using this fact in the special case of Eq. (4.2) with $\alpha = 1$ and $\beta = 2$, we obtain $z_{ij} = 0$ for all i, j .

For any given $h = \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}$ in L_p , we have

$$\pi_0(Ad_h(Q_{i,1})) = 0, \quad \pi_0(Ad_h(Q_{i,2})) = \pm\pi_0(Q_{i,2}).$$

Inspecting the first relation in (4.3), one sees that we may require $V_0(p, z, h)$ to be an irreducible module for the Clifford algebra spanned by the elements $\frac{Q_{i,2}}{\omega}$. In fact, every irreducible unitary representation π_0 of $(L_p \ltimes \mathbb{R}^{1,2}, \mathfrak{g}_0)$ can be obtained by lifting an irreducible unitary representation of L_p to the supersubgroup. That is, π_0 has the following properties: $\pi_0^{(\pm,t)}(Q_{\alpha,2})$ is essentially an irreducible representation of a Clifford algebra as discussed above but $\pi_0^{(\pm,t)}(Q_{\alpha,1}) = 0$, and the restriction of π_0 to L_p is $\phi^{(\pm;t)}$.

We shall write $\pi^{(\pm;t)}$ for the irreducible unitary representation π of the Poincaré supergroup induced by π_0 , and $V(p, z, t, \pm)$ for $V(p, z, h)$. For nonzero t , $\pi^{(\pm;t)}$ is the three-dimensional supersymmetric analogue of Wigner’s “infinite spin” (also termed as “continuous spin”) representations.^{15,16}

C. Massive case: $p^\mu = (m, 0, 0)$

In this case, the subgroup L_p , consisting of the elements of $SL(2, \mathbb{R})$ satisfying $hh^T = I$, is the special orthogonal group $SO(2, \mathbb{R})$ in two dimensions. Its universal cover is

$$\mathbb{R} \longrightarrow SO(2, \mathbb{R}), \quad \theta \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

In order to have representations of any “spin,” we need to pass over to the universal cover and consider its representations instead. Then every irreducible unitary representation is given by $\theta \mapsto \exp(is\theta)$, where the “spin” s can be any fixed real number.

We have in this case

$$\{Q_{i,\alpha}, Q_{j,\beta}\} = m\delta_{\alpha\beta}\delta_{ij} + C_{\alpha\beta}z_{ij}. \tag{4.4}$$

Let $\Psi_i = \frac{Q_{i,1} + \sqrt{-1}Q_{i,2}}{\sqrt{2m}}$ and $\Psi_i^\dagger = \frac{Q_{i,1} - \sqrt{-1}Q_{i,2}}{\sqrt{2m}}$; then the above commutation relations can be rewritten as

$$\{\Psi_i, \Psi_j\} = \{\Psi_i^\dagger, \Psi_j^\dagger\} = 0, \quad \{\Psi_i, \Psi_j^\dagger\} = \delta_{ij} + \sqrt{-1}\frac{z_{ij}}{m}, \quad \forall i, j. \tag{4.5}$$

For any $h = h_\theta := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ in L_p , we have

$$Ad_h(\Psi_i) = \exp(-\sqrt{-1}\theta)\Psi_i, \quad Ad_h(\Psi_i^\dagger) = \exp(\sqrt{-1}\theta)\Psi_i^\dagger.$$

To analyze the irreducible representations of the supersubgroup $(L_p \ltimes \mathbb{R}^{1,2}, \mathfrak{g}_0)$, it is useful to consider independently the associative superalgebra \mathcal{C} generated by the Ψ_i and Ψ_i^\dagger as defined by (4.5). The Ψ_i generate a subsuperalgebra \mathcal{C}_+ , which has one irreducible module $\mathbb{C}|0\rangle$ with

$$\Psi_i|0\rangle = 0, \quad \forall i.$$

Construct the induced module

$$F = \mathcal{C} \otimes_{\mathcal{C}_+} |0\rangle,$$

which has a basis

$$|\mathbf{n}\rangle := (\Psi_1^\dagger)^{n_1} (\Psi_2^\dagger)^{n_2} \dots (\Psi_N^\dagger)^{n_N} |0\rangle, \quad n_i \in \{0, 1\}.$$

If $N = 1$, F is the only unitary module for \mathcal{C} .

If $N > 1$, every irreducible \mathcal{C} -module is a quotient of F . Reducibility/irreducibility of F is controlled by the real numbers $\frac{z_{ij}}{m}$ related to the central charges, and unitarity imposes conditions on the possible choices of these values.

Therefore, every irreducible module $V_0(p, z, s)$ for $(L_p \ltimes \mathbb{R}^{1,2}, \mathfrak{g}_0)$ is a quotient of the module $F(p, z, s)$, which is generated by a vacuum vector $|0; p, z, s\rangle$ satisfying

$$h_\theta|0; p, z, s\rangle = \exp(\sqrt{-1}s\theta)|0; p, z, s\rangle, \quad \Psi_i|0; p, z, s\rangle = 0, \quad \forall i,$$

and the obvious property under the translation subgroup of the extended Poincaré supergroup characterised by the momentum p . A basis for $F(p, z, s)$ is given by

$$|\mathbf{n}; p, z, s\rangle := (\Psi_1^\dagger)^{n_1} (\Psi_2^\dagger)^{n_2} \dots (\Psi_N^\dagger)^{n_N} |0; p, z, s\rangle, \quad n_i \in \{0, 1\}.$$

Note that $h_\theta |\mathbf{n}; p, z, s\rangle = \exp(\sqrt{-1}(s + \sum_i n_i)\theta) |\mathbf{n}; p, z, s\rangle$.

Remark 4.1: What happens here is different from the case of the Clifford algebra, for which the induced module is always irreducible.

Let us now consider the different cases in more detail.

We have a complete understanding of the $N = 1$ case (see also Ref. 3).

For $N = 2$, let $w = \frac{z_{12}}{m}$. The Hermitian form on V_0 is normalised so that $\langle 0|0\rangle = 1$. Here we have dropped the labels p, z, s from notations for the sake of simplicity. A straightforward calculation yields $\langle 1, 1|1, 1\rangle = 1 - w^2$. Therefore, F is irreducible if and only if $1 - w^2 \neq 0$.

In the irreducible case, unitarity requires w to lie in the open interval $(-1, 1)$ of the real line. This in fact is the necessary and sufficient condition for V_0 to be unitary. This follows from the fact that both eigenvalues of the matrix

$$\begin{pmatrix} \langle 1, 0|1, 0\rangle & \langle 1, 0|0, 1\rangle \\ \langle 0, 1|1, 0\rangle & \langle 0, 1|0, 1\rangle \end{pmatrix} = \begin{pmatrix} 1 & iw \\ -iw & 1 \end{pmatrix}$$

are positive under the condition on w .

If $w = \pm 1$, F is reducible with a two-dimensional maximal submodule spanned by $(\Psi_1^\dagger + iw\Psi_2^\dagger)|0\rangle$ and $\Psi_1\Psi_2^\dagger|0\rangle$. The irreducible quotient V_0 is isomorphic to the Fock space of the fermionic operators

$$\Psi = \frac{1}{2}(\Psi_1 + iw\Psi_2), \quad \Psi^\dagger = \frac{1}{2}(\Psi_1^\dagger - iw\Psi_2^\dagger),$$

and thus is automatically unitary. In this case, half of the supersymmetry is broken.

Therefore, the massive irreducible unitary representations of the $N = 2$ extended Poincaré supergroup with fixed s are parametrized by the closed interval $[-1, 1]$. The irreducible unitary representations $V(p, z, s)$ with w lying on the boundary arose in Ref. 2, and the supermultiplets correspond to semions when $s = -\frac{1}{4}$ or $s = -\frac{3}{4}$.

In the $N = 3$ case, the irreducible massive unitary representations with fixed s are parametrized by the unit ball $B := \{(w_1, w_2, w_3) \in \mathbb{R}^3 \mid w_1^2 + w_2^2 + w_3^2 \leq 1\}$, where $w_1 = \frac{z_{12}}{m}$, $w_2 = \frac{z_{13}}{m}$, and $w_3 = \frac{z_{23}}{m}$. Furthermore, F is reducible if and only if w lies on the boundary ∂B , which is the sphere $w_1^2 + w_2^2 + w_3^2 = 1$. In this case, $\dim_{\mathbb{C}} V_0 = 4$ with half of the dimensions even and half odd, independent of the actual position of w in the sphere. One can verify these results in much the same way as in the $N = 2$ case and continue the study for the extended Poincaré supergroups with higher values of N .

V. CONCLUSION

We have classified the irreducible unitary representations of the extended Poincaré supergroups in three dimensions. Since unitary representations are always completely reducible, this achieves a classification of all the unitary representations. Irreducible unitary representations with arbitrary spins appear, which correspond to supersymmetric anyons. We have also shown that all the irreducible unitary representations have physical energy momenta, an observation which remains valid in other dimensions too, as can be shown by similar arguments as those used here. This is essentially different from the case of the ordinary Poincaré group, which admits unitary representations also with nonphysical momenta.⁸

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