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We consider the problem of portfolio risk diversification in a Value-at-Risk framework with heavy-tailed risks and arbitrary dependence captured by a copula function. We use the power law for modelling the tails and investigate whether the benefits of diversification persist when the risks in consideration are allowed to have extremely heavy tails with tail indices less than one and when their copula describes wide classes of dependence structures. We show that for asymptotically large losses with the Eyraud-Farlie-Gumbel-Morgenstern copula, the threshold value of tail indices at which diversification stops being beneficial is the same as for independent losses. We further extend this result to a wider range of dependence structures which can be approximated using power-type copulas and their approximations. This range of dependence structures includes many well known copula families, among which there are comprehensive, Archimedian, asymmetric and tail dependent copulas. In other words, diversification increases Value-at-Risk for tail indices less than one regardless of the nature of dependence between portfolio components within these classes. A wide set of simulations supports these theoretical results.

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1 Introduction

Level-q Value-at-Risk VaR_q, also known as the level-q quantile of a distribution of losses, is a commonly used risk measure, whose popularity in a wide range of areas in finance is attributed to the recommendations of the Basel Committee on Banking Supervision. A series of recent papers studied the problem of portfolio optimization in the VaR framework, focusing on the situation when the portfolio components are independent and have a heavy tailed distribution (see, e.g., Ibragimov and Walden, 2011; Embrechts et al., 2009). An interesting conclusion from that work is that if tails are extremely heavy then diversification increases riskiness in terms of VaR.

This theoretical property of VaR known as non-subadditivity or non-coherence is often weighted against the practical considerations of the ease of calculation and backtesting and smaller data requirements, compared to subadditive risk measures such as Expected Shortfall (see, e.g., Danielsson et al., 2013; Garcia et al., 2007). Moreover, it is well established in today's finance that in practice risks are dependent in some usually unknown fashion and that the behavior of other risk measures including ES is closely related to the behavior of the tail of sums of dependent risks used in VaR analysis (see, e.g. Alink et al., 2005). Therefore, a better understanding of when VaR is non-subadditive in non-iid settings is key to continued use of VaR as a robust risk measure.

The literature on VaR for independent risks is very wide and has a long tradition (see, e.g., books by Embrechts et al., 1997; Resnick, 1987). More recently, Garcia et al. (2007); Ibragimov and Walden (2007); Ibragimov (2009b) focused on the case of iid stable random variable with infinite variance and showed that VaR is subadditive provided the mean is finite. Similar results were obtained for asymptotically large losses without the assumption of a stable distribution. However, extensions to non-independence are more recent and have been limited to specific cases. For example, Ibragimov and Walden (2007, 2011) consider dependence arising from common multiplicative and additive shocks, Embrechts et al. (2009) and Chen et al. (2012) consider Archimedian copulas, Asmussen and Rojas-Nandayapa (2008) consider the normal copula, Albrecher et al. (2006) consider several copula classes permitting explicit solutions such as Archimedian copulas. An interesting result arising from these studies is that the subadditivity property of VaR is generally affected by both the strength of dependence and the tail behavior of the marginals, however in some cases only heavy tails of the marginals matter.

In this paper, we provide several new results on subadditivity of VaR in non-iid settings. The classes of dependence structures we consider are motivated by several widely used copula families and their approximators. We start with the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copula family and show that for power law risks whose tail exponent is below one, diversification is sub-optimal (ie it increases riskiness) regardless of the value of the dependence parameter provided the loss is large enough. We proceed by providing similar results on copulas that can be viewed as generalizations and first or second order approximations of the EFGM copula. This class of copulas, which we call *power-type*, includes the power copulas of Ibragimov (2009a), polynomial copulas of Drouet Mari and Kotz (2001), copulas with cubic sections of Nelsen et al. (1997), as well as a large number of related copulas with various dependence features such as asymmetry, tail-dependence, comprehensiveness etc.

The paper is organized as follows. Section 2 sets the stage by introducing the power law distributions and discussing available results for independent risks. Section 3 reviews the basics of copulas, introduces the power-type copula class, and presents the main results on limits of diversification for this class, as well as simulation evidence. Section 4 concludes.

2 Diversification under independence

2.1 Heavy tails and power law family

It has become common in financial econometrics to use the tail index of a distribution to measure its tails (see, e.g., Embrechts et al., 1997; Gabaix, 2009; Ibragimov, 2009b). The tail index characterizes the heaviness, or the rate of decay, of the tails of the relevant univariate distribution, assuming it obeys a power law. The family of distributions obeying the power law of tail decay is known as the *power law family* and the law is usually written as follows

$$\mathbb{P}(|X| > x) \sim x^{-\alpha}, \qquad (1)$$

where α is the tail index, or tail exponent.[1]

Power law distributions permit modelling rates of tail decay that are slower than the exponential decay of the Gaussian distribution, which is important in financial applications. Such distributions

[1] Here, "$\sim$" means that the left hand side is asymptotically equivalent to a nonzero constant times the right hand side, where asymptotics is as $x \to \infty$.

often form the basis of a wider class obtained by introducing a slight disturbance to the tail behavior in the form of a slowly varying function (see, e.g., Embrechts et al., 1997; Ibragimov and Walden, 2008). The tail index α governs the likelihood of observing outliers or large fluctuations: a smaller tail index means slower rate of decay, which means that this likelihood is higher. When the tail index is less than two, the tail decay is so slow that the second moment of the underlying distribution is infinite; when the tail index is less than one, the first moment is infinite. More generally, the power law distributions have the property that absolute moments of X are finite if and only if their order is less than tail index α. That is,

$$\mathbb{E}|X|^p < \infty \quad \text{if} \quad p < \alpha; \quad \mathbb{E}|X|^p = \infty \quad \text{if} \quad p \geq \alpha.$$

A large number of studies in economics and finance have documented that financial returns have distributions with values of α ranging from significantly lower than one to above four (Jansen and Vries, 1991; Loretan and Phillips, 1994; McCulloch, 1997; Rachev and Mittnik, 2000; Gabaix et al., 2006; Chavez-Demoulin et al., 2006; Silverberg and Verspagen, 2007). Many distributions can be viewed as special cases of power law, at least for asymptotically large losses. This includes Student t distribution, Cauchy, Levy and Pareto and other Stable distributions with parameter $\alpha < 2$. We will say that a risk has *extremely* heavy tails if $\alpha < 1$.

2.2 Limits of diversification under heavy tails and independence

Consider a simple problem of optimal portfolio allocation in the VaR framework with two possibly extremely heavy tails. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be the portfolio weights such that $w_1 + w_2 = 1$. Let X_j represent a loss, $j = 1, 2$ which has a power law distribution with index α_j. Consider the tail of the aggregate loss distribution $\mathbb{P}(w_1 X_1 + w_2 X_2 > x)$, where the weighted average loss $w_1 X_1 + w_2 X_2$ corresponds to a portfolio of two risks with weights w_1 and w_2. Unless one of the weights is zero, the portfolio is diversified. A 5%-VaR of this portfolio is the value of loss x for which that probability is 0.05. More generally, the q% Value-at-Risk of a portfolio Z is $\text{VaR}_q(z) = \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq q\}$, or the $(1-q)$-th quantile of the loss distribution. The problem of interest is to minimize $\text{VaR}_q(w_1 X_1 + w_2 X_2)$ over the weights w for a given $q \in (0, 1/2)$.

When X_1 and X_2 are iid with a stable distribution, it is now well understood that, for all non-zero w's, $\mathbb{P}(w_1 X_1 + w_2 X_2 > x) \leq \mathbb{P}(X_1 > x)$ if $\alpha_j > 1, j = 1, 2$. Equivalently, the VaR of a diversified portfolio $\text{VaR}_q\left(w_1X_1 + w_2X_2 > x\right)$ is no greater than that of a not diversified, $\text{VaR}_q\left(X_1 > x\right)$, if $\alpha_j > 1$. In order words, diversification helps lower the VaR for moderately, but not extremely, heavy tailed risks. If $\alpha_i < 1$ then $\mathbb{P}\left(w_1X_1 + w_2X_2 > x\right) \geq \mathbb{P}\left(X_1 > x\right)$; that is, for extremely heavy-tailed risks the benefits of diversification disappear and the least risky portfolio has just one risk. For example, if X_j's are iid Stable with $\alpha = 1/2$, that is if they are Levy distributed, the aggregate loss of an equally weighted portfolio $\frac{X_1+X_2}{2}$ has the same distribution as $2X_1$ and thus $\text{VaR}_q\left(\frac{X_1+X_2}{2}\right) = 2\text{VaR}_q(X_1) > \text{VaR}_q(X_1)$.

Analogous statements hold for portfolios of any size and asymptotically (for infinitely large losses) even if the distribution of X is not stable. Ibragimov (2009b) showed this in a general context with any number of risks, using majorization theory. Similar results are available for bounded risks concentrated on a sufficiently large interval: for such cases, VaR-based diversification is suboptimal up to a certain number of risks and then becomes optimal (Ibragimov and Walden, 2007).

There is a growing range of empirical applications of these seemingly counterintuitive results. Ibragimov et al. (2009) demonstrate how this analysis can be used to explain abnormally low levels of reinsurance among insurance providers in markets for catastrophic insurance. Ibragimov et al. (2011) show how to analyze the recent financial crisis as a case of excessive risk sharing between banks when risks are extremely heavy-tailed. Gabaix (2009) provides a survey of empirical applications in economics and finance.

It follows from the two-risks example above that the limits of diversification results hold for iid losses regardless of the weights w_j used in construction of a diversified portfolio. Therefore, in what follows we consider an equally weighted portfolio $w_1 = w_2 = 1/2$. To state the results formally, let $(\xi_1(\beta), \xi_2(\beta))$ denote independent random variables from a power-law distribution with a common tail index β. The following theorem can be easily extended to any diversified portfolio of size n, in which case $\left(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\right)$ in inequalities (2)-(3) is replaced with $\sum_{i=1}^{n} w_i\xi_i(\alpha)$.

Theorem 1 *For sufficiently small loss probability q,*

$$VaR_q\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\Big) < VaR_q(\xi_1(\alpha)), \quad if \quad \alpha > 1 \tag{2}$$

$$VaR_q\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}\Big) > VaR_q(\xi_1(\alpha)), \quad if \quad \alpha < 1 \tag{3}$$

Proof See Appendix for all proofs.

An interesting boundary case corresponds to $\alpha = 1$. This is when diversification has no effect at all, ie it neither increases nor reduces VaR. For example, if ξ's are iid Stable with $\alpha = 1$, which means they have a Cauchy distribution, it is easy to show that $\sum_{i=1}^{n} w_i \xi_i(\alpha)$ is equal in distribution to $\xi_i(\alpha)$, so a diversified and a non-diversified portfolios have identical VaRs.

It is not obvious what happens if we relax the independence assumptions. The two extreme cases, corresponding to a comonotone and countermonotone relationships between the components do not present a consistent picture. For example if we assume that $\xi_1 = \xi_2$ (a.s.) then obviously $\text{VaR}_q\left(w_1\xi_1(\alpha) + w_2\xi_2(\alpha)\right) = \text{VaR}_q(\xi_1(\alpha))$ and so diversification has no effect regardless of the tails; while if we assume $\xi_1 = -\xi_2$ (a.s.) then $\text{VaR}_q\left(w_1\xi_1(\alpha) + w_2\xi_2(\alpha)\right) = (w_1 - w_2)\text{VaR}_q(\xi(\alpha))$ and it is optimal to fully diversify regardless of the tails.

In the next section we use copulas to allow for arbitrary dependence, which includes the two extreme cases in the limit.

3 Diversification under dependence

3.1 Dependence and copulas

Copulas are joint distributions with uniform marginals. They are useful because given the marginal distributions, they represent the dependence in the joint distribution. Specifically, let $H(x_1, \ldots, x_n)$ and $h(x_1, \ldots, x_n)$ denote the joint distribution and density, respectively, of n random variables $(X_1, \ldots, X_n)$ and suppose that the marginal density and cdf of X_j are $f_j(x_j)$ and $F_j(x_j)$ respectively, $j = 1, \ldots, n$. Then, an n-dimensional copula of $(X_1, \ldots, X_n)$ is a function $C : [0, 1]^n \to [0, 1]$ such that

(a) $C(u_1, \ldots, u_n)$ is increasing in each $u_i, i = 1, \ldots, n$.

(b) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0, i = 1, \ldots, n$.

(c) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i, i = 1, \ldots, n$.

(d) for any $a_j \leq b_j, j = 1, \ldots, n,$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1i_1}, \ldots, u_{ni_n}) \geq 0,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j = 1, \ldots, n$.

(e) $H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$, or, for absolutely continuous copulas with density $c(u_1, \ldots, u_n)$, $h(x_1, \ldots, x_n) = c(F_1(x_1), \ldots, F_n(x_n)) \prod_{i=1}^n f(x_i)$.

It is well known that c is uniquely determined if F_j is monotone. The probability integral transforms $u_j = F_j(x_j), j = 1, 2$, are the uniform random variables that form the marginals of c. So, equivalently C can be defined as a joint cdf of n random variables, each of which is uniform on $[0, 1]$. The fact that we can model F_j separately from modelling the dependence between F_j's is what makes copulas natural in the analysis of dependent power-law marginals.

A well known property of the copula function is that it is bounded by the Frechet-Hoeffding bounds, which correspond to extreme positive and extreme negative dependence. For a bivariate copula, let x_1 be a fixed increasing function of x_2, then the copula of (x_1, x_2) can be written as $\min(u_1, u_2)$ and this is the upper bound for bivariate copulas. Now let x_1 be a fixed decreasing function of x_2; then the copula of (x_1, x_2) can be written as $\max(u_1 + u_2 - 1, 0)$. So the two extreme cases when diversification does not have any effect (comonotonicity) and when it is always beneficial (countermonotonicity) regardless of the heavy-tailedness are nested within the copula framework. Joe (1997) and Nelsen (2006) provide excellent introductions to copulas.

If we return to the two-risk example above, we are interested in how the aggregate loss probability for a diversified portfolio compares to that of a single risk. That is, we are interested in the behavior of

$$\begin{aligned} \mathbb{P}\left(\frac{X_1 + X_2}{2} > x\right) &= \int\int_{\frac{z_1+z_2}{2} > x} f(z_1; \alpha) f(z_2; \alpha) c\left(F(z_1; \alpha), F(z_2; \alpha); \gamma\right) dz_1 dz_2 \\ &= \mathbb{E}\left\{c\left(F(\xi_1; \alpha), F(\xi_2; \alpha); \gamma\right) \mathbb{I}\left[\frac{\xi_1 + \xi_2}{2} > x\right]\right\} \end{aligned}$$

where $c(u_1, u_2; \gamma)$ is a copula density parameterized by γ, $f(\cdot; \alpha)$'s are power-law marginal densities, $\mathbb{I}[\cdot]$ is the indicator function and ξ_j's are independent copies of X_j's. There is no general way to express this in terms of $\mathbb{P}(X_1 > x)$ and whether diversification decreases or increases VaR depends on the copula family as well as on the interaction between α and γ. However, there exist classes of copulas for which we can make explicit comparisons.

3.2 Power-type copulas

We now discuss a class of copula families which will be used in the paper. The class contains copulas that are multiplicative or additive in powers of the margins, or can be approximated using such copulas. We call this class *power-type*. It is similar but more general than the power copula family and than the polynomial copula family which we discuss below.

The most common family in this class is the Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copula family and its generalizations. The bivariate EFGM copula family can be written as follows

$$C(u_1, u_2) = u_1 u_2 [1 + \gamma(1 - u_1)(1 - u_2)], \tag{4}$$

where $\gamma \in [-1, 1]$, and its density has the form $c(u_1, u_2) = 1 + g(u_1, u_2)$, where $g(u_1, u_2)$ is an expansion by linear functions $1 - 2\,u_j, j = 1, 2$. This is a non-comprehensive copula in the sense that it has a limited range of dependence it can accommodate. For example, Kendall's τ of an EFGM copula is restricted to $\left[-\frac{2}{9}, \frac{2}{9}\right]$.

The multivariate version of the EFGM copula introduced by Cambanis (1977) has the following form:

$$C(u_1, u_2, ..., u_n) = u_1 u_2 ... u_n \Big[1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1, i_2, ..., i_c} (1 - u_{i_1})(1 - u_{i_2}) ... (1 - u_{i_c})\Big], \tag{5}$$

where $-\infty < \gamma_{i_1,i_2,...,i_c} < \infty$ are such that $\sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1,i_2,...,i_c} \delta_{i_1} \cdots \delta_{i_c} \ge -1$ for all $\delta_i \in [-1, 1], i = 1, \ldots, n$. This copula family can be viewed as a special case of a wider family of n-dimensional power copulas introduced by Ibragimov (2009a).

The power copula family can be written as follows

$$C(u_1, \ldots, u_n) = u_1 u_2 \cdots u_n \Big[1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} \gamma_{i_1,i_2,...,i_c} (u_{i_1}^l - u_{i_1}^{l+1})(u_{i_2}^l - u_{i_2}^{l+1}) ... (u_{i_c}^l - u_{i_c}^{l+1})\Big], \tag{6}$$

where $\gamma_{i_1,i_2,...,i_c} \in (-\infty, \infty)$ are such that

$$\sum_{c=2}^{n} \sum_{1 \le i_1 < i_2 < ... < i_c \le n} |\gamma_{i_1,i_2,...,i_c}| \le 1.$$

This corresponds to using nonlinear rather than linear functions in the expansion of the copula density function.

Another relevant copula family, of which the EFGM copula in (4) is a special case, is known as a polynomial copula family (see, e.g., Drouet Mari and Kotz, 2001, p. 74). An order m ($m \ge 4$) polynomial copula can be written as follows:

$$C(u, v) = uv \left[1 + \sum_{k \ge 1, q \ge 1}^{k+q \le m-2} \gamma_{kq} (u^k - 1)(v^q - 1)\right], \tag{7}$$

where $\gamma_{kq} = \frac{\theta_{kq}}{(k+1)(q+1)}$ and $0 \leq \min\left(\sum_{k\geq 1, q\geq 1}^{k+q\leq m-2} q\gamma_{kq}, \sum_{k\geq 1, q\geq 1}^{k+q\leq m-2} k\gamma_{kq}\right) \leq 1$.

One example of this copula family is Nelsen et al.'s (1997) copula with cubic section, which is written as follows

$$C(u, v) = uv + 2\gamma uv(1-u)(1-v)(1+u+v-2uv), \tag{8}$$

where $\gamma \in [0, \frac{1}{4}]$.

Several other copula families can be written as approximations of the EFGM copula. For example, it is well known that the EFGM copula is a first-order approximation to the Ali-Mikhail-Haq (AMH) copula family. The AMH copula can be written as follows:

$$C(u_1, \ldots, u_n) = (1-\gamma)\left[\prod_{i=1}^{n}\left(\frac{1-\gamma}{u_i} + \gamma\right) - \gamma\right]^{-1},$$

where $\gamma \in [-1, 1]$.

A less known result is that the Plackett and the Frank copula families are first order Taylor approximations of the EFGM copula at independence (see, e.g., Nelsen, 2006, p. 100, 133). The n-variate Frank copula, which is comprehensive, radially symmetric and Archimedian, can be written as follows

$$C(u_1, \ldots, u_n) = \log_\gamma\left[1 + \frac{\prod_{i=1}^{n}(\gamma^{u_i} - 1)}{(\gamma - 1)^{n-1}}\right],$$

where $\gamma \geq 0$.

The n-variate Plackett copula, which is also comprehensive, is rarely discussed in the literature unless $n = 2$, in which case it has the following form:

$$C(u_1, u_2) = \frac{1}{2(\gamma - 1)}\left[1 + (\gamma - 1)(u_1 + u_2) - \sqrt{[1 + (\theta - 1)(u_1 + u_2)]^2 - 4\gamma(\gamma - 1)u_1 u_2}\right],$$

where $1 \neq \gamma > 0$. However, a way to generalize to $n > 2$ is presented by Molenberghs and Lesaffre (1994). It is also worth mentioning that for all the three copula families, there exist improved second-order approximations (see, e.g., Nelsen, 2006, p. 83).

An interesting set of approximation results are given by Nelsen et al. (1997), Cuadras (2009) and Cuadras and Diaz (2012). Nelsen et al. (1997) provide a generalization of the bivariate EFGM copula using cubic terms as in (8) and show that it can be used to approximate some well-known families of copulas, both symmetric and not, such as the copulas of Kimeldorf and Sampson (1975) and Lin (1987), as well as the Sarmanov copula. They also show that copulas in (8) are second-degree Maclaurin approximations to members of the Frank and Plackett copula families.

Cuadras (2009) studies the power series class of copulas, obtained as weighted geometric means of the EFGM and AMH copulas, and shows that the Gumbel-Barnett and Cuadras-Auge copulas can be expressed as first-order approximations to that class. Cuadras and Diaz (2012) provide approximations of the tail-dependent Clayton-Oakes copula, which also have the form of a power-type generalization of the EFGM copula.

3.3 Diversification under dependence

We start with the bivariate EFGM copula. Let (X_1, X_2) be random variables with the EFGM copula and power-law marginals. Then, for any $x \geq 1$ and for $j = 1, 2$,

$$\begin{aligned}
F_j(x) &\sim 1 - x^{-\alpha}, \\
f_j(x) &\sim \alpha x^{-\alpha-1}, \\
H(x_1, x_2) &= F_1(x_1)F_2(x_2)\big[1 + \gamma(1 - F_1(x_1))(1 - F_2(x_2))\big], \\
h(x_1, x_2) &= f_1(x_1)f_2(x_2)\big[1 + \gamma(1 - 2F_1(x_1))(1 - 2F_2(x_2))\big].
\end{aligned}$$

As before, let $(\xi_1(\alpha), \xi_2(\alpha))$ be independent random variables from power-law distributions with tail index α, often called *independent* copies of (X_1, X_2). Our key insight is that in the tail, the behavior of products and powers of power-law densities and distributions of X_j's is identical to the behavior of their independent copies. This makes it possible to provide asymptotic (with respect to the loss) comparisons between the VaR of the aggregated loss and that of a single risk. More specifically, the crucial component of $\mathbb{P}\left(\frac{X_1+X_2}{2} > x\right)$ under the EFGM copula can be written as follows

$$\begin{aligned}
&\int_{\frac{s+t}{2}>x} \alpha^2 s^{-\alpha-1} t^{-\alpha-1} (2s^{-\alpha} - 1)(2t^{-\alpha} - 1) ds dt \\
= \;& 4\alpha^2 \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(2\alpha)}{2} > z\right) - 2\alpha^2 \mathbb{P}\left(\frac{\xi_1(2\alpha) + \xi_2(\alpha)}{2} > z\right) \\
& -2\alpha^2 \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(2\alpha)}{2} > z\right) + \alpha^2 \mathbb{P}\left(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\right),
\end{aligned}$$

where the behavior of the individual summands for large z is driven by the lowest tail index of ξ_j in the portfolio.

We formalize this result in the following theorem, which generalizes to n dependent heavy-tailed random variables $X_1, X_2, ..., X_n$ with multivariate EFGM copula given in (5) and power-law marginals.

Theorem 2 *For an asymptotically large $z > 0$, and any $n, \alpha > 0$,*

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i > zn\Big) \sim \mathbb{P}\Big(\sum_{i=1}^{n} \xi_i(\alpha) > zn\Big). \tag{9}$$

The result suggests that suboptimality of diversification in the VaR framework for extremely heavy tailed losses carries over from independence to the EFGM copula. That is, diversification increases VaR of dependent extremely heavy tailed risks within this copula family. Specifically, combining the results of Theorems 1 and 2, it is easy to see that the following corollary holds.

Corollary 1 *For dependent losses with the EFGM copula and sufficiently small loss probability q,*

$$VaR_q\Big(\frac{X_1 + X_2}{2}\Big) < VaR_q(X_1), \quad \text{if} \quad \alpha > 1 \tag{10}$$

$$VaR_q\Big(\frac{X_1 + X_2}{2}\Big) > VaR_q(X_1), \quad \text{if} \quad \alpha < 1 \tag{11}$$

Another interesting corollary of Theorem 2 can be obtained by combining this result with Theorem 1 of Sharakhmetov and Ibragimov (2002). The EFGM copula family has restrictive dependence, for example, it is not comprehensive in the sense that it cannot accommodate all possible values of Kendall's τ. Yet, as shown by Sharakhmetov and Ibragimov (2002), it can be used to represent *any* joint distribution of two-valued random variables (see also de la Pena et al., 2006, p. 190). Therefore, for two-valued random variables, our Theorem 2 applies to *all* dependence patterns.

Important generalizations of Theorem 2 arise if we consider the wider class of power-type copulas introduced in Section 3.2. Most popular members of this class such as the polynomial copula of Drouet Mari and Kotz (2001), the copula with cubic section of Nelsen et al. (1997) and the power copula of Ibragimov (2009b) can be written in the following general form

$$C(u_1, \ldots, u_n) = \sum_{i_1,\ldots,i_n=0,1,\ldots} \gamma_{i_1,i_2,\ldots,i_n} \cdot u_1^{i_1} \cdot u_2^{i_2} \cdot \ldots \cdot u_n^{i_n}, \tag{12}$$

for a multiple index $i = (i_1, i_2, \ldots, i_n)$ and a set of corresponding parameters γ_i with appropriate restrictions that make $C(u_1, \ldots, u_n)$ a copula. For example, Drouet Mari and Kotz (2001, Section 4.5.2) show how to obtain the polynomial copula in (7) from function $f = u^k v^q$. The key feature of such copulas is that they and their densities can be expressed as powers of u_j's. This allows to apply similar arguments as for EFGM.

Theorem 3 *For dependent losses with a power-type copula in (12) and for an asymptotically large $z > 0$, and any $n, \alpha > 0$, the conclusions of Theorem 2 hold.*

One may argue that the class of copulas in (12) is not sufficiently general. For example, it is not clear whether it can incorporate tail dependence or comprehensive copulas. However, the power-type copulas also include copulas which can approximate or be approximated by the class in (12). And, as discussed in Section 3.2, there are comprehensive and tail-dependent copulas among these copulas. Our next corollary establishes the result for such approximations.

Corollary 2 *For dependent losses with copulas whose Taylor or Maclaurin expansions can be written as (12), for an asymptotically large $z > 0$, and any $n, \alpha > 0$, the conclusions of Theorem 2 hold but only locally at the point of approximation.*

This corollary covers all the copula families discussed in Section 3.2 including the AMH, Plackett, Frank, Clayton-Oakes, Kimeldorf and Sampson, Lin, Gumbel-Barnett and many others, but only to the extent the approximations are valid. That is, the results of Theorem 2 hold for expansions at the point at which we expand, which often coincides with independence. Clearly, they do not have to hold when the approximation error is large. Therefore, applicability of Theorem 2 to a specific copula family needs to be checked on a case-by-case basis but the class of copulas to which it can be potentially applied is quite rich – it includes comprehensive copulas (Plackett, Frank), asymmetric copulas (Nelsen et al.'s copulas with cubic sections) and tail-dependent copulas (Clayton-Oakes).

3.4 Simulations

The results so far have been asymptotic with respect to the loss, that is they are valid as $q \to 0$. The interesting values of q used in practice are 0.01 and 0.05, which are non-asymptotic. This section uses simulations to study the tail behavior of VaR at non-asymptotic levels q using selected copula families.

The setup of the simulations is as follows. We generate equally weighted two-asset portfolios using losses X_1 and X_2, which have heavy tailed marginals with tail index α, ranging between 0 and 10, and a copula with dependence parameter γ, ranging between a high negative and a high positive dependence. We then calculate the ratio $\text{VaR}_q\Big((X_1+X_2)/2\Big)/\text{VaR}_q(X_1)$ for $q = 0.05$ across various values of dependence parameter and tail index. For the fixed non-asymptotic level of loss probability, the ratio demonstrates whether the effect of diversification is susceptible to changes in the copula parameter and the degree of heavy-tailedness, as well as to non-infinite losses. If the ratio is greater than one, this is an indication of diversification failure.

Table 1 reports the results for selected copulas. The simulations are based on 1,000,000 draws. The copulas we report include tail-dependent, asymmetric, Archimedian and comprehensive copulas, both power-type (eg EFGM) and not (eg Gaussian). Perhaps surprisingly, the table seems to present a very consistent picture across all these copula families, not only power-type. The threshold value of α at which we observe the reversal of the diversification benefit is close to one – for most values of θ, the reversal happens between $\alpha = 1$ and $\alpha = 1.2$, suggesting that even under infinite moments of a fractional order above one, diversification increases riskiness.

Under extreme positive dependence, as expected we observe that diversification has no effect at all. This of course can only be seen under comprehensive copulas, which allow for comonotonicity. Under extreme negative dependence, we observe the expected result that diversification always works but we observe it only for the Frank copulas, which is surprising. Under independence, the reversal happens at the same threshold value of α as for dependence which agrees with our main results. It is also worth mentioning that the positive effect of diversification where it exists is relatively small – the risk is reduced by no more than 20% – while if diversification fails it fails spectacularly, with VaR increasing by tens and hundreds percent.

4 Concluding remarks

We have revisited the limits of diversification for dependent risks. The revisit focused on a wide class of copulas that are additive in powers of margins. This class covers some well known families such as EFGM, power and polynomial families but also contains a number of other copula classes which do not have this form but can be approximated using Taylor-type expansions. So the resulting class we consider is very wide – comprehensive, tail-dependent and asymmetric copula families can be considered within this class.

The main result of the paper is that within the class, diversification increases riskiness in a VaR framework if the power index of the individual risks falls below one. This makes dependent risks within this class no different from independent in the sense that the same threshold value of the tail index delineates the benefits of diversification.

We have looked at equally weighted portfolios with components having the same tail index. The restriction of equal tail indices can easily be relaxed because the tail behavior of the aggregate loss will be dominated by the component with the lowest tail index. The limits of diversification are determined by whether the lowest index in the portfolio is above or below one. A similar result can be shown for unequally weighted portfolios but we leave both these extensions for future work.

We have also looked at simulation-based results where losses are given more realistic, non-asymptotic, values and have confirmed that diversification stops being beneficial at a threshold value of tail index which is somewhat greater than one. The discrepancy is due to the finite losses and it vanishes as $q \to 0$. However, the fact that the threshold is so much higher than one for copulas other than Archimedian and that the same threshold applies even to non-power-type copulas are of independent interest.

Table 1: VaR of a portfolio to VaR of a single loss for selected copulas

$\theta \backslash \alpha$	**0.2**	**0.4**	**0.7**	**1**	**1.2**	**1.6**	**2**	**3**	**4**	**7**	**10**
Gaussian											
-1	15.353	2.814	1.348	1.029	0.938	0.848	0.818	0.816	0.832	0.878	0.908
-0.7	15.732	2.849	1.362	1.034	0.943	0.855	0.828	0.827	0.844	0.887	0.914
-0.4	15.324	2.803	1.360	1.055	0.955	0.883	0.858	0.852	0.867	0.905	0.929
0	14.815	2.792	1.378	1.088	1.001	0.928	0.904	0.897	0.904	0.932	0.948
0.2	13.583	2.646	1.382	1.106	1.019	0.956	0.931	0.919	0.926	0.945	0.958
0.5	9.852	2.346	1.317	1.101	1.028	0.984	0.960	0.952	0.953	0.967	0.974
0.7	6.419	1.952	1.243	1.082	1.036	0.996	0.980	0.970	0.971	0.980	0.984
0.9	2.854	1.391	1.111	1.041	1.016	1.004	0.994	0.990	0.991	0.993	0.995
1	1.003	1.001	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Student-t ($v = 5$)											
-1	15.803	2.792	1.348	1.028	0.933	0.845	0.819	0.816	0.833	0.878	0.907
-0.7	16.222	2.797	1.361	1.034	0.943	0.863	0.835	0.828	0.847	0.890	0.916
-0.4	14.966	2.784	1.379	1.043	0.965	0.883	0.859	0.856	0.870	0.907	0.929
0	12.682	2.541	1.345	1.062	0.994	0.925	0.901	0.894	0.902	0.930	0.947
0.2	10.965	2.434	1.320	1.067	1.002	0.937	0.919	0.914	0.922	0.943	0.957
0.5	7.328	2.067	1.249	1.061	1.013	0.966	0.949	0.942	0.948	0.963	0.973
0.7	4.905	1.717	1.180	1.050	1.014	0.978	0.969	0.966	0.968	0.977	0.983
0.9	2.369	1.292	1.075	1.017	1.007	0.994	0.988	0.987	0.988	0.992	0.994
1	1.001	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Frank											
-3000	0.500	0.500	0.503	0.513	0.524	0.550	0.580	0.647	0.700	0.796	0.847
0.25	14.519	2.822	1.370	1.057	1.009	0.940	0.895	0.901	0.909	0.936	0.949
0.5	15.940	2.801	1.387	1.079	1.023	0.935	0.919	0.905	0.911	0.934	0.951
1	13.606	2.796	1.403	1.077	1.016	0.948	0.931	0.918	0.926	0.940	0.955
2	15.148	2.643	1.407	1.124	1.054	0.982	0.950	0.931	0.938	0.955	0.963
5	12.343	2.589	1.424	1.151	1.095	1.005	0.993	0.978	0.971	0.976	0.984
20	5.833	1.928	1.340	1.162	1.106	1.064	1.047	1.020	1.017	1.005	1.004
100	1.409	1.134	1.045	1.021	1.024	1.017	1.007	1.005	1.003	1.002	1.001
3000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Gumbel											
1	15.188	2.910	1.397	1.073	0.999	0.935	0.900	0.897	0.912	0.931	0.948
1.25	7.715	1.988	1.230	1.045	0.994	0.943	0.929	0.926	0.939	0.953	0.966
1.5	4.950	1.802	1.147	1.020	0.998	0.945	0.948	0.946	0.955	0.969	0.978
2	3.136	1.379	1.076	1.013	0.969	0.970	0.970	0.973	0.976	0.985	0.988
5	1.293	1.084	1.011	1.008	0.996	1.002	0.993	0.997	0.997	0.997	0.999
20	1.006	1.003	1.004	0.998	1.003	1.000	1.000	1.000	1.000	1.000	1.000
100	1.008	0.999	0.998	1.000	1.001	1.000	0.999	1.000	1.000	1.000	1.000
500	1.001	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1 (Cont'd): VaR of a portfolio to VaR of a single loss for selected copulas

$\theta\backslash\alpha$	**0.2**	**0.4**	**0.7**	**1**	**1.2**	**1.6**	**2**	**3**	**4**	**7**	**10**
Clayton											
0	16.542	2.838	1.356	1.085	1.011	0.920	0.900	0.895	0.903	0.932	0.949
0.25	10.263	2.529	1.331	1.089	1.022	0.948	0.933	0.919	0.926	0.947	0.959
0.5	7.473	1.963	1.249	1.052	1.005	0.958	0.940	0.934	0.946	0.958	0.968
1	3.985	1.582	1.105	1.004	0.989	0.959	0.959	0.962	0.962	0.978	0.982
2	1.999	1.200	1.048	0.997	0.989	0.982	0.975	0.982	0.991	0.992	0.992
5	1.306	1.053	1.021	0.993	1.000	0.994	0.993	0.996	0.997	0.998	0.998
20	1.007	1.004	1.004	0.997	0.998	0.998	1.001	1.000	1.000	0.999	1.000
100	1.005	1.006	1.000	1.002	0.999	0.999	1.000	1.000	0.999	1.000	1.000
3000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
EFGM											
-1	15.813	2.922	1.389	1.072	0.979	0.888	0.866	0.866	0.877	0.909	0.933
-0.7	17.016	2.813	1.335	1.082	0.966	0.909	0.869	0.876	0.885	0.914	0.936
-0.4	16.000	2.715	1.333	1.066	0.975	0.908	0.887	0.885	0.890	0.923	0.941
0	15.312	2.856	1.368	1.098	0.997	0.932	0.902	0.903	0.904	0.932	0.948
0.2	15.889	2.747	1.378	1.086	1.014	0.947	0.912	0.907	0.914	0.936	0.951
0.5	14.504	2.916	1.383	1.121	1.034	0.945	0.920	0.912	0.922	0.943	0.958
0.7	13.526	2.728	1.359	1.116	1.035	0.955	0.932	0.930	0.930	0.947	0.963
0.9	13.667	2.794	1.403	1.132	1.021	0.966	0.950	0.931	0.933	0.951	0.961
1	14.245	2.733	1.427	1.132	1.044	0.979	0.947	0.931	0.938	0.952	0.967

A Proofs

Proof of Theorem 1

This follows from Theorems 4.1 and 4.2 of Ibragimov (2009b)

Proof of Theorem 2

We start with the case $n = 2$. Due to independence between ξ_1 and ξ_2, we have that

$$\mathbb{P}\Big(\frac{\xi_1(\beta_1) + \xi_2(\beta_2)}{2} > z\Big) = \beta_1\beta_2 \int_{\frac{s+t}{2}>z} s^{-\beta_1-1}t^{-\beta_2-1}dsdt. \tag{13}$$

Now for non-independent (X_1, X_2) under the EFGM copula, we can write using (13):

$$\begin{aligned}
\mathbb{P}\Big(\frac{X_1 + X_2}{2} > z\Big) &= \int_{\frac{s+t}{2}>z} f_1(s)f_2(t)\big[1 + \gamma(1 - 2F_1(s))(1 - 2F_2(t))\big]dsdt \\
&= \mathbb{P}\Big(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\Big) \\
&\quad +\gamma \int_{\frac{s+t}{2}>z} f_1(s)f_2(t)(1 - 2F_1(s))(1 - 2F_2(t))dsdt \\
&= \mathbb{P}\Big(\frac{\xi_1(\alpha) + \xi_2(\alpha)}{2} > z\Big) \\
&\quad +\gamma\mathbb{E}(1 - 2F_1(\xi))(1 - 2F_2(\eta))I\Big(\frac{\xi_1 + \xi_2}{2} > z\Big),
\end{aligned}$$

where $I(\cdot)$ denotes the indicator function.

Now consider the last term:

$$\begin{aligned}
\int_{\frac{s+t}{2}>z} f_1(s)f_2(t)(1 - 2F_1(s))(1 - 2F_2(t))dsdt &= \int_{\frac{s+t}{2}>z} \alpha^2 s^{-\alpha-1}t^{-\alpha-1}(2s^{-\alpha} - 1)(2t^{-\alpha} - 1)dsdt \\
&= 4\alpha^2 \int_{\frac{s+t}{2}>z} s^{-2\alpha-1}t^{-2\alpha-1}dsdt \\
&\quad -2\alpha^2 \int_{\frac{s+t}{2}>z} s^{-2\alpha-1}t^{-\alpha-1}dsdt \\
&\quad -2\alpha^2 \int_{\frac{s+t}{2}>z} s^{-\alpha-1}t^{-2\alpha-1}dsdt \\
&\quad +\alpha^2 \int_{\frac{s+t}{2}>z} s^{-\alpha-1}t^{-\alpha-1}dsdt \\
&= 4\alpha^2\mathcal{I}_1 - 2\alpha^2\mathcal{I}_2 - 2\alpha^2\mathcal{I}_3 + \alpha^2\mathcal{I}_4,
\end{aligned}$$

where $\mathcal{I}_1 = \mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(2\alpha)}{2} > z\Big)$, $\mathcal{I}_2 = \mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(\alpha)}{2} > z\Big)$, $\mathcal{I}_3 = \mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(2\alpha)}{2} > z\Big)$ and $\mathcal{I}_4 = \mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2} > z\Big)$.

Thus we obtain

$$\begin{aligned}\mathbb{P}\Big(\frac{X+Y}{2}>z\Big) &= (1+\gamma\alpha^2)\mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}>z\Big)\\ &\quad -2\gamma\alpha^2\mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(2\alpha)}{2}>z\Big)\\ &\quad -2\gamma\alpha^2\mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(\alpha)}{2}>z\Big)\\ &\quad +4\gamma\alpha^2\mathbb{P}\Big(\frac{\xi_1(2\alpha)+\xi_2(2\alpha)}{2}>z\Big).\end{aligned}$$

It is a well-known result in the power law literature (see, among others, Corollary 1.3.2 in Embrechts et al., 1997) that, asymptotically as $z\to\infty$,

$$\mathbb{P}\Big(\frac{\xi_1(\beta)+\xi_2(\beta)}{2}>z\Big)\sim 2\mathbb{P}(\xi_1(\beta)>2z)\sim 2^{1-\beta}z^{-\beta} \tag{14}$$

for all $\beta>0$. In addition, if $\beta_1<\beta_2$, then

$$\mathbb{P}\Big(\frac{\xi_1(\beta_1)+\xi_2(\beta_2)}{2}>z\Big)\sim \mathbb{P}(\xi_1(\beta_1)>2z)\sim 2^{-\beta_1}z^{-\beta_1} \tag{15}$$

It follows from (14)-(15) that, as $z\to\infty$,

$$\begin{aligned}\mathbb{P}\Big(\frac{X+Y}{2}>z\Big) &\sim (1+\gamma\alpha^2)2^{1-\alpha}z^{-\alpha}-2\gamma\alpha^2 2^{1-\alpha}z^{-\alpha}+4\gamma\alpha^2 2^{1-2\alpha}z^{-2\alpha} \quad (16)\\ &\sim (1-\gamma\alpha^2)2^{1-\alpha}z^{-\alpha}\\ &\sim \mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}>z\Big).\end{aligned}$$

We now provide a generalization for any n. Let $X_1, X_2, ..., X_n$ have a multidimensional EFGM copula

$$C(u_1,u_2,...,u_n)=u_1u_2...u_n\Big[1+\sum_{c=2}^{n}\sum_{1\le i_1<i_2<...<i_c\le n}\gamma_{i_1,i_2,...,i_c}(1-u_{i_1})(1-u_{i_2})...(1-u_{i_c})\Big], \tag{17}$$

where $\gamma_{i_1,i_2,...,i_c}$ are real constants satisfying certain inequalities that guarantee that (17) represents a proper copula.

Let $X_1, X_2, ..., X_n$ have power law distributions with the same parameter $\alpha>0$. It follows from (17) that the joint cdf of $X_1, X_2, ..., X_n$ has the form

$$\begin{aligned}F(x_1,x_2,...,x_n) &= F_1(x_1)F_2(x_2)...F_n(x_n)\\ &\quad\times\Big[1+\sum_{c=2}^{n}\sum_{1\le i_1<i_2<...<i_c\le n}\gamma_{i_1,i_2,...,i_c}(1-F_{i_1}(x_{i_1}))(1-F_{i_2}(x_{i_2}))...(1-F_{i_c}(x_{i_c}))\Big],\end{aligned}$$

Let, $\xi_1(\beta_1), \xi_2(\beta_2), ..., \xi_n(\beta_n)$ denote the independent random variables with power law distributions with tail indices $\beta_1, \beta_2, ..., \beta_n$, respectively. That is,

$$\mathbb{P}(\xi_i(\beta_i) > x) = x^{-\beta_i}, \tag{18}$$

$x \geq 1$, $i = 1, 2, ..., n,$. In particular, $\xi_1(\alpha), \xi_2(\alpha), ..., \xi_n(\alpha)$ are independent copies of $X_1, X_2, ..., X_n$.

Then, it follows that

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^n X_i > zn\Big) &= \mathbb{P}\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big) \\ &+ \sum_{c=2}^n \sum_{1\leq i_1<i_2<...<i_c\leq n} \gamma_{i_1,i_2,...,i_c} \\ &\times \mathbb{E}\Big[\big(1-2F_{i_1}(\xi_{i_1}(\alpha))\big)\big(1-2F_{i_2}(\xi_{i_2}(\alpha))\big)...\big(1-2F_{i_c}(\xi_{i_c}(\alpha))\big)I\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big)\Big].\end{aligned} \tag{19}$$

Thus, since the random variables $\xi_1(\alpha), \xi(\alpha), ..., \xi_n(\alpha)$ are i.i.d.,

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^n X_i > zn\Big) &= \mathbb{P}\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big) \\ &+ \Big(\sum_{c=2}^n \sum_{1\leq i_1<i_2<...<i_c\leq n} \gamma_{i_1,i_2,...,i_c}\Big) \\ &\times \mathbb{E}\Big[\big(1-2F_1(\xi_1(\alpha))\big)\big(1-2F_2(\xi_2(\alpha))\big)...\big(1-2F_c(\xi_c(\alpha))\big)I\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big)\Big].\end{aligned} \tag{20}$$

Now consider the last term

$$\begin{aligned}&\mathbb{E}\Big[\big(1-2F_1(\xi_1(\alpha))\big)\big(1-2F_2(\xi_2(\alpha))\big)...\big(1-2F_c(\xi_c(\alpha))\big)I\Big(\sum_{i=1}^n \xi_i(\alpha) > zn\Big)\Big] \qquad (21)\\ &= \sum_{s=0}^c \sum_{1\leq j_1<j_2<...<j_s\leq c} (-1)^{c-s} \int_{\sum_{i=1}^n x_i > zn} \prod_{k\in\{j_1,j_2,...,j_s\}} (2\alpha) x_k^{-2\alpha-1} \\ &\qquad \times \prod_{k\in\{1,2,...,n\}\setminus\{j_1,j_2,...,j_s\}} \alpha x_k^{-\alpha-1} dx_1 dx_2 ... dx_n \\ &= \sum_{s=0}^c \sum_{1\leq j_1<j_2<...<j_s\leq c} (-1)^{c-s} \mathbb{P}\Big(\sum_{k\in\{j_1,...,j_s\}} \xi_k(2\alpha) + \sum_{k\in\{1,2,...,n\}\setminus\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > z\Big),\end{aligned}$$

where $1 \leq j_1 < j_2 < ... < j_s \leq c$, $s = 0, 1, ..., c$, $c = 2, ..., n$, $(s, c) \neq (n, n)$ (and, thus, $(j_1, j_2, ..., j_c)$ is different from $(1, 2, ..., n)$).

Consequently, for large z, we obtain

$$\mathbb{P}\Big(\sum_{k\in\{j_1,j_2,...,j_s\}} \xi_k(2\alpha) + \sum_{k\in\{1,2,...,n\}\setminus\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > z\Big) \sim \mathbb{P}\Big(\sum_{k\in\{1,2,...,n\}\setminus\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > zn\Big). \tag{22}$$

In addition, by Corollary 1.3.2 of Embrechts et al. (1997), we have, for large $z > 0$,

$$\mathbb{P}\Big(\sum_{k\in\{1,2,...,n\}\setminus\{j_1,j_2,...,j_s\}} \xi_k(\alpha) > z\Big) \sim (n-s)\mathbb{P}(\xi_1(\alpha) > zn) \sim \frac{n-s}{z^\alpha n^\alpha}. \tag{23}$$

So, for for $s = c = n$, $(j_1, j_2, ..., j_n) = (1, 2, ..., n)$,

$$\mathbb{P}\Big(\sum_{k=1}^{n} \xi_k(2\alpha) > zn\Big) \sim n\mathbb{P}(\xi_1(2\alpha) > zn) \sim \frac{n}{z^{2\alpha} n^{2\alpha}}. \tag{24}$$

From (21)-(24) it follows that, with $1 \le j_1 < j_2 < ... < j_s \le c$, $s = 0, 1, ..., c$, $c = 2, ..., n$, $(s, c) \neq (n, n)$,

$$\mathbb{E}\Big[\big(1-2F_1(\xi_1(\alpha))\big)\big(1-2F_2(\xi_2(\alpha))\big)...\big(1-2F_c(\xi_c(\alpha))\big)I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big] \sim$$

$$\sum_{s=0}^{c}\sum_{1\le j_1<j_2<...<j_s\le c}(-1)^{c-s}\frac{n-s}{z^\alpha n^\alpha} = \Big(\sum_{s=0}^{c}(-1)^{c-s}C_c^s\Big)z^{-\alpha}n^{1-\alpha} - \Big(\sum_{s=0}^{c}(-1)^{c-s}sC_c^s\Big)z^{-\alpha}n^{-\alpha}, \tag{25}$$

where $C_c^s = c!/(s!(c-s)!)$ denotes binomial coefficients.

Now, by the well-known identity for binomial coefficients,

$$\sum_{s=0}^{c}(-1)^{c-s}C_c^s = \sum_{s=0}^{c}(-1)^s C_c^s = 0, \tag{26}$$

$$\sum_{s=0}^{c}(-1)^{c-s}sC_c^s = c\sum_{s=1}^{c}(-1)^{c-s}C_{c-1}^{s-1} = -c\sum_{s=0}^{c-1}(-1)^{c-1-s}C_{c-1}^s = 0. \tag{27}$$

It thus follows that $\mathbb{P}\left(\sum_{i=1}^{n} X_i > zn\right) \sim \mathbb{P}\left(\sum_{i=1}^{n}\xi_i(\alpha) > zn\right)$.

Proof of Corollary 1

The result follows trivially by inversion of the cdfs in Theorem 2 for $n = 2$ and applying equations (2)-(3).

Proof of Theorem 3

The density corresponding to (12) is a polynomial of a lower order, which we write in the following generic form:

$$c(u_1, \ldots, u_n) = \sum_{k_1,\ldots,k_n=0,1,\ldots} \phi_{k_1,k_2,\ldots,k_n} \cdot u_1^{k_1} \cdot u_2^{k_2} \cdot \ldots \cdot u_n^{k_n}, \tag{28}$$

Then, using arguments similar to Theorem 2,

$$\begin{aligned}\mathbb{P}\Big(\sum_{i=1}^{n} X_i > zn\Big) &= \mathbb{E}\Big[\sum_{k_i\in\{0,1,\ldots\}} \phi_{k_1,k_2,\ldots,k_n} \qquad (29)\\ &\quad \times F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))\ldots F_n^{k_n}(\xi_n(\alpha))I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big]\\ &= \mathbb{P}\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big) + \mathbb{E}\Big[\sum_{k_i\in\{0,1,\ldots\}\setminus\{k_i=0\forall i\}} \phi_{k_1,k_2,\ldots,k_n} \qquad (30)\\ &\quad \times F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))\ldots F_n^{k_n}(\xi_n(\alpha))I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big].\end{aligned}$$

Now consider the last term.

$$\begin{aligned}&\mathbb{E}\Big[\sum_{k_i\in\{0,1,\ldots\}\setminus\{k_i=0\forall i\}} \phi_{k_1,k_2,\ldots,k_n} F_1^{k_1}(\xi_1(\alpha))F_2^{k_2}(\xi_2(\alpha))\ldots F_n^{k_n}(\xi_n(\alpha))I\Big(\sum_{i=1}^{n}\xi_i(\alpha) > zn\Big)\Big] \qquad (31)\\ &= \int_{\sum_{i=1}^{n} s_i > nz} \sum_{k_i\in\{0,1,\ldots\}} \psi_{k_1,k_2,\ldots,k_n} s_1^{-\alpha(k_1+1)} s_2^{-\alpha(k_2+1)} \ldots s_n^{-\alpha(k_n+1)} ds_1\ldots ds_n\\ &= \sum_{k_i\in\{0,1,\ldots\}} \psi_{k_1,k_2,\ldots,k_n} \int_{\sum_{i=1}^{n} s_i > nz} s_1^{-\alpha(k_1+1)} s_2^{-\alpha(k_2+1)} \ldots s_n^{-\alpha(k_n+1)} ds_1\ldots ds_n\\ &= \sum_{k_i\in\{0,1,\ldots\}} \psi_{k_1,k_2,\ldots,k_n} \mathbb{P}\left(\frac{\xi_1(\alpha(k_1+1)) + \ldots + \xi_n(\alpha(k_n+1))}{n} > z\right),\end{aligned}$$

where the new coefficients ψ's are different from ϕ's because we have expressed $(1-s_i^{-\alpha})^{k_i}$ in terms of powers of s_i^{α}. Now, using the same arguments as for (14)-(15),

$$\mathbb{P}\Big(\frac{\xi_1(\alpha) + \ldots + \xi_n(\alpha)}{n} > z\Big) \sim n\mathbb{P}(\xi_1(\alpha) > nz) \sim n^{1-\alpha}z^{-\alpha}$$

$$\mathbb{P}\left(\frac{\xi_1(\alpha(k_1+1)) + \ldots + \xi_n(\alpha(k_n+1))}{n} > z\right) \sim \mathbb{P}(\xi_1(\alpha) > nz) \sim n^{-\alpha}z^{-\alpha},$$

for all $k_i \geq 0$. It thus follows that $\mathbb{P}\left(\sum_{i=1}^{n} X_i > zn\right) \sim \mathbb{P}\left(\sum_{i=1}^{n}\xi_i(\alpha) > zn\right)$.

Proof of Corollary 2

Clearly if a copula has a Taylor expansion of the form (12) in a neighborhood of a point, the validity of Theorem 2 will be limited to that neighborhood.

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