Appendix A

Definitions

Throughout this thesis, the following notation has been used consistently. Shown below is a brief description of the symbol, and an appropriate page reference. Other definitions used in the body of the thesis are defined where they are used.

Table A.1: Notations and definitions.

Notation	Description	Page
General		
N	Number of individuals within a group.	
t	Time.	
T	Total time.	
$\mathbf{r}_i(t)$	Position vector of individual i at time t.	
$\mathbf{v}_i(t)$	Direction vector of individual i at time t.	
$\mathbf{r}_{ij}(t)$	Unit vector pointing from individual i in the direction of neighbour j.	
$\sim$	Shorthand for 'distributed as'.	
$\approx$	Shorthand for 'approximately equal to'.	
$\xrightarrow{D}$	Shorthand for 'converges in distribution to'.	
$\xrightarrow{p}$	Shorthand for 'converges in probability to'.	
q	Number of dimensions.	
$\mathbb{R}^q$	Real space in q dimensions.	
Ω_q	Unit hypersphere in $\mathbb{R}^q$.	
$\mathbf{x}$	Random vector in $\mathbb{R}^q$.	
$\mathbf{x}_i$	Random sample of random vector $\mathbf{x}_i$.	
$\lvert\mathbf{x}\rvert$	Magnitude of vector $\mathbf{x}$.	
$\mathbf{x}^T$	Transpose of vector $\mathbf{x}$.	
$\mathrm{E}\left(\mathbf{x}\right)$	Expected value of the random vector $\mathbf{x}$.	
$\mathrm{Var}\left(\mathbf{x}\right)$	Variance of the random vector $\mathbf{x}$.	
$N\left(\mu, \sigma^2\right)$	Gaussian distribution with mean μ and variance σ^2.	

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Notation	Description	Page
		Continued from previous page
χ^2_ν	Chi-squared distribution with ν degrees of freedom.	
T	Test statistic.	
	Chapter 2	
$\mathbf{r}_{group}$	Group centre	26
p_{group}	Group polarisation.	27
m_{group}	Group momentum.	27
$\hat{\sigma}$	Spherical variance.	27
NND	Nearest neighbour distance.	27
Expanse	Expanse of group.	28
NGDR	Net to gross displacement ratio.	28
ρ_q	Spherical correlation coefficient.	30
$\widehat{\rho}_q$	Estimate of ρ_q.	30
ψ	Angle between the spherical mean and x-axis.	32
$\mathbf{S}_N$	Resultant vector of the random vectors $\mathbf{x}_i$.	26
$\boldsymbol{\mu}$	Expected value of random vector $\mathbf{x}$.	26
$\hat{\boldsymbol{\mu}}$	Average group direction (estimate of $\boldsymbol{\mu}$).	26
Σ	Covariance matrix of random vector $\mathbf{x}$.	34
$N_q(\boldsymbol{\mu}, \Sigma)$	Gaussian distribution in $\mathbb{R}^q$ with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.	34
c	Scalar random variable.	35
$\mathrm{E}\,(c)$	Concentration (polarisation) of the distribution on Ω_q about $\boldsymbol{\mu}$.	36
$\mathbf{S}_N^\perp$	Component of $\mathbf{S}_N$ orthogonal to $\boldsymbol{\mu}$.	38
H	$I_q - \boldsymbol{\mu}\boldsymbol{\mu}^T$.	38
$\boldsymbol{\mu}_0$	Hypothesised direction.	40
$\mathbf{S}_N^{\perp\boldsymbol{\mu}_0}$	Component of $\mathbf{S}_N$ orthogonal to $\boldsymbol{\mu}_0$.	40
K	Number of samples.	41
N_k	Sample size of the k-th sample.	41
$\mathbf{S}_{N_k}$	Resultant vector from the k-th sample.	41
	Chapters 3 and 4	
τ	Time interval width.	50
$\mathbf{v}_i^d(t)$	Preferred travel direction of individual i.	50
r_r	Radius of the zone of repulsion.	50
r_a	Radius of the zone of attraction.	50
r_o	Radius of the zone of orientation.	50
Δr_r	Width of the zone of repulsion.	60
Δr_a	Width of the zone of attraction.	60
		Continued on next page

Continued from previous page

Notation	Description	Page
Δr_o	Width of the zone of orientation.	60
$\mathbf{v}_i^{repulse}(t)$	Preferred travel direction due to repulsive forces.	51
$\mathbf{v}_i^{orient}(t)$	Preferred travel direction due to alignment forces.	52
$\mathbf{v}_i^{attract}(t)$	Preferred travel direction due to attraction forces.	52
δ	Field of perception (degrees).	52
γ	Turning rate (degrees).	53
s_i	Speed of individuals in collective model.	53
θ	Longitude.	55
ϕ	Colatitude.	55
κ	Concentration parameter.	56
$N_{informed}$	Number of knowledgeable individuals in group.	95
$\mathbf{r}_{informed}(t)$	Position vector of knowledgeable individual i.	95
$\mathbf{v}_{informed}(t)$	Direction vector of knowledgeable individual i.	95
Chapter 5		
$V(\mathbf{r}_i)$	Social potential function of individual i.	143
$\mathbf{F}$	Force acting on an individual ($\mathbf{F} = -\nabla V$).	139
∇V	Direction differential operator on function V.	139
$V_{repulsion}(\mathbf{r}_i)$	Repulsion potential of individual i.	144
$V_{attraction}(\mathbf{r}_i)$	Attraction potential of individual i.	144
$V_{alignment}(\mathbf{r}_i)$	Alignment potential of individual i.	146
$\mathbf{F}_{repulsion}(\mathbf{r}_i)$	Repulsion force acting on individual i.	144
$\mathbf{F}_{attraction}(\mathbf{r}_i)$	Attraction force acting on individual i.	144
$\mathbf{F}_{cohesion}(\mathbf{r}_i)$	Attraction and repulsion model.	144
$\mathbf{F}_{alignment}(\mathbf{r}_i)$	Alignment force acting on individual i.	147
$\mathbf{F}_{dissipate}(\mathbf{r}_i)$	Dissipative force acting on individual i.	151
$\mathbf{F}_{total}(\mathbf{r}_i)$	Alignment model.	152
A	Repulsion magnitude parameter.	144
B	Repulsion range parameter.	144
C	Potential function parameter.	144
D	Attraction magnitude parameter.	144
E	Attraction range parameter.	144
F	Alignment magnitude parameter.	147
a	Alignment exponent parameter.	147
G	Friction coefficient.	151
ω_1	Ratio of cohesive magnitudes (A/D).	149
ω_2	Ratio of cohesive ranges (E/B).	149
r^*	Comfortable distance.	150

Appendix B

Proofs for selected results

We present mathematical proofs for selected results in this thesis.

Proof 1. *If A is any orthogonal matrix and Σ_y is a covariance matrix, then $\Sigma_y = A\Sigma_y A^T \Rightarrow \Sigma_y = kI_q$, where k is a constant.*

If

$$\Sigma_y = A\Sigma_y A^T$$

then

$$\Sigma_y A = A\Sigma_y \qquad \text{(A is orthogonal, } A^{-1} = A^T\text{).}$$

Hence, Σ_y is a real $q \times q$ matrix that commutes with all orthogonal matrices.

Let $i, j = 1, 2, \ldots, q$. The (i,j)th entry of Σ_y is σ_{ij}. Assume that $i < j$. Let the matrix A be the $q \times q$ identity matrix, except the (j,j)th entry is -1. The (i,j)th entry of $A\Sigma_y$ is σ_{ij}. The (i,j)th entry of $\Sigma_y A$ is $-\sigma_{ij}$. The (j,i)th entry of $A\Sigma_y$ is $-\sigma_{ji}$. The (j,i)th entry of $\Sigma_y A$ is σ_{ji}. But $\Sigma_y A = A\Sigma_y$, so $\sigma_{ij} = -\sigma_{ij}$ and $\sigma_{ji} = -\sigma_{ji}$. Hence $\sigma_{ij} = \sigma_{ji} = 0$ ($i < j$). Consequently, Σ_y is a diagonal matrix.

Now, let A^* be any orthogonal matrix whose (i,j)th entry is a_{ij} (where $a_{ij} \neq 0$). Let the (i,i)th element of Σ_y be denoted σ_i and the (j,j)th be σ_j. The (i,j)th element of $\Sigma_y A^*$ is $\sigma_i a_{ij}$. The (i,j)th element of $A^*\Sigma_y$ is $\sigma_j a_{ij}$. Hence $\sigma_i a_{ij} = \sigma_j a_{ij}$ and $\sigma_i = \sigma_j = \sigma$ ($\forall i, j = 1, 2, \ldots, q$). So $\Sigma_y = \sigma I_q$.

□

Proof 2. *Justification for (2.33).*

Effectively, we want to show

$$\frac{|\mathbf{S}_N|}{N} = \frac{\boldsymbol{\mu}^T\mathbf{S}_N}{N} + \frac{1}{2\mathrm{E}(c)}\frac{|\mathbf{S}_N^{\perp}|^2}{N^2} + O\left(N^{-3/2}\right). \tag{B.1}$$

Using (2.28), we can divide by a factor of $\sqrt{N}\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)$ to obtain

$$\begin{aligned}\frac{|\mathbf{S}_N|}{N} - \mathrm{E}(c) &= \left(\frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} - \mathrm{E}(c)\right)\frac{\left(\frac{(\boldsymbol{\mu}^T\mathbf{s}_N)}{N} + \mathrm{E}(c)\right)}{\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)} \\ &\quad + \frac{|\mathbf{S}_N^{\perp}|^2}{N^2}\frac{1}{\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)}.\end{aligned} \tag{B.2}$$

Firstly, let's consider the $\left(\left(\boldsymbol{\mu}^T\mathbf{S}_N\right)/N + \mathrm{E}(c)\right)/\left(|\mathbf{S}_N|/N + \mathrm{E}(c)\right)$ term. We know $|\mathbf{S}_N|^2 = \left(\boldsymbol{\mu}^T\mathbf{S}_N\right)^2 + |\mathbf{S}_N^{\perp}|^2$. We can therefore write this term as

$$\begin{aligned}\frac{\left(\frac{(\boldsymbol{\mu}^T\mathbf{s}_N)}{N} + \mathrm{E}(c)\right)}{\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)} &= \left[\frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} + \mathrm{E}(c)\right]\left[\frac{\boldsymbol{\mu}^T\mathbf{S}_N}{N}\sqrt{1 + \frac{|\mathbf{S}_N^{\perp}|^2}{(\boldsymbol{\mu}^T\mathbf{S}_N)^2}} + \mathrm{E}(c)\right]^{-1} \\ &= \left[\frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} + \mathrm{E}(c)\right]\left[\frac{\boldsymbol{\mu}^T\mathbf{S}_N}{N}\left(1 + \frac{1}{2}\frac{|\mathbf{S}_N^{\perp}|^2}{(\boldsymbol{\mu}^T\mathbf{S}_N)^2} + \ldots\right) + \mathrm{E}(c)\right]^{-1} \\ &= \left[\frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} + \mathrm{E}(c)\right]\left[\frac{\boldsymbol{\mu}^T\mathbf{S}_N}{N} + \mathrm{E}(c) + \frac{1}{2}\frac{|\mathbf{S}_N^{\perp}|^2}{(\boldsymbol{\mu}^T\mathbf{S}_N)N} + \ldots\right]^{-1} \\ &= 1 - \frac{|\mathbf{S}_N^{\perp}|^2}{2N}\frac{1}{(\boldsymbol{\mu}^T\mathbf{S}_N)\left[\frac{(\boldsymbol{\mu}^T\mathbf{s}_N)}{N} + \mathrm{E}(c)\right]} + \ldots.\end{aligned} \tag{B.3}$$

We know that $\frac{1}{N}|\mathbf{S}_N^{\perp}|^2 \xrightarrow{D} K\chi^2_{q-1}$ (Lemma 5), where $K = (1 - E(c^2))/(q-1)$. Hence, we can express $\frac{1}{N}|\mathbf{S}_N^{\perp}|^2 = O_p(1)$. Also, $\frac{\boldsymbol{\mu}^T\mathbf{s}_N}{N} \xrightarrow{p} \mathrm{E}(c)$. Consequently,

$$\frac{\left(\frac{(\boldsymbol{\mu}^T\mathbf{s}_N)}{N} + \mathrm{E}(c)\right)}{\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)} = 1 + O_p\left(N^{-1}\right). \tag{B.4}$$

We can write (B.2) as

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} - \mathrm{E}(c) &= \left(\frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} - \mathrm{E}(c)\right)\left(1 + O_p\left(\frac{1}{N}\right)\right) \\ &\quad + \frac{|\mathbf{S}_N^{\perp}|^2}{N^2}\frac{1}{\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)}. \end{aligned} \tag{B.5}$$

Now, using Lemma 3, $\frac{1}{\sqrt{N}}\left(\boldsymbol{\mu}^T\mathbf{S}_N - N\mathrm{E}(c)\right) \xrightarrow{D} N(0, \mathrm{Var}(c))$. We can express $\frac{\boldsymbol{\mu}^T\mathbf{S}_N}{N} - \mathrm{E}(c) = O_p\left(N^{-\frac{1}{2}}\right)$. Then

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} - \mathrm{E}(c) &= \frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} - \mathrm{E}(c) + O_p\left(N^{-3/2}\right) \\ &\quad + \frac{|\mathbf{S}_N^{\perp}|^2}{N^2}\frac{1}{\left(\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c)\right)}. \end{aligned} \tag{B.6}$$

Now concentrate on the last term in the right hand side of (B.6). Recall $\frac{|\mathbf{S}_N|}{N} \xrightarrow{p} \mathrm{E}(c)$, to be more precise $\frac{|\mathbf{S}_N|}{N} = \mathrm{E}(c) + O\left(N^{-\frac{1}{2}}\right)$ (Lemma 6). So $\frac{|\mathbf{S}_N|}{N} + \mathrm{E}(c) \xrightarrow{p} 2\mathrm{E}(c)$ (a positive quantity) at a rate $N^{-\frac{1}{2}}$. Hence, we can write

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} &= \frac{(\boldsymbol{\mu}^T\mathbf{S}_N)}{N} + O_p\left(N^{-3/2}\right) + \frac{|\mathbf{S}_N^{\perp}|^2}{N^2 2\mathrm{E}(c)} + O_p\left(N^{-\frac{1}{2}}\right)\frac{|\mathbf{S}_N^{\perp}|^2}{N^2} \\ &= \frac{\boldsymbol{\mu}^T\mathbf{S}_N}{N} + \frac{|\mathbf{S}_N^{\perp}|^2}{N^2 2\mathrm{E}(c)} + O_p\left(N^{-3/2}\right)\ . \end{aligned} \tag{B.7}$$

□

Proof 3. *Justification for (2.34).*

We want to show

$$|\sum_{k=1}^{K} w_k\mathbf{S}_{N_k}| = \boldsymbol{\mu}^T\sum_{k=1}^{K} w_k\mathbf{S}_{N_k} + \frac{1}{2}\frac{|\sum_{k=1}^{K} w_k\mathbf{S}_{N_k}^{\perp}|^2}{\sum_{k=1}^{K} w_k N_k\mathrm{E}(c_k)} + O_p\left(N^{-\frac{1}{2}}\right).$$

To do this, the version of Lemma 2 we require is

$$\frac{1}{\sqrt{N_k}}\left(\mathbf{S}_{N_k} - N_k\mathrm{E}(c_k)\boldsymbol{\mu}\right) \xrightarrow{D} N_q(\mathbf{0}, \Sigma_k) \tag{B.8}$$

as $N_k \to \infty$. Let $N = \sum_{k=1}^{K} N_k$ and let $N_k = \alpha_k N$, where the α_k are positive constants that sum to 1. From (B.8), we can derive

$$\frac{1}{\sqrt{N}} \left(\sum_{k=1}^{K} w_k \mathbf{S}_{N_k} - \sum_{k=1}^{K} w_k N_k \mathrm{E}\left(c_k\right) \boldsymbol{\mu} \right) \xrightarrow{D} N_q \left(\mathbf{0},\, \Gamma \right), \tag{B.9}$$

where $\Gamma = \sum_{k=1}^{K} w_k^2 \alpha_k \Sigma_k$. A necessary condition for (B.9) to hold is $\min(N_k) \to \infty$. Equivalently,

$$\frac{1}{\sqrt{N}} \left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k} - N \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) \right) \xrightarrow{D} N\left(0,\, \boldsymbol{\mu}^T \Gamma \boldsymbol{\mu}\right), \tag{B.10}$$

as $\min(N_k) \to \infty$. Now, let $\mathbf{S}_{N_k}^{\perp}$ be the component of $\mathbf{S}_{N_k}$ orthogonal to $\boldsymbol{\mu}$ (the common mean direction under the null hypothesis). Hence, $\mathbf{S}_{N_k}^{\perp} = H\mathbf{S}_{N_k}$ and $\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp} = H \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}$, where $H = I_q - \boldsymbol{\mu}\boldsymbol{\mu}^T$. From Lemma 4,

$$\frac{1}{\sqrt{N_k}} \left(\mathbf{S}_{N_k}^{\perp} \right) \xrightarrow{D} N_q \left(\mathbf{0},\, \frac{1 - \mathrm{E}\left(c_k^2\right)}{q-1} H \right),$$

as $N_k \to \infty$. So,

$$\frac{1}{\sqrt{N}} \left(\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp} \right) \xrightarrow{D} N_q \left(\mathbf{0},\, \sum_{k=1}^{K} w_k^2 \alpha_k \frac{1 - \mathrm{E}\left(c_k^2\right)}{q-1} H \right), \tag{B.11}$$

as $\min(N_k) \to \infty$. The covariance matrix of $\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}$ is $q \times q$ and is of rank $q-1$. Consequently,

$$\frac{1}{N} \left(|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2 \right) \xrightarrow{D} \left(\sum_{k=1}^{K} w_k^2 \alpha_k \frac{1 - \mathrm{E}\left(c_k^2\right)}{q-1} \right) \chi_{q-1}^2, \tag{B.12}$$

as $\min(N_k) \to \infty$.

Now, using (2.27), it follows that

$$\frac{\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} \xrightarrow{p} \boldsymbol{\mu} \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) \tag{B.13}$$

and

$$\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} \xrightarrow{p} \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right). \tag{B.14}$$

Also notice

$$|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|^2 = \left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}\right)^2 + |\sum_{k=1}^{K} w_k \mathbf{S}^{\perp}_{N_k}|^2. \tag{B.15}$$

From (B.14),

$$\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) \xrightarrow{p} 2 \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right). \tag{B.16}$$

Using (B.10)

$$\frac{1}{N}\left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}\right) - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) \xrightarrow{p} 0, \tag{B.17}$$

as $\min(N_k) \to \infty$. Hence,

$$\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) \xrightarrow{p} 2 \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right). \tag{B.18}$$

Also, from (B.12)

$$\frac{1}{N^{3/2}}\left(|\sum_{k=1}^{K} w_k \mathbf{S}^{\perp}_{N_k}|^2\right) = O_p\left(N^{-\frac{1}{2}}\right), \tag{B.19}$$

as $\min(N_k) \to \infty$.

Now, consider

$$\begin{aligned}
\sqrt{N} & \left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)\left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) \\
= & \sqrt{N}\left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|^2}{N^2} - \left(\sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)^2\right) \\
= & \sqrt{N}\left(\left(\frac{\left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}\right)^2}{N^2} - \left(\sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)^2\right) + \frac{|\sum_{k=1}^{K} w_k \mathbf{S}^{\perp}_{N_k}|^2}{N^2}\right) \\
= & \sqrt{N}\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) \\
& + \frac{|\sum_{k=1}^{K} w_k \mathbf{S}^{\perp}_{N_k}|^2}{N^{3/2}}.
\end{aligned} \tag{B.20}$$

Taking limits in (B.20), using (B.16), (B.18) and (B.19), we can see that the limiting distribution of $\sqrt{N}\left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)$ is the same as that of $\sqrt{N}\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)$. By (B.10), this is

$$\sqrt{N}\left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) \xrightarrow{D} N\left(0,\ \boldsymbol{\mu}^T \Gamma \boldsymbol{\mu}\right), \quad \text{(B.21)}$$

as $\min(N_k) \to \infty$.

If we divide (B.20) by a factor of $\sqrt{N}\left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)$, then

$$\begin{aligned} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) &= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) \\ &\times \frac{\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)}{\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)} \\ &+ \frac{1}{N^2} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)}. \end{aligned} \quad \text{(B.22)}$$

Consider the ratio in (B.22)

$$\begin{aligned} \frac{\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)}{\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)} &= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) \\ &\times \left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)^{-1}. \end{aligned}$$

We can manipulate the second term on the right hand side of this equation

$$
\begin{aligned}
&\left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)^{-1} \\
= &\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} \sqrt{1 + \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}\right)^2}} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)^{-1} \\
= &\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) + \frac{1}{2N} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}} + \dots\right)^{-1} \\
= &\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)^{-1} \\
\times &\left(1 - \frac{1}{2N} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}\right)\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right)} + \dots\right).
\end{aligned}
$$

Using this expression, with (B.12), (B.18) and realising from (B.17) that $\left(\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}\right)^{-1} = O_p\left(N^{-1}\right)$, as $\min(N_k) \to \infty$, we can express the ratio as

$$
\frac{\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)}{\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)} = 1 + O_p\left(N^{-1}\right), \tag{B.23}
$$

as $\min(N_k) \to \infty$. We can now express (B.22) as

$$
\begin{aligned}
\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right) = &\left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) \\
\times &\left(1 + O_p\left(N^{-1}\right)\right) \\
+ &\frac{1}{N^2} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)}.
\end{aligned} \tag{B.24}
$$

Hence, (B.24) can be written as

$$\begin{aligned}\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} &= \frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + O_p\left(N^{-3/2}\right) \\ &+ \frac{1}{N^2} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)},\end{aligned} \tag{B.25}$$

due to (B.10). Consider

$$|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2 / \left(\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right) N^2.$$

By (B.16), this converges to the same limit as

$$|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2 / \left(2N^2 \sum_{k=1}^{K} w_k \alpha_k \mathrm{E}\left(c_k\right)\right).$$

Therefore, (B.25) can be written as

$$\begin{aligned}\frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}|}{N} &= \frac{\boldsymbol{\mu}^T \sum_{k=1}^{K} w_k \mathbf{S}_{N_k}}{N} + \frac{1}{2N} \frac{|\sum_{k=1}^{K} w_k \mathbf{S}_{N_k}^{\perp}|^2}{\sum_{k=1}^{K} w_k N_k \mathrm{E}\left(c_k\right)} \\ &+ O_p\left(N^{-3/2}\right).\end{aligned} \tag{B.26}$$

Multiply (B.26) by a factor of N and we have (2.34). □

Proof 4. *Elaboration on $2T$ (2.38) being distributed as a $\chi^2_{(K-1)(q-1)}$ random variable.*

Let $\mathbf{y}_r = \sum_{s=1}^{K} a_{rs} \mathbf{z}_{N_s}$, where the $\mathbf{z}_{N_s}$'s are independent and $\mathbf{z}_{N_s} \sim N_q(0, H)$ with $H = I_q - \boldsymbol{\mu}\boldsymbol{\mu}^T$ and $A = (a_{rs})$ is orthogonal. We define $Z = (\mathbf{z}_{N_1}, \ldots, \mathbf{z}_{N_K})^T$. Then $Y = (\mathbf{y}_1, \ldots, \mathbf{y}_K)^T = AZ$. Hence, $\mathbf{y}_r \sim N_q(0, H)$.

The last row of A is $(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_K})$ (as $\lambda_1 + \lambda_2 + \ldots + \lambda_K = 1$). Now,

$$
\begin{aligned}
\mathbf{y}_K &= \sum_{r=1}^{K} \sqrt{\lambda_r} \mathbf{z}_{N_r}, \\
\sum_{r=1}^{K} |\mathbf{y}_r|^2 &= \sum_{r=1}^{K} \mathbf{y}_r^T \mathbf{y}_r \\
&= \text{Trace}\left(YY^T\right) \\
&= \text{Trace}\left(AZZ^TA^T\right) \\
&= \text{Trace}\left(A^TAZZ^T\right) \\
&= \text{Trace}\left(ZZ^T\right) \\
&= \sum_{r=1}^{K} |\mathbf{z}_{N_r}|^2, \qquad \text{(B.27)}
\end{aligned}
$$

$$
\begin{aligned}
2T &= \sum_{r=1}^{K} |\mathbf{z}_{N_r}|^2 - |\sum_{r=1}^{K} \sqrt{\lambda_r} \mathbf{z}_{N_r}|^2 \\
&= \sum_{r=1}^{K} |\mathbf{y}_r|^2 - |\mathbf{y}_K|^2 \\
&= \sum_{r=1}^{K-1} |\mathbf{y}_r|^2. \qquad \text{(B.28)}
\end{aligned}
$$

Now, the $\mathbf{y}_r$ vectors are independent and each one has $|\mathbf{y}_r|^2 \sim \chi^2_{q-1}$ by Lemma 5. Consequently, $\sum_{r=1}^{K-1} |\mathbf{y}_r|^2 \sim \chi^2_{(K-1)(q-1)}$ (sum of $(K-1)$ independent χ^2 random variables). □

Proof 5. *Proof of degeneration of the Fisher distribution to a spherical Uniform distribution, as κ decreases.*

Notice

$$
\begin{aligned}
\lim_{\kappa \to 0} C_F &= \lim_{\kappa \to 0} \frac{\kappa}{2\pi(e^\kappa - e^{-\kappa})} \\
&= \frac{1}{2\pi} \lim_{\kappa \to 0} \frac{1}{e^\kappa + e^{-\kappa}} \qquad \text{(L'Hospital's Rule)} \\
&= \frac{1}{4\pi}.
\end{aligned}
$$

Hence, Equation 3.6 becomes

$$f(\theta, \phi) \quad = \quad \frac{1}{4\pi}\sin\,\theta.$$

This is the probability density function of a spherical Uniform distribution (Fisher et al. 1987). □