Chapter 5

Problems on Set Systems

5.1 Introduction

In this chapter, we describe self-reduction operations for problems on set systems. A *set system* U is composed of a set of sets $U_1, \ldots, U_n$ containing elements from a *ground set* $S(U)$.

In general, the set systems used in the present work will have a positive integer weight $w(U, U_i)$ associated with each set $U_i \in U$. An unweighted set system can be treated as a set system with all weights equal to one. Given a set of sets Y, the sum of the weights of the sets in Y will be denoted by $w(U, Y)$.

For $u \in S(U)$, $R(U, u)$ will denote the set $\{U_i | u \in U_i, U_i \in U\}$, that is, the set of sets to which u belongs. For a set $X \subseteq S(U)$, $R(U, X)$ will denote the set of sets such that each set contains at least one element of X.

For $u \in S(U)$, $U - u$ will denote the set system $\{U_i - u | U_i \in U\}$, that is, the set system with the element u removed from every set in the system.

The reduction rules listed in this chapter have appeared in various sources [28, 45, 12, 47], but little or no theoretical work has been done.

u	$w(u)$
u_1	2
u_2	1
u_3	2
u_4	3
u_5	2

Table 5.1.1: A bin-packing problem

5.1.1 The Bin Packing Problem

Let U be a set of n items $\{u_1, \dots, u_n\}$, with each item u_i associated with a size $w(u_i)$, and let the bin capacity B be a positive number. A *bin packing* of U into bins of size B is a partition $\{U_1, \dots, U_k\}$ of U such that for each U_i, $w(U_i) \le B$. The *bin packing number* $b(U, B)$ of a set U and bin capacity B is the minimum cardinality of a bin packing of U into bins of size B. A *minimum bin packing* of U into bins of size B is a bin packing requiring $b(U, B)$ bins. Table 5.1.1 shows a set of items that can be packed into three bins $(\{u_1, u_3\}, \{u_2, u_4\}, \{u_5\})$ if $B = 4$, or two bins $(\{u_1, u_2, u_3\}, \{u_4, u_5\})$ if $B = 5$.

5.1.2 The Set Covering Problem

Let $U = \{U_1, \dots, U_n\}$ be a set system on a ground set $S(U) = \{u_1, \dots, u_m\}$. A *set cover* for U is a subset C of U such that, for every $u \in S(U)$, u is an element of some set contained in C. The *covering number* $v(U)$ of U is the minimum cardinality of a set cover for U. A *minimum set cover* for U is a set cover for U with cardinality $v(U)$. An *exact set cover* C for U is a set cover such that for each $u \in S(U)$, u is an element of exactly one set contained in C. The *exact covering number* $v^*(U)$ is the weight of a minimum exact set cover for U. Table 5.1.2 shows a set system with a minimum covering of $\{U_2, U_4\}$ and an exact covering of $\{U_1, U_5, U_6\}$.

U_1	$\{u_4\}$
U_2	$\{u_1, u_3, u_4\}$
U_3	$\{u_2, u_3, u_4\}$
U_4	$\{u_1, u_2, u_5\}$
U_5	$\{u_3, u_5\}$
U_6	$\{u_1, u_2\}$

Table 5.1.2: A set system with covering number two, and exact covering number three.

5.2 Self-Reductions

Several papers have considered reductions for the bin packing problem [47], and both the exact and non-exact versions of the set covering problem, for example, [28, 45, 12]. The following summarises the reductions found in these papers, given in their most general form. The following also notes some useful restrictions on the general reduction rules.

5.2.1 Dominant Feasible Set Reduction (DF)

Martello and Toth [47] describe a self-reduction rule for bin packing based on what they call *dominance* between sets of items chosen from the problem universe.

A *feasible set* F is a subset of the items from the universe U such that the sum of the sizes of the elements of F is no greater than the bin capacity B. A feasible set F_1 *dominates* a feasible set F_2 if and only if $b(U - F_1, B) \leq b(U - F_2, B)$. If a feasible set F dominates all other feasible sets of U, clearly we can obtain an optimum solution by setting $U_1 = F$ and $b(U, B) = b(U - F, B) + 1$.

Definition 5.2.1 Let F be a feasible set containing an item u dominating all other feasible sets of U that contain u. A *dominant feasible set reduction* $\langle \mathrm{DF}, F, u \rangle$ deletes all elements of F from the problem.

Theorem 5.1 If ξ is a dominant feasible set reduction on a set U, then $b(U, B) = b(U^{\xi}, B) + 1$.

Proof. The result is immediate from the foregoing discussion, and is discussed in Section 3 of [47]. □

Obviously, such a reduction procedure requires the solution of many bin packing problems, and so is unlikely to be very attractive. Martello and Toth give a sufficient condition for dominance, which they use in their reduction algorithm.

Theorem 5.2 If a feasible set F_1 has a subset $\{u_1, \ldots, u_k\}$ and a feasible set F_2 has a partition $P = \{P_1, \ldots, P_k\}$ such that $w(u_i) \geq w(P_i)$ for all $i = 1, \ldots, k$, then F_1 dominates F_2.

Proof. This is the "Dominance criterion" in Section 3.1 of [47]. Clearly any bin packing using F_2 can be converted into a bin packing using F_1 instead (by swapping the elements of P_i for u_i), that does not use any more bins than the original packing. □

Definition 5.2.2 A *superset feasible set reduction* $\langle \mathrm{SF}, F, u \rangle$ is a special case of dominant feasible set reduction in which F dominates all other feasible sets of U by the condition of Theorem 5.2.

Testing this condition, in general, still requires the solution of a bin packing problem. Martello and Toth avoid this in their algorithm by restricting the size and form of their feasible sets. Having chosen an item u to act as a pivot and put it into a feasible set F, they then fill F with items in non-decreasing order until no more items will fit. Checking the conditions of Theorem 5.2 is trivial using feasible sets in this form. This restricted SF reduction will be denoted by SFk, where k is the maximum number of items in a feasible set that will be considered for domination.

A similar dominance can be defined for set covering, analogous to Martello and Toth's bin-packing dominance. Given two sets U_i and U_j of a set system

U, U_i *dominates* U_j if the optimum cover obtained by including U_i in the cover is no greater than the optimum cover obtained by including U_j.

If a set U_i containing u dominates all other sets containing u, then U_i must be included in the cover, as for the bin packing case. Also as in the bin packing case, this reduction is unlikely to be very attractive, as it requires the solution of many set covering problems. We note the trivial case, however, when there is an element contained in only one set.

Definition 5.2.3 A *unique column reduction* $\langle \mathrm{DF1}, U_i, u\rangle$ is a special case of dominant feasible set reduction in which $R(U, u) = \{U_i\}$.

5.2.2 Dominated Column Reduction (DC)

Let Y be a subset of a set system U, and let U_i be an element of $U - Y$ such that $w(U, Y) \leq w(U, U_i)$. If $U_i \subseteq S(Y_j)$, then we say that U_i is a *dominated column*, and if $U_i = S(Y_j)$, we say that U_i is an *exactly dominated column*.

Definition 5.2.4 An *(exactly) dominated column reduction* $\langle \mathrm{DC}, U_i\rangle$ deletes an (exactly) dominated column U_i from a set system.

Theorem 5.3 If ξ is an (exactly) dominated column reduction on a set system U, then $v(U^\xi) = v(U)$ in the non-exact case, and $v^*(U^\xi) = v^*(U)$ in the exact case.

Proof. The result is obvious, since U_i could be replaced in any set cover with the sets of Y to obtain a set cover of no greater weight. □

Note that determining such a set Y is itself a covering problem, and it is easy to see that testing for the existence of such a Y is NP-complete. For this reason, the cardinality of Y is usually limited, most often to one. If there are no weights, there can be no set Y satisfying the conditions of Theorem 5.3 with size greater than one.

In the case where $|Y| = 1$, $U_y \in Y$ dominates the dominated column U_i in the sense of the dominant feasible set reduction.

5.2.3 Dominated Row Reduction (DR)

Let u_i and u_j be distinct elements of the ground set of a set system U. If $R(U, u_i) \subseteq R(U, u_j)$, u_j is said to be *row dominated* by u_i.

Definition 5.2.5 A *dominated row reduction* $\langle \mathrm{DR}, X \rangle$ on a set system U deletes a set of row dominated elements Y from U.

Definition 5.2.6 An *exactly dominated row reduction*$\langle \mathrm{DR}, X \rangle$ on a set system U deletes a set of row dominated elements X and all the elements of $R(U, X) - R(U, u_i)$ from U.

Theorem 5.4 If ξ is an (exactly) dominated row reduction on a set system U, then $v(U^\xi) = v(U)$ in the non-exact case, and $v^*(U^\xi) = v^*(U)$ in the exact case.

Proof. The result is obvious, since if u_j is row dominated by u_i, covering u_i implies covering u_j. □

5.2.4 Disconnected System Reduction (DS)

Analogous to a disconnected component of a graph, a disconnected component of a set system U is a set $Y \subseteq U$ such that $S(Y) \cap S(U - Y) = \emptyset$.

Definition 5.2.7 A *disconnected system reduction* $\langle \mathrm{DS}, Y, U_Y \rangle$ on a set system U deletes a disconnected component Y from U.

Theorem 5.5 If ξ is a disconnected system reduction on a system U, then $v(U) = v(U^\xi) + v(Y^\xi)$ in the non-exact case and $v^*(U) = v^*(U^\xi) + v^*(Y^\xi)$ in the exact case.

Proof. The result is obvious, since the disconnected components of the system are really separate covering problems. □

5.2.5 Row Difference Reduction (RD)

Theorem 5.6 Let u_i and u_j be distinct elements of the ground set of a set system U such that $R(U, u_i) \not\subseteq R(U, u_j)$ and $R(U, u_j) \not\subseteq R(U, u_i)$. If there is a $u_s \in S(U)$ such that $R(U, u_i) - R(U, u_j) \subseteq R(U, u_s)$ and $u_s \in U_t$ for some $U_t \in R(U, u_j) - R(U, u_i)$, then U_t is not in any minimum exact set cover for U.

Proof. This is Reduction 5 of [28]. □

5.2.6 Reduced Costs Reduction (RC)

Theorem 5.7 Let q_i be an integer weight associated with each element u_i of the ground set of a set system U, and let $Q = \sum_{u_i \in S(U)} q_i$. Define a function

$$f_k(q) = \min\{f_{k-1}(q), f_{k-1}(q - \sum_{u_i \in U_k} q_i) + w(U, U_k)\}$$

for $k = 1, \ldots, |U|$ and $q = 0, \ldots, Q$. Let $\tilde{v}$ be an upper bound for $v(U)$. If $w(U, U_i) + f_{|U|}(Q) \geq \tilde{v}$, then U_i is not in any minimum set cover for U.

Proof. See Section 3 of [12]. □

Definition 5.2.8 A *reduced costs reduction* $\langle \mathrm{RC}, U_i \rangle$ on a set system U deletes a set U_i satisfying the conditions of Theorem 5.7.

5.3 Confluence

Theorem 5.8 The set $\{\mathrm{DF1}, \mathrm{DC}, \mathrm{DR}, \mathrm{DS}\}$ is confluent.

Proof. Firstly, the sets $\{\mathrm{DF1}\}$, $\{\mathrm{DC}\}$, $\{\mathrm{DR}\}$ and $\{\mathrm{DS}\}$ are obviously locally confluent.

Furthermore, it is easy to see that $\{\mathrm{DS}, \mathrm{DF1}\}$, $\{\mathrm{DS}, \mathrm{DC}\}$ and $\{\mathrm{DS}, \mathrm{DR}\}$ are locally confluent since all of the elements affected by any operation must be

u	$w(u)$
u_1	998
u_2	477
u_3	423
u_4	390
u_5	321
u_6	179
u_7	98
u_8	2
u_9	1

Table 5.3.3: The bin-packing problem for the non-confluence proof.

wholly contained within a single disconnected component, and thus any operation can be performed independently of any division into disconnected components. It is also easy to see that $\{\mathrm{DF1}, \mathrm{DC}\}$ is locally confluent since a unique column cannot be dominated, and any domination relationship is unchanged by identifying a unique column. Similarly, $\{\mathrm{DF1}, \mathrm{DR}\}$ is locally confluent.

It remains to show that $\{\mathrm{DC}, \mathrm{DR}\}$ is locally confluent. Clearly, deleting a row dominated element from the system does not change any column domination relationship since the element is deleted from all of the dominated columns as well as the dominating columns. Deleting a set has an analogous effect on the row domination relationships between the elements, so it is easy to see that any pair of DC and DR reductions can be performed in either order with no effect on the result.

Theorem 2.3 and the foregoing observations show the result. □

We will give an example to demonstrate that the set $\{\mathrm{SF}\}$ is not confluent. Consider the set $U = \{u_1, \ldots, u_9\}$ with the item sizes shown in Table 5.3.3 and a bin capacity of 1000.

If a feasible set F containing u dominates all maximal feasible sets containing

	F	$w(F)$	Candidates
F_1	$\{u_1, u_8\}$	1000	u_1, u_8
F_2	$\{u_1, u_9\}$	999	u_9
F_3	$\{u_2, u_3, u_7, u_8\}$	1000	u_2, u_3, u_7, u_8
F_4	$\{u_2, u_3, u_7, u_9\}$	999	u_9
F_5	$\{u_2, u_3, u_8, u_9\}$	903	
F_6	$\{u_2, u_4, u_7, u_8, u_9\}$	968	
F_7	$\{u_2, u_5, u_6, u_8, u_9\}$	980	
F_8	$\{u_2, u_5, u_7, u_8, u_9\}$	899	
F_9	$\{u_2, u_6, u_7, u_8, u_9\}$	757	
F_{10}	$\{u_3, u_4, u_6, u_8, u_9\}$	995	u_4, u_6
F_{11}	$\{u_3, u_4, u_7, u_8, u_9\}$	914	
F_{12}	$\{u_3, u_5, u_6, u_8, u_9\}$	926	
F_{13}	$\{u_3, u_5, u_7, u_8, u_9\}$	845	
F_{14}	$\{u_3, u_6, u_7, u_8, u_9\}$	703	
F_{15}	$\{u_4, u_5, u_6, u_7, u_8, u_9\}$	991	u_5

Table 5.3.4: The maximal sets the bin-packing problem of Table 5.3.3

u, then clearly it dominates all feasible sets containing u. If a feasible set F_1 dominates a feasible set F_2, then $w(F_1) \geq w(F_2)$, and hence only the heaviest of all sets containing u can dominate all sets containing u. Table 5.3.4 lists all of the maximal feasible sets of U.

Using Table 5.3.4, it is simple to check that F_1 dominates all sets containing u_1 and F_2 dominates all sets containing u_9. None of the other sets are candidates for reduction. Setting $\alpha_1 = \langle \mathrm{SF}, F_1, u_1 \rangle$ and $\beta_1 = \langle \mathrm{SF}, F_2, u_9 \rangle$, then, we get the two non-isomorphic sets

$$U_1^\alpha = \{u_2, u_3, u_4, u_5, u_6, u_7, u_9\}$$

and

$$U_1^{\beta} = \{u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$$

By modifying Table 5.3.4 appropriately, it is simple to show that both U_1^{α} and U_1^{β} are irreducible, and hence we have shown that the set {SF} is not confluent.