Chapter 6

Self-Reduction and Probability

6.1 Introduction

In this chapter, we analyse the probability of a reduction existing in a random graph for a widely-used definition of such a graph. It is shown that, as the order of the graph tends to infinity, the expected number of reductions described in the preceding chapters tends to zero.

Let $\mathcal{X}(\mathcal{A}, \mathcal{I})$ be the set of all possible self-reductions from a set of self-reduction operations $\mathcal{A}$ on a structure $\mathcal{I}$.

Let $\mathcal{R}(\mathcal{A}, \mathcal{I}, X)$ be the function

$$\mathcal{R}(\mathcal{A}, \mathcal{I}, X) = \begin{cases} 1 & \text{if } \langle \varphi, X \rangle \in \mathcal{X}(\mathcal{A}, \mathcal{I}) \text{ for some } \varphi \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

That is, $\mathcal{R}(\mathcal{A}, \mathcal{I}, X) = 1$ if and only if X is reducible under $\mathcal{A}$. Clearly, if our structures are graphs,

$$|\mathcal{X}(\mathcal{A}, G)| = \sum_{X \subseteq V(G)} \mathcal{R}(\mathcal{A}, G, X) \tag{6.1}$$

Analogous expressions are possible for other structures.

6.2 Vertex-Weighted Graphs

In this section, we use a widely-used definition of a random (unweighted) graph described, for example, in [23]. Other models are possible, some equivalent to the model used here; however, this model seems most natural for describing an arbitrary graph with no specific properties. Let a random graph $G_{n,p}$ be a graph of n vertices, and for each pair of vertices $uv \in V(G_{n,p})$, an edge exists with probability p. Initially, we shall consider only unweighted graphs, so in this section, the graph $G_{n,p}$ will be unweighted unless specified otherwise. We will later extend our results to weighted graphs.

If $p = \frac{1}{2}$, then all possible graphs on n vertices occur with equal probability in the sample space. Given a proposition $Q(G)$, *almost every* graph is said to have property $Q(G)$ if and only if the probability of $Q(G_{n,\frac{1}{2}})$ being true tends to one as n tends to infinity.

Lemma 6.1 The expected number of non-trivial autonomous sets in $G_{n,p}$ approaches zero as n approaches infinity.

Proof. See [49]. □

Lemma 6.2 For any $p > 0$, the expected number of subset vertices in $G_{n,p}$ approaches zero as n approaches infinity.

Proof. Let x and y be vertices of a random graph $G \in \mathcal{G}_{n,p}$. Then

$$\begin{aligned}
P(N(G,y) \subseteq N(G,x) - \{y\}) &= P(\forall z \in V(G) - \{x,y\}, \\
&\quad z \in N(G,x) \vee z \notin N(G,y)) \\
&= P(\forall z \in V(G) - \{x,y\}, \\
&\quad \neg(z \notin N(G,x) \wedge z \in N(G,y))) \\
&= (1 - p(1-p))^{n-2}
\end{aligned}$$

Hence,

$$P(\langle \mathrm{S}, \{x,y\}, x\rangle \in \mathcal{X}(\{\mathrm{S}\}, \mathcal{G})) = P(y \notin N(G,x)) \wedge$$

$$\begin{aligned} & P(N(G,y) \subseteq N(G,x) - \{y\}) \\ = \; & (1-p)(1-p(1-p))^{n-2} \end{aligned}$$

and by the additivity of expectations,

$$\begin{aligned} E(|\mathcal{X}(\{\mathrm{S}\},\mathcal{G})|) &= \sum_{x,y \in V(G)} E(\mathcal{R}(\{\mathrm{S}\},G,\{x,y\})) \\ &= n(n-1)(1-p)(1-p(1-p))^{n-2} \end{aligned}$$

If $p \neq 0$, $1-p(1-p) < 1$ and so the expected number of subset reductions tends to zero as n tends to infinity for any $p > 0$. □

Lemma 6.3 The expected number of under-constrained vertices in $G_{n,p}$ approaches zero as n approaches infinity for any $p > 0$.

Proof. Let x be a vertex of a graph $G \in \mathcal{G}_{n,p}$, and let k be an integer. The degree of x has a binomial distribution with parameters n and p, so

$$P(\deg(G,x) < k) = \sum_{j=0}^{k-1} \binom{n}{j} p^j (1-p)^{n-j}$$

Let h be the lower bound used for determining the under-constrainedness of a vertex.

$$\begin{aligned} P(\langle \mathrm{UC}, \{x\} \rangle \in \mathcal{X}(\{\mathrm{UC}\},\mathcal{G})) &= P(\deg(G,x)+1 < h) \\ &= \sum_{j=0}^{h-2} \binom{n}{j} p^j (1-p)^{n-j} \end{aligned}$$

Hence,

$$\begin{aligned} E(|\mathcal{X}(\{\mathrm{UC}\},\mathcal{G})|) &= \sum_{x \in V(G)} E(\mathcal{R}(\{\mathrm{UC}\},G,\{x\})) \\ &= n \sum_{j=0}^{h-2} \binom{n}{j} p^j (1-p)^{n-j} \end{aligned}$$

If $p \neq 0$, $1-p < 1$ and so the expected number of under-constrained vertices (for a fixed h) tends to zero as n tends to infinity for any $p > 0$. □

Lemma 6.4 The expected number of quasi-autonomous cliques in $G_{n,p}$ tends to zero as n tends to infinity for any $p < 1$.

Proof. Let x, y and z be vertices of a graph $G \in \mathcal{G}_{n,p}$. Using de Morgan's law, the probability that z is adjacent to both of x and y, or neither, is

$$\begin{aligned}P(\neg(\neg(z \in N(G,x) \wedge z \in N(G,y)) \wedge \\ \neg(z \notin N(G,x) \wedge z \notin N(G,y)))) &= 1-(1-p^2)(1-(1-p)^2) \\ &= 1-p(p-1)(p-2)(p+1)\end{aligned}$$

Let $t = p(p-1)(p-2)(p+1)$; that is, let t be the probability that a vertex z is irregular with respect to the quasi-autonomous clique $\{x, y\}$. For $\{x, y\}$ to be subject to quasi-autonomous clique reduction, there must exist either zero or one such vertices. Since the regularity of each vertex is independent,

$$\begin{aligned}P(\langle \hat{\mathrm{C}}, \{x,y\}, x/y\rangle \in \mathcal{X}(\{\hat{\mathrm{C}}\}, G)) &= p\sum_{j=0}^{1}\binom{n-2}{j}t^j(1-t)^{n-2-j} \\ &= p((1-t)^{n-2} + (n-2)t(1-t)^{n-3}) \\ &= p(1-t)^{n-3}(1-(n-3)t)\end{aligned}$$

Then,

$$\begin{aligned}E(|\mathcal{X}(\{\hat{\mathrm{C}}\}, G)|) &= \sum_{x,y\in V(G)} E(\mathcal{R}(\{\hat{\mathrm{C}}\}, G, \{x,y\})) \\ &= \frac{n(n-1)}{2}p(1-t)^{n-3}(1-(n-3)t)\end{aligned}$$

If $p \neq 1$, $1-t < 1$, and so the expected number of quasi-autonomous clique reductions tends to zero as n tends to infinity for any $p < 1$. □

Lemma 6.5 The expected number of even-pairs in $G_{n,p}$ tends to zero as n tends to infinity for any $p > 0$.

Proof. Let $z_1, \ldots, z_{k+1}$ be distinct vertices of a graph $G \in \mathcal{G}_{n,p}$. Then,

$$\begin{aligned} P(z_1 \cdots z_{k+1} \text{is a chordless path}) &= P(z_1 \in N(G, z_2) \wedge \cdots \wedge z_k \in N(G, z_{k+1})) \cdot \\ &\quad P(\forall |i-j| \neq 1, z_i \notin N(G, z_j)) \\ &= p^k (1-p)^{\binom{k+1}{2}-k} \end{aligned}$$

Hence,

$$\begin{aligned} P(\langle \mathrm{E}, \{x, y\}, x \rangle \in \mathcal{X}(\{\mathrm{E}\}, \mathcal{G})) &= P(\forall k \text{ odd}, \forall z_2 \cdots z_k, \\ &\quad x z_2 \cdots z_k y \text{ is not a chordless path}) \\ &= \prod_{k \text{ odd}} (1 - p^k (1-p)^{\binom{k+1}{2}-k})^{\binom{n-2}{k-1}(k-1)!} \end{aligned}$$

and

$$\begin{aligned} E(|\mathcal{X}(\{\mathrm{E}\}, G)|) &= \sum_{x,y \in V(G)} E(\mathcal{R}(\{\mathrm{E}\}, G, \{x, y\})) \\ &= \frac{n(n-1)}{2} \prod_{k \text{ odd}} (1 - p^k (1-p)^{\binom{k+1}{2}-k})^{\binom{n-2}{k-1}(k-1)!} \end{aligned}$$

If $p \neq 0$, $1 - p^k(1-p)^{\binom{k+1}{2}-k} < 1$, and so the expected number of even-pairs on a graph tends to zero as n tends to infinity. □

Using the foregoing proofs and the Central Limit Theorem, it is now possible to say something about graphs with vertex weights.

Theorem 6.1 The expected number of reductions from the set $\{\mathrm{A}, \mathrm{S}, \mathrm{UC}, \mathrm{E}, \hat{\mathrm{C}}\}$ on the graph $G_{n,p}$ with arbitrary vertex weights tends to zero as n tends to infinity for any $p, 0 < p < 1$.

Proof. In the case of autonomous reduction, subset reduction, underconstrained deletion and even-pair reduction, any graph with weights on its vertices clearly has at most as many reductions on it as the graph with the same structure, but with no vertex weights. Hence, the expected number of reductions on an unweighted graph is an upper bound on the expected number of reductions on a weighted graph, regardless of the distribution of the weights.

As the expected values tend to zero as n approaches infinity for the unweighted cases, they must also for the weighted cases.

For quasi-autonomous clique reduction, the above probability analysis depends on the distribution of vertex weights. For a vertex x, let $w(G, x)$ be a random variable with some probability density function $f(w)$. Let x and y be vertices. The probability that z is adjacent to y but not to x (that is, z is an irregular neighbour of y) is $p(1-p)$. Using this, define a new probability density function

$$f'(w) = \begin{cases} 1 - p(1-p) & \text{if } w = 0 \\ p(1-p)f(w) & \text{otherwise} \end{cases} \tag{6.2}$$

Hence,

$$P\left(w(G,x) \geq \sum_{z \text{ irregular}} w(G,z)\right) = P\left(w(G,x) \geq \sum_{z \in V(G)} w'(G,z)\right)$$

where $w(G, x)$ and $w'(G, z)$ are random variables with probability density functions $f(w)$ and $f'(w)$, respectively. Then,

$$\begin{aligned} P(\langle \hat{\mathrm{C}}, \{x,y\}, x/y\rangle \in \mathcal{X}(\{\hat{\mathrm{C}}\}, G)) &= P(x \in N(G,y) \wedge \\ &\quad \forall z, z \in N(G,y) \vee z \notin N(G,x) \wedge \\ &\quad w(G,x) \geq \sum_{z \text{ irregular}} w(G,z)) \\ &= p(1-p(1-p))^{n-2} \\ &\quad P\left(w(G,x) \geq \sum_{z \in V(G)} w'(G,z)\right) \end{aligned}$$

Let $\mu_{w'}$ and $\sigma_{w'}$ be the mean and variance of $w'(G, x)$, respectively, and let $Z_{w'}$ be a random variable with a normal distribution with mean $n\mu_{w'}$ and variance $n\sigma_{w'}$. Using the Central Limit Theorem, as n approaches infinity, the expression above approaches

$$p(1-p(1-p))^{n-2} P\left(w(G,x) \geq Z_{w'}\right)$$

and hence, for n approaching infinity,

$$\begin{aligned} E(|\mathcal{X}(\{\hat{\mathrm{C}}\}, G)|) &= \sum_{x,y \in V(G)} E(\mathcal{R}(\{\hat{\mathrm{C}}\}, G, \{x, y\})) \\ &= n(n-1)p(1-p(1-p))^{n-2} P\left(w(G,x) \geq Z_{w'}\right) \end{aligned}$$

It is easy to see that $P(w(G,x) \geq Z_{w'})$ approaches zero as n approaches infinity, as does $n(n-1)p(1-p(1-p))^{n-2}$, and hence the expected number of quasi-autonomous clique reductions on a graph approaches zero as n approaches infinity, for an arbitrary distribution of vertex weights. The theorem follows. □

Corollary 6.1.1 Almost every graph is irreducible.

Proof. Immediate from Theorem 6.1. □

6.3 Other Structures

To the best of the present author's knowledge, very little work has been done on random Steiner problems or set systems. In particular, it is unclear what a useful definition for random edge, set and item weights should be, and determining one is beyond the scope of this thesis. The analysis of the reductions listed in this thesis on random Steiner problems seems extremely complex, as does the analysis of the superset feasible reduction for bin packing, even if some useful definition for random edge and item weights were found.

A straightforward definition of a random set system would be to define $U_{n,m,p}$ with n sets and m elements in the ground set, and element u_i contained in set $U_j \in U$ with probability p.

The behaviour of set covering reductions on set systems as n and m approach infinity depends on the relative rates at which these two variables are increasing. Intuitively, if n is increasing sufficiently quickly compared to m, the probability of finding a dominated column increases, whereas if n is growing slowly compared to m, the opposite occurs. It is straightforward to confirm this, and similar facts about other reductions, by probabilistic analysis.

While it would be possible to avoid this by constraining n and m to increase at some analysable rates relative to each other (for example, to require the ratio n/m to be fixed), this seems artificial and of limited value to the present author. Hence the problem of describing and analysing a useful model of a random set system is left to future researchers.