Geometric Seifert 4-manifolds with aspherical bases

by

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A thesis submitted to
The University of Sydney
in fulfillment of the requirements for
the degree of Doctor of Philosophy
in the School of Mathematics and Statistics

July 5, 2005

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Preface

Seifert fibred 3-manifolds were originally defined and classified by Seifert in [Se]. Scott (in [Sc2]) gives a survey of results connected with these classical Seifert spaces, in particular he shows they correspond to 3-manifolds having one of six of the eight 3-dimensional geometries (in the sense of Thurston). Essentially, a classical Seifert manifold is a S^1-bundle over a 2-orbifold. More generally, a *Seifert manifold* is the total space of a bundle over a 2-orbifold with flat fibres. It is natural to ask if these generalised Seifert manifolds describe geometries of higher dimension. Ue has considered the geometries of orientable Seifert 4-manifolds (which have general fibre a torus) ([Ue1], [Ue2]). He proves that (with a finite number of exceptions) orientable manifolds of eight of the 4-dimensional geometries are Seifert fibred. However, Seifert manifolds with a hyperbolic base are not necessarily geometric. In this paper, we seek to extend Ue's work to the non-orientable case.

Firstly, we will show that Seifert spaces over an aspherical base are determined (up to fibre preserving homeomorphism) by their fundamental group sequence (chapter 2). Furthermore when the base is hyperbolic, a Seifert space is determined (up to fibre preserving homeomorphism) by its fundamental group. This generalises the work of Zieschang ([Zi2]), who assumed the base has no reflector curves, the fibre was a torus and that a monodromy of a loop surrounding a cone point is trivial. Then we restrict to the 4 dimensional case and find necessary and sufficient conditions for Seifert 4 manifolds over hyperbolic or Euclidean orbifolds to be geometric in the sense of Thurston (chapters 3 and 4 respectively). Ue proved that orientable Seifert 4-manifolds with hyperbolic base are geometric if and only if the monodromies are periodic, and we will prove that we can drop the orientable condition. Ue also proved that orientable Seifert 4-manifolds with a Euclidean base are always geometric, and we will again show the orientable assumption is unnecessary.

To prove the first result we construct a presentation for the fundamental group. In sections 2.2 and 2.3 we find presentations for the bundle restricted to the neighbourhoods of the singular points. In section 2.4 we then construct the presentation. To complete that part of the paper, we prove our result (section 2.5).

After exploring the two families of geometries which give rise to Seifert manifolds with hyperbolic bases ($\mathbb{H}^2 \times \mathbb{E}^n$ in section 3.1 and $\widetilde{SL}_2 \times \mathbb{E}^{n-1}$ in section 3.2), we prove the second major result: a Seifert manifold with hyperbolic base is geometric if and only if its group of monodromies is finite. To do this, we first consider when the base has no reflector curves and has T^2 fibres (section 3.3). We generalise to include reflector curves (section 3.4) and Kb fibres (section 3.5). We then prove a consequence of this result: that a virtually geometric Seifert 4-manifold with hyperbolic base is geometric (section 3.6).

The last major result we prove is that all Seifert 4-manifolds with Euclidean base are geometric. These Seifert manifolds are geometric of one of four types. The first type is $\mathbb{E}^4$ which is introduced in section 1.4 along with the Bieberbach theorems. The other three types are each introduced in their own section: $\mathbb{N}il^3 \times \mathbb{E}$ in section 4.1, $\mathbb{S}ol^3 \times \mathbb{E}$ in section 4.2 and $\mathbb{N}il^4$ in section 4.3. In section 4.5, we prove the last major result. The result is split into two theorems based on the general fibre (T^2 or Kb). The proof of the theorem where the general fibre is T^2 is split into four lemmas which prove the result for each of the four geometries.