CHAPTER 7

THE DEEP WATER LIMIT OF THE
INVERSE SCATTERING TRANSFORM
FOR THE
MODIFIED INTERMEDIATE LONG WAVE
EQUATION: SOME COMMENTS

Section 7.1: Introduction

The interrelationships between the complex-valued ILW equation,

$$U_t + \frac{1}{\delta}U_x + 2UU_x + \mathbf{T}\left(U_{xx}\right) = 0, \tag{7.1.1}$$

the complex-valued BO equation,

$$q_t + 2qq_x + \mathbf{H}\left(q_{xx}\right) = 0, \tag{7.1.2}$$

and the complex-valued KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{7.1.3}$$

have been recurring themes throughout this thesis. Several times in this thesis we have exploited the properties (7.1.1) → (7.1.2) as $\delta \to \infty$ and (7.1.1) → (7.1.3) as $\delta \to 0^+$ to either procure results of relevance to this thesis or to verify particular results we have derived.

Santini et al. have shown [103] that the IST for the *real-valued* ILW equation reduces as $\delta \to \infty$ to the IST for the *real-valued* BO equation. Ref. 103 is an important contribution to the field of integrable nonlinear singular integro-differential equations because it connects two significantly dissimilar IST schemes. In Chapter 6 of this thesis we demonstrated that the IST for the complex-valued ILW equation contracts as $\delta \to 0^+$ to the IST scheme for the complex-valued KdV equation. In this chapter we examine whether the work in Ref. 103 is applicable to the IST schemes for the complex-valued ILW and BO equations. The results in this chapter are incomplete because we show that only the *direct problem* deep water limit with the direct problem in the IST for the complex-valued BO equation. Based on our incomplete results, we anticipate that the work in Ref. 103 can be adapted to accommodate the particular features of the IST scheme for the complex-valued ILW equation that are not present in the corresponding scheme for the real-valued ILW equation. The most important problem that we leave open at the conclusion of this chapter is the deep water limit of all quantities in the IST for the complex-valued ILW equation that have at least one component whose origin resides in the discrete spectrum for the complex-valued ILW equation. The reader is invited to consult Section 4.3 of this thesis to review the differences between the discrete spectra for the real-valued and complex-valued versions of (7.1.1).

Section 7.2 of this thesis, the remaining section in this chapter, contains our analysis of the $\delta \to \infty$ IST. The work in Section 7.2 corresponds to Section III, Part A, of Ref. 103. Before we proceed to Section 7.2 we will review some relevant facts about the complex-valued ILW and BO equations, particularly their IST schemes:

- equation (2.1.21) stipulates how the transition from (7.1.1) to (7.1.2) transpires;
- $G^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda)$ is prominent in the IST for the complex-valued ILW equation, and this particular function admits [see (4.4.3)] the integral representation

$$G^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) = \frac{1}{2\pi} \int_{C_-} \frac{e^{i(x-\eta)r}}{r - \varepsilon\left[\zeta_+(\lambda) + (2\delta)^{-1}\right]\left(1 - e^{-2\varepsilon\delta r}\right)} \mathrm{d}r, \tag{7.1.4}$$

where $\varepsilon = \pm 1$, $C\_$ denotes the line from $-\infty + i0^+$ to $\infty + i0^+$ in the complex r-plane and [see (4.3.14)]

$$\zeta_{+\varepsilon}(\lambda) = \frac{\lambda}{2} + \varepsilon\frac{\lambda}{2}\coth(\delta\lambda) - \frac{\varepsilon}{2\delta}; \tag{7.1.5}$$

- in Section 5.2 of this thesis [see (5.2.13$^-$)] we showed that

$$\lim_{\delta \to \infty} G^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) = \varepsilon g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda), \tag{7.1.6}$$

where [see (5.2.14$^-$)]

$$g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) = \frac{1}{2\pi} \int_0^\infty \frac{e^{i\varepsilon(x-\eta)r}}{r - \left(\lambda - i\varepsilon 0^+\right)} \mathrm{d}r; \tag{7.1.7}$$

- equations (5.2.18a-d) specify how to retrieve the Jost functions for the complex-valued BO equation from the Jost functions for the complex-valued ILW equation.

We are now in a position to proceed to Section 7.2.

Section 7.2: The Limit as $\delta \to \infty$ of the Direct Problem for the Complex-valued ILW Equation

$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)$ associated with the IST for the complex-valued ILW equation is

$$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda) = G^{+\varepsilon}_{-\varepsilon}(x,\eta;-\lambda)e^{i\varepsilon\lambda(x-\eta)}. \tag{7.2.1}$$

Equation (7.1.4) provides an integral representation of $G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)$. Throughout this section we will make the following two assumptions about the parameter λ:

1) λ is real and positive;

2) $\lambda = O(1)$ as $\delta \to \infty$.

The two listed assumptions about λ have already been employed elsewhere in this thesis, particularly Section 5.2.

Santini et al. [103] have noted (without proof) that for the case $\varepsilon = 1$ and for *real-valued* solutions of the ILW equation (7.1.1), the $\delta \to \infty$ limit of (7.2.1) leads to a loss of information, namely we retrieve from the $\delta \to \infty$ limit of (7.2.1) the trivial equation

$$g^{+}_{-}(x,\eta;\lambda) \equiv g^{+}_{-}(x,\eta;\lambda),$$

where $g^{+}_{-}(x,\eta;\lambda)$

equation. We begin our analysis of the deep water limit for the complex-valued ILW

$\varepsilon = \pm 1)$ of the indicator and for

complex-valued solutions of the ILW equation (7.1.1), the phenomenon [in relation to (7.2.1)] observed by Santini et al. is repeated.

In Section 4.3 of this thesis [see (4.3.17b)] we recorded that $\zeta_+(\lambda)$ exhibits the symmetry

$$\zeta_+(-\lambda) = -\zeta_-(\lambda),$$

where $\zeta_{+\varepsilon}(\lambda)$ is defined by (7.1.5). When we apply to (7.1.4) the transformation $\lambda \to -\lambda$, and then use the identity $\zeta_+(-\lambda) = -\zeta_-(\lambda)$ we arrive at the following integral representation for $G^{+\varepsilon}_{-\varepsilon}(x,\eta;-\lambda)$:

$$G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)=\frac{1}{2\pi}\int_{-\infty+i0^{+}}^{\infty+i0^{+}}\frac{e^{i(x-\eta)r}}{r+\varepsilon\left[\zeta_{-}(\lambda)-(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta r}\right)}\mathrm{d}r. \tag{7.2.2}$$

Our immediate objective is to compute the limit of $G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)$ as $\delta\to\infty$. Before we commence the computation of the desired limit, let us observe that

$$\lim_{\delta\to\infty}\left\{G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)e^{i\varepsilon\lambda(x-\eta)}\right\}=e^{i\varepsilon\lambda(x-\eta)}\lim_{\delta\to\infty}G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda),$$

valid because of our assumption that $\lambda=\mathrm{O}(1)$ as $\delta\to\infty$. The term

$$\frac{1}{r+\varepsilon\left[\zeta_{-}(\lambda)-(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta r}\right)} \tag{7.2.3}$$

in the integrand of (7.2.2) is the dominant influence on the behaviour of $G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)$ as $\delta\to\infty$. We can elucidate the large-δ behaviour of (7.2.3) by means of the equation

$$\zeta_{-}(\lambda)-\frac{1}{2\delta}=\frac{\lambda}{1-e^{2\delta\lambda}}, \tag{7.2.4}$$

which we can derive by writing the $\coth(\delta\lambda)$ term in (7.1.5) as a ratio of appropriate exponentials. The equation

$$G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)=\frac{1}{2\pi}\int_{-\infty+i0^{+}}^{\infty+i0^{+}}\frac{e^{i(x-\eta)r}}{r+\varepsilon\lambda\left(1-e^{-2\varepsilon\delta r}\right)\left(1-e^{2\delta\lambda}\right)^{-1}}\mathrm{d}r \tag{7.2.5}$$

crystallizes when we substitute (7.2.4) into (7.2.2). After testing all relevant scenarios we have determined that

$$\lim_{\delta\to\infty}\frac{e^{i(x-\eta)r}}{r+\varepsilon\lambda\left(1-e^{-2\varepsilon\delta r}\right)\left(1-e^{2\delta\lambda}\right)^{-1}}=\begin{cases}\theta(-\varepsilon)\dfrac{e^{i(x-\eta)r}}{r}, & \operatorname{Re}(r)<-\varepsilon\lambda\\[2ex] \theta(\varepsilon)\dfrac{e^{i(x-\eta)r}}{r}, & \operatorname{Re}(r)>-\varepsilon\lambda,\end{cases} \tag{7.2.6}$$

where

$$\theta(x)=\begin{cases}1, & x>0\\[2ex] 0, & x<0.\end{cases}$$

The equation

$$\lim_{\delta \to \infty} G^{+\varepsilon}_{-\varepsilon}(x, \eta; -\lambda) = \frac{\theta(-\varepsilon)}{2\pi} \int_{-\infty + i0^{+}}^{-\varepsilon\lambda + i0^{+}} \frac{e^{i(x-\eta)r}}{r} \mathrm{d}r$$

$$+ \frac{\theta(\varepsilon)}{2\pi} \int_{-\varepsilon\lambda + i0^{+}}^{\infty + i0^{+}} \frac{e^{i(x-\eta)r}}{r} \mathrm{d}r \qquad (7.2.7)$$

appears when we consider the $\delta \to \infty$ limit of (7.2.5), and then use (7.2.6) to assist us in the computation of this limit. Introduce into the two nonlocal terms that constitute (7.2.7) the change of variable $r' = r + \varepsilon\lambda - i0^{+}$. The reader will find that the following equation eventuates from the use of the transformation $r' = r + \varepsilon\lambda - i0^{+}$:

$$\lim_{\delta \to \infty} \left\{ G^{+\varepsilon}_{-\varepsilon}(x, \eta; -\lambda) e^{i\varepsilon\lambda(x-\eta)} \right\} = \frac{\theta(-\varepsilon)}{2\pi} \int_{-\infty}^{0} \frac{e^{i(x-\eta)r}}{r - \varepsilon\lambda + i0^{+}} \mathrm{d}r$$

$$+ \frac{\theta(\varepsilon)}{2\pi} \int_{0}^{\infty} \frac{e^{i(x-\eta)r}}{r - \varepsilon\lambda + i0^{+}} \mathrm{d}r. \qquad (7.2.8)$$

In Section 5.2 of this thesis we were confronted with the task of fusing two integrals that are similar to those in (7.2.8). Applying to the *right hand side* of (7.2.8) an appropriately modified version of the procedure that was used to derive equations (5.2.11a) and (5.2.11b) we find that

$$\lim_{\delta \to \infty} \left\{ G^{+\varepsilon}_{-\varepsilon}(x, \eta; -\lambda) e^{i\varepsilon\lambda(x-\eta)} \right\} = \varepsilon g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda), \qquad (7.2.9)$$

where $g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda)$ is defined by (7.1.7). In the wake of the aggregate of (7.1.6), (7.2.1) and (7.2.9) we are left (as claimed) with the trivial identity

$$g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) \equiv g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda).$$

One may well inquire as to how we retrieve the identity [see (5.2.29^{-})]

$$\frac{\partial}{\partial\lambda} g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) = i\varepsilon g^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) - \frac{1}{2\pi\lambda} \qquad (7.2.10)$$

from (7.2.1). The answer to our (self-posed) question resides in the identity

$$\frac{\partial}{\partial\lambda} G^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) = i\varepsilon(x - \eta) G^{+\varepsilon}_{-\varepsilon}(x, \eta; \lambda) + \frac{\partial}{\partial\lambda} \left\{ G^{+\varepsilon}_{-\varepsilon}(x, \eta; -\lambda) \right\} e^{i\varepsilon\lambda(x-\eta)}. \qquad (7.2.11)$$

Two interconnected stages are involved in the derivation of (7.2.11):

1. **1)** differentiate both sides of (7.2.1) with respect to λ;
2. **2)** use the identity (7.2.1) to remove

$$G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)e^{i\varepsilon\lambda(x-\eta)}$$

from the differential equation at the end of stage one.

Now consider the $\delta \to \infty$ limit of (7.2.11). The deep water limit of (7.2.11) followed by a use of (7.1.6) produces the equation

$$\lim_{\delta\to\infty}\frac{\partial}{\partial\lambda}G_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda) = i(x-\eta)g_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda) + e^{i\varepsilon\lambda(x-\eta)}\lim_{\delta\to\infty}\frac{\partial}{\partial\lambda}G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)\,. \qquad (7.2.12)$$

We again remind the reader that $g_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda)$ is defined by (7.1.7). It is tempting for us to interchange the operators $\lim \delta \to \infty$ and $\partial/\partial\lambda$, but such unbridled thoughts should be resisted [103].

Let us first consider the limit associated with the *left hand side*
course we will compute the limit that appears in the right hand side of (7.2.12). The integral representation of $\partial/\partial\lambda\left\{G_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda)\right\}$ is

$$\frac{\partial}{\partial\lambda}G_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda) = \frac{\varepsilon}{2\pi}\zeta_+'(\lambda) \times\int_{-\infty+i0^+}^{\infty+i0^+}\frac{\left(1-e^{-2\varepsilon\delta r}\right)e^{i(x-\eta)r}}{\left\{r-\varepsilon\left[\zeta_+(\lambda)+(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta r}\right)\right\}^2}\mathrm{d}r, \qquad (7.2.13)$$

where $\zeta_+(\lambda)$ is defined by (7.1.5) and

$$\zeta_+'(\lambda) \equiv \frac{\mathrm{d}\zeta_+}{\mathrm{d}\lambda}.$$

Equation (7.2.13) is the derivative of (7.1.4) with respect to λ. We reiterate that throughout this section λ is a real positive parameter. Given that $\lambda > 0$, we can use (7.1.5) to show that

$$\lim_{\delta\to\infty}\zeta_+(\lambda) = \lambda \qquad (7.2.14)$$

and

$$\lim_{\delta\to\infty}\zeta_+'(\lambda) = 1. \qquad (7.2.15)$$

Equations (7.2.14) and (7.2.15) furnish us with an insight into how the integrand of (7.2.13) behaves in the deep water region. We have determined by using (7.2.14) and (7.2.15) that

$$\lim_{\delta \to \infty} \frac{\zeta_+'(\lambda)\left(1 - e^{-2\varepsilon\delta r}\right)e^{i(x-\eta)r}}{\left\{r - \varepsilon\left[\zeta_+(\lambda) + (2\delta)^{-1}\right]\left(1 - e^{-2\varepsilon\delta r}\right)\right\}^2} = \theta\left(\varepsilon \operatorname{Re}(r)\right)\frac{e^{i(x-\eta)r}}{(r - \varepsilon\lambda)^2}. \tag{7.2.16}$$

In the light of (7.2.16) one can see that the deep water limit of (7.2.13) is

$$\lim_{\delta \to \infty} \left\{\frac{\partial}{\partial\lambda} G_{-\,\varepsilon}^{+\,\varepsilon}(x, \eta; \lambda)\right\} = \frac{\varepsilon}{2\pi}\int_{-\infty + i0^+}^{\infty + i0^+} \theta\left(\varepsilon \operatorname{Re}(r)\right)\frac{e^{i(x-\eta)r}}{(r - \varepsilon\lambda)^2}\mathrm{d}r. \tag{7.2.17}$$

The explicit appearance in (7.2.17) of the real part of r suggests that we should partition r into its real and imaginary parts. Before we write r in terms of its real and imaginary parts, split the nonlocal term in (7.2.17) as follows:

$$\frac{\varepsilon}{2\pi}\int_{-\infty + i0^+}^{\infty + i0^+} \theta\left(\varepsilon \operatorname{Re}(r)\right)\frac{e^{i(x-\eta)r}}{(r - \varepsilon\lambda)^2}\mathrm{d}r = \frac{\varepsilon}{2\pi}\int_{-\infty + i0^+}^{0 + i0^+} \theta\left(\varepsilon \operatorname{Re}(r)\right)\frac{e^{i(x-\eta)r}}{(r - \varepsilon\lambda)^2}\mathrm{d}r$$

$$+ \frac{\varepsilon}{2\pi}\int_{0 + i0^+}^{\infty + i0^+} \theta\left(\varepsilon \operatorname{Re}(r)\right)\frac{e^{i(x-\eta)r}}{(r - \varepsilon\lambda)^2}\mathrm{d}r. \tag{7.2.18}$$

Let $r = -r' + i0^+$ in the *first* of the two integrals in (7.2.18), and let $r = r' + i0^+$ in the *second* of the two integrals; in both cases r' assumes all values in the interval $0 < r' < \infty$. Introducing into the right hand side of (7.2.18) the appropriate expressions for r, noting how $\theta(\bullet)$ influences each integral in the right hand side of (7.2.18), and lastly omitting primes we arrive at the equation

$$\frac{\varepsilon}{2\pi}\int_{-\infty + i0^+}^{\infty + i0^+} \theta\left(\varepsilon \operatorname{Re}(r)\right)\frac{e^{i(x-\eta)r}}{(r - \varepsilon\lambda)^2}\mathrm{d}r = \frac{\varepsilon}{2\pi}\int_{0}^{\infty} \theta(-\,\varepsilon)\frac{e^{i\varepsilon(x-\eta)r}}{\left(\varepsilon r + i0^+ - \varepsilon\lambda\right)^2}\mathrm{d}r$$

$$+ \frac{\varepsilon}{2\pi}\int_{0}^{\infty} \theta(\varepsilon)\frac{e^{i\varepsilon(x-\eta)r}}{\left(\varepsilon r + i0^+ - \varepsilon\lambda\right)^2}\mathrm{d}r,$$

which, because $\theta(\varepsilon) + \theta(-\,\varepsilon) = 1$, simplifies to

$$\frac{\varepsilon}{2\pi}\int_{-\infty+i0^{+}}^{\infty+i0^{+}}\theta\big(\varepsilon\mathrm{Re}\,(r)\big)\frac{e^{i\ \ x\ \ \ \eta)r}}{(r-\varepsilon\lambda)^{2}}\mathrm{d}r=\frac{\varepsilon}{2\pi}\int_{0}^{\infty}\frac{e^{i\varepsilon(x-\eta)r}}{\big(r-\big[\lambda-i\varepsilon0^{+}\big]\big)^{2}}\mathrm{d}r\,. \tag{7.2.19}$$

From the aggregate of (7.2.17) and (7.2.19) we deduce that

$$\lim_{\delta\to\infty}\left\{\frac{\partial}{\partial\lambda}G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)\right\}=\frac{\varepsilon}{2\pi}\int_{0}^{\infty}\frac{e^{i\varepsilon(x-\eta)r}}{\big(r-\big[\lambda-i\varepsilon0^{+}\big]\big)^{2}}\mathrm{d}r\,. \tag{7.2.20}$$

When we compare the right hand side of (7.2.20) to the derivative of [see (7.1.7)] $g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ with respect to λ we immediately arrive at the conclusion

$$\lim_{\delta\to\infty}\left\{\frac{\partial}{\partial\lambda}\,G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)\right\}=\varepsilon\frac{\partial}{\partial\lambda}g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda). \tag{7.2.21}$$

Let us continue forging a path towards the two limits in (7.2.12). Our effort will now be directed towards the computation of

$$\lim_{\delta\to\infty}\left\{\frac{\partial}{\partial\lambda}G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda)\right\}. \tag{7.2.22}$$

Equation (7.2.2) provides us with an integral representation of $G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda)$, and such information is relevant to the computation of (7.2.22). Differentiating both sides of (7.2.2) with respect to λ we obtain the equation

$$\frac{\partial}{\partial\lambda}G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda)=-\frac{\varepsilon}{2\pi}\zeta_{-}'(\lambda)$$

$$\times\int_{-\infty+i0^{+}}^{\infty+i0^{+}}\frac{\big(1-e^{-2\varepsilon\delta r}\big)e^{i(x-\eta)r}}{\big\{r+\varepsilon\big[\zeta_{-}(\lambda)-(2\delta)^{-1}\big]\big(1-e^{-2\varepsilon\delta r}\big)\big\}^{2}}\mathrm{d}r, \tag{7.2.23}$$

where $\zeta_{-}(\lambda)$ is defined by (7.2.4), and

$$\zeta_{-}'(\lambda)\equiv\frac{\mathrm{d}}{\mathrm{d}\lambda}\zeta_{-}(\lambda)=\frac{1-e^{2\delta\lambda}+2\delta\lambda e^{2\delta\lambda}}{\big(1-e^{2\delta\lambda}\big)^{2}}. \tag{7.2.24}$$

Before we consider the deep water limit of (7.2.23) we will write the complex number r in the form $r=R-\varepsilon\lambda$, where R is also a complex number. Introducing into equation (7.2.23) the translation $r=R-\varepsilon\lambda$ we find that this equation is transformed into the following:

$$\frac{\partial}{\partial\lambda}G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda) = \frac{\varepsilon}{2\pi}\frac{\left(e^{2\delta\lambda}-1-2\delta\lambda e^{2\delta\lambda}\right)}{\left(1-e^{2\delta\lambda}\right)^2}e^{-i\varepsilon\lambda(x-\eta)} \times\int_{-\infty+i0^+}^{\infty+i0^+}\frac{\left[1-e^{-2\varepsilon\delta(R-\varepsilon\lambda)}\right]}{\Delta^2}e^{i(x-\eta)R}\,\mathrm{d}R, \tag{7.2.25}$$

where

$$\Delta \;\underline{\underline{\mathrm{def}}}\; R-\varepsilon\lambda+\varepsilon\left[\zeta_-(\lambda)-(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta R+2\delta\lambda}\right). \tag{7.2.26}$$

We wish to alert the reader to the fact that we have used (7.2.24) to arrive at (7.2.25). Temporarily, let us focus on (7.2.26). In Section 4.3 of this thesis we collated several identities that involve $\zeta_\pm(\lambda)$. Two of the suggested identities are relevant to any Δ: (4.3.17a) and (4.3.17c). Intelligent use of (4.3.17a) and (4.3.17c) will reveal to the reader that Δ can be expressed in the form

$$\Delta = R-\varepsilon\left[\zeta_+(\lambda)+(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta R}\right),$$

and therefore (7.2.25) now reads as follows:

$$\frac{\partial}{\partial\lambda}G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda) = \frac{\varepsilon}{2\pi}\frac{\left(e^{2\delta\lambda}-1-2\delta\lambda e^{2\delta\lambda}\right)}{\left(1-e^{2\delta\lambda}\right)^2}e^{-i\varepsilon\lambda(x-\eta)} \times\int_{-\infty+i0^+}^{\infty+i0^+}\frac{\left[1-e^{-2\varepsilon\delta(R-\varepsilon\lambda)}\right]e^{i(x-\eta)R}}{\left\{R-\varepsilon\left[\zeta_+(\lambda)+(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta R}\right)\right\}^2}\mathrm{d}R. \tag{7.2.27}$$

Santini et al. [103] have shown that the scaling $p=\delta R$, where p is a complex number, unlocks the door that leads to the deep water limit of (7.2.27). The equation

$$\lim_{\delta\to\infty}\left\{\frac{\partial}{\partial\lambda}G_{-\varepsilon}^{+\varepsilon}(x,\eta;-\lambda)\right\} = \frac{\varepsilon}{\pi\lambda}e^{-i\varepsilon\lambda(x-\eta)}\int_{-\infty+i0^+}^{\infty+i0^+}\frac{e^{-2\varepsilon p}}{\left(1-e^{-2\varepsilon p}\right)^2}\mathrm{d}p \tag{7.2.28}$$

materializes when we apply to (7.2.27) the scaling $p=\delta R$, and then allow δ to increase without bound. Standard techniques can be applied to evaluate the integral in (7.2.28). It is a small step from (7.2.28) to the equation

$$\lim_{\delta \to \infty} \left\{ \frac{\partial}{\partial \lambda} G^{+\varepsilon}_{-\varepsilon}(x, \eta; -\lambda) \right\} = -\frac{\varepsilon}{2\pi\lambda} e^{-i\varepsilon\lambda(x-\eta)}. \tag{7.2.29}$$

Substituting (7.2.21) and (7.2.29) into their correct positions within (7.2.12) we arrive at the desired result: equation (7.2.10). The derivation of (7.2.10) from (7.2.1) completes our investigation of the the complex-valued ILW equation.

Equation (4.3.53) states that

$$M^{+\varepsilon}(x; \lambda) = A(\lambda) \bar{N}^{+\varepsilon}(x; \lambda) + B(\lambda) N^{+\varepsilon}(x; \lambda), \tag{7.2.30}$$

where [see (4.3.57)]

$$A(\lambda) = 1 - \frac{i\varepsilon}{2\delta\zeta_{+}(\lambda)} \int_{-\infty}^{\infty} U(\eta) M^{+\varepsilon}(\eta; \lambda) \mathrm{d}\eta \tag{7.2.31}$$

and [see (4.3.58)]

$$B(\lambda) e^{\delta\lambda} = \frac{i\varepsilon}{2\delta\zeta_{-}(\lambda)} \int_{-\infty}^{\infty} U(\eta) M^{+\varepsilon}(\eta; \lambda) e^{-i\varepsilon\lambda\eta} \mathrm{d}\eta. \tag{7.2.32}$$

The functions $M^{+\varepsilon}(x; \lambda)$, $N^{+\varepsilon}(x; \lambda)$ and $\bar{N}^{+\varepsilon}(x; \lambda)$ are Jost functions associated with the IST for the complex-valued ILW equation, and the integral equations that characterize these particular Jost functions are presented as equations (4.3.51) and (4.3.52).

We will bring this chapter to a close by showing that (7.2.30) $\rightarrow$ (5.2.23) as $\delta \rightarrow \infty$, the latter equation being the analogue of (7.2.30) in the IST for the complex-valued BO equation. Consider the $\delta \rightarrow \infty$ limit of (7.2.31) and (7.2.32). First, observe that (2.1.21), (5.2.18a) and (7.2.14) provide us with the information

$$U(\eta) \frac{M^{+\varepsilon}(\eta; \lambda)}{\zeta_{+}(\lambda)} = \mathrm{O}(1) \text{ as } \delta \rightarrow \infty.$$

Therefore, the deep water limit of (7.2.31) is

$$\lim_{\delta \to \infty} A(\lambda) = 1. \tag{7.2.33}$$

The computation of the deep water limit for (7.2.32) is assisted by the result

$$\lim_{\delta \to \infty} 2\delta\zeta_{-}(\lambda) = 1, \tag{7.2.34}$$

which can be verified by use of (7.2.4). Equations (2.1.21), (5.2.18a), (7.2.32) and (7.2.34) lead us to conclude that

$$\lim_{\delta \to \infty} B(\lambda) e^{\delta\lambda} = \rho(\lambda), \tag{7.2.35}$$

where [see (5.2.24)]

$$\rho(\lambda) \underline{\underline{\text{def}}}\ i\varepsilon\int_{-\infty}^{\infty} q(\eta)\, m^{+\varepsilon}(\eta;\lambda)\, e^{-i\varepsilon\lambda\eta}\, d\eta.$$

Now let us devote our attention to (7.2.30). Write (7.2.30) in the form

$$M^{+\varepsilon}(x;\lambda) = A(\lambda)\bar{N}^{+\varepsilon}(x;\lambda) + \left\{B(\lambda)e^{\delta\lambda}\right\}\left\{N^{+\varepsilon}(x;\lambda)e^{-\delta\lambda}\right\}. \tag{7.2.36}$$

Equations (5.2.18a), (5.2.18c), (5.2.18d), (7.2.33) and (7.2.35) combine in a manner that permits us to conclude that the deep water limit of (7.2.36) is

$$m^{+\varepsilon}(x;\lambda) = \bar{n}^{+\varepsilon}(x;\lambda) + \rho(\lambda)n^{+\varepsilon}(x;\lambda). \tag{7.2.37}$$

The reader who compares (7.2.37) to (5.2.23) will immediately discover that these two equations are identical, and we therefore close our analysis of the deep water limit for the