CHAPTER 2

DERIVATIONS OF THE

MODIFIED INTERMEDIATE LONG WAVE

AND

MODIFIED BENJAMIN-ONO

EQUATIONS

Section 2.1: The Origin of the MILW and MBO Equations

The Modified Intermediate Long Wave (MILW) equation is a nonlinear singular integro-differential equation. In an appropriate limit [89,112] the MILW equation reduces to the well known Modified Korteweg-de Vries (MKdV) equation.

Our objectives in this section are to elucidate the origin of the MILW equation and to collate those limiting cases of the MILW equation that are relevant to this thesis. The analysis in this section is a refinement and synthesis of the original works by Satsuma et al. [112], Nakamura [89] and Gibbons and Kupersmidt [48] on the origin and derivation of the MILW equation.

The standard Intermediate Long Wave (ILW) equation provides the catalyst for the derivations that will appear in this section. A convenient form of the ILW equation from which to begin our work is [108]

$$U_t + \frac{1}{\delta}U_x + 2UU_x + \mathbf{T}\left(U_{xx}\right) = 0, \tag{2.1.1}$$

where $U \equiv U\left(x,t\right)$, $x \in \Re$, $t \geq 0$, δ is a positive parameter, subscripts denote partial derivatives and

$$\left(\mathbf{T}f\right)(x) \underline{\underline{\mathrm{def}}} \frac{1}{2\delta}\left(\mathrm{P}\right)\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f\left(\xi\right) \mathrm{d}\xi.$$

Throughout this chapter we will suppress the dependence of all variables that depend on the positive parameter δ, for example we will write $U\left(x,t\right)$ in lieu of $U\left(x,t;\delta\right)$.

The dependent variable transformation

$$W\left(x,t\right) = \int_{-\infty}^{x} U\left(\xi,t\right)\mathrm{d}\xi, \tag{2.1.2}$$

accompanied by the boundary conditions $U\left(\xi,t\right) \to 0$ (uniformly in t) as $\left|\xi\right| \to \infty$, when introduced into (2.1.1) will convert (2.1.1) into the equation

$$W_t + \frac{1}{\delta}W_x + \left(W_x\right)^2 + \mathbf{T}\left(W_{xx}\right) = 0. \tag{2.1.3}$$

Equation (2.1.2) is motivated by the observation that $U\left(x,t\right)$ is a conserved density of (2.1.1).

transformation (BT) for equation (2.1.3) is [108]

$$\left(W' + W\right)_x = \lambda + i\mathbf{T}\left(W'_x - W_x\right) + \beta e^{i(W' - W)}, \tag{2.1.4a}$$

$$\left(W' - W\right)_t = -\left(\lambda + \tfrac{1}{\delta}\right)\left(W' - W\right)_x + i\left(W' + W\right)_{xx} - i\left(W' - W\right)_x\mathbf{T}\left(W'_x - W_x\right) \tag{2.1.4b}$$

where λ and β represent two arbitrary real (nonzero) constants. If $W(x,t)$ denotes a (known) local solution of (2.1.3), then we can construct a *new* solution, denoted as $W'(x,t)$, of (2.1.3) by solving equations (2.1.4a) and (2.1.4b).

Next, in our endeavour to elucidate the origin of the MILW and MBO equations, we shall introduce into equations (2.1.4a) and (2.1.4b) the *single* dependent variable, $V(x,t)$, defined by the equation

$$V(x,t) = i\left(W' - W\right). \tag{2.1.5}$$

The functions W and W' in (2.1.5) denote two different solutions of (2.1.3) that are connected by equations (2.1.4a) and (2.1.4b). Eliminating the $\left(W + W'\right)$ terms between (2.1.4a) and (2.1.4b) we obtain the nonlinear evolution equation

$$V_t + \left(\lambda + \tfrac{1}{\delta}\right)V_x + \beta V_x e^V + V_x\mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0, \tag{2.1.6}$$

where $V \equiv V(x,t)$. Applying the transformations $x' = x - (\lambda + \beta)t$ and $t' = t$ to (2.1.6) we find that (2.1.6) is converted into the equation

$$V_t + \beta V_x\left(e^V - 1\right) + \tfrac{1}{\delta}V_x + V_x\mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0. \tag{2.1.7}$$

or translations involving any of the variables in the original (source) equation. Hereafter, we

As mentioned in the introductory stage of this section, variants of (2.1.7) have been derived by several researchers. Nakamura [89], for example, has used the BT for the ordinary ILW equation to generate the nonlinear evolution equation

$$V_t + \left(u_0 - \mu + \tfrac{1}{\delta}\right)V_x + \beta V_x e^V + V_x\mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0, \tag{2.1.8}$$

where β retains its identity, and u_0 and μ are two new real parameters that are unique to

$$\frac{\delta}{3}V_t + \frac{1}{\delta}V_x e^V + V_x \mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0 \qquad (2.1.9)$$

as their candidate to represent the MILW equation. Also, Gibbons and Kupersmidt [see equation (9) in Ref. 48] have derived a version of the MILW equation. Equations (2.1.8) and (2.1.9) are (gauge) equivalent to (2.1.7). The reader may wish to verify the following:

1) the transformations $x' = x + \left(\mu - u_0 - \beta\right)t$ and $t' = t$ when applied to (2.1.8) will convert equation (2.1.8) into equation (2.1.7);

2) the transformations $V' = V - \ln\left(\delta|\beta|\right)$, $x' = \left(x + \beta t - t/\delta\right)\mathrm{sgn}\left(\beta\right)$ and $t' = \delta t/3$ when introduced into (2.1.7) will map (2.1.7) into equation (2.1.9).

Equation (2.1.4a) is a particularly important result because it contains an *explicit* relationship between solutions of the ILW and MILW equations. The connection between the ILW and MILW equations can be visualized through the observation that the reduction $\lambda = -\beta$, which was used in the passage from (2.1.6) to (2.1.7), necessitates that we modify equation (2.1.4a) to read

$$\left(W' + W\right)_x = \beta\left(e^{i(W' - W)} - 1\right) + i\mathbf{T}\left(W'_x - W_x\right), \qquad (2.1.10a)$$

and revise equation (2.1.4b) to read

$$\left(W' - W\right)_t = \left(\beta - \frac{1}{\delta}\right)\left(W' - W\right)_x + i\left(W' + W\right)_{xx} - i\left(W' - W\right)_x \mathbf{T}\left(W'_x - W_x\right). \qquad (2.1.10b)$$

Introducing into (2.1.10a) the information contained in equations (2.1.2) and (2.1.5) leads us to the concrete form of the relationship between the ILW and MILW equations:

$$U\left(x,\, t\right) = \frac{1}{2}\left\{\mathbf{T}\left(V_x\right) + \beta\left(e^V - 1\right) + iV_x\right\}. \qquad (2.1.11)$$

Equation (2.1.11) maps solutions of equation (2.1.7) into solutions of (2.1.1). In analogy with (1.1.5), which maps solutions of the MKdV equation into solutions of the KdV equation, (2.1.11) is the Miura transformation that maps solutions of the MILW equation into solutions of the ILW equation.

The parameter δ has two extremes: shallow water limit ($\delta \to 0^+$); deep water limit ($\delta \to \infty$). In the shallow water limit, equation (2.1.7) reduces to the *focusing* version of the MKdV equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0. \qquad (2.1.12)$$

The deep water limit of (2.1.7) is the Modified Benjamin-Ono (MBO) equation:

$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x\mathbf{H}\left(Q_x\right) + \mathbf{H}\left(Q_{xx}\right) = 0, \tag{2.1.13}$$

where $\mathbf{H}$ denotes the Hilbert transform,

$$\left(\mathbf{H}f\right)(x) \underline{\underline{\text{def}}} \tfrac{1}{\pi}(\mathrm{P})\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\mathrm{d}\xi.$$

The transition from (2.1.7) to (2.1.13) is contingent on the following conditions being satisfied:

$$\left.\begin{aligned} V\left(x,t\right) &= Q\left(x,t\right) + \mathrm{O}\left(\tfrac{1}{\delta}\right) \\ \beta &= \alpha + \mathrm{O}\left(\tfrac{1}{\delta}\right) \end{aligned}\right\} \text{ as } \delta \to \infty. \tag{2.1.14}$$

Verifying that (2.1.13) is the deep water limit of (2.1.7) requires the reader to first substitute (2.1.14) into (2.1.7), and then use the limit (1.3.4).

Considerable thought is required to prove that (2.1.12) is the shallow water limit of (2.1.7). We have determined that the transition from (2.1.7) to (2.1.12) requires:

$$\left.\begin{aligned} V &= \delta v + \mathrm{O}\left(\delta^3\right) \\ \beta &= \tfrac{1}{\delta} + \mathrm{O}\left(\delta\right) \\ x' &= \tfrac{x}{2} \\ t' &= \frac{\delta}{24}t \end{aligned}\right\}. \tag{2.1.15}$$

Substituting (1.3.26) and (2.1.15) into (2.1.7) we find that the MKdV equation appears in the leading term of the series generated.

Nakamura [89] and Satsuma et al. [112], as already mentioned in this section, have derived versions of the MILW equation that are (gauge) equivalent to (2.1.7). We have chosen (2.1.7) as our representative of the MILW equation because it is endowed with a transparent deep water limit and a shallow water limit that leads directly to (2.1.12),

equation is the Gardner equation [86], and (2.1.9) suffers from the deficiency of not being easily amenable to a determination of its $\delta \to \infty$ limit.

Equation (2.1.11) is sensitive to movements in the parameter δ. For example, substituting (1.3.26), (2.1.15) and the shallow water expansion

$$U = \frac{\delta u}{4} + \mathrm{O}\left(\delta^2\right) \tag{2.1.16}$$

into (2.1.11) we obtain a series (in ascending powers of δ) whose leading term is equivalent to the equation

$$u = v^2 + i v_x. \tag{2.1.17}$$

The variable $u \equiv u(x, t)$ that appears in equations (2.1.16) and (2.1.17) evolves according to the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.1.18}$$

whereas $v \equiv v(x, t)$ satisfies (2.1.12).

Equation (2.1.17) is the famous Miura transformation [86], and its presence in our work is a consequence of the mode by which the MILW equation was derived, that is the auto BT for the ILW equation was used to derive the MILW equation. The appearance of the imaginary unit in (2.1.17), which can be removed by the point transformation $v \to iv$, is attributed to the selection of positive coefficients throughout equations (2.1.12) and (2.1.18).

The deep water analogue of (2.1.11) is

$$q = \frac{1}{2}\left\{\mathbf{H}\left(Q_x\right) + \alpha\left(e^Q - 1\right) + iQ_x\right\}, \tag{2.1.19}$$

where $q \equiv q(x, t)$ evolves according to the Benjamin-Ono (BO) equation,

$$q_t + 2qq_x + \mathbf{H}\left(q_{xx}\right) = 0, \tag{2.1.20}$$

and $Q \equiv Q(x, t)$ satisfies equation (2.1.13). The derivation of equation (2.1.19) is completed when we substitute (1.3.4), (2.1.14) and the deep water expansion

$$U(x, t) = q(x, t) + \mathrm{O}\left(\frac{1}{\delta}\right) \text{ as } \delta \to \infty \tag{2.1.21}$$

into (2.1.11), and then balance the leading order terms in the series produced.

Equation (2.1.11) maps *real-valued* solutions of the MILW equation into solutions of the ILW equation that are *complex-valued*; the complex-valued solutions being referred to here are with respect to the dependent variable only, x and t remain real-valued. In particular, if $V_0 = V(x, 0)$ denotes a real-valued solution of (2.1.7), then according to (2.1.11)

$$U_0 \equiv U(x, 0) = \tfrac{1}{2}\left\{\mathbf{T}\left(V_{0,x}\right) + \beta\left(e^{V_0} - 1\right) + i\,V_{0,x}\right\} \tag{2.1.22}$$

defines a complex-valued solution of (2.1.1); the notation $V_{0,x}$ will be used to denote $\partial V_0 / \partial x$. If we assume that $V_0 \to 0$ and $V_{0,x} \to 0$ as $|x| \to \infty$, then (1.3.28) combines with (2.1.22) to induce the boundary conditions $U_0 \to 0$ as $|x| \to \infty$.

Suppose now that $U(x, t)$ is a *known* complex-valued solution of (2.1.1) that has evolved from the *known* complex initial value (2.1.22). In an environment where all the conditions referring to $U(x, t)$ are satisfied, we can deduce from (2.1.11) that

$$V_x(x, t) = 2\,\mathrm{Im}\,\{U(x, t)\} \tag{2.1.23}$$

at all points $(x, t) \in \Re \times \Re^+$, where $\Re^+$ denotes the set of positive real numbers and zero.

evolves from the known localized initial value V_0) of the initial value problem for the MILW equation can be constructed from the solution of the initial value problem for the

complex initial value (2.1.22). Once $U(x, t)$ is known, we then determine $V_x(x, t)$ from (2.1.23), and thence $V(x, t)$ by a quadrature in x. The function of integration that remains in $V(x, t)$ is fixed by the boundary conditions $V(x, t) \to 0$ (uniformly in t) as $|x| \to \infty$.

Equations (2.1.17) and (2.1.19) transform real-valued solutions of a modified-type

The isomorphism of equations (2.1.11), (2.1.17) and (2.1.19) suggests that the procedure to solve the initial value problem for the MILW equation can also be applied, with suitable modifications, to solve the initial value problems for the MKdV and MBO equations. In Chapter 3 of this thesis we will begin the task of investigating whether the schemes outlined to solve the initial value problems for the MKdV, MILW and MBO equations are feasible. We will devote the remainder of this chapter to acquiring by means of a *direct method* the 1-soliton solution for the MILW equation.

Section 2.2: Bilinear Form of the MILW Equation

Hirota [52,53] has developed a powerful procedure that can be used to compute *particular* solutions of many soliton-type equations. For example, multi-soliton solutions

Bilinear Transformation Method. The two linear operators D_t^m and D_x^n are central in any implementation of the Bilinear Transformation Method. The actions of the two operators D_t^m and D_x^n are defined by the equation

$$\mathrm{D}_t^m \mathrm{D}_x^n (a \circ b) \ \underline{\underline{\mathrm{def}}} \ \left(\partial_t - \partial_{t'}\right)^m \left(\partial_x - \partial_{x'}\right)^n a(x,t) b\left(x', t'\right) \Big|_{\substack{x' = x \\ t' = t}}, \tag{2.2.1}$$

where m and n are arbitrary (and uncorrelated) non-negative integers and ∂_t (for example) denotes the partial derivative with respect to t. Matsuno [76] has collected the many and diverse properties of the operators D_t^m and D_x^n, and we refer the interested reader to Ref. 76 for a comprehensive coverage of the Bilinear Transformation Method.

Nakamura [89] and Satsuma and Ablowitz [110] have successfully implemented

the MBO equation. In this section we will also apply the Bilinear Transformation Method to compute the 1-soliton solution for the version [see (2.1.7)] of the MILW equation that we are using in this thesis. However, the mode by which we will arrive at the 1-soliton solution for equation (2.1.7) differs from the approach used by the authors of Refs 89 and 110. Our method begins with the (known) results associated with the application of

appropriate manner) to the MILW equation. The authors of Refs 89 and 110 use a

to solve the associated bilinear equations. We have selected the approach based on the

connection between the MKdV and MILW equations that was elucidated in Section 2.1 of

method, but to compute (for future reference) the 1-soliton solution for the MILW equation.

A review of the essential features of the Bilinear Transformation Method is timely. The MKdV equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{2.2.2}$$

$v \equiv v(x, t)$ in equation (2.2.2). We will direct our efforts towards constructing real-valued solutions of (2.2.2) that satisfy the boundary conditions

$$v,\, v_x,\, v_{xx},\, \ldots \to 0 \text{ (uniformly in } t\text{) as } |x| \to \infty.$$

The Bilinear Transformation Method uses a *dependent* variable transformation, which involves at least one (initially unknown) auxiliary function, to express the original nonlinear evolution equation in a bilinear form that contains the auxiliary function(s) in lieu of the physical (dependent) variable. Hirota [53] has determined that the dependent variable transformation

$$v(x, t) = i\frac{\partial}{\partial x}\left\{\ln\left(\frac{f(x, t)}{f^*(x, t)}\right)\right\}, \tag{2.2.3}$$

where $f^*(x, t)$ denotes the complex conjugate of $f(x, t)$, is the correct choice to express (2.2.2) in a bilinear form that involves the auxiliary function $f(x, t)$. Several restrictions [53] must be imposed on the function $f(x, t)$. For example, to avoid the trivial solution $f(x, t)$ must be a complex-valued function of the two real variables x and t. A lengthy and complicated (nonlinear) equation for $f(x, t)$ is produced when (2.2.3) is introduced into (2.2.2), and we will not burden the reader with the algebraic details. It suffices for us to record that (2.2.1) and some of the identities [76] for D_t^m and D_x^n allow us to disentangle the terms produced and retrieve the following equations:

$$\mathrm{D}_x^2\left(f \circ f^*\right) = 0; \tag{2.2.4}$$

$$\left(\mathrm{D}_t + \mathrm{D}_x^3\right)\left(f \circ f^*\right) = 0. \tag{2.2.5}$$

Solutions for (2.2.4) and (2.2.5) are computed recursively by means of the perturbation series

$$f(x,t) = f_0(x,t) + \varepsilon f_1(x,t) + \varepsilon^2 f_2(x,t) + \ldots,$$

where the real-valued parameter ε is such that $0 < \varepsilon << 1$, but otherwise arbitrary. Pure soliton solutions of (2.2.2) are associated with a truncating series. For example, the linear function (with respect to ε)

$$f(x,t) = 1 + i\varepsilon e^{2\hat{k}(x - 4\hat{k}^2 t)} \tag{2.2.6}$$

generates the 1-soliton solution for the MKdV equation. The function defined by (2.2.6) satisfies (2.2.4) and (2.2.5) for arbitrary real nonzero $\hat{k}$. Substituting (2.2.6) into (2.2.3) we obtain the 1-soliton solution for the MKdV equation:

$$v(x,t) = -2\hat{k}\operatorname{sech}\left(2\hat{k}\left[x - 4\hat{k}^2 t + \tilde{x}_0\right]\right), \tag{2.2.7}$$

where

$$\tilde{x}_0 \underline{\underline{\mathrm{def}}} \frac{\ln(\varepsilon)}{2\hat{k}}. \tag{2.2.8}$$

equation (2.1.7), alluded to at the outset of this section. Nakamura [89] has determined that the dependent variable transformation

$$V(x,t) = \ln\left(\frac{f_+ g_-}{f_- g_+}\right) \tag{2.2.9}$$

is the appropriate choice to express (2.1.7) in a bilinear form that involves the auxiliary functions $f(x,t)$ and $g(x,t)$. The components of (2.2.9) are connected to the auxiliary functions by means of the equations

$$f_\pm \underline{\underline{\mathrm{def}}}\ f(x \pm i\delta, t)$$

and

$$g_\pm \underline{\underline{\mathrm{def}}}\ g(x \pm i\delta, t).$$

We will assume that $f(x,t)$ and $g(x,t)$ satisfy all the conditions listed in (1.3.44).

into (2.1.7). The most efficient derivation uses the observation that (2.2.9) can be expressed in the form (2.1.5) through the following choices for W and W':

$$W = i \ln\left(g_- / g_+\right); \tag{2.2.10}$$

$$W' = i \ln\left(f_- / f_+\right). \tag{2.2.11}$$

The functions W and W' represent two (distinct) solutions of (2.1.3) that are connected by equations (2.1.10a) and (2.1.10b). Substituting (2.2.10) and (2.2.11) into (2.1.10a) and (2.1.10b), and then using (1.3.40) to simplify terms of the form $\mathbf{T}(\bullet)_x$ we are able to conclude that

$$2i\mathrm{D}_x\left(f_- \circ g_+\right) = \beta\left(f_+ g_- - f_- g_+\right) \tag{2.2.12}$$

and

$$\left(i\mathrm{D}_t + i\left(\frac{1}{\delta} - \beta\right)\mathrm{D}_x + \mathrm{D}_x^2\right)\left(f_\pm \circ g_\pm\right) = 0. \tag{2.2.13}$$

of Refs 89, 110 and 112 have derived their own versions of (2.2.12) and (2.2.13).

Nakamura [89] has constructed solutions of (2.2.12) and (2.2.13) by using a perturbation series for $f_\pm$ and $g_\pm$. As promised, we will solve (2.2.12) and (2.2.13) by using

now seek *particular* solutions for (2.2.12) and (2.2.13) in the form

$$f(x,t) = 1 + i e^{\hat{\lambda}(x - \phi t + x_0)} \tag{2.2.14}$$

and

$$g(x,t) = 1 - i e^{\hat{\lambda}(x - \phi t + x_0)}, \tag{2.2.15}$$

where $\hat{\lambda}$ and x_0 are arbitrary (nonzero) real parameters and ϕ, which is independent of x and t, is at present unknown. The important task is to relate $\hat{\lambda}$ and ϕ to the (fundamental) parameter, β, in the MILW equation; the parameter x_0 is *not* essential because it can be absorbed by the transformation $x - x_0 \to x$. Substituting (2.2.14) and (2.2.15) into (2.2.12) we obtain

$$\beta = \hat{\lambda}\cot\left(\delta\hat{\lambda}\right). \tag{2.2.16}$$

Repeating the procedure used to derive (2.2.16), but this time substituting into (2.2.13), we obtain

$$\phi = \frac{1}{\delta} - \beta = \frac{1}{\delta} - \hat{\lambda}\cot\left(\delta\hat{\lambda}\right). \tag{2.2.17}$$

We interrupt our calculation of the 1-soliton solution for the MILW equation to make the following observations:

- if $\hat{\lambda} = \mathrm{O}\left(1\right)$ as $\delta \to 0^{+}$, then the expansion of (2.2.16) in the shallow water region is

$$\beta = \frac{1}{\delta} - \frac{\delta\hat{\lambda}^{2}}{3} + \mathrm{O}\left(\delta^{3}\right), \tag{2.2.18}$$

which is consistent with the assumption made in (2.1.15);

- the condition (2.1.14) is retrieved from (2.2.16) in the situation

$$\hat{\lambda} = \frac{\pi}{\delta} + \frac{\pi}{\alpha\delta^{2}} + \mathrm{O}\left(\delta^{-3}\right),$$

where $\delta >> 1$.

Substituting (2.2.14) and (2.2.15) into (2.2.9) yields us

$$V(x,t) = \ln\left(\frac{\cosh\left[\hat{\lambda}(x - \phi t + x_{0})\right] - \sin\left(\delta\hat{\lambda}\right)}{\cosh\left[\hat{\lambda}(x - \phi t + x_{0})\right] + \sin\left(\delta\hat{\lambda}\right)}\right), \tag{2.2.19}$$

which is the 1-soliton solution for the MILW equation [110]. Equation (2.2.19) must be read in conjunction with (2.2.16) and (2.2.17).

An exercise that will consolidate our position is to compute the shallow water limit for each of the equations (2.2.9), (2.2.12), (2.2.13) and (2.2.19), and then verify that each of these limits is consistent with the theory for the MKdV equation. Two pieces of information will assist the reader to repeat our calculations:

1) the conditions for the transition from the MILW equation to the MKdV equation are enumerated in (2.1.15) and (2.2.18);

2) $$f_{\pm} = f \pm i\delta f_{x} - \frac{\delta^{2}}{2!} f_{xx} \mp \frac{i\delta^{3}}{3!} f_{xxx} + \mathrm{O}\left(\delta^{4}\right),$$

where $0 < \delta << 1$.

The expansion of (2.2.9) in the shallow water region is

$$V(x,t) = i\delta\frac{\partial}{\partial x}\left\{\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right\} - \frac{i\delta^3}{24}\frac{\partial^3}{\partial x^3}\left\{\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right\} + \mathrm{O}\left(\delta^5\right),$$

and when we compare this equation with (2.1.15) we immediately obtain a generalized version of (2.2.3) for which the condition $g(x,t) \equiv f^*(x,t)$ is satisfied. Equation (2.2.12) admits the shallow water expansion

$$\mathrm{D}_x^2\left(f\circ g\right) - \frac{i\delta}{3}\left(\frac{1}{2}\mathrm{D}_x^3 - 2\hat{\lambda}^2\mathrm{D}_x\right)\left(f\circ g\right) + \mathrm{O}\left(\delta^2\right) = 0, \qquad (2.2.20)$$

whereas the shallow water expansion of (2.2.13) is

$$\mathrm{D}_x^2\left(f\circ g\right) + \frac{i\delta}{3}\left(\frac{1}{2}\mathrm{D}_t + 2\hat{\lambda}^2\mathrm{D}_x\right)\left(f\circ g\right) + \mathrm{O}\left(\delta^2\right) = 0. \qquad (2.2.21)$$

Equations (2.2.20) and (2.2.21) imply (2.2.4) and (2.2.5). In the shallow water region we find that the behaviour of (2.2.19) is

$$V\left(x,t\right) = -2\delta\hat{\lambda}\operatorname{sech}\left(\vartheta\right) + \delta^3\frac{\left(5\hat{\lambda}^3\cosh^2\left(\vartheta\right) - 8t\hat{\lambda}^6\sinh\left(2\vartheta\right) - 10\hat{\lambda}^3\right)}{15\cosh^3\left(\vartheta\right)} + \mathrm{O}\left(\delta^5\right) \qquad (2.2.22)$$

where

$$\vartheta \;\underline{\underline{\mathrm{def}}}\; 2\hat{\lambda}\left(x - 4\hat{\lambda}^2 t + x_0\right).$$

At leading order (2.2.22) agrees with (2.2.7), the latter equation being the 1-soliton solution for the MKdV equation. Therefore, the theory we have developed in this section for the MILW equation is internally consistent.