

APPENDIX A

Definitions

Throughout this thesis, the following notation has been used consistently. Shown below is a brief description of the symbol, and an appropriate page reference. Other definitions used in the body of the thesis are defined where they are used.

Table A.1: Notations and definitions.

Notation	Description	Page
General		
N	Number of individuals within a group.	
t	Time.	
T	Total time.	
$\mathbf{r}_i(t)$	Position vector of individual i at time t .	
$\mathbf{v}_i(t)$	Direction vector of individual i at time t .	
$\mathbf{r}_{ij}(t)$	Unit vector pointing from individual i in the direction of neighbour j .	
\sim	Shorthand for ‘distributed as’.	
\approx	Shorthand for ‘approximately equal to’.	
\xrightarrow{D}	Shorthand for ‘converges in distribution to’.	
\xrightarrow{p}	Shorthand for ‘converges in probability to’.	
q	Number of dimensions.	
\mathbb{R}^q	Real space in q dimensions.	
Ω_q	Unit hypersphere in \mathbb{R}^q .	
\mathbf{x}	Random vector in \mathbb{R}^q .	
\mathbf{x}_i	Random sample of random vector \mathbf{x} .	
$ \mathbf{x} $	Magnitude of vector \mathbf{x} .	
\mathbf{x}^T	Transpose of vector \mathbf{x} .	
$E(\mathbf{x})$	Expected value of the random vector \mathbf{x} .	
$\text{Var}(\mathbf{x})$	Variance of the random vector \mathbf{x} .	
$N(\mu, \sigma^2)$	Gaussian distribution with mean μ and variance σ^2 .	
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Notation	Description	Page
χ_ν^2	Chi-squared distribution with ν degrees of freedom.	
T	Test statistic.	
Chapter 2		
\mathbf{r}_{group}	Group centre	26
p_{group}	Group polarisation.	27
m_{group}	Group momentum.	27
$\hat{\sigma}$	Spherical variance.	27
NND	Nearest neighbour distance.	27
Expanse	Expanse of group.	28
NGDR	Net to gross displacement ratio.	28
ρ_q	Spherical correlation coefficient.	30
$\hat{\rho}_q$	Estimate of ρ_q .	30
ψ	Angle between the spherical mean and x -axis.	32
\mathbf{S}_N	Resultant vector of the random vectors \mathbf{x}_i .	26
$\boldsymbol{\mu}$	Expected value of random vector \mathbf{x} .	26
$\hat{\boldsymbol{\mu}}$	Average group direction (estimate of $\boldsymbol{\mu}$).	26
Σ	Covariance matrix of random vector \mathbf{x} .	34
$N_q(\boldsymbol{\mu}, \Sigma)$	Gaussian distribution in \mathbb{R}^q with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ .	34
c	Scalar random variable.	35
$E(c)$	Concentration (polarisation) of the distribution on Ω_q about $\boldsymbol{\mu}$.	36
\mathbf{S}_N^\perp	Component of \mathbf{S}_N orthogonal to $\boldsymbol{\mu}$.	38
H	$I_q - \boldsymbol{\mu}\boldsymbol{\mu}^T$.	38
$\boldsymbol{\mu}_0$	Hypothesised direction.	40
$\mathbf{S}_N^\perp \boldsymbol{\mu}_0$	Component of \mathbf{S}_N orthogonal to $\boldsymbol{\mu}_0$.	40
K	Number of samples.	41
N_k	Sample size of the k -th sample.	41
\mathbf{S}_{N_k}	Resultant vector from the k -th sample.	41
Chapters 3 and 4		
τ	Time interval width.	50
$\mathbf{v}_i^d(t)$	Preferred travel direction of individual i .	50
r_r	Radius of the zone of repulsion.	50
r_a	Radius of the zone of attraction.	50
r_o	Radius of the zone of orientation.	50
Δr_r	Width of the zone of repulsion.	60
Δr_a	Width of the zone of attraction.	60

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Notation	Description	Page
Δr_o	Width of the zone of orientation.	60
$\mathbf{v}_i^{repulse}(t)$	Preferred travel direction due to repulsive forces.	51
$\mathbf{v}_i^{orient}(t)$	Preferred travel direction due to alignment forces.	52
$\mathbf{v}_i^{attract}(t)$	Preferred travel direction due to attraction forces.	52
δ	Field of perception (degrees).	52
γ	Turning rate (degrees).	53
s_i	Speed of individuals in collective model.	53
θ	Longitude.	55
ϕ	Colatitude.	55
κ	Concentration parameter.	56
$N_{informed}$	Number of knowledgeable individuals in group.	95
$\mathbf{r}_{informed}(t)$	Position vector of knowledgeable individual i .	95
$\mathbf{v}_{informed}(t)$	Direction vector of knowledgeable individual i .	95
Chapter 5		
$V(\mathbf{r}_i)$	Social potential function of individual i .	143
\mathbf{F}	Force acting on an individual ($\mathbf{F} = -\nabla V$).	139
∇V	Direction differential operator on function V .	139
$V_{repulsion}(\mathbf{r}_i)$	Repulsion potential of individual i .	144
$V_{attraction}(\mathbf{r}_i)$	Attraction potential of individual i .	144
$V_{alignment}(\mathbf{r}_i)$	Alignment potential of individual i .	146
$\mathbf{F}_{repulsion}(\mathbf{r}_i)$	Repulsion force acting on individual i .	144
$\mathbf{F}_{attraction}(\mathbf{r}_i)$	Attraction force acting on individual i .	144
$\mathbf{F}_{cohesion}(\mathbf{r}_i)$	Attraction and repulsion model.	144
$\mathbf{F}_{alignment}(\mathbf{r}_i)$	Alignment force acting on individual i .	147
$\mathbf{F}_{dissipate}(\mathbf{r}_i)$	Dissipative force acting on individual i .	151
$\mathbf{F}_{total}(\mathbf{r}_i)$	Alignment model.	152
A	Repulsion magnitude parameter.	144
B	Repulsion range parameter.	144
C	Potential function parameter.	144
D	Attraction magnitude parameter.	144
E	Attraction range parameter.	144
F	Alignment magnitude parameter.	147
a	Alignment exponent parameter.	147
G	Friction coefficient.	151
ω_1	Ratio of cohesive magnitudes (A/D).	149
ω_2	Ratio of cohesive ranges (E/B).	149
r^*	Comfortable distance.	150

APPENDIX B

Proofs for selected results

We present mathematical proofs for selected results in this thesis.

Proof 1. *If A is any orthogonal matrix and Σ_y is a covariance matrix, then $\Sigma_y = A\Sigma_y A^T \Rightarrow \Sigma_y = kI_q$, where k is a constant.*

If

$$\Sigma_y = A\Sigma_y A^T$$

then

$$\Sigma_y A = A\Sigma_y \quad (\text{A is orthogonal, } A^{-1} = A^T).$$

Hence, Σ_y is a real $q \times q$ matrix that commutes with all orthogonal matrices.

Let $i, j = 1, 2, \dots, q$. The (i, j) th entry of Σ_y is σ_{ij} . Assume that $i < j$. Let the matrix A be the $q \times q$ identity matrix, except the (j, j) th entry is -1. The (i, j) th entry of $A\Sigma_y$ is σ_{ij} . The (i, j) th entry of $\Sigma_y A$ is $-\sigma_{ij}$. The (j, i) th entry of $A\Sigma_y$ is $-\sigma_{ji}$. The (j, i) th entry of $\Sigma_y A$ is σ_{ji} . But $\Sigma_y A = A\Sigma_y$, so $\sigma_{ij} = -\sigma_{ij}$ and $\sigma_{ji} = -\sigma_{ji}$. Hence $\sigma_{ij} = \sigma_{ji} = 0$ ($i < j$). Consequently, Σ_y is a diagonal matrix.

Now, let A^* be any orthogonal matrix whose (i, j) th entry is a_{ij} (where $a_{ij} \neq 0$). Let the (i, i) th element of Σ_y be denoted σ_i and the (j, j) th be σ_j . The (i, j) th element of $\Sigma_y A^*$ is $\sigma_i a_{ij}$. The (i, j) th element of $A^* \Sigma_y$ is $\sigma_j a_{ij}$. Hence $\sigma_i a_{ij} = \sigma_j a_{ij}$ and $\sigma_i = \sigma_j = \sigma$ ($\forall i, j = 1, 2, \dots, q$). So $\Sigma_y = \sigma I_q$.

□

Proof 2. *Justification for (2.33).*

Effectively, we want to show

$$\frac{|\mathbf{S}_N|}{N} = \frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} + \frac{1}{2\mathbb{E}(c)} \frac{|\mathbf{S}_N^\perp|^2}{N^2} + O(N^{-3/2}). \quad (\text{B.1})$$

Using (2.28), we can divide by a factor of $\sqrt{N} \left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)$ to obtain

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} - \mathbb{E}(c) &= \left(\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} - \mathbb{E}(c) \right) \frac{\left(\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right)}{\left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)} \\ &\quad + \frac{|\mathbf{S}_N^\perp|^2}{N^2} \frac{1}{\left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)}. \end{aligned} \quad (\text{B.2})$$

Firstly, let's consider the $\left(\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right) / \left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)$ term. We know $|\mathbf{S}_N|^2 = (\boldsymbol{\mu}^T \mathbf{S}_N)^2 + |\mathbf{S}_N^\perp|^2$. We can therefore write this term as

$$\begin{aligned} \frac{\left(\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right)}{\left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)} &= \left[\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right] \left[\frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} \sqrt{1 + \frac{|\mathbf{S}_N^\perp|^2}{(\boldsymbol{\mu}^T \mathbf{S}_N)^2}} + \mathbb{E}(c) \right]^{-1} \\ &= \left[\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right] \left[\frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} \left(1 + \frac{1}{2} \frac{|\mathbf{S}_N^\perp|^2}{(\boldsymbol{\mu}^T \mathbf{S}_N)^2} + \dots \right) + \mathbb{E}(c) \right]^{-1} \\ &= \left[\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right] \left[\frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} + \mathbb{E}(c) + \frac{1}{2} \frac{|\mathbf{S}_N^\perp|^2}{(\boldsymbol{\mu}^T \mathbf{S}_N) N} + \dots \right]^{-1} \\ &= 1 - \frac{|\mathbf{S}_N^\perp|^2}{2N} \frac{1}{(\boldsymbol{\mu}^T \mathbf{S}_N) \left[\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right]} + \dots \end{aligned} \quad (\text{B.3})$$

We know that $\frac{1}{N} |\mathbf{S}_N^\perp|^2 \xrightarrow{D} K \chi_{q-1}^2$ (Lemma 5), where $K = (1 - E(c^2)) / (q - 1)$. Hence, we can express $\frac{1}{N} |\mathbf{S}_N^\perp|^2 = O_p(1)$. Also, $\frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} \xrightarrow{p} \mathbb{E}(c)$. Consequently,

$$\frac{\left(\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + \mathbb{E}(c) \right)}{\left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)} = 1 + O_p(N^{-1}). \quad (\text{B.4})$$

We can write (B.2) as

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} - \mathbb{E}(c) &= \left(\frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} - \mathbb{E}(c) \right) \left(1 + O_p \left(\frac{1}{N} \right) \right) \\ &\quad + \frac{|\mathbf{S}_N^\perp|^2}{N^2} \frac{1}{\left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)}. \end{aligned} \quad (\text{B.5})$$

Now, using Lemma 3, $\frac{1}{\sqrt{N}} (\boldsymbol{\mu}^T \mathbf{S}_N - N\mathbb{E}(c)) \xrightarrow{D} N(0, \text{Var}(c))$. We can express $\frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} - \mathbb{E}(c) = O_p(N^{-\frac{1}{2}})$. Then

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} - \mathbb{E}(c) &= \frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} - \mathbb{E}(c) + O_p(N^{-3/2}) \\ &\quad + \frac{|\mathbf{S}_N^\perp|^2}{N^2} \frac{1}{\left(\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \right)}. \end{aligned} \quad (\text{B.6})$$

Now concentrate on the last term in the right hand side of (B.6). Recall $\frac{|\mathbf{S}_N|}{N} \xrightarrow{p} \mathbb{E}(c)$, to be more precise $\frac{|\mathbf{S}_N|}{N} = \mathbb{E}(c) + O(N^{-\frac{1}{2}})$ (Lemma 6). So $\frac{|\mathbf{S}_N|}{N} + \mathbb{E}(c) \xrightarrow{p} 2\mathbb{E}(c)$ (a positive quantity) at a rate $N^{-\frac{1}{2}}$. Hence, we can write

$$\begin{aligned} \frac{|\mathbf{S}_N|}{N} &= \frac{(\boldsymbol{\mu}^T \mathbf{S}_N)}{N} + O_p(N^{-3/2}) + \frac{|\mathbf{S}_N^\perp|^2}{N^2 2\mathbb{E}(c)} + O_p(N^{-\frac{1}{2}}) \frac{|\mathbf{S}_N^\perp|^2}{N^2} \\ &= \frac{\boldsymbol{\mu}^T \mathbf{S}_N}{N} + \frac{|\mathbf{S}_N^\perp|^2}{N^2 2\mathbb{E}(c)} + O_p(N^{-3/2}). \end{aligned} \quad (\text{B.7})$$

□

Proof 3. *Justification for (2.34).*

We want to show

$$\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right| = \boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k} + \frac{1}{2} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\sum_{k=1}^K w_k N_k \mathbb{E}(c_k)} + O_p(N^{-\frac{1}{2}}).$$

To do this, the version of Lemma 2 we require is

$$\frac{1}{\sqrt{N_k}} (\mathbf{S}_{N_k} - N_k \mathbb{E}(c_k) \boldsymbol{\mu}) \xrightarrow{D} N_q(\mathbf{0}, \Sigma_k) \quad (\text{B.8})$$

as $N_k \rightarrow \infty$. Let $N = \sum_{k=1}^K N_k$ and let $N_k = \alpha_k N$, where the α_k are positive constants that sum to 1. From (B.8), we can derive

$$\frac{1}{\sqrt{N}} \left(\sum_{k=1}^K w_k \mathbf{S}_{N_k} - \sum_{k=1}^K w_k N_k \mathbf{E}(c_k) \boldsymbol{\mu} \right) \xrightarrow{D} N_q(\mathbf{0}, \Gamma), \quad (\text{B.9})$$

where $\Gamma = \sum_{k=1}^K w_k^2 \alpha_k \Sigma_k$. A necessary condition for (B.9) to hold is $\min(N_k) \rightarrow \infty$. Equivalently,

$$\frac{1}{\sqrt{N}} \left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k} - N \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \xrightarrow{D} N(0, \boldsymbol{\mu}^T \Gamma \boldsymbol{\mu}), \quad (\text{B.10})$$

as $\min(N_k) \rightarrow \infty$. Now, let $\mathbf{S}_{N_k}^\perp$ be the component of \mathbf{S}_{N_k} orthogonal to $\boldsymbol{\mu}$ (the common mean direction under the null hypothesis). Hence, $\mathbf{S}_{N_k}^\perp = H \mathbf{S}_{N_k}$ and $\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp = H \sum_{k=1}^K w_k \mathbf{S}_{N_k}$, where $H = I_q - \boldsymbol{\mu} \boldsymbol{\mu}^T$. From Lemma 4,

$$\frac{1}{\sqrt{N_k}} (\mathbf{S}_{N_k}^\perp) \xrightarrow{D} N_q \left(\mathbf{0}, \frac{1 - \mathbf{E}(c_k^2)}{q-1} H \right),$$

as $N_k \rightarrow \infty$. So,

$$\frac{1}{\sqrt{N}} \left(\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp \right) \xrightarrow{D} N_q \left(\mathbf{0}, \sum_{k=1}^K w_k^2 \alpha_k \frac{1 - \mathbf{E}(c_k^2)}{q-1} H \right), \quad (\text{B.11})$$

as $\min(N_k) \rightarrow \infty$. The covariance matrix of $\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp$ is $q \times q$ and is of rank $q-1$. Consequently,

$$\frac{1}{N} \left(\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp \right|^2 \right) \xrightarrow{D} \left(\sum_{k=1}^K w_k^2 \alpha_k \frac{1 - \mathbf{E}(c_k^2)}{q-1} \right) \chi_{q-1}^2, \quad (\text{B.12})$$

as $\min(N_k) \rightarrow \infty$.

Now, using (2.27), it follows that

$$\frac{\sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} \xrightarrow{p} \boldsymbol{\mu} \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \quad (\text{B.13})$$

and

$$\frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right|}{N} \xrightarrow{p} \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k). \quad (\text{B.14})$$

Also notice

$$\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right|^2 = \left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right)^2 + \left| \sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp \right|^2. \quad (\text{B.15})$$

From (B.14),

$$\frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \xrightarrow{p} 2 \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k). \quad (\text{B.16})$$

Using (B.10)

$$\frac{1}{N} \left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right) - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \xrightarrow{p} 0, \quad (\text{B.17})$$

as $\min(N_k) \rightarrow \infty$. Hence,

$$\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \xrightarrow{p} 2 \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k). \quad (\text{B.18})$$

Also, from (B.12)

$$\frac{1}{N^{3/2}} \left(\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp \right|^2 \right) = O_p \left(N^{-\frac{1}{2}} \right), \quad (\text{B.19})$$

as $\min(N_k) \rightarrow \infty$.

Now, consider

$$\begin{aligned} & \sqrt{N} \left(\frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \left(\frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right|}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \\ &= \sqrt{N} \left(\frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right|^2}{N^2} - \left(\sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)^2 \right) \\ &= \sqrt{N} \left(\left(\frac{\left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k} \right)^2}{N^2} - \left(\sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)^2 \right) + \frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp \right|^2}{N^2} \right) \\ &= \sqrt{N} \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \\ & \quad + \frac{\left| \sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp \right|^2}{N^{3/2}}. \end{aligned} \quad (\text{B.20})$$

Taking limits in (B.20), using (B.16), (B.18) and (B.19), we can see that the limiting distribution of $\sqrt{N} \left(\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)$ is the same as that of $\sqrt{N} \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)$. By (B.10), this is

$$\sqrt{N} \left(\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \xrightarrow{D} N(0, \boldsymbol{\mu}^T \Gamma \boldsymbol{\mu}), \quad (\text{B.21})$$

as $\min(N_k) \rightarrow \infty$.

If we divide (B.20) by a factor of $\sqrt{N} \left(\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)$, then

$$\begin{aligned} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) &= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \\ &\times \frac{\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)}{\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)} \\ &+ \frac{1}{N^2} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)}. \end{aligned} \quad (\text{B.22})$$

Consider the ratio in (B.22)

$$\begin{aligned} \frac{\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)}{\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)} &= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \\ &\times \left(\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)^{-1}. \end{aligned}$$

We can manipulate the second term on the right hand side of this equation

$$\begin{aligned}
& \left(\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)^{-1} \\
&= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} \sqrt{1 + \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}\right)^2}} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)^{-1} \\
&= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) + \frac{1}{2N} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}} + \dots \right)^{-1} \\
&= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right)^{-1} \\
&\times \left(1 - \frac{1}{2N} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}\right) \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)\right)} + \dots \right).
\end{aligned}$$

Using this expression, with (B.12), (B.18) and realising from (B.17) that $\left(\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}\right)^{-1} = O_p(N^{-1})$, as $\min(N_k) \rightarrow \infty$, we can express the ratio as

$$\frac{\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)}{\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)} = 1 + O_p(N^{-1}), \quad (\text{B.23})$$

as $\min(N_k) \rightarrow \infty$. We can now express (B.22) as

$$\begin{aligned}
\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) &= \left(\frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} - \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) \\
&\times (1 + O_p(N^{-1})) \\
&+ \frac{1}{N^2} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)}.
\end{aligned} \quad (\text{B.24})$$

Hence, (B.24) can be written as

$$\begin{aligned} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} &= \frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + O_p(N^{-3/2}) \\ &+ \frac{1}{N^2} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k)}, \end{aligned} \quad (\text{B.25})$$

due to (B.10). Consider

$$|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2 / \left(\frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} + \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right) N^2.$$

By (B.16), this converges to the same limit as

$$|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2 / \left(2N^2 \sum_{k=1}^K w_k \alpha_k \mathbf{E}(c_k) \right).$$

Therefore, (B.25) can be written as

$$\begin{aligned} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}|}{N} &= \frac{\boldsymbol{\mu}^T \sum_{k=1}^K w_k \mathbf{S}_{N_k}}{N} + \frac{1}{2N} \frac{|\sum_{k=1}^K w_k \mathbf{S}_{N_k}^\perp|^2}{\sum_{k=1}^K w_k N_k \mathbf{E}(c_k)} \\ &+ O_p(N^{-3/2}). \end{aligned} \quad (\text{B.26})$$

Multiply (B.26) by a factor of N and we have (2.34). \square

Proof 4. *Elaboration on $2T$ (2.38) being distributed as a $\chi_{(K-1)(q-1)}^2$ random variable.*

Let $\mathbf{y}_r = \sum_{s=1}^K a_{rs} \mathbf{z}_{N_s}$, where the \mathbf{z}_{N_s} 's are independent and $\mathbf{z}_{N_s} \sim N_q(0, H)$ with $H = I_q - \boldsymbol{\mu} \boldsymbol{\mu}^T$ and $A = (a_{rs})$ is orthogonal. We define $Z = (\mathbf{z}_{N_1}, \dots, \mathbf{z}_{N_K})^T$. Then $Y = (\mathbf{y}_1, \dots, \mathbf{y}_K)^T = AZ$. Hence, $\mathbf{y}_r \sim N_q(0, H)$.

The last row of A is $(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_K})$ (as $\lambda_1 + \lambda_2 + \dots + \lambda_K = 1$). Now,

$$\begin{aligned}
\mathbf{y}_K &= \sum_{r=1}^K \sqrt{\lambda_r} \mathbf{z}_{N_r}, \\
\sum_{r=1}^K |\mathbf{y}_r|^2 &= \sum_{r=1}^K \mathbf{y}_r^T \mathbf{y}_r \\
&= \text{Trace}(\mathbf{Y}\mathbf{Y}^T) \\
&= \text{Trace}(\mathbf{A}\mathbf{Z}\mathbf{Z}^T\mathbf{A}^T) \\
&= \text{Trace}(\mathbf{A}^T\mathbf{A}\mathbf{Z}\mathbf{Z}^T) \\
&= \text{Trace}(\mathbf{Z}\mathbf{Z}^T) \\
&= \sum_{r=1}^K |\mathbf{z}_{N_r}|^2, \tag{B.27}
\end{aligned}$$

$$\begin{aligned}
2T &= \sum_{r=1}^K |\mathbf{z}_{N_r}|^2 - \left| \sum_{r=1}^K \sqrt{\lambda_r} \mathbf{z}_{N_r} \right|^2 \\
&= \sum_{r=1}^K |\mathbf{y}_r|^2 - |\mathbf{y}_K|^2 \\
&= \sum_{r=1}^{K-1} |\mathbf{y}_r|^2. \tag{B.28}
\end{aligned}$$

Now, the \mathbf{y}_r vectors are independent and each one has $|\mathbf{y}_r|^2 \sim \chi_{q-1}^2$ by Lemma 5. Consequently, $\sum_{r=1}^{K-1} |\mathbf{y}_r|^2 \sim \chi_{(K-1)(q-1)}^2$ (sum of $(K-1)$ independent χ^2 random variables). \square

Proof 5. *Proof of degeneration of the Fisher distribution to a spherical Uniform distribution, as κ decreases.*

Notice

$$\begin{aligned}
\lim_{\kappa \rightarrow 0} C_F &= \lim_{\kappa \rightarrow 0} \frac{\kappa}{2\pi(e^\kappa - e^{-\kappa})} \\
&= \frac{1}{2\pi} \lim_{\kappa \rightarrow 0} \frac{1}{e^\kappa + e^{-\kappa}} \quad (\text{L'Hospital's Rule}) \\
&= \frac{1}{4\pi}.
\end{aligned}$$

Hence, Equation 3.6 becomes

$$f(\theta, \phi) = \frac{1}{4\pi} \sin \theta.$$

This is the probability density function of a spherical Uniform distribution (Fisher et al. 1987). \square

